# Educational Resources in the Mathematics Classroom

**Editors**

Bożena Maj-Tatsis
University of Rzeszów
Rzeszów, Poland

Konstantinos Tatsis
University of Ioannina
Ioannina, Greece

Wydawnictwo Uniwersytetu Rzeszowskiego
2024

**Reviewers**

Jenni Back
Ineta Helmane
Edyta Juskowiak
Eszter Kónya
Esther Levenson
Bożena Maj-Tatsis
Eva Nováková
João Pedro da Ponte
Christof Schreiber
Lambrecht Spijkerboer
Konstantinos Tatsis
Paola Vighi

**Cover Artwork**
*nOne-sided story*, 2024
by Andreas Moutsios-Rentzos and Ioanna Kloni

**Layout Design**
Bożena Maj-Tatsis
Konstantinos Tatsis

ISBN: 978-83-8277-174-9



Nakład: 110 egz.

# TABLE OF CONTENTS

# INTRODUCTION

Educational resources are essential in mathematics teaching, as they play a mediating role between the teacher and the student. Their design and their use have been at the focus of many studies, and this has affected our choice for the theme of this volume. Pepin and Gueudet (2020) define curriculum resources in mathematics as:

> all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom. Hence, curriculum resources would include the following:
>
> Text resources (e.g., textbooks, teacher curricular guidelines, websites, worksheets, syllabi, tests)
>
> Other material resources (e.g., manipulatives, calculators)
>
> Digital-/ICT-based curriculum resources (e.g., interactive e-textbooks) (pp. 172–173)

The current volume contains works that refer to all the above categories. Before we present the sections of the volume, it is important to provide a brief overview of the field, in relation to current research. Rezat et al. (2021) in their survey paper, view resources as instruments of change in: the mathematical content which is taught, the innovations in teaching and the students' beliefs and attitudes concerning mathematics. Indeed, many – if not all – reforms in mathematics education are accompanied by a series of resources. Another aspect of resources is their use by the teachers and the students. Pepin and Gueudet (2020) view this as a two-way interactive process since:

> (1) the resource's features influence the subject's activity and learning (for teachers, this can lead to policy choices, drawing on resources as a means for teacher education); at the same time, (2) the subject shapes his/her resources, according to his/her knowledge and beliefs. (p. 174)

The last aspect of resources we could consider is their design; a variety of approaches have been suggested, especially on textbook and ICT-based resources. Textbook analyses may either focus on the content (Pepin & Haggarty, 2001) or on the students' experiences (Norberg, 2023). Digital resources allow for a flexible adaptation and (re)design, therefore they can adjust to the needs of various students within the same classroom. They usually come with a comprehensive assessment scheme, which makes the teacher's decisions easier. However, many factors are at play during the implementation of such tools, such as their affordances and their users' skills and attitudes concerning their use.

In the present volume, 19 chapters provide a wide range of approaches to the uses of educational resources in the mathematics classroom. These chapters are placed in three parts, based on their content.

Part 1, entitled *Using educational resources for mathematics learning* contains seven works that present ways that resources in the form of materials, textbooks, tasks or activities, can be used to enhance the learning of mathematics, usually with a focus on a specific concept or field.

Part 2, entitled *Educational resources as research-based tools* contains seven works that present ways of connecting research on resources with their implementation in practice, in order to improve students' or preservice teachers' mathematical skills.

Part 3, entitled *Digital resources in the mathematics classroom* puts to the fore the use of digital tools to support students and preservice teachers. The five works included, refer to tools that vary from digital media to online assessment and investigate topics such as the resources' potentials and how their use may affect students' performance.

Overall, the present volume provides sufficient data to support the claim that educational resources constitute an important and evolving field of research in mathematics education. At the same time, the works contained in the volume, stress the need for a careful implementation of such resources, which should be based on concrete results of research. The reader who wishes to conduct such research may find useful examples of relevant studies in this volume.

Rzeszów, Poland, June 2024

The Editors

**References**

Norberg, M. (2023). Young students' meaning-making when working with mathematics textbooks – A multimodal study focusing on the designed and the discovered. *Research in Mathematics Education, 25*(2), 194–218.

Pepin, B., & Haggarty, L. (2001). Mathematics textbooks and their use in English, French and German classrooms: a way to understand teaching and learning cultures. *Zentralblatt für Didaktik der Mathematik, 33*(5), 158–175.

Pepin, B., & Gueudet, G. (2020). Curriculum Resources and Textbooks in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 172–176). Springer.

Rezat, S., Fan, L., & Pepin, B. (2021). Mathematics textbooks and curriculum resources as instruments for change. *ZDM Mathematics Education, 53*(6), 1189–1206.

# Using educational resources for mathematics learning

# Part 1

# THE PATHWAYS OF ‘ADDITIONAL’ EDUCATIONAL MATERIALS UP TO THE MATHEMATICS CLASSROOMS

Chrysanthi Skoumpourdi

University of the Aegean, Greece

*The teaching and learning of mathematics, due to its abstract nature, are enhanced through the utilization of provided and/or ‘additional’ educational materials. The wide variety of available ‘additional’ mathematics’ educational materials requires teachers to take specific decisions regarding their integration into their teaching practice. The definition of the concepts of provided and ‘additional’ educational materials (AEMs), the reflection of educators’ decisions on the placement of these educational materials in the mathematics classroom, as well as the determination of the relationship between teachers, students, and AEMs are issues that will be discussed both theoretically and practically, using specific examples. Investigating the pathways that AEMs could follow, up to the mathematics classrooms, as fundamental aspects of their designed integration, one can recognize their multidimensional relationship with teachers, as well as the learning opportunities they may (or may not) offer to students.*

## INTRODUCTION

The abstract nature of mathematics necessitates the use of educational resources in the mathematics classroom in order to concretize the concepts. The designed integration of educational resources in teaching and learning process has gained significant attention in research and education, especially in recent years, following a period of questioning their role. The positive outcomes resulting from their use have led to a reassessment of their importance and a recognition of their contribution to both the cognitive and socio-emotional domains (Skoumpourdi, 2021). Researchers internationally examine the impact of integrating materials and other means in the educational practice and highlight their contribution to facilitating teaching and learning process (Meira, 1998), fostering positive attitudes (McCulloch Vinson, 2001), enhancing self-confidence in problem-solving (Jacobs & Kusiak, 2006), supporting communication (Domino, 2010), and improving the performance of all students (Liggett, 2017; Swan & Marsall, 2010).

## DEFINING THE CONCEPT OF ‘ADDITIONAL’ EDUCATIONAL MATERIALS (AEMS)

The haptic resources used in mathematics education can be categorized as either ‘existing materials’, which are materials that exist independently of the teaching and learning of mathematics, and/or ‘specialized materials’, designed to support specific educational goals (Skoumpourdi, 2021). ‘Specialized materials’ encompass all types of educational materials and could be either ‘provided’ and/or ‘additional’ educational materials. ‘Provided’ educational materials refer to the school teaching package (curriculum, student textbook, workbook, teacher’s guide, etc.), typically provided to educators. ‘Additional’ educational materials (AEMs) are materials selected or designed to be integrated into the educational process in addition to the ‘provided’ educational materials.

The AEMs may include educational materials from different stages of the evolution of Mathematics Education:

1. Early Materials: Educational materials originating from the beginning of Mathematics Science (such as abacus, counting frame, ruler, tangram, etc.),
2. Evolutionary Materials: Educational materials, stemming from the evolution of Mathematics Education, constructed by significant researchers (such as Cuisenaire rods, Cattegno geoboard, Dienes blocks, etc.),
3. Developmental Materials: Educational materials resulting from the development of Mathematics Education, which include:
    a. Redesigns of previous materials (such as arithmetic rack, number line, etc.),
    b. Contemporary educational materials (such as mathematical mirror, Sumblox, connecting shapes, etc.),
4. Custom Materials: Educational materials designed and constructed by those involved in the educational process (teachers, students, parents, researchers, etc.) to support the teaching and learning of mathematics.

## TEACHERS’ DECISIONS REGARDING THE INTEGRATION OF EDUCATIONAL MATERIALS INTO THE MATHEMATICS CLASSROOMS

Teachers’ decisions regarding the utilization of additional educational materials and the potential factors influencing those decisions are reflected in Casey’s model (2016). This model provides a research framework that has emerged as essential for considering the abundance of additional educational materials available from diverse sources. It illustrates the linear process of teachers’ decision-making regarding the use of such materials and the potential factors influencing it, divided into four phases: 1. motivation to consider materials, 2. discovery of materials, 3. evaluation of materials, and 4. preparation/adaptation of materials (see Figure 1).

These decisions are influenced by both subjective and objective factors. Subjective factors pertain to educators themselves and include their knowledge

(Parada & Sacristán, 2010), beliefs, experiences, memories, goals, teaching style (Clements, 1999; Moyer & Jones, 2004), teaching experience (Sherin & Drake 2009), and educational level at which they teach (Skoumios & Skoumpourdi, 2021). Objective factors relate to the specific school environment, the broader educational framework, as well as the characteristics of the provided school teaching package (Janssen et el., 2015).

If educators perceive that the provided teaching package meets their needs and those of their students, and aligns with their teaching style, they may not be motivated to seek AEMs (see Figure 1). They may choose to utilize the school teaching package without deviation or modification, or they may adapt it through simple adjustments, additions, or minor changes (Brown, 2009; Davis et al., 2016). This scenario represents common practice in many countries worldwide where the school teaching package serves as the primary means for teaching mathematics in schools.

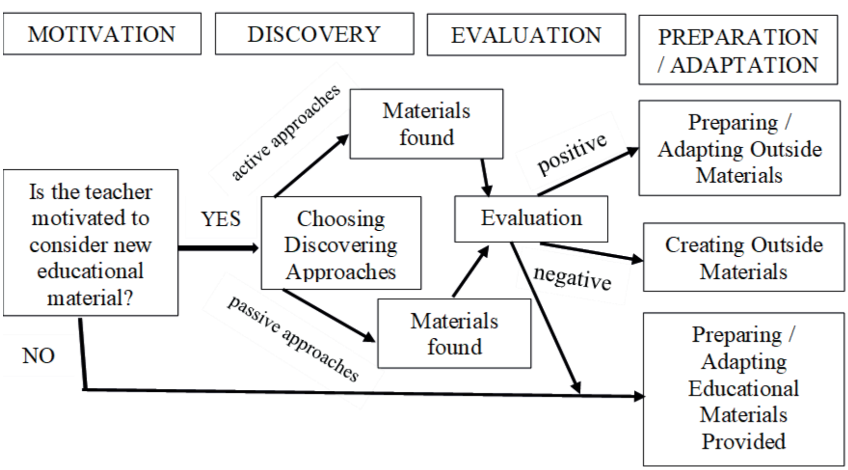



Figure 1: Teachers' decisions on outside educational materials and possible factors affecting them (Casey, 2016).

If educators find that the provided teaching package does not meet their needs, or if they are kindergarten teachers who do not have access to a teaching package, they are motivated to seek AEMs for their teaching practice and proceed to the next phases (see Figure 1). If educators, during their search, do not find suitable AEMs, or if they find such materials but do not positively evaluate them, they are inclined to adapt, prepare, or use the provided materials, create new educational materials, or continue searching for AEMs (see Figure 1).

## FRAMEWORK FOR EVALUATION OF MATHEMATICS EDUCATIONAL MATERIALS (FEMEM)

Of particular interest in the aforementioned phases is the focus on the criteria adopted by educators to evaluate AEMs for integration into mathematics classrooms. Teachers' primary criteria for positively evaluating AEMs for integration into mathematics classrooms are related to the alignment of the materials with curriculum standards (Davis et al., 2016; Roehrig et al., 2007), their teaching aims (Brown, 2009; Remillard, 2013), teaching practice (Janssen et al., 2015), and students' interests (Son & Kim, 2015). The criteria mentioned above are typically superficial in nature.

A systematic investigation into the main criteria affecting the evaluation led to the development of the Framework for Evaluation of Mathematics Educational Materials (FEMEM) (Skoumpourdi & Matha, 2021; Skoumpourdi, 2023) (see Figure 2). FEMEM analyzes the characteristics of the material and assesses its impact on the educational process and students' learning performance, as well as its metacognitive impact. It offers a systematic evaluation based on two main categories of criteria: subjective criteria and objective criteria. Subjective criteria concern the personal criteria used by educators to evaluate and select materials and are aligned with the criteria of motivation to consider additional educational materials. The objective criteria of the FEMEM are developed along three dimensions, as presented in Figure 2: 1) evaluation of AEMs before its use, 2) evaluation of AEMs during its use, and 3) evaluation of AEMs after its use.

The first dimension of the FEMEM, 'evaluation before use', is shaped by 'pedagogical' and 'mathematical validity' (see Figure 2). 'Pedagogical validity', examines factors important for attracting and using the material by many users regardless of the social context. These factors include the acceptance of the material for educational use (i.e., whether it is of high quality, safe, durable, and appropriately sized for the classroom) and its feasibility to procure (i.e., its availability, accessibility, and affordability). 'Mathematical validity', concerns the quality of the correspondence between the material and the mathematical idea/concept it represents (see Figure 2). It examines whether the material covers the mathematical idea/concept, the mathematical accuracy of the represented idea/concept, and the visibility of the material's function for the specific mathematical idea.

The degrees of mathematical accuracy (2nd phase of "mathematical validity") could be analyzed according to the three following levels:

Level 1. The material has a low degree of accuracy because it represents the concept in a way unrelated to the mathematical definition (e.g., incorrect representations or names of shapes).

Level 2. The material has a moderate degree of accuracy because it represents the concept with relative relevance to the mathematical definition (e.g., tangible materials, even with a negligible 3rd dimension, which represent plane shapes).

Level 3. The material has a high degree of precision because it represents the concept with complete relevance to the mathematical definition (e.g., mathematically precise iconic representation of plane shapes).

Similarly, the visibility of the material's function for the specific mathematical idea (3rd phase of "mathematical validity"), could also be analyzed across three levels:

Level 1. Low connection of the material and the mathematical concept when the image of the material does not refer to the mathematical concept (e.g., geoboard).

Level 2. Moderate connection of the material and the mathematical concept when the image of the material indirectly refers to the mathematical concept (e.g., tangram).

Level 3. Strong connection of the material and the mathematical concept when the image of the material directly refers to the mathematical concept (e.g., 3d models of solid shapes).

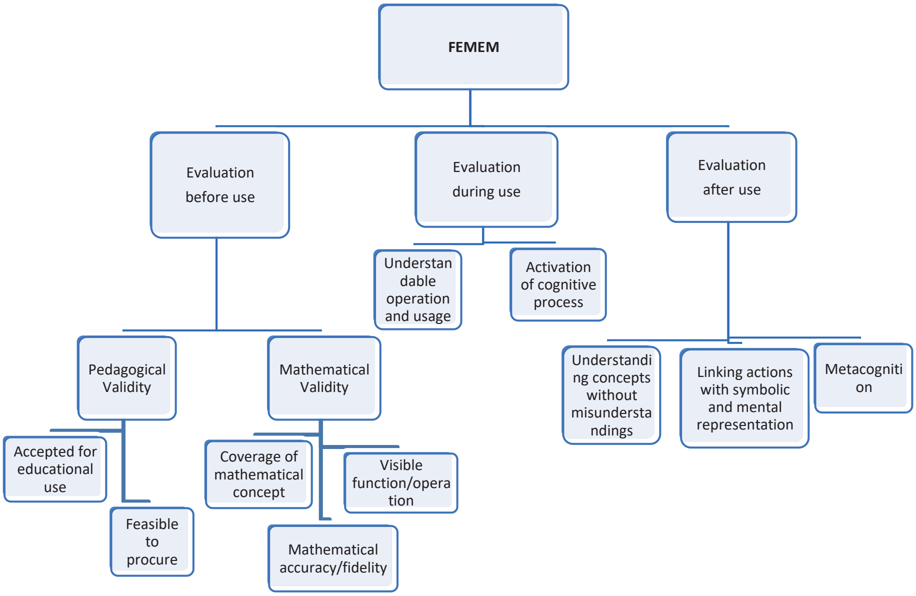



Figure 2: Framework for Evaluation of Mathematics Educational Materials (FEMEM) (Skoumpourdi, 2023).

The second dimension of the FEMEM, 'evaluation of material during its use' concerns the understanding of its operation and the way of using it to discover its internal mechanism. It also involves the activation of students' cognitive process that could emerge from its use.

The third dimension of the FEMEM, 'evaluation after its use', assesses the learning effectiveness for the material' users within a particular educational context. It evaluates the connection of actions with the material through symbolic and conceptual representation, understanding of the concept without misconceptions, and the enhancement of metacognition.

Before using the above framework, it must be clear: i) which material will be evaluated? ii) for which mathematical concept will this material be evaluated? iii) for which age group will this material be evaluated?

**AEMS FOR THE DEVELOPMENT OF SHAPES' CONCEPT**

According to van Hiele's theory (1986), the development of children's geometrical thinking is more influenced by the teaching methods and educational materials used than by age or biological maturity. Clements and Sarama (2007) suggest that young children can better understand geometric shape when their learning environment includes four key characteristics: 1) varied examples and counterexamples of the shape, 2) discussion about the shape and its characteristics, 3) presentation of a variety of other types of shapes, and 4) dealing with topics that are interesting to young children.

For young children to construct the concept of plane geometric shapes, they must simultaneously understand the shape's dual nature – the conceptual and the schematic. This can be achieved through activities such as recognition, naming, classifying, analyzing, synthesizing, and constructing geometric shapes. These activities align with the objectives outlined in international curricula for geometric shapes (Clements & Sarama, 2007).

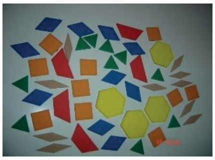


Figure 3a: Pattern blocks.

Figure 3b: Tangram.

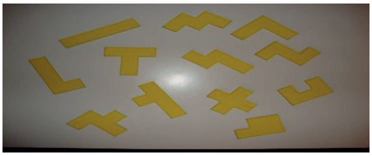


Figure 3c: Pentominoes.

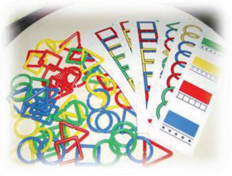


Figure 4a: Connecting shapes.

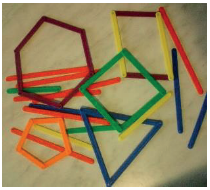


Figure 4b: Anglegs.

Figure 5a: Shape stamps

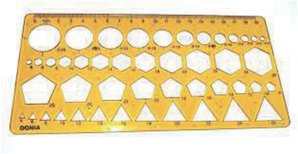


Figure 5b: Shape stencils

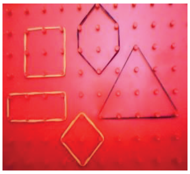


Figure 5c: Geoboard

The AEMs commonly used for approaching plane geometric shapes could be outlined as following (Skoumpourdi, 2023):

Models of shapes:

1. 'With internal filling' [like Pattern blocks (Figure 3a), Tangram (Figure 3b), Pentominoes (Figure 3c), etc.],
    a. 'Without internal filling' [like 'connecting shapes' (Figure 4a), 'anglegs' (Figure 4b), etc.],
    b. Artifacts for constructing plane geometric shapes [shape stamps (Figure 5a), shape stencils (Figure 5b), Geoboard (Figure 5c), etc.],
2. Picture books for shapes,
3. Games for shapes, as well as,
4. Shapes' cards, worksheets, art, objects, etc.

Given the wide variety of AEMs for shapes, a question arises about their suitability for teaching the concept of shapes to children of early primary school age. In other words, how can teachers evaluate these materials to decide whether to integrate them into their teaching practice?

**Indicative examples of evaluation of AEMs for shapes**

Based on the FEMEM (Table 1) we evaluated some indicative examples of AEMs for shapes[1] in early mathematics education (aimed at children aged 4-7 years old), such as pattern blocks, tangrams, connecting shapes, shape stamps, shape stencils, and geoboards. These materials are commonly used for teaching shapes and are available on the market (Skoumpourdi, 2023).

Beginning with the 'evaluation before use' we can confirm that all the materials have been positively evaluated in terms of their *pedagogical validity*. This indicates their acceptance for educational purpose, with feasibility in procurement (Table 1). They exhibit high quality, safety, durability, and

[1] The use of any tangible educational material—artifacts that possess three-dimensional quantities, albeit with a negligible third dimension—for teaching the concept of plane shapes to early elementary school children should commence with the understanding that they are selected due to their concrete nature, despite not being mathematically precise.

suitability in size for classroom use. Moreover, they are readily available, accessible, and affordable.

In assessing the *mathematical validity* of the materials, we consider whether each one covers the mathematical idea/concept, represents it accurately, and makes its function visible. Based on the evaluation, it appears that all components of the materials are visible and accurately represent one or more basic plane shapes in their stereotypical form, thus covering aspects of the mathematical concept (Table 1).

The representation of shapes is valid for all the materials, with the exception of the connecting shapes, due to the presence of notches and rounded tops, rendering their representations incomplete. Notably, not all the basic plane shapes are included, and even those that are included are limited to only one stereotypical representation.

When 'evaluating the materials during their use', it is necessary to consider research findings from their practical application. However, for the pattern blocks, the connecting shapes, the shape stamps, as well as for the shape stencils, we did not find any research results. For the tangram, based on empirical data, it appeared that children only understood its operation and internal mechanism after receiving explanations and directions from their kindergarten teacher. Although children recognized the main shapes of the tangram, except for the rectangle, they were unsure how to manipulate them, requiring explanation. This explanation was accompanied by cards with pictures that could be constructed using the tangram's seven pieces. Children's efforts to manipulate the pieces and recreate the given picture activated their cognitive processes. They were able to process information, make connections between shapes, and recreate the figures depicted in the pictures.

<table>
<tr><th colspan="2">FEMEM</th><th>Math Concept</th><th>Pattern blocks</th><th>Tangram</th><th>Connecting Shapes</th><th>Shape stamps</th><th>Shape stencil</th><th>Geo-board</th></tr>
<tr><td>Evaluation before use Pedagogical Validity</td><td>i) Accepted for educational use<br>ii) Feasible to procure</td><td></td><td>i<br>ii</td><td>i<br>ii</td><td>i<br>ii</td><td>i<br>ii</td><td>i<br>ii</td><td>i<br>ii</td></tr>
<tr><td rowspan="4">Evaluation before use Mathematical Validity</td><td rowspan="4">iii) Coverage of mathematical concept<br>iv) Math accuracy/ fidelity<br>v) Visible function</td><td>Circle recognition & construction</td><td>-</td><td>-</td><td>iii, -</td><td>iii, iv</td><td>iii, iv</td><td>-</td></tr>
<tr><td>Square recognition & construction</td><td>iii, iv</td><td>iii, iv</td><td>iii, -</td><td>iii, iv</td><td>iii, iv</td><td>iii, iv</td></tr>
<tr><td>Rectangle recognition & construction</td><td>iii, iv</td><td>-</td><td>-</td><td>iii, iv</td><td>iii, iv</td><td>iii, iv</td></tr>
<tr><td>Triangle recognition & construction</td><td>iii, iv</td><td>iii, iv</td><td>iii, -</td><td>iii, iv</td><td>iii, iv</td><td>iii, iv</td></tr>
</table>

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Shape analysis | **iii, iv** | **iii, iv** | **iii, iv** | **iii, iv** | **iii, iv** | **iii, iv** |
| | | Shapes synthesis | **iii, iv** | **iii, iv** | **iii, iv** | **iii, iv** | **iii, iv** | **iii, iv** |
| | | | **v** | **~** | **v** | **v** | **v** | **~** |
| **Evaluation during use** | **vi)** Understandable operation and usage | | | ~ | | | | ~ |
| | **vii)** Activation of cognitive process | | | **vii** | | | | **vii** |

Table 1: Evaluation of pattern blocks, tangram, connecting shapes, shape stamps, shape stencil, and geoboard, based on FEMEM (Skoumpourdi, 2023). [(i, ii, iii, iv, v, vi, vii, viii, ix, x: positively evaluated according to the corresponding criterion), (- does not match the criterion), (~ relevant after teacher's explanations), blank cell denotes the lack of research data]

Furthermore, the function of the geoboard is not evident to young children from the image of the artifact; it requires explanation. According to research results, it seemed that children did not understand its use from their first contact with the artifact (Skoumpourdi & Kossopoulou, 2011). Initially, none of them were able to use it to construct shapes. Only a few children (3 out of 15) attempted to use it, but they simply placed the rubber band on the board, leaving it in its natural shape. However, after the kindergarten teacher presented examples and guided them through various questions and activities, the geoboard transformed into an instrument for children to construct plane geometric shapes in different forms and sizes. There were three levels of difficulty in the constructions: a. Children who constructed the shapes with very few and specific movements. b. Children who constructed the shapes after making many alterations to their shapes. c. Children who found the whole process very challenging.

The children approached their constructions in three different ways: 1. They started from one nail. 2. They started from many nails. 3. They started by giving shape to the rubber, stretching it in their hands, and then placing it on the geoboard. The stretching and securing of the rubber band on the nails, as a condition for the creation of a shape, the overlapping of parts of the shapes during their composition, as well as the possibility for quick changes and/or corrections of the shapes, were not realized by the children in all cases. In other words, it appeared that while all the children integrated the geoboard as an instrument into their activity and adapted it to their needs, not all of them understood its limits and possibilities.

The 'evaluation of the materials after their use' was not possible due to a lack of research data on their long-term usage outcomes.

Although we initially agreed to select tangible educational materials for their concrete nature in teaching the concept of plane shapes, as teachers and researchers, we must reconsider the tangible forms of these artifacts. We began with models of shapes with 'internal filling,' commonly used in teaching plane geometric shapes. However, this approach raises questions about their suitability as educational materials. For instance, how would we present the square pattern block to children? Would we present it as the square geometric shape? If we present it as 'the square' and a child stacks multiple square pattern blocks, when does the square cease to be a square and become something else, like a cube or a rectangular parallelepiped? The three-dimensional nature of these materials poses a significant challenge in teaching plane geometric shapes and contributes to misconceptions. Additionally, the specificity and 'stereotypical' forms of these artifacts serve as further sources of misunderstanding.

In regard to the models of shapes 'without internal filling', one can argue that they are more suitable for teaching plane geometric shapes because they better represent the image of plane shapes compared to 'filled models'. Is this statement universally true? Perhaps not. We must consider whether these models serve as representative examples of plane shapes or as counterexamples due to factors such as unclear angles, apexes, discontinuities, and notches in the shape (as seen in connecting shapes), as well as unstable shapes (like anglegs), etc. When do we accept such a shape as an example, and when do we classify it as a counterexample? What are the boundaries between them? It appears that even in this scenario, the primary disadvantages of their three-dimensional nature and their limited variety persist, along with the lack of geometric shape accuracy.

Young children are not able to construct geometric shapes with a ruler and a compass. Thus, there are artifacts such as 'shape stamps' (Figure 5a), 'shape stencils' (Figure 5b), 'geoboard' (Figure 5c), etc., that could help them in this direction. Most of these artifacts provide children with the opportunity to easily construct and deconstruct shapes, highlighting the two-dimensional nature of the shapes. The constructions with these artifacts are either designs and imprints on paper or shapes made with rubber bands.

Of course, stamping shapes and drawing shapes with stencils require different skills and knowledge from the children than using rubber bands on a geoboard. In the former case, knowledge of the specific characteristics of the shape is not necessary, whereas in the latter case, it is essential. Furthermore, not all these artifacts facilitate the creation of shapes in different positions, sizes, and orientations. For example, the rectangle, square, and triangle are presented in their standard forms, and the circle cannot be precisely replicated on the geoboard due to its polygonal representation. Additionally, it is not easy to create compositions of shapes with these materials.

These inconsistencies should be known to educators who will use these materials so they can manage them accordingly. It is also these inconsistencies that lead to the construction of new education materials aimed at overcoming the weaknesses and limitations of the aforementioned materials (Skoumpourdi & Mpakopoulou, 2011). Furthermore, it is remarkable that the selection and integration of educational materials, even the most appropriate ones, in the teaching practice, does not automatically lead to understanding and learning (Boulton-Lewis et al., 1997). It cannot be assumed that they will highlight children's mathematical reasoning or provoke their cognitive activity. For example, if a child uses pattern blocks to construct something, it is not certain that his/her actions will acquire mathematical meaning, that their actions will lead to the desired learning outcome, or that simple or complex strategies will emerge (Drews, 2007).

The user needs to understand the function of the material and develop a relationship with it. The artifact itself has no instrumental value; it does not automatically define the role of the user. The artifact becomes an instrument when the individual is able to adapt it to his/her own needs and integrate it into their activities through the construction of personal meanings. During the process of transforming an artifact into an instrument, known as instrumental genesis (Zbiek et al., 2007), the individual interacts with the artifact and acquires knowledge that can, on one hand, shape the artifact to serve his/her goals, and on the other hand, shape his/her understanding through the instrument.

Often, the selection and use of AEMs for teaching and learning of mathematics are done without proper planning. Most of the times, their manipulation follows a routine manner. In these cases, children learn to manipulate materials in a predetermined way and cannot understand the mathematical structure involved in the process. Other times, the actions with the materials are completely free, without structure, and therefore cannot constitute a learning activity since there are no motivations for action and defined goals. In these cases, the material does not become an instrument for the children but remains an artifact to be used, resulting in children being unable to connect their actions with the material to mathematical concepts and thus not being led to understanding.

Other two factors that contribute to the positive relationship between materials and learning include: A) Students' awareness of the connection between the material and the mathematical concept being taught, as well as their understanding that the individual elements of the material relate to elements of the concept, and B) Students' ability to translate their interactions with the material into visual and mental processes for constructing the mathematical concept. This can be achieved by encouraging children not only to manipulate the materials but also to reflect on their actions, model them, describe them (Moyer, 2001), and communicate them to achieve a deeper understanding. The description should focus on presenting the actions, explaining the emerging strategies and results, and providing documented justifications for the actions. Communication should not be limited to oral and written expression but should also include other multimodal forms. Through the transfer of students' actions from empirical mathematical entities to visual and abstract ones through the organization of information, establishing appropriate connections and making generalizations, learning will occur.

## IN CONCLUSION

In examining the possible pathways of AEMs integration into mathematics classrooms, their multifaceted relationship with teachers and the learning opportunities they may or may not afford students were presented. This complicated relationship with teachers includes decisions regarding motivation, discovery, evaluation, as well as preparation and adaptation of AEMs to define their way up to the mathematics classroom. The designed integration of AEMs into the teaching and learning process, along with the strategic orchestration of their usage by teachers, establishes the groundwork for supporting children's mathematical development.

The FEMEM can serve as a valuable evaluation tool for educators and researchers. The evaluation of the 'pedagogical validity' of the material, before its use, distinguishes those suitable for educational use, feasible to procure, and therefore usable in practice. Similarly, the evaluation of the 'mathematical validity' of the material before its use identifies materials that cover the target concept with scientific validity, clear structure, operation and utility, making them suitable for practical application. The evaluation during the use of the material distinguishes those that are comprehensible, with clear operation, and effective in facilitating cognitive processes, thereby enhancing learning outcomes. Finally, the evaluation of the material after its use identifies materials that have fostered the development of mental representations linked to the target concept, have facilitated knowledge acquisition without leading to misconceptions, and can therefore be regarded as effective tools for conceptual learning.

From the evaluation of indicative educational materials for shapes, it became apparent that no single material could comprehensively address all aspects of shape construction in early years mathematics. A comprehensive grasp of each material's unique characteristics, the mathematical concepts they provoke, and the interconnections between representations is indispensable for identifying mathematical concepts, fostering generalizations, and facilitating the transition to abstract thinking. Designing why, when and how those materials would be employed, ensures the conditions for transforming them from artifacts into effective educational materials.

The above mentioned offers elements to be taken into consideration for designing and developing more sophisticated and synthetic forms of educational materials, such as multi-materials, which can be utilized to construct the mathematical concepts. By systematically evaluating AEMs based on these criteria, educators can make informed decisions about which materials are best suited for integration into mathematics classrooms to support student learning and achievement.

**References**

Boulton-Lewis, G., Cooper, T., Atweh, B., Pillay, H., Wilss, L., & Mutch, S. (1997). Processing load and the use of concrete representations and strategies for solving linear equation. *Journal of Mathematical Behavior, 16*(4), 379–398.

Brown, M. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B.-A. Herbel-Eisenmann & G.-M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). Routledge.

Casey, A. (2016). *Going beyond the provided curriculum: Teacher's investigations of outside mathematics materials* [Doctoral dissertation, University of California]. https://escholarship.org/uc/item/6h962882.

Clements, D. (1999). "Concrete" manipulatives, concrete ideas. *Contemporary Issues in Early Childhood, 1*(1), 45–60.

Clements, D. & Sarama, J. (2007). Early childhood mathematics learning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning. A project of the National Council of Teachers of Mathematics* (pp. 461–555). Information Age Publishing.

Davis, E., Janssen, F., & Van Driel, J. (2016). Teachers and science curriculum materials: where we are and where we need to go. *Studies in Science Education, 52*(2), 127–160.

Domino, J. (2010). *The effects of physical manipulatives on achievement in mathematics in grades K-6: A meta-analysis* [Unpublished doctoral dissertation]. State University of New York at Buffalo.

Drews, D. (2007). Do resources matter in primary mathematics teaching and learning? In D. Drews & A. Hansen (Eds.), *Using Resources to Support Mathematical Thinking, Primary and Early Years* (Chapter 2). Learning Matters Ltd.

Jacobs, R.-V. & Kusiak, J. (2006). Got tools? Exploring children's use of mathematics tools during problem solving. *Teaching Children Mathematics, 12*(9), 470–477.

Janssen, F.-J.-J.-M., Westbroek, H.-B., & Doyle, W. (2015). Practicality studies: How to move from what works in principle to what works in practice. *Journal of the Learning Sciences, 24*(1), 176–186.

Liggett, R.-S. (2017). The impact of use of manipulatives on the math scores of Grade 2 students. *Brock Education Journal, 26*(2), 87–101.

McCulloch Vinson, B. (2001). A comparison of preservice teachers' mathematics anxiety before and after a methods class emphasizing manipulatives. *Early Childhood Education Journal, 29*(2), 89–94.

Meira, L. (1998). Making sense of instructional devices: The emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education, 29*(2), 121–142.

Moyer, P. (2001). Are we having fun yet? How teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics, 47*(2), 175–197.

Moyer, P. & Jones, G. (2004). Controlling choice: Teachers, pupils and manipulatives in mathematics classrooms. *School, Science and Mathematics, 104*(1), 16–31.

Parada S.-E. & Sacristán, A.-I. (2010). Teacher's reflections on the use of instruments in their mathematics lessons: A case-study. In M. Pinto & T. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 25–32). Belo Horizonte, Brazil: PME.

Remillard, J. (2013). Examining resources and re-sourcing as insights into teaching. *ZDM Mathematics Education, 45*(7), 925–927.

Roehrig, G., Kruse, R., & Kern, A. (2007). Teacher and school characteristics and their influence on curriculum implementation. *Journal of Research in Science Teaching, 44*(7), 883–907.

Sherin, M.-G. & Drake, C. (2009). Curriculum strategy framework: Investigating patterns in teachers' use of a reform-based elementary mathematics curriculum. *Journal of Curriculum Studies, 41*(4), 467–500.

Skoumios, M. & Skoumpourdi, C. (2021). The use of outside educational materials in mathematics and science: Teachers' conceptions. *International Journal of Education in Mathematics, Science, and Technology, 9*(2), 314–331.

Skoumpourdi, C. (2021). *Σχεδιασμός ένταξης υλικών και μέσων στη μαθηματική εκπαίδευση των μικρών παιδιών* [Designing the integration of materials and means in young children' mathematics education]. Patakis.

Skoumpourdi, C. (2023). *Διερευνητική προσέγγιση των μαθηματικών της πρώτης σχολικής ηλικίας: Κριτήρια σχεδιασμού δραστηριοτήτων και αξιολόγησης υλικών* [An inquiry approach to early school mathematics: Criteria for designing activities and evaluating materials]. Kallipos, Open Academic Editions.

Skoumpourdi, C. & Kossopoulou, A. (2011). Ο γεωπίνακας ως εργαλείο για την κατασκευή σχημάτων από νήπια [The geoboard as an instrument for shape construction by kindergartners]. In M. Kaldrymidou & X. Vamvakousi (Eds.), *Proceedings of the 4th Conference of the Greek Association of Researchers in Mathematics Education: The classroom as a field of development of mathematical activity* (pp. 441–447). University of Ioannina.

Skoumpourdi, C. & Matha, A. (2021). Framework for evaluating math educational materials for constructing early number concept. *International Journal on Studies in Education, 3*(1), 48–60.

Skoumpourdi, C. & Mpakopoulou, I. (2011). 'The Prints': a picture book for pre-formal geometry. *Early Childhood Education Journal, 39*(3), 197–206.

Son, J.-W. & Kim, O.-K. (2015). Teachers' selection and enactment of mathematical problems from textbooks. *Mathematics Education Research Journal, 27*(4), 491–518.

Swan, P. & Marshall, L. (2010). Revisiting mathematics manipulative materials. *Australian Primary Mathematics Classroom, 15*(2), 13–19.

Zbiek, R., Heid, M., Blume, G., & Dick, T. (2007). Research on technology in mathematics education. In F. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning. A project of the National Council of Teachers of Mathe*matics (pp. 1169–1207). Information Age Publishing.

# EARLY CHILDHOOD TEACHERS' UNDERSTANDING OF A PLANE ROTATION TASK

Ewa Swoboda[*], Marta Pytlak**

[*]State Higher School of Technology and Economics in Jarosław, Poland

[**]University of Rzeszow, Poland

*In mathematics education at the early school level, "educational resources in the mathematics classroom" are understood not only as the latest technical innovations, but above all as textbooks. It is important for teachers to be able to use them appropriately, making full use of the potential of the tasks contained in the textbooks. The very teaching of mathematics at this level often requires a dynamic approach, especially in the context of considering issues related to geometry. It is important to form at this early level the intuitions of concepts that appear openly in teaching at higher educational levels. For a good educational process, it is important that teachers are well prepared and able to use the materials presented to them flexibly. Therefore, future teachers' mathematical knowledge and intuitions related to mathematical issues are important. The research conducted among future early school teachers described in the paper shows their understanding of issues related to rotational movement. The results of the research show both the great potential of students in this area, as well as areas for in-depth work on better understanding the issue of rotational motion on a plane.*

## JUSTIFICATION OF THE RESEARCH PROBLEM

When we hear the phrase 'educational resources in the mathematics classroom', we generally think that it will be related to the latest technological achievements or new forms of classroom management. Indeed, much attention is paid to the use of appropriate computer programmes, or new trends such as STEAM, which exploit the multifaceted nature of learning in a non-classroom environment. Access to technological advances is, on the one hand, highly desirable, but often disturbing. Many concerns and questions are raised about access to artificial intelligence in general: how to use it effectively in teaching, how to defend against unwanted (unethical) use by students, e.g., when solving homework assignments. Undoubtedly, any research in this area is a requirement of the modern world.

Regardless of emerging innovations, extensive research shows that what is commonly used in schools is the textbook (Haggarty & Pepin, 2002; Silverman & Even, 2015) or an e-textbook (Kamińska, 2015). Its uses can be varied (Fan & Kaeley 2000; Stylianides, 2014). But even focusing only on the proposed textbook tasks opens up a huge research area (Novotna et al., 2023; Sosniak & Stodolsky, 1993).

A very specific didactic issue is the question of textbooks for mathematics at early educational levels. In Poland, for some time there was a belief that for students at this level, exercise books would be enough because the content of the education would be presented to them by the teacher. This trend has changed, but there is still an opinion that the educational content contained in the textbook should clearly refer to what is contained in the Core Curriculum. The Ministry of National Education points out that the textbook is supposed to help students and teachers implement the core curriculum, but it should be remembered that the teacher may use a textbook but is free not to do so. You can use the textbook, but you don't have to. However, only that this form of work (without a textbook) is chosen by a small group of conscious teachers (Bieńkowska-Wójcik et al., 2014). It is very difficult, especially at this educational level, to convince teachers that textbook proposals should be looked at more broadly, more flexibly, not only through the prism of compliance with the core curriculum. In particular, it should be taken into account that the specific nature of mathematics teaching requires that intuitions of concepts that overtly appear in teaching at higher educational levels should be formed at this early level.

In our considerations, we will focus on one task addressed to early school students. The task as such is not yet included in any of the educational packages existing in Poland. It is a proposition resulting from our research on the intuitive understanding of rotation (isometric transformation on a plane) by 9-10-year-old students.

## A DYNAMIC APPROACH TO TEACHING GEOMETRY AS A DIDACTIC ISSUE

Geometric reasoning has its own specificity, which is still not fully explored. There are many indications that the formation of geometric concepts proceeds differently from that of arithmetic concepts. Researchers emphasize that the first source for basic geometric concepts is visual information, but for the development of geometric reasoning the imagery of movement is needed.

The student should be able to imagine the effect of certain actions performed *on* a geometrical object or its elements (e.g., the making of a cross-section, the extension of a height, changes in an internal angle), as well as the effect of actions performed *with* this object (e.g., the rearrangement of a block in a construction made of cubes, parallel displacement or rotation around a fixed point by a certain angle). It is only through such ideas that it will be possible to solve a large group of geometrical problems. In this way, dynamic reasoning is an essential skill for a certain class of geometric problems.

The need for specific interventions related to the formation of such skills has been voiced by researchers gathered around Milan Hejný (Hejný 2000; Hejný, Jirotková, & Slezaková, 2007, 2008; Hejný, Jirotková, Slezaková, & Michalcova, 2009; Jirotková, 2016). These opinions convince us that proposals based on the use of computer techniques are not sufficient here. The didactic problem of visualising movement in geometry teaching is often linked to the possibilities offered by the use of computer programs such as Cabri or GeoGebra, but there are also studies suggesting significant caution in their use, especially at lower educational levels (Hoyles, 1996; Jones, 1999, 2000).

Therefore, it is worth looking for teaching proposals that will allow the young pupil to act in the reality that surrounds him. Through the use of one's own experiences (the informal ones), it is possible to form intuitions regarding dynamic geometrical ideas. Such an approach is in line with the whole path that humanity has gone through, when geometrical concepts and procedures were formed on the basis of practical activities (such as creating buildings, moving in space, constructing tools or using design) (Hejny, 1990; Henderson & Taimina, 2005).

In Poland, the problem of searching for a didactic path directed at the mathematization of motion (leading students to discover the properties and description of certain isometric transformations) has a long-standing research tradition. One of the threads of this multidirectional research was the study of the possibilities of students, being at the pre-definitional level, to create the idea of plane rotation with respect to a fixed point.

## WHAT DO WE KNOW FROM PREVIOUS RESEARCH

The origins of research related to introducing students to an understanding of isometric transformations were strongly inspired by suggestions from Weyl's (1952) series of lectures on symmetries. He wrote:

> (…) symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. Beauty is bound up with symmetry. (p. 3)

The conviction that the beginnings of learning about symmetries should be connected with the construction of patterns and ornaments is very strongly rooted in the Polish didactic tradition. This was also the origin of research, which in a sense confirmed this tradition (Jagoda & Swoboda, 2010, 2011; Swoboda & Vighi, 2016). However, at the next stage, a need arose to go beyond functioning in the world of ornaments (based on the relation of object position to object). There was a need to organise such a learning environment in which the pupil himself could experience the validity of the definitions of geometric transformations learnt in the older grades. The results of the research carried out in this direction suggested that it was worth exploring the teaching strand of rotation formation. A series of experiments was created, in which students formed their idea of the position of a figure in successive stages in motion relative to a fixed point (Swoboda & Zambrowska, 2023; Szkoła, 2016).

The results of recent research (Swoboda, Maj-Tatsis & Pytlak, in press) suggest that there are some specific teaching problems to which teachers should be sensitive when working on building intuition of rotation among children. These include:

1. Maintaining a constant shape after a transformation is not a significant pedagogical problem. It is only necessary to strive to limit technical issues related to shape replication, adapting to the manual skills of a student.
2. Students should be directed towards understanding the properties of a circle (center of rotation, constant radius length) at an early stage.
3. Maintaining a trajectory along the circle for all points of the figure is one of the main pedagogical challenges.
4. Another important pedagogical issue is drawing attention to maintaining a constant orientation of the figure. This is particularly crucial for understanding the defining description of axial symmetry.

However, since the issue of building intuitions of rigid movement is not explored among the topics addressed in early school education, it is worthwhile to build these intuitions, as it were, 'by the way' of implementing other topics. Such an approach is consistent with the third principle of the H-mat conception: *Interlinking topics: not isolating mathematical patterns*. The authors of this principle state:

> When we connect the individual topics with each other, especially using our own experience, we are easily able to deduce or recall a particular piece of knowledge. (…) We learn about new concepts, processes, problem-solving strategies, and phenomena in different environments, and we gain a good understanding of them by putting the puzzle-pieces of partial knowledge from the various environments, and from the various activities, together (H-mat o.p.s., 2024).

In our opinion, an environment that will bring students closer to the idea of rotational movement could be analyzing clockwise motion. The hands are rigidly fixed in the centre of the clock face, which naturally emphasises the function of the centre of rotation. However, it is necessary to structure the task in such a way that it is possible to draw attention to the additional defining conditions for rigid rotation. In addition, teachers need to be aware of which elements to pay attention to when analysing students' work.

However, can teachers look at task proposals broadly enough to use them to build intuition and associations with various mathematical topics? This problem was the main research question described in this study.

## METHODOLOGY OF RESEARCH

### The course of the research

The research was carried out on a group of 118 female Pedagogy students (Preschool and Early School Pedagogy), during the Methodology of Mathematical Education course, in February and March 2024. Geometrical topics had not yet been covered in this course. It was a deliberate procedure. Above all, we wanted to investigate whether future female early school education teachers can use the task to form such intuitions, which are beneficial for further learning about isometric transformations.

**Research objectives**

1. Will early school teachers recognise the geometric problem in the 'clocks' task?
2. What properties of rigid rotation on the plane relative to a fixed point can they incorporate in their own solutions?
3. How do they react to inappropriate student solutions - can they specify what the inappropriateness of the solution is?

**Research tools**

The entire study involved working on two worksheets.

**WORKSHEET 1** (*in the original the students had more space for the drawing)*
Anne has glued a triangle to the hour hand. The pointer is positioned at 12 o'clock (as shown in the picture below). How will the triangle be arranged if the pointer moves to 3 o'clock? And how if the pointer moves to 6 o'clock or 8 o'clock? Represent these situations in the figures below.

Figure 1: Worksheet 1.

The aim of this task was to investigate the intuition and knowledge of female students - future teachers - related to rotation relative to a fixed point. The manipulation of the hands of a clock is an activity that is always performed in school when implementing topics related to the measurement of time. In addition, it can be used to build intuition about rotation. It naturally builds up intuitions related to circular motion (centre of rotation, constant radius length).

We assumed that the positioning of the clock hand itself would not be a problem for future teachers. What was important was the arrangement of the triangle in such a way that its shape was preserved, its orientation on the plane was correct, and its correct positioning in relation to the circle (related to the preservation of the trajectory of movement along the arc of all points of the figure). Making several drawings may have strengthened the image of continuous motion. The solutions to this task were to be the basis for answering the research question:

what properties of rigid rotation in the plane relative to a fixed point can the students incorporate in their own solutions.

After completing the tasks from the first worksheet, students were asked to reflect on the presented task. They were to write what mathematical content this task was related to, and the answers provided were to be the basis for determining whether they saw in the task an opportunity to build intuitions related to the understanding of rotation.They were then given card two, presented below.

**WORKSHEET 2** (*modified for the purpose of the article - in the original, the* drawings *were placed one below the other;* additionally, *the space required to write the verbal answer was not included here*)

The second-grade students were solving a clock task. They drew how the triangle would be arranged when the hand moved to 6 o'clock. Some solutions are shown below. Please assess their correctness. If you think the solution is wrong, try to provide the type of error and its cause.

| Solution 1: | Solution 2: | Solution 3: | Solution 4: |
|---|---|---|---|
|  |  |  |  |
| Answer: | Answer: | Answer: | Answer: |

Figure 2: Worksheet 2.

The choice of six o'clock was deliberate - in relation to the starting arrangement (12 o'clock), the triangle made a 180-degree rotation. In one of our previous studies (Swoboda, Maj-Tatsis, & Pytlak, in press), this configuration was one of the more difficult ones.

Only the third proposal is correct. In the others, typical errors, appearing in our earlier studies, are presented: the first figure shows the rotation of the triangle 'as if on a mill wheel', the second one does not preserve the orientation of the plane (the triangle is turned inside out), in the fourth proposal only one point of the figure preserves the trajectory of movement along the circle.

The aim of this task was to investigate how female students - future teachers - react to student solutions, whether they can specify what the inappropriateness of a solution is.

**Analysis of the collected research material**

A rather surprising result of the analysis of the solutions to the tasks from sheet 1 was the finding that not all arrangements were equally difficult. Hour 3 posed no problems - the triangle retained its shape and size; its position relative to the circle was correct. Perhaps this is related to the sheer size of the angle of rotation (the angle being relatively small, a right-angle alignment is typical). The erroneous solutions (if any appeared here), related to the size of the triangle drawn (too small). It was also the case that the student did not understand the task at all, since she drew a triangle by connecting the ends of the two drawn clock hands.

The second alignment - at 6 o'clock - caused more problems. There were arrangements that did not preserve the orientation of space (i.e. the triangles were 'reflected' as in axial symmetry) - this was the most common error. It was also quite common to draw a triangle with a shape far from the model.

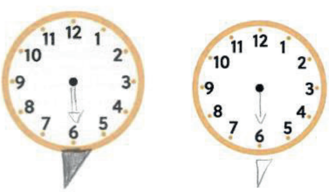


Figure 3: Typical mistakes made by student for 6 o'clock.

The most problematic was the alignment of the triangle at 8.00 a.m. Perhaps this was the result of work on the previous two positions, where the triangle made a rotation of $90^0$. In this alignment, the triangle changed shape (to isosceles, obtuse), and did not maintain an arc trajectory for all its points.

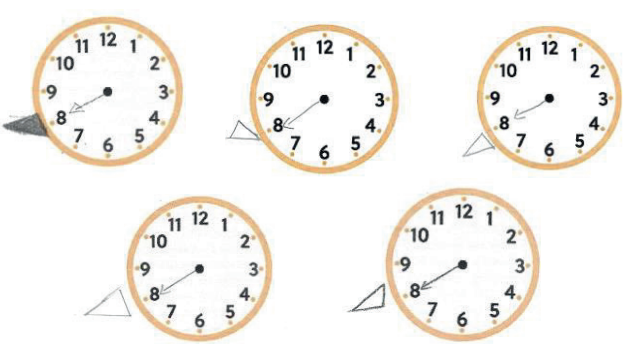


Figure 4: Example of students' solution for 8 o'clock.

A summary of these solutions is provided in Table 1.

| | 3 o'clock | | 6 o'clock | | 8 o'clock | |
|---|---|---|---|---|---|---|
| | Correct solution | Incorrect solution | Correct solution | Incorrect solution | Correct solution | Incorrect solution |
| Number | 107 | 11 | 96 | 19 | 91 | 27 |
| % | 90 | 10 | 81 | 19 | 77 | 23 |

Table 1: Students' solutions.

The analysis of the statements concerning the task itself also gave rise to interesting conclusions. The students related the task to more than one issue. This in itself is a good approach, but the types of answers are far from the expectations of geometric transformations. Most often, female students associated the task with the problem of time measurement. Regardless of the fact that the students drew the triangle in different positions in their work, when describing the purpose of the task they stated that it was about a clock and reading the hours. This approach to the task was dominant.

Equally frequent were responses related to geometric figures: *triangle, recognition of triangle, right-angled triangle and even view of triangle from different perspectives*. There was sometimes criticism about the fact that other geometric figures do not appear in the task. There were also references to the angle (which could indicate an understanding of the angle as a measure of rotation)

Wherever there was a reference to the intuition of geometric transformations, there were formulations about rotational motion: *learning geometry - how a triangle rotates*, *rotating a geometric figure in a certain way, changing the position of a figure (every hour the triangle looks differently)*, *the position of the triangle as the clock hand moves, rotating a figure, implementing a rotation*. However, there were also inappropriate, or very broad, references: *geometric displacement, movement of a triangle, displacement of figures, symmetry (*without specification - what kind*), mirror symmetry.*

Some of the statements were so general that it is difficult to classify them into a specific category. For example, students stated that this task develops the imagination, is about geometry and even about projection. And such general statements were the most numerous.

Regardless of these comments, it can be concluded that the students used an intuitive idea of rigid rotational motion, mostly realizing all its properties. However, they are not aware of the mathematical issues involved, their intuitions are deeply hidden. The quantitative distribution of these responses is shown in the Table 2.

| | A clock and hours | Geometric figures | Intuitions of geometric transformations | others |
|---|---|---|---|---|
| Number | 88 | 56 | 18 | 77 |
| % | 75 | 48 | 15 | 65 |

Table 2: Categories of students’ responses.

**Analysis of the results from Worksheet 2**

Due to the different approach to each of the presented proposals, we will separately discuss the results regarding teachers’ attitudes towards each type of error.

**Proposition 1** Some participants said that this task was well solved. They wrote: *the triangle moves around the circle, the pointer moves without changing the alignment of the triangle, the triangle's position is constant*. These opinions are consistent with the approach observed in the research described in (Swoboda et al., 2024). However, there is a fundamental difference between teachers’ and students’ approach: very often teachers wrote that *the student redrew the triangle without changing the position, the final alignment the same as the initial alignment, nothing changed* (for students it was the rotation of the triangle ‘as if on a mill wheel’). Thus, it is not clear whether the assessment of this solution assumed any movement or not.

**Proposition 2**. In these responses, participants emphasised that the triangle is in a mirror image. Thus, they compare the initial position of the triangle (12 o’clock) to the current position (6 o’clock) - such an explanation was sometimes even illustrated in the sketch drawings made by the future teachers. Thus, there is no reference to the fundamental idea of rotational motion. After such a statement, some even stated that the task was well solved. It was also quite common to omit any justification.

**Proposition 3**. In general, respondents stated that the task was correctly solved. Quite surprisingly, in some cases this solution was considered to be incorrect. The argument given was that a mirror image was used. However, there was also an answer which indicated that a mirror image was used in the task and therefore the solution was correct(!).

**Proposition 4**. Almost everyone stated that this task was solved incorrectly. A sizable group felt that here the triangle had changed its shape(!) However, mostly students had trouble justifying why this solution was wrong. It was written in very general terms*: it is ‘crooked’*, *the alignment is not correct, it is connected to the circle with the wrong vertex*, *it should be adjacent to the clock with one side*, and even *the student did not know how to draw it*. No one mentioned that each point of the triangle in rotation determines a trajectory that is an arc, i.e. that the distance of each point of the figure from the centre of rotation should be constant. A quantitative summary of the results is shown in Table 3.

| | Task 1 | | Task 2 | | Task 3 | | Task 4 | |
|---|---|---|---|---|---|---|---|---|
| | correct solution | incorrect solution | correct solution | incorrect solution | correct solution | incorrect solution | correct solution | incorrect solution |
| Number | 15 | 103 | 25 | 93 | 114 | 4 | 7 | 111 |
| % | 12 | 88 | 21 | 79 | 97 | 3 | 5 | 95 |

Table 3: Student's responses in Worksheet 2.

## SUMMARY AND CONCLUSIONS

The analysis of the collected material raises ambivalent feelings. When solving the task, most of the students - future teachers - acted correctly. They were able to maintain both the size and shape of the proposed triangle, and its orientation on the plane (without using a mirror image). Intuitively, they kept a constant distance of all points of the triangle from the center of rotation, regardless of the size of the rotation angle. This is a very promising result, suggesting that the idea for the task itself is a good one, as it triggers those elements and properties that are significantly related to the concept of rotation. These incorrect solutions that appeared turned out to be typical incorrect solutions also observed in the student group. These are perhaps the 'epistemological obstacles' associated with this concept.

However, there are also less optimistic conclusions. As can be seen from the summary in Table 2, students are unable to clearly link the issue presented in this task with the idea of rigid rotational motion. They look at the task either very narrowly (*a task about a clock)* or very broadly, enigmatically (*geometric, about the position of a figure*). When referring to the students' solutions, they feel lost. Although they are good at identifying correctly solved tasks, they are unable to specify what is wrong with the solutions they (correctly) treat as incorrect.

Regardless, the mere fact that they can see anything more than a measurement of time is taken as a good starting point for further work.

In Poland, professional training in Early Childhood Education, regards to mathematics preparation, includes two different subjects: Mathematical Education and Methodology of Mathematics Education. The objectives of the two subjects are rather different - the first one reminds and consolidates concepts from school mathematics (generally from the older grades), the second one focuses on mathematical issues implemented according to the core curriculum for the younger grades. It seems worthwhile to revise the syllabuses for both subjects in such a way that the issues of Mathematics Education are linked directly to such solutions, which fall under the issues discussed within the Methodology. Perhaps this will better open students up to a broad approach to shaping the mathematical intuitions needed by the student at later stages of schooling. It also goes without saying that such tasks should appear in textbooks for primary school students.

## References

Bieńkowska-Wójcik, W., Boroń, I., Brzyska, S., Cikorska, S., Fiertek, R., Giezek, D., Glaza, M., Gruszewska, B., Kapczyńska, E., Kazimierczak, A., Kilichowska, E., Kowal, B., Kubiak, D., Leśniewska, I., Majewska, V., Preus, D., Szcząchor, K., Tomecka, K., Wasilewska, H., & Wiewióra, E. (2014). *Bydgoski bąbel matematyczny. O wprowadzaniu zmian w nauczaniu matematyki w klasach I-III* [Bydgoszcz math bubble. About introducing changes in mathematics teaching in grades I-III] Instytut Badań Edukacyjnych.

Fan, L. H., & Kaeley, G. S. (2000). The influence of textbook on teaching strategies: An empirical study. *Mid-Western Educational Researcher, 13*, 2–9.

