# Mathematics in the Real World 

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## INTRODUCTION

Mathematics is a discipline which is related to most - if not all - other disciplines in one way or another. This fact is eloquently expressed in the phrase that mathematics is the queen of sciences, a quote attributed to Carl Friedrich Gauss, one of the most famous mathematicians. Having said that, one may notice the route of mathematics and mathematicians throughout human history and observe the various phases that the discipline has gone through: from a practical human endeavour in prehistoric times to a formal construct in Ancient Greece and then from an elitist science to a 'tool' for the informed citizen to cope with everyday life. The latter has led to new terms, such as numeracy and quantitative thinking, which showcase a shift from theory to practice. At the same time, theoretical mathematics is still progressing as a science and provides other sciences with validated theoretical constructs. From artificial intelligence and supercomputers to cosmology and nanotechnology, mathematical concepts play a prominent role and their manipulation requires very specific and very sophisticated knowledge.
Although the presence of mathematics is manifested in a multitude of scientific and everyday contexts, the anxiety associated with teaching and learning mathematics is still present. On the one hand, preservice and in-service teachers struggle with the mathematics they are expected to teach; on the other hand, students usually resort to rote memorisation of processes and decline critical thinking. So, how can mathematics become more attractive?

The present volume offers a clear response to the above question: we believe that by engaging students at all educational levels (including preservice teachers) in contextualised tasks, mathematics teaching and learning becomes interesting, meaningful and even pleasant for all participants.

The papers contained in the volume address various topics in mathematics education; in order to assist the reader, they are placed in four parts. Part 1, entitled Mathematics and other Disciplines contains papers which address the issue of the coexistence and the interrelations of mathematics with other disciplines, whether in a classroom context or in a more holistic view. Part 2, entitled Issues in Teaching and Learning Mathematics contains papers which address specific issues related to mathematics teaching at all educational levels. Part 3, entitled, Emerging Mathematics through Realistic Situations contains papers which present cases of students being engaged in meaningful mathematical tasks. Part 4, entitled Professional Approaches to Constructing Mathematics contains papers which present various approaches to the construction of mathematical concepts, mainly by preservice teachers.

Poland, June 2018
The Editors

# Mathematics and other Disciplines 

# MATHEMATICS AND THE REAL WORLD IN A SYSTEMIC PERSPECTIVE OF THE SCHOOL 

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#### Abstract

We will approach the variety of the ongoing debates about mathematics and/or reality in the framework of the interdisciplinary and institutional environments of teaching and learning mathematics. This framework is surrounded and perversely influenced by digital and networked, extracurricular mathematical educational productions. These are extracurricular practices but with very impressive and often superficial representations on the mathematical-reality link, very easily accessible in the real world of students. Thus, understanding the relationship between mathematics and the real world becomes an educational and moral responsibility for teachers. I think this makes the theme of the present volume more important.


These environments form a complexity, including and, at the same time, included in the didactic of mathematics situations. Therefore, a new variety of approaches of the relation between mathematics and reality emerges, within which the cognitive, the psychological, the social and the digital are interconnected. It is hard (or impossible) to model the interactions of the aforementioned variety with the underlying epistemological or philosophical variety, because of the complexity of the roles and intentionalities that are interwoven within and in the frontiers the school unit.

The educational need to employ various discipline sources, in order to comprehend the complex phenomena, implies a permanent presence of mathematics and this further complexifies their relationships with reality, because it let the discrete interaction of mathematics with the others disciplines to pass implicitly, which is often ignored by the formal, institutionally constituted, school reality. Within these environments, the boundaries amongst the priorities of the real world and of the noetic structures, which constituted the opposite poles in the philosophical disputes about mathematics and/or reality, become permeable and porous. In the mathematical thinking, observation and intuition, comprehension and invention, modelisation and application, adaptation and transformation seem to be synchronous.
The role of representations and symbolic languages, playing a crucial role in mathematics, becomes an obstacle in the interdisciplinary learning path of the students in the everyday school timetable across their differentiated uses in the different disciplines. Thus, the widely studied didactical transposition is effectively enriched with the praxeological transposition.

We will present concrete examples of the history and epistemology of mathematics, as well as reforms in mathematical education and, in particular, we will discuss the influence of the work of Jean Piaget, to animate the discussion between mathematics and the real world in this systemic approach to the didactics of mathematics.

## INTRODUCTION: PLURALITIES

In 1990, in Poland in the city of Szczyrk the 42nd Conference of the CIEAEM (International Commission for the study and improvement of mathematics education) was organised. The theme of the meeting was "The teacher of mathematics in the changing world". It was indeed the years that followed 1989, when change was the most tangible feature of the world, especially in Europe. A mixture of liberal politics and technology places its traces in the spirit of democratic freedom of the time, gently guiding it towards neoliberalism, globalism (mondialisation) and the financial market. Effectiveness and efficiency should be sought in all areas, including mathematics education. Efficiency was related to skills. Efficiency had to be measurable and the effectiveness to be evaluated. A globalist perspective, developed primarily by the National Council of Mathematics Teachers (NCTM) in the United States, hoped to include the diversity of mathematical knowledge and skills in the concept of skills to be used. In fact, the first recommendation of An Agenda for Action (NCTM, 1980) was that "Problem solving must be the focus of school mathematics" (p. 2). The document went on to say that "Performance in problem solving will measure the effectiveness of our personal and national possession of mathematical competence" (p. 2). It was the period of the Standards, described in the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989).

Many international commissions, as CIEAEM, have been critical, especially in Europe, where the constructivist perspective was more powerful and mixed with the French structural aspect of modern mathematics reform and with the phenomenological point of Hans Freudenthal and the perspective of realistic mathematics. Since, critical mathematical education became a special field of research (Ernest, Shiraman \& Ernest, 2016) through which the theories of Didactic of Mathematics approach crucial dimensions of real world, even the recent economic crisis (Kalavasis, 2017).
Back at Szczyrk (CIEAEM 42 Conference), the international research community of mathematical didactic practices directed its efforts towards the study of common mechanisms in the mathematical reasoning activity of the researcher and the student, in order to design adequate teaching situations for efficient schools. In fact, the first question asked by the scientific committee of this conference was: The teacher must be both educator and expert in mathematics. How does the teacher cope with the changes in emphasis in these roles?

I think that this plurality of approaches in mathematics education emerged the decade of the 90 's, implicitly diffused even in this concern for equilibrium between the two poles of mathematical expertise and teaching effectiveness, has oriented many important initiatives to edit new school books. For example, in Poland it was the "Blue Mathematics" series. We can observe in the front pages of two of them the invariant mathematical pattern background, un-influenced by the changes of the colours and of the real world images.


Figure 1: From left to right: CIEAEM 42 Proceedings and two Polish textbooks from the "Blue Mathematics" series (1996)

Now, 28 years later, the two poles of our problematic are mathematics and the real world. This means that it is not only the real world that is changing, but also the mathematics as well as the environment of mathematics education. So, our approach to learning and teaching mathematics is influenced by the interaction of these three evolutions, and we will try to view this tripolarity from a systemic and complexity point of view.
Andre Revuz (1914-2008), my first Professor in Didactics of mathematics, published in 1963 the book Mathematique moderne, Mathematique vivante (Modern mathematics, living Mathematics). His choice of wording in the title seemed strange to me: Why did he decide to use the term mathematique in singular and not the more frequent term of mathematiques in plural? In the Greek language, we only have the plural noun mathematics ( $\mu \alpha \theta \eta \mu \alpha \tau \iota \kappa \alpha ́)$. But, in contrast, we have singular nouns for Geometry, Arithmetic, as well as for Analysis and Algebra. In English, the term mathematics, although ending with the s , is a singular noun.


Figure 2: From left to right: Revuz (1965), Bourbaki (1939) and Bourbaki (1974)
Then, I noticed the strange singular noun mathématique in the title of Éléments de mathématique, the treatise on mathematics by the collective Nicolas Bourbaki; an edition started at 1939 (composed of twelve books), published by the Editions Hermann. Like Euclid's Elements (13 books), twenty-three centuries ago, the famous group Nicolas Bourbaki tried to recompose the at the time current mathematics evolution and dispersion, in line with the prototype of the Elements. They based this unification effort in the modern notion of structure. Moreover, they used the term in plural when referring to the history: Éléments d'histoire des mathématiques. Perhaps they wanted to emphasize the necessity of Bourbaki's reunification effort, because in their long history since Euclid, mathematics has become a set of scattered disciplines.

I noticed that Andre Revuz, when he approached his 90th birthday in 2002, was the protagonist of the creation of the project ActionSciences, which brings together a dozen of scientific associations for the defence of the teaching of all sciences. So, I could understand that the mechanisms of learning mathematics are involved with their environment. And that they are living mechanisms of the same kind, as the mechanisms of the evolution of mathematics themselves. And of the same kind as the mechanisms of the evolution of our reasoning. The crucial difference is that in a didactical situation all these mechanisms are interacting and are transformed, in order to create new qualities of intelligence, for and by both the student and the teachers, within the school, the family and the society. Thus, in order to study and to improve these situations, I argue that we need tools and concepts from the complexity and the system theory.
Mathematics (m) and reality (r) seem to be at the extremes of a 'tug-of-war', at multiple levels and at several historical periods. We can look for the beginning of this antagonistic game in the divergent positions of Plato and Aristotle and in the hermeneutical oppositions followed. The important thing for us is the impact of this bipolar situation to the third pole, that of mathematical education (e). In which way this phenomenally clear (m)-(r) duel influences the mathematics learning theory or/and teaching models (e)?

If we assume that the objects of mathematics exist per se in a world of ideas, out of any sensible frame, but their comprehension can use analogies in sensible frame? Or, if we assume that the objects of mathematics do not have an existence per se, that they are part of the sensible reality, but the mathematician studies them out of any sensible frame? We can see that the clearness can become obscure for the third pole. In the systemic approach, we try to understand the three poles together, as an interacting system. The complexity aspect allows as to understand that each pole is interacting with this tri-polarity.


