Critical Thinking in Mathematics: Perspectives and Challenges

Editors

Bożena Maj-Tatsis
University of Rzeszow
Rzeszów, Poland

Konstantinos Tatsis
University of Ioannina
Ioannina, Greece

Wydawnictwo Uniwersytetu Rzeszowskiego
2021

Reviewers

Jenni Back
Edyta Juskowiak
Eszter Kónya
Eva Novakova
João Pedro da Ponte
Christof Schreiber
Lambrecht Spijkerboer
Ewa Swoboda
Michal Tabach
Konstantinos Tatsis
Paola Vighi

Cover Artwork

Aftersounds of (un)defined futures, 2021
by Andreas Moutsios-Rentzos and Ioanna Kloni

Layout Design

Bożena Maj-Tatsis
Konstantinos Tatsis

ISBN: 978-83-7996-903-6

Nakład: 150 egz.

TABLE OF CONTENTS

INTRODUCTION

Critical thinking in mathematics encompasses a wide spectrum of phenomena ranging from mathematical problem posing and solving to creative thinking, abstract thinking and mathematical reasoning. The different manifestations of critical thinking pose a significant challenge to the establishment of a commonly agreed definition. However, all definitions of critical thinking contain not only aspects or habits of thinking, but also a disposition from the thinker's side. It is worth noting that although critical thinking has a prominent position in general education studies, it is rarely referred to as such in mathematics education studies. This has been an additional challenge for suggesting, preparing and editing the current volume, which is entitled *Critical thinking in mathematics: Perspectives and challenges*. The challenge came not only from the scarcity of relevant studies, but also from the diversity of the theoretical approaches which are related to – or claim that are related to – critical thinking in mathematics. This diversity is clearly demonstrated in the chapters that constitute this volume and attempt to offer a glimpse to critical thinking in mathematics, focusing either on the students or on the teachers of mathematics.

The works are categorised in three sections. The first section, entitled *Critical thinking guiding teachers' actions in the classroom*, presents works that focus mostly on preservice and inservice teachers' views and actions towards enhancing their own or their students' critical thinking skills; it also contains a chapter on adults' knowledge of children's numerical abilities. The second section, entitled *Critical thinking guiding students' actions in the classroom*, presents works that focus on analysing students' critical thinking skills, usually in relation to a learning environment designed by the teacher. The third section, entitled *Critical thinking guiding students' actions in the early years*, presents works that focus on manifestations of critical thinking in the preschool years. The volume begins by a chapter on critical thinking in mathematics, which aims to introduce the readers to the current state of the field, and also suggests possible directions for future research.

Rzeszów, Poland, June 2021

The Editors

CRITICAL THINKING IN MATHEMATICS: PERSPECTIVES AND CHALLENGES

Bożena Maj-Tatsis*, Konstantinos Tatsis**

* University of Rzeszow, Poland

** University of Ioannina, Greece

CRITICAL THINKING IN THE 21ST CENTURY

Modern societies require particular skills from the citizens who wish to lead a successful life in a world which is becoming increasingly complex and unpredictable. The unpredictable and unstable state of the world has been highlighted by events such as the financial crises of the last couple of decades and, more recently, by the CoViD-19 pandemic, which has hit every part of the world causing numerous deaths, but mostly leading countries to decisions which, sometime ago, would seem outrageous, such as curfews, strict travel restrictions and online or hybrid education at all levels (for the effect of the pandemic in mathematics education research, see, for example, Bakker, Cai, & Zenger, 2021). Moreover, a big 'wave' of misinformation, disinformation or false interpretations of data has outlined many people's (including policy makers) limited ability to handle and interpret numerical data.

Among the skills needed to deal with such situations – which are sometimes called '21st century skills' – we may identify critical thinking as one of their basic components. According to this view, critical thinking is placed next to skills such as information search and organisation, effective communication and social responsibility (Ananiadou & Claro, 2009). Due to the variety of skills related to it, there is no commonly agreed-upon definition of critical thinking. Ennis (1989) defines critical thinking as a "reasonable reflective thinking focused on deciding what to believe or do" (p. 4). Usually, critical thinking can be triggered by a problem to be solved (or even by identifying and posing a problem that needs to be solved); then, through a process that includes reasoning, the identification and implementation of appropriate tools or methods, the person is led to a solution of the problem and/or to a decision concerning the matter at hand. At the end, the person is expected to examine the proposed solution on the grounds of its suitability to address the initial problem. Although these activities may resemble the stages of problem solving (Schoenfeld, 1985) and modelling (Lesh & Doerr, 2003), critical thinking penetrates these stages and gives them a more reflective nature.

Due to the high-level mental activities involved, critical thinking is sometimes treated as equivalent to high-order thinking (Paul, 1995). In other studies, critical thinking is viewed as a more general competence, which includes components such as metacognition (including the ability to challenge one's own beliefs), intellectual perseverance and autonomy (including the freedom from egocentric and sociocentric thought), reasoning (including reflective thinking and the effective use of concepts) and the ability to identify inconsistencies and contradictions (including one own's) (Paul & Elder, 2002). Therefore, critical thinking can assist a person in his/her decision making, not only by providing the conceptual tools and the disposition to critically assess others' views, but also by providing the person with a disposition to reflect upon, challenge and, if needed, modify their own views.

The significance of critical thinking has subsequently led to efforts to include it in teaching in a straightforward manner. Such efforts have in turn led to the examination of teachers' abilities and dispositions towards critical thinking. The results have been rather mixed. Some studies report a surface implementation of a critical thinking approach (Golding, 2006), while others claim that for some teachers, critical thinking was viewed "as another sort of thinking like scientific thinking, logical thinking, problem solving, and creative thinking" (Innabi & Sheikh, 2007, p. 55).

Another issue that has emerged is whether critical thinking is – or should be – connected to particular scientific fields; for instance, whether critical thinking in sociology has inherently different characteristics than critical thinking in mathematics. This issue is multifaceted and has been thoroughly examined by scholars such as Ennis (1989), who identified at least three views on how to study the possibility of transfer of critical thinking skills from one domain to another. The first view claims that without having subject matter knowledge, one cannot think critically within that subject. Ennis's (1989) critique to this view is that, sometimes, subject matter knowledge is formed on rote-memorised facts, which deems critical thinking impossible. Secondly, what constitutes a particular domain can be a problematic issue, since the borders between particular domains are not clear. Therefore, this view is not fully justified. The second view focuses on the epistemological specificities of each domain; one may consider, for example, the differences between mathematics, social sciences and the arts. Therefore, we may agree on the fact the critical thinking varies from field to field. According to the third view "it does not even make sense to speak of critical thinking or critical thinking instruction outside of a subject-matter area" (Ennis, 1989, p. 8).

Although we may agree on the fact that critical thinking is thinking *about* something, this does not exclude the possibility that one might possess or be taught a general critical thinking ability. Following the above views, we subsequently discuss the specificities of critical thinking in mathematics.

CRITICAL THINKING IN MATHEMATICS

Features of critical thinking in mathematics

The discussion on the specific characteristics of critical thinking in particular domains, including mathematics, is still undergoing. This is manifested in a number of studies aiming to investigate critical thinking among students or teachers (e.g., Innabi & Sheikh, 2007). In such studies we may identify three components of critical thinking in mathematics, namely: *reasoning*, *problem posing and problem solving*, and *identifying the suitability of problem solutions*.

As we already mentioned, reasoning in general is considered a component of critical thinking. Within mathematics, reasoning includes the ability to construct and validate logical arguments by the use of mathematical concepts and procedures. This is sometimes opposed to the simple following of mathematical rules. Skemp (1978) has used the terms instrumental and relational understanding to describe the situations in which the students are either asked to implement the rules provided by the teacher, without questioning them (instrumental) or are requested to search for the underlying relationships among the procedures and the rules they follow (relational). Cobb, Wood, Yackel and McNeal (1992) have built upon these notions and compared two classes, which were following two different approaches or traditions: the school mathematics tradition and the inquiry mathematics tradition. The latter is related to students providing explanations and justifications of their own solutions to the rest of the class. We may claim that these activities are close to critical thinking in mathematics.

The next component of critical thinking in mathematics is posing and solving mathematical problems; these activities are interconnected and entail creativity and mathematical reasoning. Problem solving is realised in different phases which have been amply described and analysed in the literature (see, e.g., Schoenfeld, 1992). Additionally, the problem solver is expected to analyse the data provided, which may sometimes contain contradictory or inconsistent information (Applebaum & Leikin, 2007). Therefore, the problem solver needs to be able to discern data and then take a decision on the solution method; then, as soon as a solution is obtained, it should be examined concerning its suitability for the problem. This takes us to the third component, namely the identification of the feasibility of proposed solutions. This includes the ability to discern reasonable/realistic from non-reasonable/non-realistic solutions, since students engaged in problem solving do sometimes accept non-realistic solutions (Palm, 2008; Yoshida, Verschaffel, & De Corte, 1997), therefore demonstrating an absence of critical thinking.

To all these we could add metacognition, which is related to these components, especially to problem solving (Schoenfeld, 1992), and is usually viewed as the problem solver's ability to 'step back' and reflect on his or her own solution processes. The close relationship of metacognition and critical thinking with the problem solving phases is manifested in most relevant publications. A characteristic example comes from Yimer and Ellerton's (2010) work, in which the authors refer to five phases, namely engagement, transformation-formulation, implementation, evaluation and internalization. Among these, the last two contain elements which are characteristic of critical thinking in mathematics. Particularly, the evaluation phase contains the following actions:

> A. Rereading the problem whether the result has answered the question in the problem or not
>
> B. Assessing the plan for consistency with the key features as well as for possible errors in computation or analysis
>
> C. Assessing for reasonableness of results
>
> D. Making a decision to accept or reject a solution
>
> (Yimer & Ellerton, 2010, p. 250)

The internalization phase contains the following actions:

> A. Reflecting on the entire solution process
>
> B. Identifying critical features in the process
>
> C. Evaluating the solution process for adaptability in other situations, different ways of solving it, and elegance
>
> D. Reflecting on the mathematical rigor involved, one's confidence in handling the process, and degree of satisfaction
>
> (Yimer & Ellerton, 2010, p. 250)

Based on the above considerations, we may see the close relationship between problem solving actions, especially those related to self-reflection, and critical thinking. In the next section we will discuss how these features of critical thinking can be implemented in mathematics education.

Teaching and learning critical thinking in mathematics

The specificities of mathematics as a domain, which mostly come from the importance of reasoning based on logical arguments, provide critical thinking in mathematics with some distinctive characteristics. However, if we consider critical thinking as a more general competence, which aims to assist the modern citizen in his everyday decision making, we need to consider whether critical thinking in mathematics is at all possible without following an interdisciplinary approach or an approach based on Realistic Mathematics Education (e.g., Freudenthal, 1973; 1983). Within Realistic Mathematics Education the role of contexts becomes vital, thus leading educators to design authentic tasks, which are supposed to closely resemble real life situations. At the same time, the engagement of students in authentic tasks may not always be sufficient to elicit students' reasoning and sense making. Many students seem to be much affected by the didactical contract which is predominant in the mathematics classroom and this affects their decisions and leads them to superficial solution methods and 'unrealistic' solutions (Palm, 2008); in other words, to a suspension of critical thinking. Therefore, a realistic mathematics approach is not sufficient by itself to ensure critical thinking among students.

The close relationship between critical thinking and mathematical reasoning may suggest another way to insert critical thinking in mathematics teaching, namely by implementing a teaching approach based on understanding, contrary to an approach focused on the rote memorisation of facts, rules and procedures. This is possible, as we mentioned before, by implementing an inquiry approach to mathematics teaching (Cobb et al., 1992). There are two issues worth mentioning at this point. The first is whether an inquiry approach leads to the establishment of critical thinking among students in mathematics and the second is whether teachers are able to implement such an approach in their mathematics classes. Concerning the first issue, if we accept that critical thinking contains not only skills, but also dispositions, it is reasonable to assume that the implementation of an inquiry teaching approach for an extended period, may affect students' dispositions on their ways of thinking and acting in mathematics. Concerning the second issue, we have seen examples of a successful implementation of inquiry in mathematics by the teachers (Johnson, 2013; Rasmussen & Kwon, 2007), but we have also seen examples of teachers' inadequate knowledge of what constitutes critical thinking in mathematics, which in turn results in superficial implementation of a critical thinking approach in the classroom (Innabi & Sheikh, 2007).

DISCUSSION

The evidence presented in the chapter, together with the content of this volume, suggest that critical thinking in mathematics is complex and could be taught mainly by infusing it in the mathematics subject. Specific strategies, such as questioning and classroom dialogue play an important role in activating students' critical thinking skills; the role of tasks is also vital, especially interdisciplinary and authentic ones. Therefore, we may identify two main directions of future research in the field, some of which are partially covered in the current volume:

a) Studies on further specifying the distinctive characteristics of critical thinking in mathematics and how these are related to other disciplines or to critical thinking in general; it is important to not limit critical thinking studies in 'pure' mathematics classes, but to try to embed it in interdisciplinary projects, which require decision making based on data interpretation and manipulation. Such studies may focus on students' practices in particular mathematics classrooms (or educational systems) or attempt to identify commonalities among students' practices in different contexts.

b) Studies on the teachers' views, skills and practices (including task design) that enhance students' critical thinking skills in mathematics; such studies might involve the examination of the effect of different approaches (general, infusion, immersion, mixed). Additionally, studies on the effect of training courses or educational policies focused on the enhancement of critical thinking skills among teachers are also needed.

At the same time, we may refer to some issues or limitations related to critical thinking and its implementation in mathematics education. The first limitation comes from viewing critical thinking in mathematics merely as the "enculturation into dispassionate reason and analysis" (Jablonka, 2014, p. 123), therefore, not leaving space for imagination and social or political considerations of realistic mathematical problems. In that case, critical thinking comes with a negative connotation and an apathy for society – therefore it opposes its very social and responsible citizenship nature. These considerations are related to the risk of over-reliance on mathematical models and processes for dealing with problems, without any consideration – or even critique – of the models themselves, e.g., on their political or ethical aspects (see, e.g., Keitel, Kotzmann, & Skovsmose, 1993).

The second limitation comes from the approaches which – due to practical, educational or simplicity reasons – present critical thinking as merely a list of cognitive and metacognitive skills which are to be acquired by the students. Such approaches "run the risk of suggesting to treat these explicitly as learning objectives, including the assessment of the extent to which individual students use them" (Jablonka, 2014, p. 124). This is a serious risk since such limited views of critical thinking devaluate any significant attempt to infuse it in mathematics teaching and learning.

Summing up, we believe that critical thinking should penetrate mathematics teaching and learning at all educational levels, because it is a basic component of the skills required by the citizens of the modern, highly complex and everchanging world. Teaching critical thinking is a demanding endeavour, which can be realised by infusing it in the mathematics classes through authentic and eventually interdisciplinary tasks, but also by designing 'context-free' tasks, especially structured in order to foster critical thinking among students. Teachers and policy makers need to become aware of the special nature of critical thinking, therefore the need of special training and professional development courses is more than apparent.

References

Ananiadou, K., & Claro, M. (2009). 21st century skills and competences for new millennium learners in OECD countries. *Organisation for Economic Cooperation and Development. EDU Working paper no. 41*. Retrieved from http://www.olis.oecd.org/olis/2009doc.nsf/linkto/edu-wkp(2009)20.

Applebaum M., & Leikin, R. (2007). Looking back at the beginning: Critical thinking in solving unrealistic problems. *The Mathematics Enthusiast, 4,* 258–265.

Bakker, A. & Cai, J., & Zenger, L. (2021). Future themes of mathematics education research: An international survey before and during the pandemic. *Educational Studies in Mathematics, 107,* 1–24.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29,* 573–604.

Ennis, R. H. (1989). Critical thinking and Subject Specificity Clarification and Needed Research. *Educational Researcher, 18,* 4–10.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, The Netherlands: Reidel Publishing.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht, The Netherlands: Reidel Publishing.

Golding, C. (2006). From a thinking curriculum to thinking schools. *Teacher Learning Network, 13,* 3–5.

Innabi, H., & Sheikh, O. E. (2007) The Change in Mathematics Teachers' Perceptions of Critical thinking after 15 Years of Educational Reform in Jordan. *Educational Studies in Mathematics, 64,* 45–68.

Jablonka, E. (2014). Critical thinking in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 121–125). Dordrecht, the Netherlands: Springer.

Johnson, E. (2013). Teachers' mathematical activity in inquiry-oriented instruction. *Journal of Mathematical Behavior, 32,* 761–775.

Keitel, C., Kotzmann, E., & Skovsmose, O. (1993). Beyond the tunnel vision: Analyzing the relationship between mathematics, society and technology. In C. Keitel & K. Ruthven (Eds.), *Learning from computers: Mathematics education and technology* (pp. 243–279). New York, NY: Springer.

Lesh, R. A., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning and problem solving. In R. A. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective on teaching, learning, and problem solving in mathematics education* (pp. 3–33). Mahwah, NJ: Lawrence Erlbaum Associates.

Palm, T. (2008). Impact of authenticity on sense making in word problem solving. *Educational Studies in Mathematics, 67,* 37–58.

Paul, R. (1995). *Critical thinking: How to prepare students for a rapidly changing world.* California, CA: Foundation for Critical thinking.

Paul, R., & Elder, L. (2002). *Critical thinking: Tools for Taking Charge of Your Professional and Personal Life.* New Jersey, NJ: Financial Times Prentice Hall.

Rasmussen, C., & Kwon, O. (2007). An inquiry oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior, 26,* 189–194.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York, NY: Macmillan.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26,* 9–15.

Yimer, A., & Ellerton, N. F. (2010). A five-phase model for mathematical problem solving: Identifying synergies in pre-service-teachers' metacognitive and cognitive actions. *ZDM - The International Journal on Mathematics Education, 42,* 245–261.

Yoshida, H., Verschaffel, L., & De Corte, E. (1997). Realistic considerations in solving problematic word problems: Do Japanese and Belgian children have the same difficulties? *Learning and Instruction, 7,* 329–338.

Critical thinking guiding teachers' actions in the classroom

Part 1

TEACHING STRATEGIES FOR DEVELOPING CRITICAL THINKING SKILLS

Emőke Báró

University of Debrecen, Hungary

In recent years, it has been shown in several studies that pupils' interest in mathematics has been decreasing. As a math teacher, the author feels it is her responsibility to change something about this situation. The first steps to achieve this aim are based on holding some problem-based activities. In this paper, the author will discuss teaching methods as well as problem-solving strategies concerning two different activities. Pupils were solving problems using different heuristic methods, such as working backwards and pattern recognition. The observations on the classes were analyzed by different factors, such as cooperative learning, mathematical game, problem posing, and difficulties of teaching.

INTRODUCTION

The traditional way of teaching mathematics in most Transylvanian public primary schools is the teacher-centered approach. This "well-used method" usually involves repetition and memorization of previously taught material. Teachers sometimes try to fill students' minds with the knowledge of mathematics without explaining in detail the process of analyzing, evaluating, and arriving at a conclusion.

In recent years, it has been noticeable that pupils' interest in mathematics has been decreasing (Midkiff, R. B. & Thomason, R. D, 1994). Since the roots of the decline in the mathematical interest of young people are to be found mainly in the methods of teaching these subjects, therefore, this research focuses on teaching strategies and emphasis on teaching specific heuristic strategies through problem-centered curriculum processing. We analyze student and teacher behavior, students' critical thinking, and their attitude towards mathematics.

The basic principles of teaching and learning by George Pólya (1981) are the ones that I relied on: the principle of active learning, the best motivation, and consecutive phases.

According to Pólya, learning should be active. We cannot learn just by reading. We cannot learn just by listening to lectures, neither by watching. We all must add the action of our mind to learn something new. The principle of active learning says that a student learns by his or her actions. The most crucial action of learning is to discover something by oneself. One of the essential tasks for a teacher is to help their students in the process of discovery.

The best motivation principle means that for efficient learning, the learner should be interested in the material to be learned and find pleasure in the activity of learning. Nevertheless, besides these best motives for learning, there are other motives too, some of them desirable, but punishment for not learning may be the least desirable motive.

Pólya writes on the principle of consecutive phases that "learning begins with action and perception, proceeding from words and concepts, and should end in desirable mental habits." (Pólya, 1981, p. 103)

Problems and problem solving

In mathematics education, a problem is a task that requires the application of an unknown combination of tools or a novel combination of several known tools to solve a problem, and is not obvious to the problem solver (Claus, 1989; Dörner, 1983). The use of mathematical problems in mathematics education can be achieved through a variety of educational strategies. In this paper, we use the approach described by Csíkos (2010), who defines problem-based learning in mathematics as requiring students to analyze mathematical problem situations, to approach their own and their peers' minds critically, and they must learn to explain and justify their reasoning. The implementation of problem solving should take into account these principles, that is, not only create problem situations in the classroom but also allow for reasoning and discussion. Cooperative methods and the age-appropriate game as a teaching strategy provide an excellent opportunity for this.

Critical thinking

In the above interpretation, the critical attitude and thinking towards one's thoughts and that of their peers appear. Because critical thinking is a complex concept that involves cognitive skills and affective dispositions, we can find many definitions for it. Critical thinking may also involve logical reasoning and ability to separate facts from opinion, examine information critically with evidence before accepting or rejecting ideas and questions concerning the issue at hand. In other words, it makes individuals think, question issues, challenge ideas, generate solutions to problems, and make intelligent decisions when faced with challenges (Semil, 2006).

Critical thinking also involves deep reasoning and consideration of what we have received rather than forward acceptance of different ideas (Mansoor & Pezeshki, 2012).

Facione (1990) named six cognitive skills as central to the concept of critical thinking: Interpretation, Analysis, Explanation, Evaluation, Self-regulation, and Inference. Critical thinking skills, therefore, are skills that enable one to analyze and synthesize information to solve problems in a broad range of areas (Facione, 1990).

Concerning the problem solving, the research focused on learning and teaching heuristic strategies, including pattern tracking and searching, and thinking backwards through a strategy game.

RESEARCH QUESTIONS

Daily, people are faced with decisions that require reasoning, understanding, interpreting, analyzing information, and evaluating them too. These processes involve problem-solving, more critical thinking because it would enable one to make reliable and valid decisions, be able to adapt to changes in any given environment. Without question, it is recommended that teaching mathematics in primary schools, problem-solving and critical thinking skills should be infused in the curriculum.

This leads us to the following questions:

1) What teaching strategies can be used to integrate problem solving and various heuristic strategies into mathematics lessons?
2) Do educational strategies used in problem solving in our experiments, such as cooperative methods and game, contribute to the development of mathematical skills and critical thinking.

THE CIRCUMSTANCES OF THE EXPERIMENT

Participants comprised 50 seventh-grade students and 59 fifth-grade students from a Transylvanian primary school in Romania. All of them had Hungarian as their mother language.

The teacher examined her own educational practice systematically and carefully, and reflecting upon what just happened, so the teacher was herself the researcher too, delivering an action research. In order to document the experiment, different research instruments were used. The lessons were recorded by a camera, that helped the teacher in reflecting and analyzing. A written record was also made by the teacher after each lesson, every worksheet filled out by the students was photographed and their notebooks were scanned too.

Class	Number of participants	Boys	Girls	Heuristic strategy	Topic
V. A.	30	14	16	Thinking backwards	Winning strategy
V. B.	29	13	16		
VII.A.	27	12	15	Patterning	Co-variant quantities
VII.C.	24	16	8		
Total	110	55	55		

Table 1: Participants

The classes were held as part of a larger research project in which researchers at the University of Debrecen, Hungary studied the teaching of heuristic strategies in school settings (Kónya & Kovács, 2018; Kovács & Kónya, 2019). The lessons were based on learning of the two heuristic strategies mentioned above: covariant quantities as pattern tracking and pattern searching, respectively thinking backwards as a strategy game. Problem-centered learning and teaching were implemented as the pursuit of the activities was tailored to the curriculum requirements, to develop the concept of winning strategy in fifth grade, and the function concept in seventh grade. The lessons were organized through three core teaching strategies: cooperative technique Think-Pair-Share (TPS), using games, and application of students' problem-posing activity. The sketches of both lessons are divided into three main units: Immersion phase, Main activity, Problem Posing.

Immersion phase

The warm-up type of game prepared the primary heuristic strategy. In the fifth grade, the class started with a numerical game: The teacher thinks of a number and does some simple arithmetic operations on it, and then tells the result. One has to figure out the number. In seventh grade, the goal was to come up with a specific rule according to which a function machine is working. (Eureka sequences in Mason, p. 91): The teacher writes down a rule, which generates three-element sequences of numbers and provides one sample sequence satisfying the rule. The students offer three-element sequences and are given yes/no responses according to whether they do or do not satisfy the rule. All offerings are displayed. The student who guesses the rule wins.

Main activity

The main activity of the class was playing a NIM-like game in fifth grade and folding strips of paper in the seventh-grade.

The game introduced in the fifth grade is known as the subtraction game, which is the part of a broader class of games known as "Nim Games." Start with a pile of 11 of tokens (or any other collection of similar objects). Two players take turns removing 1, 2, or 3 tokens from the pile. The winner of the game is the person who takes the last token.

In the seventh grade, the folding of paper strips was the main part of the lesson. Take a long strip of paper and fold it in half from right to left. When it is opened, it has one crease and two rectangles. Fold the paper in half two times from right to left. When it is opened, it has three creases, four rectangles. After n folding operations, how many rectangles are formed, and how many creases are formed? (Mason, Burton, & Stacey, 2010)

The primary cooperative method that was used by the TPS (Think-Pair-Share) (Kagan, 1994) was, in which students think through questions using three distinct steps, encouraging individual participation. The steps: 1. Think: Students think independently of the question that has been posed, forming ideas of their own. 2. Pair: Students are grouped in pairs to discuss their thoughts. This step allows students to articulate their ideas and consider those of others. 3. Share: Student will share their ideas with a larger group, such as the whole class. Often, students are more comfortable when presenting ideas to a group with the support of a partner. The method is excellent for promoting critical thinking and communication in the classroom.

Problem posing

Understanding can be evaluated by asking students to write their mathematical problems. When a student writes a problem about their own, it helps to uncover what they know, understand, and value in the mathematical topic to which their problem addresses. Students were asked to write their math problems using the base-problem: Nim-like game (5^{th} grade) or ticker tapes (7^{th} grade).

EXPERIENCE AND DISCUSSION

Experiences regarding the game

Analyzing the warm-up game, we can see that the observations differ between the two grades. While fifth-grade students were active players from the first minute, the bigger ones received with precaution the game-like math because "either we study math or we play" – they said, if they were exclusive "or" existent in that regard. The game itself, or the game-like discovery, left only a good impression to say because the kids are very receptive to it. Although it is harder for us to bring them back from the game to the frontal conversation in fifth grade, the game was a decisive factor in the positive experience.

The most important thing to note after teaching this kind of math lesson is that everyone is given the math experience, even for the weaker students. They were happy to come forward, add their thoughts to their peers, and they were pleased to correct their classmates if they had the opportunity. The most striking concrete example of this is a fifth-grade student called S4. He was not one of the distinguished students in the fifth grade, and he was beginning to lose all mathematics when there was the lesson with winning strategy theme. S4 was so impressed with the game that on Thursday's math lesson decided to see what would happen with more tokens than one as it was in one of its tasks. On Monday, he proudly brought the 100-token game table to school, and the teacher noticed that he has been paying much more attention to mathematics lessons ever since.

The ending sticky notes also prove that the students were having fun. In the fifth grade, students made a brief point of view, and in the seventh grade, they had to use a smiley expression to express their feelings for the lesson. In fifth grade, for example, we received the following comments: "I liked the game lesson. We should do this several times.", "Let's have another game lesson." These sticky notes suggest that the kids thought we were just playing all day, so they had fun, but in the next few lessons, they found out that they had very good learned the basics of backward thinking and could use it correctly in solving mathematical problems.

In seventh grade, the feedback was mostly positive, with smiling, laughing faces appearing on the leaving notes.

Regardless of grade and class, we can say that many more students are active in these lessons than an average math class, and more people are engaged in the subject than usual. The reason was that students who usually are passive were involved in these classes. This manifestation, of course, had the consequence that students with good grades in mathematics were marginalized; or were not given as many words as in a regular math class. If we look at this phenomenon from the point of view of critical thinking, it may even suggest that "good learners" may have not necessarily well at critical thinking.

Experience regarding TPS

Thanks to the TPS method, all students were actively engaged, communicating with their peers, and since everyone had more opportunities to express their ideas and opinions than in a traditional lesson, and they felt much more liberated and better, something that revealed from their moves, reactions and the ending sticky notes.

During the TPS method, the teacher became aware of a special phenomenon in both classes of seventh grades (VII.A and VII. C). During the couple discussion phase of the method, there were both a right and wrong solution in the problem resolving of two students, the "good student" gave the wrong solution, and the "weak student" was the right one. In one class, there was a minor dispute between the two students, and in the other class, they quickly agreed. In both cases, by the end of the pair discussion, the wrong solution was accepted by the couples, as the "good student" managed to convince his partner that he was right. In this situation, critical thinking failed because the fact that a partner has a better mark in mathematics could not be separated from his own opinion, but became an influencing factor, and the review and analysis phase was missing. During the class discussion, the teacher clarified the situation, and when the right solution came out, one of the students of the couple remarked, "Well, do you see that I was right?" With this statement, we have closed this problem, which has come into its own since another positive feature of the TPS method is that if the first two phases slip away, it can be repaired in the third phase.

The TPS method places great emphasis on dialogue and classroom discussion, in which students' critical thinking and expression can be followed.

In fifth grade, a student thought as follows:

S1: Whoever has five tokens, has lost anyway, because if he takes one, four tokens remain, and then the other takes three. If he takes two, three tokens remain, and the other takes two. If he takes three, two remain, the other will take one.

When the teacher asked him from which point he started, he could not justify it. It was clear to him that with first in the case of five tokens had to be discussed the situation, not for one, as in the table asked we, math teachers. Because "who wants to play with one token?"

This issue also means that the information can be processed up to five tokens and linked to a situation treated with common sense, meaning critical thinking has been realized.

However, it is noticeable that the fifth-grade students, while still in the early stages of conceptualization, correctly justify their ideas, most of them correctly formulate and logically justify their arguments, but not always in the language of mathematics.

In seventh grade, we can observe the students' mindsets from two students' speeches:

S2: The number of rectangles is always doubled, and the crease is one less.

S3: For rectangles, the same thing as S2 says, but for the number of creases I've noticed: this always doubles the difference.

These kinds of conversations are not only right because we can find two different, correct solutions, but also that S3' feedback to the S2' idea refers to the fact that students listen to each other, compare their ideas with their peers, and if that is right, they accept it.

Experiences regarding Problem Posing

Problems posed by students in the fifth grade:

• Change the color of the tokens!

• Change the number of initially existing tokens to 100!

• Change the number of tokens that can be removed!

Invented problems in the fifth grade alone brought up new ideas that were quickly analyzed, reviewed, and revised by classmates. The first posed problem was quickly discarded, as almost all students stated that this would not give us a new problem.

By retrospectively observing the lessons, it is in this final phase that we can understand the characteristics of students' critical thinking. Students separated their opinions from the facts, critically examined all the new information, and questioned every aspect of it before realizing a new problem. Finally, some students suggested a solution plan for some new problems.

One of the problems posed in the seventh grade: A mouse eats cheese every day. Each day it eats half of the cheese that was left from the day before. Knowing that on the fourth day, the mouse eats 5 grams of cheese, calculate the size of the cheese eaten on the first day. The invented exercises also show that the students understand the mathematical structure of the problem and can correctly embed it in everyday examples.

Difficulties of teaching based on the experiences of the teacher

Already in the first (warm-up) game, the students immediately tested the teacher's critical thinking, quick responsiveness, adaptability to new problem situations, and this would be traceable throughout grade seven, especially in the class A. While discussing the "Eureka game," one student dictates a right solution. However, he expressed the same result with a different formula: the specified rule was 2y-x, but the student's solution is: "Subtract the first number from the second number and add to it the second number," which is y-x+y and gives the same result. Although the point of the game was to show that we could get the same third number from two different matching rules, it was worth pointing out to the students that the two rules are not different in any way.

The new idea required the teacher to separate his prior opinion from the facts, review what he said, mathematically analyze its correctness, and then decide whether to accept or reject the solution. All these steps are based on the definition of critical thinking defined by Semil (2006).

When the second time we face the same situation, we can make decisions much quicker, so the "problem" that we have with the second lesson is no longer a problem, so keeping more of these lessons can eliminate more mistakes and respond to different situations more quickly. This fact means that teaching in parallel classes or teacher's reflection can represent a solution to such problems.

SUMMARY

We consider the four lessons to be effective. We can claim that TPS, game, and problem posing as teaching strategies can be used to integrate problem solving and various heuristic strategies into mathematics lessons.

The study benefited students and teachers by promoting creativity in solving mathematical problems. It also explains why teachers find it difficult to infuse the concept of critical thinking into their teaching, but this way, students can learn how to think critically. As we got critical thinking requires one's effort to collect, interpret, analyze and evaluate the information to arrive at a reliable and valid conclusion so that students would rely on it at every moment in their lives.

The students had a good time at the lessons and enjoyed the math games. In addition to being engaging to them, the activities also had a useful, attractive effect, as we progressed in line with the curriculum, developed mathematical and social competences, and argued, refuted, questioned methods of solution, or developed critical thinking.

References

Claus, H. J. (1989). *Einführung in die Didaktik de Mathematik* [Introduction to didactics of mathematics]. Darmstadt: Wissenschaftliches Buchgesellschaft.

Dörner, D. (1983). Emotion und problemlösendes Denken [Emotion and problem solving]. In H. Mandl, & G. Huber (Eds.), *Emotion und Kognition* (pp. 61-64). München: Urban&Schwarzenberg.

Facione, P. A. (1990). *Critical Thinking: A statement of expert consensus for purposes of Educational Assessment and Instruction: Research Findings and Recommendations.* American Philosophical Association. Retrieved from ERIC Document Reproduction Service.

Kagan, S. (1994). *Cooperative Learning.* San Clemente: Kagan Publishing.

Kelemen, R., Csíkos, C., & Steklács, J. (2005). *A matematikai problémamegoldást kísérő metakognitív stratégiák vizsgálata a hangosan gondolkodtatás és a videomegfigyelés eszközeivel.* Magyar Pedagógia.

Kónya, E., & Kovács, Z. (2018). Let's explore the solution: Look for pattern! In B. Maj-Tatsis, K. Tatsis, & E. Swoboda (Eds.), *Mathematics In The Real World* (pp. 136-147, 12 p.). Rzeszów, Poland: Wydawnictwo Uniwersytetu Rzeszowskiego.

Kovács, Z., & Kónya, E. (2019). Implementing problem solving in mathematics classes. In A. Kuzle, I. Gebel, & B. Rott (Eds.), *Implementation Research on Problem Solving in School Settings: Proceedings of the 2018 joint conference of Promath and the GDM working group on problem solving* (pp. 121-128. , 8 p.). Münster, Germany: WTM Verlag für wissenschaftliche Texte und Medien.

Mansoor, F., & Pezeshki, M. (2012). Manipulating Critical Thinking Skills in Test Taking. *International Journal of Education 4, (1)*, 153-160.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking Mathematically (Second edition).* Pearson.

Midkiff, R. B., & Thomason, R. D. (1994). *A practical approach to using learning styles in math instruction.* Springfield, IL: C.C. Thomas.

Pólya, G. (1981). *Mathematical discovery: On Understanding, Learning and Teaching Problem Solving (Combined edition).* John Wiley & Sons.

Semil, R. (2006). Enhancing Thinking Skills in the Classroom. *Human and Social Sciences Journal 1 (1)*, 28-36.

WORD PROBLEMS DEVELOPING CRITICAL THINKING OF PUPILS AS SEEN BY PRIMARY SCHOOL PROSPECTIVE TEACHERS

Eva Nováková

Faculty of Education, Masaryk University, Brno, Czech Republic

The paper presents partial outcomes of a research aimed at analysing students' approaches to selected non-standard mathematical tasks. We studied ways of students' analysis and reflection of tasks, which – as we assume – have some potential to develop critical thinking of primary school pupils. We believe that this competence is a factor shaping professional identity of prospective teachers as reflective people of practice. Our findings suggest that the topic discussed in our contribution can be used in a wide range of activities in real-life education practice.

INTRODUCTION

Pre-graduate university training of primary school prospective teachers is multidisciplinary, with mathematical and didactic training in prominent position. The typical feature of university training is searching for ways of changing academic approach based only on theoretical knowledge. In the last decades, the idea of teachers as reflective people of practice has become dominant (Schön, 1987; Keiny & Dreyfus, 1989; Wubbels & Korthagen, 1990). The model of reflective practice as a specific (clinical) concept of professional education often appears in the context of constructivist teaching and learning, in which knowledge acquisition is usually linked to reflection of practical experience (Korthagen & Vasalos,2005; Janík et al. 2017).

Krainer (1999) proposed a model for teacher professional practice, the main component of which is the affective factor. The model focused on four factors: action, reflection, autonomy, and networking. Experience shows that teachers' practice is usually characterized by a lot of action and to a lesser extent by reflection and networking.

THEORETICAL FRAMEWORK

Teachers consider the issues of formulating tasks and of ways pupils solve them as extremely important. This is suggested, e.g., by a research collecting and analyzing experience of Czech teachers regarding "critical issues" of primary school mathematics (Vondrová & Rendl et al., 2013), or from a research into educational needs of teachers of mathematics (Bártek & Dofková et al., 2017). Teachers realize the importance of wording tasks and of ways pupils solve them for the efficiency of teaching as well as for the development of their professional competences.

Tasks and their solutions by pupils are one of traditional but still relevant research topics of didactics of mathematics (Palm, 2008; Schoenefeld.1992). In pedagogical theory and practice, tasks are viewed as mental and communicational constructs inviting pupils to actively work with their content, laying foundations for the educational situation, and determining its form, organisation and performance (Janík et al., 2017).

Forming beliefs of prospective teachers and their attitudes to tasks as a component of their professional development can be described from several points of view (Leder, Pehkonen & Törner, 2002). Wilson and Cooney (2002) put these in the context of the overall development of teacher competencies, Chapman (2002) documents these on stories of two specific teachers.

Our perspective is in accordance with the realistic approach to teacher training (Korthagen et al., 2011). We build on pre-concepts of students because these shape their point of view of the educational reality. Pre-concepts are usually linked to experience, which students themselves remember from their own education. Awareness of pre-concepts is the basis of experience learning and is the first of its three stages (see Korthagen et al., 2011, p. 186):

1) Beliefs of students regarding what the process of educating children should be, "opinions without any background or rational, which however, are rather resistant to changes" (Korthagen et al., 2011, p. 82). So far, students have been only in the position of solvers (pupils). Their goal was to solve the tasks as well, i.e. as fast, correctly, as the teacher requested, as they could, which would result in a certain kind of reward such as mark, reward by the teacher, or one's own feeling of achievement.

2) Students acquire their first professional experience, handle pedagogical situations and shape their relations to their profession. They realize that there exist numerous alternative ways of solving tasks with various level of efficiency of impact on learning and children development. Students work with various concepts and relations between them (teaching – learning). They become familiar with other aspects of tasks, especially those that can regulate children's learning and diagnose their knowledge: typology of tasks, efficient use at various stages of teaching, e.g. motivational aspect of especially word problems (Siwek, 2005). They realize the influence of wording tasks on strategy and quality of the ways of solving them (Semadeni, 1995), they learn how to assess the difficulty of tasks (Nováková, 2018), or how to identify reasons of wrong pupils's solutions (Novotná, 2000).

3) Students are able to see themselves, which significantly influences their search for their way of teaching and their beliefs (Pajares, 1992). They explicitly explain and give reasons for their actions by referencing theoretical background. This relates to the perception of their own profession and their role in it and to their use of scientific terminology. Teachers are able to analyze and reflect their

way of teaching, in which tasks are tools of teaching management and triggers of pupils' activity.

Our research aimed at identifying attitudes of prospective teachers to pupils' solutions of non-standard word problems of a specific type with a potential to develop critical thinking of pupils. The attention given to such tasks is usually only marginal (Radatz, 1983).We reflected and analyzed opinions students showed regarding word problems which did not have the usual structure: they contain redundant or unnecessary input data or – on contrary – lack some conditions, or ask questions which cannot be answered (nonsensical problems).[1] We asked whether prospective teachers believed that solving such tasks is a factor helping to develop critical thinking of pupils, which is one of the goals of teaching mathematics.

We were inspired by an older research referred to by Verschaffel, Greer and de Corte (2000), whose results correspond to numerous complaints of teachers who claim that "pupils are not willing to think".

METHODOLOGY

Our qualitatively oriented research was carried out in autumn 2019 with a group of 56 primary school prospective teachers of the Faculty of Education, Masaryk University in Brno. The research method that we used was the analysis of solutions of a set of tasks done by primary school pupils. This was followed by a joint reflection of prospective teachers in the seminar of didactics of mathematics.

We worded two research questions:

a) Are primary school prospective teachers able to analyze and reflect on pupils' ways of solving tasks which have a potential to develop their critical thinking?

b) What are the attitude and beliefs of prospective teachers regarding the usefulness of including such tasks in teaching?

The research had the following four stages:

a) Stage 1 – preliminary: Giving information about the aim of the research, discussion with prospective teachers on the role and significance of non-standard tasks and their solutions during the process of development of critical thinking of pupils. In the seminar of didactics of mathematics, prospective teachers proposed several types of such tasks. Three types were presented: a) containing information which need not be used to answer the question; b) with missing information without which the task cannot be solved; c) nonsensical problems such as asking questions regarding

[1] *"There are 26 sheep and 10 goats on a ship. How old is the captain?"* (Institut de Recherche sur l´Enseignementdes Mathématiques de Grenoble, 1980; Radatz, 1983).

something not even present in the wording of the task, or asking questions which cannot be answered from the data present in the task.

In the end, prospective teachers were asked to give such tasks to their pupils and to assess them during their pedagogical practice at the primary school.

b) Stage 2 – teaching: Prospective teachers in the role of the task givers, commentators and facilitators during the process of solving the tasks during their own teaching practice. Pupils in the 4th and 5th grade of primary school (aged 9-10) were, during the pedagogical practice of prospective teachers, given the following three tasks:

    1. Class 4D, consisting of 12 girls and 9 boys, made a trip by train and bus. Each child paid 30 crowns for the train and twice as much for the bus. How much in total did all the boys pay for the bus?
    2. How much do we pay for fencing a garden 40 metres long and 25 metres wide?
    3. In the herd there are 125 sheep and 5 dogs. How old is the shepherd? (Verschaffel, Greer, & de Corte, 2000)

c) Stage 3 – analytical: Students in the role of assessors and evaluators of pupils' solutions of the tasks. First of all, before the tasks were given to pupils, students were asked to describe their expectations regarding ways of solving the tasks and pupils' success rate. Then they were supposed to analyse the written pupils' solutions and / or their comments and to confront them with their own expectations.

d) Stage 4 – reflective: Written reflection of one's own findings and their joint presentation of opinions and attitudes on the realization stage of the activities.

RESEARCH RESULTS

A. Outcomes of the analytical stage, in which prospective teachers characterized pupils' solutions of the tasks, can be seen at two levels. The most often mentioned student responses regarding expected pupils' solutions can be summarized as follows:

Task 1: We assume that most pupils will do well, perhaps with wrong calculations in multiplications. There may be some individuals working with redundant numbers which are not necessary for answering the question.

Task 2: We assume that some pupils will try to compute the task even if they lack some necessary information. Most of them will simply add or multiply the garden dimensions. Some pupils will write that the task cannot be solved but they will not be many.

Task 3: We assume that pupils will state that the task cannot be solved without explaining why. But since they are used to solve word problems which always have a solution, they will try to start and will use multiplication or division. They will face problems with multiplication after they realize that the result is too great.

The attempts to analyse pupils' solutions by respective prospective teachers had either the form of evaluating "successfulness of solution" or interpretation of pupils' reactions to the unusual nature of the tasks both in class and in the written records.

The success rate of Task 1 was rather high (62%). The assumption of students was correct. Pupils commented that it was a "usual task with more information than we need". Mistakes were mostly in not answering the question (all boys for the bus) or in wrong numerical calculations or in unclear wording of answers.

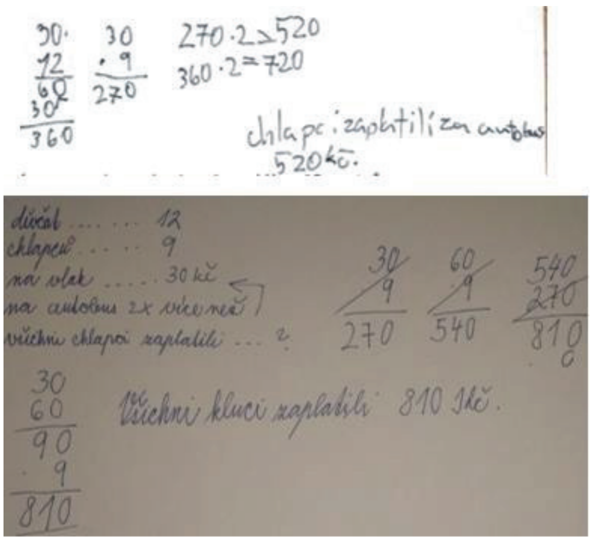


Figure1. Incorrect solutions of Task 1. Text above: "Boys paid 520 CZK for the bus." Texts below: "girls // boys // for train // for bus twice as much as // all boys paid? // All boys paid 810 CZK."

As regards Task 2, if pupils solved it at all, then they either calculated the circumference of the garden or added or multiplied the numbers without any further explanation. If they did not solve the task, they did not include an explanation why the task could not be solved. The missing price of fence per meter as a necessary condition of solution was mentioned only exceptionally. This task met student expectations as well.

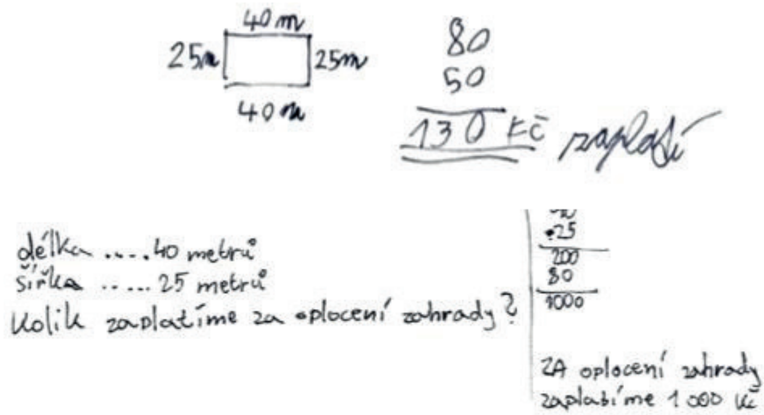


Figure 2. Incorrect solutions of Task 2 with numerical calculations. Text above: "130 CZK will pay", "How much will we pay for the fence around the garden?" "We will pay 1 000 CZK for the fence around garden."

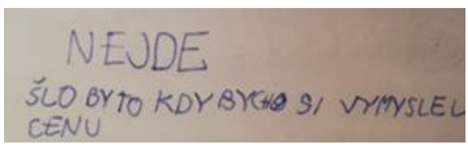


Figure 3. Explanation why the pupil did not solve the task: "Impossible. Could be done if we invented some price."

Attempts to "solve" Task 3 included almost all arithmetical operations with division being most common (most likely because age 25 seemed reasonable to solvers). Only 16% of pupils were able to give an adequate answer to the question. The answer: "Sheep and dogs have nothing in common." got the point. In some classes we had a nice discussion with those few pupils who argued why the second and third tasks have no solution – missing data, age of the shepherd does not depend on the number of dogs.

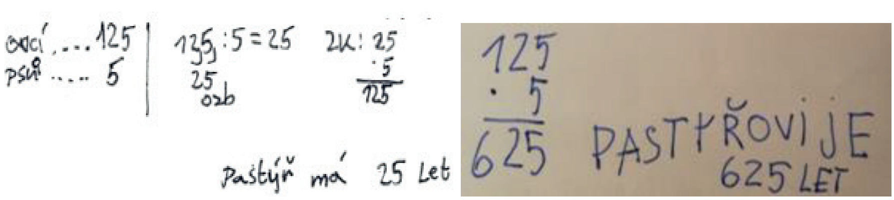


Figure 4. Calculations in the solution of the nonsensical problem. Texts: "The shepherd is 25 // 625 years old." "ZK" means verification.

B. During the joint reflection students noticed various facts. The common denominator of all of these was a surprise that pupils were not able to express the fact that the task has no solution. They believe that this is the result of teaching focusing on performance and results with minimal space left for logical thinking and argumentation. What follows are several authentic comments:

> Most solutions only confirmed my belief that pupils are accustomed to tasks which always have a numerical solution. The type of tasks we gave them was totally new for them. As a result they had problems in handling them.
>
> I think that it was something they are not used to and that this need to think differently was a bit painful.
>
> The tasks seemed unusual to the pupils. However, they nicely discussed various strategies. During these discussions they realized that not all tasks have solutions and that one needs to think well instead of just rewriting numbers and choosing a suitable arithmetical operation.
>
> I was surprised how many pupils do not think when solving tasks and simply want to find some solution, maybe because they are used to that. I think that the problem is not in children but in the style of teaching. We are trying to drill the children to be as fast as possible in their calculations but we are not trying to develop critical and logical thinking.

We extracted three levels of attitudes to the analysis of tasks from students' reflections:

a) Manifestations of reserved attitude of students to usefulness of solving such tasks and / or, in a more general sense, of the necessity of developing logical and critical thinking. What students require are clear and straightforward methodical manuals: which tasks are suitable and how to asses and evaluate them. They remain trapped in their own pre-concepts constructed when they themselves were pupils.

b) Mere statements that the pupils were not able to critically judge wording of the tasks and find out that some information was redundant or missing. Without any proposals for wider didactic use of the tasks in teaching.

c) Students start looking at the tasks and the relation between tasks and solvers from the point of view of prospective teachers. They realize the sense and usefulness of including tasks developing critical thinking for the personality development of pupils.

DISCUSSION AND CONCLUSION

We used the chance to discuss selected tasks in order to make the research participants think about themselves. We believe that a change of attitudes of students to the ways of solving tasks for primary school pupils is one of factors constructing the professional identity of teachers and their professional beliefs (Pajares, 1992). In didactically oriented subjects, the process of building the individual identity of teachers, which is affected by the belief regarding readiness of students to teach mathematics, is only initiated. Conclusions we drew from written and verbal reflections of students enable us to give a positive answer to the first question of our research. Students found out that pupils' expectations such as: "Every task given by a teacher or found in a textbook has sense and a solution; there exists only one correct and exact answer or one number; in order to solve the task one needs exactly that what is written in it."

This was evident especially during the student analysis of solutions of Task 3. This confirmed findings of Verschaffel, Greer and de Corte (2000) who state that only 12% of children aged 7-9 were able to provide a correct answer while the rest tried to "compute" the task somehow. The authors explain this phenomenon by calling it "word problem game", which includes expectations of both pupils and teachers that every word problem in every mathematical lesson will have a solution. Palm (2008) asked why pupils sometimes find solutions to word problems which are not consistent with its context and wording. He found out that all "unrealistic" answers stem from the "absence of reasonable thinking". When solving the tasks, pupils applied their own understandings of the context which agreed with their own personal experience – yet these differ from how teachers or authors of textbooks see them.

A great number of students already can identify and interpret the fact that students learned certain solution manuals and that they try to apply them to tasks without "thinking" about them. The statement of one of the students: "I had to read the task several times to find out what I was asked to do." reminds us of the crucial significance of comprehending the wording of the task for the quality of its solution. It is necessary to be able to find relevant data in the text, and in a more general sense to have an adequate level of reading comprehension.

Our findings did not result in a clearly positive answer to the second research question. As follows from TEDS-M study (Tatto, 2008), prospective teachers regard calls for critical thinking of pupils as less important than other activities (such as to clearly pass information to students, to communicate well, to effectively manage the educational process, or to use adequate means of assessment). When judging the readiness of primary school prospective teachers to teach mathematics, Dofková (2016) found out that the significance of using tasks and questions developing critical thinking increases during their training. While at the beginning of their didactical training only 57% state that it is important, at the end of the didactical training this number increases to 83%. This corresponds to our experience. Facing the educational reality, prospective teachers start asking specific questions such as whether pupils are able to use mathematical principles and techniques they had acquired to solve the tasks, whether they are able to choose adequate mathematical tools, whether the tasks were comprehensible or whether the level of their reading comprehension is enough for correct understanding of the task. Searching for such answers (and many others) is, in our opinion, part of a teacher as a true professional.

References

Bártek, K., Dofková, R., et al. (2017). *Reflexe vzdělávacích potřeb učitelů matematiky jako východisko jejich profesního rozvoje.* [Reflection of educational needs of mathematics teachers as the basis for their professional development]. Olomouc: Palacký University.

Dofková, R. (2016). *Přesvědčení o připravenosti budoucích učitelů matematiky jako didaktická výzva primárního vzdělávání.* [Beliefs Concerning the Preparedness of Future Mathematics Teachers as a Didactic Challenge of Primary Education] Olomouc: Palacký University.

Greer, B., Verschaffel, L., & de Corte, E. (2002). "The answer is really 4.5": Beliefs about word problems. In G.C. Leder, E. Pehkonen, & G. Töner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 271-292). Mathematics Education Library, Kluwer Academics Publishers.

Chapman, O. (2002). Belief structure and inservice high school mathematics teacher growth. In G.C. Leder, E. Pehkonen, & G. Töner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 171-192). Mathematics Education Library, Kluwer Academics Publishers.

Janík, T., Slavík, J., Najvar, P. & Knecht, P. (2017). *Transdisciplinární didaktika: o učitelském sdílení znalostí a zvyšování kvality výuky napříč obory.* [Transdisciplinary didactics: on teachers´ knowledge sharing and improving teaching in different subject]. Brno: Masaryk University.

Keiny, S., & Dreyfus, A. (1989). Teachers´ selfreflection as a prerequisite to their professional development. *Journal of Education for Teaching*, 15 (1), 53-63.

Korthagen, F. A. & Vasalos, A. (2005). Levels in reflection: core reflection as a means to enhance professional growth. *Teachers and Teaching: theory and practice*, 11(1), 47-71.

Korthagen, F. A., Kessels, J., Koster, B., Lagerwerf, B., & Wubbels, T. (2011). *Jak spojit praxi s teorií: didaktika realistického vzdělávání učitelů*. [Linking practice and theory: the pedagogy of realistic teacher education] Brno: Paido.

Krainer, C. (1999). Promoting reflection and networking as an intervention strategy in professional development programs for mathematics teachers and mathematics teachers' educators. In O. Zaslavsky (Ed.), *Proceedings of the Conference PME,* 23, Vol. 1, (pp. 59-168). Haifa: Technion.

Leder,G. C., Pehkonen, E. & Törner, G. (2002). *Beliefs: A Hidden Variable in Mathematics Education?* Mathematics Education Library, Kluwer Academics Publishers.

Nováková, E. (2018). Prediction and self-evaluation as a part of the process of solving non-standard mathematical tasks. In P. Błaszczyk & B. Pieronkiewicz (Eds.). *Mathematical transgressions* 2015. Krakow: Universitas.

Nováková, E. (2016). *Analýza úloh ze soutěže Matematický klokan a jejich řešení žáky primární školy. Shrnutí výsledků výzkumného šetření*. [Analysis of

Mathematical Kangaroo competition problems and their solutions by primary school pupils]. Brno: Masaryk University.

Novotná, J. (2000). *Analýza řešení slovních úloh*. [Analysis of word problem´s solutions]. Prague: Charles University.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, 62(3), 307-332.

Palm, T. (2008). Impact of authenticity on sense making in word problem solving. *Educational Studies in Mathematics*, 67(1), 37-58.

Radatz, H. (1983). Untersuchungen zum Löseneingekleideter Aufgaben. *Zeitschrift für Mathematik-Didaktik*, 4(3), 205-217.

Rendl, M., Vondrová, N., et al. (2013). *Kritická místa matematiky na základní škole očima učitelů*. [Critical places of primary mathematics through the eyes of teachers]. Prague: Charles University.

Semadeni, Z. (1995). Developing children´s understanding of verbal arithmetical problems. In *Proceedings International Symp. Elem. Math Teaching*, 27-32. Prague: Charles University.

Siwek, H. (2005). *Dydaktyka matematyki. Teoria i zastosowania v matematyce szkolnej*. [Didactics of mathematics: theory and their application in school mathematics]. Warsaw: WSiP.

Tatto, M. E. (2008). *Teacher Education and Development Study in Mathematics (TEDS-M)*. Michigan: College of Education, Michigan State University.

Verschaffel, L., Greer, B., & de Corte, E. (2000). *Making sense of word problems*. Lisse (Netherlands): Swets and Zeitlinger B.V.

Wilson, S., & Cooney, T.J. (2002). Mathematics teacher change and development. The role of beliefs. In G.C. Leder, E. Pehkonen, & G. Töner (Eds.), *Beliefs: A Hidden Variable in Mathematics Education?* (pp. 127-148). Mathematics Education Library, Kluwer Academics Publishers.

DEVELOPMENT OF ABSTRACT THINKING THROUGH HANDS-ON ACTIVITIES AND ALGEBRAIC MODELS

Ivona Grzegorczyk

California State University Channel Island, USA

In this study pre-service teachers had to modify their lessons to include interactive activities and algebraic modelling. We analyse the performance of the teachers' learning and pedagogy changes. We also evaluate performance and attitudes of sixth-graders and ninth-graders participating in the interactive lessons presented by the teachers. All participants supported hands-on modelling to routine lecture-based teaching.

INTRODUCTION

This study is a part of a larger project supported by NSF Noyce 1660521 grant aiming to prepare pre-service teachers to include active learning activities promoting mathematical modeling, engagement, excitement, discussions and students' creativity. In this paper we describe activities targeting early algebra curricular experiences including modeling, predictions, development of strategies, analysis of patterns and generalizations to other contexts that were presented to 8 pre-service teachers in order to influence their pedagogy. These future teachers were asked to design introductory algebra lessons, and then to incorporate proposed modeling activities into them. They taught the modified lessons to groups of school pupils and evaluated their experiences.

To engage learners from the start the initial questions in our activities were presented as involving magic tricks, that through explorations, discussions and predictions, led to formalization of the models and their further generalizations. At every level participants discussed the process of creating new structures and ideas, focusing on making connections and attempting different solutions. To assess the level of abstraction we have evaluated participants' creative approaches. Following (Savic et al., 2017) we defined creativity as a process of offering new solutions or insights that are unexpected for the learner, with respect to his/her mathematics background or the problems s/he has seen before, as well as discoveries made within a specific reference group that creates something new (Vygotsky, 2016).

Problem solving and creative thinking are necessary for professional success in the fast passed, technology intensive global setting of the 21st century. At every level of mathematics education, there have been criticisms about the excessive amount of structure imposed on learners, especially at the K–12 level, where students are rarely encouraged to solve open-ended problems, think creatively or pose their own questions. Already in 1989, the National Council of Teachers of Mathematics addressed the need for standards that include modeling, creativity and independent thinking, but nearly two decades later the situation in American schools is not much better, as mathematics education still concentrates on basic skills and traditional problem solving. Additionally, the worldwide emphasis on high-stakes testing brought basic skills back to the center of attention (Lesh & Sriraman, 2005). While for a long time Polya style problem-solving strategies (draw a picture, identify the givens, work backwards, solve similar problems) have been advocated as important abilities for students to develop their mathematical maturity (Polya, 1957), they are not leading pedagogy in our schools (Chazan, 2008; Drew, 2011).

Studies show that contemporary students prefer innovative rather than traditional pedagogy (Star et al 2008), learning with multiple representations (Ainsworth, 2006), through hands-on activities (Cruse, 2012) that are related to their interests (Whaley, 2012), in an engaging, playful environment (Kuh, 2003). Hence influencing future teachers to present mathematics lessons in attractive and engaging ways was our main motivation. We evaluated their lessons to see the level of engagement they incorporated after participating first in the similar interactive activities themselves. We also assessed all participants' involvement in concept modeling and understanding. The activities encouraged curiosity, explorations and creation of algebraic models, logical thinking through pattern recognition and proposing definitions for underlying rules, various representations, extensions and modifications as well as verbalizations of the thinking. Basic algebraic concepts are now introduced early in the curriculum (Stephens et al, 2015), but test results show that even high school seniors have problems understanding many of them (Kuh, 2003). We need to train teachers in bringing fun back into mathematics classrooms through pedagogy mixing contexts, explorations, and applications with new interdisciplinary connections to the abstract curriculum (Jones, 2016, Kurz, 2017, Stilianou et al. 2005, Whaley, 2012). Recently, there have been some efforts of systematically implementing new pedagogical strategies (such as inquiry-based learning or problem-based learning) to improve students' skills that are related to mathematical modeling and creativity (such as investigating ideas, providing multiple solutions, analyzing others' strategies), especially for school curricula. However, such efforts are generally not included in mathematics education for pre-service teachers at the university level.

METHODOLOGY

During this study 8 pre-service teachers in their final year at university were asked to make their pedagogy more engaging. They started by preparing lesson plans on introductory algebra concepts such as variables, equations, modeling and predictions. It turned out that all the lesson plans were lecture-based with examples worked out by the instructors on the board and some follow up questions to be solved by students individually. Next, the pre-service teachers participated in a 1- hour session with the university faculty that incorporated two interactive, exploration-based activates described below. During the session, they investigated the situations, understood the mathematical content and the related algebraic models. They were given time to master the activities and interactive pedagogy and to prepare their own modeling lessons. Later, during 1-hour sessions, they taught their engaging lessons as instructors to groups of 10 school pupils.