Haggarty, L., & Pepin, B. (2002). An investigation of mathematics textbooks and their use in English, French and German classrooms: who gets an opportunity to learn what? *British Educational Research Journal, 28*(4), 567–590.

Hejný, M. (1990). *Teória vyučovania matematiky 2* [Theory of teaching of mathematics 2]. Slovenske Pedagogicke Nakladatelstvo.

Hejný, M. (2000). Budování geometrických proceptů [Building geometric procepts]. In M. Ausbergerová & J. Novotná (Eds.), *7. Setkání učitelů všech stupňů a typů škol* [7th Meeting of teachers of all levels and types of schools) (pp. 11−17). JČMF.

Hejný, M., & Jirotková, D. (2012). Contribution of Geometry to the Goals of Education in Mathematics. *Orbis scholae, 6*(2) 57−67.

Hejný, M., Jirotková, D., & Slezaková, J. (2007). *Matematika, pro 1. ročnik základni školy, I – II dil* [Mathematics, for the 1st year of elementary school, part I – II]. Fraus.

Hejný, M., Jirotková, D. & Slezaková, J. (2008). *Matematika, pro 1. ročnik základni školy, I – III dil* [Mathematics, for the 1st year of elementary school, part I – III]. Fraus.

Hejný, M., Jirotková, D., Slezaková, J., & Michnova, J. (2009). *Matematika, pro 1. ročnik základni školy, I – II dil* [Mathematics, for the 1st year of elementary school, part I – II]. Fraus.

Henderson, D. W., & Taimina, D. (2005). *Experiencing Geometry, Euclidean and Non- Euclidean with History* (3rd Edition). Pearson Prentice Hall.

Hoyles, C. (1996) Modelling Geometrical Knowledge: the case of the student. In J.-M. Laborde (Ed.), *Intelligent Learning Environments: the case of geometry* (pp. 94–112). Springer-Verlag.

H-mat o.p.s. (2024) *Hejny method. An approach to mathematics teaching and child development.* https://www.h-mat.cz/en

Jagoda E., & Swoboda, E. (2010). Various intuitions of point symmetry (from the Polish school perspective). In B. Maj, E. Swoboda, & K. Tatsis (Eds.), *Motivation via natural differentiation in mathematics* (pp. 169−182). Wydawnictwo Uniwersytetu Rzeszowskiego.

Jagoda, E., & Swoboda, E. (2011). Static and dynamic approach to forming the concept of rotation. In M. Pytlak, T. Rowland & E. Swoboda (Eds.), *Proccedings of the Seventh Conference of the European Society for Research in Mathematics Education* (CERME 7) (pp. 558−567). University of Rzeszów and ERME.

Jirotková, D. (2016). Scheme of geometrical concepts. *Paper presented to Topic Study Group 4 (TSG4) at the 13th International Congress on Mathematical Education (ICME-13)*. Hamburg, Germany, July 24–31.

Jones, K. (1999). Student interpretations of a dynamic geometry environment, In I. Schwank (Ed.), *European Research in Mathematics Education I, volume I* (pp. 245−258). Forschungsinstitut für Mathematikdidaktik.

Jones, K. (2000). Providing foundation for deductive reasoning: student's interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics, 44*(1-3), 55−85.

Kamińska, A. (2015). Zalety i wady e-podręcznika postrzegane z perspektywy nauczycieli [Advantages and disadvantages of the e-textbook seen from the teachers' perspective]. *HUMANITAS Pedagogika i Psychologia*, *10*, 111-121.

Novotná, J., Kozánek Kiss, T., Michal, J., & Vankúš, P. (2023). Comparison of different types of reasoning & proof present in solved tasks from geometry in Slovak and Czech mathematics textbooks for lower secondary school. *Annales Universitatis Paedagogicae Cracoviensis. Studia ad Didacticam Mathematicae Pertinentia, 15*, 49-67.

Silverman, B., & Even, R. (2015). Textbook explanations: Modes of reasoning in 7th grade Israeli mathematics textbooks. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME 9, February 4–8, 2015)* (pp. 205–212). Charles University in Prague, Faculty of Education and ERME.

Sosniak, L. A., & Stodolsky, S. S. (1993). Teachers and Textbooks: Materials Use in Four Fourth-Grade Classrooms. *The elementary school journal, 93*(3), 249–275.

Stylianides, G. J. (2014). Textbook analyses on reasoning-and-proving: Significance and methodological challenges. *International Journal of Educational Research, 64*, 63–70.

Swoboda, E., & Vighi, P. (2016). Early Geometrical Thinking in the Environment of Patterns, Mosaics and Isometries. In *ICME-13 Topical Surveys*. Springer.

Swoboda, E., & Zambrowska, M. (2023). Mathematization of rotation as a didactic issue. *Didactica Mathematicae, 45*, 33−51.

Swoboda, E., Maj-Tatsis, B., & Pytlak, M. (in press) Moving from research to teaching geometric rotation according to genetic constructivism.

Szkoła, E. (2016). *Intuicje dotyczące dynamicznego reprezentowania transformacji geometrycznych* [Intuition of a dynamic representation of geometric transformations]. Non-published master thesis. Uniwersytet Rzeszowski, Poland.

Weyl, H. (1952). *Symmetry*. Princenton University Press.

# MATHEMATICS TEXTBOOKS AS A POSSIBLE CAUSE OF STUDENTS' MISCONCEPTIONS IN PLANIMETRY

Vlasta Moravcová

Charles University, Prague, Czech Republic

*The paper deals with selected phenomena occurring in Czech mathematics textbooks, and their possible influence on the formation of students' misconceptions in the field of planimetry. The analysis of textbooks was elaborated as part of long-term research into Czech pupils' and students' conceptual knowledge in geometry. We have found that the design of the textbooks is the possible cause of some students' misconceptions. However, in our opinion, the mathematics teacher, who is well acquainted with the risks, has the tools to eliminate these misconceptions.*

## INTRODUCTION AND THEORETICAL BACKGROUND

Mathematics is one of the key parts of the school curriculum, and the level of its knowledge has a significant impact on our lives. However, mathematics is difficult for many Czech pupils and students and does not rank among their favourite subjects. This fact is repeatedly confirmed by research by the Czech School Inspectorate (e.g., Novosák et al., 2022). Many mathematics teachers also perceive their subject as difficult and unpopular with students (Rendl et al., 2013).

Geometry, especially the synthetic one, comes to the fore among the difficult topics of mathematics. Students cannot rely only on algorithmization, but often need a deeper understanding of the subject matter. Children can become familiar with elementary geometrical concepts in a natural way already at preschool age. This knowledge needs to be further developed in accordance with the age of the pupils. However, part of the teachers, especially in the primary school, struggle with some tasks in geometry themselves and therefore they do not like to teach it (e.g., Šťastná, 2012; Son, 2006; Hacısalihoğlu-Karadeniz et al., 2015). These teachers are dependent on various teaching materials.

Despite the growing trend of digitisation, recent researches show that the use of printed textbooks in Czech schools (Pešková, 2018) and abroad (e.g., Hansen & Gissel, 2017) is still prevalent. According to Sikorová and Červenková (2014), mathematics teachers primarily look for suitable tasks in textbooks. Teachers usually prefer the textbooks published by publishing houses, checked by high-quality lecturers, and, ideally, reviewed and approved by the Ministry of Education, Youth and Sports[1]. They have more confidence in the correctness of the information given in such textbooks.

[1] For more details on the approval of textbooks in the Czech Republic, see (Greger, 2005).

Textbooks can be analysed from many perspectives and in many ways, and their analysis is an integral part of pedagogical research. In the Czech Republic, two main approaches to textbook analysis come to the fore: the curricular approach focusing on the relationship between the national curriculum and textbooks, and the psychological-didactical approach putting the transformation and representation of content and methods of working with textbooks, etc. in the spotlight (Knecht & Janík, 2008).

Textbook analysis employs various methods within both mentioned approaches. An overview of these research methods is introduced by Janko (2011). His classification of methods is based on Průcha's findings (1998). Janko (2011) distinguishes seven textbook analysis methods: *quantitative method* (finding the occurrence and frequency of certain measurable units of a textbook), *content analysis* (finding and evaluating qualitative textbook properties), *questioning* (collecting and evaluating of statements about various properties of a textbook), *observation method* (researching into ways of using a textbook in teaching and its influence on students), *testing method* (testing students to determine learning outcomes caused by a textbook), *experimental method* (researching into the influence of textbook modifications), and *comparative method* (comparison between several textbooks from different points of view). The most commonly used textbook research method is presumably the content analysis. Although this method is initially a qualitative, owing to the subsequent categorization of the data obtained, it turns into a quantitative one.

A high-quality mathematics textbook should, among other things, be error-free, emphasize essential problems and respect the national curriculum (Odvárko, 2019). The geometry curriculum can be structured in various ways and is also processed differently in individual Czech textbooks (Janků, 2011). The progress from concrete conception to abstract one is essential in teaching geometry and in understanding geometrical concepts. This idea corresponds to the van Hiele's Theory. Van Hiele described five levels of student thinking in geometry: *visualization*, *analysis*, *abstraction*, *deduction* and *rigor* (van Hiele, 1986; Mayberry, 1983). Students who have reached the level of abstraction should not show the evidence of misconceptions. This level should be reached by students during lower secondary school (Budínová, 2021). Tall and Hejný work with the similar principles. Tall et al. (2001) view the teaching of geometry from the perception of shapes through the manipulation of prototypes of objects to the proof and axiomatic construction of geometry. According to the Hejný's Theory of Generic Models, students gain abstract knowledge from isolated models (Hejný, 2012).

In the years 2017 and 2018, we[2] tested the level of Czech pupils' and students' understanding of selected concepts from the field of planimetry. We discovered several students' misconceptions (Moravcová et al., 2020). In the subsequent search for causes, we found that one of the problems could be in mathematics textbooks design, which, although they are very diverse, contain common shortcomings. In this contribution, we deal in more detail with the question of whether textbooks can be one of the causes of students' misconceptions; and if so, how to minimise this negative influence.

## METHODOLOGY

The testing described above was only one part of our long-term empirical research into students' conceptual knowledge. In the first phase, three tests focused on selected concepts from the field of planimetry were prepared and administered to Czech pupils and students of different ages: Test I was solved by 505 ISCED I graduates, Test II was solved by 437 ISCED II graduates, and Test III was solved by 472 ISCED III graduates and by 44 pre-service mathematics teachers in their last two university study years. The sample was obtained on the basis of its availability and included students of 9 elementary schools (ISCED I + ISCED II), 8 grammar schools (ISCED II + ISCED III) and 5 universities in the Czech Republic. All students were tested anonymously.

The assignments were based on commonly used mathematics textbooks and with respect to the Czech national curriculum (Balada et al., 2007; MŠMT, 2017). Moreover, the long-term teaching geometry experience of the research team members was also reflected.

The clarity of all the tests and the time limits for solving them were first verified with a small sample in the form of pre-tests. Some types of answers were subsequently investigated through post-tests and guided interviews with other groups of respondents of the same age. For more details about the testing (such as the number of tasks and their formulation, time limits, etc.), see (Robová et al., 2019; Halas et al., 2019; Moravcová & Hromadová, 2020; Moravcová et al., 2021).

Results of the tests showed which concepts pupils/students have problems with, but did not answer the question of what the causes of these misconceptions are. So the second phase of research followed: the search for the causes of students' misconceptions. This phase is still ongoing.

[2] Testing and data evaluation was carried out by a research team made up of members (namely in alphabetic order: Zdeněk Halas, Jana Hromadová, Vlasta Moravcová, Jarmila Robová) of the Department of Mathematics Education, Faculty of Mathematics and Physics, Charles University.

For each tested concept (so far we have dealt in more detail with the concepts of *straight line*, *ray*, *angle*, *circle*, *disk*, *triangle*, *rectangle*, *trapezoid*,[3] *axial symmetry*, *central symmetry* and *rotation*, specifically *rotation of a straight line*), we monitor when and how it is defined in individual textbooks, in which context it is placed and whether it is adequately practiced. We focus on concept classification, visual illustration, and other factors that may influence how each student understands the concept. The results presented in this paper proceed mainly from the analysis of the concepts of triangle, rectangle, trapezoid and axial symmetry.

In this research, all the official Czech textbooks used by the teachers of the tested classes were included. These textbooks were edited by *Alter*, *Didaktis*, *Fraus*, *H-mat*, *Nová Škola*, *Prodos*, *Prometheus*, *SPN*, *Studio 1+1* and *Taktik* publishing houses (in alphabetical order). We explore complete series for individual levels of education, including the relevant collections of exercises. In total, we examined more than 80 textbooks and collections of exercises focusing on topics in planimetry. However, the results for each researched concept were based only on those textbooks in which the given concept was found. For example, the trapezoid is introduced to students in the lower secondary school. Therefore, we only followed this concept in textbooks for lower and upper secondary schools. Moreover, from the given textbook series, we chose the textbook for the grade in which the concept is firstly introduced. Thus, the number of analysed textbooks for one concept varies from 9 to 16. The exception is the concept of rotation, which is introduced in upper secondary school, and was only examined in upper secondary school textbooks.

For each examined concept, we established several criteria and, using the qualitative content analysis, we looked into whether a textbook meets these criteria, and alternatively to what extent. The criteria were always based on students' errors in our testing. For example, we set the following criteria for the trapezoid concept:

A non-prototype/non-model of trapezoid occurs in the textbook.

A trapezoid in a non-prototypical position occurs in the textbook.

Furthermore, we noticed how the trapezoid is defined and included in the classification of quadrilaterals, and whether it is treated correctly in the tasks.

Thus, our research combines the quantitative method and content analysis (Janko, 2011). We are not aware of a similar analysis of Czech textbooks that would monitor the introduction of geometric concepts to such an extent.

[3] We use the term *trapezoid* for a quadrilateral with just one pair of parallel sides, while the term *trapezium* for a quadrilateral with no parallel sides (in according with the exclusive definition, which is taught in Czech schools, and with *Oxford Advanced Learner's Dictionary*).

## RESULTS AND DISCUSSION

Based on the textbooks analysis, we observed four problematic phenomena: early graphic transition from 2D shapes to their border; the prevalent occurrence of prototypes of shapes or their prototypical positions; the absence of non-models; and the absence of atypical tasks. These phenomena are evident across all the analysed textbooks. In this section, we will describe the individual phenomena and their influence in more detail.

### Graphic transition from 2D shapes to their border

Children are introduced to basic shapes such as triangle, square, circle, rectangle, etc. already at preschool age. Then, they gradually develop their knowledge. In accordance with the van Hiele's levels, a graduate of primary school should be at the level of analysis, i.e., be able to describe the properties of a shape. On secondary school level, a student should ideally reach the level of abstraction, i.e., to be able to classify and sort shapes according to their properties and should not have fixed misconceptions (Budínová, 2021).

In our research, however, we found out that many students do not consider the inner point of a 2D shape to be a point of the shape; i.e., they have a misconception "2D shape is the same as a boundary of the shape". This problem was tested by several tasks related to various shapes in our research; among other things, we observed the students' understanding of triangle.

The "triangle misconception" appears already at the end of the primary school, when approximately half of the respondents perceive the triangle only as a closed polygonal chain (Robová, 2019). Moreover, it persists in older students, including pre-service teachers. The same misconception was also described by Budínová (2021).

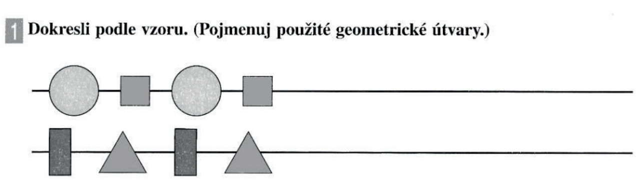



Figure 1: Illustration from the textbook (Molnár & Mikulenková, 2019; p. 7) for 2nd grade of primary school.

In our opinion, this misconception can originate in the early stages of education based on empirical materials that are presented to students in the form of triangle representations in textbooks. While pupils are presented with coloured triangles in the 1st and 2nd grade of primary school (Figure 1), they begin to construct triangles themselves and suddenly the coloured filling is missing in the pictures from the 3rd grade (Figure 2). The result of such constructions is presented as a closed polygonal chain. The absence of coloured filling in the majority of tasks was found in all analysed textbooks for the 3rd and higher grades of the primary school (a total of 15 textbooks). Moreover, this tendency is supported by the fact that, in connection with triangles, textbooks often talk only about their vertices and sides but rarely about any inner points. Then it is understandable that the inner points of the triangle do not have to be considered by the students as an integral part of the shape. Kupčáková (2017) and Budínová (2018) also draw attention to the influence of textbooks on pupils' understanding of this issue with similar results. We also observed the same issue in the case of rectangle, circle and angle.

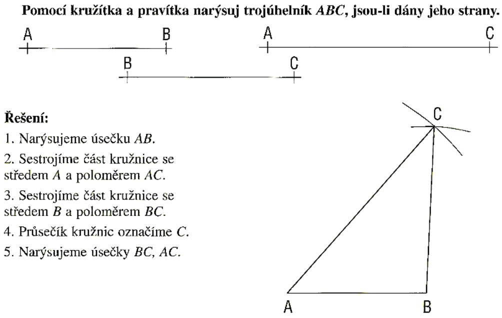



Figure 2: Illustration from the textbook (Molnár & Mikulenková, 2020; p. 7) for 3rd grade of primary school.

**Dominant occurrence of prototypes of shapes or their prototypical positions**

The students' fixation on prototypes of geometrical shapes or their prototypical positions[4] indicates that the students have not even reached the analysis level of the van Hiele's scale (Tsamir et al., 2015). We also observed this issue, and it was done through several tasks again. We found out a students' prototypical perception of square, isosceles triangle, and trapezoid (Halas et al., 2019). Furthermore, we encountered a strong fixation of the axial symmetry axis in a vertical position (Moravcová et al., 2021). The preference for prototypes and prototypical positions causes a number of misconceptions in geometry. Problems with fixation on prototypes have also been pointed out by many other researchers (e.g., Budínová, 2017, 2018; Clements et al., 1999; Tirosh et al., 2011).

As with the previous phenomenon, there might be an influence of empirical material, especially textbooks containing predominantly images of prototypes and shapes in prototypical positions. For example, the trapezoid concept was analysed in 9 textbooks. In 4 of them, only the prototypical positions (with horizontal bases, Figure 3) are depicted. In the remaining 5 textbooks, the trapezoid occurs in a different position, but the prototypical positions strongly predominate. An obtuse trapezoid occurs only in 4 textbooks. The prevalence of

[4] The prototype of a triangle is an equilateral triangle, the prototype of a quadrilateral is a square, etc. The shapes in the prototypical position typically have a horizontally located base/bases.

prototypes and prototypical positions of shapes may negatively influence the student's conceptual understanding of geometrical shapes (Hejný, 2012), as well as result in their concept image not corresponding to their concept definition (Vinner & Hershkowitz, 1980; Tall & Vinner, 1981).

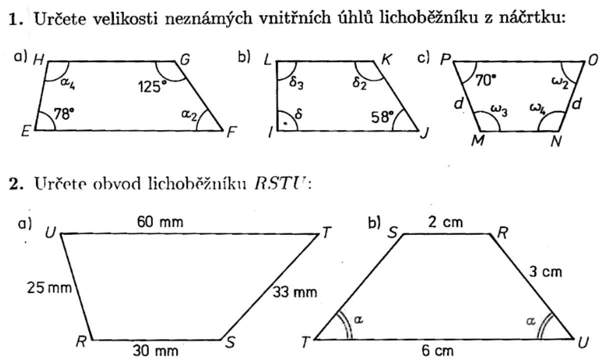



Figure 3: Illustration from the textbook (Herman et al., 1995; p. 83) for lower secondary school.

**Absence of non-models**

The analysis and abstraction levels of the van Hiele's scale are related to the students' ability to recognize a non-model, i.e., an object that does not have a declared property. In Test II, almost a fifth of the respondents identified the trapezium as a trapezoid (Halas et al., 2019). And, one of the most frequent misconceptions found out in our testing was that the students identified a rhomboid as an axially symmetric shape (Moravcová et al., 2021). This misconception was also described by other researchers (Aktaş and Ünlü, 2017; Son, 2006; Leikin, Berman & Zaslavsky, 2000; Hacısalihoğlu-Karadeniz et al., 2015).

The concept of axial symmetry was examined in 13 textbooks from the 5th grade of elementary school to upper secondary school. In 4 of them, non-models of axially symmetrical shapes do not occur. The rhomboid (which is an interesting counterexample for its central symmetry) appears only in 5 textbooks, but in 3 of them only as part of a task. We have found a similar situation in the case of non-models of other concepts.

Based on the results of our research, we can say that Czech mathematics textbooks do not contain a sufficient number of non-models of symmetrical figures, in general, they do not contain enough non-models of anything. In our opinion, students should encounter non-models of a concept as soon as they are introduced to the new concept. This is the only way they can realise which key properties distinguish the presented concept from other concepts.

**Absence of atypical tasks**

The most difficult tasks of our testing (from the students' point of view) were generally those which do not commonly occur in textbooks. However, such tasks best verify whether a student really fully understands a certain mathematical concept. Kambilombilo and Sakala (2015) also pointed out the difficulties in problem solving beyond the standard tasks from textbooks.

If a student encounters a type of task for the first time, he/she may confuse it with another one that he/she knows and is similar to. This confusion usually leads to an incorrect solution. Or, the student can proceed to a procedural approach, but, without sufficient conceptual knowledge, he/she again obtains an incorrect solution (Son, 2006).

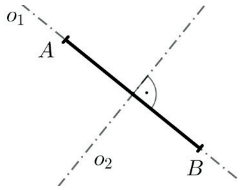



Figure 4: Symmetry axes o1, o2 of line segment AB (axis o2 is the perpendicular bisector of line segment AB).

In one of the tasks, which appeared in all three tests, students had to determine the number of symmetry axes of a line segment (the correct answer is "two", Figure 4). Most students believe that a line segment has just one axis of symmetry. In our opinion, they have confused the terms *perpendicular bisector of a line segment* ["axis of a line segment", in Czech "osa úsečky"], which is commonly explained in textbooks, and *symmetry axis of a line segment* [in Czech "osa souměrnosti úsečky"], which, with one exception, we have not found in any textbook. In some cases, respondents tried to solve the problem procedurally – they drew possible axes into the picture, but found only one solution (Moravcová et al., 2021).

The described phenomena satisfactorily explain most of the student misconceptions identified by the testing. Their influence on the formation of these misconceptions does not matter whether students work with the textbooks themselves, or only their teacher uses them in the class.

From the given information, we deduce the following conclusion: mathematics textbooks can influence the formation of students' misconceptions.

This result can also be supported by the fact that many students and teachers refer to textbooks as an authority. The most straight-forward solution is a revision of mathematics textbooks. However, this is a lengthy and non-trivial process in many respects. Aktaş & Ünlü (2017) pointed out that textbooks are indispensable for the learning environments as well as teachers. We believe that a teacher can help with the elimination of the negative textbooks influence. However, the necessary condition is that the teacher has a developed pedagogical content knowledge (Shulman, 1986). Problems with prototypes can be eliminated by using visual models, e.g., in a dynamic geometry software. Several researches confirmed that technology can help to better geometrical concept understanding (Hollebrands, 2004; Jonson-Gentile, Clements & Battista, 1994; Lobato & Ellis, 2002). Furthermore, demonstrating a sufficient number of non-models and exploring the differences between a model and a non-model of an object also can help students to gain the necessary conceptual knowledge.

Last but not least, teachers should assign a variety of tasks, including complex, open-ended and problem-based, and should use various teaching methods. For example, peer instruction method is suitable for a deep concept understanding (Mazur, 1997; Vickrey et al., 2015). However, the key factor is that the teacher must be aware of possible problems.

## CONCLUSION

Geometry is an important part of school mathematics. Their study contributes to the proper development of visualisation skills, critical thinking, problem-solving, logical argument and so on (Jones, 2002). School geometry is not an easy subject, it places considerable demands on both the student and the teacher.

The purpose of mathematics textbooks is to help both the student and the teacher. Above all, they should inspire teachers, and provide students with basic information and practice tasks. However, the strict use of only textbooks carries certain risks and may lead to the formation of students' misconceptions in the field of planimetry.

These risks can be eliminated by the teacher provided that he/she is aware of them and is able to choose appropriate methods and didactic aids in order to increase the students' conceptual knowledge.

### References

Aktaş, G. S., & Ünlü, M. (2017). Understanding of eight grade students about transformation geometry: Perspectives on students' mistakes. *Journal of Education and Training Studies*, *5*(5), 103–119.

Balada J. et al. (2007). *Rámcový vzdělávací program pro gymnázia* [Framework educational programme for secondary general education]. VÚP, 2007.

Budínova, I. (2017). Vytváření představ základních geometrických pojmů u žaků prvního stupně základní školy [Building ideas of basic geometric concepts in pupils of the first grade of primary school]. *Učitel matematiky*, *25*(2), 65–82.

Budínova, I. (2018). Vytváření představ základních geometrických pojmů u žaků prvního stupně základní školy: trojúhelník a kruh [Building ideas of basic geometric concepts in pupils of the first grade of primary school: a triangle and a circle]. *Učitel matematiky*, *26*(1), 1–11.

Budínová, I. (2021). Vývoj představ žáků o geometrických pojmech v průběhu základní školy [The development of pupils' ideas about geometric concepts during primary school]. *Učitel matematiky*, *29*(1), 1–25.

Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education*, *30*(2), 192–212.

Greger, D. (2005). Proces schvalování učebnic v historickosrovnávací perspective [The process of approving textbooks in a historical-comparative perspective]. *Pedagogická orientace*, *15*(3), 112–117.

Hacısalihoğlu-Karadeniz, M., Baran, T., Bozkuş, F., & Gündüz, N. (2015). Difficulties of prospective elementary mathematics teachers' regarding to reflection symmetry. *Turkish Journal of Computer and Mathematics Education*, *6*(1), 117–138.

Halas, Z., Robová, J., Moravcová, V., & Hromadová, J. (2019). Pupils' concepts of the trapezoid at the end of lower secondary level education. *Open Education Studies*, *1*(1), 184–197.

Hansen, T. I., & Gissel, S. T. (2017). Quality of learning materials. *IARTEM e-Journal, 9*(1), 122–141.

Hejný, M. (2012). Exploring the cognitive dimension of teaching mathematics through scheme-oriented approach to education. *Orbis scholae, 6*(2), 41–55.

Herman, J., Chrápavá, V., Jančovičová, E., & Šimša, J. (1995). *Matematika pro nižší třídy víceletých gymnázií; trojúhelníky a čtyřúhelníky* [Mathematics for lower grades of multi-year gymnasiums; triangles and quadrilaterals]. Prometheus.

Hollebrands, K. F. (2004). High school students' intuitive understandings of geometric transformations. *Mathematics Teacher*, *97*(3), 207–214.

Janko, T. (2011). Nonverbální prvky v učebnicích: Výsledky analýzy [Visuals in textbooks: Results of the analysis]. In T. Janík, P. Najvar, M. Kubiatko et al. (Eds.), *Kvalita kurikula a výuky: výzkumné přístupy a nástroje* (pp. 79–95). Masarykova univerzita.

Janků, M. (2011). Jak učit geometrii [How to teach geometry]. *Metodický portál: Články*. Retrieved February 28, 2024, from https://clanky.rvp.cz/clanek/12567/JAK-UCIT-GEOMETRII.html

Johnson-Gentile, K., Clements, D. H., & Battista, M. T. (1994). Effects of computer and noncomputer environments on students' conceptualizations of geometric motions. *Journal of Educational Computing Research*, *11*(2), 121–140.

Jones, K. (2002). Issues in the teaching and learning of geometry. In L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 121–139). Routledge Falmer.

Kambilombilo, D., & Sakala, W. (2015). An investigation into the challenges in-service student teachers encounter in transformational geometry, "reflection and rotation”. The case of Mufulira College of Education. *Journal of Education and Practice*, *6*(2), 139–149.

Knecht, P., & Janík, T. (2008). Učebnice z pohledu pedagogického výzkumu [Textbooks from the perspective of educational research]. In P. Knecht, T. Janík et al. (Eds.), *Učebnice z pohledu pedagogického výzkumu* (pp. 9–17). Paido.

Kupčáková, M. (2017). Geometrické kurikulum na 1. stupni [Geometric curriculum in the 1st stage]. In K. Sebinova, L. Gerova & P. Voštinar (Eds.), *Primárne matematické vzdelávanie – teória, výskum a prax* (pp. 76–80). UMB.

Leikin, R., Berman, A., & Zaslavsky, O. (2000). Learning through teaching: The case of symmetry. *Mathematics Education Research Journal*, *12*(1), 18–36.

Lobato, J., & Ellis, A. B. (2002). The focusing effect of technology: Implications for teacher education. *Journal of Technology and Teacher Education, 10*(2), 297–314.

Mayberry, J. (1983). The Van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education*, *14*(1), 58–69.

Mazur, E. (1997). *Peer instruction: a user's manual*. Prentice Hall.

Molnár, J., & Mikulenková, H. (2019). *Matematika a její aplikace, 2. ročník, 1. díl* [Mathematics and its applications, 2nd year, Vol. 1]. Prodos.

Molnár, J., & Mikulenková, H. (2020). *Matematika a její aplikace, 3. ročník, 3. díl* [Mathematics and its applications, 3rd year, Vol. 3]. Prodos.

Moravcová, V., Halas, Z., Hromadová, J., & Robová, J. (2020). Žákovské koncepty geometrických pojmů [Students' concepts in geometry]. In B. Bastl & M. Lávička (Eds.), *Setkání učitelů matematiky všech typů a stupňů škol 2020* (pp. 83–88). Vydavatelský servis.

Moravcová, V., & Hromadová, J. (2020). Straight line or line segment? Students' concepts and their thought processes. *Teaching Mathematics and Computer Sciences*, *18*(4), 327–336.

Moravcová, V., Robová, J., Hromadová, J., & Halas, Z. (2021). Students' understanding of axial and central symmetry. *Journal of Efficiency and Responsibility in Education and Science, 14*(1), 28-40.

MŠMT (2017). *Rámcový vzdělávací program pro základní vzdělávání* [Framework educational programme for basic education]. NÚV.

Novosák, J., Suchomel, P., Dvořák, J., Zatloukal, T., & Pražáková, D. (2022). *Vyhodnocení výsledků vzdělávání žáků 5. a 9. ročníků základních škol a víceletých gymnázií; tematická zpráva* [Evaluation of the educational results of students in the 5th and 9th grades of elementary schools and multi-year gymnasiums; thematic report]. Česká školní inspekce.

Odvárko, O. (2019). O čem přemýšlí autor učebnic [What an author of textbooks is thinking about]. *Učitel matematiky, 27*(3), 187–197.

Pešková, K. (2018). Dnešní učebnice pohledem výzkumníků a studentů učitelství [Current textbooks from the perspective of researchers and pre-service teachers]. *Komenský*, *143*(1), 25–30.

Průcha, J. (1998). *Učebnice: Teorie a analýzy edukačního média* [Textbooks: Theory and analysis of the educational medium]. Paido.

Rendl, M., Vondrová, N., Hříbková, L., Jirotková, D., Kloboučková, J., Kvasz, L., Páchová, A., Pavelková, I., Smetáčková, I., Tauchmanová, E., & Žalská, J. (2013). *Kritická místa matematiky na základní škole očima učitelů* [Critical points in mathematics at elementary school through the eyes of teachers]. Univerzita Karlova v Praze, Pedagogická fakulta.

Robová, J., Moravcová, V., Halas, Z., & Hromadová, J. (2019). Žákovské koncepty trojúhelníku na začátku druhého stupně vzdělávání [Pupils' concepts of the triangle and the rectangle at the beginning of lower secondary school]. *Scientia in educatione*, *10*(1), 1–22.

Sikorová, Z., & Červenková, I. (2014). Styles of textbook use. *The New Educational Review, 35*(1), 112–122.

Son, J. W. (2006). Investigating preservice teachers' understanding and strategies on a student's errors of reflective symmetry. *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 145–152). PME.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Šťastná, J. (2012). *Výuka geometrie na 1. stupni ZŠ – aneb "Paní učitelko, mě to baví"* [Education of geometry for primary school or "Ms. teacher, I am enjoying this work"] [Unpublished master's thesis]. Univerzita Jana Evangelisty Purkyně v Ústí nad Labem.

Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M., & Yusof, Y. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Science, Mathematics and Technology Education*, *1*(1), 81–104.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Tirosh, D., Tsamir, P., Tabach, M., Levenson, E., & Barkai, R. (2011). Geometrical knowledge and geometrical self-efficacy among abused and neglected kindergarten children. *Scientia in educatione*, *2*(1), 23–36.

Tsamir, P., Tirosh, D., Levenson, E., Barkai, R., & Tabach, M. (2015). Early years teachers' concept images and concept definitions: triangles, circles, and cylindres. *ZDM Mathematics Education*, *47*(3), 497–509.

Van Hiele, P. M. (1986). *Structure and insight*. Academic press.

Vickrey, T., Rosploch, K., Rahmanian, R., Pilarz, M., & Stains, M. (2015). Research-based implementation of peer instruction: A literature review. *CBE—Life Sciences Education*, *14*(1):es3, 1–11.

Vinner, S., & Hershkowitz, R. (1980). Concept images and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education* (pp. 177–184). California University.

# SIMILARITY IN GREEK MATHEMATICS TEXTBOOK SERIES

Panagiota Kaskaouti, Andreas Moutsios-Rentzos

Department of Pedagogy and Primary Education, National and Kapodistrian University of Athens, Greece

*In this study, we investigate the opportunities for the students' learning of similarity in mathematics textbook series (textbooks, workbooks, teachers' guides) in compulsory education in Greece. Overall, we identified 176 similarity tasks, which were analysed with respect to the included type of similarity, their context, and, and their links with proportional reasoning. The results of the analyses revealed that most of the similarity tasks are "finding" or "reflecting" exercises, around one third of the tasks are situated in real-world context, whilst the students have the opportunity to develop their proportional reasoning working in around one third of the tasks.*

## INTRODUCTION

Textbooks are at the core of most educational systems with respect to mathematical teaching and learning (Glasnovic Gracin, 2018), as they are resources both for the students' investigations and for the teachers' organisation of the mathematics instruction (Lepik et al., 2015; Weinberg & Wiesner, 2011). Thus, the mathematics learning opportunities provided to the students are closely interwoven with the qualities of the textbooks (Van Zanten & Van Den Heuvel-Panhuizen, 2018). The term "opportunity to learn" has been conceptualised in several ways, depending on the educational dimensions of interest. For example, it may refer to content coverage, content exposure, content emphasis, and quality of instructional delivery (Wang, 1998), while it may also refer to a broader perspective including what schools and teachers offer but also how schools and teachers conduct instruction (Liu, 2009; Stein, 2000).

In this study, which is part of a PhD project, the focus is narrowed to the opportunities to learn similarity as communicated by the institutional discourse that is present in the didactical relationship in the mathematics classroom (teacher, students, mathematical knowledge) and appears to significantly influence students' opportunities to learn (Hadar, 2017): the *mathematics textbook series*. 'Mathematics textbook series' (Son & Diletti, 2017) refer to a set of curricula resources that teachers use for the daily teaching: student textbooks, workbooks and teacher's guide. Given a figure on a plane, a similar figure can be produced by dilation and/or translation and/or rotation and/or reflection. Dilation is defined by a centre point and a scale factor, translation is defined by a vector, rotation is defined by a centre point and an angle, and reflection is defined by a line (Mammana, 2016; Yao & Manouchehri, 2019); they are all planar transformations that are one-to-one mapping from a set of points on the plane into itself. The concept of similarity was chosen as a central mathematical concept (as, for example, is the Pythagorean Theorem; Moutsios-Rentzos et al., 2014) that may potentially allow for the meaningful linking of crucial geometrical and arithmetical/algebraic ideas and forms of reasoning (Cox, 2013), notably proportional reasoning.

Consequently, we investigate the opportunities to learn similarity as offered in the textbook series in compulsory education in Greece. Though similarity is introduced as a geometrical relationship, one of the difficulties that students face is that similarity requires them to search for quantities and multiplicative relationships, which is closely related to the development of proportional reasoning (Lamon, 2020). Proportional reasoning involves understanding that if one quantity in a ratio is multiplied or divided by a factor, then the other quantity must be multiplied or divided by the same factor (Lobato et al., 2010). We consider the following questions: a) What type of tasks offer students the opportunity to learn similarity?, b) Do these tasks include real-world context?, and c) Do these tasks also offer the students the opportunity to develop their proportional reasoning?

## SIMILARITY IN TEXTBOOK SERIES

The students start constructing the ideas of geometrical congruency and similarity early in their school life (in Greece from kindergarten). While congruent shapes are those that exactly match, similar shapes are those that are related to the embodied sensations and the notions of “magnifying” or “shrinking”. Later, for example in the middle grades, students have the opportunity to extend their understanding of similarity noticing, for example, that in similar figures their corresponding angles are congruent and their corresponding sides are related by a scale factor (National Council of Teachers of Mathematics, 2000). However, as already noted, the students’ opportunities to appropriately construct ideas about similarity are greatly affected by the quality of the textbook series (Törnroos, 2005), as textbook series often guide the teacher’s didactic and pedagogic decisions about the adopted practices and strategies (Wijaya et al., 2015).

Researchers have investigated the opportunities that textbook series provide to students to learn the concept of similarity. Barcelos Amaral and Hollebrands (2023) considered textbooks from Brazil (for students of Grade 9) and the United States (for students of Grade 9 and 10) and found that, although there are differences, all textbooks begin the discussion of the concept of similarity in general, then apply the concept to similar triangles, with the similarity theorems and formal definitions to appear later.

The same researchers, in another study (Barcelos Amaral & Hollebrands, 2017) focussing again on Brazil (for students of Grade 9) and the United States (for students of Grade 9 and 10), accounted for few contextual problem tasks among the similarity tasks, many of which were of low cognitive demand.

Wijayanti (2019) analysed Grade 7 and Grade 9 Indonesian mathematics textbooks and revealed that the textbook authors focus more on how to use similarity to solve tasks, than on treating in the notion itself. Techniques from proportion were often used to deal with the similarity tasks. All of the textbooks treated similarity into two parts: polygon similarity and triangle similarity. Furthermore, polygon similarity always appeared first, with an informal definition of what it means for two (general) polygons to be similar and then was worked within tasks.

In the study of Lo et al. (2006), the focus was on the concept definitions and the concept images of similarity in three middle grade intended curricula (Grades 6-8). It was found that in two curricula, similarity was introduced in the context of scaling, while the third one provided two different definitions of the concept. Moreover, three major types of similarity activities were identified: differentiating (to determine/identify the similarity of given figures based on the intuitive notion of similarity), measuring (to measure a variety of attributes and use those measurements to explore similarity relationships), and constructing (to use specific tools and/or follow instructions to construct similar figures).

## METHODS AND PROCEDURES

Compulsory education in Greece lasts 11 years and includes pre-school (2 years), primary school (6 years; Grade 1 – Grade 6) and low-secondary school (3 years; Grade 7 – Grade 9). It should be stressed that all Greek schools have the *same* mathematics textbooks series provided by the Greek Ministry of Education, which further elevates their role in the Greek mathematics textbook series. Thus, the textbook series of all grades of compulsory education were included in our study. In Greece, similarity is explicitly introduced in Grade 5 and is further discussed in Grade 6, Grade 7, and Grade 9. The unit of analysis was the task. If a statement or a question had more than one part, then each part was a separate task. Overall, 176 similarity tasks were identified. Each task was coded separately for each of the three research questions and was recorded in a Microsoft EXCEL database.

## RESULTS

Amongst the 176 selected similarity tasks, the vast majority of the tasks (n=126) appeared in the textbook series of the Grade 9 (111 in the student textbook, 15 in the teacher's guide), 19 tasks Grade 5 (12 in the student textbook, 7 in the student workbook),15 tasks in Grade 6 (10 in the student textbook, 5 in the student workbook), and 16 tasks in Grade 7 (14 in student textbook, 2 in teacher's guide).

## Types of tasks offering students opportunities to learn similarity

The Greek textbooks are organised in chapters and units. The units have expository sections and exercises sets. In expository sections, new concepts are presented, statements are justified, and worked examples are presented. Each unit usually concludes with a set of exercises to be solved by the students (cf. Bergwall, 2021). The selected tasks of the present study were either in an *expository section* or in an *exercise*. The analysis has identified three types of expository section similarity tasks: a) *theory* (definitions, statements, criteria, conclusions about the notion of similarity), b) *history* (notes about the historical development of the similarity concept), and c) *worked-out examples* (solved exemplary tasks). Furthermore, exercises were categorised as (building upon Lo et al., 2006): a) *differentiating* (to identify the similar figures or determine if a pair of figures are similar), b) *measuring* (to measure and use the measurement results to explore relationships), c) *constructing* (to construct similar figures given a scale or a ratio), d) *finding* (to find the missing length of a side, the missing angle measure, the scale, the ratio, by using algebraic operations; see Figure 1 left), e) *reflecting* (to explain, to argue, to justify, to prove, to evaluate if a statement is true or false, to conclude; see Figure 1 right), and f) *communicating* (to communicate their mathematical ideas on the similarity to others).

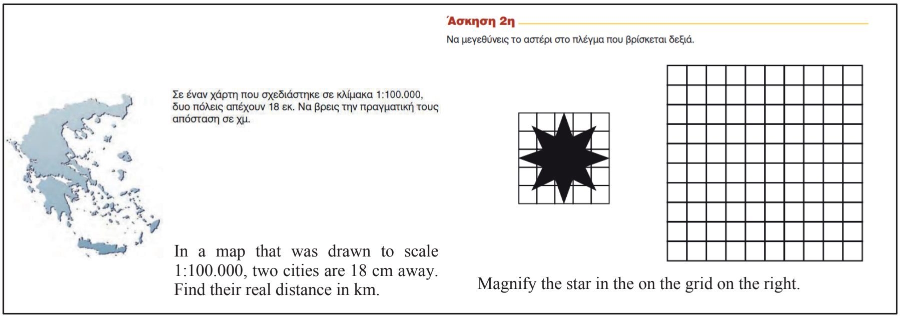



Figure 1. Finding task in real-world context (left; Grade 5, Workbook-B, p. 37), reflecting task (right; Grade 6, Workbook-D, p. 15).

The results of the analyses are outlined in Table 1.

| *Task Type* | | *f* | *%* |
|---|---|---|---|
| Exercise | | 144 | 81.8% |
| | communicating | 6 | 3.4% |
| | constructing | 21 | 11.9% |
| | differentiating | 5 | 2.8% |
| | finding | 60 | 34.1% |

| | | | |
|---|---|---|---|
| | measuring | 3 | 1.7% |
| | reflecting | 49 | 27.8% |
| Expository section | | 32 | 18.2% |
| | history | 2 | 1.1% |
| | theory | 15 | 8.5% |
| | worked example | 15 | 8.5% |

Table 1: Types of similarity tasks.

Most of the tasks are exercises (144 out of 176; 81.8%). Among the 144 exercise tasks, the finding tasks are 60, and the reflecting tasks are 49. In the finding tasks, students are asked to find, for example, the scale of an architecture design, the real distance of two cities, the length of the homothetic square or triangle sides, the similarity ratio, the angles with the same size in similar polygons, the missing length side of similar triangles, the corresponding sides of similar triangles. When students are engaged in reflecting tasks, they are required, for example, to explain their work on doubling/tripling a geometrical object, to investigate the properties of similar polygons, to explore the relationship between similarity ratio and area ratio, to justify why two given triangles are similar, to decide the true/false of the sentence "equilateral triangles are similar", to prove the similarity of two triangles. There are fewer yet noteworthy (11.9%) constructing tasks. For example, students are asked to construct a figure on a grid using a scale, a mechanical tool called "pantograph" that is used for scaling a figure, a similar polygon with certain similarity ratio when another polygon is given. Moreover, there are very few tasks that provide students with other opportunities to deal with the concept of similarity, such as, to identify/determine the similar polygons, to measure the lengths of two similar images (bees) so as to find their scale, to communicate their ideas about how similar triangles can be used to make astronomic observations.

Considering the expository sections, there are only 2 out of the 32 expository tasks that present a similarity problem from the history of mathematics (they are about how Thales of Miletus found the height of Egyptian pyramid and how Heron of Alexandria calculated the distance between two unreachable points), while the remaining 30 are theory and worked examples of the same number, through which the authors define the concept of similarity and its properties, and illustrate its use.

**Opportunities to learn similarity in real-world context**

Mathematics researchers appear to converge in about the benefits on the students learning when mathematical tasks are based on real-world context, related to students' personal day-to-day activities (even imaginary) or to broader public/community situations (Van Den Heuvel-Panhuizen, 2005). When students are engaged in such tasks, they are motivated to organize the solution according to the relevant mathematical concepts (Organisation for Economic Co-operation and Development, 2004).

The findings of our analysis with respect to this aspect of the similarity tasks in Greek textbook series is outlined in Table 2 (see also Figure 1, left).

| | | *Real-world context* | |
|---|---|---|---|
| | *Similarity tasks* | *f* | *%* |
| Grade 5 | 19 | 17 | 89.5% |
| Grade 6 | 15 | 9 | 60.0% |
| Grade 7 | 16 | 10 | 62.5% |
| Grade 9 | 126 | 18 | 14.3% |
| Total | 176 | 54 | 30.7% |

Table 2: Similarity tasks in real-world context.

Almost all of the similarity tasks in Grade 5 (89.5%, 17 out of 19), while over half of the similarity tasks in Grade 6 are situated in a real-world context. These are tasks for which, for example, students are asked to use the scale to find the real distance or the map distance, to discuss the truth or falseness of the statement "to design something with 1:1000 scale, I divide the real length by 1000", to evaluate whether a map scale can be used for a room design, to draw given figures using scale. The corresponding percentage for Grade 9 (where the vast majority of similarity tasks were identified) is only 14.3% (18 out of 126). Thus, it can be argued that the textbook series authors consider real-world context not to be so suitable for the older students. This conclusion adds to the earlier study of Wijaya et al. (2015) that showed that only about 10 % of the tasks in the Grade 8 textbooks are real context-based mathematics tasks.

**Opportunities to develop proportional reasoning**

In Table 3 we outline the results about the similarity tasks that appear to also promote students' proportional reasoning (see also Figure 2).

| | | *Proportional reasoning* | |
|---|---|---|---|
| | *Similarity tasks* | *f* | *%* |
| Grade 5 | 19 | 17 | 89.5% |
| Grade 6 | 15 | 13 | 86.7% |
| Grade 7 | 16 | 14 | 87.5% |
| Grade 9 | 126 | 69 | 54.8% |
| Total | 176 | 113 | 64.2% |

Table 3: Similarity tasks that develop proportional reasoning.

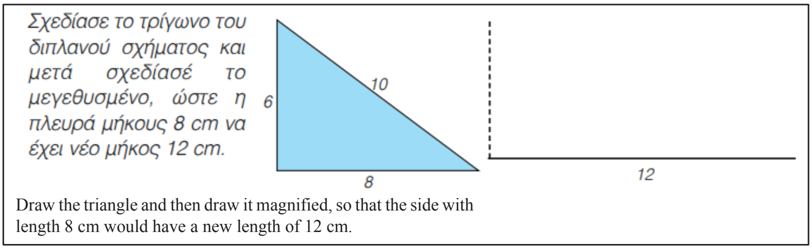



Figure 2. Similarity task with opportunity to develop proportional reasoning (Grade 7, student textbook, p. 90).

In Grade 5, Grade 6 and Grade 7, most of the similarity tasks (more than 85%) appear to also provide the students with the opportunity to develop a multiplicative way of thinking. In Grade 9, the students have the opportunity to develop their proportional reasoning, while they are engaged with the 54.8% of the similarity tasks they are provided with. For example, students' proportional reasoning is enhanced when they are explicitly asked to consider magnitudes and measurement when asked to shrink or enlarge a figure, to find the real dimensions of a room having the design scale, to explore how the microscope works, to explore how very long distances can be measured, to discover the properties of similar polygons, to evaluate if polygons with given lengths of sides are similar, to find the missing length of the side in similar triangles. All these examples, suggest that the authors of the Greek textbook series have chosen to strongly link geometrical similarity with aspects of proportional reasoning. Importantly, this appears to largely decrease in the last Grade of compulsory education (Grade 9). This may be linked with the fact that, in Greece, Grade 9 and Grade 10 are very close in terms of mathematical content, but in Grade 10 the students are for the first time engaged geometrical proof that is closely related to the Euclidean type of proof. Thus, this finding may suggest the authors of the Greek textbook series attempt to communicate to the students that Geometry has its own techniques and approaches that do not necessarily require the numerical aspects and focus on the geometrical properties (cf. Moutsios-Rentzos et al., 2014).

## DISCUSSION AND CONCLUDING REMARKS

The aim of the present study was to highlight the opportunities to learn the concept of similarity Greek mathematics textbook series offer to students. Our study draws upon and expands prior studies on similarity (e.g., Barcelos Amaral & Hollebrands, 2017; Barcelos Amaral & Hollebrands, 2023; Wijayanti, 2019) that focus only on textbooks, including also workbooks and teachers' guides. In our study, the focus was on the types of similarity tasks and their connection with the real-world, as well as the opportunities to develop the proportional reasoning through the geometric concept of similarity.

The results of the analyses revealed the need to expand on the categorisation of Lo et al. (2006), since the Greek textbook series appear to give the students the opportunity to quantify similarity, to 'argue' (explain, justify, prove etc.) about similarity, and to communicate their ideas about similarity. This suggests that the Greek students are provided with the opportunity to focus on linking geometrical ideas with algebraic relationship and to provide mathematical arguments to support their ideas, which seems support students to construct a sophisticated concept of similarity.

Moreover, in most of the tasks of primary and Grade 7, students have the opportunity to learn similarity in a real-world context in the vast majority of the tasks and to develop their proportional reasoning in more than 85%. This radically changes in Grade 9 where real-world context appears in almost half of the tasks, while proportionality appears in only about 15% of the tasks a little more than half of their grade tasks require a deep understanding of proportionality. We posit that this may echo the fact that in the following grade (Grade 10) the students are required to think about geometry in a 'Euclidean-like' way, focussing on the properties of the geometrical objects, and less about the arithmetic or algebraic relationships.

Furthermore, these findings appear to be in line with the recommendations of NCTM (2000): primary students should have the opportunities to use maps and make simple scale drawings either using a grid or not, focusing on proportionality to begin to think about similarity in terms of magnifying or shrinking transformations of geometrical figures. Most of the tasks of the primary school grades in Greece appear to require students to connect their algebraic reasoning with the geometrical reasoning and express their thinking either through words or through the production of a new figure, thus giving the students the opportunity to explore the concept of similarity in line with the above recommendation. On the other hand, almost half of the tasks of the middle school grades in Greece appear to emphasise geometric ideas of similarity (such as corresponding angles and/or sides) and not to focus on arithmetic/algebraic operations and reasonings. These seem to accord with the NCTM recommendations that middle grades students should extend the earlier intuitive notion of similarity to be explicitly linked with geometrical thinking and ideas, investigating the properties of, and the relationships among, similar shapes.

Finally, in the present study, the textbook series have been examined only as human artefacts, which does not necessarily imply their actual utilisation in the Greek classrooms. Our analysis is a first level analysis that provides a snapshot of the way similarity is addressed in textbook series (Shield & Dole, 2013). Consequently, our current research builds on these findings to consider more complex questions (Fan, 2013; Rezat, 2009) about the role of textbooks in the didactical relationship in everyday teaching, investigating the ways that in-service students utilise the textbook series tasks with the purpose to design appropriate support for pre-service and in-service primary school teachers about the teaching of similarity.

## References

Barcelos Amaral, R., & Hollebrands, K. (2017). An analysis of context-based similarity tasks in textbooks from Brazil and the United States. *International Journal of Mathematical Education in Science and Technology*, *48*(8), 1166–1184.

Barcelos Amaral, R., & Hollebrands, K. (2023). An analysis of similarity concept presented in textbooks in Brazil and the United States. *Educação Matemática Pesquisa Revista Do Programa de Estudos Pós-Graduados Em Educação Matemática*, *25*(2), 356–393.

Bergwall, A. (2021). Proof-related reasoning in upper secondary school: Characteristics of Swedish and Finnish textbooks. *International Journal of Mathematical Education in Science and Technology*, *52*(5), 731–75.

Cox, D. C. (2013). Similarity in Middle School Mathematics: At the Crossroads of Geometry and Number. *Mathematical Thinking and Learning*, *15*(1), 3–23.

Fan, L. (2013). Textbook research as scientific research: Towards a common ground on issues and methods of research on mathematics textbooks. *ZDM Mathematics Education*, *45*(5), 765–777.

Glasnovic Gracin, D. (2018). Requirements in mathematics textbooks: A five-dimensional analysis of textbook exercises and examples. *International Journal of Mathematical Education in Science and Technology*, *49*(7), 1003–1024.

Guiton, G., & Oakes, J. (1995). Opportunity to Learn and Conceptions of Educational Equality. *Educational Evaluation and Policy Analysis*, *17*(3), 323–336.

Hadar, L. L. (2017). Opportunities to learn: Mathematics textbooks and students' achievements. *Studies in Educational Evaluation*, *55*, 153–166.

Lamon, S. J. (2020). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers (4th ed.).* Routledge.

Lepik, M., Grevholm, B., & Viholainen, A. (2015). Using textbooks in the mathematics classroom–the teachers' view. *Nordic Studies in Mathematics Education*, *20*(3–4), 129–156.

Liu, X. (Ed.). (2009). *Linking Competence to Opportunities to Learn*. Springer.

Lo, J. J., Cox, D., & Mingus, T. (2006). A conceptual-based curricular analysis of the concept of similarity. In S. Alatorre, J. L. Cortina, S. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* (pp. 221–228). Universidad Pedagógica Nacional.

Lobato, J., Ellis, A. B., & Charles, R. I. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6–8*. NCTM.

Mammana, M. F. (2016). Homothetic transformations and geometric loci: Properties of triangles and quadrilaterals. *International Journal of Mathematical Education in Science and Technology*, *47*(7), 1103–1119.

Moutsios-Rentzos, A., Spyrou, P., & Peteinara, A. (2014). The objectification of the right-angled triangle in the teaching of the Pythagorean Theorem: an empirical investigation. *Educational Studies in Mathematics, 85*(1), 29–51.

National Council of Teachers of Mathematics (Ed.). (2000). *Principles and standards for school mathematics*. NCTM.

Organisation for Economic Co-operation and Development. (2004). *Learning for Tomorrow's World: First results from PISA 2003*. OECD.

Rezat, S. (2009). The utilization of mathematics textbooks as instruments for learning. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of CERME 6* (pp. 1260–1269). Institut National De Recherche Pédagogique.

Shield, M., & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. *Educational Studies in Mathematics*, *82*(2), 183–199.

Son, J.-W., & Diletti, D. (2017). What Can We Learn from Textbook Analysis? In J.-W. Son, T. Watanabe, & J.-J. Lo (Eds.), *What Matters? Research Trends in International Comparative Studies in Mathematics Education* (pp. 3–32). Springer International Publishing.

Stein, S. J. (2000). Opportunity to learn as a policy outcome measure. *Studies in Educational Evaluation, 26*(4), 289–314.

Törnroos, J. (2005). Mathematics textbooks, opportunity to learn and student achievement. *Studies in Educational Evaluation*, *31*(4), 315–327.

Van Den Heuvel-Panhuizen, M. (2005). The role of contexts in assessment problems in mathematics. *For the Learning of Mathematics, 25*(2), 2–23.

Van Zanten, M., & Van Den Heuvel-Panhuizen, M. (2018). Opportunity to learn problem solving in Dutch primary school mathematics textbooks. *ZDM Mathematics Education*, *50*(5), 827–838.

Wang, J. (1998). Opportunity to Learn: The Impacts and Policy Implications. *Educational Evaluation and Policy Analysis*, *20*(3), 137–156.

Weinberg, A., & Wiesner, E. (2011). Understanding mathematics textbooks through reader-oriented theory. *Educational Studies in Mathematics*, *76*(1), 49–63.

Wijaya, A., Van Den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics*, *89*(1), 41–65.

Wijayanti, D. (2019). Analysing textbook treatment of similarity in plane geometry. *Annales de Didactique et de Sciences Cognitives. Revue Internationale de Didactique Des Mathématiques*, *24*, 107–132.

Yao, X., & Manouchehri, A. (2019). Middle school students' generalizations about properties of geometric transformations in a dynamic geometry environment. *The Journal of Mathematical Behavior*, *55*, 100703.

# HANDS-ON ACTIVITY AS A CLASSROOM RESOURCE IN THE PRESERVICE PRIMARY SCHOOL TEACHERS' LESSONS IN HUNGARY AND INDONESIA

Linda Devi Fitriana*, Zoltán Kovács**, Christiyanti Aprinastuti*/***

*University of Debrecen, Hungary

**Eszterházy Károly Catholic University, Hungary

***Sanata Dharma University, Indonesia

*This study explores the effect of integrating creative writing with mathematical problem-solving tasks through a hands-on activity to address a gap in the literature on interdisciplinary approaches. Focusing on Hungarian and Indonesian preservice primary school teachers, we investigate characteristic writing outputs. The findings contribute insights into the interdisciplinary approach, shedding light on the potential impact of each country's pedagogical or cultural background on mathematics education.*

## INTRODUCTION

In this case study, our primary objective was to implement a mathematical topic for preservice primary school teachers within a learning environment designed to reflect the complexity of the teaching profession. We aimed to examine the success of this approach, focusing on learning outcomes and the inner motivation of the participants. The topic was number sequences, and the problem solving was based on the heuristic strategy of finding patterns. In addition to mathematical problem solving, the activities included creative writing, problem posing, and art making, the latter as homework. This classroom practice demonstrates interdisciplinarity, as it combines mathematics with creative writing to motivate individuals to express their experiences and thoughts using an integrated framework. This method may improve comprehension of mathematical ideas and boost verbal and communication abilities, demonstrating the integration of analytical and imaginative thinking in a coherent educational setting.