Figure 3: An interacting system of three poles
We could follow the evolution of the conceptualization of these essential Plato's and Aristotle's ideas, to approach the divergences in the philosophy of mathematics, the aspects of the logicism, the formalism, the intuitionism or the constructivism. And so, to study their impact in the mathematics education. Another way to the tripolarity could be to follow the learning theories, opposing for example behaviorism to radical constructivism. Or, even more concretely, we could enter into the problematic of the history of mathematics education; from the modern mathematics reforms, the realistic mathematical initiatives, to the international standards in problem solving and the STEM (science, technologies, engineering, mathematics) trends.

However, I think the most important perspective is that we can conceive the three poles as a coexistence model per se, in a unified perception in interaction with the human act of the mathematician, of the learner or of the teacher. This could be described as the Borromeo interconnected Rings, in which if one ring is cut, the two others are automatically disconnected. In mathematical knot theory, the Borromean rings are a simple example of a Brunnian link: although each pair of rings is unlinked, the whole link cannot be unlinked. (Karl Hermann Brunn (1862-1939))


Figure 4: Borromean rings
So, by englobing the phenomenology of the internal opposition of the two poles and by connecting them with the actors in a scientific or learning project (the
third pole), we could oversee the polarity in a more systemic framework. Through this systemic approach, a field of questions related to the essential opposition can be studied from the point of view of the space of the phenomenology of the ordered pairs ( $\mathrm{m}, \mathrm{r}$ ) and ( $\mathrm{r}, \mathrm{m}$ ) in interaction with the educational pole (e). This approach could allow and motivate the phenomenal opposition to interact with the anthropological aspect of the socio-cognitive activity of learning and teaching system and, thus, could allow our rethinking its poles within the framework of Didactics of mathematics. The Borromean interconnection mode means that, if one ring is absent, the meaning is simultaneously lost for all rings. In isolation, mathematics is dehydrated, the real world seems superficial, learning and teaching becomes denervated. If we want to study not each pole isolated but their function in a didactical situation or in an educational project, it is impossible to perceive each ring as being independent.

Thus, the opposition as a couple/pair, incorporating the conjunction and the disjunction of its elements, at the metacognitive level, offers us and reveals to us the human unique capacity to construct his perception of reality. So, to separate and to unite, to disjoin and to conjoin, to divide and to rejoin the elements of the real world. Or, in terms of complexity and systemic formulation, the human ability to conceive together what seems to be disjointed, and at the same time to distinguish what seems to be conjoined. In mathematics, this means thinking and acting in the space between presentation and comprehension, intuition and reason, between the part and the whole, the discrete and the continuous. In mathematics education, this means thinking and acting in the emerging space actively constructed by the interactions between the couples of a mathematical activity, on the one hand and the other, their scientific construction in history, their reflexive construction in learning processes and their transformations in teaching situations (Kalavasis \& Moutsios-Rentzos, 2015).

In this multi-space, we may recognize the semantic and symbolic traces of the interdisciplinary approach in mathematical learning and teaching processes. The interdisciplinarity enriches the access to reality, because of the high level of variety in coherence. The interdisciplinarity allows the deep intellectual visit into mathematics, because of the high level of logical coherence in the variety of their fields. My point is that the interdisciplinary approach may enrich the mathematical learning, by re-considering more clearly its own intellectual fields: by relating separated cognitive frameworks, using same symbolic/language with a variety of meaning and interactions by relating separated actors, in their variety of interactions and meanings. Especially in our times, in which we can easily claim that the mathematization of the system of disciplines (the use of concepts and mathematical symbols) is confused with a kind of mathematification of disciplines (the transformation of their concepts into mathematical concepts and mathematical symbols) in a digital environment of the real world. This tendency is evident when considering the transformations in
the representations and descriptions of the natural or even the social and financial phenomena (Kalavasis, 2017).

It is argued that it is crucial to understand in this interdisciplinary and digital complexity from the point of view of the dipole "mathematics and real world" interconnected in the third pole of mathematics education, as the essence of the intellectual activity (Moutsios-Rentzos, Kalavasis \& Sofos, 2017). And, subsequently, to identify the important role of mathematics education for mathematics and for society and to review its content and methods.
At this point, I think it is useful to recall four areas that Piaget refers to in explaining his genetic approach to scientific knowledge, as they are important to our interdisciplinary didactic approach (Packer, 2017, p. 414):

- The transition from logico-mathematical operations of the manipulation of sets of objects to the formal operations of mathematics, as rigorously deductive reasoning, independent of the real but reflecting: Beyond reality, but preparing a deeper knowledge of this reality, providing better conceptual tools.
- The passage from infralogical operations to axiomatic geometry and abstract models of physics.
- The problems of explanation in science.
- The trends in the evolution of science and the role of scientific communities.

Piaget distinguished the infralogical operations which are used to deal with continuous objects (e.g. liquids) and are based on judgements of proximity and separation in space and time. In contrast classification and counting are logicomathematical operations, applied to distinct objects. He used the term infralogical, "because they related to another level of reality and not because they develop earlier". The last area is part of what we experience by contributing in the present volume and in other international or local initiatives and commissions.

## FIRST EXAMPLE

Let us look at an example, which seems to dissociate the mental world and the material world, starting in a nominalist manner, but with a beautiful historical journey. We will enter this opposition by the name of the different sets of numbers. We recall the categories, often presented by an inclusion set relation, $\mathrm{N}, \mathrm{Z}, \mathrm{Q}, \mathrm{R}, \mathrm{I}, \mathrm{C}$, leaving aside the technical details of the zero and other elements necessary for these constructions.


Figure 5: Number sets
Why do we call as natural the simplest, most integer and perfectly disjointed numbers, while we call real, their being more of a mental invention, their transformation into continuous, including their irrational and transcendent nature? The mathematics education specialists seem to converge that it is easier to begin the teaching of numbers starting from the natural numbers. It is easier. Enumeration, cardinality, order.

But which category of numbers is the best approximation of reality? It is easy to answer that it is the real numbers, if not the complex numbers closer to the reality and to the phenomena of nature. Moreover, specialists and pedagogues seem to agree that it is easier to conceive of the real world than mathematical theories. Therefore, we should give priority to teaching real numbers and operations with continuous quantities (infralogic operations). Nevertheless, if we admit that the basis of the construction of mathematical structures and of logic of propositional calculus, in which is based the construction of set theory and the construction of the continuous of the real numbers, is related to the construction of the set natural numbers, then what is simpler and easier to approach in education: mathematics or the real world?

Natural, ..., imaginary, complex. All sets are contained to the last. But the construction of the last requires the first. So, its conception is contained in the theory of the first construction, of the infinite set with the minor cardinality.

## SECOND EXAMPLE

Pythagoreans conceived the natural numbers from a mental view (theory) in the ordering and counting of discreet, but similar, almost equal, material objects. Broad beans or pebbles, from which came the Latin word calculus, a pebble or stone used for counting. At the same time, they re-presented the numbers with geometrical shapes and gave them corresponding names. The figurate numbers. Starting from the distinction between the even (if they can be ordered in two equal collections, covering same space) and the odd (if a piece remained a little separate making the difference, bringing the inequality, bringing the difference of the one between the two), they evolved their representational constructions using the practical and noetic instrument of the gnomon. Triangular numbers, rectangular numbers, square numbers, polygon numbers, etc.


Figure 6: Triangular, square, pentagonal and hexagonal numbers
The Pythagoreans also conceived the natural numbers and the analogies, the ratio of natural numbers, but the ratio was not necessary to have an arithmetic value. They could see in their constructions the repetitive pattern and they used the cardinal and the ordinal version of the concept of number to describe and calculate the next or the successive number of the same category.

| $1^{\text {st }}$ | $1=1{ }^{2}$ |  |
| :---: | :---: | :---: |
| $2^{\text {nd }}$ | $1+3=4=2^{2}$ | The first 6 squares of natural |
| $3^{\text {rd }}$ | $1+3+5=3^{2}$ | numbers as the sum of consecutive |
| $4^{\text {th }}$ | $1+3+5+7=4^{2}$ | odd numbers |
| $5^{\text {th }}$ | $1+3+5+7+9=5^{2}$ |  |
| $6^{\text {th }}$ | $1+3+5+7+9+11=6^{2}$ |  |

In this story we may also notice the reflective process, the aller retour, the round-trip among: the concrete form of a disposition of points in the space, the area of the spatial form, and the conceptualization of the number in this disposition, the conjunction of its geometrical construction and its algebraic proprieties.

## THIRD EXAMPLE

Euclid uses the concept of ratio between continuous quantities. He could even write about the ratio between the side and the diagonal of the square, to show
that no couple of natural numbers exists that could be describe this ratio. So, within his line of thinking, he could incorporate a seemingly paradoxical way without confusing what should be distinguished, the discrete magnitude as multitude of elements and the continuous magnitude as length. The ability (or the need) to conjoin and disjoin was still important in the conceptualization of what exists in the real world even though not (yet) visible to our experience.

We may recall at this point Plato's dialogue Menon, where Socrates tried to persuade Menon that the learning process is a kind of remembering. He did an experiment with a slave who had never been educated. By employing the famous Socratic maieutic or questioning method, he wanted to show Menon that it was possible to lead the slave to find the solution of the duplication of the square, to find the side of a square the area of which is double the area of the initial square. After two trials (the first one being to double the side, thus constructing a four times bigger area square, and the second being to take a side and a half, thus constructing a triple area square), the slave was left very sceptic. Then Socrates said to Menon: "You see now, he is hesitating to make a new trial, he is in a state of mystification" (in the original text the term used was aporia; $\alpha \pi o \rho i ́ \alpha)$. This is the more important stage of the learning process. At this moment the slave's thought is in the intermediate space between one and two dimensions, between intuition and logic, between analogic thinking using obvious elements and inventing or discovering emerging alternative relationships among the elements and the whole. He was then led by Socrates's questioning to mentally disjoin the diagonal in the figure and construct based on it the new square, the one with having double area. This example is considered as the first training lesson in didactics of mathematics.

## FOURTH EXAMPLE

At this crucial point, we must travel across time directly about two thousand years later, to the Cantor's perception and construction of the infinities. Cantor, in order to compare two infinite countable sets, used the 1-1 correspondence (bijection), to prove the unaccountability of infinite sets by employing his diagonal argument.