Each lesson started with a magic math trick, which was discussed to uncover underlying patterns, rules and/or optimal strategies to be modeled in algebraic language. Small teams of pupils implemented simple modifications and/or generalizations to the model and presented them to the rest of the group. Then they designed their own patterns creatively, hence variety and complexity were added to individualized patterns, which were later modeled using algebraic language. Further discussions and explorations lead to even more generalized problems, which often were formalized as formulas that included several different variables. Most of the hands–on tasks were done individually or in small teams/pairs and shared with larger audiences for comparison and discussions.

Some of the participants were openly frustrated at first to be asked to work in unfamiliar contexts and with initially undefined variables. While the activities proved to be quite challenging, learners were fruitfully engaged at trying to design the models through logic and reasoning. The activities supported teamwork, discussions and mathematical perseverance when challenged. Several pupils presented some of the activities as magic tricks to their teachers and parents, who gave them enthusiastic reviews.

Description of the activities

Magic tricks with dice activity starts by building various towers consisting of two dice. The instructor performs magic by guessing the sum of the hidden faces on each tower. By discussion, students figure out how the trick works. They build towers with three dice and try to figure out how to guess the sum. Then they use four and five dice. Now they study the patterns to come up with the linear algebraic formula depending on the number of dice and the number on the top of the tower modeling the situation.

Next, pupils put two, then three, then four dice in a row touching each other by one side (a tower lying down). At each level they try to figure out the formula for guessing the sides that cannot be seen. After discussing, they come up with the general formula depending on the number of dice and the sum of the visible faces, which is unexpected, as it introduces several variables, sequence summations and generalizations. In the exploration part, students create their own designs using increased number of dice and trying to relate the geometric and numerical patterns. Then they generate algebraic models for the sums of the hidden faces or for other generalized questions.

Guess my number activity starts with each pupil picking its own secret natural number and then following a set of algebraic operations given by the instructor. Participants share the results of their final calculations, and the instructor guesses their individual secret numbers. Through discussions, pupils try to figure out the underlying rules and use algebra to make computational shortcuts. The activity can be repeated several times with different instructions. Once the group understands the underlying algebraic models, they design their own guessing games by creating new sets of rules and calculating the answers algebraically. They play out their scenarios in small groups. Note that the underlying equations could be linear, quadratic or of any complexity.

We collected all participants work and structured instructor observations as well as administered a survey asking about their attitudes towards learning through interactive activities and modeling.

RESULTS

We analysed data from all participants using their involvement levels, models and follow up surveys. These groups included: 40 pupils aged 11-12 (we denote this group P) working in groups of 10, 40 high school students (group S) working in groups of 10 and 8 pre-service mathematics teachers (T) who participated in one interactive session and then prepared and presented their lessons as instructors. We start here by reporting interesting results from the post-experiment evaluation of data. The analysis of surveys shows that the vast majority of participants preferred the unfamiliar situations and modelling uncertainty to routine lecture-based teaching (91% out of total 88 (pupils and teachers combined)). Their attitudes toward mathematics improved significantly and the level of scientific language used increased.

Table 1 displays data for engagement and creativity of all participants during the interactive sessions. Engagement levels were ranked based on instructors' reports and survey answers. During each modelling activity participants worked in pairs or teams of three. The generalized more complex models and their creativity were evaluated by the instructors based on the following scale:

Model is a direct generalization of the introductory model (Low)

Model introduces some new ideas into the generalized model (Moderate)

Model introduces creative/unexpected ideas to generalized model (High)

For example, in the dice activity, the generalized model involving building another simple dice tower was considered as low creativity, designing a simple 2D or 3D pattern with dice and working out the algebraic formula was marked as moderate, while proposing interesting geometric 2D or 3D patterns generating interesting formulas (possibly with parameters) was considered as highly creative.

Modelling Dice	Sample Size	Algebra Level	Engagement	Creativity
P - Pupils	40	Introductory	High	Moderate
S - students	40	Learners	High	High
T- Teachers	8	Proficient	Moderate	High
Total	88			

Table 1: Participants engagement and creativity by groups.

It is worth noticing that all groups were engaged in the activity at least moderately, each group provided generalized models, and participants more proficient in algebra provided more sophisticated and creative models. The first two groups were taught by the pre-service teachers, and the engagement level generated in the classroom was high in both cases. Pupils reported that they really enjoyed the activities.

The following results for each activity are based on the individual written work of participants and coded observations of the instructors during the sessions. Note that for teachers, this was a session before they prepared their lessons. In dice activity the 2-dice model was generalized to the 3-dice model and to the n-dice model by teamwork and the instructor lead discussions.

Majority of participants mastered and were able to use the n-dice model for different numbers of dice. Large percentage in each group was able to generalize the model and solve the specific problem (note that pupils' models were less sophisticated than models for the other two groups). Over three quarters of students and teachers were able to provide the accurate algebraic formulas (Mastered the Model) and discuss the parameters involved.

Modelling Dice	2-dice model	3-dice model	Generalized Model	Mastered Gen. Model
P - Pupils	90%	90%	65%	48%
S - students	100%	90%	80%	75%
T- Teachers	100%	100%	87%	75%

Table 2: Models for the dice activity

The initial Guess My Number activity in each session involved the entire group and the underlying linear model was uncovered by discussions. Then small teams designed their own guessing tricks. Table 3 shows the complexity of the models, where underlying formula such as $(2x+4)/2 - 3$ was considered simple,

$(x\text{-}1)^2\text{-}1$ was classified as the use of quadratic functions. Mastering the model meant that students were able to simplify the algebra of their formulas to create a quick answer for guessing the original number *x*. Some of the models used more than 5 steps.

Modelling	Simple algebra	Quadratic	Created model	Mastered
P = Pupils	70%	40%	50%	40%
S - students	80%	80%	90%	75%
T- Teachers	100%	87%	100%	87%

Table 3: Algebra use in Guess My Number activity.

Table 4 includes the comparison of (average) time spend on various models included in the pre-service teachers' lessons. Note that the initial lesson plans did not include explorations and all the models proposed to study had simple linear equations of type $y= ax+b$ as an underling concept. Their prepared interactive lessons very closely followed the activities they were exposed to during their activities session with the university faculty. Overall, in the new designed lessons they proposed more time for generalizations of linear models including for example more than one variable or more than one equation, and some quadratic models; however, they did not include complex models (mixed equations, complex formulas) even though some of them were presented to them during their interactive activities session, where 20% of time was spend on linear models, 30% on their generalizations, 20% on quadratic models and 30% on complex models. Hence teachers avoided complexity while preparing their lessons. However, when they were conducting their lessons, pupils' explorations generated more complex models than expected.

Strategies	Linear Models	General Linear	Quadratic Model	Complex
Initial lesson plans	100%	0%	0%	0%
Interactive lesson plans	60%	20%	20%	0%
Implemented lessons	50%	20%	20%	10%

Table 4: Time for algebraic models used in lessons by pre-service teachers.

These results suggest that participating in active learning sessions significantly help future teachers plan more interactive and engaging lessons, where pupils may explore and come up with unexpected algebraic models. However, teachers try to avoid this uncertainty by not planning to ask more complex questions or proposing more complex models. Further comments from the teachers showed that they were afraid of complexity that may be confusing to them and to their students.

Some interesting comments

Below we quote some of the comments from the pre-service teachers on the activities, lesson plans and the learning process from the survey.

> Teacher 1: I liked the idea of presenting algebra as magic. I used the dice tricks to design my own tricks for my students and they were really enthusiastic about solving them.

> Teacher 2: I enjoyed the interactive session. It showed me how simple my initial lesson plan was. However, my improved interactive lesson was too complicated for elementary students. We had to use *n, m, l, k* to make the correct formulas and this made them a bit confused. But all were engaged in the discussions.

In general, pre-service teachers were more confident about the linear algebraic models and enjoyed the analysis of patterns. Many comments referred to the satisfaction of being able to handle a complicated situation they have created.

> Teacher 3: The algebraic problems posed as magic were fun and during the explorations I was trying to figure out something all the time. I never though I can come up with interesting complicated formulas by myself, but I did. And I was able to include them in my teaching.

> Teacher 4: I plan to continue building more complicated dice patterns, finding patterns for patterns. Generalizing situations was interesting. I wish my math courses were taught that way, more interactive. I plan to include activities in my lessons.

> Teacher 5: I liked explorations and debating the ideas freely. Talking about strategies was interesting. I will include activities, carefully, as stuff can get complicated quickly.

> Teacher 6: Activities were involving and innovative, many discussions. I learned not to be afraid to ask generalized questions. Some of the tricks were hard and require some preparation.

> Teacher 7: I participated in activities before, but I never thought about creativity in problems. I'm worried about some generalizations form pupils, as they may come up with something too difficult for their level. I don't want to confuse them with complex equations.

All teachers (100%) were satisfied with their interactive lesson plans and implementations. All comments regarding the activities, discussions and the learning process were positive across the groups. The initial tasks inspiring curiosity were liked the most, as was the task of designing generalized models. Participants showed perseverance analysing these models and expressed concerns about their own ability to formalize them in algebraic language.

CONCLUSIONS

In order to support student thinking in algebra, it is important to provide activities requiring critical thinking and original model building in various contexts during regular school sessions. It is beneficial to them to struggle a bit and discuss the possible solutions before coming out with the correct model. They should have an opportunity to make sense of algebra as a tool for predictions and modeling patterns. Our study provided teachers with a chance to improve their pedagogy and opportunities for implementing interactive learning. All groups (P, S, T) engaged fully into the proposed activities, discussions, modeling processes and generalizations. Almost all participants showed their potential for generalizations using multiple representations. They performed well when exposed to the uncertainty and the difficulty of creating mathematics. Teachers comments indicate the suitability of the provided activities for regular classrooms (with appropriate preparation) and the need for further development and testing of activities on other topics for school and college level mathematics curricula that use variety of tools (such as manipulatives, technology, games, art and science concepts, etc.). Hopefully, this pedagogy will become more common and we can prepare more creative, open minded and thoughtful teachers that can meet future demands of the society. Interestingly, many participants tried to come up with ‘nice’ formulas, paying attentions to aesthetic, i.e. beauty of the mathematical models. Some of their comments suggest that they would like to learn more about their own cognition and the regulation of the creative processes. These suggest that we should study not only learners’ creative actions, but also their meta-cognitive skills.

References

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16*(3), 183-198.

Bates, E., & Wiest, L. (2004). Impact of Personalization of Mathematical Word Problems on Student Performance. *The Mathematics Educator, 14*(2), 17-26.

Chazan D. (2008). The shifting landscape of school algebra in the United States. In Greenes C. E., Rubenstein R. (Eds.), *Algebra and algebraic thinking in school mathematics* (pp. 19-33). Reston, VA: National Council of Teachers of Mathematics.

Darley, J., & Leapard, B. (2010). Connecting arithmetic to algebra. *Teaching Children Mathematics, 17*(3), 184-191.

Drew, D. E. (2011). *STEM the Tide: Reforming Science, Technology, Engineering, and Math Education in America.* Johns Hopkins University Press.

Driscoll, M. (1999*). Fostering Algebraic Thinking: A Guide for Teachers, Grades 6-10.* Heinemann.

French, D. (2002). *Teaching and Learning Algebra.* London: Continuum.

Graham, K., Cuoco, A. & Zimmermann, G. (2010). *Focus in High School Mathematics: Reasoning and Sense Making in Algebra*. National Council of Teachers of Mathematics.

Grzegorczyk, I. (2018). Generalizing Algebraic Models Through Interactive Activities. *Proceedings of Contemporary Mathematics Education 2018.*

Gun, B. (2014). Making Sense of Experienced Teachers' Interactive Decisions: Implications for Expertise in Teaching. *Interantional Journal of Instruction 7*(1).

Jones, A. (2016*). Secondary school mathematics teachers' views and use of manipulatives.* Lambert.

Kotsopoulos, D. (2007). Unraveling student challenges with quadratics: A cognitive approach. *Australian Mathematics Teacher, 63*(2), 19 – 24.

Kuh, G. D. (2003). What We're Learning About Student Engagement. NSSE: Benchmarks for Effective Educational Practices. *The Magazine of Higher Learning, 35*(2), 24–32.

Kurz, T. (2017). Finding linear and quadratic equations in Integer patterns. *Mathematics Teacher, 110*(6), 454-460.

Karakok, G., Savic, M., Tang, G. & El Turkey, H. (2015). Mathematicians' views on undergraduate students' creativity. In K. Krainer and N. Vondrova (Eds.), *Proceedings of the 9th Congress of European Research in Mathematics Education,* (pp. 1003–1009).

Polya, G. (1957). *How to Solve It, 2nd ed.* Princeton University Press.

Savic, M., Karakok, G., Tang, G., El Turkey, H. & Naccarato, E. (2016). Formative assessment of creativity in undergraduate mathematics: Using a creativity-in-progress rubric (CPR) on proving. R. Leikin and B. Sriraman (Eds), *Creativity and Giftedness: Interdisciplinary Perspectives from Mathematics and Beyond* (pp. 23–46). Springer.

Star, J., Smith, J., & Jansen, A. (2008). What Students Notice as Different Between Reform and Traditional Mathematics Programs. *Journal for Research in Mathematics Education, 39*(1), 9-32.

Stephens, A., Blanton, M., Knuth, E., Isler, I., & Gardiner, A. M. (2015). Just say yes to early algebra! *Teaching Children Mathematics, 22(2),* 92-101.

Stylianou, D., Grzegorczyk, I. (2005) Symmetry in Mathematics and Art: An Exploration of Art Venue for Mathematics, *PRIMUS 15*(1), 30-44.

Whaley, K. (2012). Using Students' Interests as Algebraic Models. *National Council of Teachers of Mathematics, 17*(6), 372.

Vinson, B. (2001). A comparison of preservice teachers' mathematics anxiety before and after a methods class emphasizing manipulatives. *Early Childhood Education Journal, 29*(2), 89-94.

Vygotsky, L.S. (1987) *The Collected Works*. Rieber, Robert W., Carton, Aaron S. (Eds.) Springer.

AN ANALYSIS OF DIAGRAMMATIC ACTIVITY AND COMMUNICATING ABOUT IT IN INDIVIDUAL LEARNING SUPPORT

Barbara Ott*, Annika M. Wille**

*St. Gallen University of Teacher Education, Switzerland

**University of Klagenfurt, Austria

Preservice teachers need to learn how to support children with learning difficulties. This requires a critical perspective on one's own teaching. In this paper a method is presented in order to analyse how preservice teachers support elementary school students who mainly use counting strategies when solving arithmetic problems. Using this analysing method, it becomes visible what mathematical sign activity takes place during the support and how it is intertwined with the communication about it. In this way patterns become visible in how a preservice teacher supports a child and how the child reacts. Finally, the resulting visualisation can serve as a basis for preservice teachers to reflect on their teaching.

INTRODUCTION

For learning and doing mathematics, activity with signs is necessary. From the beginning, students have to learn how to use different representational systems and how to combine them. Among other things, grade 1 students need to learn to solve arithmetic problems without counting strategies. The counting strategies are first elementary approaches, but a solidification of them can lead to difficulties in learning mathematics (Scherer & Moser Opitz, 2010). To counteract this, it is important, that learners develop a structure sense (Lüken, 2012). For this purpose, mathematical tools are used, such as the twenty field (see Figure 1). In this case, in addition to learn how to use the representational system of natural numbers, the children must also learn how to use the representational system twenty field and make connections between the two in both directions. To achieve this, some children need special support. In order to enable future teachers to provide such assistance, at the St. Gallen University of Teacher Education, the individual support of children to learn to solve arithmetic problems without counting strategies is already taken into account during the teacher education. The associated project *MaL1zu1 - Learning and Teaching Mathematics in One-to-One Support*, which is a subproject of the of the project *MALKA - Learning and Cooperating in Mathematics from the very Beginning* (PHSG, 2018), investigates how preservice teachers support grade 1 and 2 students in an individual setting. An analysis of the supportive interactions will provide information on how sign activity is induced in children by the preservice teachers and which interactional patterns arise with regard to the sign activity in different representational systems. The long-term goal is to use these visualisations as a basis for reflective discussions with preservice teachers about their teaching. In this article the method of analysis is presented by means of three exemplary scenes.

THEORETICAL FRAMEWORK

Solving arithmetical problems without counting strategies

In order to enable children to solve arithmetical problems without counting strategies, they must be supported in the development of sustainable ideas about numbers and operations (Häsel-Weide, 2016). In this context, it is particularly important that the children develop the part-whole-schema (Gerster & Schultz, 2004). Therefore, the children should be supported in recognizing numbers as structured quantities in combination with decomposing, representing and describing them (Häsel-Weide, 2016, p. 32). Thus, the learners should be able to perceive and determine cardinality of a quantity by structural subitizing (Schöner & Benz, 2018). That means that the quantities are perceived in structures and that the determination of the quantity is based on known facts without using counting strategies (Schöner & Benz, 2018, p. 127). Structured materials such as the twenty field are suitable for this (Häsel-Weide, 2016; Scherer & Moser Opitz, 2010).

Diagrammatic Activity and Communicating about it

In mathematics, diagrammatic inscriptions are of particular importance. In this article, *diagrams* will be considered from the perspective of the American philosopher Charles Sanders Peirce (1839-1914) as signs with a relational character, whose perceptible basis is an inscription (Dörfler, 2008). Several characteristics qualify an inscription as a diagram (Dörfler, 2016). A main characteristic of diagrams is that they are not individual, isolated inscriptions, but belong to a representational system. Thus, there are certain means and rules for their creation, reading and use. In the following, these activities with diagrams given by a representational system are called *diagrammatic activities* (Wille, 2020). Gestures as quasi-materialized inscriptions can be part of a diagram (Huth, in press). Thus, gesturing can also be part of diagrammatic activities. However, no diagram is a diagram by itself, but can be interpreted as such, if an appropriate representational system is known (Wille, 2020). Activities like constructing, experimenting, observing, noting, and assuring with the inscriptions help to clarify, structure, and coordinate thinking processes (Hoffmann, 2007). Thus, diagrammatic inscriptions themselves become the objects of argumentation processes. Furthermore, communication about them is possible (Dörfler, 2008). *Communication about diagrams and diagrammatic activity* includes both spoken and gestural expressions. As sign activity itself, communicating about it is an inevitable part of mathematical activity. It provides the use of denotations for diagrams that belong to different representational systems and in addition interpretations of diagrammatic reasoning (Wille, 2020). Furthermore, communicating about sign activity can lead to reflection. Reflection can be understood as a change of position (Freudenthal, 1991). This enables reinterpretations or the adoption of the perspectives of others. It can be caused, for example, by moments of irritation (Schülke, 2013) and can lead to new diagrammatic activities or a different interpretation of the diagram.

RESEARCH INTEREST

How does a preservice teacher support a child to overcome counting strategies for solving arithmetical problems and, within this support, how do diagrammatic activity and the communication about it intertwine?

SETTING

In an elective subject, preservice teachers support children of the first and second grade in learning to solve arithmetical problems without counting strategies in an individual support: One preservice teacher supports one child approximately 30 minutes per week during the spring semester. They work with support activities that were developed in the MALKA project (Wehren-Müller et al., 2018). At the beginning and at the end of the semester the competencies of the children are diagnosed. The individual support is videotaped. The supporting lessons are accompanied by a seminar at the university. There, theoretical aspects, reflections on diagnosis and support as well as practical experiences from the individual support are reflexively linked using video-based case studies.

METHOD

The analysis takes place in several steps. In a first step, an interaction analysis is carried out to reconstruct the interaction processes in detail (Krummheuer & Naujok, 1999). In a second step, the diagrammatic activity and the communication about it is analysed. For this purpose, an analysis method developed by Wille (2020) for imagined dialogues was adapted for interactions in the two representation systems twenty field and natural numbers. An analysis sheet is filled in for this purpose (see Figure 3): If a *diagram* is used in a turn, a *filled circle* is set in the column of the corresponding representational system. If *communication about* diagrams is used, a *dashed circle line* is set. If both take place, both are noted together. The filled circles or dashed circle lines are connected to each other by *solid lines* if a connection is made by *diagrammatic activities*. The line is *dashed* when the connection is made by *communicating about diagrams*. If both occur, both are noted together. If, in a turn, *diagrams of different representational systems correspond* with each other, they are connected by an *arrow*. The direction of the arrow indicates which representational system is used as the starting point. Communication that cannot be assigned to either one or the other representation system is noted as “others”.

Activities of the preservice teacher are noted in red, activities of the child in blue.

ANALYSIS

The three scenes analysed in the following are taken from a support situation between a preservice teacher, and a girl who is a student at the beginning of the second grade. In the following, we name them Tom and Samira. The transcripts were originally in German. Tom and Samira work on a task to the part-whole-schema, in which the different arrangements of chips in a row or in a block on the twenty field are to be discussed (see Figure 1).

Figure 1: Thirteen chips arranged in a row (a) with a full upper row of ten and in a block (b) with a full block of ten on the left side on the twenty field

Tom uses a magnetic twenty field with magnetic chips. When a chip is placed, there is a clicking sound. In the transcript, the fields of the twenty field are numbered as shown in Figure 2.

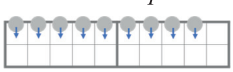
P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
P11	P12	P13	P14	P15	P16	P17	P18	P19	P20

Figure 2: Numbering at the twenty field

Scene 1

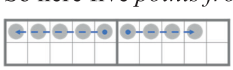
1 Tom: And how would you put the nine now on the field of points? *Pushes the chips together in a pile and closer to Samira.* Do it.

2 Samira: *Places the chips one by one from P1 to P9.*



3 Tom: Exactly. And how can you recognize it‘s nine now? Without counting?

4 Samira: So here five *points from P5 to P1* plus 4 *points from P6 to P9*



5 Tom: *Nods.*

6 Samira: equals nine.

7 Tom: Exactly

Summarizing interpretation

By asking Samira how she would put the nine chips onto the twenty field (turn 1), the preservice teacher Tom suggests that it depends on the arrangement of the nine chips. The arrangement is not given by him but allows Samira to find her own way. Samira places the chips in a row arrangement (turn 2). She begins on the left in the top line, which indicates that she is familiar with this convention for handling the twenty field. With the question in turn 3 Tom wants to find out, whether Samira can perceive the quantity by structural subitizing. Samira's answer (turn 4 and 6), in which she explains the decomposition of quantities and clarifies it through gestures, seems appropriate for him, as he confirms it (turn 5 and 7). This shows that decomposition seems essential for the preservice teacher.

Semiotic analysis

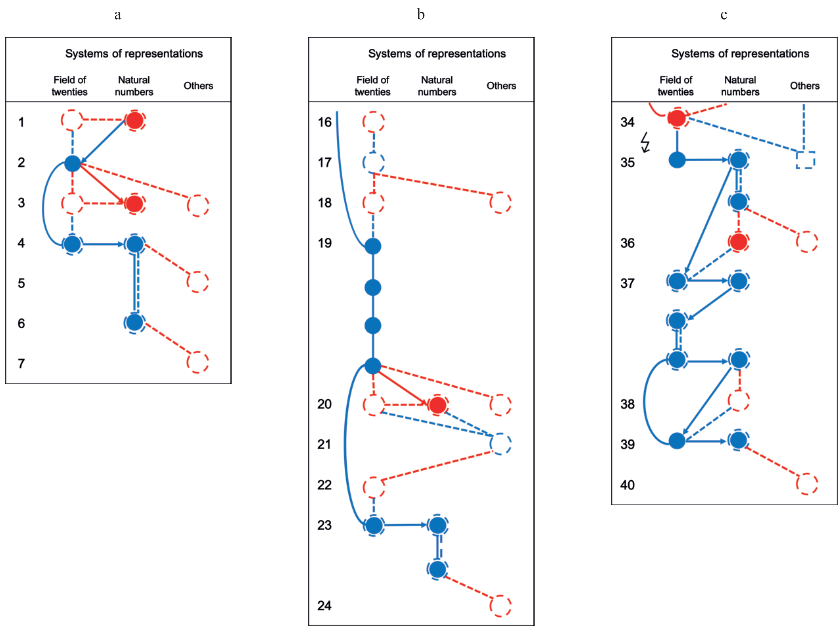


Figure 3: Analysis sheet of scene 1 (a), scene 2 (b) and scene 3 (c)

The semiotic analysis shows that diagrammatic activities take place in both systems of representation (see Figure 3a):

In the *representational system natural numbers*, *diagrams* are used both by Tom and Samira. Samira connects the diagrams in the natural numbers by her own *diagrammatic activities* and *communicates about* them (turn 4-6). Tom uses a diagram for the initiation of the task (turn 1) and for a request in combination with communicating about a diagram in the twenty field (turn 3).

In the *representational system twenty field*, *diagrams* are used exclusively by Samira (turn 2 and 4). Tom only communicates about diagrams in the twenty field. Samira's diagrams are connected by *diagrammatic activity*. In contrast, Samira's diagrammatic activity in the twenty field only takes place upon request of Tom (turn 3). In the same way, *communication about* the diagrams takes place only upon Tom's request (turn 3) by communicating about the diagrams and establishing a correspondence with the natural numbers (turn 3).

A look at the *correspondences* between the two representational systems shows that Tom creates a correspondence from the twenty field into the natural numbers that Samira has not used (turn 3). Furthermore, Tom initiates that Samira creates a correspondence between the representational systems natural numbers and twenty field (turn 1). In turn 4, Samira creates a correspondence from the twenty field into the natural numbers immediately after Tom created a correspondence with the same direction in turn 3.

Scene 2

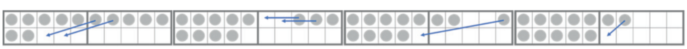
Samira has arranged twelve chips in a row on the field of twenties. This arrangement has been discussed. Then a theoretical repetition of the terms "row" and "block" has been carried out.

16 Tom: What is this now? *Points to the field of twenty.* Row or block?

17 Samira: Row.

18 Tom: Mhm. How would that look as block?

19 Samira: *Moves two chips from P6 and P7 to P13 and P14, two chips from P8 and P9 to P6 and P7, one chip from P10 to P15 and one chip from P7 to P16.*



20 Tom: *Nods slightly.* Mhm . and can you recognize that there are twelve of them real quick?

21 Samira: .. Yes.

22 Tom: Why?

23 Samira: *Points from P1 to P6* six *points from P11 to P16* plus six equals twelve.

24 Tom: Mhm.

Summarizing interpretation

After a repetition of the terms row and block arrangement, and a correct connection of the terms with the present arrangement on the twenty field (turn 16 and 17) Tom wants to find out Samira's abilities with regard to a block arrangement. He therefore asks her to change the present row arrangement, that has been chosen by herself, into a block arrangement (turn 18). Samira reacts by moving chips so that the block of tens on the left side is filled, and all the chips lie on the field as a block (turn 19). The way she moves the chips suggests that Samira does not decide what she wants to do until she is pushing. The question of whether Samira can determine the quantity of twelve by structural subitizing seems again important to Tom (turn 20). He attaches importance to a non-counting procedure by asking to do it “real quick” (turn 20). Samira hesitates briefly and only affirms his question (turn 21). Either she does not take his question as a request or she does not know the answer or is unsure of the answer. Tom concretizes his question by asking “why” (turn 22), whereupon Samira answers by explaining the decomposition, supported by gestures (turn 23). In her answer, Samira sticks to her focus on the rows by recognizing six in each row.

Semiotic analysis

The semiotic analysis (see Figure 3b) shows many similarities to scene 1. *Diagrammatic activities* take place in both systems of representation. In the *natural numbers*, *diagrams* are used both by the preservice teacher Tom and Samira. Samira connects the diagrams in the natural numbers by her own *diagrammatic activities* and *communicates about* them (turn 4-6). Tom uses a diagram for a request in combination with communicating about a diagram in the twenty field (turn 20).

In the *twenty field*, *diagrams* are used exclusively by Samira (turn 2 and 4). Tom only communicates about diagrams in the twenty field. Samira’s diagrams are connected by *diagrammatic activity*. In contrast to the system of natural numbers, her diagrammatic activity in the twenty field takes place upon Tom’s request (turn 23) except for turn 19. In turn 19, she connects several diagrams by diagrammatic activity. In the same way, *communication about* the diagrams takes place only upon Tom’s request (turn 20, 22), that is made by communicating about the diagrams and establishing a correspondence with the natural numbers (turn 20).

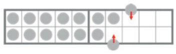
A look at the *correspondences* between the two systems of representations shows that the preservice teacher creates a correspondence to the system of representation that Samira has not used (turn 20), that is a correspondence from the twenty field into the natural numbers. Samira creates a correspondence only from the twenty field into the natural numbers, too (turn 23). These correspondences are created by Tom immediately before (turn 20).

A difference to scene 1 occurs at the very beginning by communicating about diagrams. The next difference occurs in turn 19: on the twenty field several diagrammatic activities are carried out directly one after the other by Samira. From turn 20 to 22 the request for further diagrammatic activities and communicating about them is done in two steps by communicating about diagrams in the twenty field and other communication.

Scene 3

The block arrangement has been discussed with thirteen chips. The arrangement of the twelve as double six from scene 1 was used by pushing the thirteenth chip away and back in again.

34 Tom: *Places two chips audibly on P8 and P17, briefly lifts the chips of P6 and P7 and audibly puts them back in place.* And how many are there now?



35 Samira: *Looks up briefly with the eyes, makes slight nodding movements with the head.* Eight. Äh sixteen.

36 Tom: Mhm. *Swallows.* Why sixteen?

37 Samira: I counted here *points from P1 to P8* that it is eight but there *points from P1 to P17* it can't be eight because there *points to P18* should be one more. So it is not sixteen.

38 Tom: .. But?

39 Samira: *Makes slight nodding movements with the head.* Fifteen.

40 Tom: Mhm.

Summarizing interpretation

By producing more clicking sounds when laying the chips (turn 34), Tom wants to avoid that Samira can continue counting from thirteen on. Samira needs a longer time to determine the cardinality of the quantity (turn 35). The slight nodding movements can be an indication that she is counting. It is also possible that she has counted the clicking sounds and is now confused. When determining the cardinality, she seems to concentrate on the rows, as she first gives the number of chips in the upper row as answer. She corrects herself immediately with another incorrect answer. Samira does not seem to have determined the cardinality of sixteen on the field, but rather to have obtained it by doubling the number of eight in her head. Tom appears surprised by the wrong answer and manages to think for a short time (turn 36). By asking the "why", according to his requests in previous scenes, Tom gets the chance to understand Samira's mistake and gives her the opportunity to justify or revise her answer. Samira takes up this possibility and explains that she has determined the cardinality of eight by counting (turn 37). This, again, shows her concentration on the rows. By comparing the number of chips in the upper and bottom row, she argues with the complete block of sixteen as a double eight. However, she does not seem to be able to determine the correct cardinality immediately, as she does not indicate it. By asking Samira to name the correct result (turn 38) Tom is directly following up on her previous utterance and makes no further comments. Samira, again makes slight nodding movements which indicate a counting procedure (turn 39). This means that she cannot determine the cardinality in the block arrangement without counting. It is also possible that she is still confused by the additional clicking sounds (turn 34) or that she cannot determine the result because she is thinking of an addition task ("there … should be one more" (turn 37)), but would have to subtract.

Semiotic analysis

The semiotic analysis (see Figure 3c) shows, that Samira uses *diagrams* in both representational systems. In the natural numbers as well as in the twenty field she connects diagrams by *diagrammatic activities* or *communicating about it* by herself (turn 35, 37). In turn 37, after the Tom's response to her mistake (↯), Samira establishes *correspondences* from natural numbers to the twenty field. To do this, she uses a diagram from earlier, when the mistake happened. In turn 39, after Tom's *communication about* Samira's diagram in the natural numbers, she creates for the second time a *correspondence* from natural numbers to the twenty field. Tom only uses *diagrams* at the beginning of the task in the twenty field, and one in response to Samira's mistake in turn 36. Here, he remains in the representational system of the natural numbers that she used before. Furthermore, he only *communicates about* the diagrams, whereby he also remains in the system she used before (turn 38) or communicates about other things. This scene is characterized by the fact that Samira switches back and forth between the two systems by herself when using the diagrams.

CONCLUSIONS

In scene 1 and 2 a pattern is visible: In both scenes the diagrammatic activity and communicating about it on the twenty field are stimulated by Tom by a request and a correspondence to the natural numbers (turn 3). The diagrammatic activity in the twenty field consists of gestures (Huth, in press) and shows the possible decomposition for a structural subitizing (Schöner & Benz, 2018). This seems to be essential for Tom and he does not go further into the diagrammatic activity with natural numbers. Changing the arrangement in scene 2 changes almost nothing in the pattern, except Samira's diagrammatic activities in turn 19. This indicates, that the pattern is determined less by the task than by the preservice teacher's requests and Samira's reactions. This is also shown by the fact that a slight reformulation of the request (turn 20) changes the pattern slightly, since the child no longer perceives it as a request for diagrammatic activity.

In scene 3, Samira's error in determining the number of chips and the minimal reaction of Tom in the form of a repetition of her answer as a query (turn 36) causes a change in the pattern. From this moment of irritation (Schülke, 2013) Samira starts to be diagrammatically active by her own, uses both representational systems more flexibly and communicates about it in order to reflect on her previous approach and to find new possibilities. Tom only supports this by asking her to continue her thoughts (turn 38). It can be assumed that the correspondences between the two representational systems carried out by the preservice teacher in scenes 1 and 2 have also contributed to the fact that Samira now also uses correspondences between the two representational systems after the irritation.

References

Dörfler, W. (2008). Mathematical reasoning: Mental activity or practice with diagrams. In M. Niss (Ed.), *ICME 10 Proceedings, Regular Lectures, CD-Rom* (pp. 17). Roskilde, Denmark: Roskilde University and IMFUFA.

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, O. Schmitt, R. Bruder, & W. Dörfler (Eds.), *Theories in and of mathematics education. Theory strands in German speaking countries (ICME 13 Topical Surveys)* (pp. 21-31). Berlin: Springer.

Freudenthal, H. (1991). *Revisiting mathematics education*. Dordrecht: Kluwer Academic Publishers.

Gerster, H.-D., & Schultz, R. (2004). Schwierigkeiten beim Erwerb mathematischer Konzepte im Anfangsunterricht [Difficulties in acquiring mathematical concepts in grade 1]. Retrieved from http://nbn-resolving.de/urn:nbn:de:bsz:frei129-opus-161

Häsel-Weide, U. (2016). *Vom Zählen zum Rechnen* [From counting to calculating]. Wiesbaden: Springer Spektrum.

Hoffmann, M. H. G. (2007). *Cognitive conditions of diagrammatic reasoning*. Retrieved from http://works.bepress.com/michael_hoffmann/1/

Huth, M. (in press). Gestische Darstellungen in mathematischen Interaktionen [Gestural representations in mathematical interactions]. In *Beiträge zum Mathematikunterricht 2020*.

Krummheuer, G., & Naujok, N. (1999). *Grundlagen und Beispiele Interpretativer Unterrichtsforschung* [Basics and examples of interpretative teaching research]. Opladen: Leske + Budrich.

Lüken, M. (2012). Young children's structure sense. *Journal für Mathematik-Didaktik, 33*, 263-285.

PHSG (2018). Mathematik lernen und kooperieren von Anfang an – MALKA [Learning and cooperating in mathematics from the very beginning – MALKA]. Retrieved from https://www.phsg.ch/de/forschung/projekte/mathe-lernen-und-kooperieren-von-anfang-malka

Scherer, P., & Moser Opitz, E. (2010). *Fördern im Mathematikunterricht der Primarstufe* [Promotion in primary mathematics education]. Heidelberg: Spektrum.

Schöner, P., & Benz, C. (2018). Visual structuring processes of children when determining the cardinality of sets. The contribution of eye-tracking. In C. Benz, A. S. Steinweg, H. Gasteiger, P. Schöner, H. Vollmuth, & J. Zöllner (Eds.), *Mathematics education in the early years. Results from the POEM3 conference, 2016* (pp. 123–144). Cham: Springer International Publishing.

Schülke, C. (2013). Mathematische Reflexion in der Interaktion von Grundschulkindern [Mathematical reflection in the interaction of primary school children]. Münster: Waxmann.

Wehren-Müller, M., Widmer, J., Hepberger, B., Moser Opitz, E., Ott, B., & Vogt, F. (2018). *Fördereinheiten für die individuelle Förderung (FEI). Intervention im*

Rahmen des Projektes MALKA [Support units for individual support (FEI). Intervention within the MALKA project]. St. Gallen: Pädagogische Hochschule St.Gallen.

Wille, A. M. (2020). Activity with signs and speaking about it: Exploring students' mathematical lines of thought regarding the derivative. *International Journal of Science and Mathematics Education, 18*, 1587–1611.

COMPARATIVE ANALYSIS OF TEXTBOOKS AS A WAY TO DEVELOP CRITICAL THINKING IN MATHEMATICS TEACHERS

Barbara Pieronkiewicz*, Małgorzata Zambrowska**

*Pedagogical University of Cracow, Poland

**The Maria Grzegorzewska University, Poland

Researchers agree that one of the main objectives of education is to develop students' ability to think critically. To support the development of such a skill in their students, teachers must first become critical thinkers themselves. In this paper we elaborate on the possibility of developing critical thinking skills of pre-service mathematics teachers through the means of comparative analysis of textbooks. We illustrate the potential of this activity providing a review of different textbook treatments of the common fractions division.

CRITICAL THINKING – PRELIMINARY THOUGHTS

According to the Foundation for Critical Thinking[1] the word 'critical' stems from two Greek words: *kriticos* which means 'discerning judgment' and *kriterion* for 'standards'. Etymology of this word could suggest that the development of critical thinking means developing discerning judgments based on standards. However, Halonen (1995) emphasises the fact that although this term has been widely used by the researchers who agree that the development of critical thinking should be promoted, there is still a lot of uncertainty regarding the exact meaning of this phrase. For instance, Moore (2013) has found seven interpretations attached to this term by different scholars. His respondents understood critical thinking as (1) judgement, (2) scepticism, (3) a simple originality, (4) sensitive reading of texts, (5) rationality, (6) an activist engagement with knowledge or (7) self-reflexivity.

In the daily practice of a teacher, critical thinking is absolutely essential. It is needed at each stage of lesson planning (e.g., selection of the teaching content, choosing appropriate methods of work and forms of communication, evaluation of the quality, validity and usefulness of teaching materials) as well as for making thoughtful decisions during the classes. In this article we focus our attention on the fact that in the current flood of information where a multitude of materials supporting the work of a teacher are available, taking a critical stance toward them is necessary.

[1] http://www.criticalthinking.org/pages/our-conception-of-critical-thinking/411

RESEARCH REPORTS ON THE ROLE OF TEXTBOOKS IN MATHEMATICS TEACHING

Textbooks are said to be the most pervasive and influential among all the resources available to the teachers and students (Howson, 1995; Usiskin, 2018). They are widely used by the teachers at the stage of both planning and conducting a lesson (e.g. Pepin, Haggarty, 2001; Remillard, 2005; Johansson, 2006; Nicol, Crespo, 2006; Tarr et al., 2006). They mediate between the intended and implemented curriculum by suggesting not only what to teach, but also when and how. For example, Huang et al. (2014) found that their sample of teachers stick to the textbook with respect to the:

> conceptualization of concepts and algorithms, the topic coverage, the sequence of content presentation, the approach to developing the concepts and algorithms, and the selection of problems and exercises. (p. 460)

On the other hand, researchers acknowledge (e.g., Kilpatrick, 2003) the fact that different teachers may pursue the same core curriculum in different ways and make different uses of the same textbook. Bütüner (2020) makes an important remark:

> A textbook, well-designed or not, should come to life in the hands of a well-equipped teacher. Such a teacher can identify the deficiencies in textbooks and enrich classes with content that allows students to more easily learn the underlying meaning of mathematical concepts via questioning. Such a teacher can also bring to the classroom original problems that are not present in the textbook but serve to measure conceptual understanding. S/he may use solution strategies that do not feature in the textbooks. (p. 289)

What makes a teacher give up on the faithful realization of the didactic proposal of the textbook authors? Instead of giving prospective teachers ready-made answers, it is worth offering them an opportunity to prepare a lesson plan (or play) on a chosen topic with the use of different textbooks. In such a context our understanding of 'critical thinking' includes - borrowing from Moore (2013) - judgement, scepticism and sensitive reading.

THE DIVISION OF FRACTIONS AS A CONCEPTUALLY CHALLENGING TOPIC

Primary school curriculum requires mathematics teachers to teach many fundamental mathematical concepts and procedures that form the basis for further learning. Among the topics covered by the curriculum, there are some known to be especially difficult, not only to the students but also to their teachers (e.g., Tirosh, 2000; Gichobi, 2019). The division of fractions is one of such topics. It has been shown that many pre- and in-service teachers find it difficult to address this topic conceptually and explain why the well-known algorithm ‘invert-and-multiply’ works. It is thus not surprising that this topic continues to attract the attention of many scholars. Some researchers focus particularly on how the textbooks authors address the division of fractions. For instance, Avcu (2018) explored the characteristics of three Turkish and three US mathematics textbooks in terms of type and frequency of context-based tasks on division of fractions. Li, Chen and An (2009) examined the treatments of fraction division in three Chinese, three Japanese and four American textbooks. Bütüner (2020) compared two Turkish and two Singaporean textbooks with respect to the instructional content of the unit on fraction division. According to our knowledge there has been no study on Polish textbooks approaches to the division of fractions that would be published in English.

THE DIVISION OF FRACTIONS IN POLISH PRIMARY SCHOOL MATHEMATICS TEXTBOOKS

Research problem

In our study we conducted a work similar to that of a teacher preparing a lesson on the division of fractions with the use of different mathematics textbooks. In our research questions however, we went beyond the scope of interest of a typical teacher. We wanted not only to compare the different textbook treatments of this topic (in a manner that could be adopted by a teacher), but also to examine the potential advantages of this work for a teacher.

Methodology

We have analysed chapters devoted to the division of common fractions (for brevity: DoF) found in four popular Polish series of primary school mathematics textbooks. In the text we denote the textbooks that we refer to with letters A, B, C and D, and full references are given at the end of this paper.

Results

Textbook A (5th grade)

The chapter on DoF begins with a brief introduction of reciprocals. The authors state that the numerator of one of the two reciprocal fractions equals the denominator of the other. Then the following numbers are considered: $\frac{5}{3}, \frac{3}{7}, \frac{1}{4}$ and 10, and their reciprocals are found. Next, the first example given to the students refers to a package with $\frac{3}{4}kg$ of chilli that was poured equally into 10 bags. The authors provide two solutions: in one, they convert kilograms into decagrams, get the answer in decagrams, and then convert the result into kilograms; in the second one they say that to divide by 10 means to take $\frac{1}{10}$ of the whole. After this task, there come three examples, involving division by a fraction which is not a whole number.

> Example 1: Four kilograms of nuts were separated into $\frac{1}{5}$ kg bags. How many bags were needed for these nuts?

Two different solutions are offered to the students. In the first one, the authors turn kilos into decagrams: $\frac{1}{5}kg = 20\ dag$ and $4kg = 400\ dag$. It is now easy to calculate the number of needed bags: $400dag: 20dag = 20$. The second solution is based on proportional reasoning: $1 = \frac{5}{5}$, and 1 kg consists of 5 portions of $\frac{1}{5}$ kg. Since we have 4 kg we will have five times as many portions. The authors notice that $4: \frac{1}{5} = 4 \cdot 5$, and that 5 is the reciprocal of $\frac{1}{5}$.

> Example 2: For $2\frac{3}{4}$ kg of candies, Robert paid 60zł50gr [auth.: 1 zł = 1 Polish złoty, 1 gr = 1 grosz; 1zł = 100 gr]. How much do you have to pay for $6\frac{1}{2}$ kg of such candies?

Again, the authors come up with two solutions. In one of them they change 60zł50gr into 6050gr, divide the amount of 6050 by 11 (since $2\frac{3}{4} = \frac{11}{4} = 11 \cdot \frac{1}{4}$), which allows them to calculate the cost of $\frac{1}{4}$ kg of candies, then multiply the result by 4 and get the cost of 1 kg. In the last step they determine the price of $6\frac{1}{2}$ kg of sweets. In the second solution the authors state that since $2\frac{3}{4}kg = \frac{11}{4}kg$ of candies cost 6050gr then the cost of 1kg can be calculated in the following way: $6050: \frac{11}{4} = 6050 \cdot \frac{4}{11} = (\dots) = 2200[gr]$.

> Example 3: During one hour, Susie collected $3\frac{3}{4}$ kg of strawberries. How much would she pick in three hours? How much time would she need to pick 16 kg of these fruits under the same conditions? Give an exact and estimated result.

Regarding the exact result, the authors solve the task as follows: $a \cdot 3\frac{3}{4} = 16$, $a = 16: 3\frac{3}{4} = 16: \frac{15}{4} = 16 \cdot \frac{4}{15} = \frac{64}{15} = 4\frac{4}{15}$.

We can see that in this textbook, introductory examples are solved in two ways. All the examples presented in the chapter on DoF are put in a practical context, which gives more freedom in searching for answers to the questions (e.g. converting units allows to 'workaround' the problem of dividing by a fraction). We find the tasks presented in textbook A extremely difficult. The subsequent examples do not introduce the student gradually and gently into increasingly complex calculations. Example 2 requires operating with numbers we can hardly call student friendly. Consider a task "For 2kg you paid 6zł. How much did 1kg cost?". Here the student will easily find out how to calculate the price per kilogram. Now think of the textbook task "For $2\frac{3}{4}kg$ you paid 60zł50gr. How much do you have to pay for $6\frac{1}{2}$ kg?". Calculating the price of 1 kg (which is even not the final goal in this task) is much more difficult, for it does not evoke the same intuitions as the previous task. Also we notice three different interpretations of DoF (partitive division, measurement and determination of a unit share respectively) provided in a series of subsequent examples within a single chapter. Our main concern here is whether the use of such conceptually and computationally difficult tasks and applying these different approaches to DoF almost at the same time is a good way to introduce a new, difficult topic.

Textbook B (6th grade)

The chapter on DoF begins with a catchy frame introducing reciprocals and three examples are considered: $3, \frac{3}{4}$ and $2\frac{1}{3}$. Next the authors focus on how to divide a common fraction by a natural number. The students are given a task where their role is to read from arithmetical graphs (as shown on Figure 1) results of some operations. Later on they are asked to check that two operations give the same result, e.g., $\frac{4}{5} : 2 = \frac{4}{5} \cdot \frac{1}{2}$.

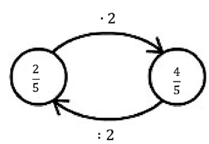


Figure 4

Then the students have to create their own graphs showing the results of some other operations, e.g., $\frac{8}{9} : 4$. The students are expected to observe or check that not only $\frac{8}{9} : 4 = \frac{8}{9} \cdot \frac{1}{4}$, but also $\frac{8}{9} : 4 = \frac{8:4}{9}$. A subsequent frame summarizes the results that should be obtained by that time, with a statement that when dividing a fraction by a natural number we simply multiply the fraction by the reciprocal of that number or, in specific cases, we may divide the numerator of the dividend by the given natural number. This part is followed by several tasks to be solved by the students. One of the tasks (below) is already an introduction to the common fractions division. (A picture of a tart is provided, and its parts can be easily counted).

> Mom divided the apple tart into 8 equal pieces. Ola, Jagoda and their brothers ate the whole cake. Each of the girls ate one piece, and the rest of the tart was divided among the boys: each of them got $\frac{1}{4}$ of the whole tart. How many brothers have Ola and Jagoda got? What part of the tart is one piece? What part of the tart did the boys divide among themselves? What part of the tart did each of them get? How many boys have divided the rest of the tart among themselves? Justify that in order to calculate how many brothers there were, you need to divide $\frac{6}{8}$ by $\frac{1}{4}$. See if instead of dividing the number by $\frac{1}{4}$ it is enough to multiply it by 4.

The next task offers two series of calculations – the students' role is to analyse them and create another three series of this kind on their own. Here is an example of such a series: $6:3 = 2$, $\frac{24}{4}:3 = 2$, $\frac{24}{4} \cdot \frac{1}{3} = \frac{24}{12} = 2$, $\frac{24}{4}:\frac{3}{1} = 2$. This task is followed by a frame making a clear statement that in order to divide two common fractions, we multiply the dividend by a reciprocal of the divisor. This is illustrated with several examples where the rule is applied, like: $\frac{2}{5}:\frac{1}{2}$, $\frac{4}{21}:\frac{9}{7}$.

From the introductory examples textbook B authors quickly infer that the observe behaviour holds true in any case: 'calculate in two ways, note that the results are the same', 'check that instead of dividing by a fraction, you can multiply it by its inverse'. The content of the frame ('invert and multiply') can hardly be regarded a summary of former investigations. In our opinion, the authors of this textbook are quite inept at introducing DoF. Their examples do not explain anything. Moreover, the tart task introduces unnecessary confusion with the fraction $\frac{6}{8}$. The students will rather think of it as $\frac{3}{4}$, but the authors force them to follow unnatural and more difficult computations with $\frac{6}{8}$.

Textbook C (5th grade)

In this textbook, finding the reciprocals and division of fractions by natural numbers are the content of a separate unit, preceding the chapter on DoF which starts with the following task:

> Draw three circles. Imagine they are pizzas. Cut them out and divide them into portions each equal to the $\frac{1}{4}$ of pizza. Count the number of portions. Write down operation $3:\frac{1}{4}$ and give the result.

In the next step, the student has to cut out three new round pizzas again and divide them into portions, each being a $\frac{3}{4}$ of pizza, answer how many portions are received, write down operation $3:\frac{3}{4}$ and give its result. Next, two pizzas are shown on a picture. Each of them is divided into 6 equal parts. The pupil has to say how many people can be served if everyone gets $\frac{2}{6}$ of pizza. Then three examples are given.

> Example 1: The pizza was divided into portions of $\frac{1}{6}$ of a pizza. How many portions were received?

This example is simple and does not require any additional explanations, also the result is obtained easily: $1:\frac{1}{6} = 6$.

> Example 2: Three apples were divided into portions of $\frac{3}{4}$ of an apple. How many portions were received?

The picture provides sufficient explanation. It shows three parts, each being a $\frac{3}{4}$ of an apple. It is also shown that three pieces, each equal to $\frac{1}{4}$ of an apple, that were cut off, form now the fourth required part, that is $\frac{3}{4}$ of an apple. Hence the result can be read directly from the picture: $3:\frac{3}{4}=4$. It is worth noting that this is exactly the same task that was already used in the introduction - only now the pizzas have been replaced by apples.

> Example 3: Hania has six bars [auth. - the picture shows that each bar consists of 5 identical pieces]. She wants to divide them into portions equal to $\frac{4}{5}$ of a bar (that is 4 pieces). How many portions will she get?

This example is not as easy as the two former. One fifth of each bar is cut off, and these removed parts form another one and a half of a required portion. The missing two pieces of the incomplete bar are drawn in a different colour, so that the students would know they are not there in fact. The picture is the only explanation provided to the operation: $6:\frac{4}{5}=7\frac{1}{2}$

Then the authors summarize the three examples, juxtaposing three pairs of different operations. In a single pair the two operations give the same results: $1:\frac{1}{6}=6$ and $1\cdot 6=6$, $3:\frac{3}{4}=4$ and $3\cdot\frac{4}{3}=4$, $6:\frac{4}{5}=7\frac{1}{2}$ and $6\cdot\frac{5}{4}=7\frac{1}{2}$. This is followed by a statement saying that the division by a number can be replaced by the multiplication by its reciprocal. This claim is immediately illustrated with several examples applying the formulated rule, for instance: $\frac{3}{4}:\frac{5}{7},\frac{4}{15}:\frac{2}{5}$.

We think that the introductory pizza-tasks may provoke a discussion in the classroom. However, we are concerned about the introduction of the invert-and-multiply algorithm which is based on the authors' observation that dividing by a given fraction gives the same result as multiplying by its inverse. In this context, especially the example with bars seems to be potentially difficult for the student for it does not explain why the division by $\frac{4}{5}$ could be replaced by the multiplication by $\frac{5}{4}$. Only in the juxtaposition of the operations the student can see that both operations give the same result. But why this is so? This is not explained.

Textbook D (5th grade)

In this textbook, unlike in others, reciprocals are addressed in a chapter on the multiplication of common fractions, which is followed by a chapter devoted to the division by a natural number. The authors introduce the algorithm for DoF with three exercises.

> Exercise 1: Match the sentence with appropriate operation: How many $\frac{1}{4}m$ sections fall into a $2m$ section? The operations given next to the statement are: $2\cdot\frac{1}{4}, 2:4, 2:\frac{1}{4}$.

Exercise 2: How many $\frac{1}{4}m$ sections fall into a 6m section? How many $\frac{3}{4}m$ sections fall into a $6m$ section?

Exercise 3: How many $\frac{1}{3}m$ sections fall into a 6m section? How many $\frac{2}{3}m$ sections fall into a $6m$ section?

Each of the tasks given in the last two exercises is accompanied by a picture where appropriate lengths are marked on a segment said to be $6m$ long. The students' task is to count the number of segments of the fractional length contained in the long one, and then write the results of relevant operations (e.g. $6:\frac{1}{4}=?$). The students are also asked to compare their results with those obtained when the dividend is multiplied by the inverse of the divisor. The authors conclude that it is easy to notice that the division by a fraction can be replaced by the multiplication by its reciprocal. This is followed by additional examples like, for instance: $\frac{2}{3}:\frac{3}{4}, 2\frac{1}{2}:\frac{4}{5}$ and $2\frac{1}{6}:1\frac{2}{3}$ illustrating (yet not explaining) the application of the rule.

The examples used in the textbook are very simple and analogous. Each of them involves dividing a whole number by a common fraction.

Results analysis

In each of the analysed textbooks there are two topics that precede the introduction of DoF: reciprocals and division of fractions by natural numbers. Looking at different textbooks we may see that their authors, although in different ways, strive to formulate the 'invert and multiply' algorithm and quickly move on to its application. Typically the algorithm is introduced through an observation that in the considered examples the same result is obtained regardless of whether the dividend is divided by a fraction or multiplied by its inverse. Furthermore, all the examples in which the authors attempt to explain where a given result comes from, are in the form of either *a fraction divided by a whole number* or *a whole number divided by a fraction*. Once made by the authors, the observation that a given division can be replaced with the multiplication by a reciprocal is immediately generalized to all cases. The problem is that this observation is made on relatively simple examples, and taken as a general rule is then immediately applied to far more difficult examples of the form: *a fraction divided by a fraction*.

In our opinion, textbooks C and D make the topic most accessible to the students: they offer easy-entry tasks and non-overburdening computations. The examples that they provide are conducive to proportional reasoning, which is not developed in the textbooks, but may become the subject of reflection and discussion of prospective teachers. However, the original approach of the textbook A authors shows that it is possible to offer other, perhaps more interesting examples. It may encourage the teachers to come up with their own tasks that will be accessible as much as those from textbooks C and D, but will contain some elements of the reasoning found in the first analysed manual. Textbook A, instead of showing that division by a fraction gives the same result as multiplication by its reciprocal, not only leads the student through interesting reasoning that, if extended and discussed in the classroom, might serve as justification to the algorithm, but also encourages school students to be creative in their search for other, easier ways to solve the given problems.

Working on excerpts from various textbooks can help to develop critical thinking in pre-service teachers. It makes a room for a constructive evaluation (judgement) by provoking some reflection: which textbook presents the topic in the most accessible way? Which approach would serve my students best and why? Does the textbook I use at school address all the aspects of the topic that are important to me? Experiences of working with textbooks can also develop the ability of careful and sensitive reading: what issues are not addressed by one textbook but appear in another? what different interpretations of a particular concept (e.g., DoF) occur in various textbooks? to what kind of mathematical activities do the textbook authors invite the students? Finally, after analysing various materials, a prospective teacher can realize that most of them are not free from deficiencies and thus it is reasonable to be sceptical toward any kind of ready-made material.

CONCLUDING REMARKS

We want to point out that we respect the work and hardship of textbook authors and we realize that writing a textbook, especially a good one, is extremely difficult. Someone could say it is easy to criticize, but our experience of working with pre-service teachers shows that constructive criticism does not come easily. Many times we have seen uncritical reception of teaching materials in prospective teachers. The most common argument that we have heard was that these materials had been prepared by some experts and the students were not competent enough to contest their work. We want our students to be aware that they not only have a right, but also a moral duty to be critical of the texts they read and all kinds of teaching materials they encounter.

Acknowledgement

This work is supported by the National Science Centre, Poland [grant number: 2018/31/N/HS6/03976].

References

Avcu, R. (2018). A Cross-National Comparison of Turkish and American Mathematics Textbooks in Terms of Fraction Division Task Contexts. *International Online Journal of Educational Sciences*, *10*(4).

Bütüner, S. Ö. (2020). A comparison of the instructional content on division of fractions in Turkish and Singaporean textbooks. *International Journal of Mathematical Education in Science and Technology*, *51*(2), 265-293.

Gichobi, M. N. (2019). Complexity of (Re)Learning the Meaning of Multiplication of Fractions in the Context of a Mathematics Content Course: An Exploration Study. In *Handbook of Research on Critical Thinking Strategies in Pre-Service Learning Environments* (pp. 385-409). IGI Global.

Halonen, J. S. (1995). Demystifying critical thinking. *Teaching of psychology*, *22*(1), 75-81.

Howson, A. G. (1995). *Mathematics textbooks: A comparative study of grade 8 texts*. Pacific Educational Press.

Huang, R., Ozel, Z. E. Y., Li, Y., & Osborne, R. V. (2014). Does classroom instruction stick to textbooks? A case study of fraction division. In *Mathematics curriculum in school education* (pp. 443-464). Springer, Dordrecht.

Johansson, M. (2006). *Teaching Mathematics with Textbooks. A Classroom and Curricular Perspective*. Luleå University of Technology, Luleå.

Kilpatrick, J. (2003). What works? In S. L. Senk, D. R. Thompson (Eds.), Standards-based school mathematics curricula. What are they? What do students learn? (pp. 471-493).

Li, Y., Chen, X., & An, S. (2009). Conceptualizing and organizing content for teaching and learning in selected Chinese, Japanese and US mathematics textbooks: The case of fraction division. *ZDM*, *41*(6), 809-826.

Moore, T. (2013). Critical thinking: Seven definitions in search of a concept. *Studies in Higher Education*, *38*(4), 506-522.

Nicol, C. C., Crespo, S. M. (2006). Learning to teach with mathematics textbooks: How preservice teachers interpret and use curriculum materials. *Educational studies in mathematics*, *62*(3), pp. 331-355.

Pepin, B., Haggarty, L. (2001). Mathematics textbooks and their use in English, French and German classrooms: A way to understand teaching and learning cultures. *Zentralblatt für Didaktik der Mathematik 33*(5), 158-175.

Remillard, J. T.: 2005, Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211-246.

Tarr, J. E., Chávez, Ó., Reys, R. E., & Reys, B. J. (2006). From the written to the enacted curricula: The intermediary role of middle school mathematics teachers in shaping students' opportunity to learn. *School Science and Mathematics*, *106*(4), 191-201.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for research in Mathematics Education*, 5-25.

Usiskin, Z. (2018). Electronic vs. paper textbook presentations of the various aspects of mathematics. *ZDM*, *50*(5), 849-861.

Textbooks

Textbook A: Lewicka, H., Kowalczyk, M. (2018) Matematyka wokół nas 5. Wydawnictwa Szkolne i Pedagogiczne, Warszawa.

Textbook B: Dubiecka A., Dubiecka-Kruk B., Malicki T., Piskorski P. (2019). Matematyka 6. Wydawnictwa Szkolne i Pedagogiczne, Warszawa.

Textbook C: Braun, M., Mańkowska, A., Paszyńska, M. (2018). Matematyka z kluczem 5. Część I. Nowa Era, Warszawa.

Textbook D: Dobrowolska M., Jucewicz M., Karpiński M., Zarzycki P. (2018). Matematyka z plusem 5. Gdańskie Wydawnictwo Oświatowe.

KNOWLEDGE AND SELF-EFFICACY OF MATHEMATICS TEACHERS IN SPECIAL EDUCATION CLASSES FOR LEARNING-DISABLED STUDENTS: THE DIFFERENCES BETWEEN MULTIPLICATION AND DIVISION

Rachel Filo, Iris Schreiber

Kibbutzim College of Education, Israel

This article describes a study investigating teaching multiplication and division in special education classes for learning disabled students. The study explored 64 teachers regarding two factors which affect teaching-learning processes: teachers' knowledge (common content knowledge, specialized content knowledge, knowledge of content and teaching, knowledge of content and students) and their self-efficacy regarding their knowledge. The findings indicated that the teachers have greater knowledge and higher self-efficacy regarding multiplication than they have regarding division.

INTRODUCTION

In the past few decades researchers have attempted to define the specific knowledge required for teaching in general and for teaching mathematics in particular (Ball, Thames, Phelps, 2008; NCTM, 2000; Shulman, 1986). They claimed that effective teaching of mathematics requires content knowledge (e.g. the ability to solve, knowledge of terms, concepts, laws and axioms) as well as pedagogical content knowledge (e.g. methods of presenting a problem, students' correct or incorrect perceptions). Mathematics teachers' knowledge has been found to be a significant factor in teaching-learning processes. It affects students' attainments in mainstream education as well as in special-education (Brownell, Sindler, Kiely & Danielson, 2010; Van-Inger, Eskelson & Allsopp, 2016).

Another factor that was found important for the quality of the teaching-learning process is self-efficacy. Self-efficacy is defined as people's belief in their ability to successfully organize and perform a series of actions that are necessary for achieving a desired result (Bandura, 1977). Studies show that the higher teachers' self-efficacy, the better relations they have with students, parents and colleagues, and the higher job satisfaction they have (Skaalvik & Skaalvik, 2010).