This paper details the outcomes of the creative writing part of the lesson. The lesson was conducted in two countries with different mathematics education and cultural traditions: Hungary (N=19) and Indonesia (N=89). We also investigated how these traditions were reflected in students' creative writing outputs.

## PRIMARY SCHOOL TEACHER TRAINING SYSTEM IN THE TWO COUNTRIES

Hungary and Indonesia use different pedagogical approaches to mathematics education that reflect their respective educational traditions and objectives.

Hungary emphasizes talent nurturing and guided discovery approach, integrating various tools, games, and interactive technology to make mathematics learning more engaging (Győri et al., 2020). This approach aims to enhance students' mathematical proficiency by creating meaningful learning experiences. Indonesia upholds the principle of implementing relevant learning (Indonesian ministry of education, culture, research and technology, 2022), allowing the ongoing implementation of Realistic Mathematics Education (RME), which emphasizes contextual problem solving and integrating mathematical concepts with real-life situations (Zulkardi et al., 2020).

In both countries, training programs for preservice primary school teachers are part of the higher education system and span four years, leading to a bachelor's degree. Primary school teacher training programs in Hungary and Indonesia aim to prepare candidates for the complex role of teaching in primary school. In Hungary, students are prepared to teach all the subjects in the first four years of primary school (including mathematics) and one subject (depending on students' choice) in grades 1 to 6. In Indonesia, primary school teachers are responsible for multiple subjects, especially in grades 1 to 3, where the lesson is thematically integrated, combining the content of several subjects. In higher grades, the distribution of subjects may vary based on the specific curriculum adopted by the school. Both countries offer programs integrating a significant portion of practical content, including teaching practice. Both countries' training programs underscore the importance of adapting teaching practices to diverse student backgrounds, with particular emphasis on the country's cultural diversity in Indonesia.

The study is framed by the following question: How is the mathematics education tradition in the two countries manifested in their creative writing outputs?

## LITERATURE BACKGROUND

Despite the growing body of literature on integrating mathematics and science, see, e.g., (An, 2017), there remains a gap in research concerning the interdisciplinary approach combining creative writing with mathematical problem solving. Our research aims to fill this void by investigating the impact of incorporating creative writing activities into mathematics problem-solving tasks.

Drawing on their expertise as a mathematics teacher, Morgan (1998) underscored the integration of writing within mathematical investigation tasks. In this context, students are allowed to emulate the authentic experiences of mathematicians, engaging in the resolution of relatively substantial and frequently original problems. Concurrently, the act of writing serves the purpose of articulating and persuading others regarding the accuracy of their mathematical findings. O'Kelley (2013) concurs with this assertion, as articulated in their statement emphasizing the interdependence of the processes of writing and engaging in mathematics. O'Kelley posits that the writing process has the potential to propel students further into the exploration of mathematical concepts.

In contemporary education, integrating writing into mathematics instruction has been acknowledged as a powerful tool for documenting information and fostering a deeper understanding of mathematical concepts among students. Alvermann (2002) and Pugalee (2004) provided converging perspectives on the significance of writing in mathematics learning. Alvermann emphasized its role in elevating cognitive abilities and critical thinking, while Pugalee highlighted its support for mathematical reasoning, problem solving, and the internalization of effective communication skills. Siegler (2007) emphasized the importance of engaging in mathematical talk to build understanding. The author suggested encouraging students to use verbalization, write reflective notes, and create stories that include mathematical problems, creating conditions for learning mathematics while participating in social classroom processes.

Free-form writing might be started by hands-on activities like paper folding, which utilizes paper as an educational resource. Encouraging students to engage in paper folding promotes conjecture, and simple observations linked to the paper-folding process make the results convincingly plausible (Coad, 2006). The synergy between mathematical and writing skills is evident as students translate three-dimensional spatial manipulations into written expressions, which challenges students to think critically about the mathematical principles at play. Thus, it provides a unique avenue for students to integrate mathematical and writing skills into a cohesive and complex learning experience, going beyond traditional teaching methods.

Using free-form writing in mathematics education research can capture the nuances of students' thought processes, shedding light on their understanding and potential misconceptions in mathematical learning. Incorporating free-form writing into research methodologies fosters a deeper engagement with mathematical content, encouraging students to articulate their reasoning, problem-solving strategies, and reflections more personally and expressively. This provides researchers with rich qualitative data and enhances students' metacognitive awareness and communication skills.

Finally, free-form writing offers a multifaceted approach to researching mathematics education, providing a window into students' cognitive processes, promoting metacognitive awareness, and fostering a deeper understanding of mathematical concepts through expressive communication (Urquhart, 2009).

## METHOD

The study participants comprised 108 preservice primary school teachers: 89 Indonesians and 19 Hungarians. We took the Indonesian sample (20 men and 69 women) from a university in Yogyakarta. The Hungarian sample consisted of 19 students (1 man and 18 women) from three universities in Nyíregyháza, Eger, and Vác. Both groups averaged 21 years old, completing a professional mathematics course in their teaching training program.

The task elaboration and lesson organization are as follows:

1. Each participant in each group was given a sheet of paper for a preliminary activity, i.e., folding the paper in half twice until it could no longer fold (see Figure 1 for an illustration of the folding). At the same time, they were instructed to find the pattern of the folding lines on the $1^{st}$, $2^{nd}$, $3^{rd}$, and so on until the $n^{th}$ fold (Mason et al., 2010). First, the participants worked individually, then had a partner discussion, and finally, a class discussion.

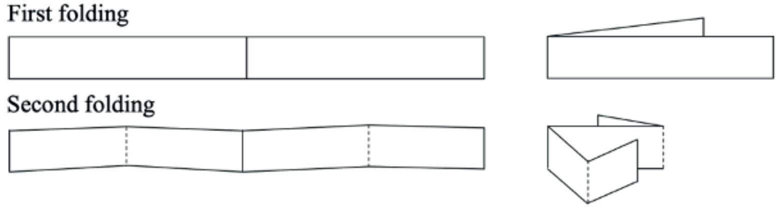



Figure 1: Illustration of the folding.

2. By reflecting on the folding paper activity, the participants were encouraged to work on the following task:
    a. Write a story related to the folding paper activity.
    b. Complete the previous story into a word problem for elementary school students that matches the folding activity.
3. Participants completed the motivation questionnaire.

The task aligns with the school curricula, incorporating patterning and recreational mathematics elements, fostering algebraic thinking through an enjoyable activity. It allows participants to develop a story problem by interpreting the paper folding activity in an open environment with a clear starting point but without specific restrictions (Stoyanova & Ellerton, 1996). The students had an opportunity to reflect on the mathematical aspect of the paper folding activity and express it into a story. Thus, the task design provides synergy to enhance mathematical and writing abilities.

To investigate the writing tendencies, the written outputs were systematically categorized, distinguishing between factual narratives representing real-life stories and artistic forms encompassing fictional motives, stories in the form of a tale or poem. Two authors coded individually the corpus and later, the disagreements were decided based on all authors' consensus.

## RESULTS AND DISCUSSION

The creative writing outputs among Indonesian participants showed that 72% of them fulfilled the task, while 28% did not provide a story. Of those who did, 100% centred their stories on factual portrayals of real-life experiences, with none presenting artistic expressions such as tales or poems. For example, in the following quotation, the student accurately reconstructed the classroom activity, and the task faithfully reflected the original problem. However, no specific creative element appears in the text.

> In the classroom, Mr. Rahmat introduced folding to his students. They folded a piece of paper in half, making it half the width of the original. Then, they folded the paper into four equal parts, making it one-fourth the size of the original.
>
> Mr. Rahmat gives each student a square piece of paper the same size. Each student is asked to fold the paper in half, making it half the width. How many layers of paper will they have after folding?

This manifestation implies at least three points: 1) It contains a real-life context that emphasizes the rootedness of RME in the minds of students, (2) The task encourages the appearance of metacognitive awareness as the students reflect on what they did before writing the story (Urquhart, 2009), and 3) It corroborates the findings of the inner motivation analysis, specifically in the "Value" subscale, where students perceived the sequence of activities as advantageous for their future careers. In this instance, the students transferred their learning experience into contexts that enabled them to apply it in their students.

Another participant focused on the Indonesian artifact, the "besek," a traditional woven basket commonly used by Javanese that embodies both practical utility and cultural importance in effectively displaying the integration of mathematics and ethnography. Ethnography appears as an approach that provides a basis for applying mathematical principles to traditional, culturally-rooted real-world phenomena. Ethnography is a possible component of RME, deeply embedded in the problem environment through cultural aspects (Prahmana, 2023). Despite the openness of the problem and its detachment from the pattern of the original task, we consider the endeavor instructive, which combines a real-life story with Indonesian culture, particularly in the context of Yogyakarta.

> Make a besek using origami paper, then count how many folds occur from the beginning of the creation until it's finished! How many fold lines are there in the shape of the besek?

Thirty-two percent of Hungarian participants wrote stories related to creative expression (artistic forms), i.e., fairy tales and poems, while the remaining 68% produced real-life stories. Cultural history may have an impact on the creative expressions of Hungarian students. Gosztonyi (2019) states that a historical perspective on mathematics plays a crucial role in a specific Hungarian tradition of mathematics education. Earlier, Hersh and John-Steiner (1993) stated that

Hungary has an artistic writing legacy, including mathematics education. The following Hungarian student's work reflects a form of artistic writing related to the tradition, which supports what Hersh, John-Steiner, and Gosztonyi mentioned.

> Once upon a time, there was a grandmother. The grandmother lived in a lovely place, in the Kerekerdő. One day, she heard that her grandchildren were coming and wanted to treat them to something nice. So, she baked them some strudel and folded the pastry several times. ["Kerekerdő" is a frequent motif in Hungarian folk tales; literally, it is translated to "Round Forest."]
>
> Grandma folded the pastry in half first, so she had two layers. How many times did Grandma have to fold it to get 15 folded lines?

This artistic writing reflects the principles of primary education set out in the Hungarian National Core Curriculum (Government of Hungary, 2020), namely that students should be encouraged to develop their problem-solving skills while experiencing joy through fun activities, which can be related to music or literature. In addition to the folk-tale motifs in the text, the problem clearly reflects the class activity.

In Indonesia, with a strong emphasis on RME, preservice primary school teachers provided a real-life context for word problems. Conversely, in Hungary, where history is deeply rooted, a preference for artistic writing styles appeared.

## CONCLUSION AND PEDAGOGICAL IMPLICATION

The incorporation of hands-on activities and the creative writing task resulted in the conclusion and pedagogical implications as follows:

The Hungarian group's creative writing products were more varied, including real-life stories, tales, and poems, while the Indonesian group's writing was merely real-life stories. This tendency reflects each country's tradition in mathematics education that is still in effect today: Hungary with its guided discovery approach, which incorporates various tools, games, and interactive resources to make learning mathematics more engaging for students, and Indonesia with its realistic mathematics education tradition, which integrates mathematical concepts and real-world situations and has been recognized for its effectiveness in enhancing mathematical literacy and fostering a deeper conceptual understanding of mathematics.

Given that the early-grade curriculum in Hungary follows a multidisciplinary system where subjects are taught separately while Indonesia follows an interdisciplinary system through its integrated thematic approach, it is expected that Indonesian preservice teachers' creative writing outputs were also in another form of literature, not merely real-life stories. A result contradicting this fact requires further investigation.

In summary, it is noteworthy to underscore the discernible advantages derived from the amalgamation of hands-on activities and creative writing. The undertaking clarified the impact of different educational backgrounds in each country and demonstrated that, despite these varying academic foundations, both groups universally found the activity intriguing and beneficial.

## Acknowledgment

This study was funded by the Research Program for Public Education Development of the Hungarian Academy of Sciences (KOZOKT2021-16).

## References

Alvermann, D. E. (2002). Effective literacy instruction for adolescents. *Journal of Literacy Research*, *34*(2), 189–208.

An, S. A. (2017). Preservice teachers' knowledge of interdisciplinary pedagogy: The case of elementary mathematics–science integrated lessons. *ZDM - Mathematics Education*, *49*(2), 237–248.

Coad, L. (2006). Paper folding in the middle school classroom and beyond. *Australian Mathematics Teacher*, *62*(1), 6–13.

Government of Hungary. (2020, September 2). *A Kormány 5/2020. (I. 31.) Korm. rendelete a Nemzeti alaptanterv kiadásáról, bevezetéséről és alkalmazásáról szóló 110/2012. (VI. 4.) Korm. rendelet módosításáról* [Government Decree No. 5/2020 (I. 31.) amending Government Decree No. 110/2012 (VI. 4.) on the publication, introduction and application of the National Core Curriculum]. https://njt.hu/jogszabaly/2020-5-20-22

Gosztonyi, K. (2019). History of mathematics, mathematical histories, storytelling in Hungarian mathematics education. In É. Vásárhelyi & J. Sjuts (Eds.), *Auch wenn A falsch ist, kann B wahr sein. Was wir aus Fehlern lernen können* [Even if A is wrong, B can be true. What we can learn from mistakes] (pp. 137–148). WTM-Verlag.

Győri, J. G., Fried, K., Köves, G., Oláh, V., & Pálfalvi, J. (2020). The traditions and contemporary characteristics of mathematics education in Hungary in the post-socialist era. In A. Karp (Ed.), *Eastern European mathematics education in the decades of change* (pp. 75–129). Springer International Publishing.

Hersh, R., & John-Steiner, V. (1993). A visit to Hungarian mathematics. *The Mathematical Intelligencer*, *15*, 13–26.

Indonesian ministry of education, culture, research and technology. (2022, June 22). *Perubahan atas keputusan menteri pendidikan, kebudayaan, riset, dan teknologi nomor 56/M/2022 tentang pedoman penerapan kurikulum dalam rangka pemulihan pembelajaran* [Amendment to decree of the minister of education, culture, research and technology number 56/M/2022 on guidelines for curriculum implementation in the context of learning recovery]. https://jdih.kemdikbud.go.id/detail peraturan?main=3156

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically second edition*. Pearson Education Limited.

Morgan, C. (1998). Writing mathematically: The discourse of investigation. In P. Ernest (Ed.), *Studies in mathematics education series* (9th ed., pp. 232). Falmer Press.

O'Kelley, S. K. (2013). Helping teachers connect writing to doing mathematics. *STRATE Journal*, *23*(1), 18–23.

Prahmana, R. C. I., Arnal-Palacián, M., Risdiyanti, I., & Ramadhani, R. (2023). Trivium curriculum in Ethno-RME approach: An impactful insight from ethnomathematics and realistic mathematics education. *Jurnal Elemen, 9*(1)*,* 298–316.

Pugalee, D. K. (2004). A comparison of verbal and written descriptions of students' problem solving processes. *Educational Studies in Mathematics*, *55*(1–3), 27–47.

Ryan, R. M., Koestner, R., & Deci, E. L. (1991). Ego-involved persistence: When free-choice behavior is not intrinsically motivated. *Motivation and Emotion*, *15*(3), 185–205.

Siegler, R. S. (2007). Microgenetic analyses of learning. In D. Kuhn, R. S. Siegler, W. Damon, & R. M. Lerner (Eds.), *Handbook of child psychology: Cognition, perception, and language* (6th ed., pp. 464–510). John Wiley & Sons, Inc.

Stoyanova, E., & Ellerton, N. (1996). A framework for research into students' problem posing in school mathematics. In P. C. Clarkson (Ed.), *Technology in mathematics education* (pp. 518–525). Mathematics Education Research Group of Australia.

Urquhart, V. (2009). *Using writing in mathematics to deepen student learning*. McREL.

Zulkardi, Z., Putri, R. I. I., & Wijaya, A. (2020). Two decades of realistic mathematics education in Indonesia. In M. van den Heuvel-Panhuizen (Ed.), *ICME-13 monographs. International reflections on the Netherlands didactics of mathematics. Visions on and experiences with realistic mathematics education.* Springer Open.

# CUSTOMIZED LEARNING PATHS: NAVIGATING COMBINATORICS FOR DIVERSE LEARNERS

Anna Kuřík Sukniak

Faculty of Education, Charles University in Prague, Czech Republic

*Teaching resources for combinatorics offer a diverse array. However, there are no universally applicable teaching materials suitable for all pupils. Creating resources that will meet each pupil's unique learning needs is desirable. The finite range of cases allows tailoring materials to suit the typical pupil's characteristics. In my research, I assigned various tasks involving computations of problems of type "5 over 2" to different pupils. They autonomously sought solutions, and when encountering tasks, they searched for correlations and isomorphisms among them. My goal is to present and discuss the findings arising from these inquiries.*

## INTRODUCTION

Understanding the learning theories and their application is paramount in education. Research findings suggest that pupils across various educational levels struggle when learning combinatorics. Further insights into combinatorial thinking, the inherent complexity of these problems, common problem-solving strategies, and frequent difficulties encountered are essential for effective teaching. The research aims to delineate the development of combinatorial thinking in lower secondary school pupils aged 11-12, focusing on appropriating and extending combinatorial thinking by exploring solutions to “5 over 2” type problems. While isomorphism is evident in some word problems, it remains concealed or proves challenging to discern in other instances. The research questions include examining the problem-solving strategies employed by pupils for “5 over 2” type word problems, tracking the development of these strategies, and investigating how pupils recognize isomorphism between these problems. This research has the potential to uncover the cognitive processes involved in navigating complex combinatorial scenarios, offering insights into the evolution and application of combinatorial thinking and problem-solving strategies. Our main aim is to uncover the cognitive processes of pupils’ minds while solving combinatorial problems.

## THEORETICAL FRAMEWORK

Combinatorial problems encompass mathematical or practical scenarios that involve combinatorial activities. They consist of a base set, a working set, and an organizational principle (Hejný, 1990). According to the implicit model (Dubois, 1984), these problems are categorized as selection, distribution, or partition, reflecting the thought operations involved. Further, they can be classified based on the operation employed for the solution, the context, and the number of combinations (Fishbein & Gazit, 1988).

Pupils at all levels face difficulties even when solving elementary combinatorial problems (e.g., Hadar & Hadass, 1981; Fishbein & Gazit, 1988; Batanero et al., 1997; Lockwood & Gibson, 2016). Researchers (e.g., Fishbein & Gazit, 1988; Godino & Batanero, 2005; Lockwood, 2013) and teachers (Vondrová, 2019) agree that the main problem in solving combinatorial problems is the lack of competence to identify the combinatorial operation. Pitfalls also include the inability to fully understand the word problem, the choice of inappropriate notation for the information, the incapability to decompose the problem into a series of sub-problems, the choice of an unsystematic computing method, the inability to add constraints to the original problem to solve it, difficulties with generalization (Hadar & Hadass, 1981) and the inability to see two isomorphic combinatorial problems as equivalent (Batanero et al., 1997; Maher et al., 2011).

Among common strategies for solving combinatorial problems are classification; systematic enumeration; use of inclusion/exclusion principle; use of recurrence; drawing of tree diagrams and graphs (Figure 1); construction of tables (Figure 1); use of addition, multiplication, and division principles; use of combinatorial and factorial numbers; use of Pascal's triangle (Batanero et al., 1997).

Combinatorial problems can be solved graphically in three fundamentally different ways. Figure 1 shows how the following problem of football matches can be solved: *Five teams entered the football tournament. It is played on a one-on-one basis, one match each. How many matches will there be?*

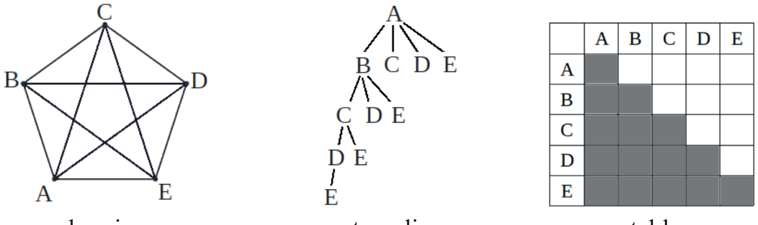



Figure 1: Three different graphical solutions to the football matches problem.

These diverse problem-solving approaches highlight the range of tools available. Additionally, considering the ability to perceive isomorphism as another valuable strategy adds depth to our understanding of combinatorial problem-solving.

Isomorphism refers to structural similarity between entities that may appear different at first glance but share identical patterns or relationships. Regarding this research, an illustration of isomorphism will be given for two problems of type “5 over 2”, tasks Towers and Party from the research problem list (Table 1).

To show that these problems are isomorphic, we will match elements from one problem to the corresponding elements in the other problem (Figure 2). We can assign the position of each cube to one of the girls. Specifically, Diana is assigned the cube on the 1st level, Ema is assigned the cube on the 2nd level, and so on. If the cube is blue, the girl will tap her glass. The first tower shows Ema’s and Diana’s toast, the second tower shows Mona’s and Diana’s toast, etc.

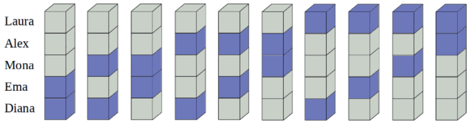



Figure 2: Illustration of isomorphism between the Towers and Party tasks.

## METHODOLOGY

During the first half of the school year were selected pupils in 6th grade from a random school. This specific time of the year and school grade increases the chance that pupils have not encountered similar combinatorial tasks during mathematics classes yet. There were no other characteristics required among pupils to be selected. Pupils split into groups of two to four according to their time possibilities. The experiment took place in school after teaching hours and consisted of three to four 90-minute sessions during which pupils gradually solved 24 problems. Their discussions or individual explanations of solutions were video recorded. This paper analyzes the first meeting with the group of three boys – Erin, Will, and Jay during which they solved five tasks (Table 1).

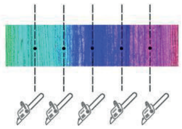
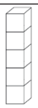
| | |
|---|---|
| Board | In a wooden toy factory, a machine cuts the same-colored boards. To make it clear which way the cut goes, each board has 5 small black dots on it. Now the machine is set up to cut each board into 3 pieces. How many different ways can it cut the board?  |
| Towers | We build towers from 5 cubes as shown in the picture. Two cubes are colored blue, the rest is gray. How many different towers can be built?  |
| Party | Diana throws a party. Ema, Mona, Alex, and Laura come to the party one by one. Everyone who comes taps a glass of lemonade with everyone there. How many of those taps there will be? |
| Stones | In the table, the number 102 is formed using 3 stones. How many different numbers can be created using 3 stones?   |
| Cups | Tom has two identical cup stickers and five differently colored cups on which he can place them. He does not want to put more than one sticker on the cup. How many ways can he put the two stickers on the cups? |

Table 1: Series of five isomorphic tasks.

## ANALYSES

For each word problem, the solving process can be decomposed into three parts:

1. Understanding the problem.
2. Grasping the task.
3. Problem-solving strategy.

Understanding the task can be shifted. The solver could understand the task differently than the author intended. It can also be an alternative when the solver finds two or more understandings. The best way to find out how the solver understood the problem is to ask him to explain the task in his own words to a peer. At the pupils' request, the experimenter added verbal explanations to make the task as clear as possible. Comprehension of the problem begins with the choice of language the solver uses to describe the objects and relationships of the problem. The strategy for problem-solving means finding an organizing principle. The analysis of the meeting attempts to replicate the timeline concerning the continuity and interconnection of individual ideas.

### Task 'Board'

Erin: 1) There were two types of objects in the problem - the cuts and the parts of the board that the cuts created. From a computational perspective, only the cuts were important. For Erin, the parts of the board were dominant, so the idea of a cross-section also arose in his mind, and the numbers 26 and 32 appeared in his results in addition to the number 20. He then had to displace the idea. 2) He recorded the various possibilities using only his fingers and stored these in his memory. 3) He fixed one cut and took each other option as the second cut of this solution. To each cut belonged 4 other cuts. That led to the result $5 \cdot 4 = 20$. The absence of a coding system made his explanation on the whiteboard unclear. However, it was here that he realized that taking the first two cuts would take this option twice in his counting. This led him to correct his result, but the idea that the original erroneous result of 20 could just be divided by two did not occur to him on the whiteboard. At the desk, he then devised a second strategy. He found 4 possibilities for the first cut, 3 for the second cut, etc., and quickly arrived at the correct number 10.

Will: 1) He understood that only one part (one-sixth) of the board was being cut into three pieces, but even this was not entirely clear to him, so he did not progress further with the problem.

Jay: 1) He clearly understood the problem. 2) He found the coding system. He used the same digit to indicate the two cuts that make up one solution. 3) Jay found a good strategy thanks to a clear grasp of using codes. Figure 3 on the left side shows how he proceeded. It was not completely systematic, but it was clear enough. Then, when explaining his solution on the whiteboard (Figure 3 on the right side), his strategy was consistently systematic.

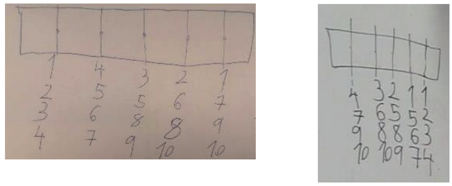


Figure 3: Jay’s solution in the worksheet and on the whiteboard.

**Task ‘Towers’**

Erin: 1) The understanding of the task was immediate. 2) He grasped the task by manipulating cubes. He stored individual cases in his mind and did not record them. During manipulation, he also discovered that harder-to-manipulate towers could be laid down and made into “trains”. 3) His organizing principle had two steps: 1. Move the lower cube up by one to the point where repetition occurred, and 2. Change the relative position of the two blue cubes so that in the first round there were two blue cubes next to each other, in the second round there would be one grey cube between the blue ones, and in the third round, there would be two grey cubes between the blue ones.

Will: 1) The understanding of the problem was immediate. 2) He grasped the task by manipulating cubes. He grasped the problem graphically. Probably inspired by Jay, he captured each of the towers with a rectangle divided into 5 parts (Figure 4, left side). The rectangles were horizontal, and Will immediately saw that a lying rectangle (train) or a standing rectangle (tower) was the same. The boy showed the ability to transform geometrically vertical and horizontal positions. He first drew a series of “trains” and then colored two blue cubes. 3) He was not looking for any organizing principle to arrange the individual towers and believed he could create the whole set by the “looking for what is missing” system. He arrived at number 9, presented on the whiteboard (Figure 4, in the middle).

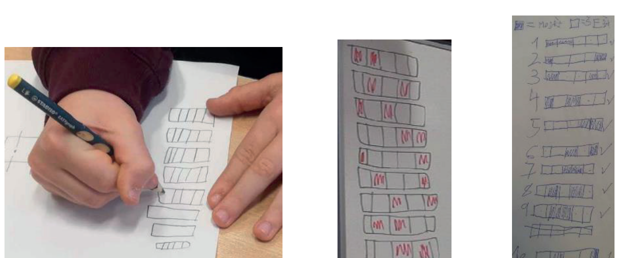


Figure 4: Solutions to the Towers task.

Erin examined Will’s solution and did not object but suggested a more appropriate organizing principle. He tried to demonstrate this by interspersing the cases in Will’s solution. Still, he then got lost in it and demonstrated his idea of moving one cube at a time using a tower of 5 cubes in a manipulative design. Eventually, he searched for the missing element of the solution and discovered the symmetry of the set of all solutions. Erin completed Will's solution with the 10th missing tower case. He looked at the whiteboard (Figure 4, in the middle) and noticed only 3 squares colored in the first and fourth columns of Will's list, whereas 4 squares colored each time in the other columns. This allowed Erin to find the missing case. In terms of insight into the combinatorial situation, this insight of Erin's was the most profound. It showed that the solution system is subject to symmetry. He applied this brilliant insight once more, in a problem with stones. This moment showed the quality of his thinking.

Jay: 1) The understanding of the task was immediate. 2) He grasped the task by manipulating cubes. He drew "towers" as "trains" right at the beginning. We could not pinpoint the cause. Life experience may have played a role here, as we wrote various lists in a column, such as things to buy. He used colors in his coding. 3) He used two organizing principles (Figure 4, right side). He found the first four cases by moving one cube to the left. Then he found cases 6 – 9 by moving the left blue cube one box to the left, and the left blue cube to the second blue cube. We did not know when he discovered he was missing one more case or how he found it. He stated that the organizing principle for the first five towers was the same as Erin's. However, this was not accurate. While Erin moved the cube from the first floor to the fifth floor each time, Jay moved the same cube one position higher. Further, Jay, in his own words, did not find an organizational strategy and searched for other towers by random selection. According to his record, there was an organizing principle.

**Task 'Party'**

Will: 1) Understanding the task was immediate. 2) He grasped the task through names. He did not write anything down and kept everything in his mind with the help of fingers to count. 3) The organizing principle was determined by process, first comer tapped once, second twice, etc. So, the result is $1 + 2 + 3 + 4 = 10$.

Jay: 1) He made sure he understood the assignment well. He asked if they all tapped together each time. 2) and 3) Neither the grasp nor the organizing principle was clear. When explaining his solution, he stopped and said he got it wrong. He made a mistake somewhere and only realized it when he should have been describing his organizing principle. We did not find enough text in written records to infer it.

Erin: 1) Understanding the task was immediate. 2) and 3) He nodded that he solved it the same way as Will did.

**Task 'Stones'**

Jay: 1) He needed to make sure he understood the assignment. Namely, whether all three stones always had to be used and how big the table was. 2) He grasped the task through the table. 3) His organizing principle was to move the stones from left to right. He started in the 3-0-0 triplet and ended in the 0-0-3 triplet. In this way, he got 7 cases. He realized he missed a combination of ones and hundreds, so he found solutions 1-0-2 and 2-0-1. This gave him a total of 9 cases. He was likely applying the same organizing principle here, i.e., shifting this time in the opposite direction, from ones to hundreds. One stone was moved from the last column to the first, and another stone in the next step. He missed the solution to 1-1-1. Erin immediately wrote it on the whiteboard. In Jay's solution on the paper, we found two other, but repeating solutions, 0-2-1 and 0-1-2. Jay did not realize he already had these solutions and was convinced that all of the solutions were 11.

Will: 1) He clearly understood the problem. 2) He grasped the problem through the numerical values of the cases. However, he did not take this fact as an organizing principle. 3) He searches the file using the "looking for what's missing" system. The repetition of case 300 on his paper in the first and second places was probably the result of inattention. He was only missing case 102, but this was a number from the assignment, so it was possible that he was not looking for it.

Erin: 1) Understanding the task was immediate. 2) and 3) We didn't know how he grasped the task, what organizing principle he used, or if he used it at all.
Erin quickly filled in Jay's solution with the 10th missing case 1-1-1. We could not determine how he found this case. Probably he had been looking for this for a long time, therefore it became dominant in his mind. This could be why it was easy for him to discover quickly that this case was missing. Erin did not articulate his thoughts on this, nor did he make any written record. This too was probably the reason for his first incorrect result that the solution was 12. At the whiteboard, he already believed the solution was 10, which he argued was due to the symmetry of the columns. In each column, were once three stones, twice two stones, and three times one stone. Again, he found the structure of all ten cases, as he did for the task Towers.

**Task 'Cups'**

Will: 1) Needed to clarify the assignment. The experimenter repeated the assignment in other words. Then Will understood the assignment clearly. 2) Will saw the isomorphism between the situation with the towers and the situation with stickers on the cups. He could describe this isomorphism at the object level. He accurately assigned five cups to each tower (the object of the set of towers) by putting a sticker on the cups corresponding to the blue cubes on the tower. It was clear to him that this representation is bijective. The nature of the isomorphism that Will had discovered was conceptual. 3) There was no need to find an organizing principle as the task was already solved thanks to the isomorphism with task Towers.

Jay: 1) He understood the assignment clearly. 2) He found the coding system. He used the same digit to label the two cups that make up one solution. He discovered the isomorphism between the Board and the Cups tasks due to the solving process. When asked if the simile to cups Will explained made sense, he broke down the steps one by one and nodded. This was how class osmosis occurred, with one trying to understand the other's vision. It required a welcoming atmosphere, effort, and willingness to understand what Will was saying. In doing so occurred a projection of Will's thoughts onto Jay's structure. This process was markedly different from the process where the teacher puts his thoughts into the structure of the pupil. Here, the pupil does not need to insert new ideas into his structure, but needs to remember these ideas, thus the autonomy of the pupil's thinking is at a very low level. This highlights the mistake teachers make when they do not allow discussion between children because a classmate would not say it accurately enough. This eliminates the pupil's intellectual autonomy; the idea is not inserted into an existing structure, but new memory information is inserted, in a belief that this is how the pupil will learn it. The nature of isomorphism that Jay discovered was procedural. 3) The organizing principle was the same as in the Board task.

Erin: 1) Understanding the task was immediate. 2) and 3) Discovered an isomorphism between the Board the Cups tasks. However, we did not know if this was due to the solving process or if he saw a bijection between structures. We believed he suspected the isomorphism at a conceptual level and then verified it procedurally.

**Didactical conclusions**

Erin: He had not yet demonstrated the ability to grasp the situation using a coding system. He had not needed it yet, but coding would make his solution more transparent. The second strategy he used to solve the Board task did not require coding, but his initial grouping could have been eliminated using appropriate codes. The "look and see" strategy was an organizing principle that relies on trusting that his mind could run through all the possibilities without writing them down. Erin had a strong short-term memory and could retain structural links, which was already seen in the organizing principle in the Towers task. There he took the transfer of the bottom cube up and then the transformation of the adjacent cube of blue and grey. In our judgment, the effort to further build this memory does not contribute substantially to the intellectual growth of the pupil. The desirable thing is to teach him to grasp the situation by notation. To support his ability to encode, two ways are suggested: 1. Asking Erin to explain a combinatorial problem to a classmate; 2. Give the boy sufficiently challenging problems that he cannot grasp without coding. This can be done, for example, by increasing the number of objects. Instead of looking for the number 5 over 2, we might look for the number 6 or 7 over 2. Erin's thinking could detect patterns. We can increase this ability when we switch to graphical grasping. A good way to do this is to offer him tasks in which he can discover other structural relationships. For example, a series of tasks: 1. We have 4 blue, 1 red, and 1 green cube. How many different towers can be created? 2. We have 4 blue and 2 red cubes. How many different towers can we create? 3. How are the previous two problems related?

Will: The Board task was currently elusive for Will. This was probably due to a combination of two factors. First, the task was long, and its context was outside Will's experience. Then the objects he perceived here were semantically diverse - cuts and parts of the board. In the Towers (Cups) task, the objects were homogeneous - blue cubes (cups with a sticker) and grey cubes (cups without a sticker) so he had no problem finding a solution here. Comparing his solutions to the Party and Towers tasks, the conceptual assignment of Towers offered Will no organizing principle, and he was looking for an enumeration of all possibilities. This raised the question of whether a problem could be found between this procedural and conceptual assignment. There were two currents in the development of his thinking. Grasp a situation through an organized set of signs (such as numbers) and build a structure. In both cases, he can be helped by classmates. The teacher can give Will tasks that already help structure the assignment. These can be tasks with procedural assignments, as in the Party task. Or to tasks with conceptual assignments, appropriate instructional tasks can be added. For example, add two pre-tasks to the task on sections: 1. If the first cut is placed to the left, how many options will there be for the second cut? 2. If the first cut is placed in the middle, how many options will there be for the second cut? There is also the possibility of giving Will the task of discovering the organizing principle in an already organized set of solutions.

Jay: A certain shortcoming of Jay's solution to the Board task was the complexity of the coding. However, there was usually spontaneous improvement here as Jay solved other problems or explained his solution to classmates. No teacher intervention was needed here. Interestingly, Jay solved the more difficult problem with cuts systematically and the problem with towers chaotically. If it were not 5 over 2 but 10 over 2, Jay would have discovered a strategy based on more sophisticated coding. This is advice to the teacher to give Jay a problem about towers with more cubes, which motivates the boy to discover the organizing principle. Life experience seemed to interfere with Jay's grasp of the simple party problem. The case was resolved by discussion with classmates. He alerted us that the Stones and Party tasks were not explicitly given. If the goal is to solve the 5 over 2 problems, it would be more appropriate to add in the Stones problem that we are just placing 3 stones into a table as in the picture, and for the Party problem choose one greeting kiss instead of tapping glasses.

## RESULTS

When a pupil had difficulty understanding the task, it was important to identify the source of the problem. We looked for possible factors such as: the type of objects, the experience, the pupil's understanding of the situation in the task, the situation's relevance with life experience, the format of the task (procedurally or conceptually), pupil's possibility to model the situation, and the pupil's ability to model the situation.

Most pupils who had difficulty grasping situations in writing or graphically were usually weaker students. However, it may be that the pupil is bright, like Erin. The strong short-term memory of such a pupil does not make it necessary to record a given problem in writing. Research is ongoing and we are trying to identify other types of learners.

## DISCUSSION

When there is a problem with pupils understanding a task the advice for teachers is to find out how it's understood and discuss it. Teachers can help pupils understand a task by giving them simple short-text tasks where the objects are familiar to the pupil. When helping weak pupils grasp the situation the advice for teachers is to solve easier problems with pupils and to support them in discussions with classmates. If pupils have strong short-term memory, the advice is to increase the task difficulty, so that their solution cannot be handled by a short-term memory anymore. Another option is to ask such a pupil to help a classmate solve the problem. Both options aim to induct a natural motivation to write the solution down.

## References

Batanero, C., Navarro-Pelayo, V., & Godino, J. D. (1997). Effect of the implicit combinatorial model on combinatorial reasoning in secondary school pupils. *Educational Studies in Mathematics*, *32*(2), 181–199.

Dubois, J. G. (1984). Une systematique des configurations combinatoires simples. [A system of simple combinatorial configurations]. *Educational studies in mathematics*, *15*(1), 37–57.

Fischbein, E., & Gazit, A. (1988). The combinatorial solving capacity in children and adolescents. *Zentralblatt für Didaktik der Mathematik*, *5*, 193–198.

Godino, J. D., Batanero, C., & Roa, R. (2005). An onto-semiotic analysis of combinatorial problems and the solving processes by university students. *Educational studies in mathematics*, *60*(1), 3–36.

Hadar, N., & Hadass, R. (1981). The road to solving a combinatorial problem is strewn with pitfalls. *Educational Studies in Mathematics*, *12*(4), 435–443.

Hejný, M. (1990). Kombinatorika, štatistika, pravdepodobnosť. [Combinatorics, statistics, probability]. In M. Hejný et al., *Teória vyučovania matematiky* 2 [Theory of mathematics education 2] (pp. 472–487). SPN.

Lockwood, E. (2013). A model of student's combinatorial thinking. *The Journal of Mathematical Behavior, 32*, 251–265.

Lockwood, E., & Gibson, B. R. (2016). Combinatorial tasks and outcome listing: Examining productive listing among undergraduate students. *Educational Studies in Mathematics*, *91*(2), 247–270.

Maher, C. A., Powell, A. B., & Uptegrove, E. B. (Eds.) (2011). *Combinatorics and reasoning: Representing, justifying, and building isomorphisms*. Springer.

Vondrová, N. (2019). *Didaktika matematiky jako nástroj zvládání kritických míst v matematice* [Didactics of mathematics as a tool for coping with critical points in mathematics]. Univerzita Karlova, Pedagogická fakulta.

# PRECISE MATHEMATICAL LANGUAGE AND LEARNING OF QUANTIFIERS

Ivona Grzegorczyk, Eric Bravo

California State University Channel Island, USA

*In this paper we address the need of developing precise language when learning mathematics. We describe and assess different methods of teaching quantifiers: language-based and symbolic approaches. We analyse the performance by three groups of learners introduced to statements with quantifiers. Then we compare their achievement with more advanced students. Our results show that introducing quantifiers early in the school curriculum is necessary.*

## INTRODUCTION

At all educational levels students need to use precise language when talking about mathematics (Dawkins, 2017). Broadly speaking, they should be able to formulate definitions of basic concepts and to be able to justify or prove that their solutions or statements are true (Schoenfeld, 1992). However, research shows that even at the college level many lack the understanding of more complicated mathematical statements, especially when they include quantifiers (i.e. expressions such as *for all* or *every* (universal quantifiers) or *exits* or *for some* (existential quantifiers). Statements in precise mathematical language and their various possible interpretations in English language cause problems in understanding them. The results in (Piatek-Jimenez, 2010) show that statements of the form *There exists…. for all …* evoked a larger variety of interpretations by students that the statements of the form *for all… there exists….*

The study described in (Chung, Shin 2023) shows that processing of ambiguities in English sentences may depend on the learners' fluency in the language and the familiarity of the context. Especially, when two or more quantifiers are included, or the sentence is negated, learners have problems with extracting the correct meaning. Many do not understand the difference between the importance of the order of quantifiers in statements such as: *Every student has a unique identification number,* and *Every number has a student assigned to it as his/her identification.* Other study (Epps, 1994) finds that undergraduates struggled trying to write symbolically statements (given to them in English) such as *Everyone is older than someone*, however when the proper order of symbolic quantifiers was provided $\forall\, x \in$\{people\} $\exists\, y \in$\{people\}.... vast majority was able to complete the task. These results underline the need for providing firm logic rules that govern mathematical discourse that go beyond understanding of individual words and often clash with everyday language which students tend to rely heavily on.  Overall, many attempts were made to improve pedagogy, methodology or technology for introducing quantifiers at the university level in various contexts, such as computer programming (Dubinsky,1997,1988), proving theorems (Epps, 2009) or through interesting activities (Kyeong, 2011). However, the assessment of effectiveness of these methods (repeated at other venues) did not show major advances in student achievements. The suggestion was made in (Dubinsky & Yiparaki, 2000) that symbolic forms may be better for introducing quantifiers (*traditional way*).

In this study we assess three different methods of introducing statements involving quantifiers at the introductory logic university courses. We worked with three groups of freshmen enrolled in mathematical logic courses (called Logic 1, Logic 2, and Logic 3 that had no systematic background related to quantifiers yet. The pre-test administered at the beginning of the study showed that the groups were compatible. We used different methodology for introducing quantifiers in each group: Logic 1 started with interactive *language-based* activities, then learned symbols for quantifiers; Logic 2 was introduced to symbolic approach first, and then translated mathematical statements into English (*traditional* method); and participants form Logic 3 followed the plan for Logic 2 group with the addition of hands-on activities based on reading a map for a small town (*traditional plus hands-on*). At the end of the study all participants were administered a post-test that included translation of various statements and their negations, as well as drawing a Venn diagram for underlying sets. To assess retention of the knowledge of quantifiers, we compared the post-test results of these three groups with a group of more advanced mathematics students who have used quantifiers in several other courses (Advanced group). Note that most of the assessment in previous research studies of comprehending quantifiers was based on questionnaires not related to underlying teaching methods. Most of the questions asked the participants to decide if the statement given was *true* or *false* and to justify their reasoning. Some tasks involved translations between symbolic and language-based forms of statements. In general, the previous results showed that learners at every level struggle with quantifiers regardless of the teaching methodology they were exposed to.

## METHODOLOGY

Note that at the beginning of our study, participants in each of our three groups Logic 1, Logic 2 and Logic 3 described above were familiar with sets, operations on sets, Venn diagrams, definitions, logical sentences, and connectives between statements (such as *and, or, not, implies*). They were able to evaluate a given logical formula as *true* or *false* when the logical value of each sentence in the formula was known. There were no quantifiers or variables in these sentences when the study started (Propositional Logic only).

We designed several activities for introducing mathematical quantifiers and the underlying variables informally (through *language-based* activities) and symbolically (*traditionally*), introducing First Order Logic through different methods.

The universal quantifier *for all, every* has a symbol ∀. We usually write the name of the variable, say *x,* that is quantified as $\forall x$ (or $\forall x \in X$ if we want to specify what set *x* belongs to). For example, the symbolic statement:

$\forall x \in$ {birds} $x$ *can fly* can be written in English as *All birds can fly.*

For the existential quantifier or *exits*, *for some* we use the symbol ∃ or $\exists y$ when specifying the variable (or $\exists y \in Y$ to indicate from what set *y* is an element of). For example, the symbolic statement:

$\exists x \in$ {birds} $x$ *can fly* can be written in English as *Some birds can fly.*

Our hope was that after our experiment students will understand both forms of quantifiers (language-based and symbolic) and be able to translate between them.

**Description of the methodology for each group.**

**Logic 1 group.** We introduced quantifiers through one hour *language-based* activity. Students started by evaluating logical values and equivalences of simple sentences such as:

All dogs are poodles.

Some dogs are poodles.

There is a dog that is not poodle.

All dogs are not poodles.

And with two quantifiers.

Every person has a friend.

Some people have friends.

Some people have no friends.

Nobody has a friend.

Some of your friends have no friends.

This activity generated many discussions as students mostly understood the meaning of the sentences, but they were not sure which ones are equivalent. Two days later, this group was introduced to the symbolic notation and was asked to rewrite the statements symbolically. Then they restated some statements with symbols in English language. The participation in the second part was less enthusiastic than during the first hour, and often students worked in silence. From then on students were using statements and their negations with (one or more) symbolic quantifiers in the course for the next 6 weeks. The post-test was administered at the end of the semester.

**Logic 2 group.** We started by introducing symbolic notation for quantifiers; this method is well tested as it is often used in a *traditional* mathematics classroom. The first session took one hour, and students were asked to explain the meaning of the sentences, to evaluate their logical values and find equivalent statements (*true* or *false* for the same elements).

$\forall x \in$ {dogs} $x$ is a poodle.

$\exists y \in$ {dogs} $y$ is a poodle.

$\exists z \in$ {dogs} z is not poodle.

$\forall w \in$ {dogs} $w$ is not a poodle.

Here are examples of statement with two quantifiers.

$\forall x \in$ {people} $\exists y \in$ {people} *x is y's friend.*

$\exists z \in$ {people} $\exists w \in$ {people} *z is w's friends.*

$\forall x \in$ {people} $\exists y \in$ {people} *x is not y's friend.*

$\exists z \in$ {people} $\forall w \in$ {people} *z is w's friends.*

$\forall x \in$ {people} $\forall w \in$ {people} *x is w's friend.*

We have expected that this group will be less active as the symbolic notation was new to them. As it turned out, the exercises generated many discussions when students tried to interpret the meaning of the sentences and evaluate their logical values. Again, they were often not sure which ones are equivalent. Two days later, this group was asked to rewrite the language-based statements symbolically. Then restate some statements with symbols in English language. From then on students were using statements and their negations with (one or more) symbolic quantifiers in the course for the next 6 weeks. The post-test was administered at the end of the semester.

**Logic 3 group.** We started with the *traditional* method of introducing quantifiers for the first two hours (as for Logic 2 activities) and observed similar involvement levels. However, at the third hour this group participated in *hands-on activity* that they completed working individually or in pairs. Participants were given a map of a small town that had a school, store, several gas stations, parks, etc. and a measuring tape to estimate distances on the map. They were asked to evaluate statements similar to the ones below.

$\forall x \in$ {houses} *the distance from x to school is less than 5 miles*.

$\exists x \in$ {houses} $\exists z \in$ {parks} *the distance from x to z is more than 3 miles*.

$\forall w \in$ {gas stations} $\exists x \in$ {houses} *the distance from x to w is more than 5 miles*.

$\exists z \in$ {park} $\forall x \in$ {people} *the distance from x to z is less than 5 miles.*

$\forall x \in$ {houses} $\forall w \in$ {gas station} *the distance from x to w is less than 9 miles*.

This activity was very successful (the mean was 75%) and majority of students answered 8 questions as above in allotted time of 20 minutes. From then on, these students were also using statements and their negations with quantifiers in the course for the next 6 weeks, when they wrote the post-test.

**Advanced Group.** Participants in this group were already studding mathematics at the university level for at least 2-3 years and had taken some advanced courses requiring writing proofs (such as linear algebra, real analysis, abstract algebra, etc.). They requested a short (20 minutes) review on statements with two or more quantifiers and their negations before taking the test. After the review, they answered the questions on the post-test given to the other groups. The aim of this part of our experiment was to assess the retention of knowledge related to quantifiers and to compare this group's achievement with our other three groups consisting of students that just completed their first mathematical logic course.

## RESULTS

We collected data from pre-test and post-test from the groups that were graded on a scale 0-4 points. Note that Logic 1, Logic 2, Logic 3 pre-test means were low as expected as students were not exposed systematically to quantifiers yet (around 1.5 out of 4). Statistically speaking there was no significant difference among these groups (all $p$-values were larger than .05). Advanced students did not take the pre-test. Post-test means increased for all three groups by at least 0.7 (which means students have learned) and $t$-tests showed $p$-values smaller than .05, i.e. each group improved significantly as expected. We then compared post-test scores for all four groups. The summary is given in a table below.

| Post-test Scores | Sample Size | Mean Score Scale 0-4 | Standard Deviation | Engagement in Activities |
|---|---|---|---|---|
| Logic 1 group | 20 | 2.13 | .85 | Moderate |
| Logic 2 group | 13 | 2.61 | .65 | High |
| Logic 3 group | 26 | 2.65 | .55 | High |
| Advanced group | 21 | 2.66 | .41 | |

Table 1: Post-test data by groups.

It is worth noticing that all logic groups were engaged in the activity at least moderately, and each group improved on the post-test significantly.

We compared each group post-test scores with others (including the Advanced group) by calculating the appropriate $p$-values and we found no significant difference among the post-test sores. However, we observe that Logic 3 group has the highest mean and the smallest standard deviation among the groups. This would suggest that the *traditional method reinforced by hands-on map reading* *activity* was the most effective. Indeed, the *p*-value between Logic 1 and Logic 3 is less than 0.10. Hence at the significance level 10% Logic 3 performed much better than Logic 1 that started with *language-based* statements.

Note that all the means are below 3 (less than 75%), and there was only one advanced student who scored 4 points. This is somewhat troubling as even though each group improved, students still did not master the quantifiers as well as we hoped for. Another interesting observation is the fact that the advanced group, even though it has the highest mean, did not perform significantly better than freshmen in their first logic course. It means that the experiences with the quantifiers in other courses did not strengthen their knowledge and skills in this area. It was an unexpected result.

Here we show the post-test questions and analyse the results for each group and typical difficulties they displayed.

**Post-test questions** (each question graded on a scale 0-1 points).

1) Write the following statement using quantifiers and symbols

*Z: For every senior at this university there is another senior at this university with a different major.*

Is this statement true? Explain.

2) Write the negation of the statement *Z* from 1) in English using symbolic quantifiers.

Is this statement true?

3) Write the following statement in English.

$\forall\, 0 < x \in R \;\; \exists\, y \in Q \;\; |x - y| < \frac{1}{2}$ Is this statement true? Explain.

4) Write the statement in English and draw the corresponding Venn diagram.

$\forall\, A \in \{\text{sets}\} \;\; \exists\, B \in \{\text{sets}\} \;\; A - B = A$ Is this statement true? Explain.

The first question asked to translate a statement in English language with two quantifiers into a symbolic statement. It turns out that this was the most complicated question for all students, and vast majority of them did not answer it or answer it incorrectly. Many were not sure about the order of the quantifiers and how to attach variables to them with appropriate sets. Note that Logic 3 group (that did the hands-on activity) scored the highest. See numerical data in Table 2.

| Question 1 Post-test | Sample Size | Mean Score Scale 0-1 | Standard Deviation | Major Errors |
|---|---|---|---|---|
| Logic 1 group | 20 | 0.12 | .22 | Order of quantifiers |
| Logic 2 group | 13 | 0.14 | .33 | Order of quantifiers |
| Logic 3 group | 26 | 0.34 | .32 | Order of quantifiers |
| Advanced group | 21 | 0.19 | .20 | Order of quantifiers |

Table 2: Post-test question 1 data by groups.

Question 2 asked to negate the statement 1). The scores were higher on this question, especially in Logic 2 group that had a mean of 92% and most of the errors in this group were marked as *no answer provided*. Students struggled the most with the placement of the negation (the word *not*) and choices of proper quantifiers. Surprisingly, Logic 3 that was successful on problem 1 achieved the lowest score here, and the mean for the Advanced group was only 70%.

| Question 2 Post-test | Sample Size | Mean Score Scale 0-1 | Standard Deviation | Major Errors |
|---|---|---|---|---|
| Logic 1 group | 20 | 0.65 | .48 | Negation placement |
| Logic 2 group | 13 | 0.92 | .27 | No answer |
| Logic 3 group | 26 | 0.60 | .31 | Negation placement |
| Advanced group | 21 | 0.70 | .47 | Negation placement |

Table 3: Post-test question 2 data by groups.

Question 3 asked for translation of the symbolic statement into English.

$\forall\, 0 < x \in R \quad \exists\, y \in Q \quad |x - y| < \frac{1}{2}$

| Question 3 Post-test | Sample Size | Mean Score Scale 0-1 | Standard Deviation | Major Errors |
|---|---|---|---|---|
| Logic 1 group | 20 | 0.55 | .51 | Order of quantifiers |
| Logic 2 group | 13 | 0.76 | .43 | Language choice |
| Logic 3 group | 26 | 0.50 | .32 | Language choice |
| Advanced group | 21 | 0.70 | .47 | No answer |

Table 4: Post-test question 3 data by groups.

All groups had a mean of at least 50% on this question. Many students had the incorrect order of quantifiers or had problems with formulating the statements in

English. Note that many participants from the Advanced group did not answer this question, which was surprising. See Table 4 for summary of the scores.

Question 4 used sets as variables for the quantifiers. Majority of students were able to figure out that the statement is true when the set *B* is empty. Hence the corresponding Venn diagram was easy to draw (but many in Logic 3 did not draw it at all, hence their lower scores).

| Question 4 Post-test | Sample Size | Mean Score Scale 0-1 | Standard Deviation | Major Errors |
|---|---|---|---|---|
| Logic 1 group | 20 | 0.91 | .15 | |
| Logic 2 group | 13 | 0.90 | .19 | |
| Logic 3 group | 26 | 0.60 | .32 | No Venn diagram |
| Advanced group | 21 | 0.90 | .22 | |

Table 5: Post-test question 4 data by groups.

**Some interesting comments from participants.** Below we quote some of the comments on the quantifier activities and the learning process from post- survey.

> Student S1: I find quantifiers hard because I'm not sure what do they mean in English in many situations. But I enjoyed playing with symbols and the language.

> Student S2: I plan to be computer science major and quantifiers are used in programming. I wish more of my courses were interactive and interesting.

> Student S3: Being precise it too precise for me. Why all the quantifiers are in front? I liked debating the language options. Translating is quite complicated for me; my first language is Spanish.

> Student S4: Now, I find logic questions interesting. But I'm worried about upper division courses where the statements have many quantifiers. It can be very difficult.

> Student S5: My last logic class was 2 years ago. We used quantifiers a lot in linear algebra. And to my surprise I had some problems with translating some of the statements with quantifiers on this test.

Overall comments focussed on difficulties of finding proper words and the right order to represent the symbolic statements into English, and on putting quantifiers in correct order when translating into symbolic form. While many questions proved to be quite challenging, learners were engaged in the learning process trying to use their own reasoning to accomplish the tasks. Some students expressed interest in studding more advanced logic.

## CONCLUSIONS

Our study focused on comparing three methods of teaching mathematical quantifiers. We found that the results for *traditional* method used for Logic 2 and 3 groups had a small edge over Logic 1, and students in these two groups showed unexpected engagement (we expected the *language-based* approach group to have more student interactions, as the introductory statements proposed were immediately understandable to students). Hence, we recommend starting with symbolic approach and an interactive learning approach (involving discussions, activities, etc.) for any methodology.

The post-test results of the three freshmen groups and the additional group of advanced mathematics participants show that the students at every level still struggle to understand the concepts. The freshmen groups seemed to grasp the idea of translating quantified statements into English better than the other way around, regardless of the method they used to learn quantifiers. All students were quite comfortable working with one quantifier in a statement but not when more quantifiers were used (as well as when the order of quantifiers mattered). Both *traditional* groups that started with symbolic quantifiers performed better than the first group that started learning using *language-based* tasks. Even though results of the third group (that worked on the additional *hands-on activity*) were the highest, there was no real statistical difference between all three groups.

On the other hand, the performance of *advanced* mathematics students who had previous knowledge of quantifiers, was below our expectations as they still struggled with translation tasks and understanding of the statements with multiple quantifiers. Since their scores were compatible with the other participants that just finished their first university level course, it seems that their understanding of quantified statements did not improve over the years. We suggest that other mathematics courses require writing in precise language at every level.

Many studies, including this one, show that university level students still struggle to communicate in precise language. Since quantifiers seem to be hard to comprehend even in the language-based contexts, we recommend introducing them much earlier at school curricula, so even young children learn to pay attention to the way they state their mathematical thoughts and learn how to place quantifiers in the right order.

### References

Chung, E.E., & Shin,J_A. (2023). Native and Second Language Processing of Quantifier Scope Ambiguity. *Second Language Research*, 39(3), 785-810.

Dawkins, P.C. (2017). On the importance of set-based meanings for categories and connectives in mathematical logic. *International Journal of Research in Undergraduate Mathematics Education*, 496–522.

Dubinsky, E., & Yiparaki, 0. (2000). On student understanding of AE and EA quantification. *Research in collegiate mathematics Education,* 4, 1–64.

Dubinsky, E. (1988). On helping students construct the concept of quantification. In A. Borbas (Ed.), Proceedings *of the Annual Conference of the International Group for the Psychology of Mathematics Education* 255–262.

Dubinsky, E. (1997). On learning quantification. *Journal of Computers in Mathematics and Science Teaching archive*, 16, 335–362.

Dubinsky, E., Elterman, F., & Gong, C. (1988). The student's construction of quantification. *For the Learning of Mathematics*, 8(2), 44–51.

Epp, S.S. (2009). Proof issues with existential quantification. In M. d. V. Gila Hanna (Ed), *Proof and Proving in Mathematics Education: ICMI Study 19 Conference Proceedings*, 154–159.

Epp, S.S. (1994). The role of proof in problem solving. In A. H. Schoenfeld & A. H. Sloane (Eds.), *Mathematical Thinking and Problem Solving.* Lawrence Erlbaum Associates, Inc.

Kyeong H. R., Lee, Y. H. (2011). The Mayan activity: A way of teaching multiple quantifications in logical contexts. *Problems, Resources, and Issues in Mathematics Undergraduate Studies (PRIMUS)*, *21*, 685–698.