Figure 7: Cantor and infinite countable sets
By using the notion of 1-1 correspondence, Cantor proved that, in the case of countable infinite sets, the part can be equal to the whole. As by definition, all the elements of a countable set must correspond one by one to the elements of the set of natural numbers, we have to find the gnomon of their successive construction, following the order of natural numbers. Of course, there were other mathematicians in previous eras who had noticed that the cardinality of the odd numbers is equal to the cardinality of the even numbers (the two qualities that define two complementary halves of the naturals) and that this common cardinality seems to be equal to the whole, to the cardinality of the set of natural numbers.

So, in an ordinary, near the real world syllogism, the half is equal to the whole. But, even if this seams easily verified in some cases of infinity, it remains unthoughtable in real world. But, does this mean that the real world is a finite world? The perception of infinity, is it not a basic concept of human mental and moral condition?

A geometrical example in the same direction is the following. Let us consider the triangle ABC . If we take a point D on the segment AB and a point E on AC . Then DE have the same number of points as BC. In fact, each line starting in A and traversing the segment BC , cuts ED in a singular point, and the inverse. Of course, that holds true only if we accept that the continuum on the segments has no holes. By calculating the cardinality of the above set numbers, Cantor proved that $\mathrm{N}, \mathrm{Z}, \mathrm{Q}$ have the same cardinality aleph zero $\boldsymbol{\aleph}_{0}$, as well as R and $\mathrm{R}^{2}$, so R and C , the same cardinality aleph one $\mathbf{N}_{1}$. So, the above shape of the successive inclusions gives a wrong image of reality! This is because we can correspond 1 to 1 all the elements of N with all the elements of Q .

Historians say that when Cantor showed that the cardinality of R is equal to the cardinality of $R^{2}$ (as if the side of a rectangle had the same number of points
with its surface), he said: "I can see it, because I proved it, but I cannot believe it".

We can see the three rings connected.
So, Cantor arrived to think in between the part and the whole, in between the common sense of the reality and the mathematical invention of the construction of the real infinity. The important was to perceive the tug-of-war in a united way, escape from the frontiers of the interior oppositions so to be able to return on these oppositions from the point of view of their coexistence in a sustainable couple. Conjoin the disjoined, then disjoin the conjoined.
Then with his famous fundamental theorem that for every set A, the power set $\mathrm{P}(\mathrm{A})$ of A (the set of all subsets of A ) always has higher cardinality than the set A itself, and the evolution of his theory, lead David Hilbert to say in 1925, seven years after the death of Cantor that "From the paradise, that Cantor created for us, no-one can expel us."

## FIFTH EXAMPLE

I would like to remind you of fact that the root of the tool of comparison between countable sets dates back to Homer's Odyssey. There was a monocular giant, Cyclops Polyphemus. You most probably know the story; the giant abandoning the usual custom of hospitality, he began to kill the comrades of Ulysses. Then Ulysses surprised him by offering him to drink wine. Polyphemus, satisfied, asked Ulysses his name, so Ulysses answered the famous "Nobody", after which, while the giant slept, Ulysses blinded his single eye. Everyone made fun of Polyphemus, thinking that he was drunk and blinded by himself, when he was crying "Nobody blinded me" because it would be impossible for Nobody to exist as a person in the real world. But Ulysses had still to escape from the cellar of the giant killer and that was not easy.
Ulysses observed that Polyphemus had invented a method to verify that all his sheep returned to the cellar at night. To count the number of sheep, it corresponded to a pebble for each animal at the exit by putting the pebbles in a dish and he checked the pebbles of sheep by the same correspondence on the return. So, the ingenious king of Ithaca decided to tie all his comrades underneath the animals and so they managed to get away, each of them under a sheep, and they managed to escape without Polyphemus noticing. This is the first description of a difficult mathematical method in a real world hard environment.
After Cantor's countability, Jean Piaget would be the one to appreciate the important presence of the 1-1 correspondence in the evolving stages of the intelligence and in particular for the conservation of number of elements of a collection independently of the form of their spatial disposition.

## SIXTH EXAMPLE

Let us consider another example in the story of Eratosthenes who counted the perimeter of earth. Eratosthenes of Cyrene (Eןo Cyrene a city situated in modern Libya) was a Greek mathematician, geographer, poet, astronomer, and music theorist, director of the famous Library of Alexandria after 230 BC . Eratosthenes approached the problem to calculate the circumference of earth through analogical reasoning. His experiment and proof are now modelized and used for didactical activities in real circumstances with students of secondary school around the world.
Eratosthenes had heard from travelers about a well in Syene (now Aswan, Egypt) with an interesting property: at noon on the summer solstice, which occurs about June 21 every year, the sun illuminated the entire bottom of this well, without casting any shadows, indicating that the sun was directly overhead. He thought that if earth is spherical, these rays would be oriented to the center of earth. So, he realized that if he could calculate in the same time that noon, the angle that forms a vertical column with the ray of the sun in Alexandria, and if he could measure the distance from Alexandria in the North to Syene in the South, he could easily calculate the circumference of Earth.


Figure 8: Eratosthenes's measurement of the Earth's circuference
However, in those days it was extremely difficult to determine distance with any accuracy. Some distances between cities were measured by the time it took a camel caravan to travel from one city to the other. But camels have a tendency to wander and to walk at varying speeds. So, Eratosthenes hired bematists ( $\beta \eta \mu \alpha \tau \iota \sigma \tau \alpha i$, step counters), professional surveyors trained to walk with equal length steps. The bematists accompanied Alexander the Great as specialists in measuring distances by counting their steps. They found that Syene is about 5000 stadia of Alexandria (stadium is an ancient Greek unit of length). What Eratosthenes had in his mind was sketched, using his geometrical knowledge and transposing the vertical of Alexandria as if it was the transversal of the two parallel lines corresponding to the rays of the sun. So, he used the equality of the two elements into the couple of the alternate interior angles. Thus, Eratosthenes arrived to calculate the Earth's circumference around 240 BC by using:

- geometrical and trigonometrical approaches: Thales theorem on equality of interior and alternate angles, measurement of angles;
- astronomical observation and geographical determination: summer solstice, oriented distance between Aswan and Alexandria;
- estimations of length: stadia, the Olympic stadium of 176.4 m , gnomon to measure the height of the column
- counting methods and tools: bematists;
- assumptions: the Earth is a perfect sphere, light rays emanating from the Sun are parallel
- hypothetical-deductive reasoning: If ... so ...

We may use modern expressions to re-story what Eratosthenes did: Eratosthenes made the assumption (or used the consensus of his time) that the sun was so far away that its rays were essentially parallel, and that Alexandria is in the north of Syene. So, he could calculate the circumference of earth passing from the two poles. But as he assumed that earth is perfectly spherical, he implied that the circumference is always the same in length, independently of his direction from pole to pole or in the equator.

Now I invite you to reflect together, where in all this, is reality disjoined from mathematics? In the bematists' experience? In the assumption about the perfect spherical shape of the Earth? In the certainty of the truth of the theorem of corresponding angles? In the axiomatic existence of parallel lines? In the assumption of the sun rays being parallel? In the concept of analogy (ratio) that lead us to take the measure of angle as equipotent and transformable to the measure of a length? Or maybe in the concept of estimation-approximation (implicit convention) that the length of the cord of a circle is equal to the length of the arc? All this activity, mathematical and real, is integrated into the hypothetical-deductive reasoning, upon which the sense and the connections in the variety of the activities emerge.

My point in this article is that all this activity cannot be distinguished in different parts, neither analyzed in sub problems clearly taxonomised in more or less mathematical or real world experience. What happens does not occur in the opposition among pure mathematics and the real world, but in the relation of the hypothetical-deductive reasoning with mathematical activity and real world activity. The coherence of the variety gives meaning in each of the discrete activities and vice-versa it assumes sense from the disjunction of these activities and their connection in the hypothetical-deductive project. In a more general way, the mathematical objects and the real world conditions are conjointly disjoints in a genetic cognitive project; that can be a scientific project or a learning project.

## SEVENTH EXAMPLE

We could find more examples in the projects of Galileo, Newton, Descartes or Pascal and more recently at Von Neumann to appreciate this anthropological complexity between mathematics and reality. It is important for the educators to understand how the paths of mathematics and of the mathematicians' experience are fundamentally connected with the conceptualization of the natural phenomena in a way compatible with the philosophical ideas and beliefs.
Mathematics, with its conical curves (the form then regarded as the most advanced and abstract) was the only science capable of expressing fundamentally the law of the fall of the bodies of Galileo Galilei at the end of XVI century. According to Dhombres (2017) this was the objective factor of disinherence of the Aristotelian type physics. It allowed another natural philosophy to take hold, which became deeply structured by mathematics, to the point of giving this science (mathematics) the power to conceptualize reality.

The parable (parabola) of Apollonius - an author of the third century BC who named the conic sections or curves obtained as the intersection of the surface of a cone with a plane by the metaphorical denomination of hyperbole (exaggeration) or ellipse (lack of) - now with Galileo also refers to the uniformly accelerated motion; a mechanics idea before becoming a mathematical concept through subsequent calculation and acceleration as a second derivative. We can see at this point the fragile transition in mathematisation and/or mathematification according to Lichnerowicz (1967).


Figure 9: The conic sections
This parable makes the movement to be recognized as independent of the weight and the form of what falls and reduces the falling thing to a numerical value. This curve allows us to recognize the fundamental independence of the movement towards the initial impetus. The movement remains uniformly accelerated irrespective of the momentum, which goes beyond its previously purely numerical role and acquires both the direction of a principle of conservation (principle of inertia) and of a spatial form (directed quantity).

Newton's Philosophiae naturalis Principia mathematica in 1687 launches a new period in which mathematics dominates the most. Interactively, the science of

Euclid itself has been metamorphosed by the invention of a calculus, bearing the name of differential and integral calculus. It provides efficient means to approach the mechanics and the optics but also by the infinitesimal seems to be able to express the intimate structure of the physical objects, such as the curves of trajectory.