As far as we know, there are no studies examining systematically special education teachers' various components of knowledge, focusing on the differences between multiplication and division, and there are no studies examining teachers' self-efficacy regarding their knowledge in special education classes. In this study we examined the knowledge and the associated self-efficacy beliefs, among 64 mathematics special-education teachers. The focus in this paper is the comparison of the knowledge and the self-efficacy related to multiplication, and the knowledge and the self-efficacy related to division.

THEORETICAL BACKGROUND

Knowledge required for mathematics teaching

Shulman (Shulman, 1986), one of the most prominent researchers defining knowledge required for teaching, stated that a combination of subject-matter knowledge and pedagogical knowledge is required. In an attempt to define the required knowledge more acurately with relation to mathematics, researchers (Ball et al., 2008) classified two components of subject-matter knowledge (common content knowledge and specialized content knowledge) and two components of pedagogical content knowledge (pedagogical knowledge of content and teaching and pedagogical knowledge of content and students):

Common Content Knowledge (CCK) is a type of mathematical knowledge required also by those who do not teach, e.g. knowledge of solving or calculating. Specialized Content Knowledge (SCK) is mathematical knowledge unique for teaching, e.g. solving a problem in various ways. Knowledge of Content and Teaching (KCT) combines subject-matter knowledge with teaching, e.g. assessing the advantages and disadvantages of various tasks, and knowing different methods for representing a problem. Knowledge of Content and Students (KCS) integrates subject-matter knowledge with acquaintance of students, e.g. knowing students' common errors and possible reasons for these errors.

Studies show a correlation between teacher's knowledge and students' knowledge and attainments (Tchoshanov, 2011). In special-education classes studies indicate that the greater teachers' knowledge is, the better they cope with their students' learning disabilities, the better intervention programs they prepare and the better attainments their students have (Bronwell et al., 2010; Van Inger et al., 2016).

Self-efficacy

In order to perform a task effectively, people need both the suitable skills and the confidence in their ability to apply them as required (Bandura, 1977). Self-efficacy is a factor that might affect the teaching-learning process: teachers and students with higher self-efficacy invest more efforts and actions and are less likely to give up (Dellinger et al., 2008). Studies indicated that teachers with high level of self-efficacy have greater job satisfaction, are more involved in the preparation of personal curricula for the students and cooperate better with the parents and colleagues (Brouwers & Tomic, 2000). In special education classes, studies illustrate that teachers with a high level of self-efficacy are more willing to try different ways of teaching, are more organized in their instruction, have better relations with students and are less vulnerable to burnout (Allinder, 1994; Sarıçam & Sakız, 2014).

Multiplication and Division of natural numbers

Multiplication and division of natural numbers are a central part of the mathematics curricula in Israel and in other countries. According to the mathematics curriculum in Israel (for both mainstream and special education), teachers need to use various solution strategies and teaching methods in order to encourage numerical insight and computational competences (Ministry of Education in Israel, 2006). Adopting Ball's framework (Ball et al., 2008), it requires KCT and SCK.

Past studies emphasize the importance of teachers' pedagogical content knowledge (both KCT and KCS) regarding multiplication and division. Ma (1999), showed that students had better attainments when taught by teachers with broader knowledge, who focused on understanding the meaning of the algorithm and not on how to technically apply it. In another study, when teachers could explain the underlying principle behind the division algorithm, the students demonstrated a more thorough knowledge and created relations between operations (Takker & Subramaniam, 2018).

Various studies (e.g., Lee, 2007) emphasize the importance of consolidating the meaning of multiplication and division in the teaching process. Researchers suggest using illustrations, everyday problems, games and various solving models, for promoting the understanding of the algorithms (Cimen, 2014; Jong & Magruder, 2014; Lee, 2007). Regarding students with learning disabilities, in order to reinforce understanding over algorithms, studies emphasize the importance of teaching with demonstrations, with concrete elements and by creating connections to everyday life, (Bakker et al., 2016; Milton et al., 2019).

According to Ball et al., (2008), teachers should be acquainted with students' conceptions and misconceptions (KCS). Researchers examined and characterized students' common errors in multiplication and division and noted misunderstanding the base-ten number position and algorithmic errors (Bainbridge, 1981; Radatz, 1979). Past studies indicate that there are differences in teachers' knowledge between multiplication and division. Teachers have partial knowledge regarding division (both content knowledge and pedagogical content knowledge): they found it difficult to solve some division problems, to explain the division algorithm and to construct teaching units (Setoromo, Bansilal & James, 2018; Sitrava, 2018). In this study we tried to find the source of this finding by examining knowledge more accurately, regarding the four knowledge components suggested by Ball. According to the demands of the Israeli Ministry of Education, the curriculum in special education classes is similar to main stream education but it is adjusted to help the children to cope with their learning disabilities in different ways, such as various illustrations and demonstrations, modular learning, and an emphasis on learning strategies. Teachers in these classrooms are obligated to use various teaching methods and strategies in order to help the students overcome their difficulties.

As for self-efficacy, most of the studies related to teaching learning-disabled students investigated teachers' general self-efficacy for teaching (Sarıçam & Sakız, 2014) but not the self-efficacy related to knowledge of multiplication and division.

As far as we know, no studies have examined both the knowledge of multiplication and division and the self-efficacy regarding this knowledge among special education teachers. In particular, no studies have explored these through a comparison of multiplication and division. Therefore, the aims of this study are: a. to explore teachers' knowledge related to teaching multiplication and division and to compare the knowledge regarding these operations. b. to explore teachers' level of self-efficacy related to their knowledge of multiplication and division and to compare the self-efficacy regarding the knowledge of each operation.

METHODOLOGY

The participants were 64 mathematics teachers in special education classes for learning-disabled students. All the teachers have a Bachelor of Education degree in special education. The teachers differ in their training and we could identify 3 groups: 29 teachers specialized in mathematics teaching as part of their academic studies; 19 teachers were trained in mathematics teaching within instruction frameworks and in-service training programs; 16 teachers did not receive any special training for mathematics teaching. They also differ in their mathematics teaching experience: In each group, half of the teachers were novice teachers with less than 10 years of experience, and half were veteran with over 10 years of experience. They teach in classes with up to 15 children which is defined as special education but is part of a mainstream regular education school. All the children have normal intelligence quotient, but they have learning disabilities that cause learning gaps between them and their peers. The students study according to the curriculum, with adjustments either in the study pace or in some curriculum contents (not in the case of multiplication and division). The learning disabilities are varied – verbal disabilities, memory problems, communication problems, etc.

The research instruments were two questionnaires that were built for the purpose of this study: a knowledge questionnaire and a self-efficacy questionnaire. Instruments were validated by three mathematics teaching experts – researchers of mathematics education in elementary school and in special education. A preliminary study was conducted with 12 teachers and reliability was examined. A Cronbach's coefficient alpha was calculated, using the pilot data.

The reliability score (> 0.7) of the instruments had an acceptable level of reliability.

The knowledge questionnaire included 24 open-ended items that examined teachers' four knowledge components defined by Ball et al. (2008), in the following manner: CCK- solving multiplication and division exercises; SCK- solving the exercise in more than one way; KCT- presenting a way of teaching or illustrating an exercise; KCS- indicating typical errors students tend to make when solving an exercise. The questionnaire included items involving computations less than 100, and computations greater than 100.

The Self-Efficacy Questionnaire included 24 statements, which the teachers ranked on a 1-5 scale, according to their level of confidence in performing what the statement indicated (5=very confident and 1=not confident at all). For each item in the Knowledge Questionnaire there was a matching statement in the Self-Efficacy Questionnaire. For example: for CCK, in the knowledge questionnaire the teachers were asked to solve the exercise 9X7, and in the self-efficacy questionnaire they were asked to rank their confidence in solving multiplication exercises that involve computations less than 100. Another example: for KCS, in the knowledge questionnaire the teachers were asked to present students' common errors when solving 1407:7, and in the self-efficacy questionnaire they were asked to rank their confidence in predicting errors students make when solving division exercises that involve computations greater than 100.

RESEARCH FINDINGS

This article presents 16 questions from each questionnaire (8 multiplication questions and 8 division questions), which represent the findings of the entire questionnaire and contain exercises of two types: involving computations less than 100 and involving computations greater than 100.

Findings related to knowledge

The results (Table 1) show that teachers' knowledge differs between multiplication and division and also between the four components of knowledge: While most of the teachers solved the exercise correctly, they had partial knowledge of content and teaching, and their knowledge of content and students was found lacking.

Type of exercise	Knowledge component	Multiplication	Division	Difference (T-test)
Exercises involving computations less than 100	CCK	98	100	-0.320--
	SCK	84	82	0.551--
	KCT	76	65	2.197*

	KCS	56	46	2.308*
Exercises involving computations greater than 100	CCK	97	88	2.226*
	SCK	84	72	2.212*
	KCT	56	53	1.022--
	KCS	20	6	2.181*

Note: -- no significant difference * $p<0.05$

Table 1: Percentage of appropriate answers to knowledge questions

The main finding in Table 1 is the difference between multiplication and division: teachers have wider knowledge regarding multiplication than regarding division. It was found in most of the knowledge components and in both types of exercises. The teacher population was varied. The results show that experienced teachers have broader knowledge and higher self-efficacy than novice teachers, and that teachers who were trained in mathematics teaching were found to be more knowledgeable and more confident than teachers who were not. Nevertheless, the differences between multiplication and division, both in knowledge and in the self-efficacy, were significant in every teacher group.

The results also indicate that teachers' knowledge of teaching is partial. The percentage of teachers who presented ways of teaching or illustrating the exercises (besides the algorithmic way) was lower than the percentage of teachers who solved exercises correctly and in more than one way (CCK, SCK). Furthermore, the results show that teachers' KCS is lacking: the percentage of teachers that presented two common errors made by students was the lowest.

The following are examples of answers that the teachers provided, for two questions that examined KCT, regarding two exercises involving computations less than 100, 7X9 and 48:6. The teachers were asked to present a method for teaching or illustrating each of the exercises. The examples demonstrate that for the multiplication exercise the teachers presented more methods for teaching and illustrating than for the division exercise.

In the research, 76% of the teachers indicated a teaching method for the multiplication exercise 7X9 (other than memorizing the multiplication table) and proposed the following methods:

- Demonstrating the meaning of multiplication as a repetitive addition:
  In an exercise such as 9+9+9+9+9+9+9 or 7+7+7+7+7+7+7+7+7
  In an everyday problem such as 7 bags with 9 cookies in each one
- Using models:
  Presenting the exercise with concrete elements using disks or bottle caps
  Drawing 9 groups of 7 circles in each group
  Calculating area by drawing a rectangle with 7 rows and 9 columns

- Using operation laws such as distributive property:
  In an exercise such as 7 X 9 = 7 X (10-1) = 7 X 10 – 7 X 1 = 70 – 7 = 63
- Using various activities and games:
  multiplication songs, ball games, computer games or applications, showing special characteristics of the multiplication of 9, etc.
- Using easier exercises with smaller digits such as 3X3 instead of 9:
  7 X 9 = 7 X 3 X 3 = 21 X 3 = 63

Most of the methods the teachers suggested were recommended in the professional literature as activities that could consolidate the meaning of multiplication. However, in the division exercise 48:6, fewer teachers (65%) suggested teaching methods, and suggested less ways to illustrate the exercise:

- Demonstrating the meaning of division in terms of partition:
  In an everyday problem such as 48 candies that need to be put into 6 bags
  Drawing 48 circles in 6 rectangles, 8 circles in each rectangle
- Using distributive property: 48:6 = (12 + 36): 6 = 12:6 + 36:6 = 2 + 6 = 8
- Emphasizing the connection between division and multiplication by using a multiplication equation: 6 X _________ = 48

None of the teachers suggested a method for teaching or illustrating using concrete elements, game or a song, neither did the teachers who had previously suggested it for multiplication. None of the teachers suggested to refer to the exercise as quotative division (the teachers referred it only as partitive division-dividing into equal parts). These examples illustrate that the teachers have wider knowledge of teaching multiplication than of teaching division. It is also manifested in the teachers' self-efficacy, as will be shown in the following chapter.

Findings related to self-efficacy

In the self-efficacy questionnaire, the teachers were asked to rank their confidence in carrying out activities regarding knowledge of multiplication and division. The results (Table 2) show that teachers generally ranked their confidence levels high, between 3-5, but were less confident in their pedagogical content knowledge than in their content knowledge, and the variance of the findings was greater.

Type of exercise	Knowledge component	Multiplication M (sd)	Division M (sd)	Difference T-test
Exercises involving computations less than 100	CCK	4.94 (0.24)	4.80 (0.47)	2.857 **
	SCK	4.53 (0.90)	4.27 (1.03)	3.401***
	KCT	4.58 (0.77)	4.32 (0.95)	4.138***
	KCS	4.12 (0.90)	4.05 (0.90)	2.308 *

Exercises involving computations greater than 100	CCK	4.71 (0.58)	4.44 (0.88)	3.432***
	SCK	4.55 (0.79)	4.38 (0.89)	3.008 **
	KCT	4.55 (0.79)	4.27 (0.99)	4.088***
	KCS	3.82 (0.98)	3.78 (1.04)	2.551 *

*p<0.05 ** p< 0.01 *** p<0.001

Table 2: Mean and standard deviation of confidence level of teachers

As we can see, the teachers ranked their confidence significantly higher when referring to multiplication than when referring to division: in all the statements in the questionnaire, in all components of knowledge and in all types of exercises.

It can also be noted that teachers demonstrated the highest level of confidence regarding CCK and demonstrated the lowest level of confidence regarding KCS- the teachers were more confident in their ability to solve exercises than in their ability to teach or to predict students' errors. Furthermore, the teachers were less confident when referring to exercises involving computations greater than 100 than when referring to exercises involving computations less than 100.

DISCUSSION AND CONCLUSIONS

The present study explored knowledge and self-efficacy regarding multiplication and division, among mathematics teachers in special education classes. The study is novel for examining systematically the four components of knowledge and the associated self-efficacy sense, while comparing multiplication and division. Another novelty is conducting the study with special education teachers.

Several meaningful findings were obtained from the study, the prominent among them being the difference between multiplication and division, both in knowledge and in self-efficacy.

Regarding teachers' knowledge, findings show that in most of the questions, especially those examining KCT and KCS, the teachers demonstrated a broader knowledge in multiplication than in division. This finding is in accordance with previous studies that have found that the teachers are lacking pedagogical content knowledge regarding division (Sitrava, 2018). This finding may be due to the fact that when teaching division, most teachers depend on the connection between division and multiplication (Downton, 2013).

The importance of teachers' knowledge, especially pedagogical content knowledge, is well acknowledged. Researchers recommend using illustrations and presentations in order to enhance comprehension, especially for student with learning disabilities (Milton et al., 2019). Teachers who participated in this study revealed partial pedagogical content knowledge, especially regarding division, and that might negatively affect their students' learning process.

Regarding self-efficacy, the teachers' level of confidence in each component of knowledge of multiplication was significantly higher than that of division. A possible reason for this finding, in line with previous studies (Lee, 2007), is the difficulty in teaching and performing long-division problems, an algorithm that raises difficulty among many students. However, we believe that the results reflect a more thorough gap in teachers' knowledge because they were also less confident in exercises that involve computations less than 100, which do not involve long-division.

The teachers were least knowledgeable and least confident regarding knowledge of content and students. A possible explanation is the inclusion of teachers without specific training in mathematics, as well as novice teachers, in the study. Those teachers were integrated in the study as part of the sample, since they are part of the population of mathematics teachers in special-education classes and as such, it is crucial to examine their knowledge and their level of self-efficacy. The focus in teaching special education classes is analyzing the errors each student tends to make and identifying their source. Each student is entitled to have a personal program based on this analysis. The teachers' knowledge and self-efficacy examined in this research can be are the foundations for effective teaching in special education classes. The research findings lead to our suggestion, like Van Inger's (Van Inger et al, 2016), for counseling and guiding teachers regarding various ways of teaching. We suggest it not only but especially for novice teachers or for teachers who weren't trained in mathematics teaching. The findings of the study may be the foundations for building intervention programs aiming to promote teachers' various knowledge components. Such programs may also increase the teachers' sense of self-efficacy

References

Allinder, M. R. (1994). The relationship between efficacy and the instructional practices of special education teachers and consultant. *Teacher Education and Special Education, 17,* 86-95.

Bainbridge, R. (1981). To err is human: Towards a more positive approach to young children's mistakes in arithmetic. *Mathematics in school, 10,* 10-13.

Bakker, M., Heuvel-Panhuizen M., & Robitzsch A. (2016) Effect of mathematics computer games on special education students' multiplication reasoning and ability. *British Journal of Educational Technology, 47,* 633-648.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching- What makes it special? *Journal of Teacher Education, 59,* 389-407.

Bandura, A. (1977). Self-efficacy: Toward a unifying theory of behavioral change. *Psychological review, 84*, 191-215.

Brouwers, A., & Tomic, W. (2000). A longitudinal study of teacher burnout and perceived self-efficacy in classroom management. *Teacher and Teaching Education, 16*, 239-253.

Brownell, M. T., Sindelar, P. T., Kiely, M. T., & Danielson, L. C. (2010). Special education teacher quality and preparation: Exposing foundations, constructing a new model. *Exceptional Children, 76,* 357-377.

Cimen, O. A. (2014). How do children multiply: reflection and prospective pedagogical implications. *Procedia – Social and Behavioral Sciences, 159,* 593-597.

Dellinger, A., Bobbett J., Olivier D., & Ellett C. (2008). Measuring teachers' self-efficacy beliefs: Development and use of the TEBS- Self. *Teaching and Teacher Education, 24*, 751-766.

Downton, A. (2013). Making Connections between Multiplication and Division. Mathematics Education Research Group of Australasia, Paper presented at the Annual Meeting of the Mathematics Education Research Group of Australasia (MERGA).

Jong, C., & Magruder, R. (2014). Beyond Cookies: Understanding various division models. *Teaching Children Mathematics, 20,* 366-373.

Lee, J. E. (2007). Making sense of the traditional long division algorithm. *Journal of Mathematical Behavior, 26,* 48-59.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates.

Ministry of Education in Israel (2006). Curriculum for mathematics in kindergarten. Jerusalem, Israel: Ministry of Education. [Hebrew]

Milton, J. H., Flores, M., Moore, A. J., Taylor, J., & Burton, M. E. (2019). Using the concrete representational abstract sequence to teach conceptual understanding of basic multiplication and division. *Learning Disability Quarterly, 42,* 32-45.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, Virginia, USA: NCTM.

Radatz, H. (1979). Error analysis in mathematics education. *Journal for Research in Mathematics Education, 10,* 163-171.

Sarıçam, H. & Sakız, H. (2014). Burnout and teacher self-efficacy among teachers working in special education institutions in Turkey. *Educational Studies, 40(4),* 423-437

Shulman, L. S. (1986). Those who understand: knowledge growth in teaching. *Educational Researcher, 15*, 4-14.

Skaalvik, E. M., & Skaalvik, S. (2010). Teacher self-efficacy and teacher burnout: a study of relations. *Teacher and Teacher Education, 26,* 1059-1069.

Setoromo, M., Bansilal, S., & James, A. (2018). Lesotho grade R Teachers' mathematical knowledge for teaching numeracy. *South African Journal of Childhood Education*, *8*(2), 1-9.

Sitrava, T. R. (2018). An investigation of prospective mathematics teachers' knowledge of basic algorithms with whole numbers: A case of turkey. *European Journal of Educational Research*, 5, 513-528.

Takker, S., & Subramaniam, K. (2018). Teacher Knowledge and learning "in-situ": a case study of the long division algorithm. *Australian Journal of Teacher Education, 43,* 1-20.

Tchoshanov, M. A. (2011). Relationship between teacher knowledge of concepts and connections, teaching practice and student achievement in middle grades mathematics. *Educational Studies in Mathematics, 76,* 141-164.

Van Inger, S., Eskelson, S. L., & Allsopp, D. (2016). Evidence of the need to prepare prospective teachers to engage in mathematics consultations. *Mathematics Teacher Education and Development, 18*(2), 73-91.

DIAGNOSTIC TEACHING AND EDUCATIONAL SUPPORT IN PRESERVICE TEACHER TRAINING

Sabine Vietz, Tobias Huhmann

University of Education, Weingarten, Germany

Diagnostic teaching and educational support form the basis for adequately addressing heterogeneity in learning mathematics. A new approach for an education module in the second phase of teacher education offers trainee teachers the opportunity to acquire these competencies in a theory-based and practice-oriented way: video-recorded microteaching and accompanying supervision are two elements of the approach, which is designed to encourage critical reflection on teaching performance. At the same time, this approach is intended to enhance the professionalisation in diagnostic teaching and supporting. The results of the module-evaluation show that these specific elements encourage and foster critical reflection, and this is seen essential to develop professionalisation.

With the beginning of the 2000s, learners' outcomes and the influence of teachers have come into focus, e.g., with studies like TIMSS and PISA. As Hattie puts it, "what teachers do matters", and "self-regulatory attributes" (Hattie, 2010, p. 22) are essential for the teachers' own learning process. Critical thinking therefore is important for teaching and learning mathematics: teachers must critically reflect on their own understanding and knowledge of the subject and the learning opportunities they offer in order to create heterogeneity-sensitive lessons to equally support all children.

PROFESSIONAL COMPETENCE

Shulman (1987) identified different areas of pedagogical knowledge as components of professional knowledge. He emphasised content knowledge (CK), general pedagogical knowledge (PK) and pedagogical content knowledge (PCK). Studies have shown that both CK and even more PCK of teachers are key for the educational success of pupils (Baumert et al., 2010; Hill et al. 2005). Baumert and colleagues consider "both CK and PCK as critical professional resources for teachers, each requiring specific attention during both teacher training and classroom teaching practice." (Baumert et al., 2010, p. 164) In their study COACTIV the between-class variance of 39% could be explained with PCK, which is relevant for developing cognitive activating mathematical learning environments. Models of teacher learning like the Interconnected Model of Professional (Teacher) Growth (Clarke & Hollingsworth, 2002, p. 951) show that teachers' learning is a dynamic and interactive process that takes place in an iterative process which includes "enaction" and "reflection" between the different domains. They differentiate between the External domain which represents the systems and policies, the Personal domain where they contextualize teachers' characteristics such as beliefs, attitudes and knowledge and furthermore the domains Practice and Consequence with a focus, e.g., on teaching and students' learning. Ponte and Chapman (2016) analysed educational studies of prospective teachers. As a conclusion they state that it is important to inquire one's own teaching and learning to acquire and construct "knowledge about mathematics, about students, about teaching, and about oneself". (Ponte & Chapman 2016, p. 293) In teacher education, therefore, it is important to interweave theory with practice, through which knowledge can be acquired, activated and applied in real situations, but also critically reflected upon and thus deepened.

Mathematical Knowledge for Teaching – Diagnostic teaching

For teacher education it is important to know which aspects are relevant to teach mathematics. Ball and colleagues detailed the differentiation of CK and PCK and built the Mathematical Knowledge for teachers (MKT) framework. They show that the professional knowledge mathematics teachers need to teach in a heterogeneously sensitive way is multidimensional. Most important for diagnostic teaching are the following three domains: (1) specialized content knowledge (SCK) is described as "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400), (2) knowledge of content and students (KCS) which includes e.g. "knowledge of common students conceptions and misconceptions" and the ability to interpret students thinking (3) knowledge of content and teaching (KCT) refers to the "design of instruction" e.g. the ability to choose fitting tasks, examples, representations and methods for teaching (Ball et al., 2008, p. 401). Teachers must have meta-knowledge (SCK, KCS, KCT) to have the ability to diagnose in a competent way. The knowledge of the development of learning processes (such as the concept of numbers, the decimal number system or arithmetic operations) is fundamental, together with the knowledge of the mathematical learning subject, as well as the knowledge of learning barriers and students' misconceptions (Ball et al. 2008; Scherer & Moser-Opitz, 2010). Equally important is the knowledge of the diagnostic and competency potential of tasks and inherent difficulty characteristics. In addition, the knowledge of "diagnostic tools", their significance and range of useability is necessary to select suitable diagnostic instruments and to be able to provide effective educational support (Scherer & Moser-Opitz, 2010). If teachers individually assess their students' learning levels, they will be able to individually support them. Thus, as a first step, process-related diagnostic teaching in individual or in classroom situations is important for planning adaptive support activities based on the learning levels of the children. The purpose is to identify individual thought processes and the underlying ideas. On the basis of SCK, teachers must have the ability to interpret the content on the basis of KCS in order to analyse individual mathematical thinking and action appropriately, to create an adaptive fit between learning requirements and opportunities on the basis of KCT. This opens up learning opportunities for children to develop understanding of mathematical concepts by recognizing and interlinking basic mathematical relationships.

Microteaching in teacher education

During their everyday teaching, teachers have to deal with a variety of tasks in the classroom (Bromme, 1992). The concept of microteaching (Allen & Ryan, 1974) offers possibilities to reduce this variety: The focus is on individual aspects of teaching, such as initiating or accompanying learning processes through thought-provoking questions. The overall goal is to create selected situations from classroom lessons and thus to reduce the complexity of teaching. Analysis and repetition of the teaching situation routinise the professional didactic action and enable competence growth through critical reflection. Prospective teachers and learners can gather, deepen and reflect on their experiences in this "protected space for teaching and learning of mathematics": For example, the prospective teachers focus on individual aspects, such as on the formulation of thought-provoking questions and direct their attention specifically to their teaching behaviour. Following these teaching situations, these aspects are analysed and reflected with professional support in connection with the children's thinking procedures and processing. Direct feedback and critical self-reflection as well as being given the opportunity to repeat the identical teaching topic immediately can contribute to the further development of the individual trainee's professionalisation.

Diagnostic Teaching and educational Support

The project presented in the following is implemented in cooperation with the Laupheim State Seminar for Teacher Education and In-service Qualification. The basis is a new approach for the second phase of teacher education for "Diagnostic Teaching and Supporting in Mathematics Education" (D&S). In addition to the theoretical education module, the new approach includes phases of microteaching with process-accompanying supervision and focuses on children with learning difficulties in mathematics. The overriding expectation is to perceive, use and develop CK and PCK as a professional basis for teaching by critically reflecting on one's own teaching activities in D&S and thus contribute to one's own professionalisation. The development of competencies for D&S is of key importance to be able to deal with the heterogeneity in mathematics teaching more appropriately.

CONTEXT OF THE STUDY

In the federal state of Baden-Württemberg, Germany, teacher education for the primary level consists of university studies (first phase of teacher education) and a one-and-a-half-year teacher training (second phase of teacher education). During teacher training, the trainee teachers (TTs) teach at a school and are professionally supervised in their further education by lecturers from the Seminar for Teacher Education. The further development of the education module “Diagnostic Teaching and Supporting in Mathematics Education” takes three fields of work into account. First, the theoretical basics of PCK are taught in accompanying seminars: These are the different basic arithmetic skills and learning barriers in the areas of developing a suitable understanding of the concept “of numbers”, “of the decimal number system” and “of the basic arithmetic operations“. TTs explore and analyse “substantial learning environments” that support the development of these basic competencies. Central to all areas is intermodal transfer (Bruner, 1971; Rau & Mathews 2017; Wittmann, 1981). Children should carry out actions and reach mathematical interpretations or be encouraged to tell a suitable story or describe a term with regard to the transfer of representation. In order to develop the didactic ability to interpret in a targeted manner, process-related diagnostic possibilities such as informative tasks or the diagnostic interview are discussed, as well as tasks, task processing and evaluation of task processing, and video clips are analysed. Second, in order to use their own diagnostic and support activities in a focused and engaged manner, TTs can diagnostically assess and support one individual child with specific learning difficulties within the protected framework of microteaching. TTs have the opportunity to apply the theoretical knowledge gained in the seminars by conducting a diagnostic interview with a child. From the didactic evaluation of their diagnostics, they can derive support goals and support contents, develop initial ideas and discuss suggestions on how the respective child can be individually supported in a way that promotes their learning process. TTs support the respective child weekly for about three months individually and within class. The individual supporting situations are filmed. Third, the analysis of the teacher trainee’s own actions takes place in an accompanying supervision seminar. TTs present the status of their support, report about successes and problems and the conception of their support plan. Every two weeks, a short sequence should be analysed from all participants based on a video excerpt from the teaching situation. The focus is on analysing the teaching behaviour, the selection of tasks, the chosen demonstration materials and the development of the child’s learning progress. Lecturers from the seminar and the university accompany the TTs during supervision and guide them through their process of critical self-reflection of their own teaching activities as well as the consideration of alternative activities by means of thought-provoking questions. TTs collaborative evaluation of their teaching performance applying different elements, processes, focal points and perspectives, aims to sensitise them with regard to learning difficulties in order to professionalise their future diagnostic teaching and educational support. TTs gain insights into the variety of learning difficulties, as the difficulties of all supported children are considered. The goal of analysing TTs’ diagnosing and support actions and activities is that they will develop heterogeneity-sensitive mathematical-didactical perception and action.

METHODOLOGY AND RESEARCH QUESTIONS

The education module was developed, implemented, examined and enhanced in terms of design-based research. To evaluate (i) the education module and (ii) the aim to build up and to deepen diagnostic and support skills, qualitative data collection was conducted after the first implementation in 2019/20 by means of guideline-based interviews and in the following evaluated in terms of content analysis and categorisation (Kuckartz, 2012). Individual telephone interviews (approx. 30-40 min) were conducted with TTs (N=7) and lecturers (N=4) independently of each other – based on interview guideline with scale and open-ended questions. The questions were subdivided into the following categories: (1) acquisition of competencies for diagnostic teaching and supporting during university studies and teacher training, (2) attitude towards teaching and learning mathematics and implementation of D&S in lessons and (3) further development of the education module for TTs. For the evaluation of the interviews, transcripts were generated, deductive categories were developed on the basis of the questions in the interview guideline and further categories and subcategories were formed inductively on the basis of the transcripts. The evaluation was carried out with the software Atlas.ti. The general objective is the evaluation of the education module "Diagnostic Teaching and Supporting in Mathematics Education", which has been further developed and adapted. The findings are presented for the following research questions:

1) How do the TTs in the area of D&S describe and evaluate their competence acquisition during their own studies (first phase of teacher education)?
2) How do the TTs in the area of D&S describe and evaluate their competence acquisition in the teacher training (second phase of teacher education)?
3) (How) did the further developed education module contribute to the acquisition of competencies? (This question was addressed to TTs and to lecturers regarding the acquisition of competencies of TTs).

RESULTS

At the beginning of the interview, TTs assessed the competencies they had acquired during their studies on a scale from 1 (low competence) to 10 (very high competence) separately for the areas of D&S and after one year of the teacher training. Moreover, they were asked to give reasons for their rating, outlining which competencies they thought they had or had not acquired. The self-assessment interviews showed that the TTs rated their competencies differently. In order to analyse how the TTs think their acquired competencies, categories were formed deductively considering the already mentioned aspects relevant for teaching (MKT) and that experience plays a fundamental role in competency development (Lipowsky, 2010). The transcripts could be analysed and coded with "CK" and PCK" as no differentiation to subcategories was possible, and "practical experience", but it became evident that this subcategory had to be differentiated inductively. For the analysis narrower and wider text contexts were taken into account. The tables show the results of "acquisition of competency" with respective descriptions and deductively and inductively formed subcategories. The numbers in square brackets show how many TTs mentioned the subcategories. All TTs mentioned the different aspects ("CK", PCK", "practical experience") as being important to acquire competencies. The differences in their statements can be found by taking a closer look to their self-assessment. TTs who rated their competency after their studies on the scale between 7 and 8.5, said they had acquired profound PCK in D&S during their university studies. Furtherly, they had used opportunities to gain practical experience while being supported professionally in supervision seminars. TTs who stated that they had acquired only a small amount of PCK in their studies and gained little individual practical experience in D&S, do consider this as lack of learning for their acquisition of competencies in these areas. TTs from this group rated their competency between scores of 3-4. The quotations listed below are from these TTs. All interviewed TT and lecturers were anonymised by numbers. The quotations were translated from German.

> TT7: Well, I have diagnosed and supported an individual child before […] I have dealt with it accordingly, I just did not know whether my actions were any good or made any sense, because feedback was simply missing.

> TT6: From such a one-time diagnosis I think you cannot gain any deeper experience, simply yes, first of all it was like a shot in the dark, however a deep knowledge base was not there.

Category	Description	Codes
Acquisition of competency	Contains aspects which show how competencies were acquired	Content knowledge [6] Pedagogical content knowledge [6] Practical experience once with an individual child [3] Practical experience with an individual child over a period [2] Practical experience with an individual child over a period with guided supervision seminars [3]

Table 1: Results to "acquisition of competency" during university studies

The statements clarify that multiple diagnostic and support activities, as well a critically-reflective exchange of perspectives and professional support are key to develop professional competency during the university studies. The interviews with the lecturers validate the knowledge heterogeneity among TTs.

L3: ... that there is simply such a different prior knowledge and there are some who are coming along who are almost somewhat saying: We already had this so intensively during our studies and now I want something different – I am exaggerating now - and others who look at you like a deer in a headlight - and don't know what diagnostic teaching and support should be at all.

L1: …because we have now seen from the feedback that they have very different starting points. Some have already dealt with diagnosis and support during their university studies, and some have already had their own students to support and have already gained a great deal of experience and knowledge, while others have not even been in contact with diagnostic teaching at all.

In summary, the following can be stated: TTs and lecturers emphasize in particular the interweaving of 1) well-founded subject-related PCK, with 2) the application of this acquired knowledge in authentic teaching-learning experiences and 3) the critical examination(s) through accompanying supervision for knowledge acquisition in the university education. Subsequently, it was determined which components were assessed as important for the acquisition of competences during the teacher training. Deductively the same subcategories were formed and some subcategories were mentioned identically ("CK" and "PCK") but the subcategories for "practical experience" had to be formed and differentiated inductively ("practical experience with an individual child over a period", "practical experience in D&S in lessons"). The code "interweaving of theory and practice" was also added as the TTs emphasised that now they have the possibility to transfer the theoretical knowledge they gain in the seminars immediately into practice. Furthermore, new subcategories for "reflection" were formed and differentiated inductively ("supervision", "professional learning teams", "individual reflection") as shown in Table 2.

Category	Description	Codes
Acquisition of competency	Contains aspects which show how competencies were acquired	Content knowledge [3] Pedagogical content knowledge [3] Interweaving of theory and practice [6] Practical experience with an individual child over a period [6] Practical experience D&S in lessons [6] Supervision [3] Professional learning teams [2] Individual reflection [4]

Table 2: Results to the "acquisition of competence" during teacher training

The following quotations show that the TTs emphasised critical reflection, the interweaving of theory and practice and the transfer of D&S to the daily lessons.

TT2: New insights related to my normal lessons, because you can also transfer a lot of the individual support...

TT5: Because I simply saw the student not only now in the supporting lesson, but also in the daily lessons and because I could gain even more insights [...]. Sure, in the support lesson or when diagnosing in the diagnostic interview, […] but now in practice or through supervision and exchange with others, one has seen other perspectives, perceived the child differently and one could [...] now perhaps also support and diagnose a little more specifically and individually.

The new formed subcategory "practical experience in D&S in lessons" and the presented quotations lead to the conclusion that many of the interviewed TTs are aware that competencies in D&S are not only required in individual supporting situations, but that these situations contribute to the further development of professional competency for everyday teaching: 1. to a more detailed perception of the individual child and 2. to the transfer of findings from individual support to classroom teaching, by perceiving the diversity of mathematical individuality and by orienting the design of teaching both didactically and methodologically to this heterogeneity. Furthermore, they are aware that the critical reflection of their teaching is important to their professionalisation as it helps to gain new insights and knowledge. The complementary view of the lecturers is particularly relevant for the assessed competency acquisition of the TTs. Independently of the TTs, the lecturers confirmed the acquisition of competency to a particular degree in the TTs, who took part in all courses of this education module (mandatory theory module, individual D&S and supervision). The lecturers specified the competencies in D&S that are important to them, such as: process orientation in the acquisition of knowledge, a view of the individual learning processes of the students, constructive handling of mistakes, communication in mathematics lessons that promotes learning, the development of a diagnostic perception and CK and PCK as a basis. In the evaluation of these interviews, the following subcategories could be formed inductively to determine the acquisition of competencies: term paper, the introduction of subject-related and didactic questions in the common exchange, planning documents and the design of lessons.

L2: ... and I think those who have really dealt intensively with the term paper and the supported child have acquired a lot of what I find important.

L1: In terms of communication, I'd say definitely because it would make communication possible at all. Many, if, as I said, when they start with the teacher training, there is right or wrong. But if you go through this education module, then they know that I can get in contact with a student regarding mistakes, on the communicative level, and can derive important insights. I believe that this competence has definitely been developed and behind this competency there is of course also a certain attitude that the TTs

> have developed along with it. […] By the way, you notice this very well when you attend classes. During classes, I can already see that a student's mistake is taken seriously. That if in discussions with the teachers, mistakes are discussed, I assume that communication has improved significantly.

The lecturers also emphasised that the critical reflection in supervision seminars and in professional learning communities is essential to develop professionalisation.

> L1: Perfectly clear: D&S must be more focused on the class. Maybe more portfolio tasks in the sense of observation and in any case a continuation of the professional learning communities, because I found that elementary. I think it is important for people to learn how to deal openly with such conversations and how to bring their content into these conversations.

CONCLUSION

The conceptual interweaving of theory and practice within the framework of microteaching reduces the complexity of teaching situations and allows focusing on individual learning difficulties and learning processes of a child. TTs' experience in protected, but no less authentic teaching-learning situations that the professional planning and designing of learning processes, the careful choosing of tasks and demonstration materials and, in particular, the diagnosis- and support oriented accompanying of learning, requires the activation and intensive use of CK as well as PCK in order to substantiate the theoretical aspects. The planning, implementation, analysis and reflection is always about an individual fitting between learner and learning opportunities. In these multi-layered processes, the TTs, who used the opportunity to support students individually, succeed in better perceiving, critically reflecting and developing their own behaviour and approach as teachers. As a key factor TTs and lecturers identify the transfer of the professional competencies acquired and deepened in microteaching to everyday classroom teaching. This leads to further questions, such as how to deal with the heterogeneity in the complex class situation adequately. Supervision must be a central component for the education modules. All TTs and lecturers considered this element as an indispensable basis for the development of professional competency, not only within this education module. The critical reflection of one's own teaching in discussions with others is the essential point in order to develop one's own abilities and knowledge in diagnostic teaching and supporting in the interweaving of theory and practice.

Acknowledgements

The design research project presented is part of the research and development of the Institute for Educational Consulting from the University of Education Weingarten, which is funded by the Federal Ministry of Education and Research for 2017-2022.

References

Allen, D.W., & Ryan, K.A. (Eds.) (1974). *Microteaching* (2nd ed.). Weinheim: Beltz.

Ball, D.L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Baumert, J., Kunter, M., Blum, W., Voss, T., Jordan, A., & Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*(1), 133-180.

Bromme, R. (1992). *Der Lehrer als Experte. Zur Psychologie des professionellen Wissens* [The teacher as an expert. On the psychology of professional knowledge]. Bern: Huber.

Bruner, J. S. (1971). Über kognitive Entwicklung. [About cognitive development]. In J. S. Bruner, R. R. Olver & P. M. Greenfield (Eds.), *Studien zur kognitiven Entwicklung* [Studies on cognitive development]. (pp. 21-53) Stuttgart: Klett.

Clarke, D., & Hollingsworth, H. (2020). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*, 947-967.

Hattie, J. (2010). *Visible learning. A synthesis of over 800 meta-analyses relating to achievement*. Reprinted. London: Routledge.

Hill, H.C., Rowan, B., & Ball, D.L. (2005). Effects of teachers´ mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*, 371-406.

Kuckartz, U. (2012). *Qualitative Inhaltsanalyse. Methoden, Praxis, Computerunterstützung* [Qualitative content analysis. Methods, practice, computer support]. Weinheim: Beltz Juventa.

Lipowsky, F. (2010). Lernen im Beruf. Empirische Befunde zur Wirksamkeit von Lehrerfortbildungen [Learning on the job. Empirical findings on the effectiveness of teacher training]. In F.H. Müller, A. Eichenberger, M. Lüders & J. Mayr (Eds.). *Lehrerinnen und Lehrer lernen. Konzepte und Befunde zur Lehrerfortbildung* [Teachers learn. Concepts and evaluation of teacher training] (pp. 51-72). Münster, New York, München, Berlin: Waxmann.

Ponte, J.P., & Chapman, O. (2016). Prospective Mathematics Teachers´ Learning and Knowledge for Teaching. In L.D. English & D. Kirshner (Ed.), *Handbook of international research in mathematics education* (3rd ed., pp. 275-296) New York, NY, London: Routledge.

Rau, M. A., & Matthews, P. G. (2017). How to make 'more' better? Principles for effective use of multiple representations to enhance students´ learning about fractions. *ZDM Mathematics Education, 49*(4), 531-544.

Scherer, P., & Moser Opitz, E. (2010). *Fördern im Mathematikunterricht der Primarstufe* [Support in mathematics lessons primary level]. Heidelberg: Spektrum.

Shulman, L. (1987): Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review*, *57*(1), 1-23.

Wittmann, E. C. (1981) *Grundfragen des Mathematikunterrichts* [Fundamental questions of teaching mathematics]. Braunschweig, Wiesbaden: Vieweg.

Critical thinking guiding students' actions in the classroom

Part 2

THE ROLE OF CRITICAL THINKING IN DATA-BASED ARGUMENTATION – EMPIRICAL FINDINGS FROM STUDIES WITH PRIMARY STUDENTS

Jens Krummenauer, Sebastian Kuntze

Ludwigsburg University of Education, Germany

For argumentation based on statistical data, elements of critical thinking can be expected to be helpful, e.g. when questioning others' interpretations of data or for developing counter-arguments. However, empirical evidence in this regard is scarce, especially as far as young learners are concerned. In a re-analysis of both qualitative and quantitative data from prior studies with primary students, we investigated what role critical thinking plays in students' data-based argumentations. The results indicate that elements of critical thinking can play both a supportive, but also a non-supportive role.

INTRODUCTION

Statistical data are often used as a basis for argumentation which aims at convincing others of a certain position. As statistical data can allow for different interpretations, and as not all thinkable interpretations indeed may be supported by the data, we expect that elements related to critical thinking (CT), e.g. questioning given interpretations of data, may play an important role in data-based argumentation. However, empirical evidence concerning the role of CT elements for data-based argumentation is relatively scarce. Consequently, this paper addresses this research need. Based on questionnaire and interview data from two earlier studies with students from the first (Emhart et al., in press; Krummenauer et al., submitted; Krummenauer & Kuntze, submitted) and fourth grade (e.g. Krummenauer & Kuntze, 2018), a re-analysis focused on the question what role CT elements can play in students' data-based argumentation was conducted. The results indicate that CT elements indeed can support students' data-based argumentation, but the evidence suggests as well that CT elements can play a non-supportive role.

In the following section, we introduce the theoretical framework of the study. We then specify the research question and report on the analysed data and the methods chosen for the analysis. Subsequently, we present results and discuss them in a concluding section.

THEORETICAL BACKGROUND

As outlined in Krummenauer and Kuntze (2019), data-based argumentation can be seen as a specific form of argumentation in which statistical data are used to substantiate certain statements. Accordingly, data-based arguments contain (at least) two elements: A statement which shall be substantiated, and a reference to particular aspects of data which are intended to substantiate the statement. A central requirement when developing data-based arguments is to consistently connect both elements with each other (ibid.).

When developing such arguments, it can be expected that, as one aspect, thinking which is related to dealing with statistical data is involved, e.g. when reading data from diagrams or considering how a set of data may vary in case of a different sample; we refer to such thinking with the term *statistical thinking* (ST) (see e.g. Shaughnessy, 2007; Wild & Pfannkuch, 1999). Further, we expect that elements associated with *critical thinking* (CT) are involved (cf. Krummenauer & Kuntze, 2018) – for instance, when questioning given interpretations of data, questioning underlying assumptions or the source of data; such thinking elements are highlighted in many CT approaches (see e.g. Ennis, 1987; Siegel, 2010).

However, Kuntze, Aizikovitsh-Udi, and Clarke (2017) point at conceptual weaknesses in existing, mainly non-empirical descriptions of the intersection domain of CT and ST. They suggest a *scientific reasoning perspective* (e.g., Fischer et al., 2014; Kuhn, 1989; Kuhn & Pearsall, 2000; Zimmerman, 2007), which affords to describe key elements of the intersection domain of CT and ST. This theoretical foundation can also be used for describing data-based argumentation (Krummenauer & Kuntze, 2018) and allows to focus simultaneously both on aspects related to ST and CT. A core aspect of this scientific reasoning approach is to distinguish between *theory* and *evidence*: In terms of data-based argumentation, this means that relevant statistical data have to be considered as *evidence*, whereas statements on which argumentation is focused (such as interpretations of data) have to be treated as elements of *theory*. Strategies of scientific reasoning in data-based argumentation are, for instance, to actively challenge theories by searching in data for counter-evidence, to reject or at least to modify statements if counter-evidence is found, or to develop statements which are in line with the available data (ibid.; Kuntze et al., 2017).

To give an example, we would like to consider the task in Figure 1, in which data about recovery times related to two sorts of headache tablets is provided. Moreover, a statement is given ("Tablet 2 is better than tablet 1") and the students are asked to give arguments in favour and against the statement so that the students have to develop arguments based on the data in order to evaluate the given statement. For being able to develop arguments in this task context, students, from the scientific reasoning perspective, have to consider the statement of the doctor and the data as separate elements. Whereas the data have to be seen as the available evidence which is decisive for assessing whether the statement is valid, the statement, as it is neither confirmed nor disproved yet, has to be treated in the sense of a *theory* which potentially needs to be rejected or modified if counter-evidence is found.

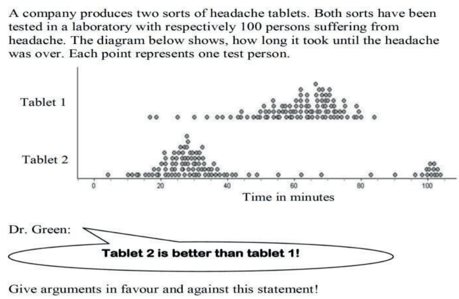


Figure 1: Task example (Kuntze et al., 2015, p. 11).

Building up on this, students have to *evaluate* the doctor's statement based on the data, which means in this case, to search within the available evidence both for aspects which support the doctor's theory and for those which may contradict it. The strategy of searching for counter-evidence is in this case particularly helpful when developing an argument against the statement. As expected according to the theoretical background, both elements related to CT and ST are part of the mentioned scientific reasoning strategies: For instance, searching for aspects in the data contradicting the doctor's statement corresponds to a CT element (questioning claims and assumptions; Ennis, 1987) and, at the same time, to a key aspect in the scientific reasoning framework. For being able to use this strategy, students have to keep a critical distance towards the statement and have to try to actively challenge the doctor's statement, which is also another key element of CT as well as of scientific reasoning. Both can be expected to influence students' thinking related to dealing with the data.

In interview studies with adults, in which the participants were asked to evaluate statements based on given data, Aizikovitsh-Udi and Kuntze (2014) and Kuntze et al. (2017) described such cases in which CT is an integral part of ST activities by the notion of *CT within ST*. However, they also found a second category, namely *CT enriching ST*, which refers to cases in which CT is not a necessary element of ST, such as, for instance, questioning the reliability of the source of data in general. Although such CT strategies are not part of ST, they may complement ST, such as seeing data in a larger context or discovering the relevance of gathering additional data, e.g. in order to control for non-reliable data sources. However, CT elements which are not part of ST do not always play a supportive role in thinking processes: Kuntze et al. (2017) and Aizikovitsh-Udi & Kuntze (2014) reported cases in which CT affected ST in a rather detrimental way (and also vice versa). For instance, if one questions data in a generalising way (e.g. "Most statistics are fake anyway"), this could lead students to not using available data even if the data actually can provide relevant information. We may thus conclude that the existing findings point to a complex interplay: On the one hand, CT elements appear to be supportive and sometimes even necessary when dealing with statistical data; on the other hand, there is evidence that CT elements also can play a rather negative role for using data as a source of evidence.

As these findings stem from studies with adults, research is needed to what extent these findings also apply to younger students, such as students on the primary level. Moreover, the focus of the existing studies has not been specific for data-based argumentation, so that research is needed in order to better understand how CT is involved in primary students' data-based argumentation.

RESEARCH QUESTION

Consequently, this study addresses this research need. In particular, the study focuses on the following research question: *What CT elements can be identified in the students' answers to data-based argumentation tasks, and what role do these CT elements play in the students' data-based argumentation?*

SAMPLE AND METHODS

In order to address this question, a re-analysis of evidence from prior studies was conducted. In this paper, we focus on data from a questionnaire study with $N = 385$ fourth-graders (e.g. Krummenauer & Kuntze, 2018) and an interview study with $N = 11$ first-graders (Emhart et al., in press; Krummenauer et al., submitted; Krummenauer & Kuntze, submitted). The questionnaire data consist of answers from students to the task in Figure 1. The data with first-graders were gathered in an interview setting in which simple data sets were used, such as shown in Figure 2. It shows the development of the height of a boy (Bob) who was measured several times during two years; the progression of time is visualised by shifting pictograms indicating different seasons. Similar to the statement of Dr Green in the questionnaire task above, statements related to the data were made by means of a glove puppet, and the participating students were asked to evaluate these statements in order to initiate processes of data-based argumentation. In the case of this data set, the statements "Bob has always grown the same amount" and "Bob's mother measured his height for more than one year" were used.

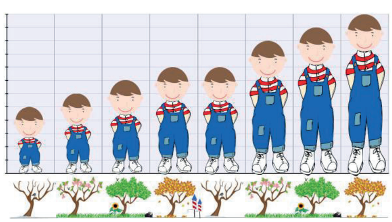
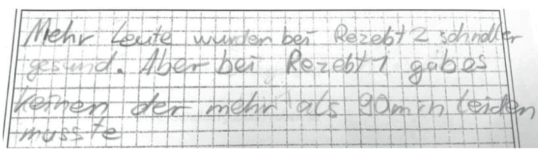


Figure 2: Task "Development of Bob's height"

In a first exploratory step of analysis, both the interview data and the questionnaire data were re-analysed in order to identify answers which contain elements which can be associated with CT, such as questioning the given statements or questioning the data (cf. Aizikovitsh-Udi & Kuntze, 2014; Kuntze et al., 2017). In a second step, we analysed the answers identified in the first step in-depth based on the scientific reasoning approach outlined above in order to reconstruct the role of CT elements in more detail (ibid.). In the following, we give examples of the analysed answers and present the corresponding interpretative analysis results.

RESULTS

First, we focus on the answers of fourth-graders to the task in Figure 1, in which the students are asked to develop arguments in favour and against the given statement. Figure 3 shows an answer which contains two data-based arguments: One in favour and one against the statement. For developing the arguments (especially in case of the argument *against* the statement), the student had to treat the statement of the doctor as a hypothesis and had to recognise counter-evidence provided by the data, which, as outlined above, corresponds to elements of CT. As the answer contains a consistent data-based argument against the statement, it can be assumed that the student, apparently, was able to successfully apply these elements of CT. As reported in Krummenauer & Kuntze (2018), about 17% of the students were able to give such answers containing consistent arguments in favour and against the statement indicating that CT was successfully applied by these students.

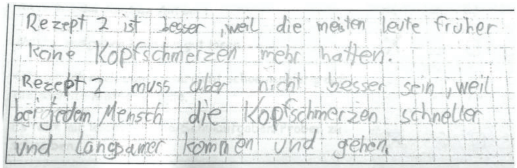


Translation:
"More people recovered faster with tablet 2. However, with tablet 1 there was nobody who had to suffer for more than 90min."

Figure 3: Answer with data-based arguments both in favour and against the statement

Besides such answers that contain data-based arguments in favour and against the statement, there were answers with only one data-based argument; many of these arguments (with a ratio of 5 to 1) were in line with the given statement (cf. ibid.), which could indicate a frequent non-availability of corresponding CT elements as the statement might not have been questioned.

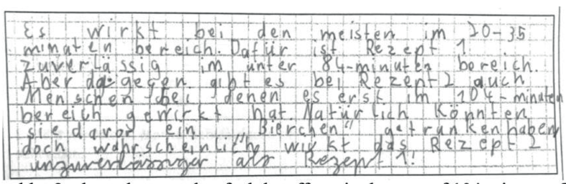
Figure 4 shows another example of an answer; it begins with a data-based argument supporting Tablet 2, which is in line with the statement of the doctor. The argument is followed by a counter-argument based on the assumption that "headaches come and go faster and slower for each person", which indicates that the student assumes that headaches come and go *randomly*. In comparison to the answer in Figure 3, the doctor's statement apparently also has been questioned, which implies the use of CT; however, in this case, questioning the doctor's statement is not based on the given data, but rather on the general assumption that headaches come and go randomly, which may correspond to questioning the relevance of the data in general. As a consequence, also the data-based argument given at the beginning of the answer is undermined. CT, therefore, appears to have played a non-supportive role for data-based argumentation in this case, as it appears that CT has hindered the student in using data as a source of evidence. Another aspect we would like to note is that the student, on the one hand, questions the statement of the doctor and the data critically; on the other hand, however, the student does not question his own assumption which is used to question the relevance of the given data.


Rezept 2 ist besser, weil die meisten leute früher keine Kopfschmerzen mehr hatten. Rezept 2 muss aber nicht besser sein, weil bei jedem Mensch die Kopfschmerzen schneller und langsamer kommen und gehen

Translation:
"Tablet 2 is better, because most people recovered earlier from headache.
Tablet 2, however, is not necessarily better because headaches come and go faster and slower for each person."

Figure 4: Example of an answer

Another answer indicating the use of elements of CT is shown in Figure 5. However, in contrast to the answer in Figure 4, CT appears in a way which, following the terminology of Aizikovitsh-Udi & Kuntze (2014), *enriches* dealing with the given data and, subsequently, also the data-based argumentation. The answer begins with two data-based arguments: The first one is in favour of Tablet 2 and, therefore, in line with the doctor's statement; the second one is in favour of Tablet 1 and questions the given statement. The student then adds another argument in favour of Tablet 1 substantiated by the information derived from the data that some people only recovered after about 104 minutes. He then states that these long recovery times potentially could be caused by drinking beer. This carefully expressed conjecture can be seen as CT which goes *beyond* the given data, but which has implications on how to *interpret* the data. Compared to the previous answer, though, the student apparently also treats his own consideration critically, as the student *weighs* both aspects (errors in measurement due to alcohol consumption vs. considering the data as they are), and concludes that Tablet 2 "probably" is indeed less reliable than Tablet 1.


Es wirkt bei den meisten im 30-35 minuten bereich. Dafür ist Rezept 1 zuverlässig im unter 84-minuten bereich. Aber dagegen gibt es bei Rezept 2 auch Menschen bei denen es erst im 104-minuten bereich gewirkt hat. Natürlich könnten sie davor ein „Bierchen" getrunken haben doch wahrscheinlich wirkt das Rezept 2 unzuverlässiger als Rezept 1!

Translation:
„It [tablet 2] works for most people in the range of 30-35 minutes. However, tablet 1 is more reliable in the range below 84 minutes. But in contrast, there are people with tablet 2 who only started to feel the effects in the area of 104 minutes. Of course, they could have had a "beer" before that, but Tablet 2 probably works less reliably than Tablet 1!

Figure 5: Example of an answer

In the following, we would like to turn the focus towards the interview data of first-graders. We report on three transcripts shown below, which refer to the task in Figure 2. The transcripts start when the statement to evaluate is presented to the students using a glove puppet.

1 Puppet (P): Bob has always grown the same amount. #00:04:52-0#
2 Interviewer (I): Do you think that too? #00:04:54-1#
3 Student (S): Hm-m (denying). #00:04:55-7#
4 I: Why don't you think so? #00:04:57-3#
5 S: Because here he is smaller and there he is the same size and there he is bigger. #00:05:04-3#
6 I: Then Sally is not right? #00:05:06-3#
7 S: Hm-m (denying). #00:05:07-6#

After the statement (which is, in this case, not in line with the data) was given in (1), the interviewer in (2) asks the student to evaluate the statement. The student, subsequently, indicates that the statement is false and substantiates this in (5) based on the given data. This shows that the student was able to reject the statement due to counter-evidence and implies that the student questioned the statement, which can be seen as an element of CT. In contrast, the following transcript shows an answer referring to the same task, in which there is no indication for elements of CT:

1 P: Bob has always grown the same amount. #00:06:13-1#
2 I: Do you think that too? #00:06:14-4#
3 S: (Agreeing) because he is getting higher and higher. #00:06:19-4#
4 I: How do you know that? #00:06:22-4#
5 S: This is winter, spring, summer (points to the diagram). #00:06:35-6#
6 I: So I got it right when you say that Sally is right, Bob has always grown the same amount? #00:06:41-6#
7 S: Yes, you are right. #00:06:43-1#

Not questioning the doctor's statement, which can be seen as a missing of CT, could be a possible reason that the student, in this case, was not able to consider the counter-evidence in the data, which is necessary to evaluate the statement successfully, and to give a corresponding data-based argument.

The following transcript shows another answer related to the other statement ("Bob's mom measured him for more than a whole year"), which is in line with the data in Figure 2. The statement to be evaluated, in this case, is expressed by the interviewer in (1).

1 I: Well, did she [Bob's mother] measure [Bob] for more than a whole year? #00:06:07-1#
2 S: Yes. Here she measured, here she measured, here she measured, here she measured, but here she did not [points successively to each data point]. #00:06:14-8#
3 I: There she did not measure? #00:06:17-0#
4 S: Yes, Bob was the same size. Or he stands on tiptoe so that he gets so tall. (..) Actually, he is so small, but he stood on tiptoe and then he

has become so tall [points to the last pictogram of Bob]. #00:06:32-6#

In (2), the student signals that he agrees with the statement. To substantiate this, the student points to the pictograms representing Bob's height at different times (see Figure 2), and comments, respectively, that Bob's mother measured at the respective data points – except the last data point, which is, however, not in line with the task context introduced to the student before. To get insight into the student's thinking, the interviewer in (3) asks for an explanation. Responding to this, the student confirms that Bob has not been measured at the last point of data, and substantiates this by stating "Bob was the same size there". Even though this is in line with the data, it does not substantiate the assertion that Bob was not measured at this point. The student, subsequently, also presents an alternative substantiation for his assertion ("Or he stands on tiptoe so that he gets so tall"), which indicates that the student questioned the given data based on a consideration about the context in which the data were gathered. While the term "or" at this point indicates that the assertion is only a *possibility* (indicating, in a sense, a critical distance), it shortly after is supposed as a fact which is not questioned anymore. The student rather uses his assertion to explain the assumed error in the data: "Actually, he is so small, but he stood on tiptoe and then he has become so tall". This implies that the student's assertion – which indicates CT – became predominant over using the given data. In (4), the student appears to reinterpret the data in a way that they seem not to contradict the assumption of an error in measurement. We may thus conclude, that elements of CT, in this case, appear to hinder the student in using the data which, as a consequence, has a negative impact on the data-based argumentation. Similar to the questionnaire answer in Figure 4, the student, on the one hand, appears to apply elements of CT to question the data but, on the other hand, does not apply CT towards his own assumptions. A critical distance towards their own assumption of an error in measurement, as it can be assumed in the questionnaire answer in Figure 5, is missing.

DISCUSSION AND CONCLUSIONS

The analysis of the answers of first and fourth-graders revealed several elements which imply that CT can be involved when primary students develop data-based arguments. The analysis of the answers further indicates that CT elements can play different roles in data-based argumentation.

As a first aspect, CT can help students to develop consistent data-based arguments, as it can be reconstructed in the answer in Figure 3, Figure 5, and in the first transcript. The categories *CT within ST* and *CT enriching ST* found by Aizikovitsh-Udi and Kuntze (2014) and Kuntze et al. (2017) in interviews with adults could be identified in a similar form in our data: Whereas CT elements in the answer in Figure 3 and in the first transcript appear to be constituting elements for developing data-based arguments, the CT elements in the answer in

Figure 5 also go beyond the given data and *enrich* thinking related to dealing with the data, which leads to a more elaborated argumentation.

The example in the second transcript as well as the answers in the interview study, which only contain an argument in favour of the given statement, indicate that *missing CT* can hinder students' data-based argumentation, as the students in these cases were not able to develop an argument contradicting a given statement.

Further answers, however, indicate that CT elements do not always play a supportive role for data-based argumentation. As seen in Figure 4 and in the third transcript, there are answers in which CT elements appear to hinder using the statistical data as the data appear to be questioned in a generalising way. At the same time, the students do not question their own assertions on the basis of which the data were questioned.

We may thus sum up, that the results show that the ambivalent role of CT in the interplay with ST found by Aizikovitsh-Udi and Kuntze (2014) and Kuntze et al. (2017) is reflected in a similar way in the analysed data-based argumentations of primary students. It, therefore, appears to depend on *how* students *use* elements of CT in data-based argumentation, which should be addressed when fostering students in this regard.

For this, fostering students with focus on *strategies of scientific reasoning* appears to be a promising approach, as these strategies, as outlined above, address elements of CT in the interplay with ST. Studies have shown that teaching scientific reasoning strategies in primary school, in principle, is possible (e.g. Sodian et al., 2002). A study currently being carried out further examines to what extent fostering strategies of scientific reasoning has effects on primary student's data-based argumentation.

Acknowledgements

This study is supported by research funds from the Senate of Ludwigsburg University of Education. The interview study including the design of the tasks used in this study was conducted in cooperation with Maria Emhart.

References

Aizikovitsh-Udi, E., & Kuntze, S. (2014). Critical thinking as an impact factor on statistical literacy. In K. Makar, B. de Sousa & R. Gould (Eds.), *Proceedings of ICOTS9*. Voorburg: ISI.