Piatek-Jimenez, K. (2010). Students' interpretations of mathematical statements involving quantification. *Mathematics Education Research Journal*, *22,* 41–56.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D.A. Grouws (Ed.), *NCTM Handbook of research on mathematics teaching and learning* (pp. 334-370). Macmillan.

# Educational resources as research-based tools

# Part 2

# EDUCATIONAL RESOURCES IN MATHEMATICS – THE INTERPLAY BETWEEN RESEARCH AND SCHOOL PRACTICE

Mirosława Sajka

Department of Mathematics, University of the National Education Commission, Krakow, Poland

*This paper presents two approaches based on the author's involvement in shaping negative numbers, algebraic expressions and functions in students, showing the interplay between research and school practice and resulting in the development of educational resources. Firstly, an algebraic approach to the 'minus' sign, understood as 'opposite' in school mathematics, is presented in the context of introducing both negative numbers and algebraic expressions to students through educational resources based on tokens. Secondly, the ways of improving students' functional thinking are briefly mentioned on the basis of educational resources developed by the international FunThink Team.*

## INTRODUCTION

Negative numbers, algebraic expressions and functions are among the most fundamental mathematical topics in schools all over the world and at the same time they cause many difficulties for students, pre-service teachers as well as educators, therefore they require well-planned didactic approaches and well-prepared educational resources based on research in mathematics teaching. The aim of this article is to present different contexts, theoretical, investigative and practical, based on own involvement, in which selected educational resources have been created in order to overcome students' difficulties in forming negative numbers, algebraic expressions and functions.

## THEORETICAL FRAMEWORK

This section presents the processes followed in designing the educational resources developed to facilitate the understanding of negative numbers, algebraic expressions and functions.

### Context I: Towards shaping negative numbers and algebraic expressions: Introducing the 'minus' sign as 'opposite to'

This section describes the cycle of creating educational resources in the form of concrete teaching tools – tokens related to the teaching of negative numbers and algebraic expressions, and then developing learning environments that detail their use, supported by a teacher's guide. The development of the educational resources was designed and co-funded within the EU project "Algebraic Approach Towards Shaping 'Minus' in School Mathematics" (AMMA). The inspiration for the project was developed theoretical approach T1 and then T2 and T3 approaches were involved to build the following theoretical framework:

- T1: analysis of the concept of negative numbers in a historical perspective, the discovery and analysis of two mathematical models,
- T2: constructivism in the teaching of mathematics,
- T3: analysis of the available didactic models for the introduction of negative numbers and algebraic expressions at school, and analysis of curricula and textbooks in Poland and Slovakia.

On this basis, the following educational resources were developed:

a) sets of tokens-models for the use of negative numbers and algebraic expressions for children and teachers,

b) lesson plans using the tokens in three languages, English, Polish and Slovak,

c) a teacher's guide in three languages, English, Polish and Slovak.

The whole cycle of the process of their creation consisted of 6 general stages, however, between stage 4 and 5 there were many interrelations, effects and rounds. The stages were:

1. Developing the theoretical framework (T1, T2 & T3)
2. Developing the first version of the learning resources (tokens and lesson descriptions)
3. Evaluation of the educational resources in the form of feedback from an external expert
4. Preparation of revised versions of educational resources (tokens and lesson descriptions)
5. Evaluation of educational resources - implementation in schools in Poland and Slovakia (4 schools, two Polish and two Slovakian).
6. Development of the final version of the educational resources (tokens and lesson descriptions and teacher's guides).

The process is illustrated in Figure 1.

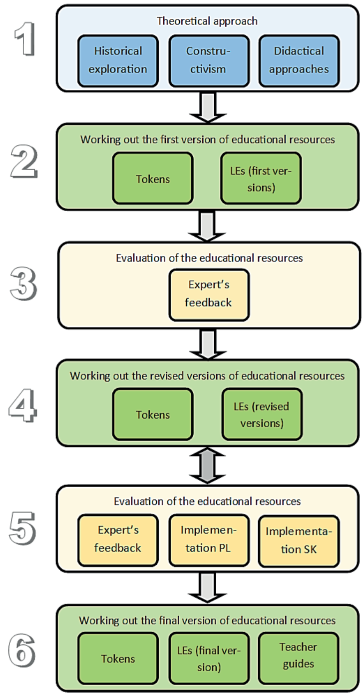



Figure 1: The process of creating educational resources in the AMMA project by Sajka, Błaszczyk and Zaręba (2022).

The whole process had its origins in historical research (T1) on the emergence of the concept of negative numbers. In this line, Błaszczyk and Sajka (2017) identified two ways of introducing negative numbers in the history of mathematics.

- In the first one, a totally ordered set $(L,\prec)$ is assumed, an element 0 in L is taken arbitrarily, and a number $a$ is negative if $a\prec 0$.
- In the second, a negative number is defined only by the formula $a+(-a) = 0$.

From a mathematical point of view, the first method involves the idea of a totally ordered group (G, +, 0, <), while the second considers only the idea of the algebraic group (G, +, 0). Through the analysis of source texts, we have shown (Błaszczyk & Sajka, 2017) that the first model comes from John Wallis's (1685) *Treatise of Algebra*, while the second comes from the theory of polynomials as developed by Descartes in his 1637 *La Géométrie* (Błaszczyk & Mrówka, 2015).

In mathematics education, the first model is overwhelmingly used. Teaching negative numbers based on this involves the conventional choice of zero on the number line, and consequently we assume at the outset that we are considering an ordered set by defining that a negative number is less than zero. The number line model attempts to support unary, binary, and symmetric integer sign functions simultaneously: -5 can represent position, -5 can represent the move '5 units to the left', -5 can also represent a number (position) opposite to 5. This multiplicity of agreements and interpretations, and the flexibility required in their use depending on the context, illustrate the limitations of this approach. However, there is another important consequence associated with the use of this model, namely that a negative number is immediately learnt to be less than zero and the student acquires the association that something preceded by a minus sign is negative, which consequently leads to a strong association of '$-x$' with a negative number, a common error and key misconception among students in later stages of their mathematics education, and causes further difficulties in understanding '$-x$' as a number opposite to $x$. This misinterpretation leads to profound difficulties in understanding, interpreting and manipulating algebraic expressions.

In contrast, teaching according to the second model involves dividing the objects – elements of the given set – into two groups, for example of different colours, such as black and white. A pair of objects of different colour: {white, black} is interpreted as 0. Note that we do not need the order of the set at this stage. We do not assume that a negative number is less than 0, but we only know that a negative and a positive form a neutral pair, and therefore from the fact that $a + (-a) = 0$ it follows that a is opposite to $(-a)$ and vice versa. The most likely model using two coloured objects was the so-called 'positive-negative charge model', described as early as the 1970s and used to interpret addition and subtraction of integers (Frand & Granville, 1978; Grady, 1978). Later, the widely quoted 'annihilation model', published in an article in *Arithmetic Teacher* by Battista (1983), showed how this model could be used to interpret all four operations on integers.

A historical review (Błaszczyk & Sajka, 2017) allowed for another conclusion: negative numbers have no concretisation in material entities. They are a purely theoretical creation that arose not to study reality, but to describe and study mathematical phenomena. Moreover, almost at the same time as negative numbers, imaginary numbers were introduced, which were accepted not much later. In everyday life, all the concretisations of negative numbers that are proposed as examples to students are artificially produced on the basis of the conventions of social life and are unavailable to many children, such as:

a) the temperature below zero,

b) the levels of buildings or car parks in certain shopping centres or railway stations; moving (by lift or stairs) to the corresponding level -1, -2 (basement), etc., and other analogous arrangements involving movement: forwards-backwards; right-left in relation to a chosen starting position (e.g., the game proposed by Thompson & Dreyfus, 1988; movement performances in Hejný, 2008),

c) depressions in the geographical sense,

d) debts (debit, loss, penalty points).

The second theoretical approach T2 is based on constructivism (e.g., Bruner, 1957, 2009; Lerman, 1989; Piaget, 1970) and the theory of Sfard (1991), according to which concepts are acquired first operationally and then structurally, and the transition to the structural form of a concept – as a mathematical object – takes place in three steps: interiorisation, condensation and reification. In the interiorisation stage, the learner becomes familiar with the processes that will eventually result in the formation of a new concept. These processes are operations performed on lower-level mathematical objects. According to the intellectual development of the Grade 5-6 pupil, it would be worth to shape the concept of negative number based on models operating in the real world and activities on concrete material, using so-called enactive representations and not only iconic or symbolic representations (Bruner, 1956, 2009). This postulate is further supported by the observation that students' specific learning difficulties in mathematics are increasing, with dysfunctions such as dyslexia, dysgraphia or dyscalculia being diagnosed more and more frequently. Moreover, this postulate is supported by research. For example, Carbonneau et al. (2013) presented a meta-analysis that combined the results of 55 studies involving over 7,000 students to show that the use of manipulatives is associated with positive effects at all levels of mathematics, and with particularly strong effects in favour of the use of manipulatives compared to instruction using only abstract mathematical symbols.

Using tokens to teach negative numbers and then algebraic expressions fulfils all these assumptions.

**Context II: Towards enhancing functional thinking**

This section describes the cycle of creating educational resources aimed at enhancing functional thinking from primary to upper secondary school. The development of educational resources was designed and co-funded within the EU project "Enhancing functional thinking from primary to upper secondary school" (FunThink), a large international project. Figure 2 shows the processes followed for the design of the educational resources, consisting of 11 stages.

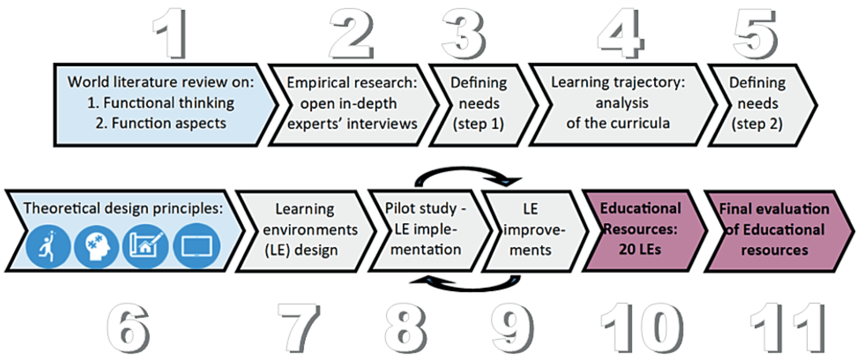



Figure 2: FunThink Learning Environments (LEs) preparation process (own interpretation).

The visible interplay between theory, research and practice is described below.

*Stage 1:* The first stage was related to the development of the theoretical background necessary for the creation of educational resources. This part consisted mainly of a review of the world literature on functional thinking and different aspects of functions. All partners were involved in a literature review. As a result of this review and numerous discussions, a theoretical background was developed by adopting four aspects of the development of functional thinking related to aspects of the concept of function (FunThink Team, 2021; Frey et al., 2022). The aspects of function are:

- Function as an input-output mapping emphasises the computational aspect of function, perceived as a requirement to perform a calculation;
- Function as a dynamic process of covariation between independent and dependent variables;
- Function as a correspondence relationship;
- Function as a mathematical object that can be studied, compared with or connected to other mathematical objects.

*Stage 2 & 3:* Empirical research was conducted in the form of semi-structured interviews with national experts (Frey et al., 2022). The research question was: *What conceptions of functional thinking do mathematics educators in the five countries Cyprus, Germany, the Netherlands, Poland and Slovakia have*? In order to answer this research question, a total of 34 semi-structured interviews were conducted with mathematics educators in all five participating countries. The case of Poland was described in Polish (Sajka, 2023). This study also resulted in a theoretical model of the experts' response categories, as they were analysed using qualitative content analysis (Mayring, 2015), which allowed for both an inductive and a deductive approach. As a result of this study, country-specific needs were formulated. In Poland, it turned out that at the secondary school level, functional covariation reasoning was not considered by the experts interviewed. This is one of the reasons why we gave further attention to this topic by preparing particular Learning Environments (LEs).

*Stage 4 & 5:* In this phase, the national curricula of the partner countries were analysed: Cyprus, Germany, the Netherlands, Poland and Slovakia, and possible learning trajectories were analysed in order to define further needs in the national contexts. However, due to the large discrepancies in the curricula, it was decided not to create explicit learning trajectories, but to prepare different LEs for the three types of schools: Primary, Lower Secondary and Upper Secondary.

*Stage 6:* This stage was the second theoretical step, related to the search for design principles, and also involved an analysis of the available approaches to the preparation and design of educational resources. As a result of the analysis, existing approaches were adopted, but it was assumed that all 4 design principles should be implemented simultaneously in each LE:

- Inquiry-Based Education (IBE): for example, Artigue and Blomhøj (2013) discuss various mathematics education frameworks that are suitable for addressing inquiry-based mathematics education, including problem solving (Schoenfeld, 1992), realistic mathematics education (Freudenthal, 1991), and the theory of didactic situations (Brousseau, 1997).
- Embodiment: because modern theories consider cognition and thinking to be embodied, "not merely located in the mind of people, but grounded in action-perception experiences with their mind and body (e.g., Lakoff & Nunez, 2000). Theories on embodied cognition, or embodiment, differ in what they consider to be activities with the body, allowing for action-perception experiences or loops" (FunThink Team, 2021, p. 12).
- Situatedness: which refers to meaningful situations in need to be organized or mathematized, in the context of *didactical phenomenology* proposed by Freudenthal (1983).
- Use of digital tools: In mathematics education, it is necessary to pay attention to the subtle interplay between the use of tools and mathematical learning. One theoretical approach that takes this into account is *instrumentation theory* (Artigue, 2002): "this theory stresses the need for a process of instrumental genesis, that a student goes through while using a tool for doing and learning mathematics. This instrumental genesis comes down to the coemergence of techniques for using the specific tool for the given task, and the development of mathematical meaning involved in the topic. This approach is key to a fruitful integration of digital technology in mathematics education" (FunThink Team, 2021, p. 14).

*Stage 7:* In this phase, the first versions of the LEs with the whole didactic package (including didactic materials, digital tools, applets, worksheets, learning videos, implementation videos, teaser videos) and descriptions of the Teacher Guides were started.

*Stage 8-10:* LEs were piloted in each partner country and the prepared educational resources were modified several times. The final versions of the 20 LEs with all the educational resources were translated into English, German, Greek, Polish and Slovak.

*Stage 11:* The final versions of the LEs were evaluated in practice, with the same pre-test and post-test prior to implementation in all countries. The evaluation reports can be found at www.funthink.eu.

The next part of the work was the preparation of courses for pre-service and in-service teachers, which will not be described for reasons of space.

**Comparison of the educational resources design cycles**

Common to both approaches are the strong theoretical underpinnings of both sets of educational resources. In the case of tokens, these are source-based historical studies, and in the case of functional thinking, a review of the available educational literature. Both contexts include a detailed analysis of the curricula in all the countries involved in the process and an analysis of the existing teaching resources available. Both approaches take into account the principles of instructional design, with Context II explicitly assuming and implementing four design principles in each LE: IBE, embodiment, situatedness and digital tools, while in Context I the main principle is constructivism with the use of manipulatives, but constructivism also assumes problem-based learning (i.e., the IBE approach), and the manipulation of tokens is a certain situational context and at the same time a kind of embodiment. So, only the tools are different: Context I assumes the use of manipulatives only, while Context II assumes the use of digital tools obligatorily. Educational resources developed in Context II are applicable worldwide due to the availability and universality of digital tools. On the other hand, the problem of accessibility of manipulatives could be overcome if we ask students to prepare their own set of cardboard cut-out manipulatives (some teachers in Poland, for example, use buttons of different shapes and colours instead of tokens) and the advantages of using manipulatives are widely described in the literature (e.g., Carbonneau, Marley and Selig, 2013).

The obvious similarities are also that both approaches included a pilot stage, improvement of developed resources and a final evaluation. The external differences were related to the territorial and content scope of the two contexts. For instance, Context I developed a coherent didactic approach aimed at the later years of primary school and has a clearly structured and linearly described learning trajectory, whereas in Context II different learning trajectories were possible, in different order and aimed at pupils of different ages. Moreover, Context I implemented only one underrepresented approach to shaping the notion of negative numbers and algebraic expressions (based on Theoretical Model II), whereas Context II deliberately implemented different approaches and shaped the notion of functions in each LE in many and preferably simultaneously in all the four aspects mentioned above. Another methodological difference is that Context II included an empirical research phase in the form of open-ended in-depth interviews with education experts, which allowed to distinguish the needs in each country; Context I did not include such research.

## EXAMPLES OF EDUCATIONAL RESOURCES

### Context I: Towards shaping negative numbers and algebraic expressions: Introducing the 'minus' sign as 'opposite to'

A set of manipulatives for the student and a set for the teacher contained 10 tokens each (60 manipulatives). It is worth mentioning that the Slovak teachers also used a multimedia whiteboard (Figure 4). The tokens are presented in Figures 3 and 4.

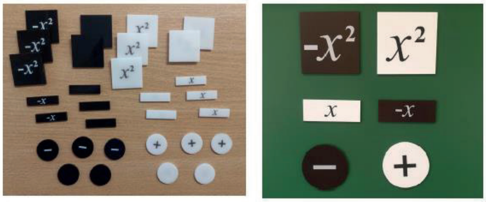



Figure 3: Tokens–left: manipulatives for students, right: magnetic tokens for the teacher.

Figure 4: Tokens and their use in AMMA project (Sajka, Błaszczyk & Zaręba, 2022).

As a result of the AMMA project, sets of tokens for pupils, teacher sets (for blackboards), complete lesson plans, teaching materials and teacher guides have been produced. All educational resources were evaluated in 4 schools (2 in Poland and 2 in Slovakia) and were found to be effective in both countries, despite significant differences in the curriculum, as in Slovakia negative numbers are introduced in grade 8, whereas in Poland they are usually introduced in grade 6. The only situation where students reported some dissatisfaction with the tokens was in the revision classes in grade 7 in Poland, where students were already familiar with negative numbers. In these classes, students sometimes expressed impatience and embarrassment with the 'childish' tokens. This effect was not observed in the older Slovak students of grade 8, where negative numbers were introduced in this way. The developed approach was also suitable for students with special educational needs and has been used with relatively young, gifted children (grade 4, Poland) and with dyslexic and even dyscalculic children in grades 7 and 8.

**Context II: Towards enhancing functional thinking**

Table 1 lists topics of modules of LEs developed for the FunThink project for primary and lower secondary education.

| Primary Education | Lower Secondary Education |
|---|---|
| 1. Distance-time version A | 1. Embodying graphs |
| 2. Distance-time version B | 2. Cooling of water |
| 3. Double number line | 3. Filling vessels |
| 4. Function machines GeoGebra | 4. Temperature |
| 5. Function machines gizmos | 5. Walking graphs |
| 6. Pattern | 6. Nomogram – Introduction and graph |
| 7. Qualitative interpretation of graph | 7. Various vessels |
| 8. Variation-covariation | 8. Marbles |
| | 9. Coordinates |
| | 10. Change is change |

Table 1: List of FunThink LEs.

Each LE was accompanied by its description, including information on the degree of implementation of the four design principles and the aspect of functional thinking developed during the lesson (Figure 5).

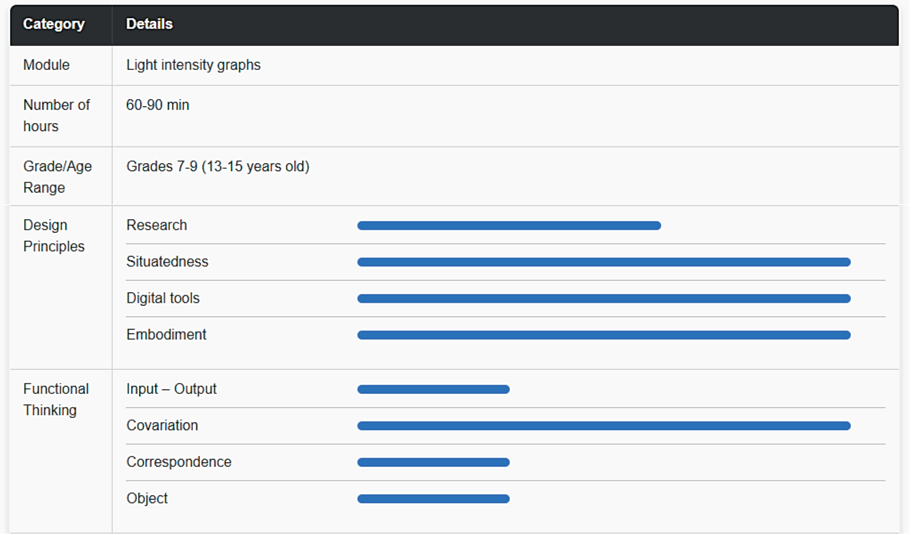



Figure 5: Extract from the description of the LE “Embodying graphs”.

## ONGOING STUDY

Covariational thinking is developed in students in the early years of mathematics, but in upper secondary school, where the concept of function is formally introduced and used, the implementation of covariational functional thinking in mathematics teaching is neglected in Polish practice, in favour of other aspects of functions, such as: function as input-output assignment, correspondence and mathematical object. However, covariational functional thinking is essential for understanding differential calculus, but also for solving modelling problems.

Research on the development of educational resources to improve students’ covariational functional thinking is being continued. Based on empirical research using methodological triangulation, including the use of eye-tracking, the difficulties in students’ covariational functional thinking have been diagnosed. To overcome these difficulties, the next approach has already started in the context of the EMPE project (Embodying Math and Physics Education).

The new educational resources will aim to overcome the diagnosed difficulties in reading and interpreting graphs and to improve covariational functional thinking. Our aim is to design, prototype and implement devices and software, using an embodied approach, involving students, their movement, experience, discovery, related to the visualisation of the concept of function, focusing on understanding, interpreting and using graphs of functions in the context of motion analysis. The educational resources developed aim to prevent such deficiencies and to support covariational functional reasoning.

**Acknowledgements**

The educational resources presented were created within the following projects:

AMMA: *Algebraic Approach Towards Shaping 'Minus' in School Mathematics*, 2022-1-PL01-KA210-SCH-000084007.

FunThink: *Enhancing functional thinking from primary to upper secondary school*, KA203-2ACBA170.

EMPE: *Embodying Math&Physics Education*, 2023-1-PL01-KA210-SCH-000165829.

## References

Artigue, M. (2002). Learning mathematics in a CAS environment: the genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245–274.

Artigue, M., & Blomhøj, M. (2013). Conceptualizing inquiry-based education in mathematics. *ZDM Mathematics Education, 45*(6), 797–810.

Battista, M. T. (1983). A complete model for operations on integers. *Arithmetic Teacher 30*(9), 26–31.

Błaszczyk, P., & Mrówka, K. (2015). *Kartezjusz, Geometria. Tłumaczenie i komentarz* [Descartes, Geometry. Translation and commentary]. Wydawnictwo Universitas.

Błaszczyk, P., & Sajka, M. (2017). On the Negative Numbers from the Historical and Educational Perspective. *Annales Universitatis Paedagogicae Cracoviensis | Studia Ad Didacticam Mathematicae Pertinentia, 9*(1), 5–36.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer.

Bruner, J. S. (1957). *Going beyond the information given*. Norton.

Bruner, J. S. (2009). *The process of education*. Harvard University Press.

Carbonneau, K., Marley, S. C., & Selig J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, *105*(2), 380–400.

Frand, J., & Granville, E. B. (1978). *Theory and Application of Mathematics for Teachers* (2nd edition). Wadsworth Publishing Company.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Springer.

Freudenthal, H. (1991). *Revisiting mathematics education*. China lectures. Kluwer.

Frey, K., Pittalis, M., Veldhuis, M., Geisen, M., Krišáková, M., Sajka, M., Nowińska, E., Hubeňáková, V., & Sproesser, U. (2022). Functional thinking: Conceptions of mathematics educators - a framework for analysis. In C. Fernández, S. Llinares, A. Gutiérrez & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, p. 349). Alicante, Spain: PME.

FunThink Team. (2021). *Vision document*. Retrieved from http://funthink.eu

Grady, M. T. (1978). A manipulative aid for adding and subtracting integers. *The Arithmetic Teacher, 26*(3), 40.

Hejný, M. (2008). Scheme oriented educational strategy in mathematics. In B. Maj, M. Pytlak, & E. Swoboda (Eds.), *Supporting Independent Thinking Through Mathematical Education* (pp. 40–48). Wydawnictwo Uniwersytetu Rzeszowskiego.

http://amma.uken.krakow.pl

http://empe.uken.krakow.pl

http://funthink.eu

Lakoff, G., & Nunez, R. E. (2000). *Where mathematics comes from*. Basic Books.

Lerman, S. (1989). Constructivism, mathematics and mathematics education, *Educational Studies in Mathematics*, *20*(2), 211–223.

Mayring, P. (2015). Qualitative Content Analysis: Theoretical Background and Procedures. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education. Advances in Mathematics Education* (pp. 365–380). Springer.

Piaget, J. (1970). *Genetic epistemology* (Duckworth, E., Trans.). Columbia University Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, meta-cognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for research on mathematics teaching and learning* (pp. 334–370). MacMillan.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1–36.

Sajka, M., Błaszczyk, P., & Zaręba, L. (2022). Algebraic Approach Towards Shaping "Minus" in School Mathematics - AMMA project. *Annales Universitatis Paedagogicae Cracoviensis | Studia Ad Didacticam Mathematicae Pertinentia, 14*, 189–193.

Sajka, M. (2023). Myślenie funkcyjne w ujęciu teoretycznym i według ekspertów – studium przypadków [Functional thinking in theoretical terms and according to experts – case studies]. In E. Juskowiak (Ed.), *Współczesne problemy nauczania matematyki. Tom 9* [Contemporary problems of teaching mathematics. Volume 9] (pp. 53–70). Adam Mickiewicz University Press.

Sajka, M., & Rosiek, R. (2020). Enhancing functional thinking from primary to upper secondary school according to "FunThink" project. *Annales Universitatis Paedagogicae Cracoviensis. Studia Ad Didacticam Mathematicae Pertinentia*, *12*, 281–284.

Thompson, P. W., & Dreyfus, T. (1988). Integers as transformations. *Journal for Research in Mathematics Education, 19*(2), 115–133.

Wallis, J. (1685). *Treatise of Algebra*. John Playford.

# STUDENTS' CONCEPT IMAGE OF FUNCTION IN CONNECTION WITH LEARNING KINEMATICS

Gergely Kardos, Eszter Kónya
University of Debrecen, Hungary

*Mathematics and physics are closely related disciplines, and many areas of mathematics play an essential role in learning physics. We aim to explore the 9th graders' concept image of functions with particular attention to the link of knowledge students have after learning about the kinematics of simple movements in physics lessons. Therefore, the students took a test consisting of two different tasks. The first task assessed their conceptual knowledge, and the second task was a physics problem with a position-time graph. The survey result shows that students likely associate the function concept with their most common activity: representing points and curves on a coordinate system. Physical tasks contribute to the concept image of function by providing activities at the action level.*

## INTRODUCTION

Mathematics and physics are closely related disciplines, and many areas of mathematics play an essential role in learning physics. In physics lessons, mathematics is always behind the basic concepts, models, expressions, and formulae, thus helping understand and describe phenomena that occur in nature (Radnóti & Nagy, 2014). It is also worth approaching the question from the other direction. On the one hand, students use many mathematical concepts earlier in physics lessons than in mathematics lessons (at least in Hungary; see textbooks (Csajági et al., 2020; Juhász et al., 2020)). This provides an opportunity to incorporate the experience gained in physics into the processing of mathematics curricula. On the other hand, physics problems often serve as a non-mathematical context in mathematical problem-solving. However, the professional experience of secondary school teachers and didactic research shows that students cannot effectively connect the knowledge acquired in these two fields. (D. Balogh, 2002; Pospiech, 2015; Ubuz et al., 2019).

In this study, we focus on a specific mathematical concept, the function, which is essential not only from the aspect of mathematics but also from several scientific fields such as physics. Ninth-grade students already have plenty of preliminary knowledge of the function concept. Some of these are acquired in mathematics lessons, some in physics lessons, or everyday life. We aim to explore the 9$^{th}$ graders' concept image of functions with particular attention to the link of knowledge students have after learning about the kinematics of simple movements in physics lessons.

We formulated the following two research questions.

RQ1. What preliminary knowledge do 9th-grader students have on the notion of function?

RQ2. How do they use their knowledge of functions when solving a physics problem?

Our survey is the first step in a more extensive teaching experiment. The experiment aimed to provide curricular knowledge of functions by building on students' experience while maintaining the link between mathematics and physics to provide students with comprehensive and applicable knowledge across multiple domains.

## THEORETICAL BACKGROUND

It is important to consider that in the case of a concept, alongside definitions, students also have some mental image of the idea. This includes all the visual representations that can be seen concerning a concept. In this respect, the first or most used examples are the most prominent. For example, students often think of functions as something described by some algebraic expression, especially by linear expression. Besides this mental picture, some properties might also be associated with the concept. These properties and the mental image constitute the "concept image" (Vinner, 1983). The concept image also has a vital role in the teaching-learning process because, in most cases, when thinking, it is not the conceptual definition but the concept image that is evoked. In Vinner's model, the concept definition and concept image are two separate cognitive structures. One or both may often be empty regarding a given concept, or even the links between them are missing. When developing conceptual knowledge, paying close attention to the concept definition, the concept image, and the links between the two in either direction is crucial.

Sierpinska (1992) described the conditions for understanding the notion of functions. Students need to move confidently in the world of (1) changes, (2) relationships or processes, (3) rules, patterns, and laws. This means that students should have (1) mathematical and non-mathematical experiences of changing objects and covariant quantities, (2) relationships, and (3) rules that determine the relationships.

> Students must become interested in variability and search for regularities before examples of well-behaved mathematical elementary functions and definitions are introduced in the classroom. (Sierpinska, 1992, p. 32)

When developing the function concept, the physics problems students are familiar with provide these worlds and this variability. The kinematics problems contribute to the initial picture of function since they deal with the process of motion, i.e., the relationship of the quantities of distance, speed, and acceleration, which vary with time. Different forms of representation are used to describe the relationship between covariant quantities. The most used representations are verbal description (written or oral), table of function, graph, and algebraic expression (Sierpinska, 1992). The latter two are common in physics problems, while the value tables are used for measuring physical quantities.

The functions defined by physical laws convey a static function picture since the relationship (physical laws) is not determined by us but discovered or just learned. A different, more dynamic image is conveyed, for example, by geometric transformations, where we create the image (Sierpinska, 1992). It can, therefore, be seen that the quantities and their relationships known from physics provide good examples of the concept of function. This is true even if it only covers part of the mathematical function concept.

Dubinsky and Harel (1992) mention four stages of understanding the concept of function. (1) *Pre-function*: In this stage, students' concept of function is vague and unsuitable for solving function-related problems. The concept only has its clues. (2) *Action*: Students think of a function as something that returns the value of an expression to a particular value of the variable. It entails substituting numbers into algebraic expressions and calculating their values. (3) *Process*: The interpretation of the function as a process includes a dynamic transformation of quantities. Students' concept of a function may show an "input-transformation-output" picture at this stage. (4) *Object*: The function can be considered an object created with a process's help. Breidenbach et al. (1992) consider these stages not development levels but ways of thinking about functions. They grouped college students' responses to the question "What is a function?" into these categories. The result showed that a significant number of responses belonged to the *pre-function* or *action* category. Most examples of functions students gave before the instruction were some algebraic or trigonometric expressions in line with the *action* way of thinking. Based on these, we expect that the features of *pre-function* and *action* will also appear in our 9th-grade students' perception of function.

## METHODOLOGY

### Sample

In Hungary, public education is divided into two main parts: an eight-year primary school, then students choosing a secondary school. Among other options, students can continue their education in four-year secondary schools, focusing on purely academic subjects, preparing them for a graduation exam. This can be followed by studying at college or university level. Each class often specializes in certain subjects, involving more lessons from them during the four years.

We conducted a teaching experiment in a ninth-grade class with a science profile in a major city in Hungary. The science profile means the students follow an advanced-level curriculum in biology, chemistry, and physics. In addition, in the ninth grade, they have 4 lessons of mathematics per week instead of the mandatory 3 lessons, which they study in subgroups of 16 students each. The class has above-average abilities, which is supported by the fact that their average score in mathematics on the central admission exam before ninth grade (34.0 points out of 50) was remarkably higher than the national average (21.1 points) (Oktatási Hivatal, 2023, p. 35). A total of 30 students participated in the survey.

**Circumstances**

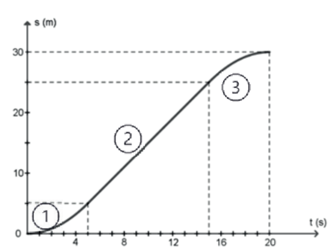
The experiment focuses on the teaching of functions. The topic was not entirely unfamiliar to the students since it is part of the curriculum in Hungarian primary schools in grades 7-8. According to the national mathematics curriculum, students must be familiar with the coordinate system, plot points in them, and read the coordinates of points. They learn about direct and inverse proportions and should be able to plot the graph of direct proportionality. They also learn the concept of function as mappings between specific sets. Physics subject is mandatory from grade 7, and the first theme students face in physics classes is the kinematics of linear motion. They learn the position, velocity, and acceleration relationship with time in this context using algebraic expressions and graphs. The experiment involves introductory lessons with one subgroup for two 45-minute lessons, where physics tasks requiring knowledge of functions are presented. Afterwards, throughout the topic, we continuously use physics examples. We process the experiment in the form of action research, and it begins with a survey.

**Measurement instrument**

At the beginning of our experiment, we wondered what kind of concept image the students possessed and how they could handle a typical kinematics problem. Therefore, the first task assessed their conceptual knowledge, and the second task was a physics problem with a distance-time graph.

Task 1. (a) What memories do you have, what we call a function? (b) Write down whatever comes to your mind about the concept.

Task 2. A car moves in a straight line at a uniform speed, accelerating or decelerating uniformly. Its movement from start to stop is shown in the following distance-time graph.




The movement can be divided into three stages (1, 2, 3). (a) How long does each stage take, and how far does the car travel during each time? What kind of movement does it make in each of these stages? (b) What is the speed of the car in stage 2? (c) Plot the speed over time in phase 1 of the motion. (d) What is the acceleration of the car in phase 1?

A code system was inductively developed to analyse the responses. The following codes were assigned to each question.

Task 1a required a content analysis. The following motifs and codes were distinguished (often more than one in the same answer). (A) a specific function is named, (B) function property (i.e., monotonous increase), (C) assignment, (D) graph or coordinate system, (E) other, and (F) no answer.

Task 1b was coded in two dimensions: the function type and the representation form. The function types students mentioned are physics functions, linear functions, quadratic functions, and others, while some of them did not answer. The forms of representation used are verbal description, ordered pair(s), graph, and algebraic expression.

Regarding Task 2, we focused on the correctness of the responses and how students determine the result: Do they connect their reasoning to the graph or calculate the result independently from them?

The two authors coded the answers separately and then decided on a few differing codes by consensus.

## RESULT AND DISCUSSION

### What do we call a function?

The motifs in the responses are summarised in Figure 1.

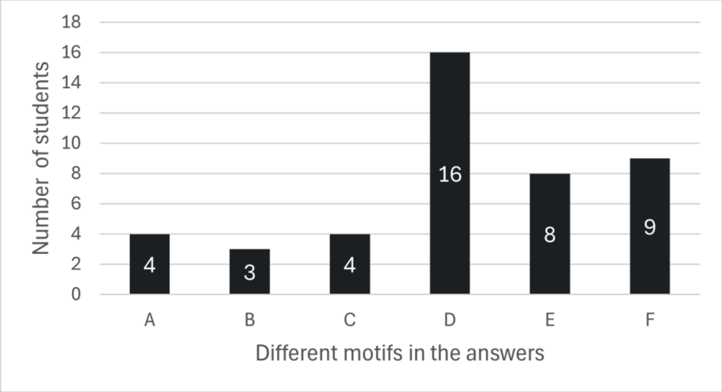



Figure 1. Frequency of different motifs about function concept.

About half of the students (16) referred to the coordinate system or the graph of a function in some form (D). Several described that functions or points can be represented in an x-y coordinate system or that the function itself is the points plotted in the coordinate system.

A specific function (A), for example, a linear function, was mentioned 4 times. Some function properties, such as “can be decreasing or increasing” (B), appeared 3 times. In 4 motifs, the function was defined as “assigning one value (usually y) to another (usually x)” (C).

There were 8 unique responses (E). For example, stating that functions "can be described with some rule," or that they can illustrate the change, or "the relationship between two pieces of data." Someone mentioned that "equations can be solved with them, and they contain letters." One described the graphing of linear functions, explaining how the slope and the intercept are determined. Another student recalled Excel functions from IT classes, while two others mentioned value tables.

From the answers, it became apparent what concept image the students have regarding functions. They likely associate the concept with their most common activity: representing points and curves on a coordinate system. Most students seem to have an empty concept definition, possibly because they encountered functions mainly in informal ways. Twenty-one students provided non-empty answers to this question. Generally, multiple different categories appeared for each student. Ten students mentioned 2, and 2 students mentioned 3 different motifs. That indicates that their concept image includes various components. Nine students either did not write anything or mentioned that they could not remember anything (F).

**Examples of function and their representations**

According to the primary school requirements, most students represented the linear function in some form, as an example (Table 1. Examples of function. Triggered by memories from physics classes, some brought up examples, including physical quantities. This meant a distance-time graph, or they wrote "the function of time," which might have originated from the assignment: "Plot the position as the function of time." In 3 cases, a parabola-like graph arising from variable movements was also mentioned.

| Physics function | Linear function | Quadratic function | Other | No answer |
|---|---|---|---|---|
| 3 | 13 | 3 | 8 | 3 |

Table 1. Examples of function.

In responses to question 1a, many also mentioned points plotted in the coordinate system as part of their definition of a function. In line with this, 7 students (among the 8 who were coded "Other") also displayed coordinate pairs in some form (Figure 2 2.). This indicates that some believe these ordered pairs also constitute functions.

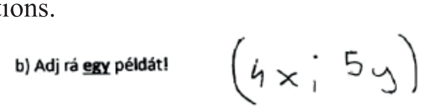



Figure 2. The ordered pair as an example of a function.

In the examples, various representations appeared (Figure 3). In line with question 1a, the most common representation (13) was the visual representation, a graph in a coordinate system. It is worth noting that except for one student, all emphasized only the first quadrant of the coordinate plane. In most cases, the negative parts of the axes were missing, or the plotted points of the function were all in the first quadrant. This perfectly aligns with the fact that in physics classes, they mostly represent positive quantities, hence only graphing the first quadrant.

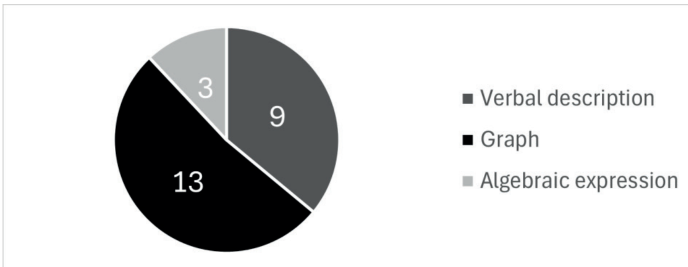



Figure 3. Frequency of different representations when the example is not an ordered pair.

In several cases (9), the example was given in some textual form, but it always meant describing a type of function (e.g., linear function). In a few cases (3), they provided an example using some rule with algebraic expressions (e.g., f(x)=x+2). Some students presented multiple representations associated with a single example, but various examples could also appear in the case of one student. In the following student work, we see a linear and a constant function named with words and depicted with a graph (Figure 4). Another example on the right side of the figure contains elements of the algebraic notation and ordered pairs in a way that is not usual. This shows that the student has a vague understanding of these concepts.

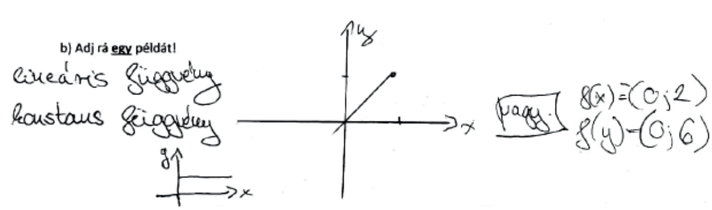



Figure 4. A student's work with different representations and various examples.

The examples align with the answers to the first question and mostly show that the concept image dominates the definition, which does not exist in most cases. This is also indicated by the fact that some students did not answer the first question but could still provide a visual example.

**Findings of the physics task**

In the physics task, almost every student could read the time and position values for each motion phase from the graph. The only issue encountered in 6 cases was that they did not calculate the distance for each segment from the graph; they just wrote down the position of the ending point as the distance. Except for 3 students, everyone accurately determined the type of motion based on the curve. This means that they can read the coordinates of the points and are also aware of the physical meaning of the graph.

In the second motion phase, the object underwent uniform linear motion, and the students had to determine its speed. From the solutions, it is evident that they solve this task separately from the given graph. In 24 cases, the students wrote down the formula v=s/t without any index or other indication that would refer to the specific part of the motion. So, to determine the speed, they used the well-known algebraic expression (formula) and substituted the values of distance and time into it. This is consistent with using the concept of function at the operational (*action*) level.

Then, they had to graph the velocity as a function of time in the first phase of the motion, which was uniformly accelerated. Figure 5 shows that 7 students solved this task perfectly.

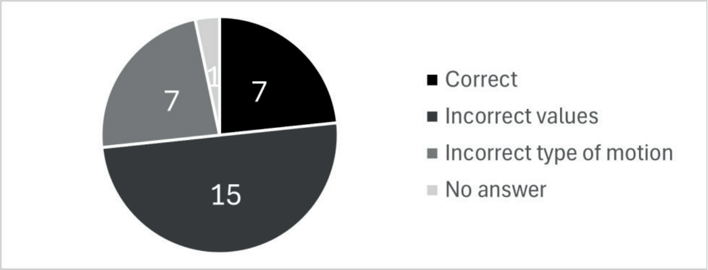



Figure 5. Frequency of different types of answers for question 2c.

In the rest of the responses, it is noticeable that the graph is separated from the previous parts of the task. In 15 cases, the final velocity was incorrectly determined, even though it had already been calculated in the previous part of the task. In Figure 6. the student drew a linear segment starting from the origin according to uniformly accelerated motion, but it does not connect to their previous results. Neither the final velocity (1 m/s) nor the time value for the ending point (4 s) matches the value calculated previously (2 m/s, 5 s).

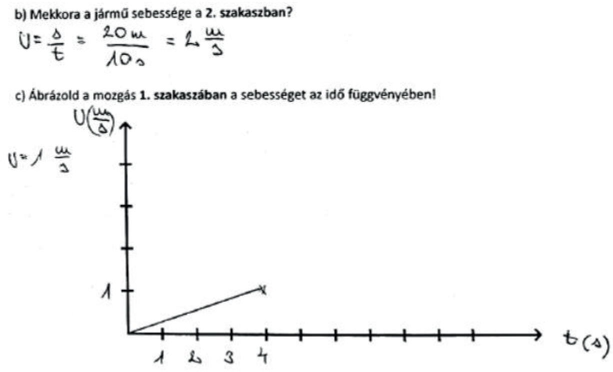



Figure 6. Different values for final velocity in the answers to questions 2.b and c.

The rest of the responses did not correspond to uniformly changing motion despite correctly describing it as such earlier when they characterized the motion. This indicates that certain factors of the motion process are overlooked during problem-solving, and students only focus on the data they need at the moment.

Physical tasks contribute to the concept image of function by providing activities at the *action* level and using a graphical representation. However, it is not enough to reach a more complex level of understanding, at least based on our experience.

## CONCLUSION

The students' responses gave us a broad picture of the first research question. Obviously, their concept image of functions draws from primary school mathematics classes and secondary school physics classes. Since they come from different schools, they have varying levels of prior knowledge. However, it can generally be said that the concept image is dominated by representation in coordinate systems and linear functions. The role of ordered pairs is also significant, but students have a vague understanding of this aspect.

Regarding the second research question, we gained partial insights. Overall, the students could read data from graphs and even qualitatively interpret what type of motion it represents. However, after extracting the data, if we ask for some calculation or further representation related to the motion, sometimes it does not match the previous graph or data. This indicates that while students are familiar with the representations involving graphs and algebraic expressions, they might have difficulty seeing the connections between them. This suggests that the confident movement between the worlds described by Sierpinska (1992) is not fulfilled. In this question, a more complete picture will be provided by the next phase of the teaching experiment.

The results supported our expectation that students think on the *pre-function* or *action* levels. Their concept of function is still vague and narrow, and during problem-solving, they are mostly stuck at the level of calculating with algebraic expressions. Teaching the topic will require a significant effort to move them towards the *process* level of understanding.

In the next step, the introductory lessons allowing us to map out the students' prior mathematical and physical knowledge even more accurately based on their reactions. Our goal in the teaching experiment is to develop a higher-level understanding of the function concept, relying on students' experiences in physics classes. Of course, this can only be done effectively if we properly assess the prior knowledge. We hope to incorporate the findings from the presented survey into teaching the topic more effectively.

## References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the Process Conception of Function. *Educational Studies in Mathematics*, *23*(3), 247–285.

Csajági, S., Fülöp, F., Póda, L., Simon, P., & Urbán, J. (2020). *Fizika 9-10. Tankönyv* [Physics 9-10. Textbook]: Vol. I. Oktatási Hivatal.

D. Balogh, I. (2002). Problémamegoldás, alkalmazás és tudásátvitel a középiskolai fizikában [Problem-solving, applicaton and knowledge transfer in secondary school physics. *Iskolakultúra*, 12(1), 51–61.

Dubinsky, E. & Harel, G. (1992). The nature of the process conception of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85–106). Mathematical Association of America.

Juhász, I., Orosz, G., Paróczay, J., & Szászné Simon, J. (2020). *Matematika 9. Tankönyv* [Mathematics 9 textbook. Oktatási Hivatal.

Oktatási Hivatal. (2023). *A 2023. évi középiskolai felvételi feladatlapok feladatainak elemzése* [Analysis of the tasks for the 2023 secondary school entrance tests]. Oktatási Hivatal. https://www.oktatas.hu/pub_bin/dload/kozoktatas/beiskolazas/meresmetodika/Kozepiskolai_felveteli_feladatok_ertekelese_-_2023.pdf

Pospiech, G. (2015). Interplay of mathematics and physics in physics education. In A. Beckmann, V. Freiman, & C. Michelsen (Eds.), *Proceedings of MACAS – 2015 International Symposium of Mathematics and its Connections to the Arts and Sciences* (pp. 36–43). University of Education Schwabisch Gmünd.

Radnóti, K., & Nagy, M. (2014). A matematika és a fizika kapcsolata [The relationship between mathematics and physics. *Iskolakultúra*, *24*(10), 102–118.

Sierpinska, A. (1992). On understanding the notion of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 25–58). Mathematical Association of America.

Ubuz, B., Gravemeijer, K., Stephan, M., & Capraro, P. (2019). An introduction to TWG26: Mathematics in the context of STEM education. In U. T. Jankvist, M. Van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education*. (pp. 4713–4720). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Vinner, S. (1983). Concept definition concept image and the notion of function. *International Journal for Mathematics Education in Science and Technology*, *14*(3), 293–305.

# TEXTS AS RESOURCES FOR POSING OPEN-ENDED PROBLEMS

Konstantinos Tatsis*, Bożena Maj-Tatsis**

*University of Ioannina, Greece

**University of Rzeszów, Poland

*Problem posing is an important aspect of mathematics and, therefore should be included in teacher education. We present a study on preservice teachers and their problem posing within a course designed to enhance their problem solving and problem posing skills. Our results show that the design of problems that are based on a given text and are open-ended has been a challenge for a considerable number of the preservice teachers involved in our study. This stresses the need for a wider inclusion of structured and semi-structured problem posing activities in teacher training.*

## INTRODUCTION

Mathematics mainly consists of solving and posing problems. This is acknowledged in mathematics education literature, especially concerning problem solving (Schoenfeld, 1992), while the last years posing problems has also gained attention by many researchers. This trend has affected curriculum design, but also teacher training. Although there is evidence that preservice teachers can pose good problems (Crespo, 2003; Leavy & Hourigan, 2020), we can also find studies reporting "irrelevant, nonmathematical, ill-formulated and sometimes unsolvable problems" (Leavy & Hourigan, 2022, p. 150). These results stress the need for a comprehensive assessment framework for the posed problems, but also for the need for systematic and wider inclusion of problem posing in teachers' training. Bearing these in mind, we designed a study involving preservice teachers, aiming to examine their problem posing skills, but also their skills in assessing their own problems. Next, we present the theoretical underpinnings of our study, followed by our research questions.

## THEORETICAL UNDERPINNINGS

Mathematical problems play a crucial role in mathematics and its learning. This fact was expressed in the relevant literature mainly by analysing the ways in which students solve problems (Schoenfeld, 1985; 1992). Gradually, the importance of engaging students in problem posing emerged, leading to relevant studies. Various definitions have been suggested in these studies; here, we present only two of them. Stoyanova and Ellerton (1996) defined problem posing as the "process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (p. 518). This definition focuses on the students' processes when they are asked to construct a problem based on a given resource; additionally, it refers to the meaningfulness of the posed problem, which is an assessment criterion. Silver (1994) defined problem posing as the activity of generating new problems and reformulating given problems which, consequently, can occur before, during, or after problem-solving. This definition focuses on the relationship between problem posing and problem solving, which has been the focus of many studies (e.g., English, 1998). A useful categorisation of problem posing situations is offered by Stoyanova and Ellerton (1996):

> In free situations students pose problems without restrictions; students are asked to "Make up a difficult problem" or, simply, "Make up a problem you like". Semi-structured problem-posing situations refer to the ones in which students are provided with an open situation and are invited to explore the structure of that situation, and to complete it by using knowledge, skills, concepts, and relationships from their previous mathematical experiences. Structured problem-posing situations refer to situations where students pose problems by reformulating already solved problems or by varying the conditions or the question of given problems. (Bonotto, 2013, pp. 39–40)

The acknowledgement of the importance of problem posing in mathematics education led to studies on teacher training, based on the premise that the teachers need to be prepared to implement such an approach in their classroom (Crespo 2003; Koichu & Kontorovich, 2013). Our own interest, as teacher trainers is aligned with these studies. One of the elements of problem posing is the strategies that can be employed by the poser, in order to pose worthwhile problems. We found that the most prevailing is the "What if not?" strategy (Brown & Walter, 1993), according to which the poser is listing the attributes of a given problem and then changes them. For example, the poser can change the numerical value of data, change the kind of data, or even eliminate one of the data (Lavy & Bershadsky, 2003).

Lastly, one of the main arguments for including problem posing in the classroom is that in the real world "many problems, if not most, must be created or discovered by the solver, who gives the problem an initial formulation" (Kilpatrick, 1987, p. 124). This argument connects problem posing with the student acting in the real world, which in turn leads to realistic mathematics education approach. According to this approach, "mathematics education should start from problem situations that are meaningful to students, which offers them opportunities to attach meaning to the mathematical constructs they develop while solving problems" (Van den Heuvel-Panhuizen & Drijvers, 2020, p. 715). These problem situations may come from resources such as objects, images, videos and texts; all of these carry their affordances, which should be considered in research and in teacher training. We have been trying to connect realistic mathematics education with problem posing in our teacher training courses; particularly, in one of our studies (Maj-Tatsis & Tatsis, 2014), we engaged preservice mathematics teachers in semi-structured problem posing and then asked them to evaluate their peers' problems, based on originality, level of difficulty and degree of being realistic. The diversity in the evaluations we acquired was an indication of the preservice teachers' diverse interpretations of what constitutes a 'good' realistic mathematical problem. An important finding of this study was the significant role of the wording of the posed problem: in some cases, misinterpretations coming from the wording of the problem led some preservice teachers to different solutions than those intended by their peers who had posed the problem.

The above considerations led us to design a study in which the preservice teachers would be asked to design and then assess problems that began with a given phrase taken from a textbook. We were also interested in studying the characteristics of the posed problems. This led us to the following research questions:

a) Were the preservice teachers able to pose open-ended problems in a semi-structured situation?

b) What were the characteristics of the open-ended posed problems?

## CONTEXT OF THE STUDY AND METHODS

The participants were 26 preservice teachers (23 women), all of whom were at the third (out of four) year of their studies. Our data were collected at the ninth of the 13 lectures of the semester. By that time, the preservice teachers have been introduced to the following content within the course entitled *Didactics of Mathematics – Teaching Practice* led by the first author of the paper: learning theories in mathematics education, problem solving (especially the use of heuristics, but also categories of problems with a special focus on open-ended problems) and realistic mathematics education (including a discussion on aspects that render a problem realistic, with a focus on the role of context). The day of data collection began with a lecture on problem posing, which contained elements and examples of the various problem posing strategies mentioned in our theoretical framework, with a special focus on the "What if not?" strategy. This was followed by examples of structured and semi-structured problem posing situations. Then the preservice teachers were asked to fill in an anonymous online questionnaire, which contained questions on their age, gender, direction of studies in secondary school, followed by these questions:

6. Write a closed realistic problem that begins with the phrase: "Two cranes unload a ship in three hours".
7. Solve your problem.
8. Rate your problem, according to its level of difficulty. (1-5 Likert scale, ranging from "very easy" to "very hard")
9. Rate your problem, according to its level of being realistic. (1-5 Likert scale, ranging from "not at all realistic" to "very realistic")

10. Write an open realistic problem that begins with the phrase: “Two cranes unload a ship in three hours”.
11. Solve your problem.
12. Rate your problem, according to its level of difficulty. (1-5 Likert scale, ranging from “very easy” to “very hard”)
13. Rate your problem, according to its level of being realistic. (1-5 Likert scale, ranging from “not at all realistic” to “very realistic”)

There were no restrictions concerning the grade that the problems should address – this was done in order to allow for a bigger variety of problems. At the same time, it was obvious to the preservice teachers, that their problems should adhere to the curriculum of the last three grades of primary school in Greece, 4, 5 and 6. After all participants completed the questionnaire – which took them approximately 30 minutes – a discussion was initiated on how they perceived the process of problem posing and whether they believed that such an approach can be implemented in their classroom. Specific problems were also selected by the instructor and discussed. There was no data collection at this point, so our analysis is based on the questionnaire responses. For the purpose of this paper, we focus on the formulations and the solutions of the open-ended problems, therefore our data come from questions 10 and 11. The responses to these questions were categorised by deploying a content analysis approach (Krippendorff, 2019). Particularly, we established codes to describe the posed problems, according to the following categories:

- *content*: the mathematical concepts involved in the problem; consisted of three respective codes, namely direct proportionality, inverse proportionality, and combination/other (this included a combination of proportionality with other concepts, such as additive relationships);
- *complexity*: the number of operations required to solve it; consisted of two respective codes, namely one operation and more than one operations;
- *openness*: consisted of two respective codes, namely open-ended and closed.

The above categories fit well with the posed problems, except two cases which were excluded, as we will show in the results. We performed a quantitative analysis of data in order to answer the first research question. Then we qualitatively analysed the characteristics of each posed problem, which was categorised as open-ended. Our results, complemented by examples of our analysis, are presented in the next section.

## RESULTS

We received 26 responses, among which one contained no text, and another one contained the following problem, which we did not include in our coding:

> What would happen if more ships arrived at the port at the same time? Do you think the unloading time is affected by the type of cargo of each ship?

Half problems (12 out of 24) were based only on direct proportionality, followed by seven problems based on other notion(s), while five problems involved only inverse proportionality. Nine problems required one operation to be solved, while the remaining 15 required more than one operation to be solved. The most interesting result though, was the number of the actual open-ended problems (among those which were categorised as such by the preservice teachers), which was lower than one would expect: 11 out of 24. Table 1 summarises these results.

| Category | | Number of problems |
|---|---|---|
| Content | direct proportionality | 12 |
| | inverse proportionality | 5 |
| | combination/other | 7 |
| Complexity | one operation | 9 |
| | more than one operation | 15 |
| Openness | open-ended | 11 |
| | closed | 13 |

Table 1: Results on the content, complexity and context of the problems.

As seen from Table 1, proportionality (direct and inverse) persevered in all problems, including the open-ended ones. This was expected, since the provided situation implied such a relationship. There were, however, some problems containing other concepts, as we will show later. Since our focus was the open-ended problems, we analysed these in respect to the categories of content and complexity. Table 2 below summarises the results of this analysis:

| Category | | Number of open-ended problems |
|---|---|---|
| Content | direct proportionality | 4 |
| | inverse proportionality | 0 |
| | combination/other | 7 |
| Complexity | one operation | 0 |
| | more than one operation | 11 |

Table 2: Results on the content and complexity of the open-ended problems

Before we present examples of the open-ended problems (signified by P), followed by the solutions suggested by the posers (signified by S), we present an example of a closed problem, which was categorised as open by the preservice teacher:

> P2: What would happen if a crane had to unload a ship? How many hours would it take?
>
> S2: One and half hour.

Although the problem is clearly closed (and the preservice teacher provides the only solution), it is clear that the problem formulation is based on the "What if not" strategy. Therefore, this strategy was interpreted as one that would surely lead to an open-ended problem, although this was obviously not the case. Next, we present examples of problems that were actually open-ended.

> P5: Two cranes unload a ship in 3 hours, a sailboat in 2 and a boat in half an hour. If we have 24 hours how many and what can we unload?
>
> S5: There are many solutions. One solution is 4 ships, 5 sailboats and 4 boats.
>
> P6: Two cranes unload a ship in 3 hours. Each ship has 20 containers, one suddenly overheated and stopped working and was replaced with one with half the yield. We have two groups of people who push them through the crane to trucks, group A unloads 1 container in 1 hour and group B 2 containers per hour what combination would you make?
>
> S6: Each crane unloads 10, so the new crane unloads 5 within three hours, so within three hours the cranes will unload 15 out of 20 and 5 will remain. Group B could be used for 2 hours and Group A for an hour.
>
> P19: Three cranes unload a ship in 2 hours. If each crane needs 400€ to operate per day, and in a week the employees work all 7 days, 8 hours a day with 3 cranes and 9 hours with 2 cranes, and the budget for the operation of the cranes is 8000€, how many ships would be likely to unload so as not to exceed the budget. Which are considered better options for the employer and which for employees and why? What would be a balanced choice for both employees and employers and why?
>
> S19: Students will make a board with 4 columns, days with 2 cranes, days with 3 cranes, money spent and unloaded ships. They find all possible answers that do not exceed 8000€. The last questions give a perspective of both the employer and the employee and [the students] can understand that although many ships could be unloaded, but the employer also wants the largest possible profit at the same time, so it may be that something that is true in mathematics is not true in reality.