## CONCLUSION: COMPLEXITIES

The problem of the relationships between mathematics and real world, as we noticed in the introduction, can be approached from the point of view of philosophy (what is mathematics, what is the real world), or from the learning theories (how the human constructs and develops mathematical concepts), or from epistemological views (the mathematical activity), or from the Mathematics Education history.

My view is based on the systemic approach from the didactical point of view of mathematics, trying to study and improve what is happening in the mathematics class between mathematics and the real world interacting with the wider framework of the interdisciplinarity in our digital age (Kalavasis \& Kazadi, 2015). In this approach of the learning project, the interdependencies between mathematical activity, the real world and the teaching of mathematics give a dynamic meaning to the phenomenal oppositions. Learning difficulties and obstacles are managed in order to enhance the variety and consistency of mathematical knowledge in didactic school situations.

Didactics of mathematics has transposed the questions of the type "What is mathematics" to the type "What do mathematicians do? What is their way of working?" because these questions could lead to two genetic approaches: that of mathematics' historical evolution and that of Piagetian epistemology of the construction of mathematical knowledge. This transition from the science as object to the project which involves the human being, transfers the question to a more participative framework, in a mixed and multi-variant environment, so closer to the real world and, speaking academically, from sciences to the field of humanities.

Piaget used the term 'Constructivism' to create a fundamental connection between knowledge and reality. His work went against the established idea of a knowledge being a static entity and something out there to be discovered, considering rather that human systems generate their own knowledge. In The construction of reality in children (La construction du reel chez l'enfant), published in 1937, Jean Piaget studies the stages by which, during the first two years, the child is able to represent a permanent objective world independent of this representation itself. This construction is carried out by two complementary movements: the accommodation of thought to things and the assimilation of new data by the previous acquisitions. Piaget highlights the complementarity of two categories of acquisitions: the organization of intelligence and the organization
of reality that take place both jointly and one by the other. This complementarity results from that which unites the accommodation of thought with things and the assimilation of new data by the acquired of the previous.

Piaget, with The Genesis of Numbers in Children (La genèse du nombre chez l' enfant) in 1941, highlights the link between the construction of the real world and the mathematical construction. He followed the construction of the whole number by the child. He emphasizes that this construction is operative, that it is carried out from groupings of classes and relations. He shows that the verbal acquisition of spoken numeration is not enough. The concept of number appears as a synthesis of classification structures and order structures, but it exceeds them both by its superior flexibility and the degree of generality obtained by successive abstractions. Then, in his Introduction to Genetic Epistemology, Piaget emphasizes that the multiple interactions between the subject and the object, both in the history of adult thought and in the genesis of cognitive functions in children, lead to the formation of knowledge which is eventually included in the scientific disciplines, characterized by their specific problems, their particular methods, their own results. Here we can see the traces of the systemic point of view that resemble the subject of learning and the object of knowledge with its interactions in the concept of the project.
It is easy to notice in this point the importance of transforming of the antagonistic relationship between mathematics and the real into a reflective relationship between the construction of mathematics and the construction of the real and, even more profoundly, the impact of this relationship process of building scientific knowledge.


Figure 10: From left to right: Piaget (1967) and Watzlawick (1984)
Piaget in Logique et Connaissance Scientifique puts himself in opposition to the positivist hierarchy of science. He argues that, although autonomous in many respects, the various scientific fields are linked by a series of connections, which makes it possible to postulate a "circle of sciences" ranging from formal sciences (logic, mathematics) to physics, then to biology, human sciences (psychology and sociology) to return to the formal sciences. Von Glasersfeld
took this further by showing how meaning is built up from experience and how we understand and construct our knowledge of the world around us through continual negotiation with the external world. His two books Construction of Knowledge (1987) and Radical Constructivism: A way of Knowing \& Learning (1995) traced the history of constructivism from Vico to Piaget and put forward the model of Radical Constructivism.

An important moment in the thoughts about the phenomenology of learning and teaching mathematics into school situations was the influence of the cybernetic theories and the systemic approach. Through the 'meeting' of constructivists and cyberneticists, we may more appropriately situate the phenomenon of learning mathematics as inter-influenced within the environment and I think that we can situate the reflexive construction of mathematical knowledge within the circumstances of the school unit as a learning organism. So, to conceive the learning and teaching mathematics as a system of internal interactions and external relations, described in the form of a pentagon-within-a pentagon.


Figure 11: A self-similar approach to the $\operatorname{Sch}(\mathrm{ool})$ Un(it) - Prot(agonists) complexity (Moutsios-Rentzos \& Kalavasis, 2016)

The interdisciplinary approach in the Didactics of Mathematics could be described as the stage of complexity. It assumes the emergence and the didactical management of the symbolic connections, the conceptual interactions, but also of the divergences and diversities in the methods and the objectives between mathematics and sciences in the school situation of learning and teaching, under and beyond the didactical transposition effect.
In this overwhelming complexity, where is mathematics and where is the real world or more precisely, the phenomenology of their connections with the human learning activity? How can we make distinctions between the whole and
its parts in a dynamic connection, between the world and the words or forms that describe the various versions of our interactive experience in it? Trying to understand the school reality, the real environment in which we teach mathematics, I soon realized that the learning of mathematics happens not only in school, but also in family situation. And more particularly in between the school and the family. In this in-between space emerges the role of the shadow education, all these structures and practices growing in parallel and at the same time in close ties with the school and of the digital and network environment. In which way may we conceive this multi-dimensional reality?

## Paul Watzlawick in The Invented Reality. How Do We Know What We Believe

 We Know? (1981 in German, 1984 in English, 1988 in French) notes that:... any so-called reality is - in the most immediate and concrete sense - the construction of those who believe they have discovered and investigated it. [...] In other words, what is supposedly found is an invention whose inventor is unaware of his act of invention, who considers it as something that exists independently of him; the invention then becomes the basis of his world view and actions. (p. 10)
Moreover, one of the contributors in this edition, Ernst von Glasersfeld stresses:
The only aspect of that 'real' world that actually enters into the reality of the experience is its constraints. (...) Radical constructivism, thus, is radical because it breaks with convention and develops a theory of knowledge in which knowledge does not reflect an "objective" ontological reality, but exclusively an ordering and organization of a world constituted by our experience. The radical constructivist has relinquished "metaphysical realism" once and for all and finds himself in full agreement with Piaget, who says, "Intelligence organizes the world by organizing itself". (p. 24)

Mathematics are in the real world and the real world is in mathematics, like in Escher's 1948 lithography Mains dessinant (Painted hands painting).


Figure 12: Mains dessinant (Escher, 1948)
This self-reference is an interference between a message and the support of this message, like a book that tells the story of the writer who writes this book. Mathematics is in the real world and the real world in mathematics, because the perception of world supposes the capacity to organize all the information that we
receive, so we do it in our mind by connecting information in a common way which is the logic-mathematic and by this procedure we construct the real world.

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# UNDERSTANDING OPTIMISATION AS A PRINCIPLE 

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Optimisation problems are classic problems in mathematics and the real world. Since the 1980s the landscape of solving optimisation problems has fundamentally changed in the era of high dimensional computing capacities as can be used today. Numerical approaches cap analytical ones since then. This shift recasts currently processes in industry as well as modelling of nature, climate change and so forth. In order to allow students to understand how mathematics and specifically optimisation is used and needed today to solve complex application problems, such as landing a spaceship on the moon, controlling robots to place objects precisely or to run a smart farm, mathematicians and mathematics educators need to work together. Inviting mathematics classes from schools to the university to learn about this, is one way of making this knowledge and these new approaches accessible to students and teachers. Principles of this approach and how these can be made accessible to students are presented in this paper.

## MATHEMATICS AS TECHNOLOGY

Mathematics is today recognised in its specific role and basis for most scientific disciplines, many fields in industry and our societies. Mathematical models and applications are used in nearly all disciplines, particularly in science and engeneering, as well as in economics and medicine. The intensive use of high performing computers and fast technological changes in the last decades have accelerated the mathematisation of many areas. Mining data in enormous quantities is possible today, which allows to simulate complex situations and to use mathematisations for an optimal feedback control of running systems (see Büskens \& Wassel, 2013). Experiments become possible by modeling and simulations which would otherwise be too cost intensive or a waste of ressources. Efficient ways of solving real world problems affords experts to work together and to acknowledge what mathematics can contribute. Powerful algorithms and their smart implementation in form of software offers solutions in robotics, autonomous driving and aeronautics (e.g. Geffken, Knauer \& Büskens, 2017). Introducing students and teachers to this field of mathematical applications and mathematical software is possible which the math fair activities by Prof. Büskens and Dr. Knauer (Knauer \& Büskens, 2018) at the University of Bremen in recent years have demonstrated.

## THE MATH FAIR EXPERIMENT ON OPTIMISATION

Providing hands-on activities with a LEGO Mindstorms vehicle, an industrial robot or a flight simulator helps school students to understand fundamental
principles of optimisation and optimal control at the math fair. These activities, complemented by the use of professional software and theoretical tasks provides them an insight into current mathematical research areas and industrial applications. At the same time it portrays possible professional paths for mathematicians, which gives students an orientation what studying mathematics can lead to.

Mathematisation of real world problems is complex; modelling and simulation are only two parts of the whole process. Allowing students to focus on these parts and other key elements in this process four components have been chosen for some of the recently organised math fairs at the University of Bremen: 1. Parameter identification, 2. Nonlinear optimisation, 3. Optimal control, 4. Optimal feedback control.

## 1. Parameter identification

Parameter identification is the focus of the first station. The relevance and meaning of parameters is introduced in the context of the long-term human interest in astronomy. It is then applied to a LEGO Mindstorms vehicle, whose hardware and software parameters are set by the students so that it follows a given path. Students investigate and experience how parameters like the distance apart of the wheels, the speed of the car etc. determine if the vehicle can follow the path or not. This allows students to practically understand the relevance of parameters and their significance in optimisation problems.

## 2. Nonlinear optimisation

In the context of a skiing problem - how to find the lowest point in a valley while avoiding trees using only local information - the theme of nonlinear optimisation is introduced. Mathematically the given problem is an optimisation problem with constraints. The mathematical conceptualisation of the skiing problem is essential at this point and introduces students to fundamental ideas of numerical solutions. The software WORHP Lab, developed by the working group Optimisation and Optimal Control at the University of Bremen, then allows the students to model, visualize and solve the given constraint problem.