Emhart, M., Krummenauer, J., & Kuntze, S. (in press). Datenbasiertes Argumentieren in der Grundschule – eine Interviewstudie mit Erstklässlern [Data-based argumentation in primary school - an interview study with first graders]. In: *Beiträge zum Mathematikunterricht 2020*. Münster: WTM-Verlag.

Ennis, R. H. (1987). A taxonomy of critical thinking dispositions and abilities. In J. B. Baron & R. J. Sternberg (Eds.), *Teaching thinking skills* (pp. 9-26). NY: Freeman.

Fischer, F., Kollar, I., Ufer, S., Sodian, B., Hussmann, H., Pekrun, R., Neuhaus, B., Dorner, B., Pankofer, S., Fischer, M., Strijbos, J.-W., Heene, M., & Eberle, J. (2014). Scientific reasoning and argumentation: Advancing an interdisciplinary research agenda. *Frontline Learning Research*, *4*, 28–45.

Krummenauer, J., Emhart, M., & Kuntze, S. (submitted). Können Kinder zu Beginn der ersten Klasse bereits mit statistischen Daten argumentieren? [Can children argue with statistical data at the beginning of first grade?]. Manuscript submitted for publication.

Krummenauer, J., & Kuntze, S. (2018). Primary student's data-based argumentation – an empirical reanalysis. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.). *Proc. of the 42nd Conf. of the IGPME* (Vol. 3) (pp. 251–258). Umeå: PME.

Krummenauer, J., & Kuntze, S. (2019). Primary students' reasoning and argumentation based on statistical data. In: U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of CERME 11* (pp. 241–248). Utrecht, the Netherlands: Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Krummenauer, J., & Kuntze, S. (submitted). Elementary students' use of context-related knowledge in data-based argumentation – an in-depth analysis of data from interviews with first and second graders. Manuscript submitted for publication.

Kuhn, D. (1989). Children and Adults as Intuitive Scientists. *Psych. Rev.*, *96*, 674–689.

Kuhn, D., & Pearsall, S. (2000). Developmental origins of scientific thinking. *Journal of cognition and development*, *1*, 113–129.

Kuntze, S., Aizikovitsh-Udi, E., & Clarke, D. (2017). Hybrid task design. Connecting learning opportunities related to critical thinking and statistical thinking. *ZDM 49*(6), 923–935.

Kuntze, S., Martignon, L., Vargas, F., & Engel, J. (2015). Competencies in understanding statistical information in primary and secondary school levels: An inter-cultural empirical study with German and Colombian students. *AIEM*, *7*, 5–25.

Shaughnessy, J. M. (2007). Research on statistics learning and reasoning. In F. K. Lester (Ed.), *The second handbook of research on mathematics teaching and learning* (pp. 957-1010). Charlotte, NC: Information Age Publishing.

Siegel, H. (2010). Critical Thinking. In: P. Peterson, E. Baker, & B. McGaw (Eds.), *Int. Encyclopedia of Education* (6th ed., pp. 142–145). Amsterdam: Elsevier.

Sodian, B., Thoermer, C., Kircher, E., Grygier, P., & Günther, J. (2002). Vermittlung von Wissenschaftsverständnis in der Grundschule. *Zeitschrift für Pädagogik*, *45*, 192–206.

Wild, C. & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review, 3*, 223–266.

Zimmerman, C. (2007). The development of scientific thinking skills in elementary and middle school. *Developmental Review*, *27*(2), 172–223.

ANTINOMIES OF PROBLEM POSING

Zoltán Kovács, Eszter Kónya
University of Nyíregyháza and University of Debrecen, Hungary

This paper is motivated by the question, "What does a classroom look like when students engage in problem-posing activities?" In particular, "What are the key features of effective problem posing and problem-posing instruction in classrooms?" The qualitative research is based on the investigation of problem-posing episodes for 5-8th graders. The authors investigate some phenomena accompanying the implementation of problem-posing activities and formulate them as antinomies. They also make suggestions for resolving these antinomies.

INTRODUCTION AND RESEARCH GOAL

An increased interest in problem posing has been shown for more than three decades. A comprehensive study by (Kilpatrick, 1987) concerns many fundamental questions of problem posing, i.e., problem structures, the process of problem formulating, and instructions in problem formulating. Problem posing as a central activity in doing and learning mathematics is discussed in (Silver, 1994). Authors of the paper (Cai, Hwang, Jiang, & Silber, 2015) collect 14 unanswered questions related to problem posing. One of them is addressed in this article, namely the question "What does a classroom look like when students engage in problem-posing activities?" In particular, the authors highlight the question "What are the key features of effective problem-posing and problem-posing instruction in classrooms?". (Unanswered Question 10, p. 9.) The present paper aims at contributing to answering this question. The authors present three antinomies of problem posing and try to resolve them. This is to suggest answers on how problem posing can be effectively managed in the class.

One of the antecedents of this research was the problem-posing working seminar of the CME'18 conference which was held by one of the authors of this paper (Kovács, 2018). There the participating researchers analyzed video footage of a problem-posing episode in a grade 6 class. The following comments were received: teacher could lose control; students are talking and arguing; open-minded activity that develops critical attitude; dynamic classroom with many interactions; students' work looks like an activity of real mathematicians. Another motif that appeared in CME'18 discussion is the critical attitude. There is a direct relationship between problem-posing and critical thinking. Sternberg (1986, p. 3) cites a broadly accepted definition of critical thinking as "critical thinking comprises the mental process, strategies, and representations people use to solve problems, make decisions, and learn new concepts."

Open questions were also raised in the CME'18 seminar, such as how to deal with the open nature of problem-posing activities in the classroom that evolves unexpected situations, and what the teacher should do with the multitude of problems that arise in the classroom. These issues are also discussed here.

THEORETICAL UNDERPINNING

The core of problem posing in a school setting is that students construct tasks that were not previously explicitly taught to them and that meet certain starting or target conditions. This approach is close to that of Pehkonen (1997), who interpreted problem posing as an open problem-solving activity, see Table 1.

	CLOSED goal situation	OPEN goal situation
CLOSED starting situation	-	structured problem posing
OPEN starting situation	semi-structured problem posing	free problem posing

Table 1. Problem posing in terms of the openness and closeness of the situations.

Just as mathematical problems need to reach an end goal from a given starting point, so mathematical problem-posing activities have a starting point and goal. The starting point may be another mathematical problem or an everyday or imaginary situation. The goal is to formulate a problem freely or under some bounding condition. Following Pehkonen, when raising problems, the start may not be well defined, but the goal is bound. In the present adaption problem posing is an activity in which we expect the student to create a question with a fixed structure after a model problem, and it is called *semi-structured problem posing*. Hashimoto (1987) finds that asking students to pose a problem in the same way as a previously solved problem can be a useful teaching technique that reflects students' understanding of mathematical concepts. If the starting point is a real-life situation that may have several interpretations and is open in this sense, and the goal situation is not specified, it is called *a free problem-posing situation*. This interpretation is in line with Stoyanova's (1997) definition: a problem-posing situation is described free when students are simply asked to pose a problem from a fictive or naturalistic situation. The third category is *structured problem posing* where the student needs to develop new problems deriving from a given problem (Stoyanova, 1997).

The type of problem-posing activity influences the educational model of the lesson design. The authors highlight two theoretical frameworks for designing the implementation of problem posing in the classroom. Ellerton (2013) proposes an Active Learning Framework (ALF) for setting problem posing in mathematics classes, see Table 2. This framework considers problem posing in classrooms to be a capstone activity that allows students to consolidate their knowledge and think critically about it.

Classroom actions			
Processing of the new content, teacher models an example	Students solve problems based on the model	Students pose problems with the same structure as the model	The class discusses and solves problems posed by students

Table 2. Classroom actions in the ALF framework

Singer and Moscovici (2008) describe a learning cycle, called IMSTRA, which includes problem posing as an extension and application of problem solving. Parallel to the role of problem posing in ALF, Singer and Moscovici characterize the role of problem posing in a constructivist approach to instruction as a method of consolidating and extending what students have already learned. The three distinct phases of the lesson are IMmersion, STRucturing, and Applying. During the immersion phase, students get immersed in the problem, e.g., address and use previous knowledge and identify tentative patterns. During the structuring phase, students move to another level of understanding when they interpret their actual results from the immersion phase and adjust the pattern. During the applying phase, students learn to use the abstract pattern that they have developed into relating and unrelating situations. They apply learned concepts and patterns to new situations by trying to solve existing problems and by creating new situations that need to be solved.

METHOD

This paper summarizes the experience of research conducted in three different schools in 2018-2019. Most of the outcomes presented in this article are the result of researcher-teacher collaborations during which the researchers (authors of this paper) and the teachers planned the classes together, and the teachers held the classes (Kovács & Kónya, 2019). We use both of the above frameworks in our design because we find that different problem-posing types require it. ALF fits in well with semi-structured problem-solving, because the problem posing and discussion is the last action in the lesson while IMSTRA for the free type because we have no model problem at all in that case.

The study is based on 23 classes in grades 5-8, out of which four ones are referenced in this paper that represents the entire research well. The approach used here is qualitative: the authors try to capture the classroom experience by the video recordings of the classes. We chose this medium because we were able to analyze the lessons from different perspectives in several rounds. On the other hand, we were able to divide the lessons into manageable, small episodes. In the first step, we identified three typical phenomena associated with the classroom implementation of problem posing. In the second step, we focused only on these phenomena. We analyzed the classroom events: assignment of the task, the way of processing of the activity, and teachers' behaviour during the classroom discussion. We coded three different types of teacher's behaviour: ignoring, incorporating, or putting aside the students' proposals. Qualitative analysis is supported by the evaluation of the students' outcomes, i.e., the success of the problem posing.

	Episode 1	Episode 2	Episode 3	Episode 4
Type of problem posing	structured	structured	semi-structured	free
School No.	1	2	1	3
Grade	6	5	8	8
Number of students	25	29	13	13
Framework	IMSTRA	IMSTRA	ALF	IMSTRA

Table 3. Features of the research.

THREE PROBLEM-POSING EPISODES

Episode 1 and 2

Two teachers in two different classes held the same lesson aiming at the introduction of the concept of the winning strategy in a game. The original problem was as follows: players take turns removing one, two, or three tokens from an initial pool of 11 tokens, and the last player to move will be the loser. Is there a winning strategy for the first player?

The lesson plan was based on the IMSTRA framework. We expect a better understanding of the model problem by examining the problems obtained by changing the attributes of the model problem. During the immersion phase, the game was played in pairs, the purpose of which was to gain and formulate experience. During the structuring phase, the whole class discussed the winning strategy of the game. During the application phase, students had to derive new problems by modifying the attributes of the original one. There are four characteristic attributes of the problem: the number of tokens is 11; the allowed amounts of tokens to be removed are 1, 2, 3; the last player to move will be the loser; players add tokens. By changing these attributes, we got new problems. This process is the well-known "What-if-not" procedure of problem posing (Brown & Walter, 1983). This activity belongs to the category of structured problem-posing where the starting point is the original problem, and the goal is reached by varying the attributes.

Students' proposed tasks (E1P1, E1P2, E1P3 in Episode 1; E2P4, E2P5, E2P6 in Episode 2):

E1P1 There are 15 tokens, the allowed numbers of tokens to be removed are two, three, or four.
E1P2 Two players take turns adding one, two, or three tokens and until reaching 12 tokens.
E1P3 We remove tokens randomly.

E2P4 Playing with 102 white tokens. (Originally, they played with colored disks.)
E2P5 Two tokens must be taken away. (The student's explanation: you will know who the winner is at the very beginning.)
E2P6 The number of tokens that can be removed is three or two, and the player who has more disks at the end of the game is the loser.

Some of the problems do not reflect the mathematical structure of the model problem. In E2P5 and E2P6, the player has no (or almost no) option of how many tokens he will remove, and you cannot talk about any winning strategy.

Episode 3

The lesson's aim was to learn new methods for solving combinatorial problems. One of these methods is the so-called *summation method.* It is a method applied to solve a type of task in which a word is read from the table, and the students need to determine the total number of possible readings. For example, *in how many ways can you read off the word GÁBOR from the table* (Figure 1). The problem-solving strategy is to add the number of possible arrivals from each direction to each letter, hence the name of the method.

B_1	O_3	R_6
$Á_1$	B_2	O_3
G_1	$Á_1$	B_1

G_1	$Á_1$	B_1	O_1	R_1
$Á_1$	B_2	O_3	R_4	
B_1	O_3	R_6		
O_1	R_4			
R_1				

Figure 1. Two tasks from the lesson plan. The model problem (L) and an application (R). The numbers indicate the number of readings to reach the letter marked.

According to the ALF, the problem-posing session followed the elaboration of the model problem and some practical exercises. The students worked in pairs. Each student created a problem that could be solved with the same strategy as the model problem, then they exchanged the problems, and everyone solved their classmate's problem. Finally, two tasks selected by the teacher were handled in class discussions.

Most of the students were able to create an analogous problem to the model problem (11 out of 13), and nine of the 11 correctly formulated problems were solved by the classmates. Figure 2 contains two examples.

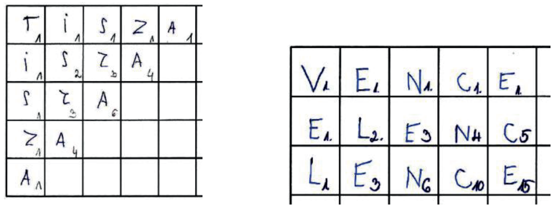


Figure 2. Two problems posed by students with attempts by classmates. The first (L, E3P1) follows the model problem. The second (R, E3P2) does not fit the model; moreover, the solution is false.

In two tasks the reading rules were too complex, i.e. the problem poser did not understand the scheme or intentionally made the problem so difficult that it could not be solved by the classmate. E3P2 is one of them. The teacher chose the well-structured E3P1 and the problematic E3P2 for the class discussion.

Episode 4

This lesson took place in two consecutive classes in grade 8. The lesson's aim is that students will understand the concept of permutation with repetitions. Students are sure to recognize permutation (without repetitions) as a combinatorial type and apply it in the right situations. After some introductory problems, the teacher presented the giftbag shown in Figure 3 and asked the students to pose mathematical questions concerning the bag. (Free problem posing.)

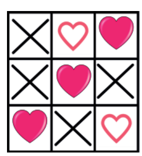


Figure 3. The pattern on the giftbag

The problem-posing activity took place in the following sub-phases: (1) they posed problems individually; (2) they discussed the problems in pairs, and if the partner did not understand the problem, they needed to revise it; (3) brainstorming: students proposed problems for "the challenge of the day" for the class; (4) they voted for "the challenge of the day," and finally, they solved it.

The 13 students made a total of 18 tasks. Eight of the tasks were appropriately formulated in a mathematically meaningful way. During the brainstorming session students selected the following four problems:

E4P1 In how many ways can X shapes (only those) be arranged in the table?
E4P2 In how many ways can these shapes be arranged in the table?
E4P3 Find a row or column in which all three shapes are different.
E4P4 In how many ways can these shapes be arranged so that each row and column has a full heart?

Finally, the students decided by voting that E4P2 was "the challenge of the day," and for the rest of the class, they worked on solving it.

The lesson was designed based on the IMSTRA model. Previous knowledge of permutation without repetition was reviewed during the immersion phase. In the structuring phase, they solved a simple problem on permutation with repetition by listing each case; however, they discussed the problem on the abstract-model level, too. In the application phase, problem-posing occurred, and they solved "the challenge of the day." It was a type of permutation with repetition that required thinking at the abstract-model level because it was impossible to list all cases (at least in the class). All the problems suggested by the students (except for E4P3) could be solved by the combinatorial model of permutation with repetition.

THREE ANTINOMIES OF PROBLEM-POSING

In this section, we summarize our observations concerning the question "What does a classroom look like when students engage in problem-posing activities." The authors identified three main phenomena (called antinomies) in the episodes. These are: (1) the appearance of unexpected (pedagogical and mathematical) situations in the classroom, (2) many students' proposals, and possibly blind paths appear, which cause (3) a dynamic interplay between no understanding and understanding of mathematical concepts.

Antinomy 1: the antinomy of the unexpected situation

Bring a problem into the class if you know the problem well, but you will get to know it better if you have brought it into the class. This issue is the antinomy of the unexpected situation.

Applying problem-solving involves the emergence of unexpected situations in the classroom. In all episodes quoted here, such cases appeared.

One source of unexpectedness is that students create a non-mathematical task, e.g., E4P3. This phenomenon is known from the literature, see, e.g., Silver and Cai (2005), where the authors suggest that teachers should set aside non-mathematical questions. Consistent application of the method also requires reaching a consensus within the class on what is considered a mathematically significant problem.

E2P5 and E2P6 are well formulated but non-content-relevant mathematical tasks. Although the teacher indicated that these problems did not meet the criteria, she decided to assign it to them for homework.

Another case of unexpectedness is when a student poses a mathematically exact problem that is unknown to the teacher, especially a blind problem (i.e., the teacher cannot solve the problem immediately). Based on previous knowledge, the teacher may or may not be able to cope with the problem in the classroom. E.g., E1P2 was unknown for the teacher; however, the mathematical structure of this game is the same as the original problem. In the classroom, the teacher dropped the task because she considered it as a blind problem. E1P3 also falls into this category. It is mathematically complex (What is the probability of the first player winning?), and the teacher rightly decided that this problem would not be dealt with during or after the class. One can only partially prepare for unexpected situations. In this case, a priori epistemological analysis (Mason, 2015) is obligatory: the situation to be brought into the class must be well known by the teacher. Moreover, the teacher's own recognized experience may help. However, despite careful preparation and a strong mathematical background, students can always present unknown problems. We argue that teacher's self-reflection is a practical skill to handle unexpected classroom events. Unexpected situations that arise during the class should be reflected on by the teacher, and after the class, they may find the right answers. As a result, the teacher will be better prepared the next time.

Antinomy 2: the antinomy of too many problems

The antinomy of too many problems means *that in problem-posing activities, we want the students to propose more problems, but the more problems arise, the more time it takes to process them*.

In Episodes 1 and 2, there was no time to address the problems in the class. The teacher assigned E1P1 for homework, but the students failed, basically because of the unclear losing position. One student solved E2P4 as homework.

In Episode 3, several problems arose, and everyone worked on a classmate's problem. This scenario was productive because the mathematical structure of most problems created was the same as the model problem.

In Episode 4, students selected the problem to be solved in the class. This settling had a positive effect on the students as they worked on their problem and not on the problem of "authority."

Applying problem-posing requires strict time management by the teacher. This method can also break the usual school framework. A possible solution to "the antinomy of too many problems" is that either the teacher selects as presented in Episode 1, or the students do as in Episode 3. There is no need for selection if students raise problems having a similar structure, and the ALF is particularly suited to this end. If the teacher feels that there is not enough time for the problem-solving phase, especially for closing it, they must deviate from the plan and adapt to the situation.

Antinomy 3: the antinomy of understanding

The antinomy of understanding means that *the prerequisite for proposing a mathematically well-structured problem is that the model problem is understandable to students. In contrast, problem-posing activity is intended to provide a better understanding of the model problem*.

The whole video of the lesson shows that in Episodes 1 and 2, many children did not understand the model problem (i.e., the meaning of a winning strategy), and the problems they created reflected it. The problem-posing phase may still make sense because it acts as a kind of indicator for the teacher: how well students can grasp the essence of the model problem. It is also an opportunity for the teacher to address inappropriately structured tasks during class discussions, and thus help to understand, as our teachers did.

The students assessed each other's problems in Episode 4, and they had to refine the created problems in pair work, which resulted in posing mathematically meaningful problems. The impact of pair work on understanding is well illustrated by the fact that although more than half of the problems initially created were not well structured, the problems proposed for "the challenge of the day" were already adequate in this respect.

In Episode 3, the problem-creation phase took place when the activity was likely to enhance the understanding of the model problem.

The actual impact of the problem-posing phase needs to be assessed by the teacher in the class, and they need to deviate from the plan if necessary, which requires adaptive skills and behavior from the teacher. Finally, the authors' experience is that pair work and teamwork cannot only be helpful in problem solving, but they help problem posing as well, by resulting in mathematically meaningful, well-structured, and content-relevant problems.

Conclusion

The authors' research question was, what are the key features of effective problem posing and problem-posing instruction in classrooms. The authors claim that the appropriate answers to the antinomies detailed above must be given. The critical issue for the teacher is 'knowing-to act' (Mason, 2015), now and not later. This ability requires a strong mathematical understanding of the topic and particular virtues that can be developed consciously. As the teacher brings the same problem several times into different classes, the schema of the problem in the teacher's mind continues to expand. The conscious incorporation of new elements into the schema requires the reflective behavior of the teacher what the authors consider most important. Other abilities are more pedagogical: adaptability, careful planning, and the use of pair and teamwork.

Acknowledgment

The research was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences. We also would like to thank Ioannis Papadopoulos for his constructive suggestions, remarks, and comments.

References

Brown, S. I., & Walter, M. I. (1983). *The Art of Problem Posing*. Philadelphia, Pennsylvania: The Franklin Institute Press.

Cai, J., Hwang, S., Jiang, C., & Silber, S. (2015). Problem-Posing Research in Mathematics Education: Some Answered and Unanswered Questions. In F. M. Singer, N. F. Ellerton, & J. Cai (Eds.), *Mathematical Problem Posing* (pp. 3-34). New York Heidelberg Dordrecht London: Springer.

Ellerton, N. F. (2013). Engaging pre-service middle-school teacher-education students in mathematical problem posing: Development of an active learning framework. *Educational Studies in Mathematics*, *83*, 87-101.

Hashimoto, Y. (1987). Classroom practice of problem solving in Japanese elementary schools. In J. P. Becker, & T. Miwa (Eds.), *Proceedings of the U S. - Japan seminar on mathematical problem solving* (pp. 94-119). Carbon Dale, IL: Southern Illinois University.

Kilpatrick, J. (1987). Problem Formulating: Where Do Good Problems Come From. In A. H. Schoenfeld (Ed.), *Cognitive Sciences and Mathematics Education* (pp. 123-147). New Jersey 07642: Lawrence Erlbaum Associates, Inc., Publishers.

Kovács, Z. (2018). Problem posing - tasks and challenges. In B. Maj-Tatsis (Ed.), *Mathematics in the real world. The conference proceedings. Abstracts, Posters, Workshops* (p. 20). Rzeszów: Wydawnictwo Uniwersytetu Rzeszowskiego.

Kovács, Z., & Kónya, E. (2019). Implementing problem solving in mathematics classes. In A. Kuzle, I. Gebel, & B. Rott (Eds.), *Implementation Research on Problem Solving in School Settings* (pp. 121-128). Münster: WTM Verlag für wissenschaftliche Texte und Medien.

Mason, J. (2015). Responding in-the-moment: learning to prepare for the unexpected. *Research in Mathematics Education*, *17(2),* 110-127.

Pehkonen, E. (1997). Introduction to the concept "open ended problem". In E. Pehkonen (Ed.), *Use of open-ended problems in mathematics classroom* (pp. 7-11). Helsinki: Universiti of Helsinki.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14(1),* 19-28.

Silver, E. A., & Cai, J. (2005). Assessing students' mathematical problem posing. *Teaching Children Mathematics, 12,* 129-135.

Singer, F. M., & Moscovici, H. (2008). Teaching and learning cycles in a constructivist. *Teaching and Teacher Education*, *24*(6), 1613–1634.

Sternberg, R. J. (1986). *Critical thinking: its nature, measurement, and improvement*. Washington, DC: National Inst. of Education.

Stoyanova, E. N. (1997). Extending and exploring students' problem solving via problem posing. *A thesis for Degree of Doctor of Philosophy in Education*. Australia: Edith Cowan University.

DOES THIS EQUATION DESCRIBE THIS SITUATION? EXPLORING THE ALGEBRAIC THINKING OF ELEMENTARY STUDENTS[1]

Esperanza López Centella*, Jana Slezáková**, Darina Jirotková**

*University of Granada, Spain

**Charles University, Czech Republic

In the present study we explore qualitatively the conceptual and semantic understanding of equations of elementary school pupils who were taught how to solve basic linear equations. We analyse the individual answers of 38 grade five students in a public primary school to a paper-based task consisting in justifying whether the equation 5+2x=13 models each one of 5 contextualized situations. Under a grounded theory approach, we provide a system of categories of the students' strategies. In particular, our findings show the abilities of the students to deal with the task and to infer true mathematical facts about equations, the variety of the students' strategies and the dependency of the strategy on the way in which the situation is presented. Teaching implications are derived.

INTRODUCTION

Since some years ago, the mathematics curriculums of primary school of a number of countries around the world have started to be enriched with algebraic activities, in accordance with the recommendations of international organizations of Mathematics Education (NCTM, 2000) and the evidences and conclusions from multiple research works on *early algebra* (Kaput, 1999). This algebraic enrichment of the curriculum aims to favor the transition of students from arithmetic—classically the main focus of elementary maths—to algebra—typically postponed to secondary school—. Getting acquainted with the notions of variables and undetermined or unknown amounts, establishing dependency relationships between variables, developing functional thinking, representing information in different systems and transferring it from one to another, and symbolizing and using meaningfully algebraic notation are part of these algebraic activities. After working some of the previous activities, equations brings a great chance to deal with several of them at once.

Aims of the research

In the context of a task involving a linear equation and five contextualized situations verbally and pictorially presented to elementary students of five grade, the aims of this research study are:

[1] Research partially funded by the Plan Propio de Investigación of the University of Granada. Esperanza López Centella gratefully acknowledges the members of the Department of Mathematics and Mathematical Education of the Charles University in Prague for their warmest hospitality during her visit from September to December of 2019 in which part of this study was performed.

(a) Explore the students' abilities to identify and justify the given equation as an algebraic transcription of a particular situation,

(b) Describe the strategies used by students to address this question.

LITERATURE REVIEW

A number of research studies in algebraic thinking have focused on epistemological analyses of scholar work with equations and the exploration of different aspects of children's performances when working with equations, discerning in such a way a demarcation between arithmetic and algebra. Herscovics and Linchevski (1994) point out the existence of a *cognitive gap* between arithmetic and algebra that they characterize *as the students' inability to operate spontaneously with or on the unknown.* Gallardo (2000) asserts that there is a *didactic cut* between arithmetic to algebra described as a change *from* working with an unknown on only one side of the equal sign of an equation (so that undoing the operation at hand is enough to know its value) *to* dealing with equations where the unknown appears in both sides and therefore has to be operated on. Filloy and Rojano (1989) suggests the terms *arithmetic equations* and *algebraic equations* to refer to these two types of equations. Balacheff (2001) understands that the shift from an arithmetic to an algebraic interpretation of equality corresponds to a shift of emphasis in the validation of the problem solution: from a *pragmatic control*, where the solution is validated arithmetically with reference to the initial context of the problem; to a *theoretical control*, where the solution is validated with reference to mathematical principles. Concerning the ways to formulate equations from verbal data, Herscovics (1989) recognises *syntactic* and *semantic* translations as different procedures, referring respectively to direct translation of key words to symbols, and the attempt to express the meaning of the problem. The first one was often observed to give rise to the so-called *reversal order*[2] error among the students (Rosnick & Clement, 1980), although some authors (MacGregor and Stacey, 1993) differ on this point. Hoch & Dreyfus (2004) asserts that any algebraic expression (for instance, an equation), represents an algebraic structure. Its "external appearance or shape reveals, or if necessary can be transformed to reveal, an internal order. The internal order is determined by the relationships between the quantities and operations that are the component parts of the structure". In this paper we will refer to these notions as *external structure* and *internal structure.*

Some other works explore how students can come to understand and use the syntactic rules of algebra on the basis of their understanding about how quantities are interrelated (Brizuela & Schliemann, 2014). These studies show

[2] The reversal order occurs when the equation of the form $\lambda x = y$ (for a number λ and variables x, y) that describes a situation is wrongly formulated as $\lambda y = x$.

that dealing with equations is not beyond ten years old students' mathematical understanding and that much more could be achieved if algebraic activities became part of the daily mathematics classes offered to elementary school children (p. 39). In this sense, Figueira-Sampaio et al. (2009) propose a constructivist computational tool to assist in learning equations of first degree in primary school, to illustrate the idea of equilibrium and properties of the equality. Otten et al. (2019) reported that the algebraic strategies such as restructuring, isolation and substitution that primary school students used when working with a hanging-mobile during their teaching experiment were later used by these students for solving linear equations in new contexts. Nührenbörger and Schwarzkopf (2016) study the processes of mathematical reasoning of equations in primary maths lessons through operations with structures of computing-terms, showing how substantial learning opportunities promote the development of a flexible and structural sustainable concept of mathematical equality.

METHODOLOGY

We performed a qualitative, descriptive and exploratory research (Erickson, 1986), based on a classroom experiment.

Participants

Thirty-eight grade five elementary students (10-11 years old) of mixed abilities in Mathematics from the same public primary school in the city of Prague (Czech Republic) participated in the study. They were two class groups, both in charge of the same teacher. According to the official Czech curriculum for primary school (VÚP, 2017, pp. 31-34), grade five students are educated in (a) numbers and numerical operations, (b) dependencies, relationships and work with data, (c) geometry in plane and space and (d) non-standard application tasks and problems. Although equations are included in second cycle (6-9 grades) of elementary school in the Czech curriculum (VÚP, 2017, p. 34), the participant students had been briefly introduced by their teacher to symbolic letters and formulation and resolution of simple linear equations at classroom. The teaching method through they had been taught was partially inspired in Hejný Method (Hejný, 2012). They had played with numbers and arithmetic operations subject to certain conditions and worked with unknown amounts. They had no prior experience with activities as the shown in our present study.

Design of the task

According to our research aims, we presented to the students the task shown in Figure 1. The worksheets for them were drafted in Czech language, the students' mother language in which they also wrote down their answers. In the five contextualized situations the same numbers (2, 5 and 13) are explicitly or implicitly in play, but neither the relations between them nor the order in which they appear are the same in all of the situations.

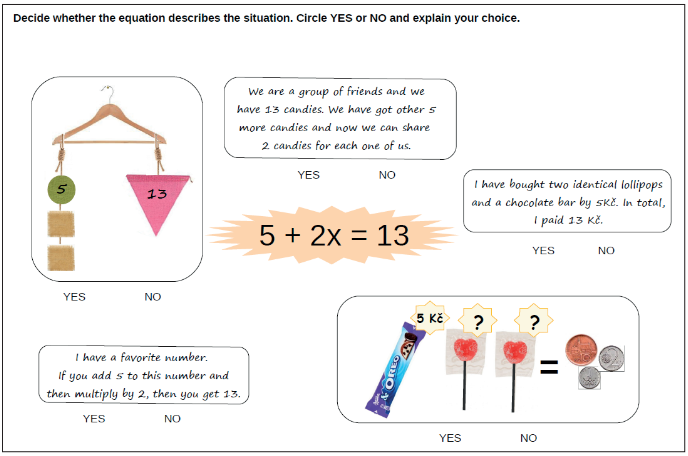


Figure 1. English version of the worksheet presented to the students

Table 1 specifies the task variables considered and the features of the situations.

	Friends	Candies	Coins	Number	Hanger
Representation	Verbal	Verbal	Pictorial	Verbal	Pictorial
Unknown	Nº. of friends	Price of lollipop	Price of lollipop	Favourite number	Weight of square
Order of app.	13, 5, 2	2, 5, 13	5, 1,1, 13	5, 2, 13	5, 1,1, 13
Possible eq.	(13+5)/x=2*	2x+5=13	5+x+x=13	(x+5)2=13	5+x+x=13
Described?	No	Yes	Yes	No	Yes

Table 1: Features of the situations in the worksheet.

Note: Situations were labelled following clockwise in the sheet. *Rational equation

Collection of data

The data collection was performed in one session at the beginning of the first semester of the two of a Czech school year, during normal Mathematics class time of the participant groups. The worksheets were given and administered by the teacher to the students with the firm instruction of doing the task individually, and writing down their explanations with pen on the sheet (and its back if they needed it). During their work, the students did not receive any feedback about the correction of their answers or suitability of their strategies. Upon completion, students' worksheets were delivered to the research team and constitute the data for this qualitative research.

Data analysis

Firstly, we transcribed all the written answers and explanations provided by the students in their worksheets, and translated them from Czech to English. Secondly, we considered as units of analysis the students' written answers to the different items of the task (namely, selection of yes/no and explanation for each situation presented). In third place, we thoroughly reviewed these data, and categorized, named and described the data following the principles of Grounded Theory (Corbin & Strauss, 1990). In agreement with our research goals, we establish categories of the focus of students' performances and of their strategies. Students' anonymity was ensured by assigning each a label: S_i where i=1,…,38. Below we show our system of categories.

Focus of performance

▪ *Resolution*. Ignoring the given equation, the student tries to find a solution for each situation. If the student can solve it, she/he circles "Yes". Otherwise, she/he circles "No".

▪ *Discrimination.* Considering the given equation, the student decides—based on one or more strategies—whether this describes mathematically each situation. If the student thinks so, she/he circles "Yes". Otherwise, she/he circles "No".

Strategies

▪ *Identification*. The student establishes a one-to-one correspondence between the contextual elements of situation and the math elements of given equation.

▪ *Equation solving.* □ *Operational*. The student follows the algebraic rules and standard solving equation process. □ *Trial and error.* The student assigns a value to the unknown and checks whether the equation holds then, repeating the process until finding a solution or stopping after some trials. □ *Memory and mental calculation.* The student does not annotate anything about the solving process, using own record of numerical facts and mental calculation.

▪ *Divisibility.* The student discusses the existence of solution relying on divisibility arguments.

▪ *Solution checking.* The student checks whether the solution of the given equation is also a solution for the situation at hand.

▪ *Equations comparison*. The student formulates an equation for the situation at hand and compares it—without solving it—with the given equation.

▪ *Solutions comparison.* The student formulates an equation for the situation at hand and solves it, comparing its solution with the solution of the given equation.

▪ *Analogy.* The student recognizes a situation as an analogy of another one, copying the answer provided in it.

▪ *Numbers comparison.* The student compares the numerical data appearing in the statement/picture of the situation at hand with those of the given equation.

RESULTS

Focus of performance

		Friends	Candies	Coins	Number	Hanger
Resolution	Yes	12	13	14	4	10
(18)	No	4	4	3	14	6
	Yes & No	2	1	1	0	1
	No answer	0	0	0	0	1
Discrimination	Yes	0	19	18	1	17
(20)	No	19	1	0	17	2
	Yes & No	0	0	1	1	1
	No answer	1	0	1	1	0

Table 2: Total number of students' answers of each type

Table 2 reports that the students participation in the task was really high, with only few non provided answers (4 over 190 potential answers). It also shows that more than the half of the students (20 of 38) focused their performances on discrimination—as requested by the task—, while the remaining students (18) did on resolution. Restricting ourselves to the first ones, it is remarkable the high rate of right answers (19, 19, 18, 17 and 17 respectively for each situation). Interestingly enough, there were who deliberately circled both "Yes" and "No" for the same situation. This is the case of S_{11} for the Weight situation:

> S_{11}: Yes, since 5+4=9+4=13. No, since 5+?+?=15 [he writes down two 5's and two arrows from each one of them to each question mark].

According to his answer apparently he forgets that the equilibrium of the hanging mobile is equivalent to the equality of the two amounts at its sides (being one of them 13) and, therefore, the numerical value of each identical square *must* be 4 (versus *could* be 4). It seems that he does not apply this reasoning only in this context, since he argues similarly in the Coins situation:

> S_{11}: Yes, 5+2x=13|-5, 2x=8|:2, 1x=4 [he writes 4 next to each question mark of the lollipop price labels]. No, because the price of lollipop could be 4.50.

He observes the solution (4) of the linear equation as a possibility (since also "it could be 4.50") instead of as the only value that satisfies the equality. The answer of S_{37} (focused on resolution) to Friends situation also illustrates this:

> S_{37}: Yes: How many people were there? 9 people. No: 1 to 8 people.

The lack of an explicit question in the statements prevented some student to identify an unknown susceptible of being described by an equation. This is the case of S_{13} in Friends:

> S_{13}: We don't have any unknowns. It is not a task, because there is no question to answer (it is all information).

Order of operations potentially matters. It turns interesting to analyse the only "Yes" answer in Number situation (restricting to discrimination as focus):

S_{17}: 5+2x=13, 5+2x=13|-5, 2x8=13|:2, x=4 [she solves the given equation].

S_{17}: If you add 5 and multiply 2x, you get 13.

S_{17}: 13-5=8, 8:2=4.

She firstly solves the given equation (getting x=4). In order to find the unknown favourite number, she tries to undo the operations that, applied on this number, give 13. She replaces addition by subtraction and multiplication by division, but she does not reverse the application order of the operations, in such a way that she gets 4 as outcome and, by solutions comparison, she circles "Yes".

Strategies

Figure 2 shows some examples of the use of the main strategies identified in our data analysis. Table 3 shows the frequencies of use of each strategy by the students who focused their performances on discrimination (20), as requested by the task.

	Friends	Candies	Coins	Number	Hanger
Identification	0	2	2	0	3
Equation solving	6	6	6	4	4
Operational	6	3	6	3	4
Trial and error	0	0	0	1	0
Mental calculation	0	3	0	0	0
Divisibility	0	0	0	7	0
Solution checking	0	0	6	1	6
Equations comparison	6	6	1	6	3
Solutions comparison	6	7	3	5	3
Analogy	2	2	9	0	4
Numbers comparison	2	0	0	0	0
Total	22	23	27	23	23

Table 3: Frequencies of the strategies used by students focused on discrimination

As evidenced by the total number of strategies used in each situation, students often involved more than one strategy in their answers. Moreover, there are notable differences in the frequency of use of each strategy depending on the presentation of each situation. In verbal situations, equations comparison together with solutions comparison and equation solving were the most used strategies, while in pictorial situations the most frequent were analogy and solution checking. The word statements awakened the students' need to formulate an equation describing the situation, while the pictures let them more easily perceive analogies with other situations and they offered a scenario (the "blanks" of the pieces of cloth and of the lollipops price labels) to implement solution checking. In Friends situation (where the unknown is the number of friends), some students misunderstood the role of the number of candies, as they compared or related it to the value (4) of the unknown for the given equation:

S_{18}: 5 + two x = thirteen → x=4, 13-5=8, 8:2=4 [she solves the given equation]. No. Everybody would have to get 4 c.

S_8: No. This equation shows that one man would get 4 candies [referring to the solution of the given equation].

In the same situation, some students' answers reveal how important is for them the external structure and numerical data, obviating the internal relations:

S_5: No. It will not work because there should be 13 of anything.

S_{33}: No. 13+5=18. This task doesn't work because it comes out of 18 candies, but we need 13 candies in total.

S_{23}: Yes, because 1,2 [referring to Friends and Candies situations] are the same because the numbers are the same.

In contrast, other students refer to mathematical properties when comparing equations. In Candies situation, S_{13} alludes to commutativity (involving symbolic letters and numbers):

S_{13}: Yes. This equation would look 2x+5=13, but 5 and 2x we can switch [comparing it with the given equation 2x+5=13].

Both disambiguation and interpretation of the statements also become crucial. For the same situation, S_{10} justifies her negative answer as follows:

S_{10}: I say no, because it says "two identical lollipops and chocolate for 5kc" so this all together written "for 5 Kc" and then total costs 13 kc. It is an illogical task. "My choice".

Some students use the same letter to symbolize different quantities. This is the case of S_{30} in his attempt of syntactic translation of the Number statement:

S_{30}: Not because it would be: x+5= x, x·2 = 13 but it is not: 5+2x=13 and also does not happen x·2=13 since 13 is not a divisor of two. [He verbally exchanges multiple by divisor].

In Weight context, this same student does not interpret the need for the numbers in the squares of being equal, but that they just need to add up to the difference (8) between the quantity in one side (13) and the quantity in the other one (5):

S_{30}: Yes, because the two numbers under 5 can be any way, so there could be 2 times 4, or 5 and 3 etc.

DISCUSSION AND CONCLUSIONS

In this work we show elementary students' abilities to recognize whether a linear equation models different contextualized situations. We shed light on the individual strategies that five grade students use to discern this. We highlight the great variety of strategies employed by the students to deal with this task as well as the different frequencies of the use of the strategies depending on the system of representation of the situations. More precisely, faced with verbally presented situations, students tended to formulate an equation based on syntactic and semantic translation of the statement, in order to compare it with the given equation. While faced with pictorially presented situations, the tendency of the students was to allude to an analogy between the situation at hand and another one of the task, concluding then the same as they did in the analogous situation. Alternatively, in this case, they also preferred to check whether the solution of the given equation was a solution for the situation at hand. This finding can be used with teaching purposes. It is also remarkable that a number of students used more than one strategy to justify their answers and that they did not systematize their strategies but adapting them at each situation. This is illustrated by the fact that many of them used divisibility arguments to discuss the Number situation.

The students' performances prove that they were able to infer mathematical truths not explicitly taught to them when briefly introduced to equations by their teacher at classroom. In particular, behind their use of solution checking and solutions comparison strategies are the following facts, connected to the internal structure of the equations: (a) If the solution of a linear equation is a solution for another linear equation, then these two linear equations are equivalent; (b) if two linear equations have the same solution, then they are equivalent.

In contrast, most of the students who used the equations comparison strategy exclusively based their strategy on the external structure of the equations. Although this did not prevent them to get right answers in this particular task, it is not a safe practice, since two equivalent equations can have different external structures (e.g. 2x+5=13 and 4x+23=2x+31). Nevertheless, there were who referred to mathematical relations (like commutativity of the sum of unknown amounts and numbers) in the justification under the use of this strategy. Another practice that stands out in the students' performances is the fact that they often mathematically check their answers (e.g. S_{18}: 1 lollipop costs 4Kc, 13-5=8, 8:2=4, 4+4+5, 8+5=13). This validation was pragmatic (Balacheff, 2001) most of times, based on arithmetic, although also a few times was theoretical, based on mathematical rules (see S_{13}, above). Concerning the use and manipulation of symbolic letters, in spite that a considerable number of students solved the given equation following a standard solving process (naming and operating with the unknown), often those same students got the value of the unknown in some situations by purely arithmetic methods, without introduce a symbolic letter to name the unknown neither operate on it. This suggests that even when they understand and know how to deal with these algebraic notations, symbolic letters are still not completely integrated in the basis of their mathematical language or they do not feel the need to use them in some contexts. Based on our findings, as many other authors (e.g. Brizuela & Schliemann, 2004), we conclude that students of this educational level are ready to address algebraic activities and tasks where the notion of linear equations is implicitly or explicitly involved, serving this to enrich their mathematical thinking.

References

Balacheff N. (2001). Symbolic arithmetic versus Algebra, the core of a didactical dilemma. In R. Sutherland, T. Rojano, A. Bell and R. Lins (Eds.), *Perspectives on School Algebra* (pp. 249-260). Dordrecht: Kluwer Academic Publisher.

Brizuela, B. M., & Schliemann, A. (2004). Ten-year-old students solving linear equations. *For the Learning of Mathematics, 24*(2), 33-40.

Corbin, J., & Strauss, A. (1990). Grounded theory research: procedures, canons, and evaluative criteria. *Qualitative Sociology, 13*(1), 3-21.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 119-161). New York: Macmillan.

Figueira-Sampaio, A. S., Ferreira dos Santos, E. E., & Arantes Carrijo, G. (2009). A constructivist computational tool to assist in learning primary school mathematical equations. *Computers & Education 53,* 484-492.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics, 2,* 19-25.

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (vol. 3, pp. 49-56).

Gallardo, A. (2000). Historical-epistemological analysis in mathematics education: Two works in didactics of algebra. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspective on school algebra* (pp. 121-139). Dordrecht: KAP.

Hejný, M. (2012). Exploring the cognitive dimension of teaching mathematics through scheme-oriented approach to education. *Orbis scholae, 6*(2) 41-55.

Hercovics, N. (1989). Cognitive obstacles encountered in the learning of algebra. In S. Wagner & C. Kieran (Eds.). *Research issues in the learning and teaching of algebra*, 60-86. Reston, VA: National Council of Teachers of Mathematics.

Herscovics, N., & Linchevski, L. (1994). 'A cognitive gap between arithmetic and algebra', *Educational Studies in Mathematics, 27*(1), 59-78.

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fenema and T. A. Romberg (Eds.), *Mathematical classrooms that promote understanding,* (pp. 133-155). Mahwah, NJ: Lawrence Erlbaum Associates.

MacGregor, M., & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations, *Journal of Math Ed., 24*(3), 217-232.

NCTM (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Nührenbörger, M., & Schwarzkopf, R. (2016). Processes of mathematical reasoning of equations in primary mathematics lessons. In N. Vondrová (Ed.), *Proceedings of the 9th CERME* (pp. 316-323). Prague: ERME.

Otten, M., van den Heuvel-Panhuizen, M., Veldhuis, M., & Heinze, A. (2019). Developing algebraic reasoning in primary school using a hanging mobile as a learning supportive tool. *Infancia y Aprendizaje, 42*(3), 615-663.

Rosnick, P., & Clement, R. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *Journal of Mathematical Behavior, 3*(1), 3-27.

Výzkumný ústav pedagogický v Praze [Research Institute of Education in Prague] (2017). *The Framework Educational Programme for Basic Education*. Prague.

TOOL-TASK DIALECTIC IN MATHEMATICS CLASSROOMS

Huey Lei

University of Saint Joseph, China

Education innovation is a distinct trend in education research, with the use of manipulatives a key research field, embracing pedagogical development in mathematics education. Tool-based pedagogy involving concrete and technological tools in mathematics classrooms helps create an interactive environment for the construction of students' mathematics knowledge, through the appropriate use of tools and the orchestration of mathematics teachers. These tools support students' abilities to construct cognitive models while harmonising contexts of the gestures made and language used by them. This empirical research proposes a new idea of Tool-Task dialectic, grounded in the analysis of mathematics lessons with designed tool-based tasks.

INTRODUCTION

Traditional theories of learning include transmission of knowledge from teachers to students through copying teachers' presentations, something that is, however, abandoned in contemporary models of school learning (Hedegaard, 2004). Mathematics educators are putting more emphasis on student-centred approaches, where students' problem-solving skills, independency and mathematical investigation are prominent (Kissane, 2016). Manipulatives, including teaching aids, apparatuses and tools, are tangible sources for inquiring about knowledge through the action of grasping it by students (Pimm, 1995), wherein using manipulatives fosters students' cognitive development, not only memory but also thought and reasoning (Norman, 1993). For example, Lopez-Real and Leung (2006) found that a dragging tool in dynamic geometry provides legitimate drag-to-fit solutions in the process of learning mathematics, developing students' mathematical reasoning. Griffiths et al. (2016) revealed exemplars drawing on practice of using manipulatives to develop understanding of arithmetic. Therefore, embedding tools into mathematics classrooms, with heuristically designed and goal-oriented activities, provides opportunities for teachers to lead students into a student-centred learning environment, conceiving the construction and reinvention of mathematics knowledge.

This paper reports on empirical research investigating the implementation of tool-based mathematics lessons from students' perspectives. Students' manipulations with tools in the process of the construction of mathematics knowledge is studied, in particular, the interplay of tools involved in lessons and the designed mathematical tasks.

THEORETICAL FRAMEWORK

Research on tool-related pedagogy has been going on for some considerable time (Trouche, 2004; Bartolini Bussi & Mariotti, 2008; Leung, 2012; Maschietto & Soury-Lavergne, 2013; Lei et al., 2018; Lei, 2019). Researchers have given different definitions to tools, instruments and artefacts. In this study, I adopt two renowned theories to initiate investigation into the implementation of tool-based mathematics lessons. The first theory, instrumental genesis, highlights psychological constructs in instruments that cultivate dual-directional interaction between instrument (also known as tool) and learner. Another theory is called tool of semiotic mediation, proposed by Bartolini Bussi and Mariotti (2008), which takes cultural factors as considerations, interfering with the production of signs generated by learners according to a Vygotskian approach.

Instrumental Genesis

Rabardel (2002) discerns instrument from artefact. The former consists of a psychological component yielding its usages important. It underpins instrumental genesis (IG), viewing a tool not only as focusing on its physical construction but also its potential and actual usage. Specifically, IG describes interrelations between the user and the tool as two directions pointing toward each other. Instrumentation process that point toward the user (i.e., from the tool), illustrate that the user is stimulated by the characteristics of the tool in order to critically utilise the tool to complete certain task. This process is strictly rooted in the constraints of the tool; that the user knows something can or cannot be done by the tool. Instrumentalisation, as a reverse directional process, reveals an idea of generation of personalised usage of the tool; that the user can discover relevant functions according to its features. The new ways of using the tool are beyond the expectation of the original designer who purposively made the tool. In addition, IG highlights utilisation scheme (Trouche, 2004); that the user manipulates the tool in a series of procedural stages blurring the distinction within the two directional processes.

Tool of Semiotic Mediation

Bartolini Bussi and Mariotti (2008) elaborate upon the tool of semiotic mediation (TSM) in terms of interrelationships between tools, mathematics knowledge in cultural bases, teachers and students. The idea of TSM emerges from the perspectives of Vygotsky; that a tool acts as a mediator between abstract mathematics and the learner who is manipulating with it. It is grounded in the idea of a cognitive tool (Norman, 1993), consisting of pragmatic and reflexive features, with the user as learner becoming smarter when using it to complete certain tasks. TSM takes social and cultural lines into consideration in classroom settings, probing interactions between tools, teachers and students. In particular, verbal and written signs generated by students in the process of manipulation with tools are premeditated. Such a process confirms the tool as a mediator, with the unique power to connect students with, or to construct, mathematics knowledge. TSM emphasises the semiotic potential of a tool where double semiotic links exist, illustrating two meanings: personal meaning to students, in particular during the process of manipulation with tool, and mathematical meanings, cognitively referring to abstract mathematics knowledge. One main role of teachers is to cultivate a learning environment for students in which to evolve the signs with personal meaning into mathematical meaning, that can be viewed as the construction of mathematics knowledge in a tool-based learning classroom setting.

These two theories provide a theoretical framework for the research and frame a way to analyse the learning progress of students from a psychological perspective. The analysis generates codes grounded in the data, initially based on IG and TSM frameworks, inquiring about tool-student interrelation and the evolution of signs generated by students respectively.

METHODS

The purpose of this study is to explore the interplay of the tools used in the lessons and the designed mathematics tasks during the implementation of the lessons, where students are producing signs with tool manipulation for accomplishing the tasks. In this regard, the study aims to answer the following research question:

What are the interplays between the tools and the tasks in the process of learning mathematics in classrooms?

To answer this question, I have adopted a qualitative research method from a positivist research paradigm (Cohen et al., 2000), in order to understand the relational meaning of tool and task as interpreted from the data. Grounded theory is known as a means to build theories inductively from data in an explorative study (Grbich, 1999). Thus, I use techniques of grounded theory to cultivate new phenomena unforeseen from analytical procedures.

Context of Study

The study invited mathematics teachers from primary and secondary schools to voluntarily design innovative mathematics lessons, embedding usage of tools as a prominent pedagogical consideration. The participating teachers fabricated mathematics tasks along with selection of tools as activities during lessons. The tool-based mathematics research lessons upon which this study focuses in the learning processes of students were conducted with tool manipulation and the assistance of mathematics tasks during the implementation of the lessons.

Data Collection and Analysis

Thirty mathematics teachers from primary and secondary schools joined the research project. Each teacher designed a tool-based mathematics lesson or series of lessons on a particular topic of their choosing. One form of guidance for selection of topics was to conceive of a meaningful and rich learning mathematics environment with the presence of the tools. Participating teachers had the autonomy to select any kind of tool, concrete or technological. A pre-lesson interview was conducted to record their perceptions, including rationale of tool selection and design of the tasks. All research lessons were videotaped and transcribed in order to capture critical episodes for analysis. One camera was set to record teachers' orchestration of the lessons, and in some cases, if resources were available, another camera was used to capture particular actions among groups or individual students. To triangulate preliminary findings based on the observation notes, post-lesson interviews with teachers and students were also conducted.

This report presents findings from a part of a larger research project, called *Tool-Task dialectic.* These findings are grounded in the data from the research lessons by the use of constant comparative method (Glaser & Strauss, 1967) involving four stages: 1) "comparing incidents applicable to each category", defined as events in the episodes; 2) "integrating categories and their properties"; 3) "delimiting the theory" and; 4) "writing the theory" (Glaser & Strauss, 1967, p. 105), which is Tool-Task dialectic. The researcher in the study plays an important role in the analytical procedure, with theoretical sensitivity (Glaser, 1998) drawn in the event categorisation and its interpretation. An open coding method is adopted in the analytical procedures for labelling and categorising events, with different blocks of data captured in its meaning (Charmaz, 2006). The dimensions of each category are identified. The final codes were generated to build up the findings while the analysis took place until the categorisation was saturated. The research lessons were analysed while the generated codes were being finalised in the comparative procedures among the lessons.

FINDINGS

I analysed students' manipulations with the tools, according to the idea of instrumental genesis, emphasising interplay between the tool and the user in the process accomplishing a certain task. The tasks played a prominent role in the context of tool-mediation, including moulding the ways of manipulation with the tools by the students. To reveal the ideas of the main findings in the study, four representable episodes are selected to depict some prominent synergies between tools and tasks emerged in the observed lessons.

Task to Tool

Episode 1 shows a direct instruction written on the worksheet viewed as a task, where a pair of students followed its commands. The task was complimented by mathematics teachers to provide guidance to the students, when having difficulty in the manipulation with the tools.

Episode 1.

[In a school playground, students are asked to use theodolites and AngleMeter installed in tablets to measure the angle of elevation of a school badge. Two students were paired to do the task.]

Student 1: Why do we need to measure this angle?

Student 2: The worksheet says that. We need to measure the angle then calculate for the height of the school badge.

Student 1: Okay. Okay.

The worksheet included a planning of action with the tools ahead of the actual manipulation. Thus, the task cultivated a sequential plan of measurement, comprising calculation and derivation to minimise the number of parameters to be measured. Episode 2 reveals the mathematical calculation and its rationale as planned by the students. Figure 1 is the capture of the task they had done before actual measurement.

Episode 2.

[In a classroom, three students in a group are explaining to the whole class about how the required parameters are being derived.]

Student 3: We use many rulers. Use one ruler to be vertical. Vertically hold the position.

Teacher 1: Please demonstrate to the whole class.

[Student 3 in the group showed to the whole class how to hold the ruler vertically.]

Student 3: It is the height of the frustum, and let the upper height [the constructed virtual cone] be ‘c’. Adding h [the height of the frustum], then over c. Can get this. [Student 4 wrote c = ah / (b – a) on blackboard.]

…

Student 4: We only need to measure these three data. [Student 4 was pointing to the derived formula.]

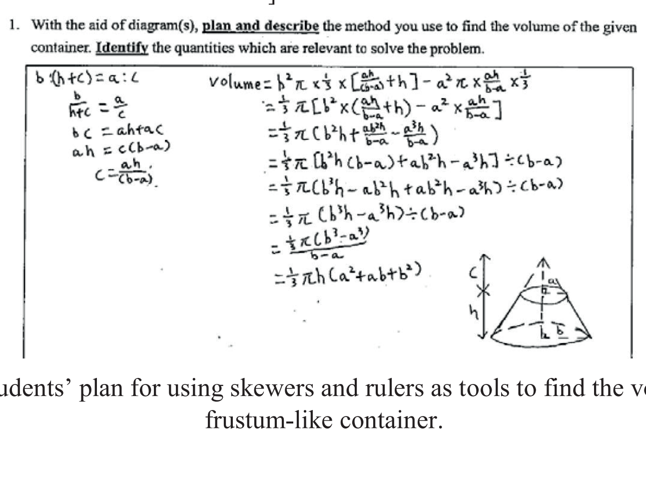


Figure 1: Students’ plan for using skewers and rulers as tools to find the volume of the frustum-like container.

The task helped the students to sort out possible ways of measurement according to significant parameters obtained in the derived formula, which closely interfered with the usage of the tools.

To briefly overview Episodes 1 and 2, I have boosted the interference of the tasks embedded in the contexts where the tools, the users and the tasks were intertwined as a triad that the designed tasks drew students' attention to, in order to plan ways of manipulation with the tools.

Tool to Task

Instrumental genesis frames a feature of a tool to stimulate students, as the users, to construct utilisation schemes in the goal of completing the tasks. The tools, in particular, demonstrate the power of generation of mission in undertaking a new role for it.

Episode 3.

	[A post-lesson interview with students]
Researcher:	In the two cycles of experiments, your group adopted two methods. Would you share with me why your group decided to change the methods?
Student 5:	At the beginning, we started a method but stopped afterward. Then we focused on the second method.
Researcher:	What made you change?
Student 5:	We just directly observed and estimated the number of marbles by wild guess of its sizes [in the first cycle of experiment as the first method].
Researcher:	Just wild guess?
Student 5:	Yes, we purely guessed by observation without using many tools.
	…
Student 5:	It was not accurate as we did not use any tools for help. It was just by wild guess.
Researcher:	Do you mean that using tools would make the estimation more accurate?
Student 5:	Since we want to find the mass of the large conical flask [in the second method], in this method we should use the balance to do that.
Student 6:	In the first method, it was too 'weak' [without much scientific support the claim]. We did not collect information about the flask. Thus, in the second method, we tried to gather more information in order to have stronger evidence to support our estimation. Also fewer errors.

Episode 3 shows how a group of students tended to use tools as a way of legitimation, to support their claim of estimation by providing objective measured values. In the first method, without any illustration but wild guessing, the students abandoned the method and made use of an electric balance to measure the weight of a conical flask with marbles in order to estimate the number of the marbles inside the container. The role of legitimation emerged in the midst of the students raising concerns about the significance of the approximation to the extent that they changed another task to measure the weights of the conical flask in a sequential manipulation, with calculation as the utilisation scheme. In short, the use of the tools not only comprised a variation of tasks contributing to judgement and mathematical reasoning, but also produced consensus ideas among the groups of students in fulfilment of communication.

Legitimation was one of the final codes generated from the analysis of the data. It was found in another scenario where a teacher provided a reference measurement to the students, using a measuring cylinder as a tool to legitimate the accuracy of estimated volumes found by the groups. Figure 2 shows the action taken by the teacher using a measuring cylinder to pour water into the container at the end of the classroom activity. Taking the measured volume obtained by the direct method as the reference of actual volume revealed that the critical power of the tool associated with the measuring task was essential. The teacher and the students were satisfied with the equivalence of the actual volume and the measured volume of the container.

Figure 2: A teacher using measuring cylinder to measure the volume of the frustum-like container.

In short, the legitimation found in the cases stimulated discussion aimed at justifying pragmatic usage of the tools, cultivating the students (and the teachers) negotiation for the mathematical contents of the tasks.

Tool-Task Dialectic

Previous paragraphs reveal directional processes from tool to task and vice versa. There was evidence supporting a sequential interplay between the tool and the task.

Episode 4.

[A post-lesson interview with a teacher]

Researcher: How can this level one activity [estimation in small conical flask] help students do the level two activity [estimation in large conical flask]?

Teacher 2: I hope that the students can learn how to estimate through level one activity with reasons, not by wild guessing. This estimation is not by wild guessing. There was a group that multiplied the estimated number found in the level one activity by a factor to act as estimation in level two. I can't say this method is no good as it may help students to have a better plan in the level two activity.

Episode 4 describes the observation by a teacher who designed two tasks requesting students to freely select an apparatus as the tool to estimate the number of marbles in small and large conical flasks as the two levels. The objective of the first task initiated the students into usage of the apparatuses, viewed as a dialect from the task to the tool. Sequentially, the concrete operations with the tools signified that the students could redesign their use of it in the second task. Moreover, the measured data obtained in the first task was used in the second task, in the meantime, the progression from one task to the other passing through the manipulation with the tools. The sequential process of Tool-Task interference denoted cognitive development of plans and actions performed by the students. Notably it is called Tool-Task dialectic in the inquiry study.

In the analysis of the research lessons, various forms of interactions between the tools and the tasks were invented among the cases. Task-oriented interaction (also known as tool to task) included the ability to develop senses, legitimate and provide feedback; while tool-oriented interaction (also known as task to tool) involved planning, acting, recording, etc. I named the inter-relationships of the dual-oriented interactions *Tool-Task Dialectic*, which was defined to explain how the tool interacts with the task and vice versa. To conclude, this finding addresses the research question in the study, which contributes to the empirical and practical interplays of both crucial parts that emerged in the tool-based mathematics classrooms.

DISCUSSION

As I proposed the idea of the triad of the tools, the users and the tasks in previous section, the users (also known as the students in the classrooms) were considered in the situations, with the interplay between the tools and the tasks contextualised in a setting viewed as the manipulators and owners of the tools and the tasks. The key focus was on interrelations between the tools and the tasks performed by the students and/or orchestrated by the teachers.

The tools and the tasks complimented each other in order to achieve certain purposes, which ultimately conceived a dialogue enabling the teachers and the students to develop mathematics knowledge in the classroom. In particular, some interplay was taken even further by not individuating the tool and the task a priori, but rather, talking about the two as one in which it is difficult to see how what emerges is usefully thought of in terms of the tools and the tasks.

Last but not least, the study includes TSM, considering the tool as the heart of mediating students and mathematics contents. For instance, Tool-Task dialectic is proposed to frame the endeavours of tools and tasks where the tools played the important role of mediating between the mathematics content and the students from the lens of the interplay between the tools and the tasks. As TSM concerning modern cultural contexts (Bartolini Bussi & Mariotti, 2008), development of signs and analysis of mathematical discourse are cardinal. For example, it is suggested that didactical cycle (Mariotti, 2009) could involve collaborative production of signs (Lei, 2019) which explicitly depicts generation of texts cooperating among students. The representation of signs is variously generated in the stages of individual and collaborative activity settings. Further study is recommended to explore the embodiment of signs overarching in Tool-Task dialectic.