The above examples present in a clear way some of the qualitative characteristics of the posed problems, which relate to the preservice teachers' views of open-ended problems in mathematics, but also of other aspects of mathematics education. The first example (P5) is a typical case of an open-ended problem, in which the student is asked to find the combinations of numbers that lead to a given sum, in this case 24. The second example (P6) contains a scheme that appeared in four other problems: according to this, additional data is inserted (e.g., the total load is broken down into containers and each container needs to be unloaded by a group of workers), which results in additional operations to be performed. This increases the complexity of the problem and renders it a realistic one, especially due to the initial condition ("one suddenly overheated and stopped working and was replaced with one with half the yield"), which resembles a realistic situation. The last example is even more interesting, because the preservice teacher (although he reformulated the initial phrase, maybe accidentally), not only inserts additional information, but also considers factors such as the total budget (and how to not overcome it) and the desired balance between the employee's and the employer's needs. We may perceive this as a feature of critical mathematics education (Skovsmose, 2020), which is considered important for teachers' training.

## DISCUSSION

Our study aimed to shed light on the problem posing of preservice teachers, which was based on a given text. Firstly, we examined whether they would be able to pose open-ended problems: our results showed that they had difficulties in doing so. Such difficulties might stem from their lack of experience, despite the fact that they had received specific instruction on what is an open-ended problem and how to design such a problem. These difficulties led them to perceive closed problems as open-ended ones.

By looking at the qualitative aspects of the open-ended problems, we were able to discern their characteristics, in order to answer our second research question. It was noteworthy that mere proportionality appeared in only 4 problems, whereas the majority of problems contained a combination of mathematical concepts; in most cases an additive relationship was used. All open-ended problems required more than one operation to be solved. In some cases, the preservice teachers inserted additional data to their problems, making them 'more' realistic, and, in some cases, adhering to the principles of critical mathematics education.

We conclude by acknowledging that the preservice teachers could have been exposed to more structured problem posing situations and to more examples of open-ended problems, before our study took place. In this line, we posit that the process of problem posing needs to be systematically implemented in teacher training, since it is one of the most important mathematical activities. Such implementation should consider the connection between problem posing and problem solving, for instance, by asking the posers to solve their own (or their peers') problems. Moreover, it should be enriched with elements such as realistic mathematics and critical mathematics education, in order to achieve a better preparation of future teachers, which will expectedly lead to a better preparation of their students.

## References


Bonotto, C. (2013). Artifacts as sources for problem-posing activities. *Educational Studies in Mathematics, 83*(1), 37–55.

Brown, S. I., & Walter, M. I. (1993). Problem posing in mathematics education. In S. I. Brown & M. I. Walter (Eds.), *Problem posing: reflection and applications* (pp. 16–27). Lawrence Erlbaum Associates.

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics, 52*(3), 243–270.

English, L. D. (1998). Children's problem posing within formal and informal contexts. *Journal for Research in Mathematics Education, 29*(1), 83–106.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123–147). Lawrence Erlbaum Associates.

Koichu, B., & Kontorovich, I. (2013). Dissecting success stories on mathematical problem posing: A case of the Billiard Task. *Educational Studies in Mathematics, 83*(1), 71–86.

Krippendorff, K. (2019). *Content Analysis: An Introduction to its Methodology*. Sage.

Lavy, I., & Bershadsky, I. (2003). Problem posing via "what if not?" strategy in solid geometry: A case study. *The Journal of Mathematical Behavior, 22*(4), 369–387.

Leavy, A. M., & Hourigan, M. (2020). Posing Mathematically Worthwhile Problems: Developing the Problem Posing Skills of prospective Teachers. *Journal of Mathematics Teacher Education, 23*(4), 341–361.

Leavy, A. M., & Hourigan, M. (2022). The Framework for Posing Elementary Mathematics Problems (F-PosE): Supporting Teachers to Evaluate and Select Problems for Use in Elementary Mathematics. *Educational Studies in Mathematics, 111*(1), 147–176.

Maj-Tatsis, B., & Tatsis, K. (2014). Problem posing by preservice teachers. In M. Pytlak (Ed.), *Communication in the mathematical classroom* (pp. 152–165). Wydawnictwo Uniwersytetu Rzeszowskiego.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic.

Schoenfeld, A. H. (1992). *Learning to think mathematically: problem solving, metacognition, and sense making in mathematics*. In D. A. Grows (Ed.), Handbook of research on mathematics teaching and learning (pp. 334–370). Macmillan.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.

Skovsmose, O. (2020). Critical Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 154–159). Springer.

Stoyanova, E., & Ellerton, N. F. (1996). A framework for research into students' problem posing in school mathematics. In P. Clarkson (Ed.), *Technology in* *mathematics education* (pp. 518–525). Mathematics Education Research Group of Australasia.

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 713–717). Springer.

# NON-STANDARD PROBLEMS AS RESOURCE TO VERIFY MULTIPLICATION UNDERSTANDING IN PRIMARY SCHOOL

Eva Nováková*, Paola Vighi**

*Faculty of Education, Masaryk University, Brno, Czech Republic

**University of Parma, Italy

*This paper presents the results of a research project focused on understanding of multiplication in primary school. An experiment was carried out in Czech Republic and in Italy in two classes with pupils aged 8-9 years. Pupils solved two arithmetic tasks based on the rectangular model of multiplication, its role was studied as a resource to promote multiplication understanding. Pupils' solutions were analyzed in detail during interviews and by examination of submitted worksheets. The results show that suitable tasks can promote the transition from the additive to the multiplicative field.*

## INTRODUCTION

Multiplication is considered to be more difficult than addition and subtraction (Clements & Sarama, 2007). From a conceptual point of view, these operations are very different, even if in the usual tasks they are reduced to calculations with numbers. In school, various models are used to introduce multiplication of whole numbers. It is important to be aware of the different features and potentiality of these models, and to take into account the possible problems connected with to the transition from a model to another.

In the Czech Republic (CR), elementary mathematics traditionally bases the methodology of multiplication on the manipulation of concrete objects arranged in rows and columns. This arrangement is described as '*a* rows of *b* elements' or '*a* groups of *b* objects', later simplified to the expression '*a* by *b*', and finally expressed as '*a* times *b*'. Pupils can also solve multiplication examples using a square grid or by cutting them out of square paper. It is emphasized that this initial phase focuses solely on the pupils' understanding of the essence of the multiplication operation, with the goal of performing calculations by heart is reserved for the second phase (Divíšek et al., 1989; Nováková & Blažková, 2022).

In Italy (IT), usually the first approach to the multiplication of whole numbers is based on making *a* groups of *b* objects, or on arrays of objects and the teacher poses the problem of counting them. Pupils observe the presence of equipotent rows and columns, and they use repeated addition to count the totally of objects. Subsequently, other representations are introduced and utilized, but soon the models are neglected, and multiplication becomes only an activity with numbers.

## THEORETICAL FRAMEWORK

The transition from the additive to the multiplicative field is complex because their structures are very different. As documented by many researchers (Mariotti & Maffia, 2018), presenting multiplication as repeated addition can hinder the understanding of multiplicative structure.

In mathematical terms, when we write $a{\cdot}b$ the symbols $a$ and $b$ represent numbers, while when we say "$a$ repeated $b$ times" the symbol $a$ denotes a number, while $b$ represents a numeral adjective, in the sense of an operator: the first is an element of the 'language', the second of the 'metalanguage' (Marchini, 2001/2002, p. 13).

Briand (1993) showed that 7-8-year-old pupils, when dealing with an arrangement of objects in rows and columns, utilize multiplication for counting them, but if the arrangement is incomplete and it becomes necessary to uncover or reconstruct its structure (see the following Tasks 1 and 2), the calculation procedures undergo a complete transformation. One possible explication is that the procedures learnt in class to enumerate a row-column arrangement are not interiorized; they become destabilized when the arrangement's conditions change. Consequently, some researchers suggest working on row-column arrangements starting from kindergarten (Rozek & Urbanska, 1998).

In his theory of semiotic representation, Duval (2006) emphasizes the role of the transition from one representation to another, distinguishing between two types of transformation, 'treatment' and 'conversion', which correspond to different cognitive processes. In our study, treatment occurs when we perform calculations as $a{\cdot}b = c$ remaining within the arithmetic register, while conversion involves, for instance, transforming a visual representation of a rectangle into a linguistic expression, such as "it is a rectangle $a{\cdot}b$" and subsequently conducting the relative calculation. According to Duval's theory, in the latter second case, transitioning from the geometrical to the arithmetical register could promote the understanding of multiplication.

Thus, it can be useful to work with various approaches on multiplicative structures, such as hopping along the number line, creating grids, generating areas, and more. These models have distinct features and should be utilized in complementary ways.

Another possible model is the Laisant's [1] table, sometimes employed by teachers as a tool for introducing multiplication. This table, also known as the "decanomial" in Montessori's (1934/2016) activities, provides a new semiotic

[1] Charles-Ange Laisant, French mathematician who invented this table (Laisant, 1915).

representation for multiplication. In Laisant's table, both columns and rows increase by one, moving respectively from left to right and from top to bottom. Maffia & Mariotti (2020, p. 28) note: "Laisant's table incorporates the rectangular model, presenting any rectangle as an ordered multiplication. Such possibility constitutes the core of the semiotic potential of this artifact."

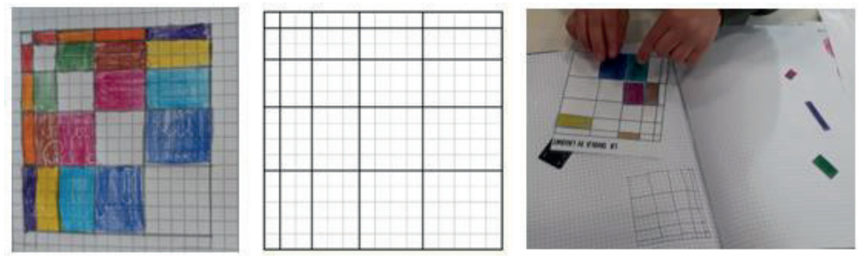


Figure 1: Laisant's table.

Initially, the construction of the table can be a drawing activity, a task assigned by the teacher based on the respect of some rules. In fact, the table allows a geometrical introduction of the multiplication, enhancing a visual perception of quantities. Essentially, as the table is realized, it immediately reveals rectangles (or squares), that appear during the construction of the table itself. Consequently, it feels natural, for example, to observe the pink rectangle and describe it as "a rectangle three times four" using everyday language, pre-empting the linguistic expressions usually employed whit multiplication.

Later the teacher can move pupil's attention on the small squares that form the pink rectangle and he can ask to count them (twelve in our example). The next step involves connecting the two initial numbers with the third: $3 \cdot 4 = 12$.

It is important to underscore a significant difference between working on multiplication with arrays and Laisant's table. When we work on arrays, $a$ and $b$, and $a \cdot b$ represent numbers. However, with Laisant's table, the scenario changes entirely: $a$ and $b$ represent linear measures, the lengths of the rectangle sides, while $a \cdot b$ represents the number of squares that forming this rectangle. Thus, there is a transition from the additive field to the multiplicative field, from linear measures to area measures. It can prepare the work in geometry with linear or two-dimensional geometrical figures.

Another positive aspect of this table is the possibility to cut rectangles and to superimpose appropriately them onto the rectangles drawn in the table, using manipulation (Figure 1). The table maintains the structure while if we manipulate objects in an array their disposition changes.

**Research questions**

In the present research, we employ two tasks, designated as Task 1 and Task 2, with the aim of investigating the following questions:

1. Are Task 1 and Task 2 a resource for diagnosing pupils' preconceptions and/or internalization of multiplication?
2. Can Tasks 1 and Task 2 serve as valuable resources for exploring the transition from additive structure to multiplicative structure?
3. Is the Laisant's table a resource for constructing multiplication structure?

**Task 1 and its a-priori analysis**

*How many tiles will be on the floor when it will be finished?*

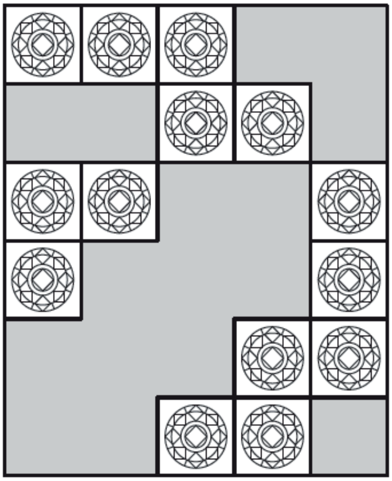


Figure 2

Task 1 originates from an assessment question presented by INVALSI (Italian National Institute for the Evaluation of Instruction and Formation Educational System)[2], the authority responsible for conducting periodic and systematic tests on pupils' knowledge and abilities. These tests are administered in all Italian schools, in the same day at the end of the school year.

The Authors of the current paper utilized the figure of the task D9 (Figure 2), which was originally presented to 7-8-year-old pupils in the year 2019. However, they modified the question in alignment with their research inquiry. Specifically, the original test question focused on determining the number of omitted tiles, while the aim of the present research is to observe whether pupils utilized multiplication, such as 6·5 or 5·6, when facing Task 1 or not. We can suppose that pupils had to mentally visualize the omitted tiles and count them with 'mental eyes'. Alternatively, they could draw them, but in this case the

[2] More detailed information see: https://invalsi-areaprove.cineca.it

drawings must be accurate. The solution could also be reached by counting ‘in horizontal’, or ‘in vertical’, or ‘in groups of tiles’.

**Task 2 and its a-priori analysis**

*How many tiles will be on the floor when it will be finished?*

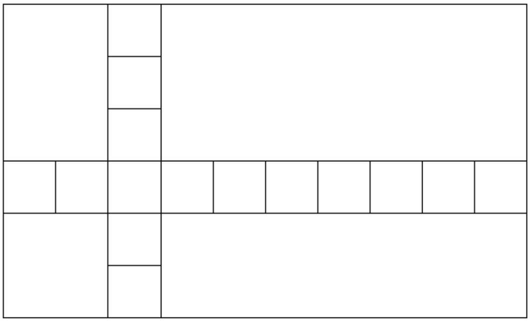


Figure 3

Task 2 presents the same question of Task 1, the ‘floor’ once again is a rectangle (Figure 3), which includes an interior ‘cross’ created by two intersecting square lines and four empty white rectangles (the drawing comes from (Briand, 1993)). We hypothesize that Task 2 can serve as an educational resource to stimulate the necessity of multiplication and promote its understanding. When presented with an array of objects, pupils tend to utilize multiplication. However, in Task 2, the conditions of enumeration are different, allowing pupils to organize their calculations in diverse ways.

Several strategies can be employed, including:

- Calculation by multiplication: 6·10 or 10·6
- Addition: 10 +10 +10 +10 +10+10 or 6+6+6+6+6+6+6+6+6+6
- Addition of blocks of tiles and the drawn tiles: 6 + 4 + 21 + 14 + 15.

**METHODOLOGY**

The research was conducted in the second grade of primary school with pupils aged 8-9 years old.

In classroom, we presented Task 1 by a worksheet. Immediately after, individual interviews were made by the researcher, prompting each pupil to explain their answer and their reasoning behind it. Thus, it was possible to use artefacts of a dual nature for further research: written problem-solving responses and subsequent interview records, which were documented in writing. Both sets of research data were then analyzed systematically.

The interview commenced with the following question: “What object was suggested from the drawing presented in the worksheet?”. This question was designed to put the pupils at ease, as they provided various responses such as “floor”, “wall paintings”, “tablecloth”, and so on.

Subsequently, the pupils explained their solutions, providing insights into their thought processes and reasoning.

Task 2 was introduced in the classroom using a multimedia interactive blackboard. Pupils were asked to observe the projected figure and provide a written answer to a question identical to that presented in Task 1, but in this case without having the possibility of draw the floor in a paper. This choice is motivated by the intention to discourage drawing and instead encourage observation of the figure and reasoning skills. In this way, we want observe if the recourse on multiplication appears. We believe that this choice may have increased the task’s difficulty, which may have more incorrect solutions than these furnished for Task 1.

In CR, the pupils were not yet familiar with the operation of multiplication. In particular, they never used the scissors and square grid. This context allows for a clearer observation of pupils thought processes and problem-solving approaches.

In IT, the teacher clarified that despite being in the third grade, the pupils' competencies were similar to those of second-grade pupils due to disruptions caused by the COVID-19 epidemic, which had slowed down the execution of usual activities. Multiplication had been recently introduced in the current school year, and the pupils had limited experience with it. However, in the previous year, pupils worked with Laisant’s table. A week later, the researcher went in classroom submitting again Task 2 by blackboard, and giving a white paper to each pupil asking to write not only its answer to the question, but also her/his reasoning. The aim was to verify if pupils use multiplication or not.

On the contrary, in CR researcher presented the Task 2 furnishing also to the pupils a paper for writing their solutions since when the task was given on the board (as well as IT) the pupils asked to redraw the picture. They were allowed to draw.

## RESULTS

### Analysis of pupils’ solutions to Task 1

In CR 18 pupils, 9 girls and 9 boys, are involved in the experiment. In IT, the total number was 22, 12 boys and 10 girls.

Two basic phenomena emerged from the analysis. The first was the need for the drawings of missing tiles, as an integral part of the solution to the problem. By sketching vertical and horizontal lines, all tiles were visible on the floor, allowing them to focus on determining the total number of tiles, both present and missing. The second phenomenon was the method of determining the number of tiles.

From the pupils' solutions, their written comments and the subsequent interview, we traced five different strategies.

a. After drawing the missing tiles, pupils proceeded to count the tiles in each row one by one, numbering them sequentially from 1 to 30. To facilitate the counting process, each tile was marked, either with a dot or a circle or a number.
b. Calculation of the number of tiles drawn on the floor and the number of missing tiles and addition 14+16 (four pupils).
c. Addition of all tiles in the rows 5+5+5+5+5+5 (eight pupils), or of all tiles in the columns 6+6+6+6+6 (two pupils).
d. Multiplication $6 \cdot 5$ or $5 \cdot 6$ (two pupils). One of them knew multiplication from older sibling.

In CR, only two pupils did not take advantage of the opportunity to draw the missing tiles in their solutions. In one instance, a pupil determined the number of tiles by mentally counting them one by one, row by row. Another girl counted the current number of tiles shown; while counting the missing tiles, she pointed to the locations of each missing tile. She then added the two counts together.

One boy did not solve the problem correctly. He made a mistake when reciting the series of natural numbers, omitting the number 16. The pupil who did not solve the problem correctly used a functional strategy.

In IT, 7 pupils used multiplication (strategy d), 11 used addition (strategy a), and 3 counted only the missing tiles.

We want to note that the question 'How many …?' suggests the use of counting, influencing the chosen strategies of solution. Moreover, the possibility to draw on the worksheet appears to promote counting one by one of the tiles. Some pupils separately calculated the number of drawn tiles (14), the number of omitted tiles (16) and subsequently they added them: 14+16=30 (strategy b). Sometimes they stop after the counting of omitted tiles. In particular, a girl finished with this strange statement: "The tiles will be available after 15 days". Pupils who used multiplication without hesitation, explain in this way: 5 in horizontal, 6 in vertical, so $5 \cdot 6=30$. This language was employed in the previous year during the activities with Laisant's table. Some of them confused 'horizontal' with 'vertical'.

**Analysis of pupils' solutions to Task 2**

In CR the development of the research investigation was the same as for the first task. After solving the task independently (about 10 minutes) each child was again interviewed by the researcher. In individual interviews pupils verbally explained, justified and commented their procedures recorded in writing.

From the pupils' solutions, their written comments and the follow-up interview, we again identified different strategies. Only two girls did not develop any solution strategy. We can distinguish four strategies for solving the problem:

a. A group of four pupils chose a procedure based on counting one by one to determine the number of squares, often reaching an incorrect conclusion. Some pupils failed to redraw the picture correctly. One boy gave an incorrect result because he made an image of seven rows instead of six.
b. Five children first noticed the number of tiles in one row and then realized that there would be the same number of tiles in all the rows. Consequently, they counted the number of rows using the number of tiles in the drawn column and calculated the resulting number of tiles by repeatedly adding the number of tiles in one row (10+10+10+10+10+10).
c. Another strategy, also based on addition, was chosen by three pupils. They noticed that except for the third column with marked tiles, there were 5 tiles missing in each column. In the floor there are 9 such columns, calculating this they found that there are 45 missing tiles. To obtain 45 they used a memory addition (5+5+5+5+5+5+5+5+5). Later they added the 15 tiles that are marked in the figure to obtain the total number of tiles. They were aware of the necessity to avoid counting the same tile twice, so they added 10+5.
d. Four children utilized the same initial situation, counting 10 tiles in a row and 6 tiles in a column. However, these children approached their solution focusing on the relationship between the number of tiles in the columns and rows and they intuitively arrived at determining the result through multiplication. When expressing their solution orally, they articulated the number of tiles as the result of the $6 \cdot 10$ reasoning.

In IT only seven pupils chose strategy (a), two chose strategy (b), two chose strategy (c), six chose strategy (d). Sometimes counting occurred by imagining the omitted tiles and mentally counting them, obviously with various and approximate results such as 57, 58, 64, 52, 88.

It is interesting to observe that a new strategy appears: counting of the tiles of white rectangles (4 + 6 + 21 + 14), counting of drawn squares (15) and then adding them together (4+6+21+14+15= 60). We suppose that the previous work with Laisant's table influenced their performances moving to observe the 'white rectangles' in Fig.3. Some pupils mistakenly counted the square placed at the intersection of the horizontal and vertical lines twice, obtaining a total of 61 tiles. Additionally, four pupils considered only the omitted tiles obtaining 45. This indicates that they applied their multiplication knowledge on the 'small rectangles' and not on the biggest. The majority of pupils used multiplication, probably influenced from a revision made in classroom by the teacher.

## CONCLUSIONS

At the end of the activities, pupils mentioned that initially the tasks seemed trivial or easy, but they found difficult to explain their reasoning.

In CR only two pupils used multiplication to solve the first problem, four pupils used multiplication in the second. One boy remarked: "Rows and columns have something in common. There are 10 squares in a row and 6 in a column. Six times ten is sixty". This observation suggests that the second task prompted the pupils to think differently.

In IT only five pupils employed multiplication in both tasks, indicating that for them this operation appears internalized. When we reintroduced Task 2 one week later, the percentage of Italian pupils who used multiplicative strategy passed from 32% to 58%.

Here we want to underline the role of the activities proposed in classroom. We believe that the difficulties and obstacles presented by the proposed tasks prompted the pupils to see the multiplication as a useful tool for organizing calculations of objects in an array. In other words, we think that our tasks provoked the need for a link between the existing understanding of multiplication and its mental representations, promoting a deeper understanding of the concept.

With reference to research question 1, we can affirm that both tasks led to the identification of pupils' preconceptions in the area of multiplication. Additionally, we observed that for some children who used multiplication in both tasks, the employment of this operation came later. For the other pupils the understanding begins slowly, step by step, as the figures drawn in Tasks 1 and 2 confuse their visualization of rectangular models.

With reference to research question 2, we can observe that perhaps after numerous attempts children come to see multiplication as better tool to solve the problems. The transition from additive to the multiplicative field must be promoted, but it is important to emphasize that each pupil has his/her own time of understanding, which must be respected.

With reference to research question 3, in Italian classes we observed the influence of the previous work on Laisant's table, particularly in some pupils.

The teachers of the classes involved in the experimentation initially considered the tasks difficult and hard to solve. However, by the end, they were surprised by the performance of pupils, particularly they observed a deeper understanding of multiplication. As mentioned earlier, Task 2 played this role, as we can verify during the last intervention in classroom. In other words, as Barmby et al. (2009, p. 219) state, "representations were used earlier on, but only for the purpose of illustrating multiplication and rarely for the purpose of supporting calculation", while our experience documents the need for continuous, rather than episodic use of the rectangular model when working on multiplication.

**Acknowledgement**
We would like to thank the teachers Lenka Sýkorová (Bosonožská school, Brno) and Patrizia Coppola (P. Maupas school, Vicofertile Parma) for the collaboration during the experiment.

## References

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K. & Borrow, C. V. A. (1998). Student's spatial structuring of 2D arrays of squares. *Journal for Research in Mathematics Education, 29*(5), 503–532.

Briand, J. (1993). *L'énumération dans le mesurage des collections: un dysfonctionnement dans la transposition didactique* [The enumeration in the measurement of collections: a dysfunction in didactic transposition]. Thèse de doctorat en Didactique des mathématiques.

Clements, D. H., & Sarama, J. (2007). Early childhood mathematics learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol.1, pp. 461-555). New York: Information Age Publishing.

Divíšek, J., Bušil, Z., Hájek, J., Križalkovič, K., Malinová, E., Zehnalová, J, & Vasilková, E. (1989). *Didaktika matematiky pro učitelství 1. stupně ZŠ* [Didactics of mathematics for 1st grade elementary school teachers]. SPN.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103–131.

Laisant, C. A. (1915). *Initiation mathématique: ouvrage étranger à tout programme, dédié aux amis de l'enfance* [Mathematical initiation: a foreign work, dedicated to childhood friends]. Hachette & cie.

Maffia, A., & Mariotti, M. A. (2018). Intuitive and formal models of whole numbers multiplication: Relations and emerging structures. *For the Learning of Mathematics*, *38*(3), 30–36.

Maffia, A., & Mariotti, M. A. (2020). From action to symbols: giving meaning to the symbolic representation of the distributive law in primary school. *Educational Studies in Mathematics*, *104*, 25–40.

Marchini, C. (2001/2002). *Appunti delle lezioni di Matematiche Complementari* [Notes of the lessons of Complementary Mathematics]. 1–156.

Montessori, M. (1934/2016). *Psychoarithmetic*. Montessori-Pierson Publishing

Nováková, E., & Blažková, R. (2022). *Rozvíjení matematické gramotnosti s využitím inovativního modulu systematického rozvoje žáků 1. až 3. tříd. Metodický text pro učitele* [Developing mathematical literacy using an innovative module for the systematic development of 1st to 3rd grade students. Methodical text for teachers]. Masarykova Univerzita.

Rożek, B. & Urbanska, E. (1998). Children's understanding of the row-column arrangement of figures. In F. Jacquet (Ed.), *Relationship between classroom practice and research in mathematics education, Proceedings of CIEAEM 50* (pp. 303–307), Neuchâtel.

# THE DEVELOPMENT OF THE SIMPLE STRATEGY FOR SOLVING MATHEMATICAL WORD PROBLEMS

Qendresa Morina

Charles University, Faculty of Education, Prague, Czech Republic

*This study draws on the literature focusing on successful interventions to improve word problem-solving. Two important factors of such interventions have been shown to improve pupils' performance: visual representations and the development of metacognitive skills. Based on these factors, a specific teaching strategy ("the SIMPLE Strategy") is designed to suit the educational context in Kosovo. The study's main goal is to determine its suitability for developing Grade 8 pupils' ability to solve word problems using iterations of design-based research. The results of the first cycle were encouraging, demonstrating the strategy's impact in solving word problems and developing visual skills.*

## INTRODUCTION

Word problems (WPs), an important aspect of mathematics education, remain an essential topic of discussion in pedagogy and research endeavours. They are considered challenges that require pupils to bridge the gap between abstract mathematics and real-world situations. Because of the challenges in their solution, various educators and researchers have investigated different strategies and approaches focused on improving pupils' skills in solving such tasks. In this paper, we investigated contemporary viewpoints on WPs as well as the innovative strategies that have emerged to handle the complexity involved in their solution. Authors like Polya (1945) and Schoenfeld (1979) believed that teaching and learning heuristic strategies would help pupils progress on the solution of WPs and help them enhance their problem-solving abilities. When solving WPs, teachers encourage pupils to read, point out important information, guess the results, consider alternative solutions, and reflect on the processes. This helps pupils to apply metacognition even if it is not consciously (Kusaka & Ndihokubwayo, 2022). Besides metacognition, another critical aspect of the WP solution is visualization. Many authors suggest that visualization is a crucial tool for problem-solving (e.g., Csikos et al., 2011; Hembree, 1992; Gani et al., 2019), and it is neglected as a method by teachers and educators. In our study, we combine visualization—precisely, the block model method—and a few metacognitive strategies presented by other authors to produce a new innovative approach known as the SIMPLE strategy. Through a carefully planned intervention, the study seeks to assess the development and effectiveness of this strategy in enhancing pupils' problem-solving skills. This study took place in Kosovo, and the participants were eighth-grade pupils. By carefully examining the intervention's progression, we aim to investigate whether pupils are able to solve WPs easily after practising the strategy within a short intervention time.

## THEORETICAL FRAMEWORK

WPs are important in school mathematics as they link mathematics with the real world. According to Vondrová et al. (2019), these are problems in which numerical data are presented, some are not, and a question is given for pupils to answer using strategies, mathematical knowledge, and computational skills.

Previous research has explored various variables that contribute to WP difficulties. Researchers began exploring the challenges pupils face while solving various WPs, starting with simple arithmetic WPs (Cummins et al., 1988; De Corte & Verschafel, 1987) and proceeding to more complicated ones (Verschafel & De Corte, 1997). Daroczy et al. (2015) show that many linguistic and mathematical variables and their combination affect students' strategies, mistakes, and success in solving WPs.

Following the difficulties, many researchers tried to organize an intervention that would result in a better understanding of WP. An important focus was on developing pupils' metacognition and visual representation.

The first factor is based on the assumption confirmed by research (e.g., Csikos et al., 2011; Hembree, 1992) that visual representations help pupils solve WPs as they bridge the challenging step from reading a WP to creating a mathematical model (Gani et al., 2019). The block model method is one of them. It originates from Singapore and is one of the ways pupils are introduced to visual representations of the structure of WPs. Pupils depict unknown quantities and their interactions in a WP by drawing a pictorial model using strips, bars, or rectangular areas.

The second factor is the development of metacognitive skills (e.g., Perry et al., 2018; Teong, 2003). Metacognition helps pupils comprehend when, why, where, and how to apply their own knowledge to solve problems successfully (Carr & Jessup, 1995). Positive results for the effect of metacognitive strategy training were achieved by Özsoy and Ataman (2009), showing that these strategies had improved pupils' skills.

In Kosovo, WPs are rarely used in mathematics lessons, and similarly to other countries, pupils face difficulties solving them. Berisha et al. (2013) found that problem-solving strategies are not employed in Kosovo's mathematics textbooks; teachers seldom use problem-solving heuristics and are presented implicitly and indirectly. This was further confirmed by Morina (2022), who showed that despite the new textbooks presented, they still do not give enough opportunities to develop pupils' abilities in the solution of WPs. Considering this context, we devised and implemented a specific teaching strategy that combines the two factors above into what we will call the SIMPLE strategy. Our main aim is to investigate whether the strategy will help pupils solve WPs with understanding. The research questions are as follows:

1. How does pupils' learning of word problem-solving progress through the SIMPLE strategy?
2. How is the SIMPLE strategy related to the pupils' problem-solving success in the post-test?

**The "SIMPLE" Strategy**

The SIMPLE strategy has been developed to support pupils in solving mathematical WPs based on the results of research (Powell & Fuchs, 2018; Teong, 2003; Xin, 2018; Ho & Lowrie, 2014). The name is an acronym from the initial letters of the stages presented in Table 1. Our strategy is considered a "scaffolding" process that enables a pupil to solve a problem, carry out a task, or reach a goal beyond his unassisted efforts (Wood et al., 1976). This scaffolding consists of the teacher "controlling" those elements of the solution that are initially beyond the learner's capacity.

| | |
|---|---|
| Study the problem | Read the problem and paraphrase it.<br>Ask yourself: Have I understood what I am supposed to find? |
| Involve block model | Draw a diagram. Use the block model method. |
| Monitor the process | Ask yourself: Is the block model method helping me reach the solution? Am I getting any closer to my objective? Am I following the right steps, etc? |
| Prepare/Present the solution. | Make the expression/calculation/equation, write the solution, and present it. |
| Look over the solution again. | Check the answer. Consider if the result is correct. |
| Evaluate your answer | Evaluate whether your solution makes sense and if there is a better way to solve the problem. |

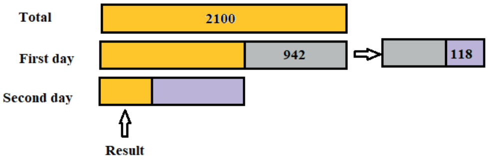
Table 1: The SIMPLE strategy's steps.

Example of a WP solution with the application of the SIMPLE strategy:

**WP1:** From 2100 kilograms of potatoes in the warehouse, 942 kilograms were sold on the first day, while on the second day, 118 kilograms less than on the first day. How many kilograms of potatoes are left in the warehouse?

**S** (Study the problem): I have to find the kilograms of potatoes left in the warehouse after two days which are still unsold.

**I** (Involve the block model method):



**M**(Monitor): Yes, I think I am on the right path and getting closer to the objective since now I know what and how much I must subtract to find the final answer.

**P** (Present the solution) – First day: 2100 – 942 = 1158

Second day: 1158 – (942-118) = 1158 – 824= **334**

**L** (Look over the solution again): 334 + 824 + 942 = 2100

**E** (Evaluate your answer): The result makes sense since we arrived at the warehouse's total kilograms of potatoes by working backwards.

## METHODOLOGY

This study implements design-based research (DBR) (DBSC, 2003). Design-based research is a methodology that helps to increase education research's impact, transfer, and translation into better practice. It underlines how important it is to build theories and design principles that may guide, inform, and improve both practice and research in the field of education (Anderson & Shattuck, 2012). Our study consists of three stages, pre-test, intervention, and post-test, repeated in three cycles. In this study, we present only the results from the first cycle.

The study population included 8th-grade pupils of Kosovo's schools and their mathematics teachers. We chose 8th-grade pupils, particularly since they know introductory algebra and can solve more complex linear equations. We began with Selami Hallaqi's school in the city of Gjilan. There were 25 pupils in this class; however, we only included 20 as participants because 5 of them missed lessons during the intervention phase.

### The experimental design

Design experiments are the primary method for acquiring data in design research methodologies (Cobb et al., 2003). We started with the first design experiment cycle of intervention, and its experiences will be used to inform the following class until we see that the intervention is well-developed and enhanced.

Pre-test. We began with a pre-test purposefully created to assess pupils' baseline. The idea was to examine pupils' knowledge of WPs, identify their strategies, and determine whether their solutions contain any visual representation.

Intervention. *Model method*: After the pre-test, pupils received two lessons using the block model method. To become comfortable with the visual part of the strategy, they practised it in four WPs (e.g., **WP1, the "I" step**) and two as homework.

*Describe*: In this stage, the SIMPLE strategy and its benefits in WP solving are explained in general.

*Model it:* After explaining the SIMPLE strategy, the teacher demonstrated how to execute each step by thinking aloud in one WP (e.g., WP1). In this lesson, pupils practised two WPs with the SIMPLE strategy and solved two others as homework.

*Practice:* In the last lesson of the intervention, pupils were assigned six WPs and were asked to use the SIMPLE strategy in the solution. As pupils were practising, the researcher was allowed to assist them when they were confused.

Post-test. The pupils solved six other WPs in a post-test phase. In this phase, we aimed to evaluate the strategy's effectiveness and its effect on the WP solution.

## DATA ANALYSIS

The data were analysed in both qualitative and quantitative ways.

*Pre- post-test analysis.* We started the analysis by examining pupils' performance in a pre-post-test using the criteria in Table 2.

| Criteria | The incorrect solution, no solution at all | Partial understanding | The correct idea, but slight errors in the calculation | Entirely correct solution |
|---|---|---|---|---|
| | Incorrect solution | | Correct solution | |
| Points | 0 | 1 | 2 | 3 |

Table 2: The criteria for the evaluation of the pre-test.

Next, we investigated whether pupils used visualizations in the problem-solving process.

For the post-test analysis, we applied the same criteria as above, and we also determined whether there was some evidence that the pupil applied the SIMPLE strategy.

The analysis was carried out in the Atlas.ti software. Each written solution was number-coded. First, we analyse the data based on three categories: „Clearly used the SIMPLE strategy, "Partially used the SIMPLE strategy", and "No indication of the usage of the SIMPLE strategy". For the first criterion, we evaluated every solution that clearly used all the strategy's steps. We assessed as the partial application of the strategy every solution that incorporated some elements of the strategy into it. With the third criterion, we assessed any solution where there was no indication, or at least we did not see any element of the strategy applied to it.

Furthermore, we investigated the link between strategy usage and solution success. Six categories emerged (see the arrows in Figure 1 for the categories analyzed).

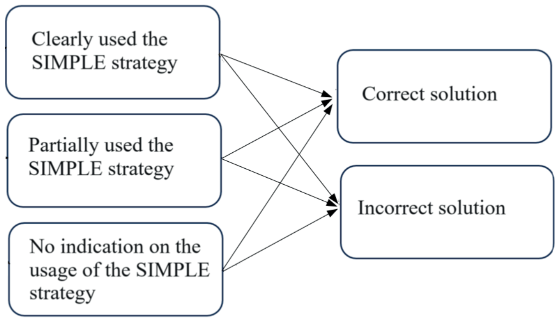



Figure 1: Framework for the usage of the SIMPLE strategy.

## RESULTS

*Pre-test:* The pre-test analysis gave a baseline understanding of the pupils' problem-solving abilities before any intervention. The findings were satisfactory, as 55.8% of the pupils solved the problems correctly. The distribution of scores among the participants is detailed in Table 3. We noticed that none of the pupils did any visualization to solve the WPs. Their primary focus was solving them by direct calculation or converting words into equations using certain keywords.

| Points/WPs | 1 | 2 | 3 | 4 | 5 | 6 | N | % |
|---|---|---|---|---|---|---|---|---|
| Incorrect solutions (0,1) | 10 | 6 | 11 | 3 | 14 | 9 | 53 | 44.2 |
| Correct solution (2,3) | 10 | 14 | 9 | 17 | 6 | 11 | 67 | 55.8 |
| Total | 20 | 20 | 20 | 20 | 20 | 20 | 120 | 100 |

Table 3: The results from the pre-test of the first cycle.

*Post-test*: In analyzing the post-test data, we first classified pupils' solutions based on predefined criteria (Table 2). The results below reveal progress in pupils' problem-solving proficiency. 74.2% of the solutions provided were correct, while 25.8% were deemed incorrect (Table 4). These results were analyzed considering 120 tasks and the 20 pupils participating in all study phases.

| Criteria/WPs | 1 | 2 | 3 | 4 | 5 | 6 | N | % |
|---|---|---|---|---|---|---|---|---|
| Incorrect solution (0,1) | 2 | 5 | 5 | 6 | 7 | 6 | 31 | 25.8 |
| Correct solution (2,3) | 18 | 15 | 15 | 14 | 13 | 14 | 89 | 74.2 |
| Total | 20 | 20 | 20 | 20 | 20 | 20 | 120 | 100 |

Table 4: The results from the post-test of the first cycle

As for using the SIMPLE strategy, 40.8% of pupils clearly used it in post-test WPs, while 35.8 % partially employed it. Conversely, 23.4% of solutions showed no clear indication of strategy usage (Table 5). When we analysed the link between the strategy's use and the solution's correctness, we can say that when pupils clearly used the SIMPLE strategy, they always arrived at the correct solution. There were also times when pupils partially used the strategy, but again, they got the correct answer. Differently, we can see that pupils mostly got incorrect solutions when they did not indicate the strategy's usage. This can be a good indicator that the SIMPLE strategy helps pupils to improve their problem-solving skills (Table 6).

| Criteria/WPs | 1 | 2 | 3 | 4 | 5 | 6 | N | % |
|---|---|---|---|---|---|---|---|---|
| Clearly used the SIMPLE strategy | 12 | 9 | 8 | 9 | 8 | 3 | 49 | 40.8 |
| Partially used the SIMPLE strategy | 5 | 7 | 7 | 7 | 7 | 10 | 43 | 35.8 |
| No indication of the usage of the SIMPLE strategy | 3 | 4 | 5 | 4 | 5 | 7 | 28 | 23.4 |

Table 5: The use of the SIMPLE strategy in WPs of post-test.

| | Incorrect solution | Correct solution | Total | % |
|---|---|---|---|---|
| Clearly used the SIMPLE strategy | 0 | 49 | 49 | 40.8 |
| Partially used the SIMPLE strategy | 13 | 30 | 43 | 35.8 |
| No indication of the usage of the SIMPLE strategy | 18 | 10 | 28 | 23.4 |

Table 6: The correlation between the SIMPLE strategy and the problem-solving success.

Furthermore, we analyzed the presence of the visualization, more specifically, the application of the block model method. The data shows that most pupils applied it within the WP's solution (Table 7).

| Criteria/WPs | 1 | 2 | 3 | 4 | 5 | 6 | Total | % |
|---|---|---|---|---|---|---|---|---|
| Do not visualize the problem | 4 | 5 | 5 | 6 | 5 | 10 | 33 | 27.5 |
| Visualize the problem using the block model approach | 16 | 15 | 15 | 14 | 15 | 10 | 87 | 72.5 |

Table 7: The presence of the visualization within the WP's solution.

These results highlight the importance of applying the SIMPLE strategy to achieve correct solutions. While the strategy proved beneficial for many pupils, other pupils may rely on different problem-solving approaches. Further investigation into these characteristics may be necessary to improve problem-solving proficiency.

## CONCLUSION AND DISCUSSION

The data collected showed that pupils were familiar with WPs and used them in their mathematics lessons. While familiarity with WPs was evident, the initial lack of strategy implementation, particularly in visualization, was notable. This supports the viewpoint arrived at by Berisha et al. (2013) and Morina (2022) that pupils in Kosovo lack familiarity with the visual aspects of problem-solving.

However, through a targeted intervention focusing on the SIMPLE strategy, significant improvement was observed. Pupils demonstrated rapid mastery of the strategy, leading to a noticeable enhancement in their WP solution accuracy and their application of the SIMPLE strategy. When pupils clearly used the SIMPLE strategy, they always arrived at the correct solution. There were also times when pupils partially used the strategy, but again, they got the correct answer. This finding corresponds to the findings that visual representation accuracy and metacognition are important for the success of the solution (Csikos et al., 2011; Hembree, 1992; Carr & Jessup, 1995). Unlike these two categories, we can see that pupils mostly got incorrect solutions when they did not indicate the strategy's usage. This can be a good indicator that the SIMPLE strategy helps pupils improve their problem-solving skills.

First, while the SIMPLE strategy proved beneficial for many pupils, it is important to mention that these findings cannot be generalized for all. The sample is small, limited to one school grade and one intervention cycle. Also, there were times when pupils partially used the strategy's step, leading them to the incorrect solution. We suppose that the limited time of the intervention influenced these solutions.

Therefore, we can conclude that the pupils quickly adapted to the SIMPLE strategy. This adaptation allowed them to improve the solution, resulting in better performance. This fact leads us to believe that implementing this strategy in early mathematics classes can positively impact pupils' outcomes and improve their problem-solving skills.

**Acknowledgement**

The article is supported by the Charles University Grant Agency (GAUK) project No. 166823.

## References

Anderson, T., & Shattuck, J. (2012). Design-Based Research: A Decade of Progress in Education Research? *Educational Researcher, 41*(1), 16–25.

Berisha, V., Thaçi, X., Jashari, H., & Klinaku, S. (2013). Assessment of Mathematics Textbooks Potential in Terms of Student's Motivation and Comprehension. *Journal of Education and Practice, 4*(28), 33–37.

Carr, M., & Jessup, D. L. (1995). Cognitive and metacognitive predictors of mathematics strategy use. *Learning and Individual Differences*, *7*(3), 235– 247.

Csíkos, C., Kelemen, R. & Verschaffel, L. (2011). Fifth-grade students' approaches to and beliefs of mathematics word problem solving: a large sample Hungarian study. *ZDM Mathematics Education, 43*(4), 561–571.

Cummins, D., Kintsch, W., Reusser, K., & Weimer, R. (1988). The role of understanding in solving word problems. *Cognitive Psychology, 20*(4), 405–438.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design Experiments in Educational Research. *Educational Researcher, 32*(1), 9–13.

Daroczy, G., Wolska, M., Meurers, W. D., & Nuerk, H. C. (2015). Word problems: a review of linguistic and numerical factors contributing to their difficulty. *Frontiers in Psychology, 6*(348), 22–34.

De Corte, E., & Verschafel, L. (1987). The effect of semantic structure on first graders' strategies for solving addition and subtraction word problems. *Journal for Research in Mathematics Education, 18*(5), 363–381.

Design-Based Research Collective (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5–8.

Gani, M. A., Tengah, K. A., & Said, H. (2019). Bar Model as Intervention in Solving Word Problem Involving Percentage. *International Journal on Emerging Mathematics Education*, *3*(1), 69–76.

Hembree, R. (1992). Experiments and relational studies in problem-solving: A meta-analysis. *Journal for Research in Mathematics Education*, *23*(3), 242–273.

Ho S. Y., & Lowrie T. (2014). The model method: Students' performance and its effectiveness, *The Journal of Mathematical Behavior. 35*(1), 87–100.

Kusaka, S., Ndihokubwayo, K. (2022). Metacognitive strategies in solving mathematical word problems: a case of Rwandan primary school learners. *SN Social Science 2*(9), 186.

Morina (2022). Word problems in Kosovo's mathematics textbooks for grade 8. *Proceedings of the ERIE International Scientific Conference*. Czech University of Life Sciences Prague.

Özsoy, G., & Ataman, A. (2009). The effect of metacognitive strategy training on mathematical problem solving achievement. *International Electronic Journal of Elementary Education, 1*(2), 68–82.

Perry J., Lundie D., & Golder G. (2019). Metacognition in schools: what does the literature suggest about the effectiveness of teaching metacognition in schools? *Educational Review, 71*(4), 483–500.

Pólya, G. (1957). *How to solve it: A new aspect of mathematical method*. Princeton University Press.

Powell, S. R., & Fuchs, L. S. (2018). Effective Word-Problem Instruction: Using Schemas to Facilitate Mathematical Reasoning. *Teaching Exceptional Children, 51*(1), 31–42.

Reusser, K. (1985). *From situation to equation. On formulation, understanding and solving situation problems*. Technical Report no. 143. Institute of Cognitive Science, University of Colorado

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press.

Teong, S. K. (2003). The effect of metacognitive training on mathematical word-problem solving. *Journal of Computer Assisted Learning, 19*(1), 46–55.

Verschafel, L., & De Corte, E. (1997). Teaching realistic mathematical modelling in the elementary school: A teaching experiment with fifth graders. *Journal for Research in Mathematics Education, 28*(5), 577–601.

Vondrová, N., Novotná, J. & Havlíčková, R. (2019). The influence of situational information on pupils' achievement in additive word problems with several states and transformations. *ZDM Mathematics Education, 51*(1), 183–197.

Wood, D., Bruner, J. S. & Ross, G. (1976). The Role of Tutoring in Problem-Solving. *Journal of Child Psychology and Psychiatry, 17*(2), 89–100.

Xin, Y. P. (2019). The effect of a conceptual model-based approach on 'additive' word problem solving of elementary students struggling in mathematics. *ZDM Mathematics Education, 51*(1), 139–150.

# HOW TO DEAL WITH MISCONCEPTIONS - CAN MATHEMATICAL DIALOGUE BE HELPFUL?

Vida Manfreda Kolar

University of Ljubljana, Faculty of Education, Slovenia

*We present the reasons for students' cognitive barriers when dealing with more complex mathematical topics, focusing on the dual nature of some abstract mathematical concepts that require judgements about which perspective to use in a given situation. We focus on mathematical dialogue, which we see as an approach that provides insight into students' misconceptions while raising awareness of one's own thought process in solving a mathematical problem. Using the method of mathematical dialogue with secondary school students, we investigated the metacognitive processes present in the student who was instructed and the student who led the dialogue. The analysis of the dialogues confirmed that the approach can be considered promising in the light of overcoming misconceptions about mathematical concepts and raising awareness of the students' own metacognitive processes.*

## INTRODUCTION

When looking at an individual's difficulty in understanding mathematical concepts, we need to be aware of a wider range of possible reasons. These may be genetic and psychological, didactic or of an epistemological nature (Cornu, 1991; Brousseau, 2002; Sidik, Suryadi & Turmudi, 2021). Genetic and psychological barriers are due to the personal development of the learner, didactic barriers are due to the teacher's instructions and teaching style, and epistemological barriers occur when the cause of the difficulties is not related to the objects, but to the nature of mathematical subjects, which, due to their specificity and abstractness, can be difficult for learners to understand. Understanding a concept involves much more than just knowing the definition. Only when we are able to identify examples and counterexamples of a concept, when we are aware of how it is related to other concepts that we have known before, and when we understand its position within a given theory as well as its application, can we say that we understand the concept (Sierpinska, 1992). A systematic building up of such a hierarchy of concepts and ideas needs to be understood and woven together in order for concepts to build on one another (Ashlock, 2002; Sarawadi & Shahrill, 2014). If the old way of looking at a concept prevents us from moving to the new, then we can speak of epistemological barriers in the development of the concept or a hierarchy of concepts. This occurs when knowledge that works well in one field of activity no longer works satisfactorily in another context and creates contradictions.

In this paper we will focus on a group of epistemological difficulties rooted in students' prior knowledge and the conflict that arises due to the gradual modification of the way they perceive some abstract mathematical concepts during their schooling. Sfard (1991) refers to the dual nature of these concepts, which means that they can be conceptualised in two fundamentally different ways: structurally, as objects, and operationally, as processes. The structural conception of a concept means seeing a mathematical entity as an object: we recognise the idea 'at a glance' and manipulate it as a whole without going into detail; the operational conception, on the other hand, involves treating the concept as a sequence of steps, paying attention to details. Gray and Tall (1994) see the duality between process and object in mathematics in that a single symbol is often used to represent both the process and the object. The authors introduce the term procept, which consists of three elements: a process that generates a mathematical object (or concept), and a symbol that represents either the process or the object. The development of abstract mathematical concepts has been the subject of much debate and theories over the last 40 years. According to Sfard mathematical concepts follow a progression from the operational to the structural conception (Sfard, 1991; Sfard 1994, Sajka, 2003), which means that the concept as a process is the starting point for the formation of the concept as an object. Under APOS theory the development of a concept also follows a progression from process to object. As an individual repeats and reflects on an action it may be interiorized into a mental process without having to execute each step explicitly. If one becomes aware of a process as a totality and realizes that transformations can act on the totality, then we say the individual has encapsulated the process into a cognitive object (Dubinsky et al, 2005). In the context of recent theories, we highlight the theory of objectification (Radford, 2011, 2018, 2021) and the ontosemiotic approach (Godino et al., 2007, 1019; Font et al, 2013). According to Radford (2021, p. 88) algebraic knowledge as such cannot be shown in itself. In order for knowledge to be something that human consciousness can perceive or feel, individuals must perform a teaching–learning activity in order for knowledge to occur. The activity itself is a process: one that materialises knowledge into something intelegabile – knowing, which is the object. A slightly different view of the evolution of abstract concepts is presented by the ontosemiotic approach (Font et al, 2013). Every entity that intervenes in a mathematical activity is considered an object, and to operationalize this broad notion of object, some categories of objects are proposed (conceptual, propositional, procedural, argumentative, linguistic, situational, etc.) (Vegel et al., 2023). It focuses on the role of mathematical objects as fundamental building blocks for understanidng mathematical concepts, which then inform actions and the formation of schemas.

Both Radford and Sfard emphasise the social and cultural aspects of learning: the importance of participation in mathematical discourse and social interaction in the formation of mathematical concepts. Accoring to Radford (2021, p. 92)

> a process of objectification occurs when students and teachers, through their joint labour, materialise the knowledge— transform it into something susceptible to be an object of consciousness—and the students start noticing or becoming conscious of it through such materialisation.

Unfortunately, practices often show that teachers prefer to choose tasks and activities that do not encourage students to engage in mathematical discussion. They do not use mistakes to promote inquiry, analysis, or learning (Schleppenbach et al., 2007; Tulis, 2013). Engaging students in fixing errors without substantial analysis of different types of errors and the conceptual (mis)understandings behind them can limit students' opportunities to learn from their errors and prevent students from developing a sense of themselves as powerful mathematical thinkers (Alvidrez et al, 2022). Hansen (2017) emphasises that the first step is to change the view of the role of learner errors, which are not a failure of teaching and learning, but part of the learning process, enabling progress in understanding. The learning environment in which mathematics teaching and learning takes place needs to be supportive for the learner in the sense that it allows for interaction between participants and for mathematical dialogue.

According to Ryan and Williams (2007), mathematical dialogue should involve a cycle of articulation, reformulation, reflection and resolution: students should have the opportunity to communicate and exchange their opinions.: *I think that because ...* (articulation); *I listened to what X said and now I think that ...* (reformulation). There should be criteria for assessing what is a good mathematical argument. In this phase, the teacher should guide the students by asking questions such as: *Is this a good argument? How can you prove that this is a correct argument? Is there a more convincing argument? Which was the best argument?* Then students should be given the opportunity to reflect on the discussion: *What we thought, what we think now, what made us change our minds?* After discussing the different points of view and reflecting on the debate, a resolution phase should follow*: Now I think..., because...*

It can be observed that metacognition plays an important role in the process of conducting mathematical dialogue and resolving students' misconceptions, as it guides us in the selection, evaluation of cognitive tasks, correction of errors, selection of appropriate goals and strategies, assessment of one's own abilities in relation to the task (Bakračevič Vukman, 2000). Hattie (2009) conducted a meta-analysis of research examining the effect of different factors on students' academic achievement. He identified the development of metacognitive skills as one of the most important factors for improving students' achievement. One of the main purposes of metacognition is for students to learn how to understand their own thought processes and to use this knowledge to improve their learning and understanding (Dunlosky & Metcalfe, 2009). Schoenefeld (1987) stresses the importance of creating an environment that allows for interaction and confrontation of viewpoints, perspectives, ways of thinking. He defines self-regulation as one of the main broad areas covered by the term metacognition. According to Schoenefeld, monitoring, assessing progress and acting on progress assessments are key elements of self-regulation. Lester et al. (1989) conducted a large-scale study to investigate the role of metacognition in mathematical problem solving. They found out that there is a dynamic interaction between mathematical concepts and the metacognitive processes used to solve problems with these concepts. This means that control processes and awareness of cognitive processes develop in parallel with the understanding of mathematical concepts. Teaching problem solving, and in particular teaching metacognition, is likely to be most effective if it is carried out in a systematically organised way under the guidance of a teacher.

Our research primarily aims to explore the role of mathematical dialogue in identifying learners' understanding of mathematical concepts and the importance of the quality of mathematical dialogue in triggering metacognitive processes in participants. The topic of understanding mathematical concepts will be explored in the field of secondary school mathematics, which, due to the complexity and abstractness of some concepts, poses a challenge both to students who struggle with misunderstanding them, and to teachers who are struggling to find ways of teaching that would allow for a more permanent and better quality of students' mathematical knowledge.

## EMPIRICAL PART

### Methodology and problem definition

The empirical part is based on a descriptive, non-experimental method of pedagogical research (Hartas 2010), the approach is qualitative. We focus on the importance of constructive mathematical dialogue as one of the ways that we see as having the potential to contribute to a higher quality of mathematical knowledge for individuals and to removing epistemological barriers in the individual's understanding of the concept.

### The research aims and research questions

The aim of the research is to investigate the role of mathematical dialogue from different perspectives: from the perspective of identifying misconceptions and from the perspective of deepening students' metacognitive processes.

The following research questions are posed:

1. Does the quality of the mathematical dialogue contribute to the identification of students' misconceptions?

2. To what extent does the mathematical dialogue stimulate metacognitive processes in the dialogue participants?

**Survey sample**

The study was conducted at the Faculty of Education, University of Ljubljana, Slovenia in the academic years 2022/23 and 2023/24. The data for the study were collected in the course Mathematical topics with didactics at the 2nd level of the university study programme Teaching - Subject Teaching. It encompassed 21 research reports, written by 42 students studying to become mathematics teachers. Students worked in pairs to prepare and carry out a research assignment related to a selected mathematical concept from a secondary school mathematics content. To make it easier to distinguish between a mathematics student and a secondary school student, in the rest of this paper we will use the terms student for a mathematics student and pupil for a secondary school student.

**Measuring instruments and data collection procedure**

The two students, working in pairs, went through the following steps of the research project:

1. Study of scientific papers on pupils' understanding of the chosen concept and related misconceptions
2. Creation of a measurement instrument - tasks to test pupils' misconceptions
3. Justification - the rationale for the choice of the task: why they chose it, what they predict will happen, where they expect the pupils' difficulties to be identified and why...
4. Critical examination of the measuring instrument
5. Conduct a mathematical dialogue with the pupils (transcript of the dialogue)
6. Reflection by the student on the dialogues carried out, referring to the theoretical points of misconceptions in the chosen mathematical concept

In the first three steps of the research project, the two students collaborated and jointly developed a measuring instrument for the mathematical dialogue, which typically included 3 to 5 different tasks. These were sequenced in a meaningful order, each with a clearly defined objective which the student had to state in the instrument description. Step 4 refers to the exchange of measuring instruments between pairs of students who critically analysed each other's instrumentation. The last two steps were carried out by each of the students on their own, each with his/her pupil. A pair of students produced a joint research report at the end of the research project.