## 3. Optimal control

Optimal control is experienced and thought through at the third station, where a parking manoeuvre of an autonomous car is discussed. The students then use WORHP Lab to calculate the optimal trajectory for an industrial robot, and experience how balancing a table tennis ball is impossible manually while perfectly easy using WORHP Lab. Sending the results to the real robot the students understand how mathematisation results in time-dependent optimisation. Last but not least, students are introduced to problems of feedback control at station

## 4. Optimal feedback control

Given a dog's problem of traversing a river with a current in the most direct way, students are introduced to central ideas of feedback control. This allows students to successfully manoeuvre and land on the moon in a flight simulator. Besides playing, conceptualising and mathematising the situation supports students to understand how and why feedback control is a key element of optimisation.

## MATHEMATICAL MODELING AND BEYOND

Mathematical Modeling (e.g., Blum, Galbraith, Henn, \& Niss, 2007, Stillman, Blum, \& Salett Biembengut, 2015) has been discussed in mathematics education as an important component of mathematisation for a long time, yet optimal control and optimal feedback control has not yet played a prominent role in this discussion to our knowledge. Realistic Mathematics Education (e.g., de Lange, 1996, Treffers, 1987) has conceptualised and examined mathematisation as a didactic principle for nearly half a century now, based on fundamental thoughts of Freudenthal (Freudenthal, 1973). Introducing students to 'mathematizing unmathematical matters' (ibid., p. 133) was and is a key concern of this approach. Meanwhile the 'mathematisation-of-the-world', e.g. in modern Information Technology and other high end technologies, has extended modeling and included simulation in engineering and industry. High performance computing made this possible, but the widespread trial and error approaches that followed had high costs as an implication. Limiting financial resources in the industrial and economic world led to yet another turn and in recent years has given mathematicians back a stronger voice and more prominent roles in industry. Optimisation - as a mathematizing principle became an indispensable component, being more rapid, fruitful and efficient in solving problems than mere simulation.
While Mathematical Modeling has been introduced into school mathematical activities since about the 1980s, the above described activities go beyond modeling as discussed in mathematics education. Simulation and real time optimisation are core elements in these activities and lead to an integrated threefold approach of Modeling-Simulation-Optimisation (MSO). The MSOcycle is discussed for mathematics applications in Engineering, Information Technology as well as Natural, Economic and Social Sciences since a while (see Wets, 1976) and is also fundamental for the math fair activities at the University of Bremen.

Finding sophisticated solutions to a wide range of discrete, continuous or stochastic problems is the motivation for the math fair activities which are presented to the school students. Even though progressive developments in mathematics require more complex cycles than even the MSO approach offers, making essential elements of such a cycle accessible to students is a good start.

So far this is only marginally discussed in mathematics education, this paper is an attempt to open the discussion. To overcome a narrow view of mathematics as an abstract discipline, which seems still to be prominent in schooling, we as mathematics educators need to support the efforts of our colleagues in mathematics and particularly optimisation to portray a rich and more vivid picture of mathematics. New groundbreaking mathematical methods and ideas have not been popularized enough so far. This seems odd as challenging problems such as how to use our limited natural resources on Earth sensibly or building smart farms seem to be important global issues. Why not also approach these challenges together with students in mathematical ways?

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# THE CROWN OF THE HIMALAYAS 

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This article presents a partial result of research that aimed to verify suggestions on an application of interdisciplinary relations of mathematics, biology, national history, and geographical studies at the primary level of education. One of the teaching situations, which is inspired by information published in media, is being analysed. The situation indicates how fifth graders may discover mutual relations among different subjects and simultaneously apply their mathematical knowledge in suitable real-life problems and situations.

## INTRODUCTION

Learning tasks are often considered as part of the core of a teaching situation (Janík, 2013). Tasks based on media texts are placed in various social, geographical, historical, artistic or technical contexts and invite pupils to solve a particular complex problem. Therefore, the intention of such tasks is to motivate pupils to discover, with the ultimate aim of developing their competencies. A certain level of reading literacy is a prerequisite for understanding the content of every text. Such tasks also require sufficient concentration and attention. Pupils should always be able to precisely understand the tasks and to draw conclusions from the information included in them.

## THEORETICAL FRAMEWORK

We choose the theory of Realistic Mathematics Education (Freudenthal, 1973; 1991) as our theoretical starting point. This theory works with the idea of realistic word problem, i.e. a word problem which translates real-life situations and context into the language of mathematics. Also, Toom (1999) mentions "real-world problems" and stresses their importance in teaching mathematics. Usually, the importance of wording problems is mentioned (Siwek, 2005). The research of Brown, Collins \& Duguid (1989) confirms that thinking and learning are not processes "locked in thought" but rather processes which are interactive in nature and as such are placed in a number of authentic and rich contexts. Palm (2008) studied the influence of context on the process of solution of tasks. He used "authentic" tasks, where authenticity was defined as having at least some of the following features: the tasks are related to events which can happen in real life, they contain questions which could be asked in real life, students must see their aim as obvious (just as in real life), data included in the tasks must be available or easily obtainable, the tasks are worded in plain words, the tasks must be realistic in the sense that they are related to a specific event. When such
tasks are solved, pupils tend to make use of their knowledge and real-life experience.

The set of tasks in our research was chosen so that it could show the mutual relations of mathematical, reading and media literacy as a condition and assumption of a successful solution by pupils. It is almost impossible to separate school and media. Media is a factor that must be taken into account - it is significant and we cannot ignore or neglect it. On contrary, it should be incorporated into the educational process (Jůva, 1999; Frau-Meigs, 2014). In our case we used media as information pool - both for wording the tasks and looking up correct answers. Tasks, based on up-to-date topics followed by pupils, can be well used in this respect, because pupils find them interesting and like their "stories". Pupils should be invited to analyze the texts and draw conclusions from them in order to successfully solve the tasks (Fuchs \& Zelendová, 2015).

## METHODOLOGY

The aim of this article is to present the issue of potential use of media texts in primary mathematics and to verify the possible use of popular-scientific style in mathematical education of the fifth grade of primary school as a tool for discovering knowledge from different subjects and apply mathematical knowledge at the same time.

The methodology approach is inspired by the method of critical didactic incidents (CDI), which is part of a qualitative methodology. CDI is based on the idea that practice is the basis for theory. This is applied in the immediate observation of pupils working which is supplemented by reflection after the action (Slavík et al., 2014). The author of this article took part in the observation of lessons in the fifth grade of the primary school and also in a collegial reflection with the teacher, which had the form of a discussion after sitting in the class. A basic condition of the approach was a complex and explicit description of the observed situation. Thus, we used a pedagogical-psychological and subject didactical approach in which the teacher was an observer and a coordinator of the activities for the pupils (Slavík, Janík \& Najvar, 2016).

The activities of individual participants - pupils, groups, and the teacher were reordered by two different means - photography and video recording. The microanalysis of educational situations is based on the constructivism approach to learning and knowledge formation and also on the ultimate goal, which is achieving the highest possible rate of pupils' understanding and cognitive activation (Rowland, Turner \& Thwaites, 2014). However, we also learn which particular problems and epistemological obstacles are manifested in the teaching and learning of a distinctive learning content (Lech, Ametler \& Scott, 2010). In this case, it was the relation between mathematical concepts and geographic knowledge. One of such stories is analysed in the article.

## The introductory text adopted from media as a basis for the analysis of a didactic situation



On returning to the base camp, the whole team, especially Jaroš, for whom it was a double victory, had a reason to celebrate. Not only did he finally succeed in scaling K2, but he also finished his mission that he started 15 years ago: scale all 14 summits above 8,000 meters. All eight-thousanders can be found in Himalaya and the person who climbs them all earns the imaginary Crown of the Himalayas. The club of the crowned has 33 members of which only 15 managed to climb without any oxygen mask support. This fifteenth member is Radek Jaroš, the first Czech who climbed all of the 14 tallest mountains.


Figure 1: The introductory text for a subsequent processing of the tasks. (Adapted from: http://www.honzatravnicek.cz/layout/images/file/abc20-S20-21_K2.pdf).
Pupils encounter the Himalayan mountain range and learn the names of particular summits in the text. When solving the tasks, pupils utilize the information in the coherent (linear) text as well as information found in the chart (nonlinear source of information).
The nature of the text and the related activities create a convenient environment for group work. Groups were created spontaneously based on relationships in the class. Each group got a worksheet with instructions of the tasks for each pupil. The six tasks were either solved as group work or individually, based on how the pupils divided the work. However, in the end, the whole group had to present their solution together and check each other's work to prove that each member was able to describe the research process and their conclusions.

## Instructions for pupils

1. Read the text carefully and answer the following questions:
a) When did Radek Jaroš decide to climb all of the 14 eightthousander summits (the text was published in 2014)?
b) How many mountaineers can claim the Crown of the Himalayas?
c) How many of them used oxygen mask support?
d) In what way did Czech mountaineer get to the summit?
2. Carefully write the names and altitudes of all of the eight-thousanders on cards of a suitable format. Check the correctness of data. Sort the cards by to their altitude, first, in ascending order, then, in descending order. Glue one of the orders on a piece of paper and write down in what order the cards are sorted.
3. Choose some pairs of the summits and compare their altitudes. Then choose three summits. Which is the highest and which is the lowest one?
4. It may be interesting to "play" with the altitudes by using various metric units. Try this. Convert the altitude of at least one of the eight-thousanders to $\mathrm{km}, \mathrm{dm}, \mathrm{cm}$ and mm and observe the changes of the number.
5. Calculate the difference of height between the highest and the lowest mountain. Choose at least two other pairs of summits and find the differences in their altitudes. Write down your calculations.

## Task 1

When answering the first question, pupils easily worked out that Jaroš decided to climb all of the 14 eight-thousander summits in 1999. While searching for the correct answer in the texts, some of the students came up with a different year which fell into a wider context of the task. From the chart, they learnt that he set his goal only after scaling Mount Everest. One pupil pointed out that: "First, he scaled one mountain and then he started to like it, so he decided to scale them all." The group accepted his opinion.