CONCLUSION

The finding is that Tool-Task dialectic contributes to the innovation of tool-based mathematics teaching and learning in classroom settings. It encourages students not only to focus on the utilisation of tools, but also provides an alternative lens for them to concentrate upon in the theme of tasks. For mathematics teachers, design of tool-based tasks is indispensable and the harmonisation of tools and tasks for students to manipulate and to carry out should be critically considered in order to synergise construction of mathematics knowledge. It is suggested to inclusively consider the roles of the tools and the tasks simultaneously.

Acknowledgement

I would like to thank the teachers and the scholars who have contribute to this research. I am grateful to the University Grant Committee, Hong Kong SAR that supported and funded for conducting this research.

References

Bartolini Bussi, M.G., & Mariotti, M.A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education, Second Revised Edition* (pp. 746-805). Lawrence Erlbaum, Mahwah, NJ.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*. London: Sage Publications.

Cohen, L., Manion, L., & Morrison, K. (2000). *Research methods in education*. London: Routledge Falmer.

Glaser, B.G. (1998). *Doing grounded theory: Issues and discussions*. Mill Valley, CA: Sociology Press.

Glaser, B.G., & Strauss, A. (1967). *The discovery of grounded theory.* Chicago, IL: Aldine.

Grbich, C. (1999). *Qualitative research in health: An introduction*. London: Sage.

Griffiths, R., Back, J., & Gifford, S. (2016). *Making numbers: Using manipulatives to teach arithmetic*. Oxford: Oxford University Press.

Hedegaard, M. (2004). A cultural-historical approach to learning in classrooms. *Outlines: Critical social studies, 6*(1), 21-34.

Kissane, B. (2016). The scientific calculator and school mathematics, *Southeast Asian Mathematics Education Journal, 6*(1), 33-55.

Lei, H. (2019). Use of apparatuses in science laboratory: Developmental characteristics of didactical interactions. *The International Education & Learning Review, 1*(2). Madrid: Global Knowledge Academics.

Lei, H., Chan, Y.C., & Leung, A. (2018). Implementation of tool-based mathematics lesson: A duo of material and digital tools. In L. Deng, W.K.W. Ma, & C.W.R Fong (Eds.), *New Media for Education Change – Selected paper from HKAECT 2018 International Conference* (pp. 125-139). Hong Kong: Springer.

Leung, A. (2012). Variation and mathematics pedagogy. In J. Dindyal, L.P. Chen, & S.F. Ng (Eds.), *Mathematics Education: Expanding horizons, The 35th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA)* (pp. 435-442). Singapore: MERGA Inc.

Lopez-Real, F., & Leung, A. (2006). Dragging as a conceptual tool in dynamic geometry, *International Journal of Mathematical Education in Science and Technology, 37*(6), 665-679.

Mariotti, M.A. (2009). Artifacts and signs after a Vygotskian perspective: The role of the teacher, *ZDM Mathematics Education, 45*, 427-440.

Maschietto, M., & Soury-Lavergne, S. (2013). Designing a duo of material and digital artifacts: The Pascaline and Cabri Elem e-book in primary school mathematics, *The International Journal on Mathematics Education, 45*(7), 959-971. ZDM.

Norman, D.A. (1993). *Things that make us smart: Defending human attributes in the age of the machine*. New York: Perseus Press.

Pimm, D. (1995). *Symbols and meanings in school mathematics*. London: Routledge.

Rabardel, P. (2002). *People and technology: A cognitive approach to contemporary instruments*. (English translation of Les hommes et les technologies: Une approche cognitive des instruments contemporains). Paris: Amand Colin.

Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations, *International Journal of Computers for Mathematical Learning, 9*, 281-307.

ENCOURAGING DISCOVERY IN SUBSTANTIAL LEARNING ENVIRONMENTS: DESIGNING PLAY AND DOCU ROOMS

Tobias Huhmann, Ellen Komm

University of Education, Weingarten, Germany

Despite the established status of discovery learning as a fundamental teaching principle it is still lagging behind in everyday teaching. The concept substantial learning environment offers opportunities to constructively respond. After years of accompanying teacher-students in practical teaching we have developed a concept to design the so called "play and docu(menting) rooms" within substantial learning environments. Our objective is to support students in processes and products of discovering by using their own documentations. In this context, questions rise about the conditions for successfully designing documentation and dynamization processes that support individual learning as well as questions about the ways of using such a designed learning environment.

INTRODUCTION

The paradigm shift towards discovery learning based on a constructivist attitude (Piaget, 1975), was founded a long time ago in didactic terms (cf. Freudenthal, 1973; Kühnel, 1922; Winter, 1989, 2016; Wittmann, 1974b, 2000). Building upon this, critical thinking of learners is gaining importance as the student role shifts from receptivity to activity. The concept of substantial learning environments provides a suitable framework to meet the paradigm of exploratory learning in elementary school mathematics teaching, and to promote not only content-related competencies but also general skills such as critical thinking interlinked with the process-related competencies of "communicating, arguing, modelling and presenting". The established status of discovery learning as a fundamental teaching principle in didactic discourse and its anchoring in curricula is still lagging behind in everyday teaching practice. For good reasons: No "algorithmic concept" can be found for the implementation of discovery learning; rather, the design and implementation are prerequisite and complex (Krauthausen, 2018). To face this situation teaching with substantial learning environments offers various opportunities. One is the design of so called play and docu rooms (Wollring, 2008). Wollring introduces the term play room as a "space to design" and the term docu room as a "space to keep" to emphasize the importance of acting and dealing with documentations in teaching and learning situations. The play room is to offer opportunities for stimulating actions and the docu room captures action results and products. Starting from this and in an effort to encourage and enable students to investigate mathematical tasks our research project focuses on the (re-)creation, handling and usage of students´ self-made documentations in context of their discoveries.

THEORETICAL BACKGROUND

Discovery learning

Historically, the concept of discovery learning originates in various disciplines of educational science. Bruner's view on discovery learning, developed from psychology, influences the understanding in various didactics to this day, including mathematics. According to this model, discovery is

> … in its essence a matter of rearranging or transforming evidence in such a way that one is enabled to go beyond the evidence so reassembled to additional new insights. It may well be that an additional fact or shred of evidence makes this larger transformation of evidence possible. But it is often not even dependent on new information. (Bruner, 1961, p. 22)

Freudenthal made essential contributions to the development of discovery teaching and learning concepts for mathematics didactics. He shaped an understanding of discovery learning by focusing on individual re-invention of mathematics (Freudenthal, 1973). In German-speaking countries this idea was decisively influenced by Winter and Wittmann (Winter, 1988, 2016; Wittmann, 1995). The latter particularly coined the term active-discovery learning, thus highlighting the receptivity of learners in traditional arithmetic lessons to be overcome. Winter has shaped the discourse on discovery learning in all school levels up to the present and has put forward the following main thesis:

> Learning mathematics is all the more effective [...] the more it is pursued in the sense of one's own active experiences, the more the progress in the learner's knowledge, ability and judgement is based on independent exploratory ventures. (Winter, 2016, p. 1)

With reference to Neber (Neber after Winter, 2016) we consider discovery learning as a theoretical construct that embodies the idea that acquisition of knowledge and skills does not happen through the transfer of information from outside, but rather through one's own perception, action and, based on this, through analysis and reflection with reference to already existing knowledge structures, usually stimulated by external impulses (Huhmann, 2013). Overall the understanding of the concept of discovery learning ranges from the pole of free discovery on the one hand and learning by instruction on the other. Thus Winter (2016) characterizes the concept of discovery learning as the antithesis of learning by instruction, which is characterized by observing, exploring, trying and asking by the students and which the teacher tries to support by providing aids, so the learners can discover for themselves, whereby "the creation of possibilities of experience does not automatically have to produce corresponding realities of experience" (Winter 2016, p. 4). According to Winter (1988) and Freudenthal (1973) discovery in mathematics lessons is primarily a local re-invention of mathematics constructed by students, which, despite the objective limitations of these discoveries, is important for the individual learning process. This takes an understanding of mathematics as an activity into account in which

> intuition, imagination and creative thinking are involved, one can gain insights and understanding through individual and collective thinking, and make discoveries independently, thereby set up confidence in one's own ability to think and enjoy thinking. (Spiegel & Selter, 2003, p. 47)

Substantial learning environment

The concept of learning environment can be understood in different ways from a pedagogical and didactical perspective. In mathematical didactics, the term is characterized by a content-related understanding (Krauthausen, 2018). In this sense, learning environments represent an extension of the concept of the "good task" (Wollring, 2008). In the context of the paradigm shift towards a discovering view of learning based on a constructivist attitude, since the 1970´s until today the core task of mathematics didactics is to design and implement mathematically rich, so-called substantial learning environments (Krauthausen, 2018; Wittmann, 1974a, 1992). The term "substantial learning environment" was coined in the German-speaking world by Wittmann (1974a, 1995, 1998) in connection with a view of mathematics didactics as an application-oriented design science and based on a Piagetian understanding of learning. Based on this Wollring (2008) develops a concept for teaching and learning with substantial learning environments, which appears particularly significant to investigate students´ discoveries regarding their self-made documentations. He defines the term learning environment as a "flexible big task. It consists of a network of smaller tasks that are bound together by certain guiding ideas" (Wollring, 2008, pp. 12-13). The learning environment refers to the concrete implementation of the task in class. Considering discovery learning in connection with students´ documentations we highlight the guiding idea "articulation" that includes three forms: action, speaking and writing. This also implies that learning environments should offer opportunities "to present processes and results in a fleeting and non-volatile way, so that discoveries are enabled and supported" (Wollring, 2008). In this context he introduces the terms "space to design" (play room) and "space to keep" (docu room). In the play room the pupils are acting with mathematical objects and in the docu room they document their activities and discoveries. This is particularly important against the background of the volatility of representation of action processes and products (Huhmann, 2013). Documentations support learners in their explorations of the learning environment (Wollring, 2008).

Discovering in substantial learning environments

"Discovery, like surprise, favors the well prepared mind" (Bruner, 1961, p. 21). This statement is taken into account by designing substantial learning environments considering articulation as a guiding idea. Thus, a well prepared mind for discoveries can be supported by enabling action-based working through and penetration of the mathematical subject in the play room. In addition, documentations recording these actions create a docu room that counteracts the representational volatility of these actions (Huhmann, 2013). A crucial role in this context is the function of and the dealing with students' self-created documents. Carrying on Selters (1994) understanding concerning functions of students' own productions, documents can feature as instruments directly in order to discover one's own solutions. In addition, documents can be used as a means of communication to relay one's own understanding and to promote critical thinking by in the examination of one's own actions and thus possibly also indirectly contribute to one's own learning.

However, it seems to be important to keep in mind, that the creation of opportunities to experience does not automatically lead to realities of experiences for the learners (Winter, 2016). Which experienced realities actually emerge on the basis of the new possibilities for experience created by designing play and docu rooms is to be investigated.

REASEARCH OBJECTIVE AND DESIGN

Background

The interest for the topic and the research objective has arisen in connection with the one-semester didactic internships since 2015: University students for elementary education make teaching experiences with substantial learning environments. The focus is on (assisted) planning math lessons in the sense of discovery learning, actual teaching accompanied by university lecturers and school mentors as well as joint critical reflection on these experiences. In this context we frequently observed that the handling and use of students' own documentations contributes significantly to the implementation of discovery learning. Especially at the beginning of the internship most teacher students lack sensitivity and knowledge regarding the importance of learner documentations.

Objective

Our first research objective is to investigate how play and docu rooms and their reciprocal relationship can be designed within substantial learning environments and how discovery learning can be made possible in a supportive way by this. Secondly we ask how children discover mathematics within substantial learning environments regarding their dealing with and usage of self-made documents.

Design

In the first research part, which we are currently working on, documents on teaching-learning processes in maths lessons are analysed regarding design and interrelation of play and docu rooms to identify activities which support or hinder discovery within these settings. The focus is on the role of students' self-made documentations. The data analysis is carried out using a structured content analysis (Kuckartz, 2014) with mixed deductive-inductive procedure. Deductive categories were derived from the concept of substantial learning environments including play and docu rooms (Wittmann, 1998; Wollring, 2008), the role of documents in view of the volatility of actions (Huhmann, 2013) and the function of students' self-productions as documents (Selter & Sundermann, 2005). Besides their confirmation these categories were also inductively differentiated and supplemented. Understanding mathematics didactics as an application-oriented design science the results of the first research part will be used for (re)designing substantial learning environments in different content-areas focusing on the layout of play and docu rooms. In the second research part we analyse in qualitative case studies how students work within these (re)designed learning environments. The aim is to find out about their use of self-made documents with regard to their individual discoveries. The data basis for the first research part originates from one-semester internships. It includes classroom observations of teacher trainers, teaching plannings and reflections of trainees and children's self-made documents. These are actual products of the children or photographs of them in various stages. Figure 1 gives an overview of the whole research project.

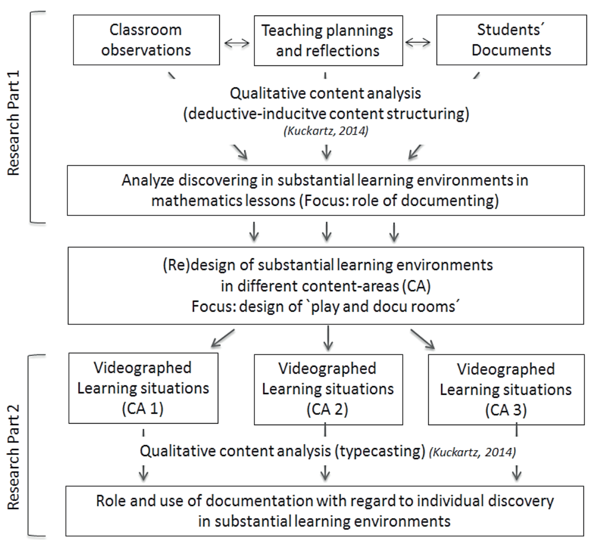


Figure 1: Overview of the research design

FINDINGS

The following results relate to the first part of the study (see Fig. 1). Regarding documentation from a general perspective these findings can be stated:

1. Task-adequate documentations enable activities in the docu room that support or directly generate discoveries as they counteract representational volatility. These include (1) repeated and permanent perceiving of processes or products emerged from activities in the play room, (2) recognizing and (re)focusing the recognized and, based on this, (3) explaining to oneself and others (see fig. 6).

2. Elaborate documentation requirements can lead to termination of the discovery process and also impede its developmental potential as the focus of the learner's attention shifts to the documentation process that is to the creation of the documentation or because the documentation process itself breaks off and therefore is insufficient for further discoveries.

3. Missing or insufficient documentation can hinder the developmental potential of discovery processes as beneficial activities in the docu room (see 1.) are not or only partially possible.

Shifting from the general to a more detailed perspective on documentations the following results relate to different types.

1) We distinguish between two different types of documentations. Both types capture mathematical action processes and products to form the document space and counteract representational volatility:

The first type of documentations is characterized as "static and not able to dynamize". They remain static in the form they are originally created. This type is often found in classroom settings. The second type of documentations is characterized as "static and able to dynamize": The originally created form can be changed by breaking it up into smaller documentation units. These units represent mathematical objects for further transformation actions to build new creations in the play room.

2) Based on the second type, there opens up a spectrum of further detailed types from "static and not able to dynamize" at the one end to "static and hard to dynamize" to "static and easy to dynamize" at the other end. "Hard to dynamize" means that the effort to change the originally created form by breaking it up into smaller documentation units is very high. "Easy to dynamize" means that this effort is very low. Figure 2 (blackboard), figure 3 (sheet of paper) and figure 4 (sheets of paper as smaller documentation units) show examples of student´s documents which relate to one and the same task[1] and represent these three different types of documentations.

[1] The task within the content area "Data, frequency, probability" was to investigate, which and how many different matches there are, if you played a tournament with four teams (red, green, orange and blue) and each team shall play exactly once against each other team.

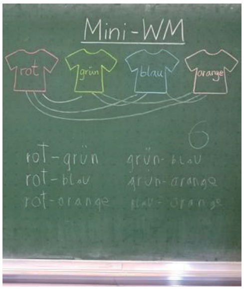


Fig. 2: Not able to dynamize

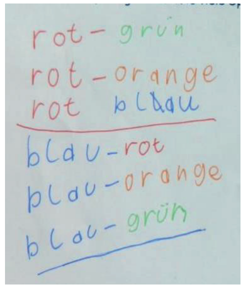


Fig. 3: Hard to dynamize

Fig. 4: Easy to dynamize

3) We identify a general potential of the documentation type “static and able to dynamize” as these documentations can be subjects to various revisions and in this process this enables renewed opportunities in renewed play and docu rooms. The opportunities in the play room include (new) experiences in the (new) scope for (1) perception, (2) “re-ordering” as “re-serializing”, “re-sorting” as “re-classifying”, and “re-structuring” and (3) experiences of action (see also Fig. 7, p. 9). So besides the possibility to gain experiences in the play room by sensory and haptic actions with mathematical objects (rotating, mirroring, pushing, building, rebuilding ...) dealing with documentation units (spatially arranging iconic or symbolic objects) arise from the docu room through dynamization.

To illustrate these new experiences in the play room figure 5 shows how an ordering activity[2] with document units structures the solutions found in a serializing way (from top to bottom or vice versa). Thereby critical thinking was triggered and new discoveries with regard to the number of possible solutions were supported: Possible solutions that were not found in an initial processing phase could now be discovered due to the order, which allows gaps to be perceived and recognized. Similar to this, figure 6 shows how a sorting activity[3] with the documentation units structures the discovered quadrangles, as they are arranged horizontally in equivalence classes. In both cases, new insights were gained after first solutions were found by dynamizing the documents and thereby individual learning paths were continued. These new insights were, in Bruner’s sense, created independently of new information, but are based on a reflective approach to the already existing own documents.

[2] The Substantial task format ‘Calculation Squares with Ears’ (see Fig. 5) is based on the following rules: The relationship between the base numbers (inner numbers): The sums of the base numbers of each row must be identical: a+b=c+d. The relationship between the base numbers and the outer numbers: The sum of the base numbers of a column is entered as a result in the adjacent outer number field: x=a+c and y=b+d. The numbers for a, b, c, d, x and y are natural numbers. The specific task to be explored in the content area number and operations was: How many calculation squares do exist with the outer numbers 10 and 130?

[3] The task within the content area space and form (see Fig. 6): How many quadrangles do exist on the 3x3 geo board?

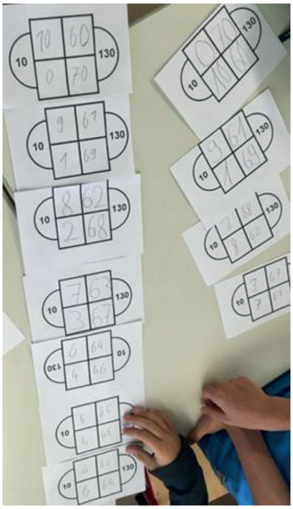


Figure 5: Ordering activity

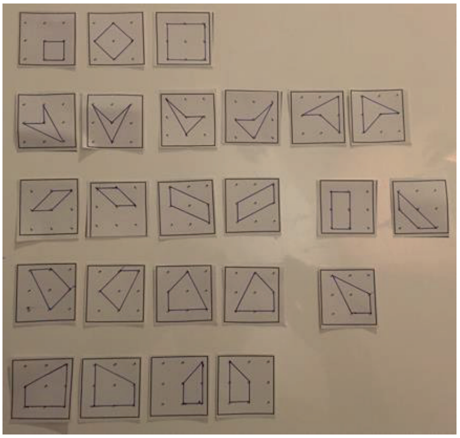


Figure 6: Sorting activity

These activities, opened up by dynamizing documents, represent a reciprocal relationship between play and docu rooms: The play room describes a temporal-successive dimension, which is characterized by dynamics in process-like, fleeting, acting representation. The docu room describes a spatial-simultaneous dimension, characterized by the static of non-volatile representation. The model in figure 7 illustrates this relationship and summarizes activities in the play and docu room that can promote critical thinking and discovery learning.

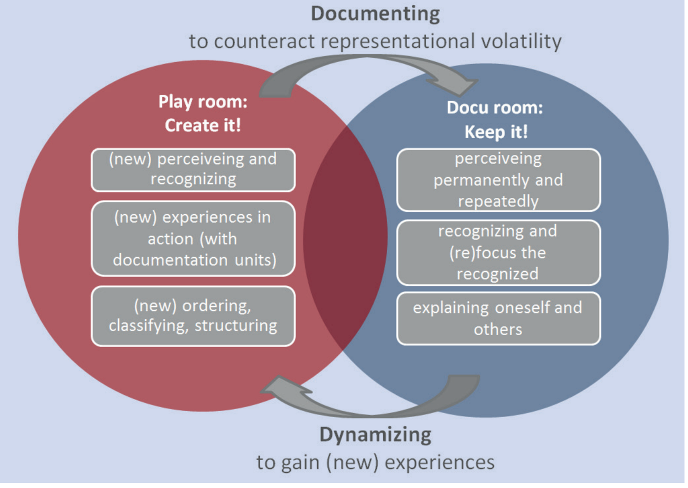


Figure 7: Model “Reciprocal design of play and docu rooms”

As documents are dynamized and actions with documentation units are carried out an intersection between docu and play room emerges. This intersection results on the one hand from the fact that documentation units of the docu room become manipulable objects of the play room. On the other hand it is due to partly direct or unflinching transformation of the objects into documentations.

DISCUSSION

The findings concerning missing, effortful and insufficient documentation from the general perspective as well as missing, effortful and insufficient dynamization of documentations from the detailed perspective show negative factors for the activation and deepening of critical-mathematical thinking and discovery learning: The developmental potential of discovery processes cannot become effective, since features and structures cannot be perceived and recognized due to missing possibilities of mathematical “re-ordering” as “re-serializing”, “re-sorting” as “re-classifying”, and “re-structuring”. In contrast, a promising potential lies in the reciprocal design of play and docu rooms: By creating “easy to dynamize documents” new possibilities to follow up the individual learning paths and opportunities to stimulate and support critical-mathematical thinking and discoveries arise. Thus new questions rise which demand a new mathematical approach. This also includes in particular new discoveries that are independent of new information (Bruner, 1961). These findings seem to be of particular importance since in every day teaching documenting usually takes place in a way that remains static: Students document their solutions and findings, but these cannot be used to continue individual learning paths. As these documents remain static after discussing the first solutions individual learning paths break off since further learning takes place by giving new tasks to the students. We identified two types of documentations that lead to this situation: “static and not able to dynamize” and “static and hard to dynamize”. So dynamization of documentations is an important design element for learning environments. For research and educational practice we see great potential for design based research to develop and investigate “easy to dynamize documents” in teaching-learning processes. As a design element dynamization of documentations raises further research questions about the conditions for successful learning: Which conditions for success can be identified to design “easy to dynamize documents”? How and under which conditions can “hard to dynamize documents” be developed into “easy to dynamize documents”? Whether and how do teachers and students use this theoretical potential in teaching-learning processes? Whether and how does this support discovery learning and critical thinking about the students’ own and others’ thoughts, ideas and learning products with the subject matter? What is the role of the size of the documentation units? We formulate the following hypothesis: Documentation units should be of such size as to enable new perception, action and refection by new ordering, classifying and structuring. By answering, the model of “reciprocal design of play and docu rooms” will be further developed. With a focus on the role of documentation for the individual learning process this will be explored in part two of the research project on the basis of further developed learning environments in different mathematical content areas. Thereby the question of *how* and *which discovery realities* arise in fact through the design of *discovery possibilities* in substantial learning environments shall be investigated.

References

Bruner, J. S. (1961). The act of discovery. *Harvard Educational Review, 31*, 21–32.

Freudenthal, H. (1973). *Mathematik als pädagogische Aufgabe* [Mathematics as a pedagogical task]. Stuttgart: Klett - Studienbücher Mathematik.

Huhmann, T. (2013). Einfluss von Computeranimationen auf die Raumvorstellungsentwicklung [Influence of computer animation on the development of spatial perception]. *Dortmunder Beiträge zur Entwicklung und Erforschung des Mathematikunterrichts: Vol. 13.* Wiesbaden: Springer Spektrum.

Krauthausen, G. (2018). Einführung in die Mathematikdidaktik – Grundschule [Introduction to mathematics didactics – Elementary school] (4th ed.). *Mathematik Primarstufe und Sekundarstufe I + II.* Berlin, Heidelberg: Springer Spektrum.

Kühnel, J. (1922). *Neubau des Rechenunterrichts: ein Handbuch für alle, die sich mit Rechenunterricht zu befassen haben* [New teaching of numeracy: a handbook for all those involved in teaching numeracy]. Leipzig: Klinkhardt.

Kuckartz, U. (2014). *Qualitative Inhaltsanalyse: Methoden, Praxis, Computerunterstützung* [Qualitative Content Analysis: Methods, practice, computer support] (2nd ed.). Weinheim, Basel: Beltz Juventa.

Piaget, J. (1975). *Der Aufbau der Wirklichkeit beim Kinde* [The construction of reality in the child] (2nd ed.). Stuttgart: Klett Cotta.

Selter, Christoph (1994). *Eigenproduktionen im Arithmetikunterricht der Primarstufe* [Students' own productions in arithmetic classes at elementary level]. Wiesbaden: Dt. Univ.-Verl.

Spiegel, H., & Selter, C. (2003). *Kinder & Mathematik: Was Erwachsene wissen sollten* [Children & Mathematics: What adults should know] (1st ed.). Seelze-Velber: Kallmeyer.

Sundermann, B., & Selter, C. (2005). *Lernerfolg begleiten – Lernerfolg beurteilen. SINUS-Transfer Grundschule* [Monitoring learning success – assessing learning success. SINUS-Transfer Primary School]. Retrieved from www.sinus-an-grundschulen.de

Winter, H. (1988). Lernen durch Entdecken? [Learning by discovering?] *Mathematik Lehre*n, 28, 6–13.

Winter, H. (1989). *Entdeckendes Lernen im Mathematikunterricht: Einblicke in die Ideengeschichte und ihre Bedeutung für die Pädagogik* [Discovery learning in mathematics classes: Insights into the history of ideas and their significance for education]. Braunschweig: Vieweg & Sohn.

Winter, H. (2016). *Entdeckendes Lernen im Mathematikunterricht: Einblicke in die Ideengeschichte und ihre Bedeutung für die Pädagogik* [Discovery learning in mathematics classes: Insights into the history of ideas and their significance for education] (3rd ed.). Wiesbaden: Springer Spektrum.

Wittmann, E. C. (1974a). Didaktik der Mathematik als Ingenieurswissenschaft [Didactics of mathematics as engineering science]. *ZDM, 3,* 199-121.

Wittmann, E. C. (1974b). *Grundfragen des Mathematikunterrichts* [Basic questions of mathematics teaching]. Wiesbaden: Vieweg+Teubner Verlag.

Wittmann, E. C. (1992). Mathematikdidaktik als "design science" [Didactics of mathematics as "design science"]. *Journal Für Mathematik-Didaktik*, *13*(1), 55–70.

Wittmann, E. C. (1995). Unterrichtsdesign und empirische Forschung. [Teaching design and empirical research.] In K. P. Müller (Ed.), *Beiträge zum Mathematikunterricht: Vorträge auf der 29. Tagung für Didaktik der Mathematik vom 6. bis 10. März 1995 in Kassel* (pp. 528–531). Hildesheim: Franzbecker.

Wittmann, E. C. (1998). Design und Erforschung von Lernumgebungen als Kern der Mathematikdidaktik. [Design and exploration of learning environments as the core of mathematics didactics.] *Beiträge zur Lehrerinnen- und Lehrerbildung, 3,* 329–342.

Wittmann, E. C. (2000). Aktiv-entdeckendes und soziales Lernen im Rechenunterricht: - vom Kind und vom Fach aus [Active-discovery and social learning in numeracy classes: - from the child and from the subject]. In G. N. Müller & E. C. Wittmann (Eds.), *Beiträge zur Reform der Grundschule: Vol. 96. Mit Kindern rechnen* (2nd ed., pp. 10–41). Frankfurt am Main: Arbeitskreis Grundschule.

Wollring, B. (2008). Kennzeichnung von Lernumgebungen für den Mathematikunterricht in der Grundschule [Labelling of learning environments for the teaching of mathematics in elementary school]. In Kasseler Forschergruppe (Ed.), *Lehren - Lernen - Literacy: Vol. 2. Lernumgebungen auf dem Prüfstand: Zwischenergebnisse aus den Forschungsprojekten* (pp. 9–26). Kassel: Kassel Univ. Press.

DO STUDENTS ANALYZE AND EVALUATE THE RESULT OF THEIR PROBLEM SOLVING ACTIVITY?

Márton Kiss, Eszter Kónya

University of Debrecen, Hungary

In this paper, we investigate 9th grade students work while they solve mathematical problems where more possible cases or even an impossible case appear. We planned a developing experiment which aimed at accustoming students to handle a mathematical problem in a conscious way and from a wider perspective. Furthermore, another aim was to change the inaccurate belief that each mathematical problem has an answer and that is the only answer.

INTRODUCTION

The critical thinking skills of mathematics have appeared in the curriculum of mathematics education in many countries (Mason, Burton, & Stacey, 2010) and in Hungary as well. When students think critically in mathematics, they make reasoned decisions about what to do and think and do not simply guess or apply a rule without assessing its relevance. So, it is reasonable to include problem solving as a means of a better understanding of mathematical concepts into the mathematics classrooms. While the students analyse problem situations, they critically adapt to their thinking and learn to explain and justify their idea. In our previous research, when we investigated the typical mistakes students make, we realised that most errors could have been avoided if the students had consciously analysed their solution (Kiss & Kónya, 2018).

In this paper, we investigate 9th grade students work while they solve mathematical problems where more possible cases or even an impossible case appear. The starting point of our investigation was one of our earlier research in which we tried to get an insight into students' metacognitive activities (Kiss, 2019).

THEORETICAL BACKGROUND

According to Facione & Facione (1992) the core critical thinking skills are in general: interpretation; analysis; evaluation; inference; explanation and self-regulation. From mathematics education's point of view, we can say that these skills are essential for someone to be successful in learning mathematics and to become a good problem solver. These required skills are in close relation with the four phases of the mathematical problem solving process described by Polya: (1) Understanding the problem; (2) Devising a plan; (3) Carrying out the plan; (4) Looking back. (Polya, 1957). Understanding the problem means interpretation and analysis as well. Devising and carrying out a plan is in line with the evaluation and inference, i.e. evaluation of the given information, making connections between them furthermore identifying and securing elements needed to draw a reasonable conclusion. Explanation and self-regulation are parts mainly of the fourth phase. In this study, we pay attention mainly to the fourth Polya-phase.

Problem solvers should represent and justify their results and be able to monitor their problem solving activity. The self-regulation skill cannot be separated from the term of metacognition. Metacognition is a conscious control and regulation of representing and processing information (Zsigmond, 2008). When using metacognition, students become aware of their style of learning and can recognize and implement different solving strategies. As Henningsen and Stein pointed out “Self-monitoring can increase students' feelings of competence and control and, in turn, their motivation to remain engaged with a task at a high level.” (Henningsen & Stein, 1997, p. 527).

We agree with Aizikovitsh-Udi and Cheng (2015), "critical thinking is constituted through both dispositions and abilities. While the abilities may be developed through direct instruction, the dispositions are better thought of as “habits of mind” and their development requires long-term participation in learning environments conducive to reflection and argumentation.” (p. 455) and “a structured program should start with the promotion of appropriate dispositions and progressively move to the development of critical thinking abilities.” (p. 456) In our teaching experiment, we used the generally accepted infusion approach of teaching critical thinking, where critical thinking skills are taught explicitly using the discipline's content. The mathematical content we chose for our experiment was the plane geometry in grade 9.

Another aspect that should be considered in the developmental experiment is the difference in the students’ existing beliefs regarding mathematical problem solving. Here, we highlight only one: several people have an inaccurate belief that every mathematical problem has a solution and it is the only solution (Schoenfeld, 1992). As Ennis (1985) pointed out, critical thinking is a reflective and practical activity the purpose of which is to moderate action or belief. We claim that a carefully planned instructional program aiming at developing students’ critical thinking can help to change the above-mentioned belief as well.

Vygotsky's theory on the zone of proximal development (Vygotsky, 2000) supports our research on classroom instruction as follows. The theory which is often characterised also by the notion of scaffolding represents the distance between a student’s actual and potential level of development. According to Vygotsky, a student can get only to the zone of proximal development. The initial phase is to determine the actual level of student’s development through the tasks that he/she solves independently. Thereafter, we investigate how this student handles a more difficult task than the previous one. We assist in demonstration, leading questions, or worked examples. "The difference between the child’s actual level of development and the level of performance that he achieves in collaboration with the adult, defines the zone of proximal development." (p. 272-273) In this study, we analyse four students' developmental processes concerning the topic of plane geometry and try to determine their zone of proximal development.

RESEARCH QUESTION

We planned a developing experiment aimed at accustoming students to handle a mathematical problem in a conscious way and from a wider perspective. Another aim was to change the inaccurate belief that each mathematical problem has an answer and that is the only answer. We wondered whether the students realise that there are more possible cases or recognise the contradiction while solving the problem. Our aspects in the analysis:

Q1 Does it depend on students' content knowledge whether they recognise more cases and impossible cases or not?

Q2 To what extent can critical thinking of students with different levels of development be improved?

METHODOLOGY

The developing experiment took place in a 9th grade class consisting of 29 boys (age of 14-15 years). The class had an average performance in Mathematics, and they had three Mathematics lessons weekly which is the minimum in compulsory education in Hungary. The first author of this article was the teacher during the experimental period.

The 5-week-long experiment was built around a whole geometrical topic (triangles, quadrilaterals). During this time, we adhered to the existing curriculum, the only new element was that besides the routine tasks we also discussed geometric problems that have more possible solutions, or which lead to contradictions. The program started with a pre-test on 11 March 2019. This was followed by 11 lessons; on the 5th lesson, a second test was written. We finished the experiment with a post-test on 15 April. One and a half months later after the post-test, a delayed test was written.

In the tests and during the lessons, students were given the following and similar type of problems:

Four children problem (Pre-test)

Anna, Béla, Cili, and Dani are standing along a line. Béla is 5 m far from Anna, Cili is 3 m far from Béla, Dani is 1 m far from Cili. How far can Dani be from Anna?

There are four different possible cases depending on the order of the children. Usually, the answer given by the students is 5+3+1=9 m only. (Kovács & Kónya, 2019).

The height problem (Second test, Delayed test)

Two sides of an isosceles triangle are 6 cm and 16 cm. What is the measure of the height belonging to the base?

In this problem, two different cases have to be dealt with, since the text does not tell which item belongs to the base and the legs. However, one of the two cases is impossible which can be proven by using the triangle inequality theorem (6+6<16).

The kite problem (Post-test)

The measures of two interior angles of a kite are 70° and 150°. What are the measures of the other two interior angles of the kite?

In this problem, three different cases have to be taken into consideration, since the text does not tell where each angle is located. However, one of the cases is impossible which can be proven by using the theorem about the sum of the triangle's interior angles (150°+150°+70°>360°).

RESULTS AND DISCUSSION

The achievement of the class

After analysing the tests, it seems that around half of the group worked with more cases during the post-test and delayed test, while at the beginning (*Four children problem*) only two students worked with more cases. They also realised the contradiction, and some change is noticed in the students' approach. Figure 1 gives a detailed overview of students' achievement. We distinguish 4 categories as follows. "More cases": Students handled more than one case; "Impossible case" shows how many students dealt with this case at all; "and realises that it is impossible" is a subcategory of the previous one, while "and explains" refers to the students from the previous category who even explained why the case is impossible. The results of the *Four children problem* are not demonstrated here since there were only two students who realised more cases.

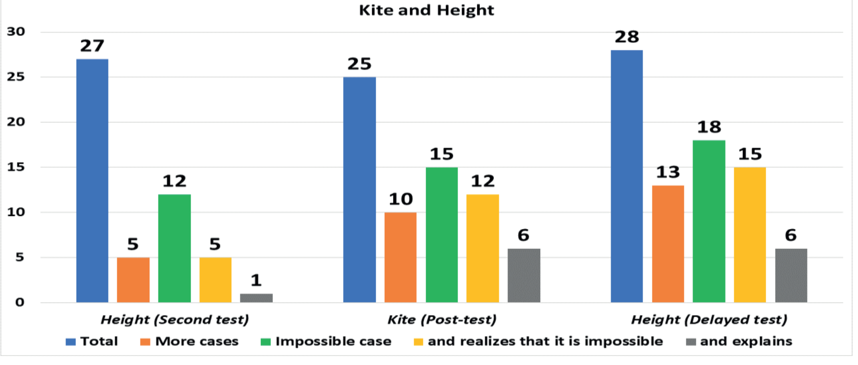


Figure 1. Summary of the achievement of the class.

What can be the reason behind the experience that the development did not take place in the whole class? The rate of students' development was significantly influenced by the lack of students' content knowledge. For example, it is much harder to realise that a case is impossible if they cannot use the Pythagorean theorem properly, which meant difficulty for some students. Also, the zone of proximal development can be considered here as well. Maybe the level we expected from the students was not always consistent with their proximal development zone.

To deeper analyse our findings, we divide the students' developmental process into 4 stages: (1) Understanding the task; (2) Acquiring content knowledge needed in this topic; (3) Recognising more cases and contradiction; (4) Handling those issues consciously and explaining them. The level originally we expected is at least the (3).

The achievement of the four chosen students

After analysing the work of the class, we picked up four students who managed to reach a certain level of development and whose actual level was all different. We wanted to observe the level they can reach during the same developing process. Regarding their achievement on the pre-test and in the lessons, students S1, S2, S3, and S4 were chosen.

S1: He was the only student who was quite close to our expected level. He found all the four answers to the *Four children problem*.

S2: It was visible that he was uncertain, he thought of more cases, but he crossed them out and left only the usual case (9 m). Later in the lessons, we realised that he has an appropriate content knowledge in geometry.

S3: He only wrote the usual answer, which means that he was not accustomed to such type of problems, but he had clever comments in the experimental lessons.

S4: He had similar results to S3; he was not accustomed to such type of problems, but he did not show any remarkable results during the lessons.

We described the developmental process of the four students on the basis of the second- post- and delayed tests.

S1's work reflected our "expected" level (4). On each test, he examined the existence, dealt with more cases and explained his answers precisely, e.g. he referred to triangle inequality in the *Height problem*. Comparing his pre- and second tests with his post- and delayed tests, in the latter two, he worked more precisely and went into details. So, the development is apparent in his case.

S2 immediately dealt with the possible case in the second test (*Height problem*) (Figure 2). His first try was crossed out because he got a negative number as a square. Secondly, he may have realised that one leg was longer than the hypotenuse. Anyway, he judged his results, since he had a third try and accepted the result of that try. This is a kind of metacognitive activity, self-checking. It seems that S2 has the content knowledge needed in solving the task. However, he took no notice of the open start of the problem and therefore, he did not realise the two possible cases. And that is what the development was aimed at.

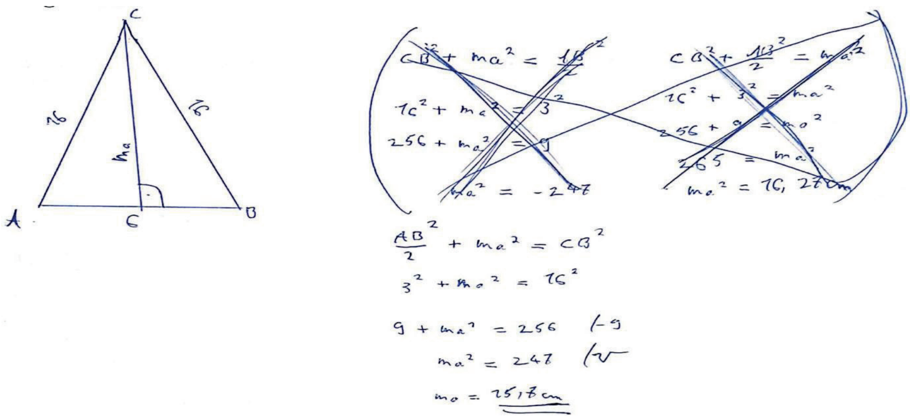


Figure 2: Finally S2 found the correct answer.

In the post-test (*Kite problem*), he had a clear systematic solution with analysing three cases. (Figure 3)

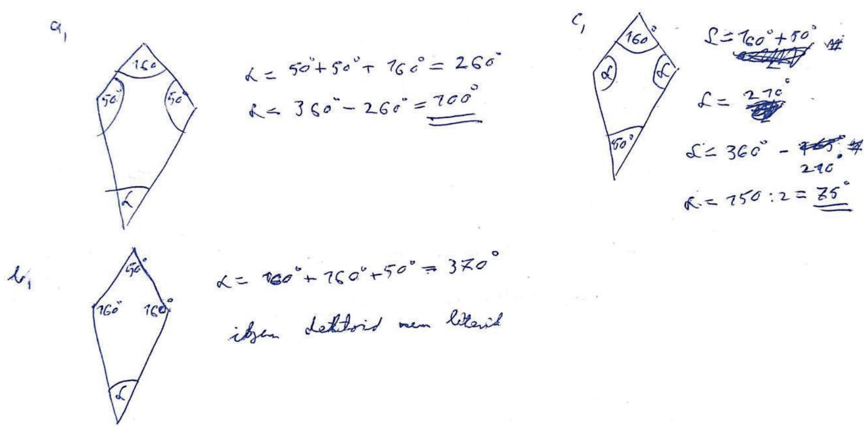


Figure 3: a). possible b). "there is not such a kite" c). possible

He possesses the content knowledge necessary for successful problem solving (angle properties of a kite). The effect of the development is visible: he realised that there are more cases and gave an explanation too.

In the delayed test (*Height problem*), S2 did not stop working after finding a possible case, but he examined another case as well. Then he crossed that out, concluded that there is only one triangle, but the justification is missing. (Figure 4)

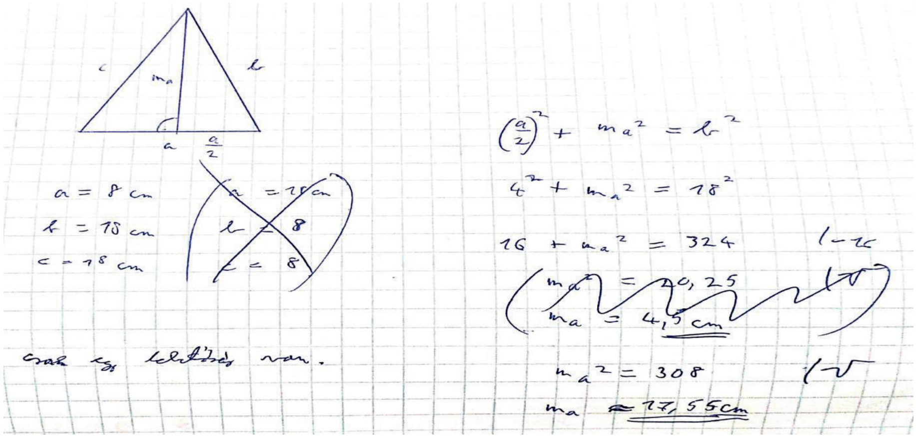


Figure 4: "There is only one case."

Based on the post-test, we can establish that the zone of proximal development of S2 coincides with the level we set out. Nevertheless, when analysing his delayed test, it seems that he has not reached this level permanently since he did not give any explanation there. However, the recognition is present. We can conclude that out of the two potential steps (recognition and explanation), he can perform the first one on his own (picked out from the context) after more than a month.

In the second test, **S3** got two different answers for the lengths of the height but did not compare them. Firstly, he constructed the triangle and measured the height (13,5 cm). Then using the Pythagorean formula incompletely, he counted it (16,27 cm). He did not feel that constructing is not a good method for finding the answer. Furthermore, he did not feel that he had to get the same results while constructing and counting. There is no trace of the other case. It seems that his content knowledge is incomplete.

In the post-test, he gave his answers about different cases in a table. (Figure 6)

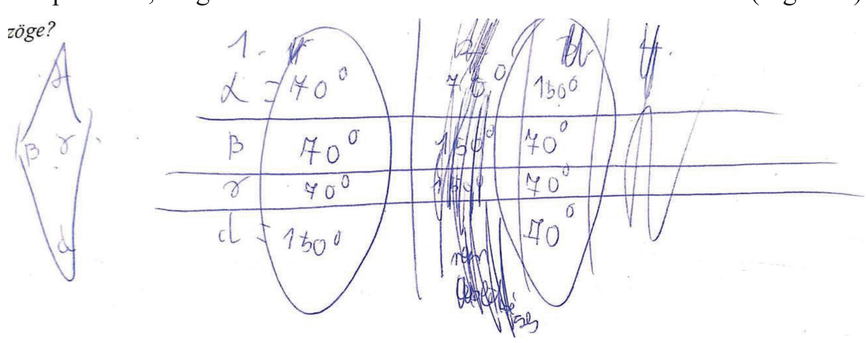


Figure 5: Different cases in a table

After finding a possible and an impossible case correctly, he looked for more cases; however, he simply crossed out the impossible case without explaining it. He did not realise that there are two cases among his answers which are the same. Creating a table in his solution may refer to consciousness, clever thinking and planning (metacognitive activity). He managed to find the impossible case, however, the way he handled it was not precise.

In the delayed test, his content knowledge is visible. He used the Pythagorean theorem correctly; he did not want to construct. Although he wrote down the two cases, he did not explain. He counted the existing case properly.

S4 similarly to S3 could not apply the Pythagorean formula in the second test and did not deal with more cases. In the post-test, he solved the problem for one kite (50°; 50°; 160°; 100°). In the delayed test when he drew the first diagram, he realized that this case is impossible, but instead of a detailed explanation, he simply changed the lengths of the sides (see the arrows in Figure 6). Thereafter, he calculated the height of the existing triangle correctly and what is more, he was able to correct himself, which means that his content knowledge related to the Pythagorean theorem has developed.

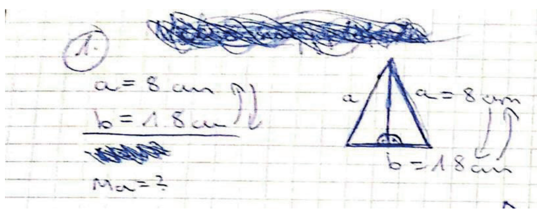


Figure 6. S4 realizes that such a triangle doesn't exist.

We summarize the achievement and the development of the investigated four students in Table 1.

	Pre-test	Second test		Post-test		Delayed test	
	More cases	More cases	Contra diction	More cases	Contra diction	More cases	Contra diction
S1	✓✓	✓✓	✓✓	✓✓	✓✓	✓✓	✓✓
S2	hesitate	-	✓	✓✓	✓✓	✓✓	✓
S3	-	-	-	✓✓	✓	✓✓	✓
S4	-	-	-	-	-	✓	✓

Table 1: The development of the students

✓ means development related to the pre-test, and ✓✓ means that presumably, the students handle and look for more cases (they do not stop working after one answer) and impossible cases (they write down that a case is impossible and may explain it) consciously.

Although all the three students got ✓✓, it does not mean that they reached the same level. It rather means that more or less, they had the results we expected from the development (though in the case of S2 and S3, explaining did not improve totally). There is a difference in *how* they handle the recognised facts.

S1 had the highest actual level of development; however, he did not develop as much as S3 did. Although S3's actual level of development is lower, his rate of development seems to precede S1. The four students' zone of proximal development enabled improvement at this level. This is also true for some students in the class, but our expectation would have been a huge step for the other students.

CONCLUSION

As analysis of the work of S3 shows clearly, learning of the appropriate content knowledge should precede the recognising more or even impossible cases. Moreover, if someone possesses the required content knowledge it is not enough for critical thinking. The result of S4's delayed test gives an example of this.

We find that the rate of development is significantly influenced by the zone of proximal development. There were some students whose zone of proximal development did not open the door of development at the level we wanted to reach. Based on the pre-test, S1 was really close to the expected level. That is why in contrast with S3, S1 could not get hold of such a huge development compared to himself. So from this aspect, the developing experiment was not significant for students with low or high actual levels of development. The zone of proximal development differs from student to student, and it cannot be determined objectively.

Acknowledgment

The research was funded by the Content Pedagogy Research Program of the Hungarian Academy of Sciences. We also would like to thank Ioannis Papadopoulos for his constructive suggestions, remarks, and comments.

References

Aizikovitsh-Udi, E., & Cheng, D. (2015). Developing Critical Thinking Skills from Dispositions to Abilities: Mathematics Education from Early Childhood to High School. *Creative Education, 6*, 455-462.

Ennis, R. H. (1985). The Logical Basis for Measuring CT Skills. *Educational Leadership, 43*, 44-48.

Facione, P. A., & Facione, N. C. (1992). *The California Critical Thinking Disposition Inventory.* Millbrae, CA: California Academic Press.

Henningsen, M., & Stein, M. K. (1997). Mathematical Tasks and Student Cognition: Classroom-Based Factors That Support and Inhibit High-Level Mathematical Thinking and Reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Kiss, M. (2019). *Metakognitív tevékenységek a problémamegoldás során* [Metacognitive activities during problem solving] - master thesis. Debrecen, Hungary: University of Debrecen.

Kiss, M., & Kónya, E. (2018). Mi a sorsa egy megtanított ismeretnek? [What happens to a previously taught knowledge?]. In *Használni akartam, nem tündökölni - A 75 éves Ambrus Andrást köszöntik tanítványai, tisztelői [I wanted to be useful, not shine - on the occasion of the 75th birthday of András Ambrus]* (pp. 163-175). Kiskőrös, Hungary: Druk-ker Kft.

Kovács, Z., & Kónya, E. (2019). How do novices and experts approach an open problem? In Z. Kolar-Begovic, R. Kolar-Super, & L. J. Matic (Eds.), *Towards new perspectives on mathematics education* (pp. 245-260). Osijek, Croatia: Element d.o.o.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically.* Harlow: Pearson.

Polya, G. (1957). *How to solve it* (2nd ed.). Princeton, New Yersey: Princeton University Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Vygotsky, L. S. (2000). *Thought and Lamguage [Gondolkodás és beszéd]* (Hungarian. kiad.). Budapest: Trezor.

Zsigmond, I. (2008). *Metakognitív stratégiák* [Metacognitive strategies]. Kolozsvár, Romania: Scientia.

PROBLEM SOLVING: HOW DO STUDENTS WITH DIFFERENT PERSONALITY TYPES SHOW THEIR CRITICAL THINKING WHEN SOLVING A MATHEMATICAL PROBLEM?

Linda Devi Fitriana

University of Debrecen, Hungary

Personality becomes a factor which indirectly influence student critical thinking. This research aims to investigate how students with different personality types show their critical thinking when solving a mathematical problem, as identified by the application of Keirsey's theory. The subjects are junior secondary school students who represent each personality type by Keirsey. Qualitative data were collected through a problem solving test and interview. The result underlines that all students propose different conclusions for the problem and show their critical thinking according to the nature inherent in their personality type. This condition can be a beneficial input for teachers in designing instruction for preparing students to be better in critical thinking by considering their character.

INTRODUCTION

Problem solving and critical thinking are interrelated; critical thinking affects one's ability to solve a problem (Jacob & Sam, 2008; Butterworth & Thwaites, 2013). As important capabilities in mathematics (De Lange, 2006; OECD, 2013), both skills are used to solve problems in daily life. Therefore, it is not astonishing that one of the essences of mathematics learning is to develop those skills.

Problem solving can be interpreted as an effort to find a solution to the situation at hand (Polya, 1973; Krulik & Rudnick, 1989). There are four steps in problem solving: understand the problem, devise a plan, carry out the plan, and look back (Polya, 1973). Meanwhile, critical thinking is a skill which comprises mental processes, strategies, and representations people use to solve problems, make decisions, and learn new concepts (Sternberg, 1986). The mental processes involved in critical thinking are metacomponents, performance components, and knowledge-acquisition components (Sternberg, 1985). Metacomponents are used to plan, monitor, and evaluate the plan when solving a problem. Performance components are used to carry out the instructions of metacomponents. Knowledge-acquisition components are used to learn how to solve a problem, then controlled by metacomponents and solved by performance components. In addition, critical thinking also defined as to think logically and accurately when evaluating reasons as the basis for taking action (Carrol, 2004;

Jacob & Sam, 2008; Lai, 2011). Therefore, when solving a problem, students with a higher level of critical thinking tend to perform better.

Although personality is not the main factor which directly affects critical thinking when solving a problem, the significant correlation appears between thinking and personality (Budsankom et al., 2015). Personality influences efforts to solve a problem, affects the learning strategy, and thinking process. Keirsey (1998) classified human personality into 4 types: guardian, artisan, rational, and idealist. The classification was leaning on the brainchild of Myers Briggs Type Indicator (MBTI) personality type proposed by Myers & Briggs (1962) that the real differences of each individual can be observed through differences in 4 aspects.

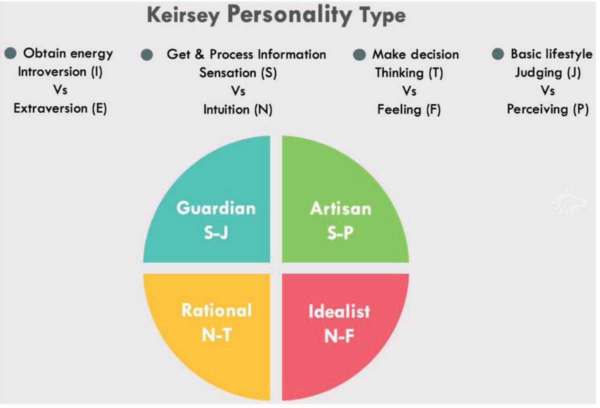


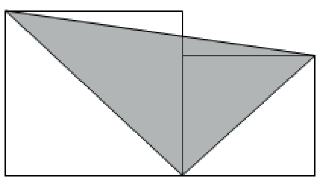
Figure 1: Keirsey personality type.

The second and the third categories refer to mental powers or cognitive dimensions and are often considered as the most two important dimensions, while the first and the fourth refer to attitudes which describe the way to get energy and to deal with the outside world (Tyagi, 2008).

In this case, related to four aspects already mentioned, the way to collect information, make decisions, and basic lifestyle, students with different personality types may exhibit critical thinking differences in receiving and processing information, choosing a strategy, and implementing strategy when solving a problem. The rationale above leads to the aim of this study: to investigate how students with different personality types show their critical thinking in solving a mathematical problem. In addition, referring to Threeton (2008) that temperaments outlined in Keirsey's theory identify characteristics which relate to preference in the learning process, the result of this study can be a beneficial input for teachers in designing instruction to train critical thinking.

METHOD

The subjects are 8^{th} graders aged 13-14 years. They were chosen purposively by selecting one junior high school in Surabaya-Indonesia, choosing a class in which the students have higher average of mathematics ability compared to 2 other classes by considering that critical thinking are more visible to students above the average, giving all students Keirsey personality questionnaire and mathematics ability test (obtained 1 rational and 1 idealist students with higher average of mathematics ability in the class and the rest are guardian and artisan students with various abilities), and selecting 4 students with the same mathematical ability (above the average in the class) and represent each type of Keirsey's personality. All subjects were given the problem below and interviewed based on their answer.



The side length of the big square is 5 cm and the small one is 3 cm.

A. Determine the area of the shaded shape!
B. If the side length of each square is changed to 2 times before, is the area of the shaded shape also be 2 times the original area?
C. If the side length of each square is changed to k times from the original size ($k \neq 1$), is the area of the shaded shape also be k times the original area?

RESULT AND DISCUSSION

Understanding the Problem

Guardian (S-J)

On his answer sheet, the guardian student wrote: the side length of big and small squares are $5\ cm$ and $3\ cm$, consecutively. He put relevant information which indicates knowledge-acquisition components are involved (Sternberg, 1986). When asked for the information provided, at first it was a little bit challenging to make him answered more detail instead of pointing to the picture on the question sheet. He was more focused on the picture, purely explained what presented without linking basic information provided. This condition underlines Woosley (2001), sensing type prefers to take concrete and tangible information, clearly presented objects. Furthermore, he explained the questions completely but looked like reading and stated that the shaded shape is a triangle which has different color.

Artisan (S-P)

The artisan student did not write down anything on his answer sheet regarding the provided information. Having the same tendency as the guardian student, he also pointed to the picture at the beginning of the interview, but could explain the presented condition. This shows that the artisan student did screening relevant information which involves knowledge-acquisition components (Sternberg, 1986). His explanation indicates that he was processing information by relying on what he saw clearly. Amplify the statement of Tyagi (2008), sensing people prefer to see objects by relying on their senses. In addition, he explained the questions completely and realized there is a changed condition, the side length of two squares. For the shaded shape, he confidently stated: “It is a dark triangle”.

Rational (N-T)

The rational student shows knowledge-acquisition by screening relevant information (Sternberg, 1986). He wrote something similar to the guardian student. Nevertheless, the way he explained the information he got is quite different from the guardian and the artisan students. While other students explained the information given is two squares with their side length, he stated: “The information given is all the sides length of both squares even though it says only one side, because the square has the same side length”. About the shaded shape, he revealed a right triangle because one of its angles is 90° formed by the diagonals of both squares. He inferred relations between stimuli and mapped out relevant information, which show the involvement of performance components (Sternberg, 1986). The performance above reinforces Woosley (2001) and Tyagi (2008), intuitive people prefer to focus on connections and relationships among the fact, information, or concept. They tend to see patterns and connect some known information when taking and processing new information (Russo, 2012).

Idealist (N-F)

The idealist student did not write down anything on his answer sheet, but during the interview, he could explain the problem using his own words. The involvement of knowledge-acquisition components shown by putting relevant information (Sternberg, 1986). Nevertheless, in contrast to the rational student who also has an intuitive nature, he said that the shaded shape is a triangle instead of a right triangle as the rational student stated. Besides, although the explanation by him is complete, the exposure less shows the character of an intuitive person who tends to connect some information and fact (Woosley, 2001; Tyagi, 2008).

Devising a Plan

Guardian (S-J)

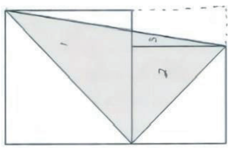


Figure 2: Strategy by the guardian student.

The guardian student found only one strategy. To determine the area of the shaded shape, initially he thought of subtracting the rectangle area by the area of triangles 1, 2, and 3 as in figure 1. But he realized this step is not appropriate because he would obtain the area of the unshaded shape.

He kept trying to look for alternatives and finally aware that the shaded shape area could be obtained by calculating the rectangle area and subtracting it by the area of three triangles surrounding the shaded shape. According to Sternberg (1986), this effort shows metacomponents indicated by planning and evaluation. Asked about the reason for choosing the strategy, he stated it is easy to apply and he only found 1 strategy as an alternative because the previous one is not suitable.

Artisan (S-P)

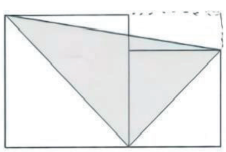


Figure 3: Strategy by the artisan student.

The artisan student found only one strategy as revealed by the guardian student. The area of the shaded shape is determined by subtracting the area of the rectangle with the area of three triangles surrounding the shaded shape.

The effort to make a plan for getting a solution shows metacomponents in critical thinking (Sternberg, 1986). Regarding the reason for choosing the strategy, he also proposed the same reason as the guardian student.

Rational (N-T)

The rational student did not put streaks on the figure. He found three strategies: the strategy used by the guardian and the artisan students, using the formula of the triangle area, and using Heron formula. He claimed all three strategies are appropriate and the second strategy is the most efficient. He bears out his characteristic as a thinking person who tends to decide after analyzing and weighing the evidence, likes to give critical analysis, and considers the "right vs wrong" principle (Brownfield, 1993). This trait brings prominence to his mental process during the planning and evaluating strategy, especially on metacomponents aspect.

Idealist (N-F)

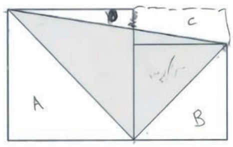


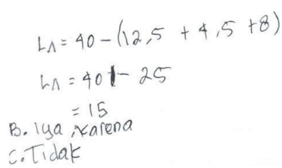
Figure 4: Strategy by the idealist student.

According to the idealist student, there is only one strategy: the same as that proposed by the guardian and the artisan students. The area of the shaded shape is determined by subtracting the area of the rectangle with the area of 3 triangles A, B, and C.

Choosing appropriate steps to solve the problem implicates metacomponents of mental process in critical thinking (Sternberg, 1986). In contrast to the rational student, when asked the reason for choosing the strategy he clarified that he did not know exactly his reason, just thought of and wanted to apply the strategy, and extremely sure it could be easily applied. Referring to Keirsey & Bates (1984) and Woosley (2001), feeling people are less being objective. They base their decisions on what is important to themselves and tend to think with their hearts (Brownfield, 1993; Tyagi, 2008).

Carrying Out the Plan

Guardian (S-J)



The area of triangle $= 40 - (12{,}5 + 4{,}5 + 8)$
$= 40 - 25$
$= 15.$

B. Yes, because
C. No

Figure 5: Answer by the guardian student.

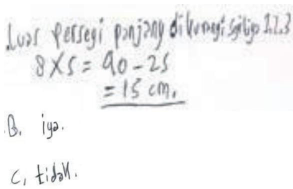
Performance components of mental process in critical thinking are used to carry out the instructions of metacomponents (Sternberg, 1985). In this case, the guardian student took steps according to the strategy that has been set previously. Judging people are accustomed to being on the track in the pre-determined decision (Keirsey, 1998). He got the area of the shaded shape is 15 without unit. For questions B and C, he explained as follows.