The purpose of the present research is to analyse students' research reports. Selected examples of mathematical dialogues are analysed from different perspectives:

1. Analysis of the quality of the mathematical dialogue and the chosen task:
   i. At what level was the dialogue guided (narrowing or guiding questions (Hattie et al., 2017)?
   ii. Did the task achieve its purpose of revealing and correcting misconceptions?
2. The presence of metacognitive elements in the student: self-reflection after the dialogue: did I guide the pupils well, did I take advantage of the right moments in the conversation, misconceptions?
3. Presence of metacognitive elements in the pupil: how did I understand the problem at the beginning, how do I understand it at the end, did anything change in between, what convinced me…?

## RESULTS AND DISCUSSION

In this section we present a more detailed analysis of two selected mathematical dialogues, the first concerns the topic of absolute value, and the second concerns combinatorial situations. Each dialogue is analysed according to the criteria defined above. We have selected an example of a dialogue where the weakness of the student's guidance and the poorer awareness of their own thought processes in both participants is shown, and an example of qualitative guidance with a higher degree of self-regulatory processes present.

### Mathematical topic: absolute value

The analysis of the first example - understanding absolute value in inequalities - is presented in the following order: first, we introduce the task, its objectives and students' expectations (see Table 1), followed by the transcription of the dialogue and then its analysis in terms of the quality of the dialogue and the students' and pupils' metacognitive processes.

| | |
|---|---|
| **Participants** | Student (S) and pupil (D). |
| **Task** | Solve the equation $\lvert x - 5 \rvert \leq 5$. |
| **Objective of the task** | The task tests the level of understanding of the concept of absolute value:<br>– Operational: absolute value as a process of manipulating an algebraic expression<br>– Structural: absolute value as distance/absolute value as a function |
| **Expectations** | The student expects that the problem will encourage the pupil to solve algebraically, using the calculation procedures that |

| | |
|---|---|
| | follow from the definition of absolute value. However, the pupil is not expected to come up with a graphical solution to the inequality on his/her own, where the notation would be understood as a comparison of two functions. |

Table 1: Absolute value - task, objectives and expectations.

*Dialogue:*

1 D: (reads) Solve the equation $|x - 5| \leq 5$. So this means that... (D draws a number line, labels zero and units from $-5$ to 5. Since it is less than or equal to 5 we can get to 9.

2 S: How did you work out that all numbers up to 9 are appropriate?

3 D: Oh no, we can go up to 10 because $|10 - 5| = 5$. On the other hand, we can go down to $-5$. But no, it can't be greater than $-1$ because $|-1 - 5| = |-6| = 6 \geq 5$.

4 S: What about the case $x = -0.5$?

5 D: Am, yes it still is … so the bound is 0. So $x$ is greater than or equal to zero and less than or equal to 10.

6 S: Yes, that's right. Now I wonder if you would be able to solve this equation graphically?

7 D: Am... I don't know exactly how to go about it.

8 S: What if we imagine that the left-hand side of the equation represents a linear function and the same for the right-hand side, would you be able to plot these two functions in a coordinate system?

9 D: Yes, for this example (shows the right hand side of the equation, $y = 5$) we have a line passing through 5 (draws a line). But for $y = |x - 5|$ (thinks for a moment, then continues), if I draw a line $y = x - 5$ (thinks for a moment) and insert 0 I get $-5$, that is here (draws the point $(0, -5)$), if $x$ =1 we get $-4$ (draws the point (1,$-4$), so for the next one it will be like this (draws the points $(2, -3), (3, -2), (4, -1), (5,0)$ correctly, then draws a line through the points drawn).

10 S: Great. We have a function in absolute value, so what will the graph be?

11 D: If it's in absolute value, then that's how it will be mapped (the student uses his hand to map the part below the abscise axis above the abscise axis).

12 S: That's right.

13 D: (draws the mapped part of the line)

14 S: So what is the solution if we look at the two drawn lines?

15 D: Hm, since it must be less than or equal to 5, then the solution is this triangle.

16 S: Do the edges of this triangle belong to the solution?

17 D: Yes, because we have less than or equal to.

18 S: Do the values you determined earlier on the number line match the resulting graphic?

19 D: Yes, because these vertices of the triangle are between 0 and 10.

*Analysis of the quality of the dialogue:*

In examining the quality of the dialogue presented, we will refer to Herbel-Eisenmann and Breyfogle (2005) who distinguish between two patterns of teacher-student interaction: questions that narrow the conversation (funneling questions) and questions that guide the conversation (focusing questions). The key difference between the two is who does the cognitive work in the learning process: in the case of narrowing questions it is the teacher, in the case of guiding questions it is the learner. Narrowing questions limit learners' answers to short answers, guiding questions, on the other hand, support learners in finding their own ways to solve a problem.

In the dialogue described above, we can recognise the presence of both types of questions. At the beginning of the dialogue, there is an open-ended question to gain insight into the pupil's way of thinking: "How did you work out that all numbers up to 9 are appropriate?" (line 2), but later the dialogue switches to the use of narrowing questions: "What about the case $x = - 0.5$?" (line 4), where the student himself points out the value of the variable $x$ at which the pupil will notice his mistake, or e.g. "What if we imagine that the left-hand side of the equation represents a linear function and the same for the right-hand side, would we be able to plot these two functions in a coordinate system?" (line 8), where the pupil is guided to a new strategy of solving by graphing functions: the student has a clear goal of guiding the conversation and, by asking questions, leads the pupil to solve the problem by graphing. However, we may notice that the new method of solving triggers a new misconception that would be worth exploring further, but which the student fails to exploit: where on the graph can we see the solution to the inequality? It seems that the pupil sees the solution in the area defined by the triangle and not in the interval on the abscissa axis (line 15).

*Analysis of metacognitive processes in a pupil:*

The pupil follows the student's guidance to a new solution procedure by drawing a graph of the two functions in the inequality notation, but there is no evidence that the pupil is aware that the two procedures are equivalent in determining the solutions. This can be inferred from the end of the dialogue when the student points out that in both cases the same solution to the inequality was obtained (line 18).

The representation in the form of a number axis, which the learner initially drew, suggests that he associates the concept of absolute value with distance and that he will use the drawn number axis to help him. However, we notice later that this is not the case, because instead of finding the solution to the equation by identifying the numbers that are less than or equal to 5 from $x = 5$, the pupil proceeds to use a computational procedure - inserting concrete numbers into the equation and checking whether or not the statement is true or false. It can be concluded that the pupil's understanding of the concept of absolute value is at the level of operational understanding. The same applies to the understanding of the concept of function, which is reflected in the way the graph of a linear function is drawn (inserting values and drawing points on the graph) (lines 9 and 10). While the task could act as a trigger to reflect on one's own way of solving and to see the equivalence of three different solution methods, this is only on condition that the pupil has reached an appropriate level of understanding of the concept, i.e. understanding the notation of an absolute value inequality in a structural way - as a function. Without this insight, even metacognitive processes are reduced to a basic level of checking one's own computational procedures, which can be observed during the actual process of computing concrete values (line 3). Findings from other studies (Almog & Ilany, 2012) also confirm that students prefer an analytical approach, based on algebraic manipulation of symbols, to a geometric interpretation - seeing mathematical notation as a relation between two functions - when solving inequalities with absolute values.

*Analysis of metacognitive processes in the student:*

Based on the student's research report, it can be summarised that the student's reflection on the dialogue with the pupil focused mainly on the analysis of the reasons for the pupil's difficulties. There is no reflection on whether he guided the pupil well by asking questions or whether he identified critical moments in the conversation. The student is aware of the importance of presenting different ways of solving such problems, which should allow the student to move flexibly between different ways of solving them.

**Mathematical topic: combinatorics**

As already noted, the second dialogue concerns the understanding of combinatorial situations. The description of the example follows the same sequence of steps as the previous example: The task is presented in Table 2.

| **Participants** | Student (S) and pupil (D). | |
|---|---|---|
| **Task** | Task 3:<br><br>Matic has four cars of different colours: black, orange, white and grey. He has decided to distribute them among his friends Sara, Nejc and Neza. How many different ways can he divide the cars? An example: Sara can get | Task 5:<br><br>Jure, Katja, Luka, Marta and Nika will stay overnight at their grandmother's house. She has two different rooms (green and yellow) where she could put all or some of the grandchildren. How many different ways can the grandmother put her grandchildren in two different rooms? For example, a grandmother could use |

| | | |
|---|---|---|
| | all three cars. | just one room to accommodate all five grandchildren… |
| **Objective of the task** | To examine the strategies that pupils will use when solving problems of variations by repetition: whether they will rely on formula, or whether they will start from the context of the problem and move away from the use of formulas | |
| **Expectations** | Errors may occur in understanding the verb 'to distribute', as the pupil might understand that he has to give at least one car to everyone. If the pupil relies on formulas, it may also happen that he/she confuses base and power (write $4^3$ instead of $3^4$).<br><br>Two tasks of the same type were chosen in order to see if the pupil will recognise the similarity between the two tasks and therefore might use the same strategy to solve them. | |

Table 2: Combinatorics - task, objectives and expectations.

*Dialogue:*

1 D: Is it easier to draw lines for cars or for friends? Is it easier with a calculation?

2 S: You have to think about what makes sense.

3 D: But do lines help at all?

4 S: They do help, but you have to think about what they are for. They are for one thing, but not for another.

5 D: If I do 4 lines for cars. And who can get a black one? Three can get it. Who can get the orange one? (draws 4 lines and writes the numbers 3 on them)

6 S: Okay, for cars it makes sense to you.

7 D: And then here comes $3^4$. If I give the lines to my friends. e.g. How many cars can Sara get? Nejc can also get 4 and Neza can get 4.

8 S: Does the number of cars Sara gets have any effect on the number of cars Nejc gets?

9 D: Yes. If Sara gets them, Nejc doesn't.

10 S: What if the lines are cars? Do we have any interdependencies there too?

11 D: So three people can get the black car. 3, 3, 3 (as she wrote earlier).

12 S: Is there any dependency between these threes? For example, if the black car goes to Neza, does that have any effect on who the orange car goes to?

13 D: No. So it's 3 again.

14 S: Yes, now you just have to work out whether there are pluses or minuses between the 3s.

…

15 D: (reads the text of the task) U, this task seems to me to be similar to task 3. Here we have 5 people and 2 rooms. So we have 5 lines belonging to one person each. Jaka can choose between the green or the yellow

room, so he has 2 choices. The same goes for Katja and everyone else.

16 S: Are these choices related to each other?

17 D: No, which means that there are times and no plus between them.

18 S: Yes, it is.

19 D: So $2^5$, which is 32.

20 S: Are you sure that's OK?

*Analysis of the quality of the dialogue:*

The student showed very good expertise in managing the dialogue. The questions are open-ended and guide the pupil to make his own decisions about whether to draw lines for cars or for people (line 2), nor does he give an explanation of why the lines are useful, because he wants the pupil to come to his own understanding (line 4). The student also asks conceptual type of questions, which encourage the pupil to think about whether it makes more sense to find out how many people can choose a car of a certain colour, or vice versa: how many different cars can one person choose (lines 8 and 10). When solving task 5 (lines 15 to 20), the student does not actually have much cognitive work to do, as the pupil recognises the similarity of the two problems and transfers the strategy already used to the new situation. We can conclude that the management of the conversation is based on deepening understanding, not on finding the appropriate formulas.

*Analysis of metacognitive processes in the pupil:*

The pupil questions the usefulness of the line-drawing procedure (line 3), which is an indication of self-regulation, and we also notice that thinking aloud and articulating the procedure helps her to judge which of the two possible approaches is appropriate (lines 5 and 7). In the second part of the dialogue (solving task 5), the pupil immediately shows an understanding of the problem: she recognises that it is the same type of problem-situation as in Task 3, although the context has changed. To summarise, the pupil has transferred the knowledge used in one example, to a new, related example. Again, the degree of metacognitive processes perceived in the dialogue is related to the degree of understanding of the problem and the development of mathematical thinking, which is in line with the findings of Lester et al. (1989), who state that control processes and awareness of cognitive processes develop in parallel with the understanding of mathematical concepts.

*Analysis of metacognitive processes in the student:*

The student's report on his own perspective on how he saw the conversation was interesting. After the discussion was over, he was satisfied with his guiding and questioning, but his opinion changed drastically after listening to the recordings of the dialogue and realising that he had actually helped the student much more than he had thought he had. Listening to his own recording provided a tool that stimulated his awareness of his own role in guiding the dialogue with the pupil.

## CONCLUSION

The difficulties learners have in understanding abstract mathematical concepts and the search for ways to overcome or at least alleviate these difficulties are the source of countless debates among didactic mathematics experts. As this is an epistemological problem stemming from the complexity and abstractness of the concept, the solution to the problem is not necessarily linked to the teaching method, but to the development of the individual's mental schema. We are aware that we cannot bypass the individual's current stage of development, but we can, through appropriate pedagogical approaches, help the individual to move more quickly between the stages of understanding a concept and to reach a level of objectification of the concept. Thompson et al. (2014) point out that understanding the concept is the cognitive state of an individual, which represents the balance of all acquired knowledge that is created in the process of knowledge assimilation. A thoughtfully guided mathematical dialogue that leads the learner to carry out his or her own mental inferences can contribute to deepening the connections between different representations of a concept, which is crucial for concept reification. In this paper, we analysed in more detail two dialogues between a mathematics student and a secondary school pupil. The two cases differ, among other things, in the level of the pupil's understanding of the abstract concept. While the pupil in the absolute value case reaches the level of operational thinking, the pupil in the second example shows a higher, structural level of understanding, which also allows her to reflect more easily on her thought process and to make connections between different contexts.

Analysis of the research reports has highlighted another important fact. Flexibility in the way the dialogue is conducted depends on the student's mathematical competence and on his/her being prepared for the dialogue in advance (i.e. having prepared possible actions in advance, having thought about how to respond to a particular situation). Some students reported that it was only during the interview itself that they realised they were not well prepared and did not know what to ask or how to ask the question - they were not happy with the way they asked, but could not think of anything else.

In summary, the quality of the dialogue depends on the interplay of different stakeholders: the level of the learner's reasoning and reflection, the selection of a good, productive task that will stimulate cognitive conflict and, above all, the teacher's competence to conduct a quality dialogue which is based on professional and didactic preparation of the teacher.

**Acknowledgment**

The author acknowledges the financial support of the Slovenian Research Agency under the project Developing the Twenty-first-century Skills Needed for Sustainable

Development and Quality Education in the Era of Rapid Technology-Enhanced Changes in the Economic, Social and Natural Environment (Grant no. J5-4573) funded by the Slovenian Research Agency.

## References

Almog, N., & Ilany, B.-S. (2012). Absolute value inequalities: high school students' solutions and misconceptions. *Educational Studies in Mathematics, 81*(3), 347–364.

Alvidrez, M., Louie, N., & Tchoshanov, M. (2024). From mistakes, we learn? Mathematics teachers' epistemological and positional framing of mistakes. *Journal of Mathematics Teacher Education, 27*(1), 111–136.

Ashlock, R. B. (2002). *Error patterns in computation: Using error patterns to improve instruction*. Pearson Education, Inc.

Bakračevič Vukman, K. (2000). *Razvoj mišljenja v odrasli dobi: kognitivni, sociokognitivni in metakognitivni aspekti* [The development of thinking in adulthood: cognitive, sociocognitive and metacognitive aspects]. Faculty of Education.

Brousseau, G. (2002). Th*eory of Didactical Situation in Mathematics*. Kluwer Academic Publisher.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153 – 167). Kluwer Academic Publishers.

Dubinsky, E., Weller, K., McDonald, M. A., & Brown, A. (2005). Some historical issues and paradoxes regarding the concpet of infinity: An Apos-based analysis: Part 1. *Educational Studies in Mathematics, 58*(3), 335–359.

Dunlosky, J., & Metcalfe, J. (2009). *Metacognition.* Sage.

Font, V., Godino, J. D., & Gallardo, J. (2013). The emergence of objects from mathematical practices. *Educational Studies in Mathematics, 82*(1), 97–124.

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM International Journal of Mathematics Education, 39*(1), 127–135.

Godino, J. D., Batanero, C., & Font, V. (2019). The onto-semiotic approach: implications for the prescriptive character of didactics. *For the Learning of Mathematics, 39*(1), 37–42.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25*(2), 116–140.

Hansen, A. (2017). *Children's errors in mathematics*. Sage –Learning matters.

Hartas, D. (2010). *Educational research and inquiry, qualitative and quantitative approaches*. Continuum International Publishing Group.

Hattie, J. A. (2009). *Visible learning: A synthesis of meta-analyses relating to achievement.* Routledge.

Hattie, J. A., Fisher, D., & Frey, N. (2017). *Visible learning for mathematics: grades K-12: what works best to optimize student learning*. Corwin Mathematics.

Herbel-Eisenmann, B. A., & Breyfogle, M. L. (2005). *Questioning our patterns of questioning. Mathematics Teaching in the Middle School, 10*(9)*,* 484–489.

Lester, F., Garofalo, J, & Kril, D. (1989). *The role of metacognition in mathematical problem solving: A study of two grade seven classes*. Final report to the National Science Fundation of NSF project MDR.

Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early Algebraization. A global dialogue from multiple perspective*s (pp. 303–322). Springer-Verlag.

Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5-to 12- year-olds: The global evolution of an emerging field of research and practice*. Springer.

Radford, L. (2021). *The Theory of Objectification: A Vygotskian Perspective on Knowing and Becoming in Mathematics Teaching and Learning.* Brill.

Ryan, J., & Williams, J (2007). *Children's mathematics 4-15: learning from errors and misconceprions.* Open University press.

Sajka, M. (2003). A secondary school student's understanding of the concept of function - A case study. *Educational Studies in Mathematics, 53*(3), 229–254.

Sarwadi, H. R., & Shahrill, M. (2014). Understanding students' mathematical errors and misconceptions: the case of year 11 repeating students. *Mathematics Education Trends and Research, 2014*, 1–10.

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and U.S. mathematics classrooms. *The Elementary School Journal, 108*(2), 131–147.

Schoenfeld, A. H. (2016). Learning to think mathematically: problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education, 196*(2), 1–38.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on the processes and objects as differents sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification - The case of algebra. *Educational Studies in Mathematics, 26*(2), 191–228.

Sidik, G., Suryadi, D., & Turmudi, T. (2021). Learning obstacle on addition and subtraction of primary school students: Analysis of algebraic thinking. *Education Research International,* 1–10.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The concept of function – Aspects of epistemology and pedagoogy* (pp. 25 – 58). Mathematical Association of America.

Sierpinska, A., Bobos, G., & Pruncut, A. (2011). Teaching absolute value inequalities to mature students. *Educational Studies in Mathematics, 78*(3)*,* 275–305.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 1–24). University of Wyoming.

Tulis, M. (2013). Error management behavior in classrooms: Teachers' responses to student mistakes. *Teaching and Teacher Education, 33*, 56–68.

*Vergel, R., Godino, J. D., Font, V., & Pantano, O. L. (2023). Comparing the views of the theory of objectification and the onto-semiotic approach on the school algebra nature and learning. Mathematics Education Researh Journal, 35(3), 475–496.*

Vidakovic, D. (1996). Learning the concept of inverse function. *Journal of Computers in Mathematics and Science Teaching, 15*(3), 295–318.

# MATHEMATICAL LITERACY: THE CONCEPT AND ANALYSIS OF STUDENTS' PERFORMANCE IN PRIMARY SCHOOL

Tatjana Hodnik

University of Ljubljana Faculty of Education, Slovenia

*This paper introduces the concept of mathematical literacy as a starting point for developing mathematical literacy competences at the national level, from kindergarten to secondary school. We have made an important addition to the otherwise fairly well-known concept of mathematical literacy (e.g., in the context of PISA international mathematical literacy assessment) within the NA-MA POTI project by defining in detail two cornerstones of mathematical literacy: 1) mathematical thinking, the understanding and application of mathematical concepts, procedures and strategies, and communication as the basis of mathematical literacy, and 2) problem-solving in a variety of contexts (e.g., personal, social, professional and scientific) that allow for mathematical treatment. The latter also highlights mathematical modelling, which is generally about explaining observations of the material world in conceptual (mathematically structured) language. The concept of mathematical literacy was the starting point for the design of mathematical literacy tasks that were used to measure mathematical literacy competences in Slovenian primary schools in 2020 and 2021. We present the results of students' achievements according to the cornerstones and learning outcomes of mathematical literacy. A total of 30 primary schools (10 schools in each of the three educational periods) or 1,380 students were included in our study. The students' results in a knowledge test, which were analysed qualitatively, showed a statistically significant correlation between mathematical literacy and conceptual mathematical knowledge.*

## INTRODUCTION

Achieving competences in mathematical literacy enables individuals to respond to the challenges of the modern world, especially in areas where mathematics is involved or where there is a need to deal with a situation mathematically. This means that school mathematics is taking on new dimensions, or that the understanding of mathematical knowledge is changing to some extent (perhaps even more than we would like or can yet manage). Goos and Kaya (2020) note that, in a large number of countries, the concept of mathematics is being reconsidered as part of curricular reform in terms of changes to the selection and organisation of mathematical content. It is quite clear that the complexity of the changes we are witnessing and the associated rapid growth of technology mean that there are many definitions of mathematical literacy competences.

In our view, the OECD definitions of mathematical literacy within the framework of the PISA survey from 2003 to 2017 (OECD, 2003; OECD, 2017) best correlate with the current understanding of mathematical literacy. They define mathematical literacy as the activity of an individual who is able to formulate, use and interpret mathematical content in different contexts. This definition makes it clear that it is not only about the individual's recognition and understanding of the role of mathematics in everyday life, but also about their ability to interpret and articulate mathematical content in more complex contexts.

Niss and Hojgaard (2019) define mathematical literacy as the individual's insightful mathematical performance and response to the challenges of a given situation. It is important to note that the situations do not have to be mathematical. Suciati et al. (2020) define a mathematically literate person as someone who is sensitive to identifying mathematical concepts that are inherent to situations that are not mathematical in the starting point. A mathematically literate person thus understands, analyses, interprets, evaluates and synthesises the data of a problem situation, builds a mathematical model and determines a solution, while effectively managing mathematical concepts.

According to Stacey and Turner (2015), mathematical literacy is the individual's ability to formulate, apply and interpret mathematical concepts in a variety of contexts, including mathematical reasoning and using mathematical procedures, facts and tools to describe, explain and predict phenomena, which helps the individual to respond to the challenges of the world and to reflect on his or her choices. Suciati et al. (2020) add that mathematical literacy can be seen as the individual's mastery of reasoning, concepts, facts and mathematical tools and strategies in solving everyday problems. The problems that are investigated within mathematical literacy are so-called life problems, which require real-world data and mathematical modelling (Kula Unver et al., 2018; Manferda Kolar & Hodnik, 2021). Such definitions of mathematical literacy can be a good starting point for developing a concept of mathematical literacy in the context of a particular education system. For us, they provided a basis for conceptualising mathematical literacy in the project *Scientific and Mathematical Literacy: Promoting Critical Thinking and Problem Solving* (the NA-MA POTI project), which is presented in detail below.

**THE NA-MA POTI PROJECT**

The NA-MA POTI project was implemented in Slovenia from 2016 to 2022, with the primary aim of developing mathematical literacy competences at the national level, from kindergarten to secondary school (NA-MA POTI - Zavod RS za šolstvo (zrss.si)). In the NA-MA POTI project, we defined mathematical literacy and elaborated this concept into cornerstones, sub-cornerstones and descriptors, so that it can be applied as effectively as possible in the classroom.

This elaboration was developed for the whole vertical: kindergarten, primary school and secondary school.[1]

The main objective of the NA-MA POTI project was to develop mathematical literacy in students and to train teachers to reflect on mathematical literacy and integrate it appropriately into the learning process. The first phase of the project aimed at defining mathematical literacy. It is not possible to simply copy the definition and the related competences of mathematical literacy from other projects. We advocate the view that the formulation of such important concepts, which may have long-term implications for the Slovenian education system, should actively involve researchers in the relevant research field and practitioners who have a reflective and therefore qualitative knowledge of the functioning of the Slovenian education system and its development, as well as of the problems of education, and who are familiar with the documents that define the education system. In the NA-MA POTI project, we followed this approach to the greatest extent in defining mathematical literacy, which we defined as the ability of an individual to use mathematical thinking and mathematical knowledge in order to:

- apply mathematical concepts, procedures and tools in different structured environments;
- analyse, justify and effectively communicate their ideas and results in formulating, solving and interpreting mathematical problems in different structured environments;
- perceive and be aware of the role of mathematics in everyday and professional life, relating it to other areas and making responsible decisions based on mathematical knowledge, and have a willingness to accept and co-create new mathematical insights.

Starting from the definition of mathematical literacy, we identified two fundamental cornerstones (CS) of the concept:

CS 1 *Mathematical thinking, the understanding and application of mathematical concepts, procedures and strategies, and communication as a basis for mathematical literacy*; and

CS 2 *Solving problems in a variety of contexts (personal, social, professional and scientific) that allow mathematical treatment*.

Both of the CSs of mathematical literacy were elaborated into learning outcomes (LOs). CS 1 was elaborated into seven LOs and the CS 2 into three LOs.

[1] This article presents selected results of the NA-MA POTI project exploring mathematical literacy in kindergarten, primary school and upper secondary school, in which researchers Tatjana Hodnik, Zlatan Magajna and Vida Manfreda Kolar (University of Ljubljana, Faculty of Education) conceptualised the research project, developed the tasks and interpreted the results. The project produced ongoing annual reports and the results were also published in Slovenian (Magajna et al., 2022).

Under CS 1, students: LO 1.1 understand information containing mathematical content; LO 1.2 know and use mathematical discourse and symbology; LO 1.3 present, justify and evaluate their own thought processes; LO 1.4 recognise, understand and apply mathematical concepts in a variety of contexts; LO 1.5 know and apply the relevant procedures and tools in different contexts; LO 1.6 predict and evaluate results, and justify claims, procedures and decisions; LO 1.7 apply different strategies to solve mathematical problems.

Under CS 2, students: LO 2.1 address a wide variety of life problems (problems that do not require mathematical modelling); LO 2.2 handle situations with mathematical modelling; LO 2.3 understand mathematical practices in different contexts.

Each LO was further refined with descriptors for each level of schooling: for kindergarten, for the final grade of each three-year period of nine-year primary school ($3^{rd}$, $6^{th}$ and $9^{th}$ year) and for the final year of upper-secondary school.

In a sense, the LOs of CS 1 summarise the learning objectives that are much more operationally defined in the curriculum (Curriculum, 2011), and it could even be argued that they bring nothing fundamentally new. What needs to be highlighted, however, is the role of the LOs of CS 1 in the implementation of CS 2 and in the development of mathematical literacy in general. The two components CS 1 and CS 2) of mathematical literacy differ in certain elements, but above all there is a relationship of interdependence between them: the strengthening of one component contributes to the development of the other.

In CS 1, we placed special emphasis on the learner's *understanding and use of mathematical discourse, the learner's role in the interpretation of different mathematical representations, communication, and the critical evaluation and recognition of mathematics in different contexts*. The aim is to contribute to the realisation of focusing on the aspect of 'rigor', which in teaching, as stated by Hattie et al. (2017), means a balance between conceptual knowledge, procedural knowledge, and fluency and application of knowledge. CS 1 is rounded off by LO 1.7, which highlights the individual's ability to use different strategies in solving mathematical problems. The coherence of all of the learning outcomes of CS 1 can be summarised according to Dubinsky (2001); namely, that an individual's mathematical knowledge is reflected by solving a variety of mathematical problems in which s/he uses complex thought processes to manipulate the mathematical processes and objects needed to solve the problem s/he is working on.

The key goal of CS 2 is to *develop* the student's *ability to deal with mathematical concepts in different structured environments*. Within CS 2, we have identified three LOs, with mathematical modelling certainly being the biggest innovation in terms of mathematics teaching in Slovenia. Let us first examine LO 2.1 *the learner considers a variety of life problems* (which do not require mathematical modelling). How do these differ from the mathematical problems (LO 1.7) we have considered within CS 1? The answer is relatively simple: whereas, in a mathematical problem, the mathematical content of the problem is clear and transparent, in a life problem it is yet to be discovered. An example of such problems is Fermi problems, which are characterised by the very fact that one first has to figure out how to break down the problem in order to use the data to get an answer, after which one uses (depending on the problem) the skills of data estimation, results, computation, reasoning, extending the situation, etc., to arrive at a solution to the problem in the form of a rough estimate (Manfreda Kolar & Hodnik, 2023). For instance, a Fermi problem may be to find out how many bricks there are in the walls of a given school building. In dealing with life situations, we are not only interested in the solution, but also in the process of solving and translating the situations into the mathematical symbolic world, after which we move into the field of mathematical modelling. Mathematical modelling is understood as the process of translating a life situation/problem into a mathematical model, which is then used to solve the problem (Greefrath & Vorhölter, 2016). When translating a life situation (realistic problem), we apply certain mathematical procedures to the mathematical model and design rules to derive the mathematical calculations. This process of transformation into mathematical discourse is called mathematisation, a process guided by the question: What mathematical knowledge can be used to solve a realistic problem (Stillman, 2012)? LO 2.3 *understand mathematical practice in different contexts* is linked to modelling.

## PROBLEM DEFINITION

In the 2019/20 school year, the NA-MA POTI project tested the mathematical literacy of preschool children and primary and secondary school students by means of knowledge tests that included tasks to assess their knowledge of CS 1 and CS 2. The tests were designed to measure the participants' progress in mathematical literacy, and were planned to be administered twice: at the beginning and at the end of the school year. However, due to the Covid-19 pandemic, the pre-test was not conducted; only the final test was conducted in 2022. In the meantime, preschool, primary and secondary mathematics teachers were systematically developing the children's and students' mathematical literacy competences in the classroom, supported by project collaborators through the project's planned activities (seminars, workshops, exchange of good practices, etc.).

In our research, we focused mainly on the relationship between the learning outcomes that are particularly emphasised in relation to mathematical literacy and represent a novelty in terms of a systematic approach to the development of mathematical literacy in Slovenia, as well as focusing on the other learning outcomes. The first group of tasks measure achievements in LO 1.6 and LO 1.7, and all of CS 2 (especially LO 2.1 and LO 2.2), which are associated with mathematical literacy in a narrower sense. The second group of tasks measure achievements in four learning outcomes of CS 1 (LO 1.1 to LO 1.4), in which we identify elements of conceptual knowledge, and LO 1.5 in the area of procedural knowledge. We are aware that the two types of knowledge (conceptual and procedural) cannot be completely separated, but are complementary. When researching the issue, we were interested in the question:

Is the level of conceptual and procedural knowledge related to the level of knowledge of mathematical literacy in primary school students?

## RESEARCH METHODOLOGY

We used a quantitative research approach with descriptive and causal non-experimental methods. The results form part of a larger study on the monitoring progress in scientific and mathematics literacy in the NA-MA POTI project, which was co-funded by the Republic of Slovenia and the European Union from the European Social Fund.

### Sample

The tests in mathematical literacy were administered in the final year of each of the three educational periods (VIO1, VIO2, VIO3) of primary school. The number of participating schools and students is given in Table 1.

| | VIO1 | VIO2 | VIO3 |
|---|---|---|---|
| Number of participating schools | 10 | 10 | 10 |
| Number of students | 487 | 491 | 402 |

Table 1: The sample of students.

### Instruments

The knowledge tests used to investigate the issues were piloted with primary school students and evaluated by university teachers/researchers and school teachers. The teachers' comments and the difficulty and discriminability indices calculated from the pilot tests were taken into account in the final design of the tests used to assess mathematical literacy. The tests included multiple-choice and short-answer tasks, which could be either correct or incorrect. Below is a brief presentation of the tests for each VIO.

The tests included a different number of tasks for each VIO. Most of the items contained several parts (assessment units), which were answered separately by the students and then independently categorised (according to sub-items) and evaluated. Thus, the test for VIO1 contained a total of 39 assessment units in 20 items, VIO2 students also solved 20 items containing a total of 30 assessment units, and VIO3 students solved 13 items containing a total of 36 assessment units.

**Presentation of selected mathematical literacy tasks**

Below we present short descriptions of selected tasks to illustrate the interpretation of learning outcomes in the tasks of mathematical literacy and the performance of the students in these tasks. In doing so, we will limit ourselves to tasks that represent a certain novelty in terms of systematically developing mathematical literacy.

LO 1.6: *predict and evaluate outcomes, and justifies claims, procedures and decisions*

VIO1: Certain information is underlined in a word problem. The students had to judge whether the underlined information in the word problem was too much, too little or just right for the problem. The task was solved correctly by 35.7% of the students.

VIO2: The task provided information about passengers on a ship and the number of passengers who visited certain places. The students had to find out whether each passenger had visited at least one city by a complex comparison of the number of total passengers with the number of visits. The task was solved correctly by 32% of the students.

VIO3: The students determined whether a quadrilateral in which the diagonals intersect at right angles is necessarily a rectangle, a square, a kite or none of these. The success rate for this task was only 12.7%.

LO 1.7: *apply a variety of strategies to solve mathematical problems*

VIO1: The students were given the total number of legs and the total number of heads of foxes and penguins. They were asked to choose the correct answer from the given answers about the number of one and the other. The task was solved correctly by 37.4% of the students.

VIO2: The students were given the perimeter of a rectangular garden and the information that the measurements of the lengths of the sides of the garden are natural numbers. The task was for them to think about how long the sides of a rectangle can be, how many different possibilities there are, and what the maximum area of the garden can be. Slightly less than 24% of the students were successful in this task.

VIO3: The task we have chosen as an illustration requires students to be able to generalise. In it, the elements of a sequence (a picture pattern made of matchsticks) is represented graphically. The pattern was continued correctly by 60.4% of the students, but the general rule was identified correctly by only 22.9% of them.

LO 2.1: *addresses a variety of life problems that do not require mathematical modelling*

VIO1: The task required the students to read a table with the number of siblings for each child. The students had a great deal of difficulty reading the table or answering the questions (e.g., How many children do not have any brothers or sisters? How many children have the same number of brothers as sisters? etc.). Less than 25% of the students were successful in this task.

VIO2: The life context of the task is taking medication, while the mathematical context is finding the lowest common multiple. The task gives information about various medicines that a patient has to take at different time intervals. The question was: If she takes all of the medicines at the same time the first time she takes them, when is the next time she will take all of the medicines at the same time again? Only 29.7% of the students were successful in this task.

LO 2.2: *handle situation through mathematical modelling*

LO 2.2 is made up of four parts. Part 2.2.1 concerns the meaningful linking of the situation under consideration to mathematical objects or to the field of mathematics. Part 2.2.2 refers to the construction of a mathematical model for the situation at hand. Part 2.2.3 concerns the application of the new model given and part 2.2.4 concerns a critical judgement on the appropriateness of the model. The last three parts were tested with tasks only for VIO3.

Part 2.2.1: *put the situation into a mathematical context*

VIO1: In the task, the students were asked to identify the most suitable way to cut and sew Slovenian flags using three strips of the same width but different lengths in the colours of the Slovenian flag. The task was solved correctly by 31.6% of the students.

VIO2: In one of the tasks, the students linked statements (e.g., a student scored at least 8 points on a knowledge test, fewer than 8 people can go on a boat, etc.) to inequalities. Less than 30% of the students were successful in this task. The terms 'at most' and 'at least' were the most difficult for them.

VIO3: The task asks for the length of a ramp for the physically handicapped that would be built along a staircase that includes a platform. Only just over 30% of the students related the situation to the Pythagorean theorem; of these, two thirds were inaccurate in their use of the theorem.

Part 2.2.2: *develops mathematical models for a given situation*

VIO 3: In one of the tasks, the students were asked to determine whether the forms given in the task give a value that is too large or too small for the area of a non-standard character (a chicken egg viewed from the side, as if it were a shape). The task items essentially required modelling the area of different shapes with areas of circles, rectangles and triangles. The percentage of students who provided correct answers to the various task items was in the range of 40–61%.

Part 2.2.3: *uses mathematical models*

VIO3: In one of the tasks, the students were presented with a realistic model to calculate the number of fire extinguishers needed in buildings. The model was presented in the form of a longer instruction, a description of the parameters taken into account, and two tables with the necessary data. The students were relatively successful in reading the tables or observing the individual conditions, but the performance in combining the data from the tables was much lower, with a success rate of 14% of the students.

**Data collection and processing process**

The student tests were administered in the first months of the 2019/2020 school year. The results of the tests were analysed descriptively, measures of mean and variance are presented, and Pearson correlation coefficients were used to determine the correlation between performance in each LO.

**Results with interpretation**

Analysis of the test results

Table 3 shows the average performance in the assessment units of the mathematical literacy tests for the students by VIO and by LO. Some of the entries in the table are blank because we did not test achievements in all LOs in all VIOs.

| LO | VIO1 | | VIO2 | | VIO3 | |
|---|---|---|---|---|---|---|
| | M | SD | M | SD | M | SD |
| 1.1 | 0.550 | 0.237 | 0.407 | 0.212 | 0.740 | 0.441 |
| 1.2 | 0.427 | 0.254 | 0.500 | 0.501 | 0.593 | 0.373 |
| 1.3 | | | | | 0.427 | 0.332 |
| 1.4 | 0.326 | 0.282 | 0.760 | 0.428 | 0.629 | 0.191 |
| 1.5 | 0.343 | 0.342 | 0.465 | 0.263 | 0.300 | 0.458 |
| 1.6 | 0.497 | 0.376 | 0.260 | 0.442 | 0.423 | 0.273 |
| 1.7 | 0.230 | 0.261 | 0.293 | 0.247 | 0.420 | 0.323 |
| 2.1 | 0.432 | 0.170 | 0.292 | 0.230 | | |
| 2.2.1 | 0.320 | 0.465 | 0.309 | 0.243 | 0.100 | 0.300 |
| 2.2.2 | | | | | 0.426 | 0.229 |
| 2.2.3 | | | | | 0.489 | 0.241 |
| 2.2.4 | | | | | 0.143 | 0.296 |
| 2.3 | | | | | | |

Table 2: Average student achievement by LO and VIO.

In Table 2, we have shaded the performance of the lower half of the students' achievement in the LO for each VIO. The shading of the cells indicates that, in general, the performance in the LO whose development in teaching was emphasised in the project is lower than the performance in the LO related to conceptual and procedural knowledge. This is particularly evident for VIO 2 and VIO 3.

In the description of the LOs, we emphasised that LOs 1.1–1.4 refer to conceptual knowledge, LO1.5 to procedural knowledge, and LO 1.6 and LO 1.7 to problem-based knowledge, while CS 2 is entirely concerned with modelling or applying knowledge in contexts that are structured differently from school contexts. In the project, the importance of LO 1.6, LO 1.7 and CS 2 was specifically emphasised in relation to mathematical literacy. We examined their correlation with the LOs related to conceptual knowledge (LO 1.1–1.4) and the LOs related to procedural knowledge (LO 1.5). Table 3 shows the correlation coefficients between the achievement in the LOs under consideration (mathematical literacy) and the average achievement of the participants in the LOs of a conceptual or procedural nature. The calculated correlations that are not statistically significant at the 5% risk level are in brackets; all of the other calculated coefficients are statistically significant. In the table, we have slightly shaded cells with correlations between 0.200 and 0.300; cells with correlations above 0.300 are shaded more strongly; all other cells are unshaded.

| **LO** | **VIO1** | | **VIO2** | | **VIO3** | |
|---|---|---|---|---|---|---|
| | 1.5 | 1.1-1.4 | 1.5 | 1.1-1.4 | 1.5 | 1.1-1.4 |
| 1.6 | 0.140 | 0.350 | (0.076) | 0.103 | 0.249 | 0.350 |
| 1.7 | 0.183 | 0.448 | 0.096 | 0.018 | 0.219 | 0.359 |
| 2.1 | 0.155 | 0.494 | 0.170 | 0.230 | | |
| 2.2.1 | (0.054) | 0.104 | 0.190 | 0.172 | (0.037) | (0.024) |
| 2.2.2 | | | | | 0.225 | 0.229 |
| 2.2.3 | | | | | 0.142 | 0.182 |
| 2.2.4 | | | | | 0.209 | 0.187 |
| 2.3 | | | | | | |

Table 3: Pearson correlation coefficients between average student achievement in selected LO by VIO.

The interpretation of the coefficients is based on the fact that, in the classroom, students are mainly confronted with situations related to the understanding of concepts (conceptual knowledge) and procedures (procedural knowledge). In Table 4, we can see relatively high correlations of problem knowledge (LO 1.6 and LO 1.7) with conceptual domain building blocks and lower correlations with procedural domain building blocks. This is particularly evident for VIO1 and VIO3. Given that problem situations are present in the process of learning mathematics, albeit perhaps to a lesser extent, the difference in the magnitude of the correlations is attributed to the nature of the domains or LOs. Procedural knowledge is, of course, an important basis for problem solving, but the role of conceptual knowledge is more crucial. If the problem solver does not possess the necessary conceptual knowledge to solve the problem, s/he cannot proceed using the appropriate mathematical procedures (Hodnik Čadež & Manfreda Kolar, 2015; Manfreda Kolar & Hodnik Čadež, 2013). A similar observation applies to LO 2.1, which is concerned with solving simple life problems that do not require modelling. Such problems are, of course, a standard part of existing mathematics teaching and their success correlates reasonably well with the level of conceptual knowledge.

Another important observation concerns the correlation of parts of LO 2.2 with the LOs related to conceptual and procedural knowledge. In no case are the correlations high, and there are no differences between the correlations with the conceptual and procedural LOs. This suggests that students have less and unsystematic exposure to modelling in the classroom.

In summary, the lower correlation of a given LO with the LOs of the conceptual or procedural domain can be attributed to the modest and non-systematic presence of the development of mathematical literacy in the learning process.

## DISCUSSION

In this discussion, we would like to highlight the key findings of our research on mathematical literacy in the Slovenian context within the NA-MA POTI project. The aim of the project was to develop mathematical literacy in preschool children, and primary and secondary school students. Therefore, we first came up with a concept of mathematical literacy, taking into account the findings and research in this field. The concept is defined by two cornerstones with a number of learning outcomes, each of which has descriptors for the specific educational period. Our research confirms that the concept of mathematical literacy is appropriate for the purpose for which it was developed, i.e., to develop mathematical literacy in primary school students. This is justified by the following findings:

- the concept is conceived both in the separation of basic mathematical knowledge (conceptual and procedural) and mathematical literacy, and in their interconnection;
- the concept allows for the creation of a variety of tasks covering different areas of mathematical literacy;
- the concept brings a significant difference to mathematics education compared to the existing curriculum, which is reflected in the performance of students in mathematics tests, as well as in the frequency with which the teacher includes mathematics activities (lower achievement correlates with a lower occurrence of mathematical literacy activities in mathematics education).

The mathematical literacy tests developed in the project were administered in all of the VIOs. The results suggesting that conceptual and procedural knowledge are related to mathematical literacy are encouraging, as we see opportunities for raising mathematical literacy in students in the interdependence of knowledge. Good mathematical knowledge, both conceptual and procedural, is a prerequisite for developing mathematical literacy. The real challenge is to find contexts in which a student who is well equipped with mathematical knowledge can demonstrate mathematical literacy competences.

A more detailed analysis of the correlation between students' performance in mathematical literacy tests and procedural and conceptual knowledge shows that students perform less well in problem-solving and modelling tasks than on tasks that assess conceptual or procedural knowledge. The results also show that good conceptual knowledge is more important than procedural knowledge for the development of problem-solving skills.

Some of the learning outcomes of the cornerstones of mathematical literacy developed in the project are already present in the existing curriculum, but they are not sufficiently recognised at the level of curriculum implementation. These include areas such as problem-solving strategies, critical thinking, modelling strategies and procedures. Emphasising these processes for literacy development by no means involves reconstructing mathematics education. Although these are peripheral skills, they need to be made visible and systematically developed in mathematics teaching with well thought-out tasks. They are skills that enable preschool children and primary and secondary school students to think more easily and effectively mathematically in out-of-school contexts.

Knowing that innovations in mathematics education were introduced in the NA-MA POTI project – meaning that they also represent innovations for the teacher's practice, especially in terms of selecting appropriate contexts for developing mathematical literacy, which are structured differently from those in school – we can conclude that the results of our research demonstrate that the project achieved its main objective, i.e., to create an environment for the systematic development of mathematical literacy in primary school. The students' test results reflect the fact that the NA-MA POTI project provided teachers with knowledge that they have applied to their teaching, as well as providing guidelines for further development in this area. However, we would like to stress once again that mathematical literacy objectives can complement mathematical knowledge to a very limited extent, but they cannot replace existing mathematical content.

Could the results of our survey be linked in any way to the results of the 2022 PISA survey, where our 15-year-olds scored an average of 485 (OECD average 472)? The highest average score was achieved in the subject of shapes and solids (492 points), while in mathematical competences the highest average score was achieved in the competence 'interpreting' (487 points), although there were no significant differences in the average scores for the other competences: formulating (482 points), applying (483 points) and reasoning (485 points). The lowest scores achieved by Slovenian pupils were in tasks on change and relationships, probability and working with data (Educational Research Institute, 2023). Although we have not had the opportunity to analyse the tasks in depth, the PISA results suggest that the mathematical literacy performance of Slovenian 15-year-olds may correlate with their conceptual knowledge rather than their procedural knowledge, where procedural knowledge is not simply about using familiar procedures, but about using a procedure in a context that is less structured than a school context (in our case CS 2). It is not possible to make more detailed comparisons of the PISA results with the NA-MA POTI project results. Although it seems clear to everyone what mathematical literacy is, it turns out that this is not the case in tasks that test mathematical literacy. It can only ever be said that students have demonstrated a certain level of mathematical literacy in the selected tasks.

## Study limitations

Finally, let us mention again some of the limitations of our study. The first is that the tests were designed to measure students' progress in mathematical literacy, but due to the Covid-19 pandemic, they were not administered in the planned way. The second limitation is that the tests include tasks that are not balanced in the sense of having the same number of tasks of similar difficulty for each learning outcome defined in each CS.

## References

Ministry of Education and Sport (2011). *Curriculum. Primary School Curriculum. Mathematics.* https://www.gov.si/assets/ministrstva/MIZS/Dokumenti/Osnovna-sola/Ucni-nacrti/obvezni/UN_matematika.pdf

Dubinsky, E. (2001). *Using a theory of learning in college mathematics courses*. University of Warwick.

Educational Research Institute (2023). *Pisa 2022: Nacionalno poročilo s primeri nalog iz matematike* [Pisa 2022: National report with examples of mathematics tasks] Educational Research Institute.

Goos, M., & Kaya, S. (2020). Understanding and promoting students' mathematical thinking: A review of research published in ESM. *Educational Studies in Mathematics, 103*(1), 7–25.

Greefrath, G., & Vorhölter, K. (2016) Teaching and learning mathematical modelling: Approaches and developments from German speaking countries. In *Teaching and learning mathematical modelling. ICME-13 topical surveys*. Springer.

Hattie, J., Fisher, D., & Frey, N. (2017). *Visible learning for mathematics*. Sage.

Hodnik Čadež, T., & Manfreda Kolar, V. (2015). Comparison of types of generalizations and problem-solving schemas used to solve a mathematical problem. *Educational Studies in Mathematics, 89*(2), 283–306.

Kula Unver, S., Hidiroglu, C. N., Tekin Dede, A., & Bukova Guzel, E. (2018). Factors revealed while posing mathematical modelling problems by mathematics student teachers. *European Journal of Educational Research, 7*(4), 941–952.

Magajna, Z., Manfreda Kolar, V., Metljak, M., & Hodnik, T. (2022). Matematična pismenost v slovenskih šolah in vrtcih: koncept pismenosti in analiza stanja [Mathematical literacy in Slovenian schools and kindergartens: The concept of literacy and an analysis of current situation]. In T. Hodnik, S. Hudovernik, A. Lipovec, & M. Slapničar (Eds.), *Koncept in analiza matematične in naravoslovne pismenosti v slovenskih šolah in vrtcih* [The concept and analysis of mathematical and science literacy in Slovenian schools and kindergartens] (pp. 7–23). University of Ljubljana Faculty of Education.

Manfreda Kolar, V., & Hodnik Čadež, T. (2013). Dependence of the problem solving strategies on the problem context. In M. Pavleković, M. Kolar-Boegović, & R. Kolar-Šuper (Eds.), *Mathematics teaching for the future* (pp. 162–172). Element.

Manfreda Kolar, V., & Hodnik, T. (2021). Mathematical literacy from the perspective of solving contextual problems. *European journal of educational research*, *10*(1), 467–483.

Manfreda Kolar, V., & Hodnik, T. (2023). Fermi problems and their value for developing students' mathematical literacy. In A. Žakelj, M. Cotič, D. M. Kadijevich, & A. Lipovec (Eds.), *Selected topics in the didactics of mathematics* (pp. 69–90). University of Primorska Press.

Niss, M., & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9–28.

Organisation for Economic Co-operation and Development (2003). *The PISA 2003 assessment framework.* Mathematics, reading, science and problem solving knowledge and skills. OECD.

Organisation for Economic Co-operation and Development (2017). *PISA 2021 mathematics: A broadened perspective*. OECD.

Stacey, K., & Turner, R. (2015). The evolution and key concepts of the PISA mathematics frameworks. In K. Stacey, & R. Turner (Eds.), *Assessing mathematical literacy* (pp. 5–33). Springer.

Stillman, G. A. (2015). Applications and Modelling Research in Secondary Classrooms: What Have We Learnt?. In S. J. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 791–805). Springer.

Suciati, Munadi, S., Sugiman, & Febriyanti, W. D. R. (2020). Design and validation of mathematical literacy instruments for assessment for learning in Indonesia. *European Journal of Educational Research, 9*(2), 865–875.

# Digital resources in the mathematics classroom

## Part 3

# POTENTIALS OF DIGITAL EDUCATIONAL RESOURCES IN THE MATHEMATICS CLASSROOM – DIDACTICAL CONSIDERATIONS AND EMPIRICAL FINDINGS

Daniel Walter

TU Dortmund University, Germany

*Digital media is increasingly finding its way into the classroom as educational resources. However, the question of how digital media can be used beneficially and integrated into subjects is by no means new. While there are certainly a number of positive examples of the use of digital media, it can be stated that their integration has not yet been as successful as one might have hoped. The reasons for this can certainly be manifold. In order to specifically address concepts for teaching and learning mathematics with digital media, this paper will focus on subject-specific potentials which arise primarily through the availability of digital media. These potentials are first illustrated with examples and then, based on empirical studies, their use by learners will be discussed.*

## INTRODUCTION – DESIGNING MATHEMATICS TEACHING

The shared goal of researchers, teachers and all other people who contribute directly or indirectly to the design of teaching mathematics is to create high-quality mathematics lessons. Given the sometimes-sobering findings in school assessment studies, such as TIMSS (Mullis et al., 2020), at least with regard to the performance of students in Germany, the question of enhancing subject-related teaching quality is more pressing than ever. Especially in recent times, there has been a growing interest in examining how digital educational resources can contribute to this effort. However, this interest is not new; Freudenthal (1981) articulated the following major problem of mathematics education over four decades ago: "How can calculators and computers be used to arouse and increase mathematical understanding?" (p. 146).

In order to find solutions, it is first necessary to clarify what constitutes good mathematics teaching. In this context, there is often a focus on three generic dimensions of teaching quality, which are formulated as follows and their integration has been shown to promote the design of quality instruction (Praetorius et al., 2018):

- *Classroom management*
- *Student support*
- *Cognitive activation*

Regarding the contribution of digital media to high-quality teaching, it appears that in the general societal and educational policy discourse, aspects of classroom management are predominantly emphasized, with only occasional references to subject-specific didactic aspects of teaching. While it is certainly helpful and welcome that common tablet apps are intuitive to use, this alone is by no means a guarantee of the presence of a digitally rich learning resource with substantive subject-specific didactics.

However, in order to address the two dimensions of *student support* and *cognitive activation*, which are more influenced by subject-specific didactics, and at the same time to unfold the complexity of these two dimensions for subject-specific learning, the following five principles of effective mathematics instruction have been developed at the *German Centre for Mathematics Teacher Education (DZLM)* [1]:

| | | |
|---|---|---|
|  | *Principle of Conceptual Focus* | Establishing concepts, strategies, procedures |
|  | *Principle of Longitudinal Coherence* | Enabling long-term learning |
|  | *Principle of Enhanced Communication* | Talking about mathematics |
|  | *Principle of Student Focus and Adaptivity* | Addressing learning levels |
|  | *Principle of Cognitive Demand* | Encouraging active learning processes |

Table 1: Five principles for high-quality mathematics teaching (Prediger et al., 2022).

This article aims to demonstrate to what extent specific subject-specific potentials of digital media can support the implementation of these five principles and report empirical findings on their usage. To achieve this, the following potentials will be examined section by section:

- *Understanding and structure representations*
- *Support representation processes*
- *Aligning concrete und mental actions*
- *Provide informative feedback*
- *Outsource calculus*

[1] Homepage of the DZLM: https://dzlm.de/en/international-visitors

## UNDERSTANDING AND STRUCTURE REPRESENTATIONS

### The potential and how it can support learners

The first potential, *Understanding and structuring representations*, focuses on the in-depth understanding and structuring of representations. Understanding representations can be supported by digital media insofar as a mathematical object can be represented through various forms (enactive, iconic, and symbolic representations). Furthermore, these representations can be designed so that changes made to one representation *automatically* adjust the other representations accordingly (Drijvers & Sinclair, 2023). This aspect of design is internationally established under the term *multiple equivalent linked representations* (MELRs) (Harrop, 2003).

Accordingly, number representations in the 'Calculation field' app can be changed on a symbolic level, for example, by increasing a summand by +1 using a swipe gesture over the number sign. The iconic representation automatically adapts synchronously to the enactive change of the symbolic representation level by displaying ten additional counters (see Figure 1). Moreover, a complementary approach is also feasible, whereby a counter is added, prompting adjustment in the symbolic representation.

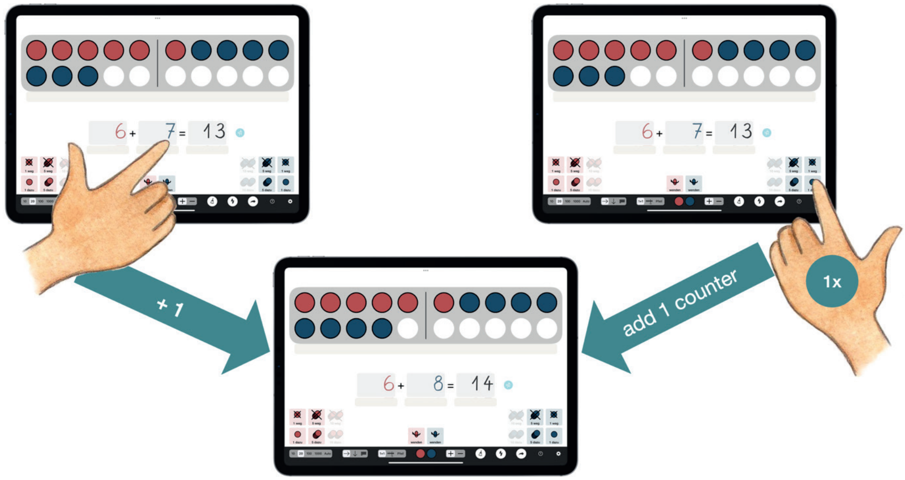



Figure 1: Multiple equivalent linked representations in 'Calculation field'.

Meanwhile, the *structuring* of representations is automated by the software. For example, if another red counter is added, it appears directly to the right of the sixth red counter. The blue counter positioned there would be automatically shifted to the second row immediately next to the last placed blue counter. Thus, the counters are consistently arranged in a decimal manner (Walter, 2018). If individual counters are moved, altering the structure, a simple touch on the 'Calculation Field' can prompt a restructuring of the counters.

The illustrated potential primarily addresses the *Principle of Conceptual Focus*. In order to enable learners to establish foundational concepts, strategies, and procedures, it is essential that the representations used in respective grade levels are thoroughly understood. This requires the ability to not only utilize mathematical objects on a single representation level but also to relate various representations – even in different number ranges. The above software facilitates this by allowing not only number representation up to 20 but also up to 100 or 1000 when additional counters are added, as the 'Calculation Field' automatically adjusts. In this way, the continuity of educational resources is also ensured.

### Selected empirical findings regarding the potential

Various studies have examined *how* and *which* MELRs are used by learners in the process of representing numbers. This includes the study conducted by Walter (2018), in which the 'Twenty Frame' software – a precursor to the 'Calculation Field' app – was utilized in a qualitative investigation involving 19 learners experiencing difficulties in learning arithmetic at the beginning of the second grade. It was observed that learners often demonstrated isolated perspectives on individual representations when representing tasks within the number range up to 20, rather than considering the various representations in their process. For instance, when representing 8+7, learners would either focus solely on the symbolic representation at the bottom of the screen and place tiles until the desired number was visible. Another approach involved a narrow focus on the iconic representation in the centre of the screen, with counters being added until they reached a desired field (in this case, the eighth field for the first summand). Finally, a third approach was observed, wherein some children focused on the button for counter selection ('add 1') and tapped it multiple times while concurrently counting in increments of one.

After the individual representation processes focusing on specific representations were completed, the children were asked to relate their approaches to the other available representations, which they were able to do in most cases. Consequently, it became apparent that, at least in the group of learners classified as relatively low-performing, the utilization of MELRs is not solely guaranteed by their availability. Targeted stimuli are required to initiate appropriate usage patterns.

## SUPPORT REPRESENTATION PROCESSES

### The potential and how it can support learners

The second potential, *Support representation processes*, can help learners to represent mathematical objects in a skillful manner. This will be illustrated below with an example from early arithmetic instruction, where a central hurdle for children is to understand numbers cardinally as compositions of other numbers, as they often perceive them solely in an ordinal manner. This fact frequently leads to many learners adhering to non-sustainable counting procedures, such as laying down eight individual counters one after the other, while structures remain unused (e.g., laying down a group of five and three single counters) (Gaidoschik, 2019).

In this context, touch-based digital devices can promote the more skillful representation of numbers by utilizing multitouch technology (Meletiou-Mavrotheris et al., 2015). For example, in the 'Math Tablet' app, it is possible to touch the screen surface with multiple fingers simultaneously, so that counters appear simultaneously exactly where the fingers touch the screen (here: 4 and 2 fingers), resulting in the appearance of tiles simultaneously (here: 6, see Figure 2). Simultaneous representation of numbers can support thinking about numbers cardinally (Segal, 2011).