The answers to questions b) and d) can be easily found in the text. Thirty-three mountaineers earned the Crown of the Himalayas (by 2014, when the text was published) and Jaroš managed to scale the summit without any oxygen mask support. Pupils commented on their different solutions, which may be because of the various interpretations of the phrase "in what way". Examples of how the phrase was interpreted were: "He believed that he could manage", "With lots of effort", "With a group of other mountaineers.". Not only did pupils consider the technical aspect of climbing the eight-thousanders (without any oxygen), but also other aspects of such an extremely demanding situation (i.e. support of the team, motivation). These aspects might promote discussions over the correct answer among the pupils as well as with the teacher.


Figure 2: Example of pupil's work, task 1
Pupils used the subtraction $33-15$ to work out the answer to the question c). 18 mountaineers needed an oxygen mask support.

## Task 2

The cards containing the names of the mountains might be put in a descending order: Mount Everest ( $8,848 \mathrm{~m}$ ), K2 ( $8,611 \mathrm{~m}$ ), Kangchenjunga ( $8,586 \mathrm{~m}$ ), Lhotse ( $8,516 \mathrm{~m}$ ), Makalu ( $8,463 \mathrm{~m}$ ), Cho Oyu ( $8,201 \mathrm{~m}$ ), Dhaulagiri ( $8,167 \mathrm{~m}$ ), Manaslu ( $8,162 \mathrm{~m}$ ), Nanga Parbat ( $8,125 \mathrm{~m}$ ), Annapurna ( $8,091 \mathrm{~m}$ ), Gasherbrum I ( $8,068 \mathrm{~m}$ ), Broad Peak ( $8,047 \mathrm{~m}$ ), Shishapanagma ( $8,046 \mathrm{~m}$ ), Gasherbrum II ( $8,035 \mathrm{~m}$ ).

The instructions asked pupils to glue the cards to a piece of paper in one of the orders. In Figure 3 ("swan") we can see an example of the result. The teacher assessed this work with the pupils and together they discussed its suitability, practicality, usability, functionality and design. It emerged that the task was easier for the pupils who put their cards into a line from left to right or right to left. Groups which chose to glue the cards into columns arranging the cards from the top of the page to the bottom had problems assessing the ascending or descending order of the cards. Several times they have done the mistake by assessing the data as if they would read and sort them as text (in rows). The problem of the ordering the cards is on the gluing, since the pupils cannot try and eventually change, but it can be also in the 'swan' since it is difficult read the altitude in this artefact.


Figure 1: Ordering cards with the names of the mountains

## Task 3

Pupils used their knowledge of comparing two (three) four-digit numbers using the decimal system. If the numbers are written by placing the digit in the thousands place (8) it is easier to see the significance of the digits in the places behind the decimal place. Pupils suitably used the descending order from the previous task.

## Task 4

Let us choose Makalu ( $8,463 \mathrm{~m}$ ) as an example. If we convert its height from meters to kilometres, we get 8.463 km , which can be rounded to 8.5 km . The height of the mountain can be converted to decimetres ( $84,630 \mathrm{dm}$ ), centimetres $(846,300 \mathrm{~cm})$, and to millimetres $(8,463,000 \mathrm{~mm})$.

The difficulty of this task (converting the altitude to kilometres, decimetres, centimetres and millimetres) is to decide which number should be rounded. Numbers containing the digit 0 proved to be more complicated for pupils. Figure 3 shows how one pupil did not consider the digit 0 in the hundreds place and incorrectly rounded the answer when determining the mountain's altitude in kilometres. Then he was asked by the teacher to read his answer. He said: " 8 thousand kilometres and 35 meters". The mistake was clarified when the teacher drew the pupil's attention to it and encouraged the pupil to convert his number back to meters.


Figure 4: Incorrect conversion to kilometres
Various methods of solving the task occurred when converting the altitude to kilometres. The answer is a decimal number; however, pupils of the primary school have little experience with this concept. They might get the height 8 km if they used rounding. Sometimes the metric units were combined (Figure ):


Figure 5: Converting the altitude of the mountain into different metric units
The remaining two tasks are focused on calculating the difference of altitude between the highest and the lowest mountain and making a graph of journey of Jaroš over the Himalayas.

## Task 5

The height difference between the highest and lowest mountain, i.e. Mount Everest and Gashenbrun II, is calculated as $8,848-8,035=813$. Most of the pupils subtracted the numbers in writing. The teacher suggested a very suitable simplification of the task as both mountains are eight-thousanders, i.e. it is enough to calculate $848-35$ (or even 48 - 35). In Figure 6 we can see the calculation of a student who compared the altitudes of Mount Everest and K2.


Figure 6: Numerical calculation of height difference

## CONCLUSIONS

Our research deals with a selection of topics based on media reports (Nováková et al., 2015). Analysis of its outcomes made use of participation in lessons and after-lesson feedback. In our article we discussed tasks and student solutions of one specific topic during one specific lesson. We focused on the issue of didactical construction of the educational content of tasks and at the same how to use the activity of pupils to form their knowledge (Slavík, Janík \& Najvar, 2016)

By means of the introductory text, pupils are invited to learn about a specific sport-climbing - as they learn new geographic terms. By understanding a text from printed media or a magazine, pupils prove that they have sufficient level of reading literacy. By solving the tasks, pupils prove their mathematical literacy. Moreover, reading and mathematical literacy are interconnected through the analysed text.
The development of mathematical and reading literacy is one of the basic aims of primary education. Nevertheless, these literacies are usually developed separately, in the lessons of mathematics and Czech language. The approach of the research enables teachers to develop mathematical and reading literacy simultaneously, "hand in hand", using activities aimed at specific topics taken from media (newspapers, magazines, etc.) which are highly motivating.

The research showed that, in order for complex tasks involving problem solving based on media texts to be successful, it is crucial that pupils are able to:
a) Decode a text from various fields and contexts (social, geographical, technical, historical, etc.), read with understanding and employ existing
knowledge and experience, deduce conclusions, and look up information needed for solving the task in the text,
b) Transfer the situations and problems in the wording of the task into mathematical language,
c) Use drafts, graphs and diagrams to represent the task,
d) Read linear and non-linear sources (charts, graphs), interpret data and use the data for solving the problems,
e) Properly use mathematical terms and symbols to express themselves adequately, both in writing and orally.

My experience showed some problems as well. I consider pupils' motivation problematic. For teachers, the problems may be caused by the demands on their mathematical knowledge and skills and their professional teaching competencies. The demands of the lesson planning and organization are significant as well as material sources and relating limited possibilities of realization in the everyday educational reality. Despite the fact that the teacher had devised the scenario, it is inevitable that the realization of the activity need not meet the expectations resulting in the aims not being fulfilled. For these reasons, the creativity and flexibility of the teacher to use this situation in a lesson are also significant.

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# USING MUSIC TO LEARN MATHEMATICS 

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Music and mathematics - these two significantly different subjects belong to the two seemingly separate areas of science and humanities - culture and art. Despite this, multiple common elements and analogies between the two are known, as well as the use of mathematics in music. The opposite is, however, rarely discussed - the use of music in mathematics, including the aspect of teaching. This article proposes an exemplary approach, including the results of an initial study concerning the use of music in mathematics education. The research presented in this article shows the possibility and effectiveness of using music to teach geometric transformations of the plane (reflection symmetry and point reflection) at 6th-7th elementary school grade levels. The research also presents a preliminary diagnosis as to whether teachers of mathematics and music theory as well as elementary-level music school students realize the possibility of knowledge transfer between mathematics and music. The mathematics teacher working in the music school has not made use of the possibility of transferring musical knowledge to mathematics before.

## INTRODUCTION - MUSIC AND MATHEMATICS

Mathematics is the language of many branches of science, it is therefore not surprising to find possible applications of mathematics in music. These uses concern multiple areas of music and are the subject of multiple scientific studies. For instance, the research monograph Matematyczna koncepcja muzyki [Mathematical concept of music] (Sudak, 1992) presents i.a. the issue of considering music to be a mathematics-based science, a number-based approach to sounds and intervals, the mathematical aspects of musical systems, and the mathematical classification of intervals. Mathematics is also present in musical composition - both in the general structure of a piece of music as well as its mathematical compositional techniques, such as dodecaphony. An example of using mathematics to analyze music, especially by making use of set theory in the analysis of musical works is Lindstedt's (2004) monograph as well as other publications, concerning e.g. the analysis of the works of Mozart (Grębski, 2014). Sudak (1992) also discusses the mathematical and aesthetic aspects of this issue and the fall of the old definition of music, including an analysis of the new one. Attempts were also made at a comprehensive definition of music theory as axiomatic theory, including the use of advanced notions and algebraic structures (Wille, 1985).
Various publications concerning the relations between mathematics and music can also be found in English-language works.

## MUSIC AND GEOMETRIC TRANSFORMATIONS

In this work, special attention is given to the parallels between point reflection and reflection symmetry and music. Brożek's (2004) monograph is an example of a rare analysis. The mathematics didactics equivalent would e.g. be one of the CIEAEM 57 conference presentations (Galante, 2006).
The use of mathematics in geometric transformations teaching is inspired by musical pieces which make significant use of polyphony and imitation. Polyphony is a type of musical texture consisting of two or more simultaneous melodic lines called voices. Imitation is the repetition and transformation of the melody of a given voice. The most elaborate polyphonic musical technique which makes use of imitation is the fugue, and the undisputed master of polyphony was Johann Sebastian Bach, whose masterful precision and compositional skills amaze to this day.
In such pieces, a theme is presented at the beginning by one of the voices. It is a short, usually two-bar melody. Further in the piece, the melody recurs in different voices and is also transformed in each of them, all in compliance with various strict rules of harmony and musical structure as well as melodic transformation techniques.
In music theory, types of melodic transformations are e.g. inversion, retrograde, and retrograde inversion. These terms are taught to modern music school students during such music-related subjects as "rules of music" or, in the later years, "musical forms," where the students also analyse particular fugues.
The following research is based on the fact that the aforementioned melodic transformations have mathematical equivalents - transformations of the plane. Retrograde and inversion correspond to reflection symmetry (vertical and horizontal, respectively), while retrograde inversion is analogous to point reflection - the composition of two axial symmetries, with the perpendicular axis. The aforementioned melodic transformations are presented in Figure 1.