Interviewer: What do you think about questions B and C?

Guardian: If the side length of two squares is changed to 2 and k times the original, the area of shaded shape becomes 2 and k times the original because it is influenced by the side length of both squares.

Although he understands that the area of the shaded shape is influenced by the side length of both squares, he could not draw a correct conclusion. He did not give reason to support his conclusion, like taking several examples through writing as well as giving an explanation during the interview. In addition, there is an inconsistent answer. For question C, on the answer sheet, he concluded "No" but during the interview he stated "Yes". This condition appears likely because sensing people tend to process clear objects (Woosley, 2001), while k particularly is not clear enough for him.

Artisan (S-P)



The area of the rectangle subtracted by the area of triangle 1, 2, 3

$$8 \times 5 = 40 - 25$$
$$= 15\ cm.$$

B. Yes

C. No

Figure 6: Answer by the artisan student.

By doing the calculation, the artisan student got the area of the shaded shape is 15 cm. He made a mistake in writing unit for the area that should be "cm^2". During the interview, he admitted that he had not been focused on his planned strategy. He was curious to apply Pythagorean theorem but did not continue it because he was aware, he would get the hypotenuse length of the shaded shape, not the area. According to Russo (2012), perceiving types are less planned and may prefer to keep their options open, they may be more spontaneous, relying on their ability to adapt to the changed situation. In this case, perceiving nature seems to affect performance components in critical thinking.

For questions B and C, he only answered "Yes" and "No" without presenting evidence.

Interviewer: What is your conclusion about questions B and C?

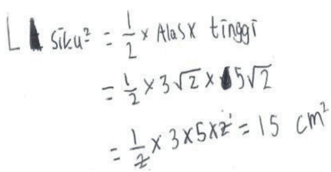
Artisan: The area of shaded shape becomes twice if the side length of the square is changed to 2 times, but it does not become k times the original if the side length of the square is changed to k times.

Interviewer: What did you notice about k?

Artisan: Actually, k is abstract and I find it difficult to determine k.

He could not show a logical argument and evidence for supporting his conclusions. Following Tyagi (2008), sensing types tend to process information that is seen clearly by their sense.

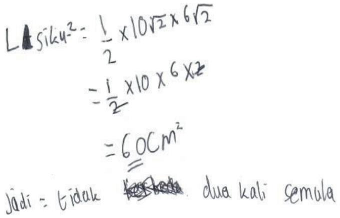
Rational (N-T)



$$\text{The area of right triangle} = \frac{1}{2} \times base \times height$$
$$= \frac{1}{2} \times 3\sqrt{2} \times 5\sqrt{2}$$
$$= \frac{1}{\cancel{2}} \times 3 \times 5 \times \cancel{2}$$
$$= 15\ cm^2$$

Figure 7: Answer to question A by the rational student.

Performance components in critical thinking shown by the rational student when executing the predetermined strategy. For question A, he got the area of the shaded shape is $15\,cm^2$. Meanwhile, the answer to questions B and C as follows.



The area of right triangle $= \frac{1}{2} \times 10\sqrt{2} \times 6\sqrt{2}$

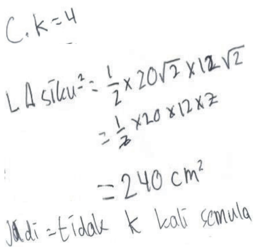
$$= \frac{1}{\not{2}} \times 10 \times 6 \times \not{2}$$

$$= 60\ cm^2$$

So: not twice before

Figure 8: Answer to question B by the rational student.

For question B, he got the area of the shaded shape is $60\,cm^2$. According to him, it means 4 times, not twice the original area.



C. $k = 4$

The area of right triangle $= \frac{1}{2} \times 20\sqrt{2} \times 12\sqrt{2}$

$$= \frac{1}{\not{2}} \times 20 \times 12 \times \not{2}$$

$$= 240\ cm^2$$

So: not k times before

Figure 9: Answer to question C by the rational student.

Interviewer: Please explain, how did you draw the conclusion?

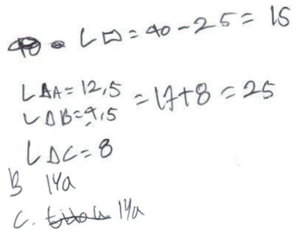
Rational: For example, I took $k = 4$, then I got the area of shaded shape $240\ cm^2$, not 4 times but 16 times before. The ratio is 1:16. On question B, if $k = 2$ the area of shaded shape becomes 60 cm^2, it means 4 times, not twice the original area. The ratio is 1:4. So, if the side length of the square is changed to k times before, the side length of the shaded shape which is triangle also becomes k times before, then the area of shaded shape becomes k^2 before. The ratio is $1{:}\,k^2$. So, my conclusion if the side length of the square is changed to k times the original, then the area of shaded shape does not become k times but k^2 the original.

In this case, it is not quite sure that rational student understands the logical structure of the question or not. Because if we see from the point of view of refusing a statement, it is necessary to take only one counter example. If the counter example does not meet the statement, it means the statement is not valid. But for this problem, the rational student took $k = 2$ and $k = 4$.

On the other hand, during the interview, he also explained about right triangles with varying side lengths followed by the area. He concluded by associating several concepts, such as the area of triangle and ratio. Not only answered the question but also tried to investigate the exact value of the shaded shape area.

He seems very curious. Perhaps, because he is an intuitive person who tends to be good at observing patterns and continually looking for new possibilities (Woosley, 2001).

Idealist (N-F)



The area of the shaded shape $= 40 - 25 = 15$

$$\left.\begin{array}{l}\text{The area of triangle A } = 12{,}5 \\ \text{The area of triangle B } = 4{,}5 \\ \text{The area of triangle C } = 8\end{array}\right\} = 17 + 8 = 25$$

B. Yes
C. ~~No~~ Yes

Figure 10: Answer by the rational student.

The idealist student got the area of the shaded shape is 15. He wrote on the answer sheet without unit, but during the interview he emphasized it must be cm^2. Performance components in critical thinking shown through not answering all questions. He only arrived at question A. As the result, he could not conclude correctly for questions B and C. Compared to rational student who has the same nature, what idealist student does less shows the character of intuitive person who tends to make decision after analyzing the problem and focus on cause-effect relationships (Brownfield, 1993; Tyagi, 2008).

Looking Back

In the last step, students show differences in their own metacomponents. The guardian student examined several carried out steps, checked the calculation step, and realized the previous result was wrong. He tried to recalculate but didn't recheck the whole process because of limited time. The artisan student claimed not to check the whole steps after getting the final answer but always checks every step he does. The rational student rechecked all the steps taken. The idealist student checked all the steps indicated by the streaks and corrections on his work.

CONCLUSION

The result of this study are consistent with the result of the research on correlation between thinking and personality (Budsankom et al., 2015). Students show different performance according to the character tendencies inherent in their personality type. The differences arise in the way students process information, plan, monitor, and evaluate their strategies and reasons as the particular part of critical thinking.

	Guardian (S-J)	Artisan (S-P)	Rational (N-T)	Idealist (N-F)
Understanding the problem	Focus on the picture	Focus on the picture	Connect some information	Explain simply using own words
Devising a plan	Propose subjective reason	Propose subjective reason	Propose objective reason	Propose subjective reason
Carrying out the plan	Consistently	Flexibly	Consistently	Consistently
Looking back	Several steps	While calculating	Overall checking	Overall checking

Table 1: Conclusion.

This research is confined to the data of several students in one school that shows similarity with the data of several students in another school, therefore the result cannot be generalized. On the other hand, this result could be a reinforcement for teachers to apply cooperative learning so that students, with their character inclinations, will cooperate each other to exchange and discuss their own ideas.

References

Brownfield K. M. (1993). The Relationship between the Myers-Briggs Personality Types and Learning Styles. *Education Resources Information Center (ERIC), 381 577*, 1-15.

Budsankom P., Sawangboon T., Damrongpanit S., & Chuensirimongkol J. (2015). Factors Affecting Higher Order Thinking Skills of Students: A Meta-Analytic Structural Equation Modeling Study. *Educational Research and Reviews, 10*(19), 2639-2652.

Butterworth J. & Thwaites G. (2013). *Thinking Skills. Critical Thinking and Problem Solving Second edition*. New York: Cambridge University Press.

Carroll R. T. (2004). *Becoming a Critical Thinker.* Scepdic.

De Lange. (2006). Mathematics for Literacy. *Quantitative Literacy: Why Numeracy Matters for Schools and Colleges*, 52-80.

Jacob S. M, & Sam H. K. (2008). Measuring Critical Thinking in Problem Solving through Online Discussion Forums. *Proceedings of the International MultiConference of Engineers and Computer Scientists* (Vol. 1).

Keirsey D. & Bates M. (1984). *Please Understand Me: Character & Temperament Types*. Carlsbad: Prometheus Nemesis Book Company.

Keirsey D. (1998). *Please Understand me II: Temperament, Character, Intelligent*, Carlsbad: Prometheus Nemesis Book Company.

Krulik S. & Rudnick J. A. (1989). *Problem Solving: A Handbook for Senior High School Teachers*. Philadelphia: Temple University.

Lai E. R. (2011). Critical Thinking: A Literature Review, *Pearson's Research Reports.*

Myers I. B. & Briggs K. (1962). *MBTI Manual: A Guide to the Development and Use of the Myers-Briggs Type Indicator.* Unknown Publisher

OECD. (2013). *PISA 2012 Assessment and Analytical Framework: Mathematics, Reading, Science, Problem Solving, and Financial Literacy.* Paris: OECD Publishing.

Polya G. (1973). *How to Solve It*. New Jersey: Princeton University Press.

Russo C. J. (2012). The Impact of Personality Type and Gender on Students' Performance in a Business Capstone Course. *Academy of Educational Leadership Journal*, *16,* 1.

Sternberg R. J. (1985). Teaching Critical Thinking Part 2: Possible Solutions. *The Phi Delta Kappan*, *4*, 277-280.

Sternberg R. (1986). *Critical Thinking: Its Nature, Measurement, and Improvement.* Washington DC: EDRS.

Threeton M. D. (2008). The Relationship between Personality Type and Learning Style: A Study of Automotive Technology Students. *Journal of Pennsylvania State University.*

Tyagi A. (2008). Personality Profiles Identification Using MBTI Test for Management Students: An Empirical Study. *Journal of the Indian Academy of Applied Psychology*, *3*, 1.

Woosley E. R. (2001). *A Look at the Relationships between MBTI Psychological Types and Mode Conflict Styles*. Wisconsin: University of Wisconsin-Stout.

IMMERSING IN A DIGITAL STORYTELLING IN MATHEMATICS: THE STUDENTS' REFLECTIVE ACTION

Giovannina Albano*, Anna Pierri*, Maria Polo**

*University of Salerno, Italy

**University of Cagliari, Italy

This paper illustrates an online activity design for promoting the critical mathematical thinking among the students. The activity foresees the students to participate as active protagonist and as observers of the protagonists during their problem solving activity. In the latter case, the students are expected to respond to reflective demands. The first results of a pilot point out that the role of observer makes students participate in a deeper way and therefore can be of greater help for the development of critical thinking.

INTRODUCTION AND THEORETICAL FRAMEWORK

This work refers to a didactical methodology, named DIST-M, which allows to design online competence-oriented activities, making use of narrative and social approaches, developed within an Italian PRIN research project[1]. In this paper we will focus on the analysis of a case study, concerning an activity devoted to promote the "culture of theorems". It is intended as the development of those competencies needed to conjecture and prove within a theory of reference (Boero, 1999; Mariotti, 2006). Such development is strictly related to foster critical thinking in interactions among students while exploring (mathematical) situations and argumenting their findings. This involves both constructively providing justification of mathematical claims and critically analyzing and assessing existing or proposed justification attempts.

The whole PRIN project is framed in a Vygotskian approach in which social interaction and communication play a key role in the evolution of students and language has been widely recognized as fundamental in learning mathematics. Students' participation in social interactions and in communication is the key to their evolution and the appropriation of cultural tools, such as language. (Vygotskij, 1934). According to Sfard (2001), thought is a form of communication and languages do not are just couriers of pre-existing meanings, but they are builders of meanings themselves. Learning thus becomes the participation in a particular discourse which is "mathematical discourse". In our case, this discourse occurs through a mainly written communication, being on digital platform.

---

[1] This work is part of the project PRIN 2015 "Digital Interactive Storytelling in Mathematics: A Competence-based Social Approach", Prot. 20155NPRA5, funded by MIUR, effective from 5 February 2017.

The idea of implementing a teaching/learning process into a digital interactive storytelling, where students are immersed as actors or observers, fits the idea of expanding the context of mathematics education beyond the classroom (Engerstrom, 2017). This expansion occurs along the layer of bridging elements from societal practices in mathematics instruction: indeed, students are more and more involved in digital games/stories outside the school. The DIST-M would bring such elements into mathematics education practices, bridging the agency of experts' mathematics knowledge and the motivation of the students. It also allows to add on the school activity system with new societal components, such as rules, community and division of labour (Lazarou et al., 2016).

In the presented case study, students and teacher are engaged in a suitable designed online activity, where the interactions among students and students-teacher are well organized. According to activity theory (Leontiev', 1978), each participant has a role and specific actions to perform. The organization of the activity foresees that the roles can be played as actors or as observers (Albano et al., 2020). Nevertheless, "the very subject of learning is transformed from isolated individuals to collectives" (Engerstrom, &Sannino, 2010, p. 5). Thus learning does not depend on a single role and the related actions, but on the whole group, as a system. In this paper we will analyze the outcome of a pilot, with particular reference to the students who played the roles played as observers.

THE METHODOLOGY

The design of the activity

The activities are framed in a narrative, in which there are characters that will act as avatars for the various participating students.

The scheme of the whole activity is divided in the following phase (see Figure 1): (a) *Inquiry*: after an inquiry stage, the students are expected to produce a summary of what they observed; (b) *Conjecture:* the students are required to put into a statement the findings of the previous phase, generally expressed in a verbal form; (c) *Formalize:* the students manipulate the produced statement in order to have a representation functional to prove it; (d) *Prove:* the students are engaging in producing a 'reasoned calculation', that is organization of arguments in deductive chain and justification of each deductive step; (e) *Reflect*: at the end, the students are asked for critically revise the whole experience played. This last phase is divided into two parts. The first one consists in a collective assessment: it is a re-reading of history that produces an awareness of what happened. The latter is a self-assessment: each student returns to the episode or episodes played as an actor and reviews/evaluates how she played her role.

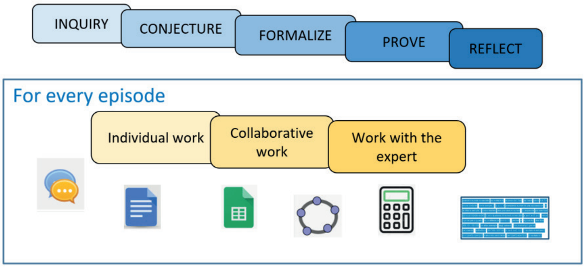


Figure 1: The design of the activity

The design includes moments of both individual work and collaborative work and discussion with the expert. The expert acts as a mediator by intervening when necessary. Her mediation concerns both mathematics and communication. Different tools have been considered: *Communication tools*, like chat, through which we can observe the use of different kinds of registers (colloquial vs. literate); *Collaboration tools*, like shared files; *Supporting tools* for inquiry, formalization and proof, like spreadsheets, CAS, calculators, word blocks (Albano & Dello Iacono, 2019). Each phase of the design corresponds to a specific episode of the story illustrated in the following section.

The story and its characters

The story is defined by starting to an interesting mathematical problem, that aims to develop argumentative skills in the students. The problem is the following: *choose four consecutive natural numbers, multiply the two intermediate numbers, multiply the two extremes, and subtract the results. What do you get?* (Mellone & Tortora, 2015).

The students are immersed in a story that evolves according to the interaction of the characters with herself. In the context of the story, a group of four friends, Marco the Boss, Clara the Pest, Federico the Promoter and Sofia the Blogger receive mysterious messages from the aliens and collaborate with each other to understand what it is. The messages, noted by the promoter on a sheet, is the reconceptualization of the mathematical problem within the story. Indeed, the sheet, received from aliens shows some quadruplets and operations corresponding to the subtraction between the product between the second and third term and the first and fourth term (see Figure 2). For solving the enigma, they ask for help from Federico's uncle, Gianmaria (the Guru).

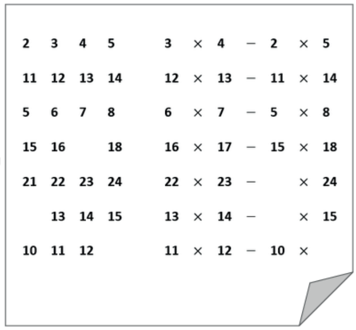


Figure 2: The sheet

Each of them take a specific role: the *Boss*, she is the leader of the group of friends, who takes care of organizing the group work in order to solve the task in the best way; the *Pest*, she intervenes within the peers group posing doubts and questions about the mathematical problem; the *Blogger*, she loves to write and has a blog craze and summarizes the shared answers of the group; the *Promoter*, she is the initiator, the one who launches the ideas, opens a track. A further character is involved in the story, that is the *Guru* (Gianmaria in the story). She is the expert, acting as a mediator, intervening during the interactions, encouraging students to better clarify what they said and to improve communication. She is the one who asks “what did you want to say?”, who says: “Please explain, clarify, expose ... complete the sentence, ... did you mean this or that?”.

The experiment

The experiment carried out and analysed in this paper involved 26 students from the second year of high school. All students are enrolled in a Moodle course, divided into groups of 4 or 5 if the total number of students is not a multiple of 4. So, we have four groups of four students and two groups of five students. Group 1 is the protagonists group, composed by Marco (Boss), Federico (Promoter), Sofia (Blogger) and Clara (Pest). This last role is duplicated in the case of Group of 5 students. Groups 2, 3, 4, 5, and 6 are the groups of the Observers. The generic group of the Observatories (named Group i, with i=1,...,6) is constituted by the following students: Boss Observer, Promoter Observer, Blogger Observer and Pest Observer. The Observer is a student whose objective is not to respond to the requests dictated by history, but to observe the protagonists during their problem-solver activity.

In subsequent episodes, groups and roles are exchanged. Each observer can see the discussions of the protagonists within their chat but they can't take part in the discussion; on the other side they have a chat with the other observers belonging to the same group and a private personal diary.

During the experimentation, we observed that on average the students used the chat between them while watching the actors' chat and immediately afterwards they had time to complete their diary about the episode observed. In this work, for space reasons, we analyze the personal diary in relation to the actors' chat. Students worked in presence on Episode 1 (*Inquiry*) and Episode 3 (*Formalize*), while they worked in remotely on the rest of the story. Each Episode is associated to a chapter of the story implemented in Moodle platform.

DATA ANALYSIS AND DISCUSSION

Our analysis is focused on the first chapter of the story, entitled "Arrivanogli Alieni 1" and corresponds also to the first episode of the design (*Inquiry*). So, the students belonging to Group1 take the roles of the protagonist. The other groups observe and annotate, in a personal diary, everything about the actions, communications, of the protagonist they are observing. We have predisposed a template of the personal diary in order to address the students in writing. Some questions have posed them, as: *How do you think the character you observed played his/her role? What was his/her contribution? Do you think that his/her interventions in the story have been useful to achieve the goal? Why? What would you have done in his/her place?*

The analysis of the interactions in chat crossed with the observations of the groups and their individual diaries is aimed at highlighting whether the "external" observation of the protagonists allows both to participate in the resolution of the problem and to reflect on the behaviour of the different actors with respect to the role of each one in the story.

In the following we report an excerpt from the chat among the protagonists.

[...]

1 Boss: Yes…the result is always 2

2 Pest: So, did we solve the problem?

3 Blogger: I don't know

4 Promoter: so having 4 successive numbers and making the difference between the multiplication of the extremes and the extremes the result is 2

5 Promoter: extremes and middles

6 Boss: Blogger what do you think: Do we have reasoned well?

7 Pest: I think that's right. What do you think? Do you have any other ideas?

8 Boss: Also me, I think the reasoning is correct.

[…]

9 Boss: I think that is (n+2*n+3)-(n+1*n+4)

10 Blogger: Here it is

[…]

11 Pest: Do you have any other ideas on how to write the formula?

12 Blogger: (n+1)*(n+2)-(n)*(n+3) can you see if that's good for you?

13 Promoter: When we agree we have to move forward in comics so the Blogger can give the answer to Gianmaria.

14 Pest: Boss formula appears correct

15 Blogger: Are the same thing only that at the Boss the smallest number is n+1 while at my point it's n

[...]

16 Gianmaria: The one you sent me is this (n+1) *(n+2) -(n) *(n+3)

17 Gianmaria: Can you explain it better?

18 Gianmaria: What does this "formula" have to do with the sheet you showed me?

19 Boss: we thought that by taking four consecutive numbers and multiplying the extremes and the averages and subtracting the multiplication of the extremes from the multiplication of the averages, the result will always be 2

The chat excerpt reported is useful to better understand the notes of the various observers in their personal diary. In this analysis we have considered only the most representative contributions of the observers' reflections.

Boss- Observers. We report some transcripts of the students who played the role of Boss-observer:

20 S9-G3: I think the Boss, the role I have observed, has played well his role, involving the whole group and making useful statements, explaining the reason for his ideas. He has contributed a lot taking the situation under control, making everyone participate, explaining why the formula given by the Pest is wrong and talking with Gianmaria about the final solution, even if in the final questions he didn't intervene at all, leaving everything in the control of the Pest. I also think that his interventions were very useful and that in his place I would have behaved in the same way.

21 S22-G6: I think the Boss has played his role well, because he has been very active and has been able to answer every question asked. He was also the one who asked for advice and considerations regarding the formula to be sent to the uncle. He was not afraid to express his doubts and uncertainties about the formula and also advised what to add and what not. In his place maybe I would have helped a little bit the others in the end when Gianmaria asked so many questions to better understand.

S9 focused her attention on the explanation [see excerpt #20: ...explaining the reason for his ideas…. explaining why the formula given by the Pest is wrong …], used both for to endorse correct things but also to refute ideas. S22 point out an interesting affective component of the Boss through the expression [see excerpt #21: ...He was not afraid to express his doubts and uncertainties about the formula...]. That it’s important for reflecting in a critical way.

Promoter- Observers. We report some transcripts of the students who played the role of Promoter observer:

22 S11-G3: The promoter decently played his role by allowing the group to notice an important detail for the future solution of the problem, namely that the operations took place between averages and extremes, advising the group to use a spreadsheet to verify the accuracy of the calculations. After recommending the spreadsheet, the participation of the promoter in the problem is decreasing, however, we do not overlook that his considerations and advice were important to achieve the solution of the problem. In his place I probably would have tried to keep the group focused on one hypothesis at a time instead of 3 at the same time so as to improve the answer then given to Uncle Gianmaria making it complete so as to eliminate all his doubts about it.

23 S24-G6: The promoter discusses little with his classmates, but is decisive in the actions. The promoter states the correct formula. The promoter is not very involved in the discussion. The group has taken long time to complete the question, they're probably much insecure. In his place I probably would have been more involved and decisive, so as to reassure the classmates and remove doubts and perplexities.

S11 provides an interesting suggestion on the way to work how [see excerpt #22: ...In his place I probably would have tried to keep the group focused on one hypothesis at a time instead of 3 at the same time so as to improve the answer then given to Uncle Gianmaria making it complete so as to eliminate all his doubts about it…]. S24 is much focused on affective aspects (see excerpt #23: he writes "much insecure") and for consequence he points out his desire "reassure his classmates". This is particularly interesting because it seems that S24 recognizes the well-known relationship between cognitive and affective levels in being successful in mathematics activities (Di Martino & Zan, 2011). This kind of interaction about the affective aspects has been already observed in a previous experimentation of the Project (Albano et al., 2019).

Pest- Observers. We report some transcripts of the students who played the role of Pest observer:

24 S6-G2: Initially he played well; asking questions but also contributing to the resolution of the problem (always raising some doubts). But in the end he spoke little and the questions he asked were very simple, so there was no doubt. He found more than one solution with his questions. Yes, by asking questions she was able to find another solution. Her classmates, not being sure of the first solution, looked for another solution and succeeded. I would have done the same thing but maybe with a few more questions at the end (before entering into conversation with the uncle).

25 S10-G3: I think the character I observed could have done better, but I understand it wasn't very easy. His part was crucial, even though he almost wrote more statements than questions. I think his interventions were important for reflection. If I were him, I would

have asked more questions, maybe even simpler questions, to get them to the answer earlier.

S6 is interesting respect to the critical thinking [see excerpt #24: ... found more than one solution with his questions…. Yes, by asking questions she was able to find another solution]. The questions promote the development of critical thinking. S10 individuates in the reflection an important element "I think his interventions were important for reflection" (see excerpt #25).

Blogger-Observers. We report some transcripts of the students who played the role of Blogger observer:

26 S8-G2: The blogger, although initially a little absent, managed to find a second solution to the question, making it even simpler than initially thought. I think, therefore, that her interventions have been quite useful in achieving the objective. I believe that in her place I would have behaved in the same way

27 S12-G3: 1) At first she was behaving like the blogger then at the end she started asking questions like the pest. 2) She has given a mediocre contribution. 3) In some cases she has been very useful in achieving the objective. Because she started to understand the concept of the message by reasoning on how to reach the conclusion. 4) I would have done the same thing as her only at the end I would have wanted to give a concrete explanation

28 S25-G6: She didn't play badly, but she could have done much better. At the beginning she wasn't very active but in the end she helped to give the formula for the final answer, even though she tried several times before but with failed attempts. Finally, when she sent the email to Uncle Gianmaria she wasn't very precise, she didn't write an important part of the formula. In her place I would have reasoned more and finally given the solution in less time.

S8 is more oriented to the mathematical content expressed by the Blogger [see excerpt #26: ... find a second solution to the question, making it even simpler than initially thought…]. S25 is much focused on the reasoning, when we can observe in the sentence "she wasn't very precise, she didn't write an important part of the formula" (see excerpt #28). He recognizes a strong bug in the formula.

The personal diaries show that all students/observers identified and reflected on the actions of the character playing the observed role. For some observers, looking at a particular role from the outside, allowed to identify inconsistencies or overlaps in the interpretation of a role, as reported by an observer blogger belonging to G3 group, that writes: "*At first she was acting like the blogger, then eventually she started asking questions like the pest*".

At the conclusion of our analysis, performed by crossing the observers' personal diary and the conversation in the chat of the protagonists, we can extract at least three dimensions of student's reflective thinking. So, *a posteriori*, we provide a first synthetic description of the possible categories individuated.

Category 1*: dimension of thinking and acting on the mathematical content*

To observe from the outside an action of others while they are solving the mathematical question and to reflect in real time in relation to the task but also to the behavioural modalities of those who perform it, as pointed out by S12-G3. We note how from the conditional form "I would have wanted" appears a "heartfelt involvement in action and history". One learns not only to answer the question but also how one could have done or acted in a different way.

Category 2*: reflect and/or develop critical thinking*

To observe from the outside an action of others while they are solving the mathematical question and reflect in relation to the comparison between conjectures/solutions/ different behaviours more or less functional to the task, as pointed out by the students S6-G2 and S8-G2.

Category 3: *reflecting and/or grasping emotional aspects of one's own or others' behaviour*

To observe from the outside an action of others while they are solving a collective task and to reflect to capture emotional aspects of the action of others, as student S24-G6 or the student S22-G6.

The pilot has confirmed that active observation can be functional with respect to a certain way of acting. As well as the setting up of a learning environment favourable to "learn to learn" and to make inferences about how one could have acted (without the danger or the consequences of real action...). We can now put forward the hypothesis that observation or rather the role of observer makes students participate in a deeper way and therefore can be of greater help for the development of critical thinking.

Acknowledgment

We would like the teacher Piera Romano and her students of the I.S. "Genoino" Institute in Pagani (Italy) and her students who participated in the experimentation.

References

Albano, G., Pierri, A., & Polo, M. (2019). Engagement in mathematics through digital interactive storytelling. *Proceeding of Eleventh Congress of the European Society for Research in Mathematics Education (CERME11)* (pp. 1501-1508).

Albano, G., & Dello Iacono, U. (2019b). A scaffolding toolkit to foster argumentation and proofs in Mathematics. *International Journal of Educational Technology in Higher Education, 16*(4), 1-12.

Albano, G., Dello Iacono, U., & Fiorentino, G. (2020). A technological storytelling approach to nurture mathematical argumentation. To appear in *Proceedings of CSEDU 2020, the International Conference on Computer Supported Education*.

Boero, P. (1999). Argumentation and mathematical proof: A complex, productive, unavoidable relationship in mathematics and mathematics education. In *International newsletter on the teaching and learning of mathematical proof* (July/August 1999).

Di Martino, P., & Zan, R. (2011). Attitude towards mathematics: A bridge between beliefs and emotions. *ZDM - International Journal on Mathematics Education*, *43*, 471-482.

Engeström Y. (2017). Expanding the Scope of Science Education: An Activity-Theoretical Perspective. In: Hahl K., Juuti K., Lampiselkä J., Uitto A., Lavonen J. (Eds.), *Cognitive and Affective Aspects in Science Education Research. Contributions from Science Education Research*, *3*. Springer, Cham

Engerström, Y., Sannino, A. (2010). Studies of expansive learning: Foundations, findings and future challenges. *Educational Research Review*, *5*, 1–24.

Mariotti, M. A. (2006). Proof and proving in mathematics education. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education*, (pp. 173–204). Rotterdam: Sense Publishers

Mellone, M., Tortora, R. (2015). Ambiguity as a cognitive and didactic resource. In Krainer K., Vondrová N. (Eds.), *Proceeding of Ninth Congress of the European Society for Research in Mathematics Education (CERME 9)*, pp. 1434-1439.

Sfard, A. (2001). There is more to discourse than meets the ears: looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, *46* (1–3), 13–57.

Vygotsky, L. S. (1934). *Thought and language.* Moscow-Leningrad: Sozekgiz.

Lazarou D., Sutherland, R., & Erduran, S. (2016). Argumentation in science education as a systemic activity: An activity theoretical perspective, *International Journal of Educational Research*, *79,* 150-166.

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. EnglewoodCliffs: Prentice-Hall.

CRITICAL THINKING OF STUDENTS IN THE PROCESS OF GENERALIZATION

Anna Pyzara

Maria Curie-Skłodowska University, Lublin, Poland

Every teacher should have critical thinking skills. This research aims to examine whether prospective teachers of mathematics use critical thinking. To this end, I analysed their activities related to the generalization process. The results of my study show that students used critical thinking in their work, but not enough. Students were able to follow a pattern and recognize the rule of conduct in simple situations, but they had difficulties with the description and justification of the rule, especially when it required the consideration of different cases.

INTRODUCTION

The quality of teaching clearly depends on the skills of teachers, which in turn depend, inter alia, on their education. The preparation of a student for the teaching profession should include a thorough understanding of the subject being taught. Mathematics teaching is oriented – among others – to the development of mathematical activity by students. This goal is considered one of the most important in mathematics teaching (Krygowska, 1977). One of the elements of mathematical activity is generalization (Ellis, 2011), which in turn is regulated by critical thinking. These facts have inspired the present study, which aims to explore issues of critical thinking of students in the process of generalization.

THEORETICAL FRAMEWORK

Let us begin by what is understood by mathematical activity. According to Nowak (1989) "The mathematical activity of a pupil is a work of mind oriented on learning of concepts and on mathematical reasoning, which is stimulated by the situations that lead to formulation and solving theoretical and practical problems." (p. 110) According to this definition, searching for a general formula and formulating theorems requires the student's mathematical activity.

Generalization is one of the most important processes that occur during the construction of mathematical concepts, discovering theorems, and solving mathematical problems (Ciosek, 2012). This process can be analysed as the mathematical activity of individuals. Creative mathematical activities are particularly important for the students' development. One type of creative mathematical activity is discipline and criticality of thinking. Creative mathematical activities also include method transfer as well as putting and verifying hypotheses (Maj, 2011). These activities are very helpful in discovering theorems and solving mathematical problems, i.e., in generalizing.

Generalization is a process that requires many mathematical activities. Ellis (2011) defines generalizing as an activity in which people in specific sociomathematical contexts engage in at least one of three actions: (a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, or (c) deriving broader results from particular cases.

By 'generalization process' we mean, briefly, a sequence of acts of thinking which lead a subject to recognize, by analysing individual cases, the occurrence of common peculiar elements; to shift attention from individual cases to the totality of possible cases and extend to that totality the common features previously identified. Detecting patterns, identifying similarities, linking analogous facts are all at the base of generalization processes; the key element in these processes is not the detection of similarities between cases, but rather the shift of attention from individual cases to all the possible ones, as well as the extension and adaptation of the model to any of them (Malara, 2012).

We can see generalization as a transition from individual cases to patterns, relationships, and structures on them. In this process, similarities must be discovered, individual cases should be taken into account to combine them into a general concept (Ellis, 2011). Generalization requires solving problems by analysing information, setting and verifying hypotheses, searching for a strategy of action and evaluation of evidence and arguments. All these activities are elements of critical thinking (Firdaus et al., 2015; Huitt, 1998; Krulik & Rudnick, 1999; Sukmadinata 2004; TC, 2013).

Paul Chance, who is a cognitive psychologist, defines critical thinking as the ability to analyse facts, generate and organize ideas, defend opinions, make comparisons, draw conclusions, evaluate arguments and solve problem (Huitt, 1998). In line with the previous, Sukmadinata (2004) states that critical thinking is a skill of reason on a regular basis, systematic skills in assessing, solving problems, appealing the decision, give confidence, analysing assumptions and scientific inquiry. When students think critically in mathematics, they make reasoned decisions or judgments about what to do and think. In other words, students consider the criteria or grounds for a thoughtful decision and do not simply guess or apply a rule without assessing its relevance (Ennis, 1996; TC, 2013).

According to Facione (2011), the most basic element of critical thinking is the ability of interpretation, analysis, evaluation, inference, explanation and self-regulation. The assessment of critical thinking skills in non-routine mathematical problem solving consists of three parts; the identification and interpretation of information, information analysis, and the evaluation of evidence and arguments (Firdaus et al., 2015; Krulik & Rudnick, 1999). Therefore, we can study the critical thinking of students by analysing their mathematical activities regarding generalization, because generalization tasks are not routine and require critical thinking.

Generalization in a Polish school

Analysis of school textbooks showed that in the Polish primary school students make generalizations quite rarely. Students are asked to generalize by induction. They analyse the initial specific cases and then look for a rule of conduct to find the general formula for the n-th element in the form of an algebraic expression. Sometimes generalization is used to find finite sum values.

Tasks that require generalization skills usually appear as part of trivia - they are not standard tasks. Examples of such tasks from handbooks for grades 7 and 8 of primary schools are shown on Figures 1 and 2. These tasks were an inspiration to create worksheets, which were used as our research tools.

1. Look at the drawings. Find the sum given.

a) $1 + 3 + 5 + ... + 11 + 13 = ?$

b) $1 + 3 + 5 + ... + 99 = ?$

c) $1 + 3 + 5 + ... + (2n + 1) = ?$

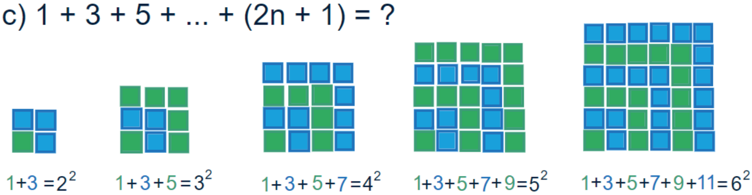


Figure 1: Task for grade 8 that requires generalization (Braun et al., 2019, p. 85).

2. The strip of land separating the lawn from the bike path is divided into equal L-shaped flower beds, as shown in the figure below.

Using the information given in the figure, write in the form of an algebraic expression the length of this belt when the flower beds are:

a) 12 (as in the picture) b) 912 c) 1000 d) 2n

Figure 2: Task for grade 7 that requires generalization in the form of an algebraic expression (Braun et al., 2020, p. 196).

THE AIM OF THE RESEARCH AND METHODOLOGY

The research aims to examine whether students of mathematics (future mathematics teachers) use critical thinking during their generalization process.

The participants were asked to make generalizations. Particularly, they were asked to find a formula for the nth step based on the analysis of the drawing of the first steps and to justify their actions (Ciosek, 2012; Ellis, 2011; Malara, 2012). An analysis of students' responses was carried out to assess their critical thinking skills. This was possible because many mathematical activities associated with generalization are part of critical thinking (Firdaus et al., 2015; Krulik & Rudnick, 1999; TC, 2013). By analysing the elements of critical thinking of students, I examined their ability to identify and interpret information, their ability to analyse information and evaluate evidence and arguments. I examined how the students justified their conclusions and verified their answers. I was also interested in whether they were able to notice irregularities in the given solutions (Firdaus et al., 2015; Huitt, 1998; Krulik & Rudnick, 1999; Sukmadinata 2004; TC, 2013).

Another aim of this study was to verify whether the critical thinking ability of beginning students is significantly different from that of fifth year students who are in-service teachers. Studies on mathematical activities (also those associated with generalization) have shown that these activities do not emerge spontaneously, but should be developed (Ellis, 2011; Maj, 2011). One of the methods to develop students' critical thinking is to pose a variety of problems, for example, problems with conflicting data (Semadeni, 2008). The tasks provided to our students were such.

The study was conducted at the beginning of 2020. The participants were 42 students who can be categorised in two groups. The first group (29 participants) consisted of first-degree mathematics students with a teaching specialization (future mathematics teachers). The second group (13 participants) consisted of second-degree mathematics students with a teaching specialization, who were already working as teachers.

The research tool was a worksheet designed for individual student work. The students had 60 minutes to complete the tasks, which were designed to assess various aspects of critical thinking. The analysed elements of critical thinking are presented in the list of detailed research questions at the end of the section. Next to each question, the number of the task (or tasks) used to test the given student skill (critical thinking element) was presented. The worksheet contained seven tasks referring to students' generalization skills, mainly induction type generalization (Ciosek, 2012). The tasks required finding the rules of conduct in the next steps based on the drawing, and describing the formula for the nth element. The first four tasks contained tables for entering data that could help students find the general formula, while in the next two tasks students had to create such tables themselves. The last task contained a table in which some data was incorrect. The purpose of this task was to check whether the students verified the correctness of the initial data, thus, whether they recognized irregularities.

The worksheet completed by the students was quite extensive, therefore the content of the tasks will be presented when discussing the results of individual tasks. Each task was related to specific research questions, as shown below.

a) Do students follow the pattern? (Task 1, 2, 3, 4)

b) Do students use the same strategy, or do they create their own? (Task 1, 6)

c) Do they use the learned strategy to solve other problems? (Task 2, 3, 5)

d) Can they discover the rule of conduct and write it as a general formula? (Task 1, 2, 3, 4, 5, 6)

e) Do they justify their decisions? (Task 2, 3, 5, 7)

f) Can they explain their course of action? (Task 2, 3)

g) Do they consider different criteria? (Task 4, 6)

h) Do they guess the answers or verify them? (Task 6)

i) Do they check the correctness of the rule or apply it uncritically? (Task 6, 7)

j) Do they analyse the information provided in the task? (Task 1, 6, 7)

k) Do they provide conclusions? (Task 7)

ANALYSIS OF STUDENT SOLUTIONS

The results will be presented firstly in relation to the tasks, and then collectively.

Task 1 results

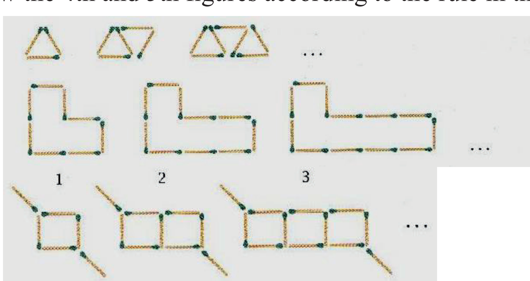
Task 1 concerns the research questions a, b, d, j and was the following:

I. Look at the following figures. They are made of matches according to a certain rule. Draw the 4th and 5th figures according to the rule in the given pattern.

a)

b)

c)



II. The following tables show the number of matches needed to build the subsequent figures in each of the designs. Complete the tables. Table for questions a) and b):

Figure number	1	2	3	4	5	6	7	10	20	n
Number of matches	3	5								
Number of matches										

Table for question c):

Figure number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**10**	**20**	**n**
Number of matches	6	9	12	15						
Number of matches	6	6+3	$6+2\cdot 3$							

The analysis of Task 1 solutions has shown that students could follow the pattern and use the strategy presented, but not everyone could read the content of the task accurately. Forty participants out of 42 correctly drew the additional figures from matches, but six students presented only the 4th figure (fifth missing), and one did not make the drawing; these participants did not analyse the correctness of the solution in relation to the content of the task.

The same strategy to supplement the match table was used by 38 participants: 80% of the respondents correctly provided the number of matches and set a general formula. Most students correctly discovered the rule of conduct, but eight students gave the formula for the nth element incorrectly.

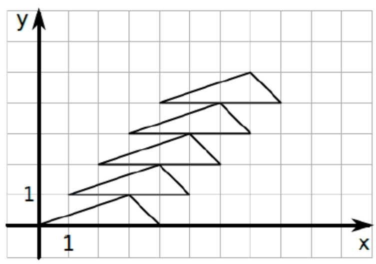
Task 2 results

Task 2 concerns the research questions a, c, d, e, f and was the following:

Ewa drew a triangle located in a coordinate system as in the figure. She drew subsequent triangles adjacent to it in such a way that the center of the base of the drawn triangle was the apex of the previous triangle (look at the figure).

a) Complete the table:

Triangle number	Coordinates of the "upper" vertex	Triangle number	Coordinates of the "upper" vertex
1	(3, 1)	6	
2	(4, 2)	7	
3		8	
4		9	
5		10	



b) Look at the table, think about it and write what the coordinates of the "upper" vertex in the triangle will be with the number 20, with the number 217, with the number n.

c) The "upper" vertex of one of the triangles has coordinates (a + 4, b - 2). Write the coordinates of the "upper" vertex of the previous triangle.

d) How to determine the coordinates of the "upper" vertex? Describe the rule of conduct.

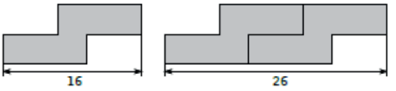
The analysis of Task 2 solutions has shown that everyone has recognized the rule of procedure and can apply it in a simple situation: all have correctly completed the table and 41 participants have successfully performed the command b) (followed the pattern). Only one person incorrectly determined the vertex coordinates using the triangle number. Definitely worse students coped with the application of the rule in an unusual (abstract) situation: 12 participants gave wrong answers, and three participants did not give any answer. The situation is similar with the justification for the rule: 29 students were able to fully explain the course of action, while the other participants did it partly, imprecisely or not at all.

Task 3 results

Task 3 concerns the research questions a, c, d, e, f and was the following:

The drawings show the shape and method of laying tiles and some dimensions in centimetres. a) Complete the table:

Number of tiles	The length of the placed figure	Number of tiles	The length of the placed figure
1	16	5	
2	26	6	
3		7	
4		8	



a) Look at the table, think and give the length of the figure made of 12 tiles, of 52 tiles, and of n tiles.

b) How did you calculate the length of the figure? Describe your reasoning.

d) The length of a figure made of some tiles is 486 cm. How many tiles has it been laid (show your calculations).

The analysis of Task 3 solutions has shown that most students were able to recognize the method of conduct: 41 participants completed the table well (followed the pattern) and 34 correctly defined the general formula for n tiles. Four students well determined the length of the figure built of 12 and 52 tiles, but they gave the wrong number.

Similarly to the previous task, a large proportion of students had a problem with a detailed description of the rule: 40% of the respondents fully justified it, 43% of the students gave a partial explanation, and 17% of the students did not give reasons. The students were able to apply the opposite of the known rule in an untypical situation: 83% of respondents carried out command d) well.

Task 4 results

Task 4 concerns the research questions a, d, g, j and was the following:

Consider the number 0,5(843).

a) Complete the table that shows what number is at the k-decimal place.

k	Digit	k	Digit	k	Digit
1	5	2	8	3	4
4	3	5	8	6	
7		8		9	
10		11		12	

b) In which decimal places does the number 4 appear? How can you say that precisely?

c) Give the number, which is on the 99-th, the 1326-th, the 3n-th decimal place.

d) Enter the number which is on the 101st, on 1327th, on the 256th, and on the 3n + 1st decimal place.

e) Can you give a rule, what number is on the nth decimal place?

The analysis of Task 4 solutions has shown that students cope differently with different aspects of critical thinking.

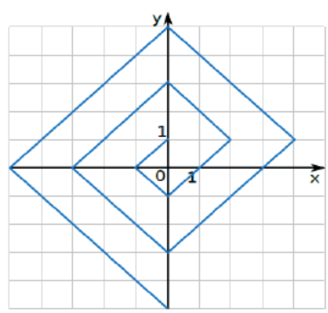
Students filled out the table well and most of them (39 out of 42) noticed that the number 4 appears in places with multiples of 3 (followed the pattern and written a general formula). Interestingly, only 36 participants have used this observation in answering point c).

Thirty students recognized the other rules and correctly described in point d), but only 22 participants gave the rules determining what number is on the nth decimal place, with only eight of these persons making the appearance of the number 5 in the first place (initial criteria included). Fourteen students did not attempt to give the rules.

Task 5 results

Task 5 concerns the research questions c, d, e and was the following:

Starting from point (0,1), we build a broken line, the part of which consists of 10 sections is shown in the figure. The next sections of the broken line are numbered with consecutive natural numbers. The first section of the broken line is equal to the square root of 2.



Answer the following questions. Create the appropriate table by yourself to help you notice the dependencies.

Each of the sections is parallel to the first or second section.

a) To which episode is the parallel episode number 20, number 99, number n?

b) What is the length of episode number 20, number 99, number n?

c) Write down your reasoning.

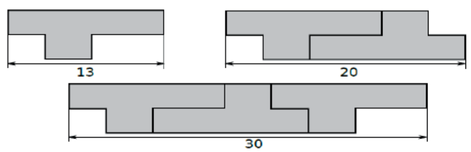
In tasks 1 to 4, students had to complete the table, and only then look for a rule of conduct. In Tasks 5 and 6 they had to create the appropriate tables themselves.

In Task 5, students had to discover relationships regarding the location of individual sections and their length. The analysis of Task 5 solutions has shown that 31 of 42 subjects created their own tables (they used the learned strategy to solve other problems): 26 of them concerned the length of sections, one parallelism, and four showed both the length and location of the sections. Thirty-two participants recognized the rule regarding the location of the sections (a), including 12 imprecisely determined the location of the nth section. Also, 32 students discovered the rule regarding the length of sections (b), including 12 incorrectly specified the length of the nth section or did not provide a general formula. Twenty students who made good generalizations, took into account parity and odd numbers section. Only 12% of the respondents gave justification for both rules, 31% clearly explained one rule, while 28% of students partly justified one of the rules. Others did not provide an explanation.

Task 6 results

Task 6 concerns the research questions b, d, g, h, i, j and was the following:

> The drawings show the shape and method of laying tiles and some dimensions in centimetres.
>
> Create an appropriate table and try to answer the questions about the length of a figure consisting of: a) 50 tiles, b) 61 tiles, c) n-tiles.
>
> d) Write down your reasoning.



Task 6 required distinguishing between an even and an odd number of tiles (considered different criteria). The analysis of Task 6 solutions has shown that 21 of 49 students made this distinction. The strategies developed in previous tasks were helpful in solving it. Despite this, only six students fully correctly recognized and justified the rule. Twenty-four participants made the table, but 19 students gave only numerical values for the initial elements; they could not generalize the formula. The length of the figure consisting of 50 and 61 tiles was well determined by 30% of students, but only 17% could write the general formula. Almost half of the respondents (48%) did not answer at points a) and b) at all, while at point c) as many as 55% of answers were empty. This task also showed that many students do not verify the correctness of their answers: 21% of the answers in each of the points were incorrect. Students showed, however, elements of critical thinking that can be seen in the analysis of the drawing: 32 students wrote their observations in the drawings (they analysed the information provided in the task), in which 10 additionally drew further pieces of the puzzle and subjected them to analysis (they used the learned strategy to solve other problems).

It is worth noting that in this task one can see a big difference in the work of beginning students and students who are already teachers. The correct general formula was given by 23% of second-cycle students, while another 23% presented patterns slightly different from the correct ones, while 10% of first-grade students gave a good generalization. This is due to the fact that teachers much more often (54%) had used mark strategies earlier (they made calculations as in Task 1), and only a few first-degree students used the same strategy (10%).

Task 7 results

Task 7 concerns the research questions b, d, g, h, i, j and was the following:

A table on prisms is presented below. Complete it.

The number of sides of the polygon at the base of the prism	3	4	5	6	10	n
S – number of prism walls	5	6	7			n+2
K – number of prism edges	9	12				2n+3
W – number of prism vertices	6	9	10	12		2n

b) Make an analogous table for pyramids.

The number of sides of the polygon in the base of the pyramid	3	4	5	6	10	n
S – number of walls of the pyramid						
K – number of edges of the pyramid						
W – number of vertices of the pyramid						

c) Prove that for each pyramid and prism there is the formula $S + W – K = 2$.

d) Write down your observations.

In Task 7, the two information in the first table were incorrect (the number of vertices of the cuboid and the general formula for the number of edges). Students had to complete the tables and prove a pattern that did not agree to the incorrect number of edges. The analysis of Task 7 solutions has shown that 16 respondents noticed and corrected both incorrect information, and 11 persons recognized the incorrect formula. In contrast, nine students filled in the table according to the wrong formula, moreover three other students noticing the incompatibility correct information for incorrect. The evidence of dependence was clearly conducted by 70% of the respondents, but only 43% wrote conclusions or observations.

The solutions of five participants have shown that the inconsistency in the taking of evidence caused the analysis of initial data and led to the finding of incorrect data. Interestingly, five other students, despite the impossibility to prove the relationship, did not recognize the errors in the table.

Summing up all the students' works, it can be seen that they used critical thinking in the course of their work, but to varying degrees. Every student made some corrections in their work (they often concerned calculation errors), so we may say they employed critical thinking during the calculations. On the other hand, verification of solutions with regard to the content of a task rarely occured. In half of the works, calculations, records or analysis of drawings was visible, which shown the analysis of a given problem, but this did not always coincide with the correctness of the answer. Students did not guess the answers, but they were not always able to justify them. Most students used the same strategy, only three students presented their own original solutions. Seventeen students, by creating their own tables, were able to follow the rule (they recognized the rule), but were unable to generalize it in the form of a formula when it required considering various criteria.

CONCLUSIONS

Research has shown a diversity in the skills of mathematics students with teaching specialization regarding critical thinking.

Students were able to follow the pattern and recognize the rule of conduct in simple situations, but they had a problem with the description and justification of the rule, especially when it required consideration of different cases.

Second-cycle students were much more likely to use previously learned strategies than first-cycle students.

In summary, it can be claimed that students used critical thinking in their work, but not enough. Mathematical activities proving critical thinking will not develop by themselves, appropriate actions are needed to support their development. This conclusion is consistent with the results of other researchers on mathematical activities (Ellis, 2011; Maj, 2011). One should undertake activities leading to the development of critical thinking of future teachers so that they would be able to teach their students this skill.

References

Braun, M., Mańkowska, A., & Paszyńska, M. (2020). *Matematyka z kluczem Podręcznik do matematyki dla klasy siódmej szkoły podstawowej* [Maths with the key Maths textbook for the seventh grade of primary school]. Warsaw, Poland: Nowa Era.

Braun, M., Mańkowska, A., & Paszyńska, M. (2019). *Matematyka z kluczem Podręcznik do matematyki dla klasy ósmej szkoły podstawowej* [Maths with the key Maths textbook for the eighth grade of primary school]. Warsaw, Poland: Nowa Era.

Ciosek, M. (2012). Generalization in the process of defining a concept and exploring it by students. In B. Maj-Tatsis & K. Tatsis (Eds.), *Generalization in mathematics at all educational levels* (pp. 38-56). Rzeszów, Poland: Wydawnictwo Uniwersytetu Rzeszowskiego.

Ellis, A. (2011). Generalizing-Promoting Actions: How Classroom Collaborations Can Support Students' Mathematical Generalizations. *Journal for Research in Mathematics Education, 42*(4), 308-345.

Ennis, R. H. (1996). *Critical Thinking*. USA: Prentice Hall, Inc.

Facione, P.A. (2011). *Critical Thinking: What It Is and Why It Counts.* Millbrae, CA: The California Academic Press.

Firdaus, F., Kailani, I., Bakar, M., & Bakry, B. (2015). Developing Critical Thinking Skills of Students in Mathematics Learning. *Journal of Education and Learning, 9*(3), 226-236.

Huitt, W. (1998). Critical Thinking: An Overview. *Educational Psychology Interactive*. Valdosta, GA: Valdosta State University. Retrieved from http://www.edpsycinteractive.org/topics/cogsys/critthnk.html

Krygowska, Z. (1977). *Zarys dydaktyki matematyki* [Outline of mathematics didactics], *Part 2*. Warsaw, Poland: WSiP.

Krulik, S., & dan Rudnick, J. A. (1999). Innovative Tasks to Improve Critical and Creative Thinking Skills. Dalam Developing Mathematical Reasoning in Grade K-12. In L. V. Stiff & F. R. Curcio (Eds.), *1999 Yearbook*. Reston, Virginia: NCTM.

Maj, B. (2011). Developing Creative Mathematical Activities: Method Transfer and Hypotheses' Formulation. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proceedings of the Seventh Conference of the European Society for Research in Mathematics Education (CERME 7)* (pp. 115-1124). Rzeszów, Poland.

Malara, N. (2012). Generalization processes in the teaching/learning of algebra: students' behaviours and teacher role. In B. Maj-Tatsis & K. Tatsis (Eds.), *Generalization in mathematics at all educational levels* (pp. 57-90). Rzeszów, Poland: Wydawnictwo Uniwersytetu Rzeszowskiego.

Nowak, W. (1989). *Konwersatorium z dydaktyki matematyki* [Seminar in didactics of mathematics]. Warsaw, Poland: PWN.

Palinussa, A. (2013). Students' Critical Mathematical Thinking Skills and Character. *IndoMS. J.M.E, 4*(1), 75-94.

Semadeni, Z. (2008). Factors hampering independent mathematical thinking. In B. Maj, M. Pytlak, & E. Swoboda (Eds.), *Supporting Independent Thinking Through Mathematical Education* (pp. 18-25). Rzeszów, Poland: Wydawnictwo Uniwersytetu Rzeszowskiego.

Siwek, H. (2005). *Dydaktyka matematyki Teoria i zastosowania w matematyce szkolnej* [Mathematics didactics Theory and applications in school mathematics]. Warsaw, Poland: WSiP.

Sukmadinata, N. S. (2004). *Kurikulum dan Pembelajaran Kompetensi*. Bandung: Kesuma Karya Bandung.

TC2 (2013). *Critical thinking in elementary mathematics: What? Why? When? and How?* The Critical Thinking Consortium. Retrieved from www.tc2.ca.

PROBABILITY KNOWLEDGE EFFECT ON CRITICAL THINKING IN YOUNG AGES

Michail Zorzos, Evgenios Avgerinos

University of the Aegean, Greece

Critical thinking is an indispensable advantage for any rationally thinking citizen (Aizikovitsh & Amit, 2008). Mathematics is a characteristic domain of science, that promotes the development of critical thinking. Specifically, Probabilities, because of their nature and their impact to everyday life, may be the domain of mathematics that can directly benefit critical thinking processes. This paper investigates the relation between critical ability and probability thinking. The research was conducted on primary school students. The results show that students appear to improve their critical skills after a quick probability lesson.

INTRODUCTION

Mathematics is an integral part of curricula around the world. Their usefulness is obvious and cannot be doubted. The basic knowledge of Classical Arithmetic, Measurement, Geometry, Statistics and Probability are essential for one' s daily survival. Of course, the use of mathematics is not limited to actions and assessments. The construction of rational ideas, creativity, reasoning and critical ability, are benefits of mathematics teaching (Harel & Stylianides, 2017). Thus, one of the main goals of mathematics teaching is to create people with the knowledge and skills they will need in their future lives (Czocher, 2016).

Chances are a domain of mathematics that has an immediate impact on everyday life and attracts the interest not only of the mathematics specialists (Ross, 2010). The great mathematician-astronomer Marquis de Laplace believed that Probabilities had the specifications to become the most important field of human knowledge (Ross, 2010). Their theory deals with the measure of certainty about the appearance of an outcome and is therefore directly related to decision making (Howard, 1988). In other words, it is connected to the processes of thinking and critical ability.

The subject of the present paper deals with the correlation of Probability theory with critical ability. In particular, the development of critical thought of elementary school students is studied through knowledge of Probability Theory. The purpose of the research is to examine students' critical competence in daily Probability Problems and how this is influenced by basic knowledge of the theory.

DECISION-MAKING, RISK AND PROBABILITY

Decisions are becoming increasingly part of everyday life (Aizikovitsh & Amit, 2008). It is a fact that every decision, important or not, does not tend to be made on the spot (Bonnett, & White, 2018). Understanding the importance of decision making and creating the sense of responsibility, are the starting points for using the Probability theory. In other words, enhancing the ability to deal with risk situations can lead to problems being avoided (Turner, Macdonald, & Somerset, 2007). Thus, it is common for everyday decisions to have a significant influence on probability theory (Engel & Orthwein, 2018).

Strategic reasoning and decision making are a key factor in everyday business practices (Batanero & Chernoff, 2017). This gives the advantage to those who know probability practices. Theory and its evidence-based methods provide the learner with the knowledge and skills to choose the situation with minimal risk (Batanero & Chernoff, 2017). They also improve reasoning about random events and supply the person with unbiased judgment skills (Turner, Macdonald, & Somerset, 2007).

PROBABILITIES AND CRITICAL ABILITY

According to the above, the immediate consequence of the right choice of a decision, is the development of critical thinking. Critical competence is mainly regarded as the rational assessment of a situation, free from intuitive perceptions or empirical interventions (Paul, 1992). Critical thinking skills are vital to the success of the modern world (Aizikovitsh & Amit, 2008).

In the field of education, the learner is required to be open-minded and confident, in order to evaluate information and react to the event as a critical thinker (Aizikovitsh & Amit, 2008; Aizikovitsh & Amit, 2010). The school must prepare the student to think, challenge and look for alternatives. Past researches have shown that with appropriate teaching intervention it is possible to construct a learning environment that promotes critical thinking (Aizikovitsh & Amit, 2010).

Knowledge of Probability Theory encourages students to overcome their causal thinking (Batanero & Chernoff, 2017) and through the development of complex reasoning, they encourage the development of critical ability (Aizikovitsh & Amit, 2008). This is achieved by disputing the assumptions (Langrall, 2018) and by learning to see the difference between the subjective and the objective interpretation of the results (Kyburg, 1966), as the pupil now gains his own judgment and the misunderstandings are avoided.

In 2018, Borovcnik and Kapadia, in a study, attempted to capture the stages of probabilistic thinking. The general conclusion of this recording is the direct correlation between probability reasoning and critical thinking. Specifically, in dealing with a probability problem, the student organizes his or her thinking, realizes the abilities he / she needs to address the problem, investigates the theory and finally answers the problem (Borovcnik & Kapadia, 2018). In short, a strictly structured process of critical scepticism.

METHOD

The present study was conducted on primary school students. The main reason for this choice was the virgin ground in the students' knowledge of Probability Theory. In this way, it will be possible to determine whether students' critical ability is improved by a small supply of probabilistic knowledge. This research represents a small step towards the development of critical capacity through Probability Theory. Research questions deal with the development of students' critical ability through probabilistic knowledge.

The study was conducted on 210 students from fourth grade to sixth grade. Specifically, the sample was divided into two groups. The first group consisted of the experimental group with 90 students and the second one, the control group with 120 students. For the purpose of the research, a questionnaire was designed, which was also the mean of data collection. The resulting data were analysed with the statistical analysis program S.P.S.S. and are presented in the form of tables.

The researchers distributed the questionnaires to the control group without any teaching intervention. On the contrary, before the questionnaires were given to the experimental group, a small didactic intervention was followed. One could characterize it as a discussion of probabilities, their basic concepts and the classic definition of Probability. More specifically, a twenty-minute introduction to basic concepts and classical definition was made and some examples of everyday life were discussed. The questionnaires were distributed to the experimental group one week after the researchers' teaching intervention.

The questionnaires distributed to the two groups consisted of one question and two daily exercises. A characteristic of the exercises, in order to better control students' reasoning and critical thinking, was that the students had to justify their response. The question deals with the interpretation of a luck experiment (rolling the dice) and the student is asked to answer a few questions about the experiment and its continuation. The first exercise deals with the classic definition of probability and its use (the wheel of fortune). Finally, the second exercise involves a problem that could be a daily thought for a student (a surprise gift between 2 possible outcomes).

Below, are presented the question and the exercises that constituted the questionnaire, as they were given to the students.

Question 1

We have thrown a dice three times. All three brought six. We intend to do it again. Mark the suggestions as right or wrong:

a) It is impossible to bring six again.

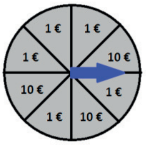
b) With our luck, we are going to bring six.

c) The probability is 1/6, as in any cast.

d) It's 50% to bring six and 50% not to bring.

Exercise 1

There is a wheel of fortune. Which of the following amounts are we most likely to win on the following wheel and why?



Exercise 2

Going to the supermarket, we bought two cereals boxes from a well-known brand that gives a gift in each box. The gift is a card or a sticker of the national basketball team. If we find the card in the first box, then in the second box will we find the sticker? Please justify your answers.