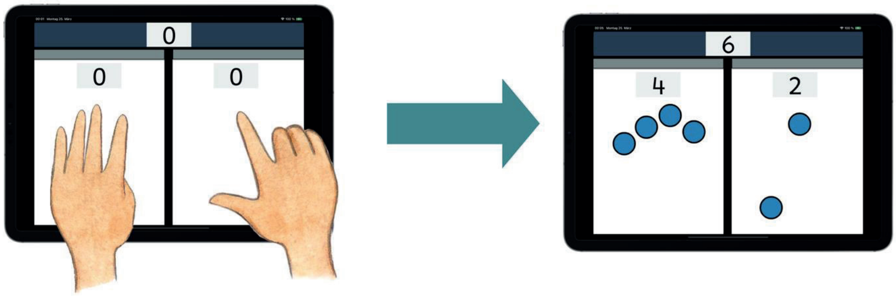



Figure 2: Multitouch representation when using the 'Math Tablet' app.

The described potential can primarily contribute to the implementation of the *Principle of Longitudinal Coherence*, which aims at the necessity of long-term learning. It is stated that in earlier grade levels, it is essential to establish the concepts and strategies necessary for further mathematical development. In relation to the example, this means that the skillful representation of numbers in early mathematical instruction can be considered as a fundamental foundation, allowing learners to adeptly represent large numbers and understand their relationship to the decimal number system.

**Selected empirical findings regarding the potential**

Although multitouch technology is a relatively recent technological development, several studies have already explored the extent to which the resulting potential to support representation processes is utilized by learners. In their experiments using the multitouch table, Ladel and Kortenkamp (2014) investigated how internalization and externalization processes unfold. They observed that the formulation of a task influences children's approaches. For tasks structured as 'Please put x counters on the table', children tended to sequentially place individual counters. However, when children were additionally encouraged to represent the counters 'all at once', many learners changed their approach by quasi-simultaneously representing the quantities with their fingers (Ladel & Kortenkamp, 2014).

This finding was also confirmed in the studies conducted by Walter (2017), where children used the 'Math Tablet' for number representation. Here, the role of fingers as the primary medium of representation was also evident: It was often observed that learners initially represented numbers on their fingers sequentially to subsequently position all outstretched fingers simultaneously on the screen.

Difficulties that learners encounter when using multitouch technology are also documented. For example, learners may position multiple fingers too closely together, causing the digital device to recognize them as a single finger (Sinclair & Heyd-Metzuyanim, 2014; Walter, 2017). Thus, it remains to be noted that the use of multitouch technology does not guarantee that numbers were represented cardinally on the fingers.

## ALINGNING CONCRETE AND MENTAL ACTIONS

### The potential and how it can support learners

The third potential, *Aligning concrete and mental actions*, aims to orientate the actions conducted on digital educational resources closely to the intended mental actions. This is particularly important because the transition from concrete actions on materials to mental actions can be supported if there is a structural match.

In this context, virtual materials offer the potential to closely align actions with normatively desired mental operations. This alignment can be even closer with certain tablet apps than with their respective physical counterparts (Peltenburg et al., 2009). This is illustrated using the example of the task 11–2 with the assistance of (physical and digital) tens system materials.

With physical tens system materials, the number 11 is initially represented using a ten rod and one unit cube, and then two unit cubes are subtracted. To visualize the difference, it is essential to substitute the ten rods with ten unit cubes. Only then is it possible to subtract the last unit cube.

Without physical materials – purely mentally – the task would probably not be solved in this way. For 11–2, it seems somewhat unreasonable to perform a mental substitution process of the ten for ten ones after calculating 11–1=10 before subtracting a one. The more intuitive approach seems to be to subtract one unit *directly* from the ten – without the detour of a substitution process.

Digital equivalents of base-ten-block material, as used in the app 'Practicing place value', can realize this aspect: The number 11 is represented using a ten rod and one unit cube, after which the ten rod is moved to the ones column and automatically unbundles into ten-unit cubes (see Figure 3). Sarama and Clements (2006) assess this by stating that "[s]uch actions are more in line with the mental actions that we want students to carry out.” (p. 113; Thompson, 1992)

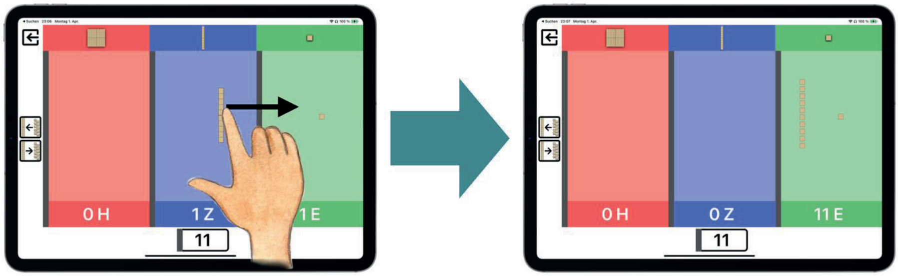



Figure 3: Aligning concrete and mental actions in ‘Practicing place value’.

The described potential relates to the *Principle of Enhanced Communication*, which emphasizes the importance of productive discourse in mathematics instruction, so that mathematics is primarily understood as a process, not just a product. The question of which actions with the material best align with intended mental operations can serve as such a discourse opportunity. For example, the approach outlined in Figure 3 can be compared with actions using physical tens system materials, and discussed with the children to explore which approach might be considered more skillful and why.

**Selected empirical findings regarding the potential**

Schulz and Walter (2019) conducted an interview study with 29 second and third graders, investigating the extent to which children utilize the potential of *aligning concrete and mental actions* when representing numbers. After working with the children to understand how to use the software (especially unbundling hundreds and tens), the children were asked, among other tasks, to first represent the number 200, which all children accomplished using two hundred-squares. Following this, the children were asked to ‘take away twenty’. Some children utilized the potential by unbundling one hundred-square and then deleting two ten-rods. However, most children imitated the actions familiar to them from their previous instruction: they deleted one hundred-square, added ten ten-rods, and then deleted two ten-rods. This finding suggests that previous experience with concrete physical material overlaps and influences the intended use of the implemented feature.

These findings align with those of Litster et al. (2019). Based on their interview study with 100 primary grade children using the Montessori Number Base-10 Blocks app, they suggest that children do not automatically incorporate the existing potentials into their usage patterns.

However, various studies also indicate that working with virtual tools “at certain phases of learning may be more efficacious than their physical counterparts”

(Clements, 1999, p. 56; Thompson, 1992). For instance, Burris (2013) found in an app with similar features that children could generate more efficient and diverse representations of numbers through bundling (depicted by materials sticking together) and unbundling (depicted by materials being hammered apart).

## PROVIDE INFORMATIVE FEEDBACK

### The potential and how it can support learners

Especially in the context of digital educational resources, the fourth potential, *Provide informative Feedback*, plays a significant role. For example, educational software can provide immediate feedback to a child right after completing a task, indicating whether it was solved correctly or incorrectly. Given a class size of nearly 30 children, this is not feasible for a single teacher to accomplish. From a teaching organisation perspective, computer-generated feedback can therefore be seen as having great potential.

However, the majority of available software seem to primarily include feedback that is purely product-oriented, merely informing learners whether they have provided a correct or incorrect answer. Feedback that also incorporates the processes of the children and provides specific feedback on specific responses is only found in isolated cases (Walter & Schwätzer, submitted). Accordingly, most apps seem to have a rather limited understanding of feedback, which is minimally informative and learner-centered, and not designed as postulated by Pardo (2018): “A process to positively influence how students engage with their work in a learning experience so that they can improve its overall quality with respect to an appropriate reference and increase their self-evaluative capacity” (p. 433).

An example of how digital educational resources can provide informative feedback is implemented in ‘Practicing place value’. In Figure 4, children are tasked with verbally stating the displayed number (here: 49) after touching the microphone icon at the bottom of the screen. The software then recognizes the spoken number word (in German: neunundvierzig; in English: nine-and-forty) and provides feedback on whether it is correct or incorrect. If it is incorrect, the app also indicates which number it understood (here: “incorrect (understood as 94)”). This additional information allows the child, for example, to realize that they have spoken the number inversely. The correct digits are included in the spoken number, but not in the correct order – a mistake often observed due to the inverse way of speaking numbers in German (Möller et al., 2015).

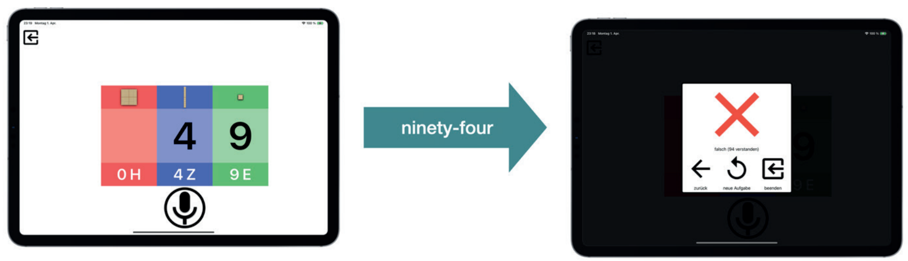



Figure 4: Informative feedback in 'Practicing place value'.

Informative Feedback addresses primarily the *Principle of Student Focus and Adaptivity*, as it particularly takes into account the individual learning levels of children. The intention of informative feedback is to understand the child's approach and offer as adaptive feedback as possible, providing guidance for individual constructive further work.

**Selected empirical findings regarding the potential**

The extent to which the feedback function exemplified above in the 'Practicing place value' app is utilized by learners has not been empirically examined. Previous research on the use of other digital media primarily suggests that automated and computer-supported feedback is *utilized very differently* by learners and that the effectiveness of the feedback *depends on the learners' prerequisites* for learning.

For instance, Steffen (2019) found in her research with 142 preschool children on the feedback function of the 'Osmo Tangram game system' that digital feedback was perceived by learners and used for targeted work, such as repositioning incorrectly placed shapes. However, contradictory findings were also observed, as some children did not perceive the feedback functions or could not derive any benefit for their own further work. Harras (2007) examined how 60 learners dealt with error feedback when using software for arithmetic automation. She found that automated error feedback (such as offering additional representations) was only useful when learners could recognize and classify their errors themselves.

**OUTSOURCE CALCULUS**

**The potential and how it can support learners**

There is consensus that learning mathematics involves more than just determining results. Rather, it also involves providing children with cognitively challenging tasks that target not only products but also learners' processes (Mullis et al., 2020). To prevent the focus in mathematics instruction from (solely) being on routine tasks, these tasks can be delegated to software, which is where the fifth potential, *Outsource calculus*, comes into play.

The aim of this potential is to enable children to concentrate on cognitively activating tasks without being distracted by individual calculations. However, this idea is not a didactic innovation born out of tablet apps but has been established for almost three decades under the term *computational offloading* (Scaife & Rogers, 1996). In mathematics education, this approach is particularly recognized in relation to 'mathematically weak' children (Krauthausen & Lorenz, 2011). Many students, especially when dealing with routine tasks, have such great difficulties that they can hardly examine connections between tasks based on them.

An example of the practical implementation of *Outsource calculus* can be found in the digital version of the traditional NIM game (Holton, 2006). The NIM game (whether analog or digital) is a strategy game in which two players compete against each other, taking turns placing 1 or 2 counters of their color (determined at the beginning) on a game board consisting of 10 fields from left to right. The player who occupies the last field wins.

The NIM game offers rich learning opportunities for children, as they can discover, for example, that certain fields are 'special' or can be characterized as 'winning fields'. For instance, a player who occupies the seventh field with a counter will win the game because the opponent must occupy either the eighth or ninth field with a counter. From these two fields, the game can then be won directly by moving to field 10. Similarly, fields 1 and 4 are winning fields, and even the decision of who starts the game has a crucial influence on the course of the game. Thus, a player who is aware of the winning strategy can make their first move to field 1 and then be confident in occupying fields 4, 7, and 10 with skillful play.

The author of this article has had teaching experiences showing that many children gain initial experience playing and develop an intuition that field 7 is 'special'. However, many children have significant difficulties explaining why field 7 is important to occupy and understanding that fields 1 and 4 are also winning fields. Therefore, focused support is needed for the children to enable more targeted analysis.

The digital version of the NIM game can realise this by saving played game rounds in an archive in which the playing fields are arranged according to criteria (chronological, field length, starting player) so that hypotheses about winning fields can be developed (see Figure 5, left). In addition, individual games played can be selected and analysed in detail, leading to the question of whether, for example, the player with the blue counters could have won the game in Figure 5 (right) if he had played differently.

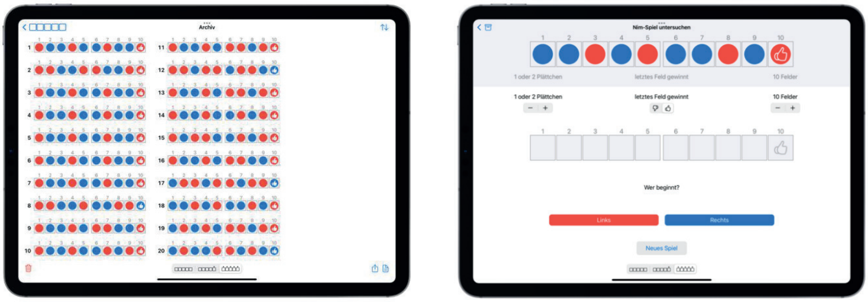



Figure 5: Outsource calculus in 'NIM'.

The illustrated examples clarify that children are not meant to merely play the NIM game without reflecting on their approaches. Instead, through the automatic compilation of the archive and the conveniently usable sorting mechanisms, opportunities are created for them to exchange ideas about clever strategies. Accordingly, the *Principle of Cognitive Demand* is addressed, as children are encouraged to reflect on their actions, develop strategies, and discuss potential solution paths.

**Selected empirical findings regarding the potential**

Krauthausen et al. (in press) used semi-standardised interviews with 14 children from third and fourth grade to investigate the extent to which Outsource calculus can support learners in discovering the winning strategy in the NIM game by focusing on the archive. They found that pupils can be supported in discovering the winning strategy by focussing on analysing individual game sequences and by analysing multiple game sequences compiled in the archive - and not just by playing the NIM game themselves several times. These findings are in line with other study results that recognise the potential of *Outsource calculus* to promote learning processes (e.g. Bezold & Ladel, 2014).

## SOME CONCLUDING THOUGHTS

In this article, five mathematics didactic potentials of digital media were characterised, related to principles of high-quality mathematics teaching and selected research findings were compiled. Overall, the outlined potentials offer new opportunities for developing mathematical competencies. Nevertheless, various ways of using them were identified, not all of which appear to be equally useful for mastering the tasks assigned to them and it is therefore unlikely that learners will automatically and intuitively utilise their potential. This leads to three desiderata for future research and development work:

- *Prepare teachers for different types of use:* To anticipate conceivable uses and to be able to make instructional planning decisions on how to deal with these uses, teachers need to receive tailored training. They must be empowered to provide appropriate stimuli and tasks.

- *Support teachers in the effective embedding of potentials:* Teachers must be able to use the respective apps in such a way that the implemented potentials are utilised. The mere existence of potentials is no guarantee of a 'good' app - they must be recognised and used by the teachers. Accordingly, mathematics edcuation needs to contribute more to the development of professional development concepts that support teachers in utilising the potentials.
- *Implementing potentials in maths apps:* There are already numerous promising maths apps that contain the outlined potentials. Some of these have been presented in this article. Nevertheless, there are relatively few apps in the extensive inventory of maths apps that contain any of the potentials (Walter & Schwätzer, under review). This highlights the need for mathematics education to become more involved in the research-based development of maths apps in the future – and thus provide more impetus for high-quality mathematics teaching.

## References

Bezold, A., & Ladel, S. (2014). Reasoning in primary mathematics – An ICT-supported environment. *Bildung und Erziehung*, *67*, 409–418.

Burris, J. T. (2013). Virtual place value. *Teaching Children Mathematics, 20*(4), 228–236.

Clements, D. H. (1999). “Concrete” Manipulatives, Concrete Ideas. *Contemporary Issues in Early Childhood, 1*(1), 45–60.

Drijvers, P., & Sinclair, N. (2023). The role of digital technologies in mathematics education: Purposes and perspectives. *ZDM – Mathematics Education*.

Freudenthal, H. (1981). Major problems of mathematics education. *Educational Studies in Mathematics*, *12*(2), 133–150.

Gaidoschik, M. (2019). Didactics as a Source and Remedy of Mathematical Learning Difficulties. In A. Fritz, V. G. Haase, & P. Räsänen (Eds.), *International Handbook of Mathematical Learning Difficulties* (pp. 73–89). Springer International Publishing.

Harrass, N. (2007). *Computereinsatz im Arithmetikunterricht der Grundschule* [Computer use in arithmetic lessons in primary school]. Franzbecker.

Harrop, A. (2003). Multiple Linked Representations and Calculator Behaviour: The Design of a computer-based Pedagogy. *Journal of Computers in Mathematics and Science Teaching*. *22*(3), 241–260.

Holton, D. (2006). Holton on Problem Solving – 2: The 21 Game. *Mathematics In* School, May 2006, 7–8.

Krauthausen, G., Scharlau, J., & Walter, D. (i. Dr.). »Ab ins Archiv«: Erkunden einer Gewinnstrategie zum NIM-Spiel. In B. Thöne et al. (Eds.), *Denkwege von Kindern* *und Inhalte gleichermaßen in den Blick nehmen* [Focusing equally on children's ways of thinking and content]. WTM-Verlag Münster.

Krauthausen, G., & Lorenz, J. H. (2011). Computereinsatz im Mathematikunterricht. In G. Walther, M. van den Heuvel-Panhuizen, D. Granzer, & O. Köller (Eds.), Bildungsstandards für die Grundschule: Mathematik konkret [Educational Standards for Primary Schools: Mathematics in Practice] (pp. 162–183). Cornelsen.

Ladel, S., & Kortenkamp, U. (2014). Number concepts – processes of internalization and externalization by the use of multi-touch technology. In C. Benz, B. Brandt, U. Kortenkamp, G. Krummheuer, S. Ladel, & R. Vogel (Eds.), *Early Mathematics Learning. Selected Papers of the POEM 2012 Conference* (pp. 237–256). Springer.

Litster, K., Moyer-Packenham, P. S., & Reeder, R. (2019). Base-10 Blocks: A study of iPad virtual manipulative affordances across primary-grade levels. *Mathematics Education Research Journal*, *31*(3), 349–365.

Meletiou-Mavrotheris, M., Mavrou, K., & Paparistodemou, E. (Eds.). (2015). *Integrating Touch-Enabled and Mobile Devices into Contemporary Mathematics Education.* IGI Global.

Möller, K., Shaki, S., Göbel, S., Nürk, H.-C. (2015). Language influences number processing – A quadrilingual study. *Cognition, 136*, 150–155.

Mullis, I. V. S., Martin, M. O., Foy, P., Kelly, D. L., & Fishbein, B. (2020). *TIMSS 2019 International Results in Mathematics and Science.* Retrieved from Boston College, TIMSS & PIRLS International Study Center website: https://timssandpirls.bc.edu/timss2019/international-results/

Pardo, A. (2018). A feedback model for data-rich learning experiences. *Assessment & Evaluation in Higher Education, 43*, 428–438.

Peltenburg, M., Van Den Heuvel-Panhuizen, M., & Doig, B. (2009). Mathematical power of special-needs pupils: An ICT-based dynamic assessment format to reveal weak pupils' learning potential. *British Journal of Educational Technology*, *40*(2), 273–284.

Praetorius, A.-K., Klieme, E., Herbert, B., & Pinger, P. (2018). Generic dimensions of teaching quality: The German framework of Three Basic Dimensions. *ZDM – Mathematics Education*, *50*(3), 407–426.

Prediger, S., Götze, D., Holzäpfel, L., Rösken-Winter, B., & Selter, C. (2022). Five principles for high-quality mathematics teaching: Combining normative, epistemological, empirical, and pragmatic perspectives for specifying the content of professional development. *Frontiers in Education*, *7*, 969212.

Sarama, J., & Clements, D. H. (2006). Mathematics, Young Students, and Computers: Software, Teaching Strategies and Professional Development. *The Mathematics Educator*, *9*(2), 112–134.

Scaife, M., & Rogers, Y. (1996). External cognition: How do graphical representations work? *International Journal of Human-Computer Studies*, *45*, 185–213.

Schulz, A., & Walter, D. (2019). 'Practicing place value': How children interpret and use virtual representations and features. In U. T. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *CERME 11 – Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 2941–2948). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Segal, A. (2011). *Do gestural interfaces promote thinking? Embodied interaction: Congruent gestures and direct touch promote performance in math*. ProQuest Dissertations Publishing.

Sinclair, N., & Heyd-Metzuyanim, E. (2015). Developing number sense with TouchCounts. In S. Ladel & C. Schreiber (Eds.), *Von Audiopodcast bis Zahlensinn* [From audio podcast to numerical sense] (pp. 125–150). WTM-Verlag.

Steffen, A. (2022). *Digitale Lernbegleitungen bei der Bearbeitung von Raumvorstellungsaufgaben* [Digital learning support in the processing of spatial design tasks]. Waxmann.

Thompson, P. W. (1992). Notations, Conventions, and Constrains: Contributions to Effective Uses of Concrete Materials in Elementary Mathematics. *Journal for Research in Mathematics Education*, *23*(2), 123–147.

Walter, D. (2017). On the representation of quantities with multi-touch at the "Math-Tablet." In J. Novotná & H. Moraová (Eds.), *Equity and diversity in elementary mathematics education: Proceedings of the International Symposium Elementary Maths Teaching 2017 (SEMT 2017)* (pp. 449–458). Charles University, Faculty of Education.

Walter, D. (2018). How Children Using Counting Strategies Represent Quantities on the Virtual and Physical 'Twenty Frame.' In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach, & C. Vale (Eds.), *Uses of Technology in Primary and Secondary Mathematics Education* (pp. 119–143). Springer International Publishing.

Walter, D., & Schwätzer, U. (under review). Mathematics apps under the magnifying glass: An analysis of the app stores' inventory. *Journal article.*

.

# THE ROLE OF NEW TECHNOLOGIES IN SHAPING VARIOUS WAYS OF SOLVING AN UNUSUAL MATHEMATICAL TASK

Edyta Juskowiak

Adam Mickiewicz University, Poznań, Poland

*The article offers an extensive literature review concerning the utilization of ICT tools within mathematics education. It integrates the author's empirical research findings and outcomes derived from sequential survey investigations among prospective mathematics and computer science educators. The primary objective of these surveys was to evaluate the attitudes and utilization patterns of these tools by students. The conclusions indicate a significant recognition among students of the importance of ICT tools in mathematics education, particularly in terms of enhancing motivation and shaping cognitive approaches. Over 80% of respondents acknowledged the significance of ICT tools, with more than half noting their positive impact on motivation and cognitive processes. Thus, the study suggests that the integration of ICT tools can enhance the attractiveness and efficacy of mathematics education.*

## INTRODUCTION AND THEORETICAL FRAMEWORK

In the process of learning and teaching mathematics, tasks play one of the most important roles. Each type of task serves different didactic purposes, provokes and enhances various skills, and presents unique challenges. Overcoming these challenges enriches the student's mathematical potential. In the concept of problem-based teaching, tasks are divided into exercise tasks, ordinary applications of theory, and problem-type tasks (Krygowska, 1977). The first serve to consolidate simple operations and schematic mathematical habits, the aim of the second is to develop the skill of choosing the solution path along with the correct application of concepts and their properties, while problem-type tasks are challenges that do not yield to a single correct solution path or familiar patterns from everyday school life; they provoke a creative attitude as well as curiosity and perseverance. This type of tasks can be classified as non-standard mathematical tasks, which according to Schoenfeld's (1980) definition are simply tasks that are unlike any previously solved. The idea of solving such tasks arises from:

- the result of many different approaches to solving, often many unsuccessful ones, which, however, ultimately allow discovering the correct course of action,
- unconsciously, seemingly out of nowhere, but simply as a result of previously acquired mathematical experiences,

- diligent use of heuristic techniques such as finding a similar problem, establishing variables, or generalizing constants, etc.,
- finally, the proper use of mathematical 'scaffolding' in the form of a teacher or educational tools such as ICT.

Problem-type tasks, as non-standard tasks, can always be solved in several ways. According to Polya (1964), an effective process of solving such tasks proceeds through four stages:

1. Understanding the task
2. Formulating a plan of solution
3. Execution of the plan
4. Looking back

It seems that the first two stages are the most demanding because they verify the mathematical potential of the solver. This potential includes knowledge of mathematical concepts and their properties, as well as the ability to apply them; the language of mathematics – the ability to read and interpret words, symbols, expressions, formulas, formulations, and finally whole sentences; knowledge of many methods of working on tasks and ways of reasoning; the ability to ask questions and verify hypotheses, and finally, the ability to select the appropriate mathematical tools that can help answer them. Executing the plan and looking back is the verification of the so-called 'good work', which Zofia Krygowska often wrote about in her publications. Polya (1964) in his work *How to Solve It?* argued that it is not the result of the task that is most important but finding and understanding all possible ways to solve it.

Many studies have already been conducted to examine and describe the role of information technology tools (ICT) in the process of solving non-standard mathematical tasks. The turn of the 20th and 21st centuries became a time of intensified research on ways to incorporate ICT into coping with mathematical challenges in the form of various types of tasks and shaping mathematical concepts. Efforts were made to discover ways and moments of their use at various levels of intellectual development, while also examining threats or limitations that should be avoided. It is worth remembering the words of Prof. Konior: (as cited in Pawlak, 2004, p. 302)

> The accelerated technological development of the world and the growing progress of science mean that the modern school is no longer able to equip its graduates with a body of knowledge sufficient for their entire period of active life and professional activity [...]. Conservative estimates suggest that in the coming decades, a person will face the need to change their profession two or even three times.

From the research conducted so far, it follows that the use of computer programs or graphic calculators, as well as other ICT tools, can improve the understanding of mathematical concepts, especially the concept of function (Dunham, 2000; Juskowiak 2004, 2010; Waits & Demana, 1996). Students using ICT, as a result of studying a larger number of representatives, have a broader base of function examples, better understand the relationships between graphical, algebraic, and numerical representations, connect graphs with equations, interpret and read graphical information. Large-scale research conducted by Polish educators has shown that the systematic use of graphic calculators, for example, has enabled the development of activities related to coding and algorithm creation (Herma, 2004; Kowalski, 2020).

Research shows that the use of calculators can lead to an improvement in mathematical problem-solving skills. Students then need to focus less on memorizing formulas and computational patterns and more on the actual problem-solving process (Dunham, 2000; Kutzler, 2000; Waits & Demana, 1996). It has also been observed that students, when working with a calculator, have a more flexible approach to problem-solving, engage more in solving tasks, and are less likely to give up in case of failure. They solve atypical tasks that cannot be solved using algebraic methods (Dunham, 2000; Demana & Waits, 2020; Waits & Demana, 1996). An important and often appreciated function of ICT tools by teachers is the graphical illustration of algebraic and numerical data, known as visualization. Graphic calculators and computer programs help students visualize problems, enabling them to create better and faster graphs, which aids in learning mathematics. Graphs generated using new techniques can be used to teach important mathematical concepts. Until recently, the ability to draw graphs of more complex functions appeared after calculus. Now, one can view graphs of such functions without introducing such advanced theory (Legutko, 1990; Waits & Demana, 1996). New technologies are also beginning to be used in situations that were previously unacceptable, such as in mathematical proofs. Classical deductive reasoning should be free from any inaccuracies and informalities. New technologies allow for the observation of details invisible to the naked eye. Working with a capable student and one who struggles with mathematics has recently been the subject of many experiments and considerations. Perhaps ICT tools will help solve many problems related to this. Kutzler (2000) notices the possibility of building ‘scaffolding’ using a graphic calculator over an incomplete floor of knowledge. It compensates for the lack of more basic knowledge and allows for the avoidance of errors. The calculator enables the introduction of far-reaching facilitations in teaching through the trivialization of certain activities, experimentation, visualization, and focusing attention on the problem of the task. Using calculators is beneficial for students with spatial imagination problems, students from special classes with reduced requirements, and those lacking self-confidence (Dunham, 2000; Waits & Demana, 1996). In Duda’s research (2011) on the mathematical creativity of talented students, a graphic calculator was used in the process of solving problems. It turned out that this tool allowed students to perceive and formulate new problems, new and subjective theorems, proofs, and problem-solving methods, learn discipline and critical thinking, and also provided an opportunity for initial contact with many mathematical concepts, as well as deepening the understanding of already known mathematical concepts. The necessity of using calculators at moments when the teacher and the student encounter various difficulties – often impossible to overcome using traditional teaching methods and previously used teaching aids – is emphasized (Dałek, 1993; Kąkol, 2002). The role of robotics and artificial intelligence has played a significant role in recent years, each properly implemented in the learning-teaching process, whether as a mandatory or additional element, allowing the development of each component of critical thinking, especially needed in the process of solving non-standard tasks. In Borkowicz's doctoral dissertation (Borkowicz, 2024), one of the aims was to examine the role of robotics classes, specifically classes using LEGO, in different age groups (from elementary school students to university students), to examine the impact of systematic work with these tools on changes in skills, attitudes, and soft competencies in solving substantive problems. One of the most significant conclusions of this study is contained in the following quote:

> Research has also shown what difficulties in students can cause any negligence in the education process, including in the areas of cooperation skills, communication, problem-solving, creativity, information searching and verification, and logical thinking. It has been observed that elements such as exchanging experiences and ideas and constructing statements about one's current actions are for students not so much difficult as essentially foreign elements in the education process. Working with robotics tools can support the formation of attitudes allowing for effective teamwork. (Borkowicz, 2024, p. 105)

Openness to various ways of solving mathematical tasks, as well as patience and cooperation, must be learned in the process of education. It turns out that incorporating ICT into the process can not only help methodologically, materially, and heuristically (as described above) but also the use of ICT tools, including robotics, can prepare for dealing with precisely these challenges. There is still little research on the role of artificial intelligence and its full implementations. We see in the attempts at implementation that AI, like other ICT tools, can help in all the above-mentioned areas, and in the teaching-learning process, it can support the teacher in organizing personalized education, adjust tasks and problems to the student's level (Alshater, 2022; Baidoo-Anu & Owusu Ansah, 2023). According to Pokryshen (2024, p. 62):

> By providing all necessary data (text) for analysis to the ChatGPT 3.5 system, it performs its analysis very well and displays results. Therefore, it is important not only to formulate suggestions correctly but also to present input data. Tasks such as paraphrasing, translating, and changing the tone of text demonstrate all the capabilities of modern AI systems.

> ChatGPT 3.5 can become a good assistant to a teacher or an assistant in preparing various documents. Tasks related to helping write lesson plans, preparing educational activities, formulating project topics, or technological project maps are performed by the system at a high level. In such queries, it is important to provide the duration of the event (45 minutes), the class of students, the name of the educational subject, and any additional parameters, if necessary.

In the article *Artificial intelligence in mathematics education: A systematic literature review* (2022), the results of the analysis of 20 research publications published between 2017 and 2021 are presented, examining how AI can influence and improve the results of mathematics students during the teaching-learning process. AI can be implemented in mathematics education through various approaches: systems, teaching agents, autonomous agents, machine learning models, digital technology devices, and comprehensive approaches. However, it seems that robotics was the most commonly used among mathematics students, teachers, and educational researchers among all these approaches. Different attitudes exist among students, and teachers towards the use of ICT tools (Franczyk & Rajchel, 2024; Shen et al., 2023); it turns out that AI is closer to those who have education in the field of exact sciences, while the rest are still looking at this tool. In the study on the use of technologies such as ChatGPT in education, it was noticed that the main disadvantage is the lack of human interaction (Terwiesch, 2023). Technology cannot understand human emotions and various behaviors, which can lead to inappropriate reactions, including potentially harmful to students. Alshater (2023) emphasizes that ChatGPT cannot verify incorrect or biased data and has limited specialist knowledge, requiring additional verification and evaluation by experts.

## GOALS, ORGANIZATION AND METHODOLOGY

The author of this article has investigated what is the role of ICT tools in the process of learning and teaching and to what extent ICT tools are effectively used in the mathematics education process:

- Research on students using a graphic calculator (2005).
- Research on future mathematics teachers regarding ways of using ICT tools in the process of solving non-standard tasks, several years of qualitative research (2014-2019).
- Research on future mathematics and computer science teachers regarding ways of using ICT tools in the process of solving non-standard tasks (beginning of 2023).

Over the past 20 years, the author of the article has conducted qualitative research several times to examine the role of ICT tools in shaping mathematical concepts (Juskowiak, 2004, 2010), solving non-standard tasks (Juskowiak, 2021), and organizing the education process – the result of these analyses became the direction of studies in mathematics and computer science teaching

(Juskowiak, 2019). After the pandemic, which both allowed most of the population to learn to use these tools and at the same time tired of its excess, the author decided to find out again what the belief of future mathematics and computer science teachers (observation will last for the next 3 years) is regarding the inclusion of ICT (all possible tools), while also examining their ideas for implementing ICT themselves in the process of solving a non-standard mathematics task. This article will present the results of the survey and ideas for the implementation of ICT by future mathematics and computer science teachers during the solution of one mathematical task. The study involved students of mathematics and computer science teaching participating in the mathematics didactics subject (22 students). This was the second year of studies, so the time was not so distant from their student experiences, but at the same time, a moment when they have already worked through the basic issues of mathematics didactics, computer science didactics, learned various ICT tools, and participated in internships at primary school as observers or supporting observers. It was therefore a time when future teachers were very open to finding the best way to organize lessons. Unfortunately, it often happens that after the first continuous internships, future teachers become discouraged from previously discovered ideas for organizing the teaching process, or simply due to the good care of the school teacher, they change their way of thinking about organizing this process. The study involved women (19) and men (3). A survey was developed by the author of the article and the students were given a week to complete it. This took place after the end of mathematics didactics classes, the aim of which was to discuss the role of the teacher in the process of solving mathematical problems in many ways using various didactic tools. During these classes (3 meetings of 1.5 hours each), students, working in groups of 3 or 4, attempted to solve several tasks in many ways themselves, and in one of the geometric tasks, they were required to consider whether it would be possible to solve them using ICT and if so, how. Each group wrote that they would use the Geogebra program to visualize the situation given in the task and animations that would allow them to cut and move individual parts of the figure.

The content of this task was as follows: *Check what part of the rectangle's area is the area of the shaded figure.*

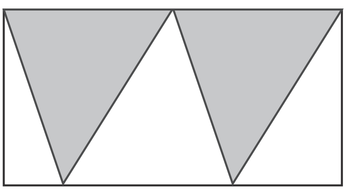


Figure 1: The drawing attached to the task.

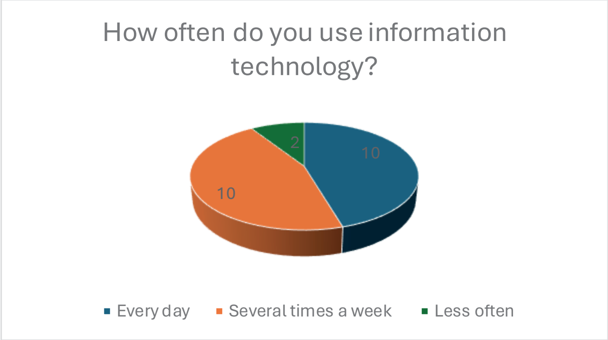
It is worth noting that none of the 6 tasks that preceded the task requiring the use of ICT were solved using ICT, nor was any idea for using ICT described during their solution.

The survey included the following questions:

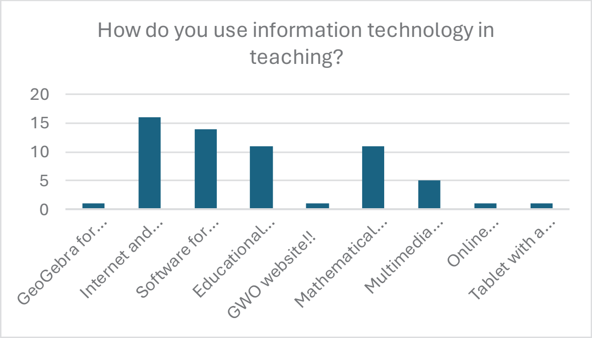
1. How often do you use information technology (e.g., computer, tablet, etc.)?
2. How do you use information technology in teaching?
3. Do you believe that using information technology facilitates solving mathematical tasks?
4. What benefits do you see in using information technology in teaching mathematics?
5. Do you think that teaching mathematical skills using information technology helps in getting students interested in mathematics? Why?
6. In solving which types of tasks do information technologies bring the most benefits?
7. In which branches of mathematics do you think tasks need support from new technologies?
8. What are your observations regarding students' reactions to using information technology in learning mathematics?
9. Do you think there are any limitations or challenges associated with using information technology in teaching mathematics? What are they?
10. Are you aware of any scientific research or good practices related to the use of information technology in teaching mathematics? If so, please provide examples.
11. What suggestions would you have for other mathematics teachers who would like to start using information technology in their work?
12. Are there any additional comments or observations you would like to share regarding the use of information technology in teaching mathematics?

Students were asked about their method of contact with students. 21 of them provide tutoring, 3 work in schools, 1 conducts classes as part of a science club, and 1 conducts extracurricular activities as part of alternative education. Some students chose multiple options.

The results of selected survey questions are presented below in the form of pie and bar charts, showing the number of responses and the percentage value.




Graph 1: The number of responses to question 1.



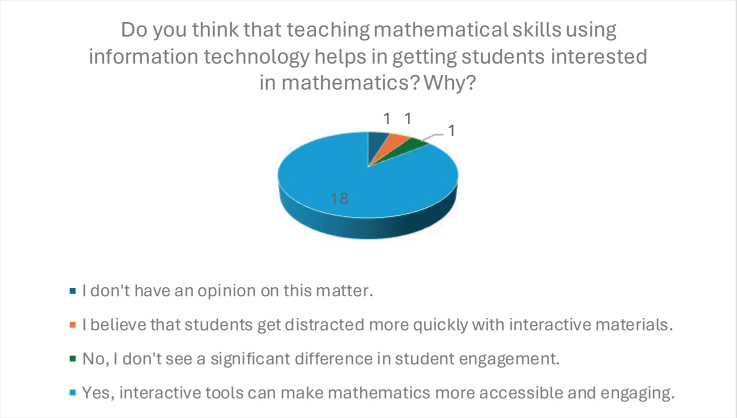

Graph 2: The number of responses to question 2.

| Do you believe that using information technology facilitates solving mathematical tasks? | |
|---|---|
| Answer | Number |
| Yes, thanks to various tools, one can better understand the problem. | 17 |
| Yes, because it allows for quick checking of results. | 15 |
| Yes, because they present several ways to solve a task, while in books authors often focus on one method. | 1 |
| Yes, it allows for quick correction of errors not resulting from misunderstanding of the material (e.g., computational errors) and focusing on the main content being conveyed. | 1 |
| They have a better impact on engaging students in the task. | 1 |

Table 1: The number of responses to question 3.

| What benefits do you see in using information technology in teaching mathematics? | |
|---|---|
| Answer | Number |
| The ability to tailor materials to different learning styles. | 13 |
| Development of technological skills alongside mathematical skills. | 2 |
| Faster assessment of student progress due to process automation. | 5 |
| Improved student engagement through interactive lessons. | 15 |
| Development of technological skills. | 2 |
| Enhancement of lessons, topics can be presented in an interesting way for students, thereby helping them remember more information. | 1 |
| Pointing students to reliable sources of information (which they will use for their own learning anyway). | 1 |

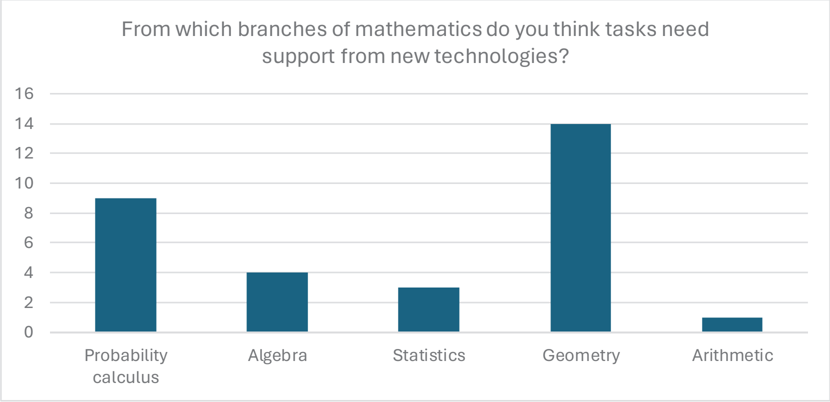
Table 2: The number of responses to the question 4.




Graph 3: The number of responses to question 5.

| In solving which types of tasks do information technologies bring the most benefits? | |
|---|---|
| Answer | Number |
| Exercise-type tasks | 11 |
| Problem-solving tasks | 8 |
| Tasks involving the application of theory | 10 |

Table 3: The number of responses to question 6.




Graph 4: The number of responses to question 7.

It is also worth noting that surveyed students clearly recognized the need for utilizing ICT primarily in the context of geometry, suggesting that this area of mathematics is particularly amenable to modern tools. The overwhelming majority of respondents emphasized that the use of ICT in the teaching and learning process of mathematics contributes to making classes more engaging and aids in better understanding of the material. Moreover, the surveyed individuals often utilize various ICT tools in their work with students, demonstrating their practical involvement in employing modern technologies in education.

Despite these positive trends, there is a noticeable discrepancy between the theoretical recognition of the value of ICT and the actual practice of its utilization. Although respondents expressed their belief in the benefits of using ICT, especially in the context of geometry and student engagement, the data indicate that the practical application of ICT is less common than one might expect. This discrepancy suggests potential barriers or limitations in the use of technology in mathematics education, which merit further investigation and understanding.

It is noteworthy that students expressed positive opinions about the facilitating role of ICT in problem-solving, citing benefits such as improved understanding, quick error correction, and increased engagement. Additionally, the data revealed preferences regarding the use of ICT in exercise-type tasks, problem-solving, and theoretical application, indicating its potential effectiveness in various mathematical domains. However, the gap between theoretical recognition and practical implementation warrants further consideration of obstacles hindering consistent utilization of ICT in pedagogical practices.

Furthermore, the diverse responses regarding perceived benefits and challenges suggest the need for individualized approaches to integrating ICT based on individual teaching contexts and student needs. Further research on the effectiveness of specific ICT tools and strategies in promoting mathematical proficiency and student engagement may provide valuable insights for teachers seeking to improve their instructional practices. It is also worthwhile to consider ways of effectively educating teachers on integrating ICT into their daily work and raising awareness of the potential benefits and limitations of this integration.

In the context of further research, it is important to identify factors that may inhibit the full utilization of the potential of Information Technology in mathematics education. Longitudinal studies tracking changes in students' attitudes toward the use of ICT as they progress in their education and pedagogical experience could be conducted. Additionally, comparative research analyzing the effectiveness of different ICT tools and strategies in various mathematics teaching contexts could yield valuable insights.

## CONCLUSIONS

From the conducted research, albeit local and conducted only on one group of students - future teachers, one can clearly see the evident need for using various ICT tools. Students believed that "Frequent use of information technology helps students visualize the task and facilitate finding the solution", Geometry and probability calculus, in which all types of tasks are equally open to the possibility of using ICT. Students admitted to frequently using these tools when working with students, which they experience, for example, during tutoring sessions. Although they acknowledged that they did not know the results of research on the benefits of ICT in the process of teaching and learning mathematics, they described exactly the same benefits from their use as those described in this article in its theoretical part. However, they rightly believed that "Not every lesson must be based on technology; it is worth taking several different courses with the same tool" and "I believe that information technology can be used in math lessons, but it should not be used all the time. It can be limited to review lessons – my suggestion here is Kahoot – I saw that students really liked the review in the quiz during my practices". They emphasized that the most important thing is to properly and comprehensively prepare future mathematics teachers during their studies to work with these tools. Students in the survey did not mention the possibility of using artificial intelligence even once.

### References

Alshater, M. (2022). *Exploring the Role of Artificial Intelligence in Enhancing Academic Performance. A Case Study of ChatGPT*. Philadelphia University, Jordan.

Baidoo-anu, D., & Owusu Ansah, L. (2023). Education in the Era of Generative Artificial Intelligence (AI): Understanding the Potential Benefits of ChatGPT in Promoting Teaching and Learning. *Journal of AI*, *7*(1), 52–62.

Borkowicz, B. (2024). *Wykorzystanie narzędzi robotyki w nauczaniu informatyki* [Use of robotics tools in computer science teaching]. Doctoral Dissertation, AMU Poznan.

Dałek, K., Dąbrowska, M., Zamek-Gliszczyński, T., Mostowski, K., Zawadowski, W. (1993). Przekonania i przeświadczenia w sprawie kalkulatorów [Beliefs and Convictions Regarding Calculators]. *NiM, 8,* 11.

Demana, F., & Waits, B. K. (2020). Calculators in Mathematics Teaching and Learning: Past, Present, and Future. In E. D. Laughbaum (Ed.), *Hand-Held Technology in Mathematics and Science Education: A Collection of Papers* (pp.1–11). The Ohio State University.

Duda, J. (2011). Twórczość matematyczna uczniów uzdolnionych a kalkulator graficzny [The Mathematical Creativity of Gifted Students and the Graphing Calculator]. *Didactica Mathematicae, 32*, 43–93.

Dunham, P. (2000). Hand-held calculators in mathematics education: A research perspective. In E. D. Laughbaum (Ed.), *Hand-Held Technology in Mathematics and Science Education: A Collection of Papers* (pp. 38–47). The Ohio State University.

Franczyk, A., & Rajchel, A. (2023). Postawy studentów wobec ChatGPT w edukacji [Students' Attitudes towards ChatGPT in education]. *Horyzonty Wychowania*, *23*(65), 89–101.

Herma, A. (2004). Wpływ kalkulatora graficznego na rozwijanie wybranych aktywności matematycznych (fragment badań wstępnych) [The Impact of the Graphing Calculator on the Development of Selected Mathematical Activities (Preliminary Research Findings)]. *Dydaktyka Matematyki, 26*, 81–94.

Juskowiak, E. (2004). Analiza pracy uczniów z kalkulatorem graficznym podczas rozwiązywania zadań (fragment badań) [Analysis of Students' Work with a Graphing Calculator During Problem Solving (Research Excerpt)]. *Dydaktyka Matematyki, 26,* 95–118.

Juskowiak, E. (2010). Graphic calculator as a tool for provoking students creative mathematical activity. In B. Maj, E. Swoboda, K. Tatsis (Eds.), *Motivation via Natural Differentiation in Mathematics* (pp. 268–280). Wydawnictwo Uniwersytetu Rzeszowskiego.

Juskowiak, E. (2019). Jak kształcić kompetentnego nauczyciela matematyki i informatyki? O doświadczeniach autorów kierunku Nauczanie matematyki i informatyki na WMiI w Poznaniu [How to Educate a Competent Mathematics and Computer Science Teacher? Insights from the Authors of the Mathematics and Computer Science Education Program at WMiI in Poznań]. In A. B. Kwiatkowska & M. Sysło (Ed.), *Informatyka w edukacji. Edukacja informatyczna a rozwój sztucznej inteligencji* [Computer Science in Education. IT education and the development of artificial intelligence] (pp. 118–126). Wydawnictwo Naukowe Uniwersytetu Mikołaja Kopernika.

Juskowiak, E. (2021). O tym, jak przyszli nauczyciele matematyki (nie) korzystają z nowych technologii w procesie rozwiązywania zadań [On How Future

Mathematics Teachers (Do Not) Use New Technologies in the Problem-Solving Process]. In T. Przybyła (Ed.), *Liczby w cyfrowym świecie. Rozmowy o współczesnej edukacji matematycznej dziecka* (pp. 75–90). Wydawnictwo Naukowe UAM.

Kąkol, H. (2002). Zintegrowane nauczanie matematyki z elementami informatyki w gimnazjum [Integrated Teaching of Mathematics with Elements of Computer Science in Middle School]. *Studia Matematyczne Akademii Świętokrzyskiej, 9.*

Krygowska, Z. (1977). *Zarys Dydaktyki Matematyki, części 1, 2, 3* [Outline of Mathematics Didactics, parts 1, 2, 3]. WSiP.

Kutzler, B. (2000). The algebraic calculator as a pedagogical tool for teaching mathematics. In E. D. Laughbaum (Ed.), *Hand-Held Technology in Mathematics and Science Education: A Collection of Papers* (pp. 98–110). The Ohio State University.

Legutko, M. (1990). Nauczyciel, reforma nauczania matematyki i mikrokomputer [The Teacher, Mathematics Education Reform, and the Microcomputer]. *Matematyka. Społeczeństwo. Nauczanie, 1,* 22–29.

Pawlak, R. (2004). Dojrzałość matematyczna [Mathematical Maturity]. *Dydaktyka matematyki, 26*, 289–311.

Pokryshen, D. (2024). Artificial intelligence in education: cases of using ChatGPT 3.5. *Physical and Mathematical Education, 39*(1)*,* 56–63.

Polya, G. (1964). *Jak to rozwiązać?* [How to solve it?]. Państwowe Wydawnictwo Naukowe.

Riyan, H., & Nurain, N. (2022). Artificial intelligence in mathematics education: A systematic literature review. *International Electronic Journal of Mathematics Education, 17*(3), 1–11.

Schoenfeld, A. H. (1980). Jak nauczać twórczego rozwiązywania zadań [How to Teach Creative Problem Solving]. In A. Góralski (Ed.), *Zadanie, metoda, rozwiązanie* [Task, method, solution]. Wydawnictwa Naukowo-Techniczne.

Shen Y., & Li, S. (2023). Pre-service teachers' usage of and beliefs about teaching mathematics with digital mathematical tools: What matters in China? In P. Drijvers (Ed.), *Proceedings of the thirteenth congress of the European Society for Research in mathematics education* (pp. 2788–2795). Alfréd Rényi Institute of Mathematics and Eötvös Loránd University Budapest, Hungary.

Terwiesch, C. (2023). Would ChatGPT Get a Wharton MBA? A Prediction Based on Its Performance in the Operations Management Course, Mack Institute for Innovation Management at the Wharton School, University of Pennsylvania, https://mackinstitute.wharton.upenn.edu/2023/would-chat-gpt3-get-a-wharton-mba-newwhite-paper-by-christian-terwiesch/

Waits, B. K., & Demana, F. (1996). A computer for all students revisited. *Mathematics Teacher, 89*(9), 712–714.

# A COMPARATIVE ANALYSIS OF MATHEMATICS STUDENTS' PERFORMANCE ON PAPER-PENCIL VS ONLINE ASSESSMENTS

Eliza Jackowska-Boryc*, Abimbola Akintounde**, Katarzyna Charytanowicz***,

*University of Marie Curie Sklodowska, Lublin, Poland

**American College of Education, IN, USA

***Paderewski International Secondary School, Lublin, Poland

*In the wake of the COVID-19 pandemic, schools around the world made dramatic changes to the modalities of their teaching and assessments. One such unprecedented change was the dynamic shift from traditional paper-pencil tests to test administration via remote learning systems and secured platforms, such as Moodle, Exam.net, Respondus and Proctorio. The purpose of this study was to compare the performance of International Baccalaureate students on paper-pencil tests with the results obtained on the same test when administered via the Exam.net computer-based platform.*

## INTRODUCTION

The unprecedented COVID-19 pandemic elicited an unraveled transformation in the landscape of teaching and learning across the globe (Engelbrecht et al., 2020). Educational institutions around the world, from K-12 schools through universities, were compelled to make pedagogical adjustments that would inadvertently affect over 1.5 billion students around the world (UNESCO, 2020). Mathematics education was not spared, as teachers had to harness expedient tools to ensure the continuity of teaching, learning and assessments for their students (Chirinda et al., 2021). Needless to say, digital technology became the panacea for mitigating the impending learning losses facing schools during the global lockdown (Gopika & Rekha, 2023). The abrupt transition to remote learning forced faculty to ponder on ways to innovate their traditionally offline summative assessments and course examinations (Sletten, 2021). Initially, many school systems quickly resorted to postponing examinations, but as the pandemic lingered beyond several months, it became increasingly urgent to seek sustainable alternatives to traditional paper-pencil assessments (Crawford et al., 2020). The pursuit of pragmatic alternatives to in-person paper-pencil assessments incited the emergent of electronic examinations via secured learning management systems, such as Moodle, Microsoft Teams, Respondus, Proctorio, Canvas, Exam.net, and Blackboard Collaborate.

## THEORETICAL FRAMEWORK

Lev Vygotsky's theory of constructivism served as the framework for this study, Constructivists theorize that adapting cognition-enhancing tools stimulates learner engagement, and motivates desired learning outcomes (Vygotsky, 1978). With regards to this study, computer-based testing (CBT) is hypothesized as a stimulus that could elicit differences in test scores when compared to traditional paper-pencil mathematics tests. Advocates of constructivism perceive that technology invigorates learning and alters the landscape of student achievement (Lotter & Jacob, 2020). Researchers have documented a higher probability of academic success with students taking online examinations when compared to their paper-pencil benchmark performances (Alonso-Conde & Zúñiga-Vicente, 2021). Other researchers have also reported that remote examinations geared learners to out-perform their paper-pencil prometric data from pre-COVID across several courses (Zheng et al., 2021).

## STATEMENT OF THE PROBLEM

Although online examinations have been tapped as credible platforms for evaluating student learning with fidelity (Ardid et al., 2015), the problem is that online assessments have remained controversial in academia (Itani et al., 2022). Skeptics argue that cheating, inequitable access to technology, internet mishaps, and concerns about test security render virtual assessments unsustainable (Crawford et al., 2020). Diverse studies have been conducted regarding user-friendliness, teachers' acceptance, and the cost-savings of e-Examinations (Eltahir et al., 2021; Ocak & Karakus, 2021; Pettit et al., 2021). However, empirical evidence to substantiate the efficacy of administering assessments using online platforms is limited, and deserves extensive investigation (Sletten, 2021; Vicario et al., 2023). Therefore, this study sought to compare mathematics students' performance on paper-pencil examinations with the performance recorded on the online version of the examination, when administered via Exam.net. The following research questions were posed to examine the objectives of this study:

1. How did the overall students' mathematics performance on 2019 paper-pencil examinations compare with overall performance in the first year of online CBT administration on Exam.net (2020)?
2. How did the overall students' mathematics performance on 2023 paper-pencil examination compare with overall performance in 2021 and 2022, where online CBT administration on Exam.net were used?

## METHODOLOGY

This quantitative causal-comparative study utilized existing verifiable archived data from the end-of-course International Baccalaureate (IB) exams that was administered to 251 final year secondary students between 2019 and 2023 from an International Baccalaureate Diploma Programme in a private high school in Poland. Causal comparative analysis is used to test for analyzing causes of differences in retrospect between two or more pre-existing groups (Schenker & Rumrill, 2004). Permission was duly obtained from the school administrators, and all prometric data gathered were cleaned, de-identified for any divulged demographic information, and sorted using Microsoft Excel. Figure 1 illustrates the enrolment of distribution of all 251 students by course levels from paper-pencil tests in 2019 and 2023 versus online Exam.net administration from 2020 through 2022.

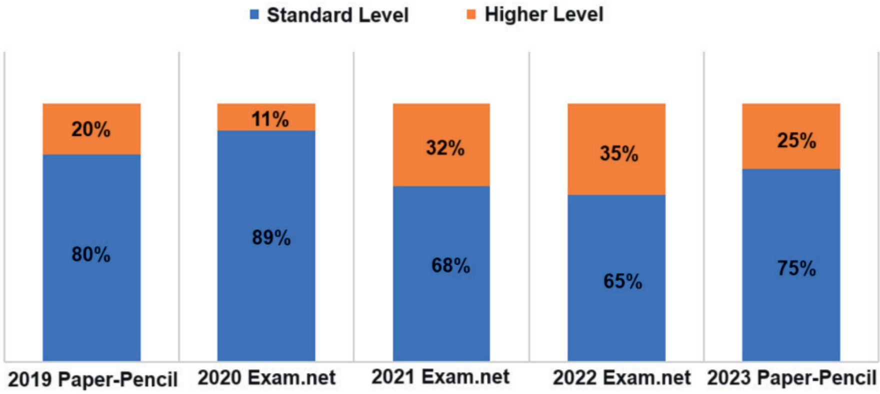



Figure 1: Student Enrolment in Standard Level versus Higher Level Mathematics.

In the light of the results presented in this study, it is worth paying attention to how the conducted tests were carried out before, after and during online education. The research covers the years from 2019 to 2023, where the results of mathematics tests in the international IB DP program were examined, during traditional and remote learning. In years 2019 and 2020, the scope of the assessed material was exactly the same, but the method of conducting classes and assessment differed. In 2019, students were taught in the classroom, while in 2020 they used remote education. Moreover, the final tests varied significantly. In 2019, students had to take the exam in person, which consisted of several components such as tests written under supervision and research work sent for evaluation (so-called Internal Assessment). In 2020, due to the outbreak of the Covid-19 pandemic, it was impossible to conduct stationary tests, so the final grades were based on one component – Internal Assessment. Therefore, the first research question concerns the comparison of grades from the last year of traditional education in 2019 with grades from the first year of CBT (year 2020). Then, the mathematics syllabus has changed in 2021, therefore the scope of the assessed material in 2021-2023 was different than in 2019-2020. Additionally, remote learning was used in 2021 and 2022, and traditional education in 2023. The way exams are conducted in these years is also an important variable. In

2021 (as in 2020), grades were awarded on the basis of works written by students, and there were no written in person examinations conducted. In 2022 and 2023, students took their examination in standard mode and their assessment included all components. For this reason, in the second research question, we decided to compare the results from three years: 2021, 2022 and 2023, using analysis of variance. It is also worth mentioning that exams written in person in 2019 has exactly the same number of questions and points as in new examination in 2022 and in 2023 where in person written exams were conducted.