Figure 1: Musical theme and its retrograde, inversion, and retrograde inversion (comp. J. Sajka)

## AIM AND METHODOLOGY OF RESEARCH

The aim of the following research is the attempt at providing a preliminary answer to three wide-ranging research question sets:

Questions 1. Do music school students, mathematics teachers in music schools, and music rules teachers notice the relations between mathematics and music? Do they notice the possibility of making use of knowledge transfer between mathematics and music? If so, is cross-referencing being used in class for both subjects?

Questions 2. How could geometric transformation teaching (e.g. reflection symmetry and point reflection) include music? Is such a didactic proposition possible to be implemented?

Questions 3. Can the demonstration of the melodic transformation model be effective in the scope of geometric transformation teaching? Can music school students correct their mistakes in the scope of reflection symmetry and point reflection after being presented with the parallels between these geometric transformations and their melodic transformation equivalents?

Empirical research was carried out in order to acquire the answers to these questions. This article presents a didactic proposition which shows how the analysis of musical works and themes to be used in music schools as a new model of facilitating the mathematical understanding of reflection symmetry and point reflection. This proposition was verified in practice.

One 7th grade elementary-level music school class was invited to take part in the study. The research was carried out in two parts, entitled the preparation phase and the main phase, respectively.
The aim of the preparation phase was mainly to acquire the answers to Questions 1, particularly: Do the students, their mathematics teachers, and their music rules teachers notice the relations between mathematics and music? Were the relations between reflection symmetry, point reflection, and melodic transformation presented during mathematics and music lessons?

The aim of the main phase was to acquire the answers to Questions 2 - showing the parallels between these topics and the possibility of using musical knowledge to teach mathematical transformations - reflection symmetry and point reflection. To this end, a lesson plan was prepared and an experimental lesson was conducted.

The impact and effectiveness of the lesson was checked by assessing the students' mathematical knowledge before and after the lesson - this was the method of acquiring the answers to Questions 3. The mathematical knowledge regarding reflection symmetry and point reflection of the students was assessed by using Research Sheet 1 before the lesson. After the lesson, Research Sheet 2 was used to assess whether knowledge transfer had occurred and whether the
students are able to autonomously make use of melodic transformations to correct their own mistakes in symmetry-related mathematics tasks.
The general schema of the research is presented in Table 1.
I. Questionnaire for 7th grade students regarding making use of the relations between mathematics and music in learning mathematics and music rules as well as being aware of the parallels between mathematics and music.

| Preparatory | II. Questionnaire for mathematics teacher regarding identifying <br> study <br> the relations in knowledge between mathematics and music as <br> well as the hitherto use of music during mathematics lessons. |
| :--- | :--- |
|  | III. Questionnaire for music rules teacher regarding identifying <br> the relations in knowledge between mathematics and music as <br> well as the hitherto use of mathematics during music lessons. |
| Main study | IV. Research Sheet 1 |
| V. Lesson  <br>  VI. Research Sheet 2 |  |

Table 1: Research schema

## RESULTS AND ANALYSIS OF PREPARATORY PHASE <br> Student questionnaire

Fifteen students took part in this part of the study. The questions included in the questionnaire were as follows:

1) Have you ever made use of mathematical knowledge or noticed anything related to mathematics during your music history or music rules lessons? YES/NO. If YES, elaborate:
2) Have you ever used musical knowledge when solving mathematics tasks? YES/NO. If YES, how?
3) Do you notice any similarities between mathematics and music? YES/NO. If YES, list them:
4) What do you think, can learning music influence learning mathematics? YES/NO. If YES, elaborate:
5) What do you think, can learning mathematics influence learning music? YES/NO. If YES, elaborate:

The analysis of the data in Figure 2 shows that 9 students declared their deliberate use of mathematical knowledge during music history or music rules lessons.

Subjects' questionnaire answers


Figure 2: Answers for further questions from the Student questionnaire ( $\mathrm{N}=15$ )
These students noted that they made use of mathematical knowledge when learning about fundamentals and overtones, frequencies, intervals, triads, and tetrachords.

However, in the case of question 2, only one student provided an affirmative answer, writing: "Only (a basic example) in relation to fractions - I compared them to intervals."

Eight students provided an affirmative answer to question 3.
Six students state that learning music could influence learning mathematics, although their reasoning is very broad, e.g. "Music helps to develop memory and intelligence, which is useful when learning mathematics."

Eight students agreed that learning mathematics influences learning music, e.g. "Mathematics helps significantly with creating chords and notation."
None of the students noticed the similarities between melodic and geometric transformations on their own.

## Questionnaires for mathematics and music rules teachers

The teachers were asked about their teaching as well as making use of musical knowledge during mathematics lessons and mathematics knowledge during music rules lessons.

Unfortunately, the teachers did not come off well when compared to the students. The mathematics teacher circled "NO" regarding all questions contained in the questionnaire, stating that he has never referenced music during mathematics lessons in the case of music school students. In particular, he has not been making use of melodic transformation when teaching about reflection symmetry and point reflection. He has also not noticed the students spontaneously refer to music during mathematics lessons.

The music rules teacher also circled "NO" regarding the questions related to referring to mathematics when teaching music rules. In particular, he has not been presenting the similarities between melodic transformation and reflection symmetry and point reflection. He did, however, provide an affirmative answer to the questions regarding the students noticing the parallels between mathematics and music on their own: when building intervals, teaching about fundamentals and overtones, and analysing the frequencies of particular sounds. He noted that the students noticed other relations between sound and mathematics.

## STRUCTURE, RESULTS, AND ANALYSIS OF MAIN PHASE

Sheet I and Sheet II were analogous and consisted of 22 and 23 tasks respectively, differing only in the numerical data and the shapes of the geometric figures. Table 2 presents the objectives of particular tasks.
Sheet 1 Sheet 2 Objectives of task
Tasks 1, 2, 14, 15, 23 The student is able to recognize geometric shapes which are
\(\left.$$
\begin{array}{cl}\text { Tasks 3, 4 } & \begin{array}{l}\text { axially or centrally symmetric. } \\
\text { Task 5 } \\
\text { The student is able to draw shapes which are reflected } \\
\text { across a given line or through a given point. }\end{array} \\
\hline \text { Tasks 6, 7, 8 } & \begin{array}{l}\text { The student is able to draw the symmetry axis of a shape } \\
\text { and decide whether it exists. }\end{array} \\
\hline \text { Task 10 } 9 & \begin{array}{l}\text { The student is able to mark the point which is symmetrical } \\
\text { across the axes OX and OY or across the beginning of the } \\
\text { coordinate system. }\end{array} \\
\hline \text { Task 11 Task 12 } & \begin{array}{l}\text { The student is able to provide the coordinates of the point } \\
\text { transformed across the axes OX or OY or across the } \\
\text { beginning of the coordinate system. }\end{array}
$$ <br>
\hline The student is able to draw a symmetry axis so that the two <br>
given points are symmetrical in relation to one another. <br>
The student is able to decide whether the drawn line is the <br>
symmetry axis of a shape consisting of two points. <br>

Additionally, in Sheet 2, the student is able to decide\end{array}\right\}\)| The student is able to decide which points are images of |
| :--- |
| point symmetry. |

## The student is able to draw a symmetry axis and define how

 many axes of symmetry there are in a:Tasks 19, 20, 21, 22

- segment, ray, circle, and rectangle, and is able to draw symmetry axes to them. (Sheet 1) - line, ray, disk, equilateral triangle. (Sheet 2)

Table 2: Objectives of the particular tasks from the Research Sheets I and II The survey which made use of Sheet 1 was carried out before the experimental lesson. The students' results are presented in the graph in Figure 3.


Figure 3: Percentage of correct answers to tasks from Sheet I ( $\mathrm{N}=15$ )
The analysis of the results in Sheet 1 shows that the students had difficulties understanding the concepts of reflection symmetry and point reflection. A common mistake was an incorrect transformation of a shape. When drawing a point-symmetrical geometrical shape, the students often transformed the shape in relation to a vertical line.
Most of the students were also mistaken in regard to the symmetry axis, considering it a line between two points (shapes). The task regarding the coordinates of the point symmetrical to the given point in relation to the axes OX and OY as well as point $(0,0)$ also turned out to be very difficult. The students provided the wrong coordinates.
An experimental music lesson followed the Sheet 1 survey, intentionally conducted in regard to the timetable of the class so as to not coincide with a mathematics or music lesson. This lesson consisted almost exclusively of
music-related content - playing, singing, notation of melodies. The types of melodic transformation (retrograde, inversion, and retrograde inversion) were revised during the lesson.

The students created their own melodic transformations and recognized them by ear. The only reference to mathematics consisted of showing the similarities between inversion and retrograde and reflection symmetry as well as retrograde inversion and point reflection. The terminology used during the lesson was, however, strictly music-related.

Following the lesson, the survey which made use of Sheet 2 was carried out. The results for particular tasks are presented in Figure 4.


Figure 4: Percentage of correct answers to tasks from Sheet $2(\mathrm{~N}=15)$
Over $75 \%$ of the students' answers were correct for most tasks contained in Sheet 2. Therefore, after the lesson concerning the inversion, retrograde, and retrograde inversion melodic transformations was carried out, the amount of properly solved tasks undoubtedly increased.

It is crucial in regard to this study that several students spontaneously related to musical terms when solving the mathematics tasks contained in Sheet 2. An example of such an answer is presented in Figure 5.


Figure 5: The student's explanation: "Yes, because retrograde inversion is used."
This proves that the melodic transformation model was useful for the student in the context of point reflection. The same student's drawing and exemplary reasoning in regard to the shapes not being symmetrical in relation to point $S$ was similar (Figure 6).


Figure 6: The student's explanation: "No, because only the retrograde transformation is used."