RESULTS

The analysis of the data collected is shown below. For the convenience of the reader, these were sorted into tables. What should be noted at this point, is that the tables show only the percentages of results to facilitate comparison and make the tables simpler. Furthermore, for the purposes of the research, only the answers that were followed by a proper justification was considered to be the correct answer. This was intended to protect the research from accidental or intuitive responses.

Question	Control team			Experimental Team		
Come 6 the first three times	Right	Wrong	Blank	Right	Wrong	Blank
Impossible to bring six again	55.8%	13.4%	30.8%	53.3%	33.4%	13.3%
Definitely six the next time	50.8%	19.2%	30%	68.9%	18.9%	12.2%
The probability is 1/6, as in any cast	12.5%	55.8%	31.7%	27.8%	58.9%	13.3%
50% to bring six and 50% to not bring	24.2%	45.8%	30%	36.7%	36.7%	12.2%

Table 1: Percentages of the Answers to the Question.

In the table above, it is observed that the students responded to a question that described a simple experiment. At first glance, the difference in the performance of the two teams is evident. Specifically, in the first question the percentages of correct answers are quite close, in the second question the experimental group answered 68.9% correctly, compared to the control group which answered 50.8%. In the third and fourth question, about the probability of the event 'six coming in the next throw', are 58.9% and 36.7% for the experimental group and 12.5% and 24.2% for the control group. Also important in this table, is to observe the gap between the two groups.

Exercises	Control team			Experimental Team		
	Right	Wrong	Blank	Right	Wrong	Blank
Exercise 1	24.1%	20.9%	55%	28.9%	16.7%	57.4%
Exercise 2	26.7%	20%	53.3%	13.3%	36.7%	50%

Table 2: Percentages of the Answers to the Exercises.

In the second table are presented the results of the exercises. In Exercise 1 (with the Wheel of Fortune), 28.9% of the experimental group appeared to respond correctly, compared to 24.1% of the control group. On the other hand, in the second exercise (the visit to the supermarket), the correct answers were 13.3% and 26.7% for the experimental and control groups respectively. Besides, it appears that the percentages of blank answers were high in both groups.

CONCLUSIONS

Probabilities are a domain of mathematics indissolubly linked to decision making, reasoning and judgment. The importance of critical thinking in today's citizens (Aizikovitsh & Amit, 2008), as well as the impact of probability theory in everyday situations (Engel & Orthwein, 2018), could be a promising combination for the development of students' competence. This study demonstrates that young learners can improve their critical thinking through probabilistic knowledge. In this way, the research of Borovcnik and Kapadia (2018) is verified, in which the stages of probabilistic thinking are directly related to those of critical thinking.

Aizikovitsh and Amit (2010) believed that with appropriate teaching intervention it is possible to increase students' critical ability. For this reason, they suggested a combination of critical thinking skills with the mathematical content of probabilities in everyday activities. This part, as it is proved through the results of this research, can be fulfilled easily even to young students without a cognitive background. However, it seems that probability theory does not require any particular intervention, in order to show its benefits to the student's critical ability. Also, the difference of the performance between the two groups agrees with Kyburg's (1966), Batanero's, and Chernoff's (2017) view, that students through probability, think more critically and minimize subjectivities. Consequently, the conclusion of the present work summarizes that students have the possibility, through theory of probability, to justify their decisions in a rigorous and clear structure, as well as to process better the information, which are provided to them. In other words, they are able to think critically.

References

Aizikovitsh, E., & Amit, M. (2010). Evaluating an infusion approach to the teaching of critical thinking skills through mathematics. *Procedia - Social and Behavioral Sciences, 2(2),* 3818–3822.

Aizikovitsh, E., & Amit, M. (2008). Developing critical thinking in probability session. In *Joint Meeting of PME 32 and PME-NA XXX.* (pp. 9-16). Mexico: International Group for the Psychology of Mathematics Education.

Batanero, C. & Chernoff, E. (2017). Teaching and Learning of Probability. In G. Kaiser, *Proceedings of the 13th International Congress on Mathematical Education* (pp. 439-442). Switzerland: Springer International Publishing.

Bonnett, L., & White, S. (2018). May the odds be ever in your favour. *Teaching Statistics, 40*(3), 94-97.

Borovcnik, M., & Kapadia, R. (2018). Reasoning with Risk: Teaching Probability and Risk as Twin Concepts. *Teaching And Learning Stochastics,* 3-22.

Czocher, J. (2016). Introducing Modeling Transition Diagrams as a Tool to Connect Mathematical Modeling to Mathematical Thinking. *Mathematical Thinking And Learning*, *18*(2), 77-106.

Engel, J. & Orthwein, A. (2018). The Six Loses: Risky Decisions Between Probabilistic Reasoning and Gut Feeling. In C. Batanero & E. Chernoff, *Teaching and Learning Stochastics* (pp. 261-275). Switzerland: Springer International Publishing.

Harel, G, & Stylianides, A. (2017). Reasoning and Proof in Mathematics Education. In G. Kaiser, *Proceedings of the 13th International Congress on Mathematical Education* (pp. 459-462). Switzerland: Springer International Publishing.

Howard, R. (1988). Uncertainty about Probability: A Decision Analysis Perspective. *Risk Analysis, 8*(1), 91-98.

Kyburg, H. (1966). Probability and Decision. *Philosophy Of Science, 33*(3), 250-261.

Langrall, C. (2018). The Status of Probability in the Elementary and Lower Secondary School Mathematics Curriculum: The Rise and Fall of Probability in School Mathematics in the United States. In C. Batanero & E. Chernoff, *Teaching and Learning Stochastics* (pp. 39-50). Switzerland: Springer International Publishing.

Paul, R. (1992). Critical thinking: What, why, and how. *New Directions For Community Colleges*, *1992*(77), 3-24.

Ross, S. (2010). *Basic Principles of Probability Theory* (V. Felouzis, Mt., ed.). Athens: Klidarithmos Publications (1976 Standard Edition).

Turner, N. E., Macdonald, J., & Somerset, M. (2007). Life Skills, Mathematical Reasoning and Critical Thinking: A Curriculum for the Prevention of Problem Gambling. *Journal of Gambling Studies, 24(3*), 367–380.

STUDENTS' CREATIVE THINKING DURING SOLVING ALGEBRAIC TASKS

Marta Pytlak

University of Rzeszow, Poland

Mathematics education is not only about teaching concepts, theorems and algorithms. It is also teaching the skills which students need in everyday life. One of them is reflective and critical thinking. It is a skill that can be successfully developed during math lessons. This kind of thinking supports many mathematical activities, such as discovering regularities and generalizations. The paper presents some results of research concerning algebraic thinking and generalization. We are trying to answer the question: does reflective thinking support the ability of generalizing and perceiving regularity.

INTRODUCTION

Nowadays, mathematics education is facing new challenges. Its goals are seen differently. It is expected that more emphasis will be placed not on equipping the student with extensive knowledge, but with skills useful in real life (da Ponte, 2008; Krygowska, 1985, 1986). Learning and teaching mathematics is primarily understood as learning to think, act and communicate mathematically (Arzarello, 2016). It is expected that a student who graduates primary school will not only demonstrate knowledge of relevant mathematical facts. In addition to substantive knowledge she or he should also demonstrate a whole range of mathematical skills. These include, above all, the ability to analyse and to make hypotheses, argument and justification ability, and creative and critical thinking. Especially critical and creative thinking is particularly important (Oldridge, 2015). Changes in the curriculum that have recently taken place in Polish education put the main emphasis on the teaching of mathematics focused on the development of thinking. The idea is to educate in such a way that the student will be a self-thinking person (MEN, 2008). Therefore, an important goal for the teachers is to teach students how to think critically. Students who can think critically grow into lifelong problem solvers. Critical thinking with students means that they can take information and analyse it, draw conclusions, formulate opinions, reflect on their work, and approach problems in a systematic way.

One of the ways to develop mathematical thinking is to discover regularity. Discovering the regularity by pupils is the most important problem and it presents the trends of teaching in the world. In many countries in the teaching of mathematics the attention is directed to the functioning of the regularity. In literature we can find many descriptions of research carried out in discovering and generalizing these rules (Carraher, Martinez, Schliemann, 2008; García

Cruz, Martinón, 1997; Littler, & Benson, 2005a, 2005b; Stacey, 1989; Mason, 1996; Orton & Orton, 1999; Sasman, Olivier, & Linchevski, 1999; Zazkis & Liljedahl, 2002a, 2002b).

Teaching to perceive and to use regularities means teaching a certain attitude to mathematics. Regularities stimulate thinking outside particular cases, they guide to the thinking of the general rules. In Poland one can also find research results that give base to the wide interest in regularities shown in mathematical education of children. Searching for regularity is extremely effective in solving mathematical problems, it is a strategy of solving tasks. As Swoboda (2006) writes:

> … noticing the regularity is a skill desired by all means. Activities in which a child notices the regularity, acts according to the rule – are those stimulating his mental development. They are also the basis of mathematical thinking at each level of mathematical competence. (pp. 51-52).

Unfortunately, this type of thinking about mathematics is not common in our school reality. The rhythm and regularity, in Polish practice of teaching mathematics, children generally meet in pre-school and in the younger grades of primary school. The most common are the geometric regularities of drawing special patterns. The task: "complete the pattern" is placed for a child to do. It is not expected of them to discover the rule governing the model, just to draw it as carefully as possible. It is mainly about training some manual abilities needed to learn shapely and learn writing carefully, not about the development of mathematical thinking. In grades 4 and 5 of primary school students sometimes meet with the tasks relating to the arithmetic regularity (e.g., number of triangular numbers, magic squares) or geometric (mosaic). These tasks are usually handled by teachers marginally – they do not appreciate the value of such tasks, do not know what purpose would those serve.

As the studies show (Gruszczyk-Kolczyńska, 2001; Urbańska 2003) noticing the rule by a child is usually a source of his/her immense satisfaction. On the other hand, one can meet the opinion, that in teaching of mathematics "too little situations, in which a student in a spontaneous way could experience both the joy of discovery and the fact of discovering something new, are being created . Students are indifferent to the mathematical issues, and an indifferent man cannot be creative." (Skurzyński, 1992). You see, then, apparently it shows that also in the area connected with learning there are great, unused possibilities.

Used tasks connected with searching and noticing dependency and regularity could support the development of students' reflective, critical thinking. On the other hand, this kind of thinking – critical and reflective thinking – can significantly support the development of mathematical thinking, and in particular algebraic thinking.

METHODOLOGY OF RESEARCH

The research lasted four years and included two stages: preliminary, diagnostic study and the main research. The goals, the tools and the research methods evolved at particular stages. Each stage of study allowed to identify the research problem better. Due to it, both the research topics and the selection of the research group could be specified. It had the influence on the formation of the final research tool.

The preliminary research

The first stage of the study, the so-called initial study was focused on the ability to perceive regularity by students. This phase of the study lasted less than two years. Twenty six students from the fourth and sixth grade of primary school were investigated (see also Pytlak, 2006).

The main research questions were as follows:

1. How do students from primary school cope with the tasks concerning noticing and discovering the regularities?
2. What is the students' way of thinking while solving the task connected with discovering the regularity?
3. Are the students at this level of education able to make a generalization?

The research tool consisted of two sheets. Each of them contained three tasks related to arranging a pattern of triangles.

<table>
<tr>
<td>Sheet I

1. How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 separated triangles, which length of each sides equals one match?
<table>
<tr><td>Number of triangles</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr>
<tr><td>Number of matches</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>
2. And how many matches do you need to construct :
a) 10 triangles
b) 25 triangles
c) 161 triangles?

3. Can you give some general rule by which you can calculate the number of matches needed to build a certain number of triangles?</td>
<td>Sheet II

1. How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 connected in one row triangles, which length of each sides equals one match?
<table>
<tr><td>Number of triangles</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td></tr>
<tr><td>Number of matches</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>
2. And how many matches do you need to construct :
a) 10 triangles
b) 25 triangles
c) 161 triangles?

3. Can you give some general rule by which you can calculate the number of matches needed to build a certain number of triangles?</td>
</tr>
</table>

Figure 1. The research tool from the preliminary study

Students were solving these tasks during one hour lesson. First, they received sheet 1, and after solving all the tasks they received sheet 2. During their work, the students had sticks from which they could arrange the patterns from the task. 18 students worked in pairs, and 8 individually. Thus, 34 students' works were assessed as: 17 sheet 1 and 17 sheet 2.

The results obtained by students during this stage of research are as follows:

		Correct (without mistakes)	Incorrect	Correct after correction
Sheet 1	Task 1	17	0	0
	Task 2	17	0	0
	Task 3	16	1	0
Sheet 2	Task 1	15	0	2
	Task 2	4	2	11
	Task 3	5	5	7
Total		74	8	20

Table 1. Results from preliminary research

The preliminary tests showed that the students are able to cope with the task consisting of discovering regularities. Although the type of task presented to them was new for them, they spontaneously approached to its solutions and reached very interesting results. At the same time they presented very different approaches and different ways of solving it.

The study also showed the importance of an interaction during the teaching and learning of mathematics, especially in the creation of the new mathematical knowledge. And it occurred both on the line for the student - teacher, as well as during interactions between students themselves. The teacher had to develop students' self-control ability by requiring from the students arguing and justifying the steps made by them during the work on the task. Thanks to this, the child faced with the problem was more aware of what he or she was doing. The need for verbalization of thoughts to explain their proceedings resulted in the creation and discovering new mathematical knowledge.

After analysing the process and the results of the work of 4th and 6th grades the following conclusions can be drawn:

- the fourth grade students need specific physical experience connected with solving the task, while the sixth grade students link directly to the abstract knowledge;
- at the level of the fourth grade students do not feel the need to make a schema and they see the relationships and dependencies only after going through a series of structurally-related activities, while older students can make a schema very well, they simplify the task, they can notice the relationships and dependencies only after a few sequences;

- older students have already quite significant experiences, with a fixed web of connections. “Generic model” for the task presented to them is available to them. So they start right away with the level of abstraction. For the fourth grade students the set of experiences and web of interconnections is only emerging;
- senior grade students use symbols to record observed relationships, while the lower grade students describe these relationships in words or by example.

Thus, is there any sense of talking about the creation of algebraic thinking in the case of the fourth grade students? At this level the real thinking process took place. Here they could find some exploration and gained experience as a basis for further learning. For these students the task was a challenge and it inspired them to new discoveries. For students of sixth grade the task was a trivial. Therefore, at the next stage of the research I focused on the 9-10 years old students.

At the next stage of the research, the research tool and the organization were changed. The main goal remained the same. In addition, the impact on interaction between students during solving the task was investigated (see also Pytlak, 2008).

The main research

The research tool consisted of four sheets and each of them consisted of two tasks. The tasks were as following: the students were making a match pattern consisting of geometrical figures – one time there were triangles and another time there were squares with a side length of one match. In the first two of the sheets the figures were arranged separately, in the second of the two – connected in one row. The next sheets concerned: (1) separated triangles, (2) separated squares, (3) connected squares and (4) connected triangles. In each of the sheets the problem was presented in a frame of the next two tasks. They were constructed in such a way in order to inspire students to search and discover occurring rules.

In the first task the students had to give the number of matches needed to arrange one after another from one to seven triangles or squares. The question was: How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 of such figures? The results should be written in the table. In the second task, there was a question about a number of matches which are needed to construct 10, 25 and 161 of such figures (Littler, 2006). In order to give an answer for these questions the students had to discover the rule occurring in the first task.

Sheet I

1. How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 **separated** triangles, which length of each sides equals one match?

Number of triangles	1	2	3	4	5	6	7
Number of matches							

2. And how many matches do you need to construct :
a) 10 triangles
b) 25 triangles
c) 161 triangles?

Sheet IV

1. How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 **connected in one row** triangles, which length of each sides equals one match?

Number of triangles	1	2	3	4	5	6	7
Number of matches							

2. And how many matches do you need to construct :
a) 10 triangles
b) 25 triangles
c) 161 triangles?

Figure 2. The first and the fourth sheet of research tool

The choice of the tasks and the order of the sheets were not random. The problem was to check if the students will benefit from their earlier experience while solving the new tasks. As already elaborated, the strategy of solving the problem will be applicable while doing the next task. Accepting this kind of strategy will prove an appropriate construction of the research tool – that is which provokes enlarging already existed cognitive web towards building a generic model.

This task and the way of its presentation (four following sessions) were something new for students. So far during maths lessons they did not solve the tasks concerned with the perception of the appeared rules and generalization of noticed regularities.

The research was carried out among students from the fourth grade of a primary school. Forty four (9-10 years old) students from two different primary schools working in pairs took part in them. The research contained four following meetings, during which students were solving these tasks. All the meetings were recorded by a video camera. After the research, the report was presented. The students were working in pairs. The researcher was talking with every group of students while they were solving the tasks.

The students had worksheets, matches (black sticks), ball pens and a calculator. Before students started their work, they had been informed that they could solve this task in any way they would recognize as suitable; their work would not be graded; teacher would be videotaping their work and that they could write everything on the worksheet which they recognized as important. The research material consists of worksheets filled by students, as well as the film with students' recorded work and a stenographic record from it.

The results obtained by students at this stage are as follows:

		Correct (without mistakes)	Incorrect	Correct after correction
Sheet 1	Task 1	22	0	0
	Task 2	21	1	0
Sheet 2	Task 1	20	0	2
	Task 2	22	0	0
Sheet 3	Task 1	19	0	3
	Task 2	6	4	12
Sheet 4	Task 1	19	1	2
	Task 2	11	2	9
Total		138	8	28

Table 2. Results from the main research

Detailed research results from both stages have already been discussed. Here I would like to present some examples of students' work, where initially students incorrectly solved the task, but thanks to reflection and critical thinking they were able to solve the task correctly.

EXAMPLES OF STUDENTS' WORK

Example 1

A student (S12) taking part in the first stage of research, was described by a math teacher as a child with average mathematical skills. She solved the tasks from the first sheet very quickly. First, she put together a fragment of the pattern. She noticed the relationship between the number of triangles and the number of matches used to build them. She also wrote verbally the noticed relationship "a given number [number of triangles] times three" Her justification was as follows: "because one triangle needs three matches ". In the second sheet student started from arranging a fragment of the pattern and after that she solved the first task correctly. To solve the second task, she first extended the table and thus obtained the number for 10 triangles. Then, using the data from the table, she calculated the number of matches for 25 triangles. She noticed first that 25 = 2x10 + 5, so if 21 matches we need for 10 triangles, and for 5 - 11, then for 25 you will need 53 matches together.

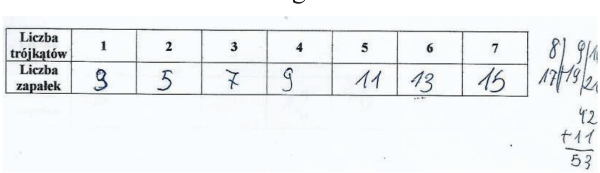


Figure 3. Student's (S12) work – extending the table

Initially, for 161 triangles, she couldn't find the right number of matches. However, referring to the method of arranging, she discovered the corresponding relationship. This verified the previous solution and she corrected the incorrect results. The following conversation between the student and the teacher shows it:

S12: Here this will not be so easy [she is pointing at the task 2c)].

T: For 161 triangles?

S12: Yes. I must count it in a different way… as it was arranged…

T: How was it arranged?

S12: Because every time we add two … but there is another one at the beginning… [she is looking at the pattern]. I think I know it now..

T: Yes?

S12: It will be like multiplying these triangles by two and add one more.

T: Willi t be always like that?

S12: Yes, it was like that before [she is counting again the number of matches for 25 triangles]. Oh no, here is a mistake [she is correcting 53 on 51]

As an answer for the third task student wrote a following rule:

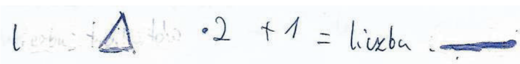


Figure 4. General rule written by the student S12

The girl initially solved the task without a lot of reflection. She used arithmetic procedures and operations known by her. Only for more number of triangles it turned out that the applied methods were not sufficient. There was a moment of hesitation and re-analysis of the task. There was a reflection how to create a pattern resulted in the discovery of an appropriate rule.

Example 2

Two students (S3 and S4) worked together on the task. According to math teacher, girls had no problems with learning mathematics. They solved the tasks from sheet 1 very quickly. They wrote all the results correctly. However, they could not write down noticed relationship. Students presented it by writing relevant numerical examples.

In the second sheet, by arranging the pattern, they completed the first task correctly. The way of arranging (adding two matches) provoked them to apply the following rule in the second task: multiply the number of triangles by 2. They explained it in a conversation with the teacher:

S4: Here it will be ten times two [pointing at the task 2a)]

T: Do you notice that dependence?

S3: Yes, we multiply by 2.

T: Maybe we can check it here [pointing at the table]

S3: For 3 triangles there are 7.

T: Seven. Is it something multiplied by 2?

S4: No, it is 3 times 2 and one more.

T: And what about here? [pointing at the table 4]

S3: Four times two are eight

S4: And one more

S3: Nine, so eight plus one

T: And what about here?

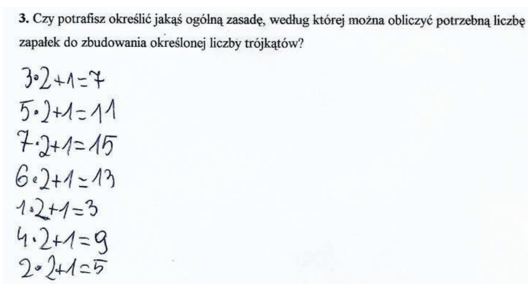
S4: Six Times two are twelve … and one more…

S3: 13

T: So how will it be in this task? [pointing at on the task 2]

S4: It will be 21 [correcting the result]

After talking with the teacher, the girls corrected the results in task 2. However, they could not write the noticed rule in words. As an answer to the question 3 they wrote only calculations for all the results obtained in the table.


3. Czy potrafisz określić jakąś ogólną zasadę, według której można obliczyć potrzebną liczbę zapałek do zbudowania określonej liczby trójkątów?

3·2+1=7
5·2+1=11
7·2+1=15
6·2+1=13
1·2+1=3
4·2+1=9
2·2+1=5

Figure 5. Students' work – justification of the rule

The girls couldn't quite connect the number of triangles with the number of matches. They noticed the relationship between these quantities, but they could not generalize it. Even when the students were talking with the teacher, they only reported applying rule to a specific number of triangles. Despite the experience with creating a pattern and talking to the teacher, they could not find a general rule. Perhaps there was no reflective approaches. The girls could not independently, critically look at the obtained results, assess their correctness. Hence maybe this is the reason of the lack of success in solving the task.

Example 3

Students (S5-II, S6-II) took part in the second stage of the research. They solved the two first research sheets without any major obstacles. In the third sheet, they first arranged a fragment of the pattern, and then quickly filled the table. When they were calculating the number of matches in the second task, they used the adding method. One student wrote down and the other one counted.

S5-II: For 10 squares it will be … 22 +3 +3+3 [he starts counting 3 to the number of matches from 7 squares]

S6-II: It will be 31

S5-II: Yes. And for 25 squares …. +3+3+3 [he is writing the results]

S6-II: [she is counting in mind] It will be 76.

S5-II: And for 161 squares …[he is writing 161+3]

S6-II: 161?

S5-II: Oh, no, we are starting from 76 [he is correcting and writing 79+3+3+3+3+3+3…, he is counting silently]

T: Do you count like this all the time?

S5-II: [he is looking at the task] No, it is possible to make it in a different way … It will be three times and add four.

T: Why?

S5-II: Because to make 161, you need to multiply 160 by 3 and add the first one, which has 4.

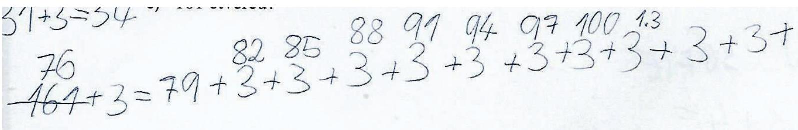


Figure 6. Students' work – solution for 161 squares

The first approach to the task was mechanically. After arranging the pattern they discovered how to create it. they used this knowledge consistently. The only improvement in counting was adding to the previous known value, not counting from the beginning. An important point was the conversation while solving the task. This forced the students to re-examine the way of their work. Then they noticed that a different approach to solve this task is possible. After that they were able to discover and write the general rule.

SUMMARY

Based on the research we have found that:

- Visual representation for the 9-10 year-old students is very strong and can stimulate to explore the relationship.
- The role of physical manipulation and gaining experience in solving the problem is very important. The structure of activity suggests the structure of thinking and shows relationships which occurred in the task.

The research have shown that simply counting a task by mechanical means does not reveal anything new. Using known, familiar procedures and algorithms does not give students a chance to be creative. Students' mistakes can be a source of discovering something new. It is important to discuss the task and reflective approach. Critical thinking not only allows us to verify our solution, but it is also a chance for a new look at the task. And this is the starting point for discovering new things and developing mathematical thinking. Hence, critical thinking and reflective approach should be promoted, because it gives measurable results in the teaching-learning process.

References

Arzarello, F. (2016). Basing on an inquiry approach to promote mathematical thinking in the classroom. In B. Maj-Tatsis, M. Pytlak, & E. Swoboda (Eds), *Inquiry based mathematical education* (pp. 9–20). Rzeszów: Wydawnictwo Uniwersytetu Rzeszowskiego.

Chamberlin, S., & Moon, S. M. (2005). Model-eliciting activities as a tool to develop and identify creatively gifted mathematicians. *Prufrock Journal, 17*(1), 37–47.

Carraher, D., Martinez, M., & Schliemann, A. (2008). Early algebra and mathematical generalization. *The International Journal on Mathematics Education*, *40*(1), 3–22.

García Cruz, J. A., & Martinón, A. (1997). Actions and Invariant Schemata in Linear Generalizing Problems. In E. Pehkonen (Ed.), *Proceedings of the 21th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 289–296). Helsinki: University of Helsinki.

Gruszczyk-Kolczyńska, E. (2009). *Wspomaganie rozwoju umysłowego oraz edukacja matematyczna dzieci w ostatnim roku wychowania przedszkolnego i w pierwszym roku szkoły podstawowej* [Supporting mental development and mathematical education of children in the last year of pre-school education and in the first year of primary school]. Warsaw: Edukacja Polska.

Krygowska, A. Z. (1985). Matematyka współczesna i nauczanie w świetle dyskusji na zachodzie Europy [Contemporary mathematics and teaching in the light of discussions in Western Europe]. In G. Treliński & H. Siwek (Eds.), *Modernizacja kształcenia matematycznego i jego wpływ na rozwój dydaktyki matematyki [Modernization of mathematical education and its impact on the development of didactics of mathematics]*. Kraków: Wyd. Naukowe PWN.

Krygowska, A. Z. (1986). Elementy aktywności matematycznej, które powinny odgrywać znaczącą role w matematyce dla wszystkich [Elements of mathematical activity that should play a significant role in mathematics for all]. *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki, 6,* 25–41.

Littler, G. H., & Benson, D. A. (2005a). Patterns leading to generalization. In *Proceedings of SEMT'05* (pp. 202–210). Prague: Charles University.

Littler, G. H., & Benson, D. (2005b). Patterns leading to Algebra. In *IIATM- Implementation of Innovation Approaches to the Teaching of Mathematics*, Comenius 2.1

Mason, J. (1996). Expressing Generality and Roots of Algebra. In N. Bednarz, C.Kieran, & L. Lee (Eds.), *Approaches to Algebra, Perspectives for Research and Teaching* (pp. 65–86). Dordrecht: Kluwer Academic Publishers.

Mason, J., Johnston-Wilder, S., & Graham, A. (2005). *Developing Thinking in Algebra*. London: Sage.

MEN (2008). *Podstawa Programowa z komentarzem* [Core Curriculum with comments].

Oldridge, M. (2015). *Critical and Creative Thinking in the Math Classroom.* Retrieved from https://thelearningexchange.ca/critical-and-creative-thinking-in-the-math-classroom

Ponte, da, J. P. (2008). Investigating mathematics: A challenge for students, teachers and mathematics education researchers. In B. Maj, M. Pytlak, & E. Swoboda (Eds.), *Supporting Independent Thinking Through Mathematical Education* (pp. 122–137). Rzeszów: Wydawnictwo Uniwersytetu Rzeszowskiego.

Pytlak, M. (2006). Uczniowie szkoły podstawowej odkrywają regularności [Primary school students discover regularities]. *Roczniki Polskiego Towarzystwa Matematycznego, Dydaktyka Matematyki, Seria V, 29,* 115–150.

Pytlak, M. (2008). Connections – as a fundamental element to constructing mathematical knowledge (exemplify of one pre-algebra task). In B. Maj, M. Pytlak, & E. Swoboda (Eds.), *Supporting Independent Thinking through Mathematical Education* (pp. 68–76). Rzeszów: Wydawnictwo Uniwersytetu Rzeszowskiego.

Sasman, M., Olivier, A., & Linchevski, L. (1999). Factors influencing students' generalization thinking processes. In O. Zaslavski (Ed.). *Proceedings of the 23th International Conference for Psychology of Mathematics Education* (vol. 4, pp. 161–168). Israel: Haifa.

Skurzyński, K. (1992). *Matematyka – nasza niedostrzegalna kultura* [Mathematics – our imperceptible culture]. Szczecin: Wydawnictwo Naukowe Uniwersytetu Szczecińskiego.

Stacey, K. (1989). Finding and Using Patterns in Linear Generalising Problems. *Educational Studies in Mathematics, 20*, 147–164.

Swoboda E. (2006). *Przestrzeń, regularności geometryczne i kształty w uczeniu się i nauczaniu dzieci* [Space, geometric regularities and shapes in learning and teaching children]. Rzeszów: Wydawnictwo Uniwersytetu Rzeszowskiego.

Urbańska, A. (2003). O tworzeniu się pojęcia liczby u dzieci [On the formation of the concept of number in children]. *Zeszyty Wszechnicy Świętokrzyskiej,16*, 51–71.

Zazkis, R., & Liljedahl, P. (2002a). Repeating patterns as a gateway. In *Proceedings of 26th Conference of the International Group for the Psychology of Mathematics Education* (Vol. I. 213–217). Norwich, UK: University of East Anglia.

Zazkis, R., & Liljedahl, P. (2002b). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49,* 379–402.

MANIFESTATIONS OF CRITICAL THINKING IN THE PROCESS OF SOLVING TASKS BY SEVENTH GRADERS

Edyta Juskowiak

Adam Mickiewicz University in Poznan, Poland

The article contains information about the preliminary results of the research conducted in a group of 14-year old students from four classes. The main research aims to check students' readiness to use formal operations to solve mathematical problems in the field of geometry. In addition, these solutions were analyzed for the characteristics of critical thinking.

INTRODUCTION AND LITERATURE OVERVIEW

Mathematical thinking is a way of understanding the world, its elements and relations between them (Devlin, 2019). It is something different from solving tasks, performing operations and applying procedures. Mathematical thinking has been developed by humankind for over 3000 years and brings many benefits not only to scientists. It is a particularly important skill in the world of the 21st century where virtually anyone can aspire to high positions in business or politics. Analytic thinking is highly valued wherever success, leadership, big money and power are mentioned. It turns out, however, that the smooth transition from mechanical performance of mathematical operations to truly mathematical thinking causes considerable problems for many people (Devlin, 2019).

Mathematical thinking is characterised by a whole set of intellectual activities undertaken by a person who solves a mathematical task. Manifestations of mathematical thinking are, e.g.:

- detecting and using analogies;
- schematisation and mathematization;
- defining, interpreting of a given definition and its rational usage;
- coding, constructing and rational usage of mathematical language;
- algorithmisation and rational usage of algorithms (Krygowska, 1986).

In the Regulation of MEN (Ministry of National Education) of December 23, 2008 regarding the core curriculum in pre-school education and general education in specific types of school, mathematical thinking is defined as an ability to use mathematical tools in everyday life and to form judgments based on mathematical reasoning. Below-mentioned Mason writes in his book that mathematical thinking is a dynamic process which broadens our understanding because it lets us manage more and more complex ideas (Mason, 2005).

The following factors have great impact on the efficiency of mathematical thinking:

- Ability to employ processes used in mathematical research.
- Control over mental and emotional states and ability to use them.
- Understanding of relevant mathematical field.
- Improvement of mathematical thinking achievable through concretisation, generalisation, putting forward hypotheses, justification.
- Provoking mathematical thinking facilitated by such activities as: creating a gap-challenge, surprise, contradiction, perceived gap.
- Supporting mathematical thinking by asking questions, posing and taking challenges, reflection.
- Maintaining mathematical thinking understood as increasing awareness of the processes, our own involvement, mental states (Mason, 2005).

Mogens Niss, however, defining the eight foundations of mathematical competences, prioritises mathematical thinking, whose one of the most important manifestations is the ability to formulate questions. The remaining foundations are:

- Formulating and solving mathematical problems.
- Mathematical modelling involving communication.
- Mathematical reasoning.
- Representing mathematical entities, i.e. understanding, interpreting.
- Using mathematical symbols and formalisms.
- Mathematical communication, i.e. understanding mathematical messages.
- Using auxiliary tools. (Niss, 2003)

Developing all aforementioned competences and activities and other manifestations of mathematical thinking should be simultaneously supported by the transferred knowledge.

Looking at all the above-mentioned features of mathematical thinking, we can notice a range of those which characterise critical thinking so, we could say, developing mathematical thinking in an appropriate and efficient manner we develop a student's/person's critical thinking. These activities are possible and desirable at every stage of intellectual development mentioned by Piaget (1966, 1977). At every of these stages, however, one has to use different methods and tools relevant to the perception capabilities characteristic for a given developmental stage.

It is difficult to define critical thinking in a clear manner. In one of their articles, Ma and Spector (2019) write that critical thinking has to be interpreted from three points of view: education, psychology and epistemology, and their impact on the development of critical thinking is illustrated in the chart no 1. It depicts a visual presentation of critical thinking in four areas:

1. skills – educational perspective,
2. predispositions – psychological perspective,
3. developmental stage – epistemological perspective,
4. time.

Taking time into account emphasises a dynamic nature of critical thinking in the context of a specific aspect and developmental approach.

In the chart we can easily see how developing competences typical for mathematical thinking influence the development of critical thinking.

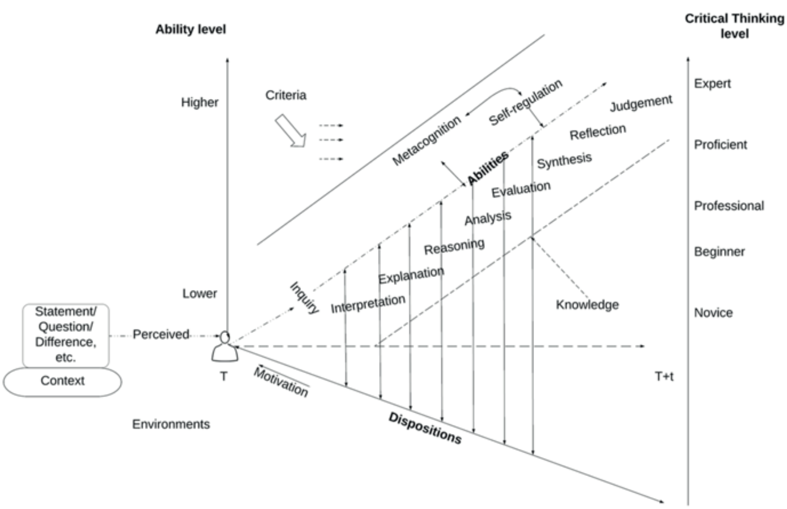


Figure1. Visual presentation of critical thinking.

Adopting the developmental approach to the critical thinking, Spector (2019) claims that critical thinking includes a range of cumulative and related skills, predispositions and other variables such as: motivation, criteria, context and knowledge, and he illustrates it on the diagram above. Developing critical thinking is based on experiences, e.g. observing something unusual or atypical and then, with the use of different forms of inference involving observation, reasoning, argumentation, proving, testing the conclusions and refection and coming to conclusions and answers.

The development of critical thinking often begins with simple experiences, such as observing differences, encountering puzzling questions or problems, questioning someone's claims which usually leads to more complex experiences requiring the ability of mathematical thinking at its higher levels, i.e. logical reasoning, questioning assumptions, considering and assessing alternative explanations. Knowledge and motivation for development are indispensable to trigger this type of thinking. If a person is not interested in what should be observed or studied, an attempt to solve the problem is not usually made. So creative reasoning and critical thinking require motivation and questioning attitude.

A research conducted by Darling-Hammond (2014) showed that in order to improve learning levels and reduce differences in pupils' achievements regular government investments in schools coping with high demands would be more efficient if the focus was shifted on the development of skills, broadening knowledge and teaching competences and the quality of core curricula. The Polish core math curriculum in primary schools includes competences which influence the development of critical thinking in the process of learning and teaching, that is:

I. Calculative ability.

2. Verification and interpretation of the results and assessment of the reasonableness of the solution.

IV. Reasoning and argumentation.

1. Conducting a simple reasoning, providing arguments justifying the correctness of the reasoning, distinguishing a proof from an example.

2. Noticing regularities, similarities and analogies and drawing conclusions based on them.

3. Applying strategies stemmed from the content of the task, devising strategies to solve the problem also in multistage solutions and those requiring an ability to combine knowledge of different fields of mathematics.

The aforementioned competences should substantively enforce and provoke mathematical thinking and critical thinking. These abilities are also verified during the tests – primary school final exam, e.g. in tasks involving inference or reasoning. A detailed analysis of most frequently used textbooks shows, however, that the number of tasks developing these abilities is far too small. There are too few open tasks at all education levels, tasks posing problems, tasks with an excess of data or their deficiency which provoke observation, experience, argumentation and research, tasks which are atypical, often new and cannot be subject to already known schemes.

The need for changes related to mathematical education, opening up to the exploitation and development of critical approach, both by students and teachers, is visible in the following statements:

> There is an obvious contradiction between a widely accepted need for training of staff having a creative approach, reflected in official documents or popular slogans and, on the other side, the teaching practice which does not prepare the youth to act creatively and does not trigger an innovative attitude, relying rather on the receptive attitude toward teaching also in the field of mathematical education (Klakla, 2003).
>
> Math lessons are usually based on the quick series of questions and answers which totally reduces time and space necessary to develop mathematical thinking. A belief that repeating the same exercises which are done quickly by the students conduces to the development of mathematical thinking is equally harmful. Practice requires time to address each problem independently and the quality of reflection depends on the possibility to carefully consider the solution, look into the alternatives and generalisations (Mason, Burton, & Stacey, 2005).
>
> I think that apart from a lack of sufficient mathematical knowledge, a shortage of educational knowledge among teachers and a tendency to limit the process to the presentation of "ready maths" might be mentioned (Klakla, 2006).
>
> Criticism is one of the manifestations of creative mathematical activity. This kind of mathematical activity consists in overcoming a conflict between the prerequisites of formal thinking and its other strands. These strands of mathematical thinking may include e.g. intuition, a firmly rooted habit or a suggestion of name whose colloquial meaning may be similar but not identical to its mathematical meaning (Klakla, 2003).

Such conflicts may be observed in the process of solving tasks including problems, in all types of reasoning or inferring, they are often incidental to geometry learning and in the processes where visualisation of mathematical content in the form of figures or diagrams plays an important role.

METHODOLOGY, RESULTS AND SHORT DISCUSSION

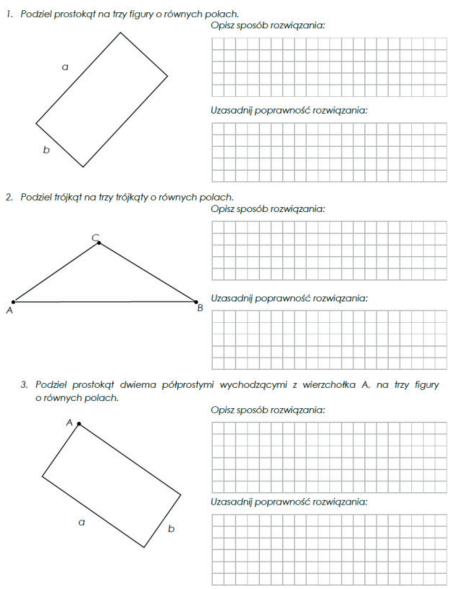
This article presents and discusses examples of the solutions to three geometric tasks provided by four seventh graders (14 years old) selected randomly out of 78 students.

The research was conducted at the turn of 2019 and 2020. The discussion will refer to manifestations of the above-mentioned mathematical activities or features specific for critical thinking perceived in the solutions. Presented works come from a more comprehensive study conducted by the author of this article in a group of almost two thousand students aged 14 who solved 6 geometric tasks requiring justification. The main goal of the research was to check the students readiness to show formal mathematical thinking.

The author of the paper in her main research made an attempt to examine the manifestations of formal reasoning, to diagnose and describe the ways of solving tasks with the instruction "show that" by the Polish seventh graders at the end of their school year.

The aim was, inter alia, to check whether the students use symbols in relation to abstract concepts logically, use the definitions correctly, use hypothetical-deductive reasoning, reason correctly, use abstract objects or build primarily on their experiences and observations and particularly on specific acts in the process of solving mathematical tasks?

The article analyses solutions to three tasks presented below.



1.Divide the rectangle into three figures with equal areas.
Describe the solution of the task:
Justify the correctness of the solution:

2. Divide the triangle into three triangles with equal areas.
Describe the solution of the task:
Justify the correctness of the solution:

3.Divide the rectangle into three figures with equal areas using two rays coming from the apex A.
Describe the solution of the task:
Justify the correctness of the solution:

Figure 2. Worksheet.

The tasks are atypical for a student of Polish school. Students are expected to provide not only a solution but also a justification of the correctness of the applied reasoning or the attempts or their judgments. The tasks at the whole worksheet are also a probe showing if the above-mentioned factors which influence the efficiency of igniting and developing mathematical thinking were present in the process of learning and teaching.

These three tasks for seventh graders are atypical because of the requirement for justification of the correctness of a previously presented solution. This requirement is seldom imposed during the classes of mathematics in Poland, a Polya’s (2009) “glance backward” is rare (Maj 2009; Klakla, 2003, 2006; Mason, 2005).

Let us recall that it indeed involves a justification of the correctness of a method or knowledge in use or a choice of another way of working or, finally, a verification of the result which is the most frequent case. In each of the three tasks a student can use one of two strategies: either properly divide a chosen side of the figure and justify that areas of the resulted figures are identical, or find an appropriate way of dividing a chosen side through modifications of the area formula for the figure. Task no 1 and task no 2 require basic knowledge about the properties of rectangles and triangles. Task no 3 is not a typical task for a seventh grader but, on the other hand, it is the one which can be solved with the use of some conclusions/ideas related to the solutions of previous two tasks.

It was decided that in the process of worksheets analysis each, even the smallest manifestations of mathematical and critical thinking are to be considered.

They were to include:

1. a proper justification of the correctness of the presented solution,
2. an attempt to justify the correctness of the presented solution,
3. a typical, i.e. unrelated to school schemes and unconventional attempts to find a solution or to justify it,
4. correct and logical use of knowledge with the aim of solving the task,
5. dilemmas and reflection when deciding about the distinction between the solution and the justification,
6. attempts to find a solution of the task despite the lack of knowledge and ideas even if they are only some draft drawings,
7. reflection on students' own knowledge or ignorance and searching for their reasons.

DISCUSSION ABOUT THE RESULTS

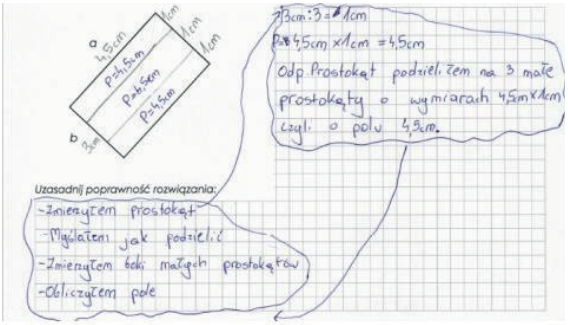
Despite the first impression that students present only the results of their ignorance on the worksheets because there was lack of solutions, partially correct and wrong answers, a deeper reflection on what was written reveals a hidden potential of the students, i.e. knowledge which they cannot use effectively, ideas which they give up and attempts to solve the task despite a lack of tools or a lack of knowledge about using those which they have.

With each task there will be students' ideas to solve the task presented with assigned names altogether with some brief comments and an attempt to assess whether they can be considered as manifestations of mathematical or critical thinking. The discussion of the results will be followed by a chart presenting the number of solutions in several categories.

Task no 1

This task was solved by the maximum number of students, 86% of all mentioned in this paper. They proposed different ways of dividing the rectangle and all of them were correct. The justification caused more trouble and either it was missing, or it was wrong or partially correct.

Task solved correctly or correctly in part (with justification)



Solution/Justification: I divided the rectangle into three small rectangles with area 4.5 cm and dimensions: 4.5 cm x1 cm.
Solution/Justification:
- I measured the rectangle.
- I was wondering how to divide.
- I measured the sides of small rectangles.
- I calculated the area.

Figure 3

The student solved the task on the basis of real object, he divided the side of the rectangle which can be divided into three equal parts without remainder. He named the set of the instructions and performed actions and solved the task in compliance with school schemes providing even an answer. He had a dilemma about the distinction between the solution and the justification.

Solution: I measured. I divided into three equal parts with sides a and b.

Figure 4

The student divided one of the sides into three equal parts and informs that he measured the side. He does not feel the need to count the area of rectangles, nor to argue in any other way that the areas of rectangles will be equal. He writes, however, that the rectangle with sides a and b will keep its area unchanged, which is true but constitutes an answer to a question which was not asked. The student adopts an active and reflective attitude towards the task.

Task solved correctly or correctly in part (without justification)

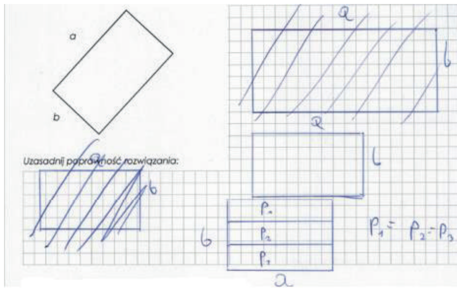
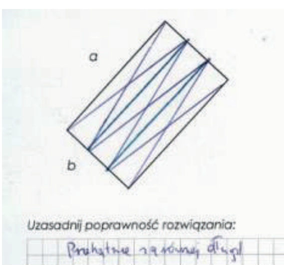


Figure 5

Sometimes the position of the figure does not allow to solve the task. This student had to see the figure horizontally on a squared sheet of paper in order to demonstrate the equality of areas in relation to the equal number of squares. This track of thought is implicit and construable from visible elimination of previous examples. There is no comment or any justification. The student is not able to “get away from” the object.

An atypical way to solve the task



Justification: Diagonals are of the same length

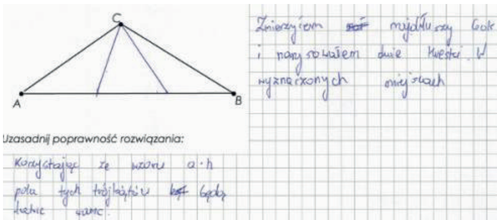
Figure 6

Non-standard tools were used to justify the correctness of this result. The justification is incomplete but the approach was correct. The student wrote that the lengths of diagonals are equal but what does it mean? There are no further conclusions.

Task no 2

Although the students found the task difficult, they made attempts to solve it writing comments or making rough notes. They divided one of the side into three equal parts but they could not justify why it is a correct solution. There were some proposals to divide the triangle into three smaller ones without dividing one of the sides and using all sides as new bases of smaller triangles. Tasks containing a justification were very rare.

Task solved correctly or correctly in part (with justification)

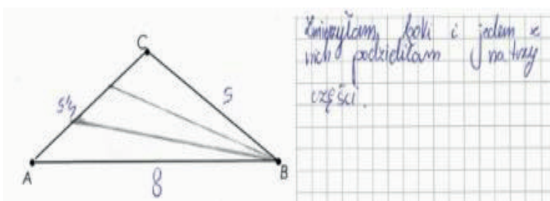


Solution: I measured the longest side and drew two lines. In the marked places. Justification: I used the formula a, x, h, the areas of these triangles will be the same.

Figure 7

The student proposes a solution and a justification which is almost correct. There is no information why the h is so crucial in the formula.

Task solved correctly or correctly in part (without justification)

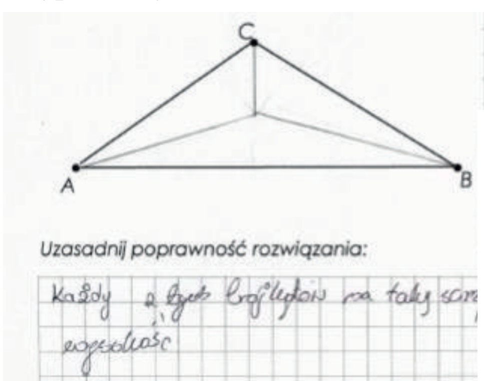


Solution: I measured the sides and divided one of them into three parts.

Figure 8

A justification of the correctness of the solution is missing but the idea produced by the student shows that he/she is aware of the fact that dividing any of the sides of a triangle into three equal parts always results in getting three triangles with equal areas.

An atypical way to solve the task



Justification: Each of these triangles is of the same height.

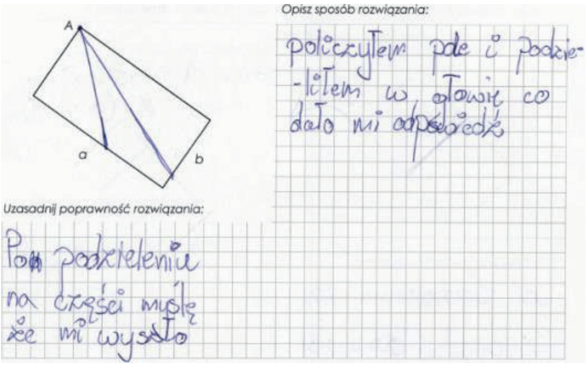
This solution of the task was very rare. Maybe the heights of the triangles are even equal but is it sufficient to satisfy all the requirements? Maybe an initial assumption that the auxiliary figure shows an equilateral triangle is completely wrong.

Figure 9

Task no 3

This task was solved by the minimum number of students, only 3 out of 78 in the research group. Students made some attempts to draw and analyse the proposed positions of the rays in relation to questions posed in the instruction.

Task solved correctly or correctly in part (with justification)



Solution: I calculated the area and divided in mind which resulted in a solution. Justification: I think I succeeded after having divided into parts.

Figure 10

The student writes – I divided in mind and I think I succeeded. Polish students often check the results of their work through checking the correctness of a solution to the task, the solutions are provided at the last pages of textbooks. If this possibility is excluded, then the reflection on the correctness of the proceeding is often found unnecessary. The presented attempt illustrates a correct approach.

A typical way to solve the task

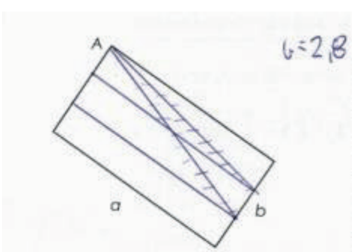


Figure 11

The student made an attempt to use the method applied in the solution to the task no 1, it is a unique example of transferring the method in considered solutions to task no 3. It is a starting point for solving the task. The first attempt was rightfully rejected. There are, however, no more signs of further trials. By his/her notes, the student manifests the analytical thinking.

Quantitative data analysis

Task no	Grade (number of students)	Task solved correctly or correctly in part	Solution with justification	Lack of solution	Wrong solution or any attempts to solve the task
1	I (20)	14	9	6	-
	II(17)	16	10	1	-
	III(21)	21	16	-	-
	IV(20)	16	7	1	3
Total	78	67	42	8	3
2	I(20)	3	1	17	-
	II(17)	3	2	12	2
	III(21)	13	5	2	6
	IV(20)	11	4	-	9
Total	78	30	12	31	17
3	I(20)	3	1	13	4
	II(17)	-	-	11	6
	III(21)	-	-	10	11
	IV(20)	1	1	7	12
Total	78	4	2	41	33

Figure 12. Quantitative data of solutions.

The research participants often left comments on the worksheets saying that they have not been faced with tasks provoking thinking conflicts almost at all although they are already seventh graders. This is reflected in data included in the chart (fig. 12). Typical tasks (task no 1), consistent with students intuition, were solved; however, they were rarely accompanied by justification. Those which required creative engagement in the form of the transfer of a method, using analogies between solutions, reflection, reasoning, inference and thinking independently from illusions and false intuition related to particular cases (task no 2 and 3) were difficult and described by students as unsolvable.

SUMMARY

Critical thinking is considered to be valuable in all disciplines but despite some provisions in different documents like curricula or core curricula about developing their manifestations, the courses focusing on this issue are rare. As it was mentioned, the process of the development of mathematical thinking, stimulated in an effective way, requires the development of critical thinking.

The analysis of the solutions to the tasks presented in this article shows the difficulties which the students may have when implying mathematical activities in the process of solving an atypical mathematical tasks, choosing an effective reasoning, justifying their judgments, reflection, choosing an appropriate method or providing comments and, finally, asking a question.

These abilities should be acquired by students in parallel with the process of acquiring mathematical knowledge and the students should face different challenges and new situations at each level of their intellectual development. It is visible, however, after a deeper analysis that students are willing to develop and provoke mathematical thinking.

References

Darling-Hammond, L. (2014). What can PISA tell us about US education policy? *New England Journal of Public Policy, 26,* 1-14.

Devlin, K. (2019). *Introduction to Mathematical Thinking*. Gliwice: Helion.

Juskowiak, E. (2019). "Using geometry, justify (…)". Readiness of 14-year-old students to show formal operational thinking. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 217-224). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Klakla, M. (2003). Dyscyplina i krytycyzm myślenia jako specyficzny rodzaj aktywności matematycznej [Discipline and criticism of thinking as a specific type of mathematical activity]. *Studia Matematyczne Akademii Świętokrzyskiej, Tom 10, Kielce,* 89-106.

Klakla, M. (2006). Gotowa wiedza i aktywność w matematycznym kształceniu na przykładzie kątow Langleya [Ready knowledge and activity in mathematical education on the example of Langley angles]. *36 Annales Academiae Paedagogicae Cracoviensis, Studia ad Didacticam Mathematicae Pertinentia I*, Krakow: Wydawnictwo Naukowe Akademii Pedagogicznej, 75-84.

Krygowska, Z. (1986). Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich [Elements of mathematical activity that should play a significant role in mathematics for all]. *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki, 6,* 25-41.

Mason, J., Burton, L., & Stacey, K. (2005). *Matematyczne myślenie [Thinking Mathematically]*. Warsaw: WSiP.

Ma, S., Spector, J. M. (2019). Inquiry and critical thinking skills for the next generation: from artificial intelligence back to human intelligence. *Smart Learning Environments, volume 6*.

Maj, B. (2009). *Rola zadań wieloetapowych w rozwijaniu aktywności twórczej nauczycieli matematyki [The role of multistage tasks in developing creative activities among mathematics teachers].* Unpublished Doctoral dissertation, Pedagogical University in Krakow, Krakow, Poland.

Niss, M. (2003). Quantitative Literacy and Mathematical Competencies. In B. L. Madison & L. A. Steen (Eds.), *Quantitative Literacy. Why Numeracy Matters for*

Schools and Colleges. Proceedings *of the National Forum on Quantitative Literacy held at the National Academy of Sciences in Washington* (pp. 215-220).

Piaget, J. (1966). *Origins of Intelligence in Children*. Warsaw: PWN.

Piaget, J. (1966). *Child Psychology*. Warsaw: PWN.

Piaget, J. (1977). *Where Education is Going*. Warsaw: Wydawnictwo Naukowe PWN.

Polya, G. (2009). *How to solve it?* Warsaw: Wydawnictwo naukowe PWN.

Spector, J. M. (2019). Complexity, Inquiry Critical Thinking, and Technology: A Holistic and Developmental Approach. In T. Parsons, L. Lin, & D. Cockerham (Eds.), *Mind, Brain and Technology. Educational Communications and Technology: Issues and Innovations* (pp. 17-25). Cham: Springer.

INTUITION AND REASONING – ANALYSIS OF CRITICAL THINKING OF SECONDARY SCHOOL STUDENTS BASED ON PARADOXES AND SOPHISMS

Mirosława Sajka

Pedagogical University of Krakow, Poland

The topic of the paper is related to the problem of examining and shaping critical thinking in secondary school students. The research was conducted to analyse the students' ability to assess the correctness of statements and reasonings, especially when influenced by intuition. With these aims in mind, the research tool was composed with the use of sophisms and paradoxes. Firstly, the students assessed the correctness of the statements, and then were asked to read a reasoning, assess its correctness, and, finally, reassess the correctness of the statements. The results of this pilot study have shown that the reasoning attached to the statements, both paradoxes and sophisms, caused the majority of students to change their assessment of the correctness of the given sentences.

INTRODUCTION

All researchers in the field of didactics of mathematics and teachers of mathematics unanimously recognize that critical thinking, to which the current monograph is devoted, is a key skill in everyday human life and a skill that should be intensively shaped in students during mathematics lessons. Our study refers to the interrelationship between reasoning and intuition in decision-making. There are many studies that identify the interrelations also in the field of mathematical education. Due to the limited size of this chapter, we cannot discuss them in detail. However, it is worth mentioning here that e.g. Kahneman (2011) presented a model of human cognition based on two modes or 'systems' of thinking: *System 1* thinking that is *fast and intuitive* and *System 2* thinking that is *slow and tedious*. This model was developed in cognitive psychology and on its basis researchers in mathematics education have offered *dual-process theory* (DPT) (Leron & Hazzan, 2009).

Much earlier Fischbein (1987) distinguished between primary intuitions which "develop in individuals independently of any systematic instruction as an effect of their personal experience" (p. 64) and secondary intuitions which "are acquired, not through natural experience, but through some educational interventions" (op. cit., p. 71). Vinner (1997) proposed an alternative theoretical framework to explain for what he calls "meaningless behaviors" in the context of mathematics. He defined *pseudo-analytic processes* in which students superficially select elements in the problem and apply a procedure relevant for a typical question due to superficial similarity with previous problems. Vinner suggested that these pseudo-processes are "simpler, easier, and shorter than the true conceptual processes" (Vinner, 1997, p. 101), thus many students unconsciously apply them. Klakla, in turn, draws attention to yet another aspect of the relationship between intuition and reasoning. His research (Klakla, 1982) on the identification of mathematically talented students states that the so-called "discipline of thinking" (Klakla, 2003) is one of the factors of mathematical activity. As defined by Klakla (1982), this concept is understood as "the ability to overcome the conflict between the requirements of formal thinking and its other tendencies, e.g. intuition" (p. 48).

SOPHISMS AND PARADOXES

Defining a paradox

Many times, fixed beliefs and assumptions can lead us to wrong conclusions. When we realise this, we speak of a paradox. More precisely, when, contrary to our intuition, the assertion made is true. A paradox can be defined as "a formulation containing a striking, surprising thought that is at odds with commonly held beliefs" (Kawałek, Bać & Pabich, 2011, p. 41). Kowal (1969) distinguishes statements that "seemingly contradict commonly accepted judgments, nevertheless are true" (p. 77) and calls them paradoxes. Pogonowski (2008) notes that "Paradoxes modify our intuitions" (p. 7), therefore we can and should use them for didactical purposes.

Defining a sophism

The history of sophisms dates back to remote times, and the concept itself has often been subject to heated debate. Originally used as a means of entertainment and exercising the mind, it involved finding a deliberately placed error in reasoning. Over time, it was used to prove a point and to prove even the most absurd theses. Teachers then began to be called sophists, who, among other things, verified the correctness of statements and arguments. The meaning and role of sophism was discussed in mathematics: "[...] consists precisely in such concealment of an intended error that it remains unnoticed for the time being, and readers or listeners notice it only when the result is absurd [...]" (Lietzmann, 1958, p. 74), Pogonowski (2008) defines a sophism as "reasoning that has the appearance of being correct, but (after appropriate analysis) turns out to be incorrect" (p. 4) and Tworak (2012) describes it as "an apparently correct argument presented with the intention of embarrassing or misleading someone, usually based on some dishonest trick" (p. 99). Thus, it is important to be able to think critically, correctly analyse and draw conclusions and argue. The activity of assessing the correctness of sophisms and finding errors support the development of critical and logical thinking in students.

METHODOLOGY

Aim of the study

The research described in this article attempts to investigate how secondary school students use critical thinking skills for analysing and evaluating statements and reasonings, and what guides the students when drawing conclusions: whether it is reasoning, intuition, or, possibly, other factors. The aim of the study is to try to acquire preliminary answers to the following questions:

a) Do reasonings in the form of sophisms or justifications of paradoxes have the effect of changing a previously-made assessment of the truthfulness of the statements?
b) Do students treat the presented reasoning (regardless of its correctness) as unquestionable evidence in assessing the truthfulness of the statements?
c) Do students recognise correct reasoning as evidence for the truth of an intuitively false statement?
d) Can students recognise correct proof reasoning?
e) Can students indicate any errors in reasoning?
f) How can the error detection of the students be described?
g) Does the mathematical experience of the students have an impact on the development of critical thinking related to the analysis of sophisms and paradoxes?

Research tool

Due to the pandemic restrictions in May 2020, the survey was conducted online, using two questionnaires (Parts I and II). Prior to the core moment of the survey, the students had answered some questions regarding their level of learning of mathematics (basic/extended) and their number in the logbook, in order to identify the answers of specific students from both parts. In the first stage, students assessed the truthfulness of six sentences, using their previous mathematical knowledge and intuition.