Inferential statistical analysis was performed on Statistica 13.1 software. To answer the first research question, we needed to analyze the assumptions of the independent *t*-test. This type of test is used to compare results for two independent groups. If the difference in means is large enough, it is assumed that the two compared groups differ statistically significantly in terms of the value of the dependent variable. The *t*-test for independent samples has the most assumptions to meet: normality of samples and the homogeneity of variances of the compared groups. It is worth noting that if this assumption is not met, it is possible to perform this test with a correction taking into account the lack of homogeneity of variances. To verify if the assumption of normality is met, we will use the Shapiro-Wilk test. The Shapiro-Wilko test is considered the best test to check the normality of the distribution of a random variable. The main advantage of this test is its high power, i.e. for a given $\alpha$ (in our case 0.05), the probability of rejecting the null hypothesis (if it is false) is higher than in the case of other tests of this type. To check the homogeneity of variances we will use Fisher's F test (used to compare variances), Welch test (statistical test of equality of expected values in two populations; it is a generalization of the Student's t-test to populations with different variances), and Levene's test (is used to check whether the assumptions necessary to conduct the analysis of variance are actually met). We excluded outliers and independence of selected groups. To answer the second research question, we needed to compare results from three groups (grades from years 2021, 2022 and 2023). In this case, we used the ANOVA to test the difference between three means. Analysis of variance or ANOVA is a linear modeling method for assessing relationships between variables. For key factors and insights related to multiple charts, the ANOVA test checks whether the average predicted value differs across categories of multiple input variables. In our case we compare three groups that is the reason why we needed to apply ANOVA instead of simple *t*-test for independent samples as before. Then, similarly to the previous study, we used the Shapiro-Wilk test to verify the normality of the distribution of the tested samples. To assess the homogeneity of variances, we used Levene's test. Finally, we used the post-hoc test - Fisher's least significant difference test (smallest significant differences last significant differences – LSD. It involves determining the smallest significant differences between data in the samples) which allowed us to assess exactly which groups of the respondents are different from each other.

## RESULTS

For the first research question, the following hypotheses were posed:

$H_0$: There is no significant difference in the overall mathematics performance (SL+HL) on the 2019 paper-pencil examinations and the overall performance in the first year of online CBT administration on Exam.net (2020)

$H_1$: There is a significant difference in the overall mathematics performance (SL+HL) on the 2019 paper-pencil examinations and the overall performance in the first year of online CBT administration on Exam.net (2020).

The Shapiro-Wilk test indicated that the condition of normal distribution was met in the two independent samples. However, a $p < 0.05$ on the Levene's test indicated the heterogeneity of variances. Hence, the Welch test, which corrects for unequal variances in the F statistic, was adapted in lieu of the independent student *t*-test. Due to the resultant $p < 0.05$, the H0 hypothesis was rejected, consequently leading to the conclusion that there is a statistically significant difference between the results obtained by overall SL+HL students on paper-pencil (2019) and the students' performance in the first year of testing on Exam.net (2020).

| | *2019 (paper-pencil) SL+HL (All students' results)* | *2020 1st year administration of the Exam.net (All students)* |
|---|---|---|
| Mean | 53.61 | 65.79 |
| Observations | $n = 46$ | $n = 38$ |
| *p*-value | 0.00017 | |

Table 1: Overall SL+HL results on the paper-pencil (2019) vs. Exam.net (2020).

For the second research question the following hypotheses were posed:

$H_0$: There is no significant difference in the overall mathematics performance (SL+HL) on the 2023 paper-pencil examinations and the overall performance in 2021 and 2022, where online CBT administration on Exam.net were used.

$H_2$: At least one of the means in the overall mathematics performance (SL+HL) on the 2023 paper-pencil examinations and the overall performance in 2021 and 2022, where online CBT administration on Exam.net were used differ from the others.

Table 2 presents the number of observations and means in each of the three groups.

| | *2021* | *2022* | *2023* |
|---|---|---|---|
| Mean | 66,4 | 47,95 | 59,4 |
| Observations | $n$=50 | $n$=56 | $n$=65 |

Table 2: Overall SL+HL mean results on the paper-pencil (2023) vs. Exam.net (2021 and 2022).

The assumption of normal distribution was satisfied via the Shapiro-Wilk test. In turn, the variances for both groups were equal, as shown by Levene's test ($p$ -value>0.05), which means that the condition regarding homogeneity of variances was met. After conducting the ANOVA, we received $p$-value= 0.000000000042<0.05, which means that we reject the H0 hypothesis once again and conclude that there is a statistically difference between the results at least two samples. Additionally, we applied the post-hoc test - Fisher's least significant difference test to find out which pairs of data is a statistically significantly different. All $p$ -values are less than 0.05, that means the differences are statistically significant between each pair of data. The results of this test are presented in Table 3.

| | 2021 | 2022 | 2023 |
|---|---|---|---|
| 2021 | - | $p$=0.000000000011 | $p$=0.0061 |
| 2022 | $p$=0.000000000011 | - | $p$=0.0000021 |
| 2023 | $p$=0.0061 | $p$=0.0000021 | - |

Table 3: Overall SL+HL mean results on the paper-pencil (2023) vs. Exam.net (2021 and 2022).

## DISCUSSION AND CONCLUSION

The aim of this study was to compare mathematics students' performance on paper-pencil examinations with the performance recorded on the online version of the examination, when administered via Exam.net in years 2019-2023. The research results indicated a statistically significant difference between the results of mathematics exams of students who took exams in a traditional way in 2019 and students who used remote learning. Students studying remotely obtained statistically higher exam grades than students in 2019 despite the same syllabus for the exams. One of the factors that influenced the results is the fact that in 2020, final exam grades were based on one component, not three as in 2019. Moreover, the uncertain epidemiological situation meant that students could feel safer working from home while learning remotely. It should be added that this was the very beginning of remote learning, so students who took their final exams in 2020 had previously spent many months on traditional education at school.

The second research question presented equally interesting results. Statistical tests conducted for three independent groups (student results in 2021, 2022 and 2023) showed statistically significant differences between each pair of data tested. It is worth mentioning that in these years the same scope of material was comparable, which differed significantly from the scope of material in 2019-2020. Therefore, we could not conduct a statistical study comparing the differences between students' results obtained in 2019-2020 and 2021-2023. After conducting statistical tests, it turned out that students taking final exams in 2021 had significantly higher results than students in 2022 and in 2023. A significant factor was the fact that in 2021 students' final grades were awarded on the basis of one component (as in 2020), while in 2022 and 2023 all components were taken into account. Moreover, it is worth noting that in 2021, just like in 2022, students used remote learning, while in 2023 they used traditional forms of learning. In terms of how the exams are administered, in 2022 and 2023 students wrote all exams in a traditional, in-person manner. In 2021, final exams were replaced by an assessment of one component that had a significant impact on the final grade. After comparing the results from 2021 and 2023, it turned out, as before, that students in 2021 achieved higher results. The reasons may be similar to those mentioned earlier. Moreover, when comparing the results of students from 2022 and 2023, it turned out that the results of students in 2023 are statistically much higher than the results of students in 2022. To sum up the answer to the second research question, statistically the highest results were achieved by students in 2021 who studied in the mode remotely, and their final grades were based on one component. Then, students in 2023, during traditional learning, achieved statistically higher results than students in 2022. The students in 2022 who learned remotely, and their final grades were based on all components, performed the worst. The comparative analysis conducted in the study concerned only those groups that were comparable in terms of the mathematics material covered. Other comparisons would not be reliable.

This study was limited in scope to the International Baccalaureate end of course examination at one Polish school, and data was only archived for one CBT platform, namely the Exam.net. The interaction of potential confounding factors, such as teaching practices, variety of rigor in test items, student motivation and demographic differences, test anxiety and sample size on CBT performance could further investigated on a larger scale.  Nonetheless, the results of this study could inform policymaking regarding technology use for assessments.

## References

Alonso-Conde, J. A., & Zúñiga-Vicente, A. B. (2021). Online Assessment in Times of COVID-19 Using Two Platforms: Moodle and Teams. *Proceedings of the 13th International Conference on Education and New Learning Technologies, July 5-6, 2021.*

Alzahrani, M. (2022). *Traditional Learning Compared to Online Learning During the COVID-19 Pandemic: Lessons Learned from Faculty's Perspectives*. SAGE Open.

Ardid, M., Gómez-Tejedor, J. A., Meseguer-Dueñas, J. M., Riera, J., & Vidaurre, A. (2015). Online exams for blended assessment. Study of different application methodologies. *Computers & Education, 81*(8), 296–303.

Chirinda, B., Ndlovu, M., & Spangenberg, E. (2021). Teaching Mathematics during the COVID-19 Lockdown in a Context of Historical Disadvantage. *Education Sciences, 11*(4), 177.

Crawford, J., Butler-Henderson, K., Rudolph, J., & Glowatz, M. (2020). COVID-19: 20 countries' higher education intra-period digital pedagogy responses. *Journal of Applied Teaching and Learning, 3(1),* 1–20.

Eltahir, M. E., Alsalhi, N. R., & Al-Qatawneh, S. S. (2022). Implementation of E-exams during the COVID-19 pandemic: A quantitative study in higher education. *PLOS ONE, 17*(5), e0266940.

Engelbrecht, J., Llinares, S. & Borba, M. C. (2020). Transformation of the mathematics classroom with the internet. *ZDM Mathematics Education, 52*(5), 825–841.

Gopika, J. S., & Rekha, R. V. (2023). Awareness and Use of Digital Learning Before and During COVID-19. *International Journal of Educational Reform.*

Itani, M., Itani, M., Kaddoura, S., & Al Husseiny, F. (2022). The impact of the Covid-19 pandemic on on-line examination: challenges and opportunities. *Global Journal of Engineering Education, 24*(2), 16.

Lotter, M. J., & Jacob, L. (2020). Using smartphones as a social constructivist pedagogical. *Journal of Teaching in Travel & Tourism, 20*(2), 1–17.

Ocak, G., & Karakuş, G. (2021). Undergraduate students' views of and difficulties in online exams during the COVID-19 pandemic. *Themes in eLearning, 14*, 13–30.

Pettit, M., Shukla, S., Zhang, J., Sunil- Kumar, K. H., & Khanduja, V. (2021). Virtual exams: has COVID-19 provided the impetus to change assessment methods in medicine? *Bone & Joint Open Journal, 2*(2), 111–118.

Schenker, J., & Rumrill, P. D. (2004). Causal-comparative research designs. *Journal of Vocational Rehabilitation, 21*(3), 117–121.

Şenel, S., & Şenel, H. C. (2021). Use of take-home exam for remote assessment: A case study from Turkey. *Journal of Educational Technology and Online Learning, 4*(2), 236 –255.

Sletten, S. R. (2021). Rethinking Assessment: Replacing Traditional Exams with Paper Reviews. *Journal of Microbiology and Biology Education, 22.*

United Nations Educational, Scientific and Cultural Organization. (2020). *COVID-19 educational disruption and response.* UNESCO.

Vicario, C. M., Mucciardi, M., Perconti, P., Lucifora, C., Nitsche, M. A., & Avenanti, A. (2024). The impact of the COVID-19 pandemic on academic performance: a comparative analysis of face-to face and online assessment. *Frontiers in Psychology, 14.*

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes.* Harvard University Press.

Zheng, B., Lin, C.-H., & Kwon, J. B. (2020). The impact of learner-, instructor-, and course-level factors on online learning. *Computers and Education, 150*(1), 103851.

.

# IMPACT OF APLUSIX ON ALGEBRA PERFORMANCE AMONG PRE-SERVICE TEACHERS IN COLLEGES OF EDUCATION IN GHANA

Marlene Kafui Amusuglo, Antonín Jančařík

Charles University, Faculty of Education, Prague, Czech Republic

*This study sought to examine the impact of teaching algebra with APLUSIX software on students' performance in a College of Education (CoE) in Ghana. Two colleges code-named A and B were assigned as the experimental and control groups. The students were taught algebra with APLUSIX and without APLUSIX respectively. The intervention sessions lasted for of six weeks. Pre and post-test were respectively conducted before and after the treatment. One way analysis of variance (ANOVA) revealed that the students at College A who were taught using APLUSIX performed higher than students at College B who were taught without using the APLUSIX. It was concluded that APLUSIX is effective in enhancing students' performance in algebra.*

## INTRODUCTION

The use of computers as an educational tool has continued to grow rapidly as a new way to teach. Computer-assisted instruction (CAI) is an instructional method that has been developing for years (Liao & Chen, 2007). This idiom equates with other modern terms such as Computer-Based Instruction (CBI) (Hannafin & Foshay, 2008, Computer-Based Learning Environment (CBLE) (Moos & Azevedo, 2009), or Computer-Aided Learning (CAL) (Santally et al., 2004). CAI allows the use of a computer to provide instructional content (Seo & Bryant, 2009). It also gives immediate feedback when being interacted with by the user. Some CAI programmes can be adjusted according to students' ability levels, and others limit advancement until skill mastery is achieved. Instruction may also involve using the stand-alone software (Seo & Bryant, 2009).

The importance of Information and Communication Technology (ICT) in education is growing in significance as the globe quickly shifts to digital media and information (Dei, 2018). ICT has grown to play a significant role in the field of education, both as a subject and as an essential component of how instruction is delivered in schools (Du Plessis & Webb, 2012). With reference to the revised curriculum in mathematics in Ghana, ICT is required to be integrated into the teaching and learning of mathematics (MoE 2019). APLUSIX provides students with quick feedback, comprehensive guidance, and individualised learning experiences, it is preferred for teaching algebra over other software (Nicaud et al., 2004). This interactive approach has been shown to improve students' knowledge and performance in algebra (Tsikliras et al., 2018). The APLUSIX software was adopted to teach algebra at a CoE and the results compared with the teaching of algebra without the APLUSIX software, in another CoE, in Ghana, to ascertain the impact of using APLUSIX to teach algebra on students' performance in algebra. This study aimed to examine the effect of teaching using APLUSIX on pre-service teachers' performance in algebra.

## LITERATURE REVIEW

Educational research has focused on using technology-based aids in mathematics instruction, especially in algebra. For example, (Bouhineau et al., 2002; Nicaud et al., 2006) assert that programmes such as APLUSIX can improve student achievement. Janvier (1987) states that use of representations in mathematical thinking is fundamental and most of the textbooks today make use of a wide variety of diagrams and pictures in order to promote mathematics understanding. Again, Vlassis (2004) asserts that students encounter difficulties with algebraic concepts, including handling the negative sign in equations and algebraic expressions. Also, Printer (2010) asserts that in elementary algebra, the fundamental symbols and methods are grasp and this aids in understanding how real-world issues can be simplified into equations and solved. The author discussed that the skill of converting complex problems into symbolic representations forms the cornerstone of advanced mathematical and scientific pursuits, showing the remarkable capability of human thought.

ICT use in the Ghanaian society is dated back to the 1990s, but the country's national ICT agenda and the corresponding legislative framework were adopted in the 2000s (Frempong, 2011). The school computerization initiative launched in 2011, aimed to massively introduce ICT resources into classrooms and provide instructors with training (Natia & Seidu, 2015). According to Quaicoe and Pata (2018), the national ICT agenda was envisioned to accelerate Ghana's transition to an information communication and technology society. As a result, the Ministry of Education established a national agenda for ICT in education in 2008 (Quaicoe & Pata, 2018). A New Education Reform (NER) that was implemented in 2007 was influenced by this goal (Quaicoe & Pata, 2018). ICT was first introduced in schools during the NER 2007 academic year, both as a stand-alone subject and was integrated in all other subjects taught in primary schools. The fundamental national ICT agenda in Ghana aims to give learners the ability to confidently and creatively use ICT resources and tools in order to develop the necessary abilities and expertise to be fully engaged in the worldwide economy of knowledge (Frempong, 2011). APLUSIX is an interactive software that allows students to work through algebraic problems step-by-step, providing immediate feedback and guidance (Bouhineau et al., 2002; Nicaud et al., 2006). This interactive nature of the software can help students develop confidence in using ICT resources. The flexibility of APLUSIX, allows students to input their mathematical expressions, this can foster a sense of creativity in using ICT tools and solve algebraic challenges (Herrington, 1999).

ICT as a tool has the ability to change how education is provided, according to Fisher (2005). It allows for the customisation of both the subject matter's delivery and content to meet each student's unique requirements and experiences (Fisher, 2005). According to Schiller and Tillett (2004), primary school teachers must incorporate and use ICT tools in their lessons since both the teachers and the pupils benefit from the experience. Similarly, Onasany (2009) asserts that ICT can be used to prepare the present generation for the future workforce, and it can also make the teacher more effective and efficient thereby increasing the school's productivity. ICT can support students in becoming autonomous learners capable of creating collaborative projects, inquiry, and critical thinking and problem-solving skills (Onasany, 2009). It also enables information searches, software programming, group collaboration, idea generation, and revision (Dei, 2018). In recent times, students' performance in mathematics has been a subject of intense discussion and research interest. Students' weaknesses have been identified in algebra. For example, the chief examiner's report on the West African Senior School Certificate Examinations for core mathematics for 2014 and 2017 revealed that among other topics, students had weaknesses/difficulties in the following areas: word problems, variations, binary operations, logarithms, and which are heavily dependent on algebra (WAEC, 2014; 2017).

Observations at College A and College B revealed that students have major issues when it comes to solving problems in algebra. Issues such as wrong manipulations and applications of mathematical operations (division, multiplication addition and subtraction), inappropriate application of laws in indices and logarithms, and wrong application of mathematical concepts easily come to the fore. This has reflected in the number of students who do not perform well during the end of semester examinations each year in University of Cape Coast, Institute of Education. Colleges of Education, Three Year Diploma in Basic Education. For instance, 20% and above are referred in Number and Basic Algebra each year (Analysed result for End of First Semester Examination 2014-2017 in College A). With reference to the revised curriculum in mathematics, ICT should be integrated into the teaching and learning of mathematics (Agyei & Voogt, 2011). It is for this reason that APLUSIX software is being adopted to teach algebra at College A to ascertain the impact it can make on the performance of students in algebra. As we already mentioned, the aim of the study was to examine the effect of teaching using APLUSIX on pre-service teachers' performance in algebra. The following research question was developed to guide the study: What is the effect of teaching using APLUSIX on pre-service teachers' academic performance in algebra?

## DESCRIPTION OF APLUSIX

APLUSIX, as introduced by Bouhineau et al. (2002), is a software which stands out as a learning platform tailored for formal algebra. It incorporates an advanced expression editor that presents expressions in their conventional format and allows seamless modification. This editor operates based on the structural properties of algebraic expressions outlined by Kieran (1991), facilitating actions like selection, cut, copy, paste, drag, and drop, wherein only valid sub-expressions can be manipulated. APLUSIX empowers students to make and learn from their errors by freely developing calculation steps represented as rectangles containing expressions. The system systematically verifies these steps for equivalence, with results promptly displayed.

These editors facilitate numerical calculations and formal algebraic tasks such as expansion, factorization, and solving equations, inequalities, and systems of equations. They also provide immediate feedback on the equivalence of consecutive expressions. The allowed domain includes numerical expressions with integers, decimals, fractions, and square roots, as well as algebraic expressions with degrees up to 4 (including polynomials and rational expressions with a sum of degrees up to 4), equations, inequalities, or systems of equations. Students receive strategic guidance about the problem's state to aid them in reaching a solution. Gauges like ‘Reduced’, ‘Sorted’, ‘Expanded’, ‘Factored’ (for numerical and polynomial factors), and ‘Equation’ can assist users (Bouhineau et al. 2002).

## METHODOLOGY

### Research design

This study’s quasi-experimental design made use of already-existing intact groups. This study used this method to assess students taught with the APLUSIX software. According to Creswell (2012), a quasi-experimental design is a form of experimental design in which both groups remain intact while an independent variable (treatment) is manipulated.

### Study area

Two CoEs were selected from Ghana and code-named College A and College B were used for the study. The colleges were code-named due to anonymity. The selected participants were already in intact groups in the selected CoEs, with 40 students in each group. Participants in College B were assigned to the control group, whereas those in College A were assigned to the experimental group.

### Data collection instruments

Two different tests but equivalent forms were used to assess participants. One was used for the pre-test, and the other was used for the post-test. Pure mathematics test items were selected from the test bank of the CoE and were used to gather the required data and a few of the items are shown below. Both tests were made up of 14 constructed-response-item formats each. Fifty minutes was allowed for the test. Students were asked to simplify, factorise, and find the truth set the following.

1. $2(4x-5)-6\ (5x+5y-4)$
2. $(-2x^2mn)(-4\ xy^2m^2n)$
3. $54x^2y^2t \div 6xy$
4. $a^2b-\frac{2}{3}\ ab^3$
5. $(2x-4)(x+1)-(x-2)(x+2)=0$
6. $2y(1-x)-3(x-1)$
7. $5x^2-13x-6$
8. $9y^2-10+1$
9. $2x^2+3x+1$
10. $5\ +8\ (x+2)=23-2(2x-5)$
11. $\frac{x}{3}-\frac{x}{5}=4$
12. $\frac{x}{3}-\frac{1}{3}(x-4)=\ 2x+\frac{3}{2}$

**Data collection procedures**

An introductory letter was collected from the Head, Department of Mathematics and ICT Education, University of Cape Coast (UCC) to the Principals of College A and College B to enable easy access to the students and classes where participants were selected for the study. The intervention was implemented during the normal classes' hours for six continuous weeks in the respective classroom for College B students and the computer laboratory for College A students.

**Intervention**

Participants were briefed on the purpose of the study and made aware of the ethical considerations regarding the confidentiality and anonymity of the results of the study. Unfortunately, they had no option to withdraw from the study since they were intact groups, and the intervention was done during regular mathematics lessons. As part of pre-intervention preparation, the APLUSIX software was installed on all the computers in the computer laboratory in College A. The control group in College B was taught algebra by the subject tutor without using the APLUSIX software while the researcher taught the experimental group in College A using the APLUSIX software. This intervention was done twice a week for six weeks during the regular mathematics class. The experimental group was introduced to the software (i.e., how to boot the device, where to locate, open, set and get the software ready for work). Intensive tuition and solving of problems with the APLUSIX software continued for the next weeks till the sixth week. After the intervention, both groups were taken through the final post-test and the test scores were recorded for comparison and analysis.

## RESULTS

The study's purpose was to examine the effects of teaching algebra with APLUSIX on students' performance in selected Colleges of Education (CoEs). Our research question was What is the effect of teaching using APLUSIX on pre-service teachers' academic performance in algebra?

ANOVA was used to compare the post-test scores of the experimental and control groups using their pre-test score. Table 1 presents the descriptive statistics of the two groups.

| | | Pre-test | | Post-test | |
|---|---|---|---|---|---|
| Groups | N | M | SD | M | SD |
| Control | 40 | 56.35 | 8.23 | 56.83 | 7.38 |
| Experimental | 40 | 62.83 | 9.64 | 66.15 | 8.14 |

Table 1: Descriptive Statistics of test results (Source: Field survey (2019)).

Table 1 shows that the mean scores for the control group before and after the intervention were 56.35 and 56.83 respectively, whereas the mean scores for the experimental group before and after the intervention were 62.83 and 66.15 respectively. The experimental group started with a higher mean score (62.83) than the control group (56.35). This suggests that, before any intervention, the experimental group might have had a higher proficiency or ability, in algebra compared to the control group. A one way between-groups analysis of variance was conducted to compare the effectiveness of teaching with APLUSIX on students' performance in algebra. The independent variable was the intervention (teaching with APLUSIX), and the dependent variable was the scores of students after the intervention. Participants' scores before the intervention (pre-test) were used as the variate in the analysis. Table 2 shows the test for the differences in the groups on their post-test scores.

| Source | Df | Mean Square | F | Sig. | Partial Eta Squared |
|---|---|---|---|---|---|
| Corrected Model | 2 | 1155.309 | 21.470 | 0.000 | 0.358 |
| Intercept | 1 | 3266.644 | 60.707 | 0.000 | 0.441 |
| Pre-test | 1 | 571.506 | 10.621 | 0.002 | 0.121 |
| Group | 1 | 958.047 | 17.804 | 0.000* | 0.188 |

| | | |
|---|---|---|
| Error | 77 | 53.810 |
| Total | 80 | |
| Corrected Total | 79 | |

Table 2: ANOVA Test of Between-Subject Effects of the Post-Test Scores (Source: Field survey (2019); *Significant at 0.05 level).

Preliminary analysis was done to ensure that there were no violations of the assumptions. As presented in Table 2, after equating groups on their pre-test scores, there was a statistically significant difference in the post-test scores for the two groups, $F(1, 77) = 17.80$, $p < 0.05$, partial eta squared = 0.19. There was a weak relationship between the pre-test scores and the post-test scores in algebra, as indicated by a partial eta squared value of 0.12. To further explain the findings, adjusted means of each group after controlling for their pre-test scores are presented in Table 3.

| Groups | Adjusted Mean | Standard Deviation |
|---|---|---|
| Control | 57.80 | 1.20 |
| Experimental | 65.17 | 1.20 |

Table 3: Adjusted Means (Source: Field survey (2019)).

From Table 3, after controlling for the pre-test scores of the groups, the adjusted mean score for the control group was 57.80, with a standard deviation of 1.20. However, the adjusted mean score for the experimental group was 65.17, with a standard deviation of 1.20.

## DISCUSSION

The results showed that the experimental group performed better in algebra than the control group after controlling for their pre-test scores. This implies that the APLUSIX intervention was effective in enhancing students' performance in algebra. The content for this study was algebra, which happens to be one of the fundamental topics in the Ghanaian curriculum for mathematics. APLUSIX is a computer application that allows students to freely build and transform algebraic expressions and solve algebra exercises like it is done on paper. It also gives feedback on the correctness of steps. This feedback is done by a function that organises and statistically analyses the data and displays the learner's results on the screen (Nicaud et al., 2006). The teacher can use these results to analyse individual learners' performances. Students were able to concentrate on comprehending the fundamental algebraic concepts since APLUSIX provided them with step-by-step coaching and fast feedback, which probably optimised the cognitive load (Tsikliras et al., 2018). This was made possible by the interactive and personalised features of the APLUSIX programme (Nicaud et al., 2004). The APLUSIX-based instruction was significantly more effective in encouraging meaningful learning and improving students' algebraic ability. In this study, the computer provided a reality of the situation in which the students may learn vicariously through interaction with the model (APLUSIX). Based on the findings of the study, it can be concluded that APLUSIX is effective in enhancing students' performance in algebra.

## References


Agyei, D. D., & Voogt, J. M. (2011). Exploring the potential of the will, skill, tool model in Ghana: Predicting prospective and practising teachers' use of technology. *Computers & Education, 56*(1), 91–100.

Bouhineau, D., Nicaud, J. F., Pavard, X., & Sander, E. (2002). A Microworld for Helping Students to Learn Algebra. *Proceedings of ICTMT*, 5, 9–22.

Creswell, J. (2012). Educational research: Planning, conducting and evaluating quantitative and qualitative research (4th ed.). Pearson.

Dei, D. G. J. (2018). Assessing the use of Information and Communication Technology in teaching and learning in secondary schools. *Library Philosophy and Practice 2*(2), 1–10.

Fisher, T. (2005). *Technology means nothing without online.* http://www.tes.co.uk/search/story/?story_id=20529.

Frempong, G. (2011). Developing information society in Ghana: How far? *The Electronic Journal of Information Systems in Developing Countries, 47*(1), 1–20.

Hannafin, R., & Foshay, W. (2008). Computer-based instruction's (CBI) rediscovered role in K12: An evaluation case study of one high school's use of CBI to improve pass rates on high-stakes tests. *Educational Technology, Research and Development, 56*(2), 147–160.

Herrington, J. (1999). Using situated learning and multimedia to investigate higher-order thinking. *Journal of Educational Multimedia and Hypermedia, 8*(4), 401–421.

Janvier, C. (1987). *Problems of representation in the teaching and learning of mathematics*. Lawrence Erlbaum Associates.

Kieran, C. (1991). A procedural-structural perspective on algebra research. In F. Furinghetti (Ed.), *Proceedings of the 15th International Conference for the Psychology of Mathematics Education* (Vol. 2, pp. 245–253). Genoa, Italy.

Liao, Y. K., & Chen, Y. W. (2007). The effect of computer simulation instruction on student learning: A meta-analysis of studies in Taiwan. *Journal of Information Technology and Application, 2*(2), 69–79.

Ministry of Education (2019). National teacher education curriculum framework: The essential elements of initial teacher education. Ghana Education Service.

Moos, D., & Azevedo, R. (2009). Learning with computer-based learning environments: A literature review of computer self-efficacy. *Review of Educational Research, 79*(2), 576–600.

Natia, J., & Seidu, A. (2015). Promoting teaching and learning in Ghanaian Basic Schools through ICT. *International Journal of Education and Development using ICT, 11*(2), 113–125.

Nicaud, F., Bittar, M., Chaachoua, H., Inamdar, P., & Maffei, L. (2006). Experiments with APLUSIX in four countries. *International Journal for Technology in Mathematics Education, 13(*2), 79–88.

Nicaud, J. F., Bouhineau, D., & Chaachoua, H. (2004). Mixing microworld and CAS features in building computer systems that help students learn algebra. *International Journal of Computers for Mathematical Learning*, *9*(2), 169–211.

Nicaud, J. F., Bouhineau, D., & Chaachoua, H. (2004). Mixing microworld and CAS features in building computer systems that help students learn algebra. *International Journal of Computers for Mathematical Learning, 9(*2), 169–211.

Onasanya, S. A. (2009). *Innovations in Teaching / Learning V: Information and Communication Technology (ICT) in Education*. http://www.unilorin.edu.ng /publications/onasanya/ICT%20in%20Edu.pdf.

Plessis, A., & Webb, P. (2012). Teachers' Perceptions about their Own and their Schools' Readiness for Computer Implementation: A South African Case Study. *Turkish Online Journal of Educational Technology-TOJET, 11(*3), 312–325.

Pinter, C. C. (2010). *A Book of Abstract Algebra*. Courier Corporation.

Quaicoe, J. S., & Pata, K. (2018). Basic school teachers' perspective to digital teaching and learning in Ghana. *Education and Information Technologies, 23*(2), 1159–1173.

Santally, M., Boojawon, R., & Senteni, A. (2004). Mathematics and computer-aided learning. *Academic Exchange Quarterly, 8*(2), 194–199.

Schiller, J., & Tillett, B. (2004). Using digital images with young children: Challenges of integration. *Early Child Development and Care, 174*(4), 401–414.

Seo, Y., & Bryant, P. (2009). Analysis of studies of the effects of computer-assisted instruction on the mathematics performance of students with learning disabilities. *Computers & Education, 53*, 913–928.

Tsikliras, K., Solomos, N., & Lazakidou, G. (2018). The use of educational software in teaching algebra: a review of the literature. *International Journal of Information and Education Technology, 8*(2), 134–138.

Vlassis, J. (2004). *Meanings and symbols in mathematics: Study of the use of the "month" sign in polynomial reductions and the solution of equations of the first degree to an unknown.* Peter Lang.

West African Examinations Council. (2014). *Chief examiners' report on WASSCE core mathematics*. https://www.waecgh.org/Exams/ChiefExaminersReport.aspx.

West African Examinations Council. (2017). *Chief examiners' report on WASSCE core mathematics*. https://www.waecgh.org/Exams/ChiefExaminersReport.aspx.

# ATTITUDES TOWARDS GRAPHING CALCULATORS AND THE SELF-EFFICACY OF MATHEMATICS STUDENTS

Abimbola Akintounde

American College of Education, IN, USA

*The purpose of this study was to analyze the relationship between students' self-efficacy in graphing calculators and their attitude towards calculator usage in mathematics instruction. Graphing calculators provide learners with the opportunity for making conjectures and bridging complex mathematical ideas. Student voices have been historically ignored in calculator research in pedagogy. Therefore, this quantitative study randomly sampled the opinions of 32 upper-level secondary mathematics students in the United States to correlate the relationship between students' self-efficacy and their attitudes towards utilizing the graphing calculator. A significant correlation was found between their self-efficacy and their dispositions regarding calculator usage in mathematics.*

## INTRODUCTION

Mathematics has remained a challenging subject for many secondary students (Clark, 2011). It was expected that the advent of technological innovation, such as the calculator would offer a glimpse of hope as a versatile tool for aiding students in problem-solving. For decades, the calculator has been revered as a vital classroom technology, yet its usage has been deterred by divergent perspectives held regarding this classroom technology (Neubauer, 1982; Parkhurst, 1979). Initially, educators were sparingly open to negotiating the use of 17th Century four-function calculators to aid the arithmetic of emerging mathematics, the advent of 19th Century graphing calculators (with automated capabilities for graphing, solving equations, analyzing statistics) further tainted the tone of the already-contentious debate (Ellington, 2003). Although, the National Council of Teachers of Mathematics credited the emergence of this new technology as a natural partner in the emergence of problem solving (NCTM, 1989), opponents argued that the demerits outweigh the potential benefits obtainable from its integration in mathematics (Sigg, 1982).

Technology avails learners with the opportunity for making conjectures and bridging complex mathematical ideas (Zembat, 2008). The National Council of Teachers of Mathematics recommends that technology should be incorporated into reinforcing the conceptual understanding of mathematics (NCTM, 2000). Advocates of calculators argue that students should be allowed to harness technology tools to problem solve (Crawford et al., 2016; Orellana & Barkatsas, 2017). On the contrary, skeptics have rejected the call as an attempt to use technology to circumvent the natural order of learning by productive struggle

(Boyle & Farreras, 2015). Others have condemned graphing calculators as a cheating device due to the possibility of maneuvering the programming app for storing formulas and algorithms (Bain, 2015; Migicovsky et al., 2014). Students are becoming more dependent on calculators to perform the most basal mathematics operations that they should have mastered with automaticity (Lightner, 1999). Over reliance on graphing calculators could adversely affect secondary students' success in college calculus, especially if they become accustomed to using this technology tool to evade the mastery of underlying mathematical (Mead, 2014). As graphing calculators are now becoming more ubiquitous, more affordable, and more sophisticated in their capabilities, the need for evidence-based guidance on the role of calculators in mathematics instruction has become imperative (Miles, 2008).

## THEORETICAL FRAMEWORK

Lev Vygotsky's Social Constructivist Theory and Social Cognitive Theory (SCT) postulated by the renowned Canadian American psychologist, Albert Bandura, were tapped to build the theoretical framework for this study. Social Constructivism opines that knowledge construction requires epistemological access to an individual via an affective stimulus (Vygotsky, 1987). The attitudes of students towards graphing calculator usage are postulated to be critical to their learning. Bandura also postulated that learning does not take place in a vacuum, but in a social setting in which variables such as self-efficacy and attitudes influence the production of effort by an individual (Bandura, 1986). Self-efficacy has been described as an inherent belief or confidence in one's own ability to fulfill a task (Osborne & Dillon, 2008; Pajares, 1996). Self-efficacy does not emanate from one's skill proficiency, rather it evolves as a product of self-judgement of one's capacity to execute, cope and persist in accomplishing a task (İnce, 2023). On the other hand, attitudes have been described as a learned favorable or unfavorable disposition towards a task or object (Fishbein & Ajzen, 1975). Enabling an attitudinal affinity towards a task or object is contingent on the individual's self-efficacy (Broekman et al., 2002).

Attitudinal studies on calculators have correlated students' dispositions with their mathematics achievement (Munger & Loyd, 1989). Previous studies have also correlated self-efficacy with students' achievement in mathematics (Pajares & Miller, 1994). The higher a student's self-efficacy, the longer they persevere through mathematical productive struggle until accurate results are obtained in problem solving (Fast et al., 2010; Peters et al., 2013). Researchers compel that graphing calculators increase students' self-confidence, lowers anxiety, and subsequently enhances learner engagement in mathematics (Waits & Demana, 2000). By identifying and mitigating any existing antipathy towards calculators, educators could alter the landscape of student achievement in mathematics (Abdullah et al., 2005).

## STATEMENT OF THE PROBLEM

The problem is that there is limited empirical evidence to substantiate a relationship between students' self-efficacy in adapting the features of graphing calculators and their attitudes towards calculators for mathematics problem solving. While an enormous number of studies have examined the academic gains ensuing from teacher resistance or adoption of calculators as a pedagogical tool, researcher have excluded students who are the eventual users of this technological tool (Hembree & Dessart, 1986). Self-efficacy and attitudinal disposition towards technology have not garnered adequate attention enough to substantiate policy making for mathematics classrooms (Thurm & Barzel, 2020). Therefore, students' attitudes and their self-efficacy regarding calculators need to be investigated.

## JUSTIFICATION FOR THE STUDY

An individual's attitude towards technology could deter the acceptance or rejection of a seemingly advantageous tool, such as the graphing calculator (Rogers, 1983). It has been reported that calculators serve as a positive motivation for students to develop a positive attitude and more confidence in mathematics as a subject (McCauliff, 2003). Their rationale was that students who were allowed to take assessments with the graphing calculator demonstrate a positive attitude towards calculator usage (Hembree & Dessart, 1986). Yet, others contend that the attitudes of students towards mathematics have remained indifferent despite their calculator affordances (Ellington, 2003). Aside from the chasm in empirical evidence regarding students' attitudes with non-usage and usage of calculators in mathematics, the efficacy of students in adapting these technology tools requires further investigation. Some studies have shown that students who have low self-efficacy in utilizing calculators were more likely to avoid using it to check their work in mathematics classrooms (Graham et al., 2008). The inconclusive nature of existing empirical evidence to correlate the relationship between mathematics students' self-efficacy and their attitudes towards utilizing the graphing calculator demands extensive investigation.

## THE PURPOSE OF THE STUDY

This study analyzed the relationship between students' self-efficacy in utilizing the typical features of a graphing calculator and their attitude towards its incorporation in mathematics instruction. The following research questions were posed for the objectives of this study:

1. To what extent does gender account for variations in students' attitudes and self-efficacy towards graphing calculator usage in mathematics instruction?

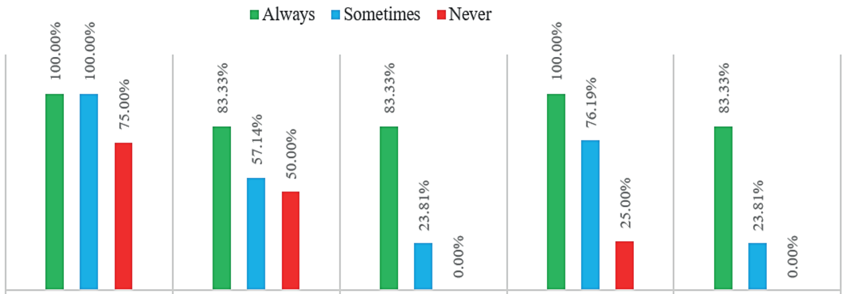
2. To what extent do students' self-efficacy in using graphing calculators differ based on the frequency of use allowed by their mathematics teachers?
3. To what extent do students' dispositions towards graphing calculators vary based on the frequency of use allowed by their mathematics teachers?
4. How are students' self-efficacy related to their attitudes towards utilizing graphing calculators?

## METHODOLOGY

This quantitative study was conducted by disseminating an anonymous electronic survey to 32 grade-12 students randomly selected from a list of 105 graduating students in the advanced mathematics track of an International Public High School in the Washington DC area of the United States. Students' use of the TI-84 Texas Instruments calculators in both curricula were strictly determined by their teachers' regulations, ranging from always, to sometimes or never allowed. All responses were coded, cleaned, sorted and analyzed using Microsoft Excel. Descriptive statistics were illustrated as frequences and charts, while the inferential analysis of the hypotheses were implemented using the Analysis of Variance (ANOVA) and Pearson's Product Moment Correlation Coefficient (PPMC).

## RESEARCH RESULTS

Five of the mathematics students in this study claimed that their teachers completely ban the use of graphing calculators, 21 students (65%) had teachers who sparingly allowed graphing calculators, while six students stated that their teachers always allowed graphing calculators. Respondents were asked to select the frequency of graphing calculator usage in their math classes. Eleven items regarding attitudes towards calculators in the mathematics classroom were posed for students to rate on a 5-point Likert scale, ranging from Strongly Disagreed through Strongly Agreed. Six items were also posed on a 5-point Likert scale regarding students perceived self-efficacy in utilizing the graphing calculator. Students whose teachers limited calculator access were the least confident in the use of programming, statistical, and calculus features of the graphing calculator. On the contrary, students whose teachers frequently integrated graphing calculators into instruction reported the highest self-efficacy in utilizing graphing calculators (see Figure 1).




| | Graphing tools | Equation Solver | Programming Apps | Calculus (Integrals/Derivatives) | Statistics/Function Writer |
|---|---|---|---|---|---|
| Always | 100.00% | 83.33% | 83.33% | 100.00% | 83.33% |
| Sometimes | 100.00% | 57.14% | 23.81% | 76.19% | 23.81% |
| Never | 75.00% | 50.00% | 0.00% | 25.00% | 0.00% |

Figure 1: Bar chart of self-efficacy in utilizing graphing calculator apps.

Students whose teachers always allowed calculator use exhibited a more positive attitude towards graphing calculators, but only 50% of them reportedly indicated an interest in pursuing more rigorous mathematics. Similarly, students who are sometimes allowed showed the highest hesitation for pursuing harder mathematics (29%) if given a calculator.

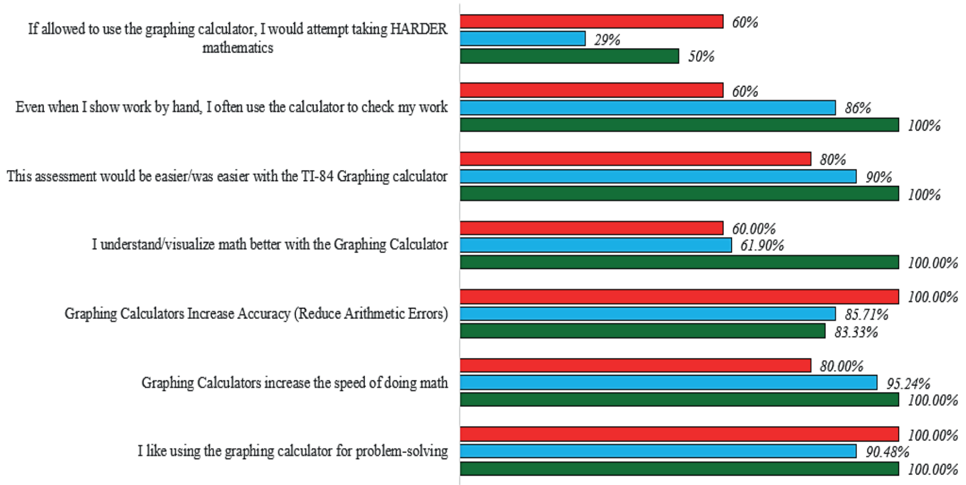



Figure 2: Histogram of students' positive disposition towards graphing calculators.

Students who were always deprived access to graphing calculators still indicated some interest in pursuing harder mathematics, if availed a calculator. Students from classrooms where graphing calculators were banned ranked highest (100%) in the disposition that calculators could salvage students from arithmetic errors. Most students did not agree to preferring showing work compared to graphing calculator, neither did they agree that using the calculator was equivalent to cheating. Many of the students perceived that calculator usage reduces the rigor expected by teachers.

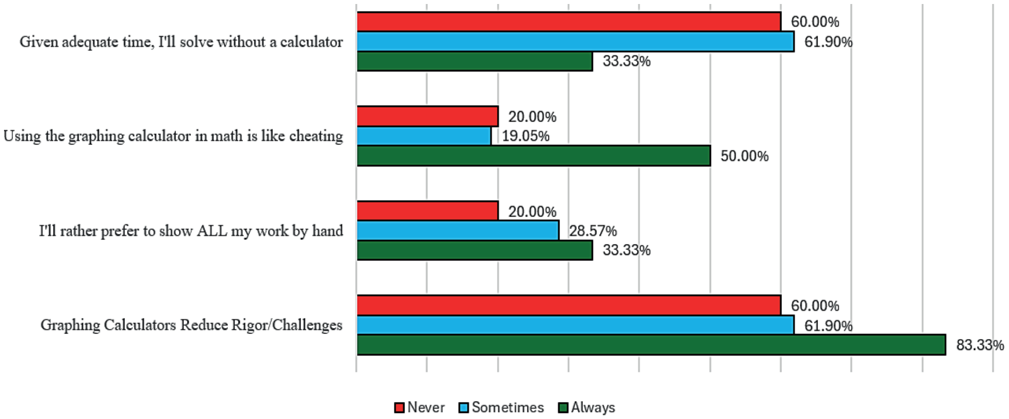



Figure 3: Negative dispositions regarding graphing calculator.

There was a significant difference in $H_1$ attitudes ($F_{(1,63)} = 5.28$, $p = 0.0077$) by gender and there was also a significant difference in $H_2$ self-efficacy ($F_{(1,20)} = 6.06$, $p = 0.02$) towards graphing calculators by gender. In contrast to females and non-binary students, males demonstrated a higher self-efficacy and positive disposition towards using graphing calculators.

$H_3$ There was a significant difference in the students' self-efficacy by the frequency of calculator usage ($F_{(2,29)} = 4.18$, $p = 0.025$). The mean self-efficacy of students from classrooms that always allowed students to use graphing calculators was greater than those of students who never got a chance to use graphing calculators.

| *Source of Variation* | *SS* | *df* | *MS* | *F* | *P-value* |
|---|---|---|---|---|---|
| Between Groups | 108.63 | 2 | 54.32 | 4.18 | 0.025 |
| Within Groups | 377.09 | 29 | 13.00 | | |
| Total | 485.72 | 31 | | | |

Table 1: Analysis of Variance for self-efficacy by frequency of usage.

$H_4$ The result of the Pearson rank correlation coefficient indicated a moderate positive relationship between self-efficacy and students' overall attitude towards using graphing calculators in mathematics ($r = 0.40$, $p = 0.024$).

Analysis of Variance (ANOVA)

| | *df* | *SS* | *MS* | *F* | *Significance F* |
|---|---|---|---|---|---|
| Regression | 1 | 42.82 | 42.82 | 5.64 | 0.024 |
| Residual | 30 | 227.90 | 7.60 | | |
| Total | 31 | 270.72 | | | |

| | *Coefficients* | *Standard Error* | *t Stat* | *P-value* |
|---|---|---|---|---|
| Intercept | 19.02 | 2.62 | 7.26 | $4.44 \times 10^{-8}$ |
| Efficacy | -0.30 | 0.13 | -2.37 | 0.024 |

Table 2: Correlation coefficients for self-efficacy and dispositions to calculators.

## DISCUSSION AND CONCLUSION

This study was limited in scope to the small sample of students from one high school in the United States. The validity of this study could have been impacted by the small sample size and potential bias in selection. However, the results of this study contribute to the existing body of knowledge regarding technology use in mathematics pedagogy. Gender differences were prominent in students' attitudes towards graphing calculator usage in mathematics instruction and also in their self-efficacy in utilizing graphing calculator features in mathematics. Boys were more confident in their ability to adapt diverse features of the graphing calculators than their female and non-binary peers. Students were mostly unanimous in their preference of calculators than showing work by hand. Students who hailed from classrooms with the most frequent usage of calculators surpassed others in self-efficacy on performing programming, statistics, calculus on TI-84 graphing calculators. Students who were always or occasionally allowed to use the device demonstrated a more positive attitude towards the integration of calculators into mathematics instruction. However, these students showed more hesitation for pursuing harder mathematics, and perceived that mathematics errors were still possible with the use of calculators than students whose teachers outrightly banned calculators.

In conclusion, this study reinforces the notion that students embrace calculators as an integral addition to the modern mathematics classroom (Clark, 2011). However, based on the respondents' attitudes towards calculators and their perception of the rigor experienced with or without calculator usage, over-reliance on the graphing calculator might be counter-intuitive to students' readiness for higher level of rigorous mathematics beyond the capacity of graphing calculators. Although, calculator usage tends to improve learners' efficacy in acquiring proficiency in the technology applications reposed in graphing calculators, teachers should ensure that graphing calculator technology does not become a deterrent to their students' ability to formulate efficient and accurate computations (NCTM, 2000). Further investigation is needed to explore how students' attitudes and self-efficacy in calculator usage influence their mathematics proficiency across diverse classrooms.

## References

Abdullah, M. L., Abdullah, W. S. W., & Osmar, A. (2005). A New Look at the Students' Attitudes toward Scientific Calculators. *Malaysian Online Journal of Instructional Technology, 2*(2), 17–26.

Bain, L. Z. (2015). How students use technology to cheat and what faculty can do about it. *Information Systems Education Journal, 1*3(5), 92–99.

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Prentice-Hall, Inc.

Boyle, R. W., & Farreras, I. G. (2015). The effect of calculator use on college students' mathematical performance. *International Journal of Research in Education and Science, 1*(2), 95–100.

Broekman, I., Enslin, P., & Pendlebury, S. (2002). Distributive justice and information communication technology in higher education in South Afric*a. Journal of South African Higher Education,16*(1), 29–35

Clark, J. W. (2011). *Mathematical Connections: A Study of Effective Calculator Use in Secondary Mathematics Classrooms*. State University of New York, Oswego.

Crawford, L., Higgins, K. N., Huscroft-D'Angelo, J. N., & Hall, L. (2016). Students' use of electronic support tools in mathematics. *Educational Technology, Research and Development, 64*(6), 1163–1182.

Ellington, A. J. (2003). A Meta-Analysis of the Effects of Calculators on Students' Achievement and Attitude Levels in Precollege Mathematics Classes. *Journal for Research in Mathematics Education, 34*(5), 433–463.

Fast, L. A., Lewis, J. L., Bryant, M. J., Bocian, K. A., Cardullo, R. A., Rettig, M., & Hammond, K. A. (2010). Does math self-efficacy mediate the effect of the perceived classroom environment on standardized math test performance? *Journal of Educational Psychology, 102*(3), 729–740.

Fishbein, M., & Ajzen, I. (1975). *Belief, Attitude, Intention, and Behavior: An Introduction to Theory and Research.* Reading, Addison-Wesley.

Graham, E., Headlam, C., Sharp, J., & Watson, B. (2008). An investigation into whether student use of graphics calculators matches their teacher's expectations. *International Journal of Mathematical Education in Science and Technology, 39*(2), 179–196.

Hembree, R., & Dessart, D. J. (1986). Effects of Hand-Held Calculators in Precollege Mathematics Education: A Meta-Analysis. *Journal for Research in Mathematics Education*, *17*(2), 83–99.

İnce, M. (2023). Examining the Role of Motivation, Attitude, and Self-Efficacy Beliefs in Shaping Secondary School Students' Academic Achievement in Science Course. *Sustainability, 15*(15), 11612.

Lightner, S. L. (1999). A Comparison of the Effectiveness of Applied and Traditional Mathematics Curriculum. *Journal of Industrial Technology, 1*5(2), 1–9.

McCauliff, E. (2003). The Calculator in the Elementary Classroom: Making a Useful Tool Out of an Ineffective Crutch. *Interdisciplinary Journal of Graduate Studies, 27* (139).

Mead, K. C. (2014). *Exploring everyday Math without using technology* [Doctoral dissertation]. Department of Mathematical Sciences Theses and Dissertations, State University of New York at Fredonia.

Migicovsky, A., Durumeric, Z., Ringenberg, J., & Halderman, J. A. (2014). Outsmarting proctors with smartwatches: A case study on wearable computing security. In *International Conference on Financial Cryptography and Data Security* (pp. 89–96). Springer, Berlin, Heidelberg.

Miles, C. (2008). The Use or Non-Use of Calculators Affects on Student's Ability to Perform Basic Mathematics Problems. *OTS Master's Level Projects & Papers, 89.*

Munger, G. F., & Loyd, B. H. (1989). Gender and Attitudes toward Computers and Calculators: Their Relationship to Math Performance. *Journal of Educational Computing Research, 5*(2), 167–177.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* NCTM.

Neubauer, S. G. (1982). *The use of hand-held calculators in schools: A review.* Indiana University, South Bend.

Orellana, C., & Barkatsas, A. (2017). The influence of teacher perceptions and teaching approaches on senior secondary mathematics students' use of CAS calculators. In *Psychology of Mathematics Education* (PME) 41 (pp. 345–352). International Group for the Psychology of Mathematics Education.

Osborne, J. F., & Dillon, J. (2008). *Science Education in Europe.* Nuffield Foundation.

Pajares, F. (1996). Self-efficacy beliefs in academic settings. *Review of Educational Research, 66*(4), 543–578.

Pajares, F., & Miller, M. D. (1994). The Role of Self-Efficacy and Self-Concept Beliefs in Mathematical Problem-Solving: A Path Analysis. *Journal of Educational Psychology, 86*, 193-203.

Parkhurst, S. (1979). *Hand-Held Calculators in the Classroom: A Review of the Research.* [ERIC Document Reproduction Service No: ED200416].

Peters, L. D., Pressey, A. D., Vanharanta, M., & Johnston, W. J. (2013). Constructivism and Critical Realism as Alternative Approaches to the Study of Business Networks: Convergences and Divergences in Theory and in Research Practice. *Industrial Marketing Management, 42*, 336–346.

Rogers, R. W. (1983). Cognitive and Physiological Processes in Fear Appeals and Attitude Change: A Revised Theory of Protection Motivation. In J. Cacioppo & R. Petty (Eds.), *Social Psychophysiology* (pp. 153–177). Guilford Press.

Sigg, P. O. (1982). *The hand-held calculator: Effects on mathematical abilities and implications for curriculum change*. Indiana University School of Education. [ERIC Document Reproduction Service No: ED218147].

Thurm, D., & Barzel, B. (2020). *Self-efficacy– the final obstacle on our way to teaching mathematics with technology?* In U. T. Jankvist, M., van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 2749–2757). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME.

Vygotsky, L. S. (1987). The collected works of L. S. Vygotsky, 1. In R. W. Rieber & A. S. Carton (Eds.), *Problems of General Psychology.* Plenum Press.

Waits, B. K., & Demana, F. D. (2000). *Calculators in Mathematics Teaching and Learning: Past, Present, and Future. Part 2: Technology and the Mathematics Classroom*. Princeton Hall.

Zembat, I. O. (2008). Pre-service teachers' use of different types of mathematical reasoning in paper-and-pencil versus technology-supported environment. *International Journal of Mathematical Education in Science & Technology, 39*(2), 143–160.

# ADDRESSES OF THE CONTRIBUTORS

**Abimbola Akintounde**
American College of Education, IN
USA
Abimbola.Akintounde4229@my.ace.edu

**Marlene Kafui Amusuglo**
Charles University
Faculty of Education, Prague
CZECH REPUBLIC
mkamusuglo@outlook.com

**Christiyanti Aprinastuti**
University of Debrecen
HUNGARY
Sanata Dharma University
INDONESIA
christiyantia@gmail.com

**Eric Bravo**
California State University Channel Island
USA
erik.bravo322@myci.csuci.edu

**Katarzyna Charytanowicz**
Paderewski International Secondary School, Lublin
POLAND
k.charytanowicz@paderewski.lublin.pl

**Linda Devi Fitriana**
University of Debrecen
HUNGARY
flindadevi@gmail.com

**Ivona Grzegorczyk**
California State University Channel Island,
USA
ivona.grzegorczyk@csuci.edu

**Tatjana Hodnik**
University of Ljubljana
Faculty of Education
SLOVENIA
Tatjana.Hodnik@pef.uni-lj.si

**Eliza Jackowska-Boryc**
University of Marie Curie Sklodowska
Lublin
POLAND
eliza.jackowska-boryc@mail.umcs.pl

**Antonín Jančařík**
Charles University
Faculty of Education, Prague
CZECH REPUBLIC
antonin.jancarik@pedf.cuni.cz

**Edyta Juskowiak**
Adam Mickiewicz University in Poznań
POLAND
edyta@amu.edu.pl

**Gergely Kardos**
University of Debrecen
HUNGARY
kardos.gergely@science.unideb.hu

**Panagiota Kaskaouti**
Department of Pedagogy and Primary Education
National and Kapodistrian University of Athens
GREECE
jkaskaouti@primedu.uoa.gr

**Eszter Kónya**
University of Debrecen
HUNGARY
eszter.konya@science.unideb.hu

**Zoltán Kovács**
Eszterházy Károly Catholic University
HUNGARY
kovacsz@science.unideb.hu

**Anna Kuřík Sukniak**
Charles University
Faculty of Education, Prague
CZECH REPUBLIC
anna.sukniak@gmail.com

**Bożena Maj-Tatsis**
University of Rzeszow
POLAND
bmaj@ur.edu.pl

**Vida Manfreda Kolar**
University of Ljubljana
Faculty of Education
SLOVENIA
Vida.Manfreda@pef.uni-lj.si

**Vlasta Moravcová**
Charles University, Prague
CZECH REPUBLIC
morava@karlin.mff.cuni.cz

**Qendresa Morina**
Charles University, Faculty of Education, Prague
CZECH REPUBLIC
qendresamoriina@gmail.com

**Andreas Moutsios-Rentzos**
Department of Pedagogy and Primary Education
National and Kapodistrian University of Athens
GREECE
moutsiosrent@primedu.uoa.gr

**Eva Nováková**
Faculty of Education
Masaryk University, Brno
CZECH REPUBLIC
novakova@ped.muni.cz

**Marta Pytlak**
University of Rzeszow
POLAND
mpytlak@ur.edu.pl

**Mirosława Sajka**
Department of Mathematics
University of the National Education Commission, Krakow
POLAND
miroslawa.sajka@up.krakow.pl

**Chrysanthi Skoumpourdi**
University of the Aegean
GREECE
kara@aegean.gr

**Ewa Swoboda**
State Higher School of Technology and Economics in Jarosław
POLAND
ewa.swoboda@pwste.edu.pl

**Konstantinos Tatsis**
University of Ioannina
GREECE
ktatsis@uoi.gr

**Paola Vighi**
University of Parma
ITALY
vighi.paola@aschig.191.it

**Daniel Walter**
TU Dortmund University
GERMANY
daniel.walter@tu-dortmund.de