PART I

Assess the truthfulness of the following sentences:

1. The solution of the equation $\sqrt{x+3} = 3 - x$ is the number 6. T/F
2. The number of points on the segment AB is equal to the number of points on the segment AC, where $AC = \frac{1}{2}AB$. T/F
3. If $a, b > 0$ and $a > b$, then $a > 2b$. T/F

4. The sum of positive numbers can give a negative number. T/F
5. The number 0,(9) is equal to the number 1. T/F
6. The probability that among 22 players on the pitch, at least two of them share a birthday is equal to approximately $\frac{1}{2}$. T/F

Among these statements there are three false sentences (1, 3, 4) as well as three true sentences (2, 5, 6), in which intuition may interfere and cause misjudgements (we treat these sentences as paradoxes).

The second stage was prepared on the basis of the same tasks, but with an attached reasoning – (pseudo)justification – so, depending on the sentence, it presented either a fallacious reasoning – a sophism (if the initial sentence was false), or a proof-explanation of a paradox, if the sentence was true. The students' challenge was to analyse the reasoning attached to these statements, assess the correctness of the reasoning presented, and reassess the truthfulness of the statements. The next activity concerned finding errors in reasoning and explaining their reasons. This question was answered by students who had marked the reasoning as "incorrect". The construction of the research tool on the example of statement no. 4 (Section IV) is shown in Figure 1. Statement no. 5 was intentionally justified using a similar method (Fig. 2). Secondary school students who were learning mathematics at the expanded level could also justify it by using the concept of convergent geometric series.

SECTION IV

Is the following reasoning correct?

1. $x = 1 + 2 + 4 + 8 + 16 + 32 + \cdots$
2. $x = 1 + 1 \cdot 2 + 2 \cdot 2 + 4 \cdot 2 + 8 \cdot 2 + 16 \cdot 2 + \cdots$
3. $x = 1 + 2 \cdot (1 + 2 + 4 + 8 + 16 + 32 + \cdots)$
4. $x = 1 + 2x$
5. $-x = 1$
6. $x = -1$

YES	NO
↓	↓
What do you think of the statement presented along with the reasoning?	On which line is the mistake and why?
	↓
	What do you think of the statement presented along with the reasoning?

4. The sum of positive numbers can give a negative number. T/F

Figure 1: Section IV on statement no. 4 from questionnaire (Part II)

1. $x = 0{,}9999\ldots$
2. $x = 0{,}9 + 0{,}09 + 0{,}009 + \cdots$
3. $10x = 9 + (0{,}9 + 0{,}09 + 0{,}009 + \cdots)$
4. $10x = 9 + x$
5. $9x = 9$
6. $x = 1$
7. Thus $0,(9) = 1$

Figure 2: Justification of truthfulness of statement no. 5 from questionnaire (Part II)

The sophism from section IV (Figure 1) is based on an incorrect assumption that the notation 1+2+4+8+16+32+⋯ represents a real number, what refers to conceptual difficulties with the infinity notion. Similarly, the justification for statement no. 2 was connected with the notion of the cardinality of a set. The two reasonings were the most difficult and challenging for secondary school students. On the other hand, statement no. 4 was so absurd that the students should have questioned this sophism, and statement no. 2 should have been a known fact, taught as part of the school course of mathematics. Verifying the remaining reasonings and finding errors was fully accessible to students at this educational level in Poland. Figure 3 shows the remaining two sophisms.

1 $\sqrt{x+3} = 3 - x$
2 $(\sqrt{x+3})^2 = (3-x)^2$
3 $x + 3 = 9 - 6x + x^2$
4 $x^2 - 7x + 6 = 0$
5 Numbers $x_1 = 1$ and $x_2 = 6$ are solutions.

1. $a > b$
2. $ab > b^2$
3. $ab - a^2 > b^2 - a^2$
4. $a(b-a) > (b+a)(b-a)$
5. $a > a + b$
6. We are adding the inequalities:
   $a > a + b$
   $a > b$
7. $2a > 2b + a$
8. $a > 2b$

Figure 3: Sophisms from questionnaire on statements no. 1 (left) & no. 3 (right)

The final stage of the questionnaire was to collect questions about the research tool as a whole:

Q1: Did any of the tasks particularly amaze or surprise you?

Q2: What reflections or comments do you have after completing this questionnaire?

Q3: What do you think was the purpose of filling in this questionnaire?

Q4: Do you have any doubts about the questionnaire?

SELECTED RESULTS

The assessment of truthfulness of statements from Part I and II of survey differed significantly and are shown in Figure 5. Firstly, Figure 4 presents the same results in the context of percentage change of respondents' opinion in assessing truthfulness of subsequent statements under influence of presented reasonings. Whereas Figure 6 presents the summary of students' assessment of correctness of the provided reasonings.

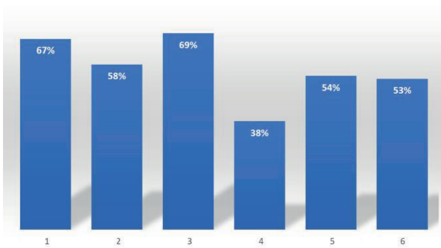


Figure 4: Percentage change of respondents' opinion in assessing truthfulness of subsequent statements under influence of presented reasoning

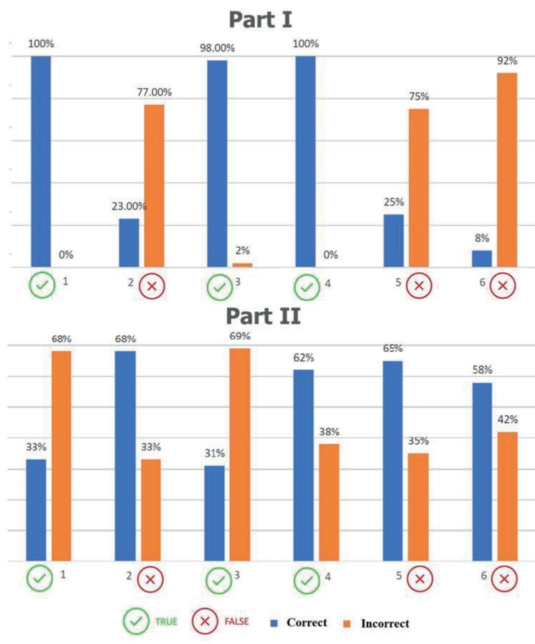


Figure 5: Percentage summary of students' responses in assessing the truthfulness of statements from Part I and II of survey

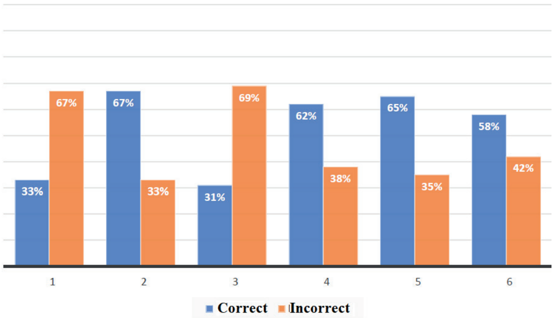


Figure 6: Percentage summary of students' correct and incorrect answers in assessing the correctness of reasoning

Students' opinions about the research tool

Interesting answers were provided to questions 1-3. We will cite some of them, starting with Q1:

> Very interesting solutions. I began to appreciate the work of the teachers who constantly have to look for our mistakes and are probably as surprised as I am when looking at some solutions.

> I would prefer assignments like this at school, where I have to look for a mistake or find out if something is true, it's a bit like solving a puzzle and not just the same tasks over and over again that need to be solved.

The answers to Q3 were unanimous among the majority of respondents. When considering the purpose of completing the questionnaire, the pupils mentioned checking their ability to find errors, analysing solutions, checking whether the included reasoning helped them decide whether it was correct. A few of the students grasped the link between the two parts of the questionnaire. The respondents' answers overlap to a large extent with the objectives set by the author of this paper. Exemplary answers include:

> Showing that tasks which are surprising at first sight can have a logical solution, and providing an explanation for this. And, on the other hand, showing that simple mathematical tasks can have solutions with errors in them, which (as I have found out myself) are not always easy to find.

> To see if we change our minds about the correctness of the tasks, after seeing the (correct) solutions. And in the tasks with incorrect solutions, to see if we can find the error.

> Showing that there can always be a solution that is wrong but the error is not visible. Therefore, checking whether someone will think that some tasks are logically incorrect or will just look at the solution and believe it.

> Looking for errors in proofs and solutions to mathematical tasks.

> Pointing out how even a small oversight can lead to big mistakes.

> As in earlier answers. Showing that anything can be proved, and the student must be careful and look for mistakes even when provided the "proof" of a statement.

Among those tested were only 5 students who misread the objectives of the study, showed a tendency to treat any reasoning presented as evidence, and, unfortunately, assimilated the incorrect solutions, e.g:, "to show that without the solution I know nothing", "That in 1 I knew nothing, and now I know because I have the solution", "tricky tasks that look false without a solution", "Yes, I think the aim was to see if we would answer the same if we had proof and if we didn't."

RESULTS' ANALYSIS AND DISCUSSION

The analysis of the results obtained from the conducted research allowed to obtain preliminary answers to the posed research questions (which will be abbreviated as Re.1, Re.2, etc.).

Re.1. The overall results from all research groups allowed us to draw a conclusion that, to a significant degree, the included reasonings, both paradoxes and sophisms, influence the students' inferences. For almost 60% of the respondents, the reasonings included in the questionnaire determined the final evaluation of the truthfulness of the statements. Slightly more changed their decision to incorrect one, the influence of sophisms was noted in 58% of the research group, and 55% for paradoxes.

Re.2. Such a sizeable change in answers in the second part of the worksheet shows that students mostly treat the presented reasoning as evidence of the truth of the statement. At this point, special attention should be paid to particular reasonings which were not proofs, but were considered as such. Some students decided on the correctness of the reasoning after noticing a familiar expression, without further analysis of its correctness, and thus considered the presented reasoning as evidence. Task one (Fig. 3, left) contained responses such as: "the way you can do it and calculate it using a delta is not that difficult", suggesting that since there is a quadratic equation, the task is easy and the reasoning is true. In this section there was also a response directly confirming the listed conclusion, i.e., "I don't remember how it was calculated, but since it is solved, then it is true". These examples correspond to Vinner's view on *pseudo-analytic processes*.

A large group treated the sophism on statement no. 3 (Fig. 3, right) as proof. The students did not analyse the posted solution for correctness, arguing that this type of task is difficult and certainly correct, since it leads to the expected result.

Even the reasoning for one of the most absurd statements (no. 4, Fig. 1) was considered by some respondents as proof. Students used expressions such as: "even something illogical can be proven" and "It seems impossible and yet it is proven".

Some reactions to the paradoxes presented in the questionnaire contributed to the clear conclusion that the students tend to treat the reasoning presented in writing as irrefutable proof of the truth of the statement. The majority of the respondents treated the statements as not obvious, untrue, or unrealistic, but eventually, under the influence of the reasoning provided, they accepted them as true. In the comments, we read, e.g., “it seems impossible, and it is shown to be so”, although sometimes this approach was accompanied by doubt: “Surprisingly, the solution, when analysed, is correct, which makes for a surprising answer, or the error is very, very well hidden”.

Re.3. The analysis of the collected results showed that in each of the given paradoxes (2, 5, 6), more than half of the respondents changed their answer to the correct one under the influence of the reasoning provided in Part II. Exactly 55% of the respondents recognized the truth of the statements, despite their initial negative assessment. This means that the students tend to treat reasonings, and not their intuition, as evidence. The students made this choice in three situations: (a) when they did not find any faults in the reasoning, (b) when they knew of or have seen a similar task of this type and (c) when they treat every solution as proof in advance, e.g. “[...] if it is proved then it must be true”.

Re.4. This question cannot be answered conclusively on the basis of this research. The results confirm the ability to identify flawless reasoning. For 62% of the respondents, the reasoning attached to the paradoxes was correct. However, a deeper analysis of the collected answers raises many doubts. As mentioned in the previous sections, the students seemed to treat the reasoning as proof of correctness. Thus, for the most part, the considerations, even incorrect at times but containing the final and expected result, were also perceived by students as correct. This is also confirmed by the fact that 58% of the respondents considered sophisms, fallacious reasoning, as correct.

Re.5. The research questionnaire included 3 tasks which should have had their correctness questioned by the students. Only 42% of the respondents considered them to be incorrect. Sometimes the respondents could not identify what the error was or where it occurred. They considered the reasoning to be wrong according to their intuition, influenced by ‘illogical’ - as reported by students - statements. One of the questions about the place and cause of the error (statement no. 4, Fig. 1) was not answered correctly by any student, which, as mentioned above, was predictable. It was the most difficult task, and the error was not directly related to the arithmetic operations contained therein, instead being related to the concept of infinity.

In the other two sophisms, there was a group of students who were able to find the error, although it was correctly defined only by some, especially for statement no. 1 (Fig. 3, left). In their answers, the students drew attention to writing down the assumptions, sometimes called them the domain of the equation. A considerable number of students found and described the error in sophism based on statement no. 3 (Fig. 3, right) without any problems, i.e., by changing the sign of inequality when dividing by a negative number on both sides, which resulted directly from the assumptions. Although some students did not have the opportunity to find the error in the remaining reasoning, as it did not exist, they tried to do so. The inconsistencies that were highlighted by these students related mainly to the lengths of the segments, e.g., "The segments are not equal to each other, so the number of points on them will not be equal to each other either", the inequality of two numbers (statement no. 5), "in the last line, because 0.(9) in the period is almost 1 but it is not equal" and an overly large and therefore improbable result.

Re.6. When looking for errors, the students either drew attention to the lack of assumptions, defining the domain, and the consequences that are obtained without including them in the evidence, or described the changes that need to be made in order to obtain the correct result. The explanations of some of the participants reveal their curiosity about the error found in the reasoning, or even pride in having located it correctly. The most emotional was the task in section IV (Fig. 1), where none of the students were able to realise what the false step was. While some were agitated by the reasoning involved, describing it as 'ridiculous' or 'unrealistic', others expressed great interest, to the point of seeking information about it in other available sources. There were some incorrect answers, where the students pointed out alleged errors that were merely due to a lack of knowledge and malformed understanding of mathematical concepts, such as:

> The error here is that only a few points are marked and the lengths of the segments are not given. Let's make the longer 6 cm and the shorter 3 cm, then 6 points will fit on this long segment, with 3 points on the shorter segment. [...]

> [...] the shortcut multiplication formula is wrong, it should be the other way around (a-b).

In summary, the collected results showed that it was much easier to find an incorrect entry in the reasoning than to give the reason for the error, and the process of finding the error itself evoked many extreme emotions.

Re.7. The huge impact of faulty reasoning on the conclusions can be seen in students in classes at the elementary level of mathematics. As many as 70% changed their answer to the incorrect one. The surveyed students learning the core curriculum in mathematics at the extended level were definitely more careful in analysing the presented reasonings. There is a huge difference between the levels in relation to the influence of paradoxes and sophisms on reasoning. Respondents from the primary level were far more likely to treat any reasoning as proof, whereas students from the extended level tended to question the veracity of even the correct reasonings, therefore there was little influence on the respondents in terms of inference and sentence evaluation. These students analysed the presented reasoning critically and drew conclusions carefully, which is certainly a useful skill in activity of this type, being able to resolve their uncertainty regarding the truth of statements and the correctness of reasoning by reaching for other sources of knowledge.

With the exception of the 2nd year of secondary school with an extended mathematical profile, the results of the other groups remain at a similar level in Sections 4-6. Sections 1 and 3 show the biggest impact of the presented reasonings on the answers of the 2nd year of secondary school group at elementary level in comparison to other groups. Concerning the three classes mentioned above, it can be inferred that, on the one hand, faulty reasoning had the greatest impact on the evaluation of the truthfulness of the sentences, and, on the other hand, correct reasoning had the lowest impact just in the second year of high school. Also, the students from the penultimate year of secondary school with an extended mathematical profile showed the lowest percentage of change in the students' answers in each task.

The above information shows that the level of implementation of mathematics content has a positive impact on critical thinking.

SUMMARY

Summing, it is difficult to unequivocally state what secondary school students are guided by in assessing the veracity of sentences. Whether they follow their intuition or reasoning, which is shown to not always be correct, depends on individual preferences and the type of task. The questionnaire was completed by students who treated each reasoning as evidence as well as students to whom even the presented evidence was not enough of an argument for the truth of some statements. The respondents manifested difficulties in defining the reason for the error, and sometimes in finding it.

Certainly, in the process of teaching-learning mathematics, more emphasis should be placed on tasks related to verifying solutions for their correctness.

Acknowledgment

This article was created on the basis of research designed and conducted under the guidance of the author by Izabela Łaskawska, a student of the Pedagogical University of Kraków, as part of her master's thesis.

References

Fischbein, E. (1987). *Intuition in science and mathematics.* Dordrecht, the Netherlands: Reidel.

Kahneman, D. (2011). *Thinking, Fast and Slow*. London, UK: Penguin Books.

Kawałek, A., Bać, W., & Pabich, B. (2011). *Zadania nie muszą nudzić* [Tasks need not be boring.]. Rzeszów, Poland: Podkarpackie Centrum Edukacji Nauczycieli.

Klakla, M. (1982). Z badań nad rozpoznawaniem uzdolnień matematycznych [From research on the recognition of mathematical giftedness]. *Dydaktyka Matematyki, 2*, 33–81.

Klakla M. (2003). Dyscyplina i krytycyzm myślenia jako specyficzny rodzaj aktywności matematycznej [Discipline and critical thinking as a specific type of mathematical activity]. *Studia Matematyczne Akademii Świętokrzyskiej, 10,* 89–106.

Kowal, S. (1969). *500 zagadek matematycznych* [500 mathematical puzzles]. Warsaw, Poland: Wiedza Powszechna.

Leron, U., & Hazzan, O. (2009). Intuitive vs. analytical thinking: Four perspectives. *Educational Studies in Mathematics*, *71*, 263–278.

Lietzmann, W. (1958). *Gdzie tkwi błąd?* [Where the error lies?]. Warsaw, Poland: PZWS.

Pogonowski, J. (2008). *Paradoksy a intuicja* [Paradoxes versus intuition]. Poznań, Poland: UAM.

Tworak, Z. (2012). *Paradoksy* [Paradoxes]. Poznań, Poland: UAM.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, *34*, 97–129.

Critical thinking guiding students' actions in the early years

Part 3

ADULTS' KNOWLEDGE OF CHILDREN'S NUMERICAL COMPETENCIES

Pessia Tsamir*, Esther S. Levenson*, Dina Tirosh*, Ruthi Barkai*/**

*Tel Aviv University, Israel

**Kibbutzim College of Education, Israel

Recognizing that young children engage with numerical activities outside of the school setting, this study investigates adults' knowledge regarding children and numerical activities. Questionnaire were handed out to 92 adults, none of whom were preschool teachers. Questions focused on the numerical competencies adults believed could be promoted during early childhood and the level of difficulty of numerical skills. Findings indicated that most participants mentioned counting, but did not necessarily differentiate between verbal and object counting. Few mentioned skip counting or counting backwards. Adults were aware of specific skills that might be difficult for children to carry out.

INTRODUCTION

In recent years, there has been increased interest in promoting preschool children's mathematics knowledge (e.g., Sarama & Clements, 2009). Studies have found that young children are capable of learning numerical concepts and that early intervention can have a positive effect on mathematics achievement in primary school (Jordan, Kaplan, Ramineni, & Locuniak, 2009). Towards this end, several educators have focused on enhancing preschool teachers' knowledge for teaching mathematics to young children (e.g., Tsamir, Tirosh, Levenson, Barkai, & Tabach, 2015). One important element of teachers' knowledge for teaching mathematics, is knowledge of their students' conceptions, misconceptions, and ways of thinking (e.g., Shulman, 1986). Ball and her colleagues (e.g., Ball, Thames, & Phelps, 2008) identified knowledge of content and students (KCS) as a sub-domain of pedagogical-content knowledge (Shulman, 1986). KCS is "knowledge that combines knowing about students and knowing about mathematics" (Ball, Thames, & Phelps, 2008, p. 401). This includes anticipating and predicting what examples students might find confusing or difficult and what tasks students might find interesting or motivating. Within the context of counting objects, for example, it might mean knowing that counting a small number of items arranged in a row is easier than counting a larger collection, with no specific order (Baroody & Wilkins, 1999).

While efforts to promote preschool teachers' knowledge to teach mathematics are commendable, we also recognize that young children spend a great part of the day outside of the school setting with other adults, such as parents, grandparents, aunts, and uncles. These adults can also play a role in children's mathematical development and may also be considered educators (Zippert, &

Rittle-Johnson, 2020). As with teachers, if we wish to support parents and other adults in their endeavours to engage young children with mathematics, we should also consider their knowledge of young children's mathematical conceptions and skills. This paper investigates adults' knowledge regarding young children's numerical competencies.

NUMERICAL COMPTENCIES FOR YOUNG CHILDREN

Several educators, researchers, and curricula differentiate between verbal counting and object counting (e.g., Israel National Mathematics Preschool Curriculum [INMPC], 2010; Sarama & Clements, 2009). Verbal counting is the skill of reciting numbers in the conventional order. However, verbal counting is more than a rote skill. It also includes knowing the principles and patterns in the number system as coded in one's natural language (Baroody, Lai, & Mix, 2006). The relationship to language may be seen in the difficulties of English-speaking (and Hebrew-speaking) children when learning the number words from 11 till 20, and going from 29 to 30 (Han & Ginsburg, 2001). Learning to count verbally occurs in phases (Fuson, 1988). The first phase is the acquisition phase. Children during this phase not only learn the conventional number words, but learn to produce them in order and consistently. Thus, a common error of children in this phase is to recite the conventional number names, in a consistent fashion, but not in order. For example, reciting 1, 2, 3, 5, 4 and when asked to count again, repeating the numbers 1, 2, 3, 5, 4 as before. The second phase is the elaboration phase when children become aware that the chain of numbers can be broken up and that parts of the chain may be produced starting from a number other than one. Thus, another counting competency is being able to count forward from some number other than one (INMPC, 2010).

Another counting skill mentioned by several curricula, including the Common Core State Standards Initiative (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and the Israel preschool curriculum (INMPC, 2010) is skip counting. Before entering first grade, children should be able to count by twos, fives, and tens, count forwards from some number other than one, and count backwards. Skip counting lays the groundwork for multiplication (Sarama & Clements, 2009). Counting backwards assists children when learning subtraction. Counting backwards can be considerably more difficult than counting forward. In one study (Howell & Kemp, 2010), although 93% of five-year-old children could count forward from 1 to 10, only 54% could count backwards from 5. Knowing the number that comes before some number is more difficult than knowing which number comes after some number (Howell & Kemp, 2010). Within numbers up to 10, some numbers are easier to tell which number comes next or which number comes before than others. For example, knowing which number comes after 9 is easier than knowing which number comes after 6 (Tirosh & Tsamir, 2008). Knowing which number comes before two, is easier than knowing which number comes before six.

Object counting refers to counting objects for the purpose of saying how many. Gelman and Gallistel (1978) outlined five principles of counting objects: the one-to-one correspondence principle, the stable-order principle, the cardinal principle, the abstraction principle, and the order-irrelevance principle. Competence in object counting may be related to the number of objects to be counted, as well as how the objects are set up (Briars & Siegler, 1984; Gelman & Gallistel, 1978). In Tsamir et al.'s (2018) study, 4-5 year old children were requested to count seven identical bottle caps placed in a circle, and then count seven different coloured bottle caps placed in a circle. All the children had previously succeeded in counting eight different objects placed in a row. It was hypothesized that with identical items, children might continuously go around in a circle, not knowing when to stop, but with different items, children might use the difference as an anchor to stop. Findings indicated that in both situations, half of the children correctly counted the caps. The most common mistake was ending the counting with the cap one started to count with, thus counting it twice. Two children, when faced with counting items in a circle, simply claimed that they did not know what to do and gave up.

Related to counting objects is the understanding that wholes consist of parts, and in general, number composition and decomposition. Several researchers have suggested that encouraging children to compose and decompose numbers, may enhance their number sense as well as their ability to solve addition and subtraction problems (e.g., Baroody, Lai, & Mix, 2006). Tsamir, Tirosh, Levenson, Tabach, & Barkai (2015) investigated kindergarten children' ability to compose and decompose the number seven within the context of two games, one using physical manipulatives and one using picture cards. In each game, children were presented with a number of items and were asked to say how many more would be needed to make seven. Results indicated that children found it easiest to compose seven from seven and zero, whereas the most difficult was three and four.

Knowing mathematics also includes knowing the symbols with which we express mathematical concepts. According to several curricula, before first grade, children should be able to identify and write the numbers from 0 to 20, with 0 representing a count of no objects (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). In studies with children, it was found that children often confuse the numerals six and nine (Tirosh & Tsamir, 2008). First, the two symbols are visually similar. Second, in Israel, the languages of Hebrew and Arabic are read from right to left, but numbers are read from left to right, possible causing even more confusion between 9 with 6.

Several studies focused on mathematical activities carried out at home. Missall, Hojnoski, Caskie, and Repasky (2015) listed 19 activities related to number and operations and asked parents to rate how often they engaged their children with those activities. Among the most frequent activities were counting aloud, counting out a number of items from a larger group, and reading numbers. Among the least frequent activities were skip counting, counting backwards, and comparing the number of objects in two sets. Similar findings were found by Skwarchuk (2009), who also found that many parents incorporated numerical concepts during natural settings at home. A different study found that parents engage their children with identifying numerals more often than comparing the magnitudes of numbers (Vandermaas-Peeler, Ferretti, & Loving, 2012).

The above studies show that parents do engage their children with number activities at home. Thus, if we wish to offer interventions for parents, as well as other adults who interact with children, we need to investigate what these adults know regarding children's engagement with numbers. We ask: What are adults' perceptions regarding number competencies that can be promoted among young children? Within various competencies, are adults aware of which tasks may be more difficult for children to complete correctly?

METHODOLOGY

A convenience sample of 92 adults, chosen based on their not being preschool teachers, participated in the study. The adults were between the ages of 20 and 60, and resided in middle to high socio-economic neighbourhoods. Over 90% had an academic degree.

A questionnaire was designed with seven open questions, based on numerical competencies mentioned by several curricula (e.g., INMPC, 2010) and our previous studies with children (e.g., Tirosh & Tsamir, 2008), as detailed in the background section. Questionnaires were filled in by participants in the presence of the researcher. The following questions were presented.

1. In your opinion, which numerical concepts/skills can be promoted among young children, before they begin first grade?

2. Which number symbols between 0 and 9, will most young children find difficult to identify?

3. It is possible to arrange 8 items in different ways (for example, in a row, in a circle, in a pile). For which arrangements will most children be able to state that there are 8 items?

4. The number 7 can be composed from two different numbers in several ways, for example, 0 and 7, 1 and 6, 2 and 5, and more.

a) Which decompositions of the number 7 will be <u>easiest</u> for children to identify?

b) Which decompositions of the number 7 will be most difficult for children to identify?

5. For which numbers, between 0 and 9, will most children be able to say the number that comes right after that number?

6. For which numbers, between 1 and 10, will most children be able to say the number that comes right before that number?

7. Children should master counting skills up till 30. What counting skills do most children carry out successfully?

FINDINGS AND DISCUSSION

This section reports on participants' responses to each of the questions in the questionnaire. Note that for each question, participants were able to offer more than one response. Participants' responses for each question are also compared to curricula standards and to findings from previous studies.

Question 1: Numerical skills that can be promoted

Participants mentioned a variety of numerical skills and concepts in response to the first question (see Table 1).

Skill/concept	Frequency (%)
Addition and subtraction	58
Counting (non-specific)	28
Counting objects (only)	11
Verbal and object counting	21
Comparing amounts	21
Identifying number symbols	15
Multiplication	8
Division, zero, fractions	4
Even and odd numbers; estimation	3
Counting backwards; knowing the number that comes before or after; number conservation	2
Skip counting	1

Table 1: Frequency of number skills mentioned by participants that can be promoted before first grade.

Some participants listed one skill, while others listed a few. In previous studies, we found that preschool teachers did not always differentiate between verbal counting and object counting (Tsamir, et al., 2014). In the current study, we also see that some participants (21%) specifically mentioned both, while other did not. For those who simply wrote counting (28%), it is difficult to know if they considered both verbal and object counting or just one or just the other. On the other hand, 21% specifically related to both verbal and object counting, showing an awareness that these are two separate skills.

Surprisingly, fewer participants mentioned counting than adding and subtracting. Perhaps adults believe that counting, both verbal and object counting, is something that comes naturally and does need to be promoted. Although none of the adults mentioned composing and decomposing numbers, those skills are related to addition and subtraction, which was mentioned by many participants (see Table 1). The INMPC (2010) does state that before first grade, children should be able to add and subtract within the range of 1-10. However, the curriculum also specifies that children should be able to carry out these operations with physical items.

Question 2: Difficult number symbols

With regard to identifying number symbols, as can be seen from Table 2, most participants believed that 9 and 6 would be difficult to identify. This is in line with previous studies with children (Tirosh & Tsamir, 2008). Two participants explicitly wrote that they do not know which number symbols children would find difficult to identify.

Number	0	1	2	3	4	5	6	7	8	9
Frequency (%)	10	4	7	5	16	23	68	24	3	76

Table 2: Frequency of responses to which number symbols are difficult to identify

Question 3: Arrangements of items to be counted

Recall that on the questionnaire, we suggested that objects to be counted may be laid out in different formations such as in a row, a circle, or in a pile. We then asked participants to name formations for which children would successfully be able to count eight items. Frequencies of participants' responses are shown in Table 3.

Formation of 8 items	One row*	Pile*	Column	Groups	Circle*	Other
Frequency (%)	86	12	11	9	5	5

*Suggested arrangements

Table 3: Which formations of 8 objects are easily counted?

Recall that in a previous study, it was found that a circle formation may be challenging for children (Tsamir et al., 2018). The low response of participants to this formation indicates that they seem to recognize this challenge.

Participants also correctly recognized that children would be able to count items set in a row. Regarding the response of "Groups," one participant wrote, "structured piles, such as four and four or two, two, two, and two." Another participant wrote, "two rows with four items in each." In fact, studies have shown that when items are organized into groups of two or three, children may subitize the amounts, and then use known facts or skip counting to count the total number of items (Baroody, Lai, & Mix, 2006). In the category of "other," one participant suggested placing the items in a square formation. It is difficult to know if participants might have considered additional or different formations if the questionnaire had not made any suggestions. None of the participants responded that they did not know the answer to this question.

Question 4: Composing and decomposing the number 7

Questions 4a and 4b dealt with ways of composing and decomposing seven. Note that the way the question is worded, the order of the numbers composing seven is not significant. That is, the question does not differentiate between decomposing 7 into 6 and 1 or into 1 and 6. Participants were also able to list more than one answer for each question. Findings (see Table 4), indicated that the majority of participants believed that decomposing seven in the extremes, that is 7 and 0, and 1 and 6, would be easier than decomposing seven into numbers that are relatively close to each other, such as 3 and 4, and 2 and 5. This observation would be correct if we were presenting six items and asking a child to say how many more is needed to make seven. Beginning with one and asking the child to complete it to seven can be more difficult (Sarama & Clements, 2009). That being said, participants' responses were in line with a previous study that found the decomposition of seven into seven and zero is relatively easy for children, while the decomposing of seven into three and four can be quite difficult (Tsamir, Tirosh, Levenson, Tabach, & Barkai, 2015). Seven participants wrote for this task that they did not know what children would be able to do.

Composing 7	(0,7)	(1,6)	(2,5)	(3,4)
Identified as simple compositions	32	76	12	4
Identified as difficult compositions	27	3	54	45

Table 4: Which decompositions of seven are easy, and which are more difficult?

Questions 5 and 6: What comes before and what comes next

Questions 5 and 6 dealt with children's ability to say which number comes before or after some number. In general, finding in Table 5 show that participants believed that for the numbers between 1 and 5, as opposed to the numbers between 6 and 9, children would know to say which number comes next and, with the exception of the number one, children would know which number comes before. One participant explicitly stated, “the smaller the number, the easier it will be to say which number comes before.” Taking into consideration that zero is not always considered by children (and some adults) to be a number (Levenson, 2013), it makes sense that few adults believed young children would be able to say that zero comes before one, and few children would be able to say what number comes after zero. Regarding other numbers, adults’ beliefs of children’s abilities were in line with previous studies that found, for example, that knowing which number comes after 9 is easier than knowing which number comes after 6, and knowing which number comes before two, is easier than knowing which number comes before six (Tirosh & Tsamir, 2008).

Number	0	1	2	3	4	5	6	7	8	9	10
After	8	77	70	58	39	30	18	18	23	24	-
Before	-	25	75	51	43	33	20	16	15	24	24

Table 5: Which number will children know the number that comes after/before?

Two participants stated that they did not know for which numbers children would be able to say the number that comes next, and eight participants wrote that they did not know for which numbers children would be able to say the number before. On the other hand, 12 participants stated that for all numbers, children would be able to say which number comes next, while five participants stated that for all numbers, children would be able to say which number comes before.

Question 7: Counting skills

The last question was an open question regarding which counting skills adults believed could be promoted among young children. This question differed from the first question in that the first question inquired about all numerical skills, while the last question investigated if participants would be aware of different counting skills, such as counting backwards and skip counting. That being said, 86% of participants only related to counting in general, without specifying if they meant verbal or object counting. Among those that wrote counting, 83% wrote until what number they thought children could count up to. Of those participants, 39% believed children could master counting up to 10, and 18% believed children could master counting up to 20. One participant mentioned counting backwards from 10 to 1. None of the participants mentioned skip counting or counting forward from some number other than one.

CONCLUSIONS

Our first research question focused on adults' perceptions regarding number competencies that can be promoted among young children. Findings indicated that adults were mostly aware of counting (some differentiating between verbal and object counting and others not) and adding and subtracting. This finding is in line with previous studies of parents that found the most prevalent activity carried out at home is counting (Missall, Hojnoski, Caskie, & Repasky, 2015). Thus, workshops for parents and other adults who wish to engage children with numerical activities, might introduce adults to additional numerical competencies, such as number composition and decomposition, set comparison, and skip counting.

Within various competencies, adults were generally aware of which tasks may be more or less difficult for children to complete correctly. This result is encouraging. On the one hand, few adults mentioned identifying number symbols as a competency to be promoted during early childhood. Yet, most of the participants were able to say that children confuse the symbols of 6 and 9. This might indicate that adults notice children's engagement with numerical concepts, and notice what children find difficult. Educators may build on this awareness to plan appropriate interventions with adults. Finally, this study opens up a discussion about provoking the mathematical thinking of children by adults who are not necessarily their teachers.

Acknowledgement

This research was supported by The Israel Science Foundation (grant No. 1631/18).

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education*, 59(5), 389-407.

Baroody, A. J., Lai, M., & Mix, K. S. (2006). The development of young children's early number and operation sense and its implications for early childhood education. In B. Spodek & O. Saracho (Eds.), *Handbook of research on the education of young children* (Vol. 2, pp. 187-221). Mahwah, NJ: Erlbaum.

Baroody, A. J., & Wilkins, J. L. M. (1999). The development of informal counting, number, and arithmetic skills and concepts. In J. V. Copley (Ed.), *Mathematics in the early years* (pp. 48-65). Reston, VA: National Council of Teachers of Mathematics.

Fuson, K. C. (1988). *Children's counting and concepts of number*. New York: Springer-Verlag.

Gelman, R., & Gallistel, C. (1978). *The child's understanding of number*. Cambridge: Harvard University Press.

Howell, S. C., & Kemp, C. R. (2010). Assessing preschool number sense: Skills demonstrated by children prior to school entry. *Educational Psychology*, *30*(4), 411-429.

Israel National Mathematics Preschool Curriculum (INMPC) (2010). Retrieved January 15, 2020 from https://edu.gov.il/minhalpedagogy/preschool/subject/math/Pages/math-curriculum.aspx

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: kindergarten number competence and later mathematics outcomes. *Developmental Psychology*, *45*(3), 850-867.

Levenson, E. (2013). Exploring one student's explanations at different ages: the case of Sharon. *Educational Studies in Mathematics*, *83*(2), 181-203.

Missall, K., Hojnoski, R. L., Caskie, G. I., & Repasky, P. (2015). Home numeracy environments of preschoolers: Examining relations among mathematical activities, parent mathematical beliefs, and early mathematical skills. *Early Education and Development*, *26*(3), 356-376.

Sarama, J. & Clements, D. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. New York: Routledge.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Skwarchuk, S. L. (2009). How do parents support preschoolers' numeracy learning experiences at home?. *Early Childhood Education Journal*, *37*(3), 189-197.

Tirosh, D. & Tsamir, P. (2008). *Starting right: Mathematics in preschool*. Unpublished research report. In Hebrew.

Tsamir, P., Tirosh, D., Levenson, E. & Barkai, R. (2018). Engaging young children with mathematical activities involving different representations: Triangles, patterns, and counting objects. *Center for Educational Policy Studies Journal, 8*(2), 9–30.

Tsamir, P., Tirosh, D., Levenson, E., Tabach, M., & Barkai, R. (2015). Analyzing number composition and decomposition activities in kindergarten from a numeracy perspective. *ZDM Mathematics Education*, *47*(4), 639-651.

Tsamir, P., Tirosh, D., Levenson, E., Tabach, M., & Barkai, R. (2014). Employing the CAMTE framework: Focusing on preschool teachers' knowledge and self-efficacy related to students' conceptions. In C. Benz, B. Brandt, U. Kortenkamp, G. Krummheuer, S. Ladel, and R. Vogel (Eds.), *Early Mathematics Learning – Selected Papers from the POEM 2012 Conference* (pp. 291-306). New York: Springer.

Vandermaas-Peeler, M., Ferretti, L., & Loving, S. (2012). Playing the ladybug game: Parent guidance of young children's numeracy activities. *Early Child Development and Care*, *182*(10), 1289-1307.

Zippert, E. L., & Rittle-Johnson, B. (2020). The home math environment: More than numeracy. *Early Childhood Research Quarterly*, *50*, 4-15.

MATHEMATICS IN THE KINDERGARTEN: CONTINUING AND COMPLETING A REPEATING PATTERN

Iris Schreiber

Kibbutzim College of Education & Bar Ilan University, Israel

This paper reports a study on the patterning knowledge of 206 Israeli children, aged 4–6. The children were given a repeating pattern (consisting of circles and squares) and were asked to continue and to complete it. For each task the children were given a set of shapes from which they chose the appropriate ones for performing the task. Some sets include exactly the necessary shapes, some sets lack the required shapes, whereas others include surplus shapes. The results show that the given set affects children's performance.

INTRODUCTION

Kindergarten is an important beginning for the child's exploration of mathematical objects and ideas (Sarama & Clements, 2009). Mathematics learning in kindergarten is essential for placing the foundations of many mathematical topics and concepts that the children will learn later in school (NCTM, 2000). An important topic, which is part of the mathematical curriculum for kindergarten in many countries, is patterns. A pattern is a series of elements arranged according to a certain rule. Each element has a single value determined by its place in the series, so that the elements appear in a predictable way. Patterns and structures are considered as the heart of algebraic thinking, which may be promoted by continuing a pattern, being able to identify and describe the 'general' element of a pattern and expressing and justifying these generalizations (Zazkis & Liljedahl, 2002; Warren, 2005).

Each patterning task has various characteristics that may affect the children's performance, such as the length of the given pattern, the way it is presented to the children, the unit of repeat, and so forth. One of these characteristics is what we provide to the children so that they can perform a given task. For example, if children are asked to continue building a tower of blocks that consists of a repeating pattern of yellow-blue-yellow-blue blocks, we can provide them: (1) only blue and yellow blocks (just the exact necessary blocks); (2) blocks in a wide range of colours, among them the blue and yellow required for the task, so the children need to identify and choose the blocks appropriate for completing the task; (3) a set of blocks lacking one or more of the needed colours: only blue blocks, or only yellow blocks, or blocks in various colours but not the needed colours. In this case the children are expected to say that it is impossible to continue building the tower due to the lack of some or all of the needed blocks.

The different options offered to the children for performing the task would be referred to in this paper as **Bank**.

In most studies related to patterns in kindergarten, the bank has not been a variable of the research, has not been examined systematically, and many articles neglect to even mention it. When the bank is mentioned, it mostly contains exactly what the child needs. In this study, the bank is incorporated as one of the variables, aiming to research the effects of the type of the bank on children's performance.

LITERATURE REVIEW

Mathematics learning in early childhood is important for developing creativity, mathematical skills, and thinking abilities (Ministry of Education in Israel, 2010; NCTM, 2000). Researchers frequently discuss mathematics curricula at pre-school ages, recommending many learning activities (Sarama & Clements 2009, 2011; Greenes, Ginsburg, & Balfanz, 2004).

The present study focuses on the mathematical topic of patterns, which is part of the mathematics curriculum for kindergartens in many countries, including Israel. Its importance is highlighted in policy documents and curricula (NCTM, 2000). Patterns may form the basis for understanding recurring structures, which promote the acquisition of various mathematical concepts - such as variables, functions and algebraic expressions (Warren, 2005). Patterns may also lead to a high level of thinking - the ability to generalize (Ministry of Education in Israel, 2010). Recommendations for early ages suggest focusing on patterns that have different characteristics, such as colour, position and quantity, and to present patterns in various ways such as pictures, concrete elements, sounds or movements.

Many studies recommend teaching the topic of repeating and growing patterns at all ages, particularly in kindergarten, and suggest various activities and patterning tasks such as describing, creating, continuing or completing a pattern (Burton, 1982; Threlfall, 1999; Papic, Mulligan, & Mitchelmore, 2011; Warren, 2005).

The present study investigates two types of tasks that, according to the curriculum in many countries (including Israel) and to many researchers, are considered ones that may promote mathematical thinking and generalization abilities:

“Continue a pattern”: produce/create/build a continuation for a given pattern (Burton, 1982; Economopoulos, 1998; Warren, 2005).

“Complete a pattern”: produce/create/build the missing elements of a given pattern (Burton, 1982; Warren, 2005).

Studies that explore the factors influencing the difficulty level of repeating patterns tasks, indicate the length and the complexity of the unit of repeat as significant factor (Kyriakides & Gagatsis, 2003; Threlfall, 1999; Tsamir et al. 2018). Past studies indicate that children spontaneously create repeating patterns while playing, and that they are naturally inclined towards them. However, it was also found that kindergarten children have difficulties providing a verbal description of patterns and recognizing the basic repeating unit (Fox, 2005; Garrick, Threlfall & Orton, 1999; Highfield & Mulligan, 2007; Tsamir, Tirosh, Barkai & Levenson, 2018). According to past studies, some of the typical unexpected answers that children make in patterning tasks are continuing a pattern randomly, repeating one element of the pattern systematically (Sarama & Clements, 2009; Threlfall, 1999), or copying the pattern (Fox, 2005; Garrick, Threlfall & Orton, 1999).

So far, the bank has not been explored as a possible factor influencing the difficulty level. Usually, it remains unreported: most studies do not even indicate the bank given to the subjects. Among studies in which the bank is mentioned, it is usually an exact bank or sometimes, a case of no bank: the children are asked to continue or complete the missing parts of the pattern either in free speech or in drawing (Papic et al. 2011; Warren 2005); or they may be given a pattern in which some parts, either in the middle or in the end, are hidden and be asked to say what are the missing part (Burton, 1982; Economopoulos, 1998). Some articles indicate using an exact bank. For example, the pattern may be a row of teddy bears in different colours and the subjects are given teddy bears only in the colours that are needed to complete the task (Burton, 1982; Clarke, Clarke & Cheeseman, 2006).

The potential influence of the type of bank on performance is neglected. Only few past studies manipulated the type of the bank and used it as one of the variables of the study. Tsamir, Tirosh, Barkai & Levenson (2018), examined the influence of the way teachers present the bank on children performance in a task of copying a repeating pattern that consisted of beads in two colours. Most of the children were given an exact bank, incorporating the two types of beads, in two ways: in the same or in separate containers. Only one of the teachers presented another type of bank, a surplus bank, incorporating - the needed beads as well as beads in other colours, in two ways - all in the same container or in different containers, one for each colour.

As far as I know, there have been no studies in which the bank was methodically constructed, and which examined the influence of different banks on the same subjects. Furthermore, there have been no studies that used and examined a third type of bank: a lacking bank, which lacks the needed elements. In this type of bank, the task cannot be performed, creating an unsolved mathematical problem, which according to a previous study, lead some children to accept the possibility that a problem may not have a solution, and to try to find a practical solution instead (Tirosh, Tsamir, Levenson, Tabach, & Barkai, 2015).

The aim of the present study is to examine children's performance in patterning tasks - percentage of right answers and the common unexpected answers - referring to the differences between the two tasks (continue a repeating pattern and complete a repeating pattern), and to the difference between the three types of banks.

METHODOLOGY

Participants

206 Israeli children aged 4 to 6. All the children attended kindergartens in the same city in Israel. All the kindergartens were of the same socioeconomic status (according to the criteria of the Ministry of Education in Israel). The children had previous experience creating and continuing repeating patterns, but no experience completing a repeating pattern. Ethical approval to the study was given by the Chief Scientist of the Ministry of Education in Israel.

Research instruments

The pattern chosen for this study is a pattern abababab, consists of the following unit of repeat, red circle- yellow square: 

Children's knowledge was examined in two tasks ("Continue a repeating pattern", "completing a repeating pattern") with various banks, regarding percentage of mathematically correct responses and common unexpected answers.

The Bank is one of the variables of the study, which is designed to examine whether there is a relation between the bank and children's performance in the two tasks. For this purpose, various questions were formulated, in which the task is the same (continuing or completing the same repeating pattern), but type of bank is different: the children were asked to continue the same repeating pattern each time with a different type of bank: an exact bank, containing exactly the items needed for solving the task; a surplus bank, containing the items required for solving the task and items that are not needed; and a lacking bank, in which items needed for solving the task do not appear. The different banks were constructed systematically regarding the two variables that characterized the pattern: colour (red, yellow) and shape (circle, square). Each variable can have four values:

(i) Lacking: when the bank does not contain the needed colours or shapes. For example, the bank does not contain the shapes square and circle.

(ii) Partial: when the bank contains only one of the needed colours or shapes. For example, the bank has only yellow colour. In this case the bank is lacking the needed elements and the pattern cannot be continued or completed.

(iii) Exact: when the bank contains exactly the needed colours or shapes. For example, the bank has exactly the two colours yellow and red.

(iv) Surplus: when the bank contains additional shapes and/or other colours. For example, the bank has the two needed colours yellow and red, as well as other colours such as green, blue or pink.

Accordingly, there are 16 different bank options as illustrated in Table 1.

Colour / Shape	Surplus	exact	partial	Lacking
Lacking	 1	 2	 3	 4
Partial	 5	 6	 7	 8
Exact	 9	 10	 11	 12
Surplus	 13	 14	 15	 16

Table 1: The different banks

Each cell in Table 1 presents a different bank which is defined by the value of each of the two variables. For example, bank 4, in the lacking-lacking cell, lacks the required shapes (a star and a triangle, but not a square or a circle) and lacks the required colours (blue and pink, but not yellow or red). Or, bank 5, in the partial-surplus cell, is a bank with part of the required shapes (contains only a square but not a circle) and with surplus colours (yellow and red as well as blue and pink).

Two questionnaires were structured this way, one for each type of task: Continue the pattern and complete the pattern. In each questionnaire there were 16 items, each with the same pattern and with one of the banks presented in table 1.

The questionnaires were presented as a computer game. The children performed in front of a computer screen using a software developed specifically for the study. Each time, one pattern appeared on the computer screen with a set of shapes below it, from which the children chose the appropriate shape. This set of shapes below the pattern constituted the bank. The children were asked to continue or complete the pattern by "dragging" the shape they chose. The bank was displayed to the children as boxes with a figure of the shape on it, so that the children could drag as many items of that shape as they wanted.

It is important to note that computer software was also used in previous studies (Highfeld & Mulligan, 2007), and that in this study all the children were acquainted with computers and were used to playing and studying using computer software.

Examples for items from the questionnaires:

Continue the pattern with an exact-exact bank (exactly the shapes and colours needed for solving the task) 

Complete the pattern with a surplus-exact bank (surplus shapes but exactly the colours needed for solving the task) 

Continue the pattern with an exact-lacking bank (exactly the needed shapes but none of the colours needed for solving the task) 

The questionnaires included items that the children could solve by continuing or completing the pattern (like the two first examples above), and items that the children could not solve (like the two last examples above). In this case the expected answer was that it was impossible to solve the task.

Research procedure

All the children participated in two sessions, one for each task, of approximately ten to fifteen minutes. Each session was devoted to one questionnaire. Each child responded to the questionnaire in the presence of the researcher in a quiet area of the kindergarten. The researcher presented the pattern and the bank to the child and asked him: “Do you think this pattern can be continued? Why?”. After continuing the pattern, the researcher asked “Do you think your work is appropriate? Are you satisfied with it? Do you want to change anything? Why?”. The child could continue playing until he was satisfied with his answer.

The children’s responses (the way they continued or completed the pattern and their oral response) were documented and recorded automatically by the computer software.

RESULTS

The results regarding the two tasks indicate that the performance of the children in continuing the repeating pattern was similar to their performance in completing the repeating pattern. No differences were found between the two tasks.

When the bank included the necessary shapes, the expected reply was a mathematically correct continuation (or completion) of the pattern according to the unit of repeat: yellow square-red circle. When the bank lacked the necessary shapes, the expected reply was that the pattern cannot be continued. Children's replies which were the mathematically correct expected ones will be referred to as “expected answers”. All other replies will be referred to as “unexpected answers”.

The results regarding the percentage of expected answers for each item of the questionnaire, defined according to the different characteristics of the bank

(Table 2), show that children performance was influenced by the type of the bank. Children performed differently in the same task with the same pattern when they had different banks.

The results in table 2 are organized in four main categories of banks in which similar results were observed: in each category similar success percentage and similar unexpected answers. The first two categories are for banks that contain what is needed for completing or continuing the pattern: the exact-exact bank and banks that include the necessary items but in which at least one variable is surplus. The other two categories are for banks that do not comprise of what is needed for completing or continuing the pattern (it does not contain the yellow square and/or the red circle): banks with one element only and banks with more than one element. In these two categories the tasks are unsolvable, and the children were expected to say that there is no solution.

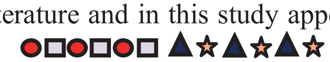
Can the task be solved?	The bank and its characteristics		Task	
	The bank	shape-colour	Continue the pattern	Complete the pattern
yes		Exact-exact	80.6	86.4
yes		Exact-surplus	58.7	67.5
		Surplus-exact	58.7	66.0
		Surplus-surplus	57.3	64.1
no		Partial-partial	90.3	90.8
		Partial-lacking	89.3	90.3
		Lacking-partial	89.8	90.8
no		Partial-exact	79.6	79.1
		Lacking-lacking	76.2	76.2
		Lacking-exact	75.2	74.8
		Exact-partial	74.3	78.2
		Exact-lacking	72.8	73.8
		Surplus-partial	80.6	80.6
		Partial-surplus	75.7	75.7
		Surplus-lacking	75.2	75.2
		Lacking-surplus	73.3	73.8

Table 2: The percentage of children who answered the expected answer

The table demonstrates that the characteristics of the pattern influenced children's replies, both the number of items in the bank, and whether it contains the required elements. The results illustrate that the success percentage was the highest when the bank included only one element. The children said that the pattern could not be continued or completed. The success percentage was the lowest when the bank was surplus but included the needed items. It is important to note that not only had the children performed similarly well in both tasks, they also made similar unexpected answers in the two tasks. Below are examples of the common unexpected answers observed in the study:

A) Continuing or completing the pattern randomly: the child continued or completed the pattern with colours or shapes that did not appear in the pattern. This unexpected answer was reported in the literature and in this study was mostly observed in items with surplus banks: 

B) Copying the bank: the child copied the shapes in the same order they appeared in the bank. This unexpected answer was not reported in the literature and in this study was mostly observed with surplus banks:

C) Continuing a repeating pattern with another repeating pattern. The child looked for an alternative solution and created another repeating pattern. This unexpected answer was not reported in the literature and in this study appeared mostly in items with 2 elements in the bank: 

The results show we can characterize three common types of replies. Type 1, children who give the expected answer. Type 2, children who give an expected answer only when the bank includes the necessary shapes and look for an alternative solution when it lacks them (as in example C). Type 3, children who tend to give an unexpected answer in most of the banks (as in examples A and B).

The participants were 4-6 year-old children who all studied at the same kindergarten together. It was examined whether age influenced the replies children gave. As expected, children aged 5-6 provide the expected answer more than children aged 4-5. It was so for all types of bank, except for banks that included only one item. In these banks there was no significant difference.

DISCUSSION AND CONCLUSIONS

This research examined the way children answer two tasks of repeating pattern - continuing the pattern and completing the pattern - when given each time a different bank (namely, a set of elements provided along the task that should use in order to solve it). While previous studies make limited reference to the concept of the bank, in this study the bank has been manipulated to see if it affects performance, noticeably presenting types of banks that are not prevalent in patterning tasks, like partial or surplus banks. This entailed uncommon answers that were expected from the children (such as “it is impossible to continue the pattern”), and according to the percentage of success the children coped well with this uncommon demand.

The results indicate that the type of the bank affects children's performance. The fact that the effect was found in the two tasks reinforces the significance of various types of banks as a factor influencing children's performance. The percentage of expected answers was the highest in items with banks which consisted of one element only, probably because the children noticed that the pattern showed two elements and the bank contained only one. The percentage of expected answers was the lowest in items with surplus banks which contained the shapes needed for solving the task, probably because the bank was loaded with confusing details. It is noteworthy that the bank size is not the only factor that effects performance: if we look at two banks that consisting of eight elements, bank 14 (see table 1) which contains what is needed for solving the task, and bank 1 which does not contain what is needed for solving the task, we can see that in bank 14 the success percentage was higher, suggesting that children might be more capable of noticing whether or not they can or cannot solve a task, but it is harder for them to solve it as expected.

The various banks leaded to unexpected answers that were not mentioned in previous studies. One of the children's answers was continuing (or completing) the pattern with an alternative repeating pattern. It was observed in items with a lacking bank that did not contain the needed shapes, in which the expected answer was that the task cannot be performed. It complies with the results of a study regarding an unsolved mathematical problem (Tirosh et al., 2015), in which the children accept the possibility that a problem may not have a solution and try to find a practical solution instead. In the present study, the children continued or completed the pattern in an alternative way - they suggested an alternative repeating pattern.

The results imply that different banks lead to various answers because they affect the way children observe the characteristics of the pattern. Based on the results, this study recommends offering children various banks in different levels of difficulty for the same task, as part of the teaching process. More broadly, it is important to vary the materials given to the children to perform a task (abstract, pictorial, concrete); to mind the importance of examples and non-examples; to mind the importance of various tasks, including unsolvable tasks. It can facilitate the observation of the pattern as a whole and to strengthen the ability to identify the repeating unit of the pattern, which, in other words, is to identify the way the pattern was structured. In addition to varying the banks, the materials and the tasks, critical thinking can be encouraged through questions such as “Do you think this pattern can be continued? Why?”, “Are you satisfied with your work? Why?”, “Would you like to change your reply? Why?”.

Acknowledgment

The study is part of a doctoral thesis conducted under the supervision of Prof. Pessia Tsamir and Prof. Dina Tirosh, TAU, Israel.

References

Burton, G. M. (1982). Patterning: powerful play. *School Science and Mathematics, 28,* 38–44.

Clarke, B., Clarke, D., & Cheeseman, J. (2006). The mathematical knowledge and understanding young children bring to school. *Mathematics Education research journal, 18,* 78-102.

Economopoulos, K. (1998). What comes next? The mathematics of pattern in kindergarten. *Teaching children Mathematics, 5,* 230–233.

Fox, J. (2005). Connecting algebraic development to mathematical patterning in early childhood. In P. Grootenboer., R. Zevenbergen, & M. Chinnappan (Eds.), *Proceedings of the 29th Annual Conference of the Mathematics Education. Research Group of Australia* (1, 221–228). Canberra, Australia: MERGA.

Garrick, R., Threlfall, J., & Orton, A. (1999). Pattern in the nursery. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 1–17). London & New-York: Cassel.

Greenes, C., Ginsburg, H. P., & Balfanz, R. (2004). Big mathematics for little kids. *Early Childhood Research Quarterly, 19,* 159-166.

Highfield, K., & Mulligan, J. (2007). The role of dynamic interactive technological tools in preschool's mathematical patterning. In J. Watson & K. Reswick (Eds.), *Proceeding of the 30th Annual Conference of the Mathematics Education Research Group of Australia* (v1, pp. 372-381). Canbara, Australia: MERGA.

Kyriakides, L., & Gatatsis, A. (2003). Assessing student problem solving skills. *Structural Equation Modeling,10*, 609-621.

Ministry of Education in Israel (2010). Curriculum for mathematics in kindergarten. Jerusalem, Israel: Ministry of Education. [Hebrew]

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, Virginia, USA: NCTM.

Papic, M., Mulligan, J., & Mitchelmore, M. (2011). Assessing the development of preschoolers' mathematical patterning. *Journal for Research in Mathematics Education, 42,* 237–269.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge.

Threlfall, J. (1999). Repeating pattern in the early primary years. In A. Orton (Ed.), *Pattern in the teaching and learning of mathematics* (pp. 18–29). London & New-York: Cassel.

Tirosh, D., Tsamir, P., Levenson, E., Tabach, M., & Barkai, R. (2015). Unsolvable mathematical problems in kindergarten: are they appropriate?. In K. Krainer, & N.

Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education -CERME 9* (pp. 2010-2016). Prague, Czech Republic.

Tsamir, P., Tirosh, D., Barkai, R., & Levenson, E. (2018). Copying and comparing repeating patterns: children's strategies and descriptions. Paper presented at POEM 2018- *A Mathematics Education Perspective on early Mathematics Learning,* Kristiansand, Norway, 29-30.

Warren, E. (2005). Young children's ability to generalize the pattern rule for growing pattern. In H. L. Chick, & J. L. Vincent (Eds.), *Proceeding of the 29th International Conference for the Psychology of Mathematics Education* (vol. 4, 305–312). Melbourne: PME.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: the tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics, 49,* 379-402.

ADDRESSES OF THE CONTRIBUTORS

Giovannina Albano
University of Salerno
ITALY
galbano@unisa.it

Evgenios Avgerinos
University of the Aegean
GREECE
eavger@aegean.gr

Ruthi Barkai
Tel Aviv University
Kibbutzim College of Education
ISRAEL
ruthi11@netvision.net.il

Emőke Báró
University of Debrecen
HUNGARY
temoke10@gmail.com

Rachel Filo
Kibbutzim College of Education
ISRAEL

Linda Devi Fitriana
University of Debrecen
HUNGARY
flindadevi@gmail.com

Ivona Grzegorczyk
California State University Channel Island
USA
ivona.grzegorczyk@csuci.edu

Tobias Huhmann
University of Education, Weingarten,
GERMANY
huhmann@ph-weingarten.de

Darina Jirotková
Charles University
CZECH REPUBLIC
darina.jirotkova@pedf.cuni.cz

Edyta Juskowiak
Adam Mickiewicz University in Poznań
POLAND
edyta@amu.edu.pl

Márton Kiss
University of Debrecen
HUNGARY
kmarci88@gmail.com

Ellen Komm
University of Education, Weingarten
GERMANY
komm@ph-weingarten.de

Eszter Kónya
University of Debrecen
HUNGARY
eszter.konya@science.unideb.hu

Zoltán Kovács
University of Nyíregyháza
HUNGARY
kovacsz@science.unideb.hu

Jens Krummenauer
Ludwigsburg University of Education
GERMANY
jenskrummenauer@msn.com

Sebastian Kuntze
Ludwigsburg University of Education
GERMANY
kuntze@ph-ludwigsburg.de

Huey Lei
University of Saint Joseph
MACAU, CHINA
lei.huey@usj.edu.mo

Esther S. Levenson
Tel Aviv University
ISRAEL
levenso@tauex.tau.ac.il

Esperanza López Centella
University of Granada
SPAIN
esperanza@ugr.es

Bożena Maj-Tatsis
University of Rzeszow
POLAND
bmaj@ur.edu.pl

Eva Nováková
Faculty of Education, Masaryk University
CZECH REPUBLIC
novakova@ped.muni.cz

Barbara Ott
St. Gallen University of Teacher Education
SWITZERLAND
Barbara.Ott@phsg.ch

Barbara Pieronkiewicz
Pedagogical University of Krakow
POLAND
barbara.pieronkiewicz@up.krakow.pl

Anna Pierri
University of Salerno
ITALY
apierri@unisa.it

Maria Polo
University of Cagliari
ITALY
mpolo@unica.it

Marta Pytlak
University of Rzeszow
POLAND
mpytlak@ur.edu.pl

Anna Pyzara
Maria Curie-Skłodowska University
POLAND
anna.pyzara@umcs.pl

Mirosława Sajka
Pedagogical University of Krakow
POLAND
miroslawa.sajka@up.krakow.pl

Iris Schreiber
Kibbutzim College of Education
Bar Ilan University
ISRAEL
irisifi5@gmail.com

Jana Slezáková
Charles University
CZECH REPUBLIC
jana.slezakova@pedf.cuni.cz

Konstantinos Tatsis
University of Ioannina
GREECE
ktatsis@uoi.gr

Dina Tirosh
Tel Aviv University
ISRAEL
dina@tauex.tau.ac.il

Pessia Tsamir
Tel Aviv University
ISRAEL
pessia@tauex.tau.ac.il

Sabine Vietz
University of Education, Weingarten,
GERMANY
vietz@ph-weingarten.de

Annika M. Wille
University of Klagenfurt
AUSTRIA
annika.wille@aau.at

Małgorzata Zambrowska
The Maria Grzegorzewska University
POLAND
m.zambrowska@gmail.com

Michail Zorzos
University of the Aegean
GREECE
mzorzos@aegean.gr