## SUMMARY AND CONCLUSIONS

The students of a 7th grade elementary-level music school class were able to notice multiple similarities between mathematics and music. They noticed mathematics in music in the scope of building intervals, the frequencies, fundamentals, and overtones of sounds, and rhymes. They stated that music influences learning mathematics by improving memory and intelligence.
However, neither the mathematics teacher nor the music teacher have ever made use of the relations between these two subjects during their lessons.
Both the teachers as well as the students did not notice the parallels between melodic transformations and geometric transformations of the plane in the form of reflection symmetry and point reflection.

An experimental music lesson which showed these similarities resulted in the students spontaneously making use of this model when solving the mathematics
tasks contained in Sheet 2. A spontaneous knowledge transfer from music to mathematics had taken place. This allowed the students to correct their mistakes in tasks concerning reflection symmetry and point reflection, greatly improving their results in the mathematics tasks. Some of the students even made use of musical terminology in their explanations concerning the mathematics tasks.
The study shows that, in the case of music school students, it is beneficial to make use of the musical model of melodic transformation as an additional, different way of presenting the concept of reflection symmetry and point reflection. In the case of the students who took part in the study, this proved to be both effective in a mathematical context as well as enjoyable. The students were very satisfied with the experimental lesson which showed them a previously unknown type of relationship between mathematics and music, considering the lesson very interesting.

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# DOES THE CURRENCY NAME MATTER? 

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#### Abstract

One of the research questions answered in this article came up as we analysed some TIMSS and PISA tasks, where a name of fictive currency (ZED) was used. Does the use of the fictive currency name influence pupils' performance? Is the effect significant? We selected 4 different tasks in two variants and tested pupils in grades 3 (task $3 B, N=173$ ), 4 and 5 (task $4 A, N=700$ ) and 7 (task $7 A$ and $7 C$, $N=255$ ). The tasks were assigned in pairs - one in the local currency (crowns or CZK) and the other in currency ZED. The results were mixed: while we found a significant difference for one task in grade 7, the differences for other tasks and grades were not significant. We also investigate the pupils' coordination of units and analyse their main solving strategies and mistakes in these tasks.


## THE AIM OF THE RESEARCH

When looking at the results of PISA and TIMSS, we can see that most of the tasks, where a currency is used, uses a fictive currency called ZED. This is done so as not to give any country the advantage of everyday familiarity with the currency while for other countries it would be almost unknown. But what is the effect of using such a fictive currency on the pupils' performance? We know from the test administrators that pupils often asked what ZED means, but is the effect significant? We are not interested at this moment in the situation when some currency exchange rate is needed. We want to investigate only those mathematical tasks that deal with the same currency and where the pupils need to coordinate units such as value per coin or price per item with the number of coins and the number of items.

Therefore, we decided to investigate the issue and look for answers to the following questions: Does the use of a fictive currency in a word problem influence performance of pupils? What strategies and problems pupils display when coordinating the units? Is there any difference in the strategies and errors based on the currency used in the task?

## THEORETICAL FRAMEWORK AND LITERATURE REVIEW

When children start developing their number sense, the first notion they get familiar with is the process and notion of count (3 cars etc.). Later, the concept develops further to encompass value or measure. This value must be always accompanied by a unit (Hejný, 2014). For example - two coins (two represents a count) can have a value of 4 crowns, because each of them has a value of two crowns. Local currency is usually the first environment where children
encounter the difference between count and value and where the concept of unit of units becomes necessary.

The concept of number thus develops from count through units composed of other units to coordination and iteration of abstract units. This is very important for many areas of mathematics: multi-digit numbers, arithmetical operations, fractions, geometrical measurement, etc. (Langrall, Mooney, Nisbet \& Jones, 2008).

Our research question is aimed at finding whether the name of the unit used in the word problem makes any difference. While there is plentiful research on the development and coordination of units (Curry, Mitchelmore \& Outhred, 2006; Wheatley \& Reynolds, 1996), units relations and transformations, units estimation when using standard or non-standard units (Jones, Gardner, Taylor \& Andre, 2012), or on how the everyday experience with various units influences the mathematical skills acquired (Resnick, 1987), we have found no research that would explicitly deal with the influence of the unit name (or the familiarity thereof) in word problems.
It is generally acknowledged that numbers and quantities are not the same thing (Nunes et al., 2016). For example, Olive and Caglayan (2008) state that a quantity is some quality of an object that can be measured (in some units). The actual magnitude (or value) of the given quantity is the number of specified units. Pupils often associate the algebraic symbol they use with the name of the quantity rather than its magnitude - see also (Thompson, 1995). The authors further distinguish between an extensive quantity, which can be counted or measured directly, and an intensive quantity, which is derived from the multiplicative combination of two like or unlike quantities (like meters per second). The task they used for their research is similar to the tasks we selected for our investigation:

Mrs. Speedy keeps coins for paying the toll [...]. She presently has three more dimes than nickels and two fewer quarters than nickels. The total value of the coins is $\$ 5.40$. Find the number of each type of coin that she has.

In this task the monetary values of specific coins are intensive quantities (they are the values per coin) and the numbers of each type of coin and total value are extensive quantities. The coordination between those quantities - number of coins, value per coin, value of all coins of given type and value of all coins represents a problem for pupils, the same holds for associating appropriate units with the different quantities. The unit coordination in this case takes place on three levels: a single coin is the first level, the value of the single coin and the number of those single coins are related at a second level (a composite unit of units), whereas establishing the value of all the coins (of different types, using only the number of nickels as unknown - i.e., considering the relationships between the number of various coins) requires a third level of unit coordination
(a composed unit of units of units) (Olive \& Caglayan, 2008). This unit coordination on three levels seems to be necessary for correctly solving this type of problem.

Looking at the problem of unit names from the language perspective, we can see the new unit name as unknown word (or unknown abbreviation). Pupils have to derive its meaning based on the context of the task - if the task says: "Peter paid 17 ZEDs for the ice cone", it suggests that the abbreviation ZED must be a currency name. When analysing task difficulty, White (2010) mentions low frequency words as one of the factors that increase the task difficulty (p. 86). The same holds for abbreviations - the task difficulty increases with the use of abbreviations. However, the meaning of abbreviations or the low frequency words can be clarified by the surrounding text and thus the impact may be reduced (p. 92).
Vincent (2009) investigated the influence of non-standard words presence and the task length on the performance of 94 secondary pupils. She found that these language factors influenced the performance significantly. Sepeng and Madzorera (2014) researched the same age group and found that the knowledge of mathematical vocabulary was quite strongly correlated with success in word problem solving.

## METHODOLOGY

## Tasks and their a priori analysis

Each of the three tasks below was used in two versions, they both had exactly the same wording only the word "crowns" was replaced by word "zeds" in the second version. The first number in the task code represents the grade, for which it was created, the last number is version -1 for crowns and 2 denotes zeds. For each task, only version 1 is shown. In version 2 tasks, there is unfamiliar / unknown word "zeds", in task 3B1 and 7A1 there are additional unknown name of coins: "two-zed coins" and "five-zed coins". Pupils in grade 7 solved two tasks of this type - i.e. each pupil solved one task in version 1 (crowns) and the other in version 2 (zeds), while in other grades each pupil solved only either version 1 or version 2. The position of the variants within the versions of the test varied to account for any undue influence of the order of tasks.

3B1: Dad had 6 coins in his pocket. These were coins with the value of 2 crowns and 5 crowns. He promised us an ice cone if we guess correctly how many of each kind of coins he has. He said that altogether he had 18 crowns. How many twocrown coins and how many five-crown coins did he have?
The pupils have to work with units of units (two-crown coins or two-zed coins). Two different quantities are present - the number of coins and their value. The task is similar in a way to task 7 C 1 , it may require the coordination of units on

[^0]three levels (i.e., formulating equation like $5 x+2(6-x)=18$ ), but as Grade 3 pupils are not familiar with linear equations, we expect them to solve the task by trial and error method.

4A1: Benjamin and Nicolas are saving money for a trip. Benjamin will put aside 3 crowns every day and Nikolas 5 crowns. After how many days will Nicolas have exactly 10 crowns more than Benjamin?
In this task, the unit of units are the number of crowns saved per day for each boy (or the difference thereof). They must be multiplied by the number of days to get the total value saved. No real objects (coins) are referred to, so the impact of the unknown currency name might not be that significant.

7A1: Benjamin is collecting only five-crown coins and Nicolas only two-crown coins. Nicolas has 10 coins more but 40 crowns less. How many crowns does Benjamin have?
It is again important to distinguish between the number of coins, value per coin and the value of all the coins. Value per coin can be deduced either from everyday knowledge - in case of crowns - or from the coin names and analogy with local coin names. The linear equation hidden in this task is: $2(x+10)=$ $5 x-40$ where $x$ represents the number of Benjmin's coins.

7C1: Joseph knows that a pen costs 1 crown more than a pencil. His friend paid 17 crowns and bought 2 pens and 3 pencils. How many crowns will Joseph need to buy 1 pen and 2 pencils? (A TIMSS task, we only did not explicitly ask pupils to write down their calculations.)
This task leads to two equations that can be reduced to one equation with a single unknown. The quantities are the cost of the objects (intensive quantity expressed as crowns per pen, crowns per pencil) and extensive quantities - the total amount paid and the number of objects (i.e., pens or pencils). Some relations are known between the price of the pen and the price of the pencil which might require the coordination of units on three levels (Olive \& Caglayan, 2008) and lead to the equation: $2(x+1)+3 x=17$. Again, we expect pupils to use mainly heuristic methods. Also, the task consists of two steps: finding the price of a pen and a pencil and the price of 1 pen +2 pencils. We expect that some pupils may forget this second step.

## Participants and procedure

The present study is part of wider research within the Grant Agency of the Czech Republic (GA ČR) project aimed at investigating variables influencing the difficulty of word problems. The participants were $1733^{\text {rd }}$ graders, $3624^{\text {th }}$ graders, $3395^{\text {th }}$ graders and $2557^{\text {th }}$ graders from four Prague primary schools purposefully sampled within the above project. These are medium size schools, with no specialisation, attended by children from their immediate surroundings. No selection of pupils was made, the whole classes participated.

To make sure that the groups of pupils that solved different versions of the test are equally able, initial testing was done within the project, each class was divided into equally abled groups based on the results if this initial testing. The tasks we focus on in this report were part of the third round of testing that took place in November 2017. The tests were assigned by trained helpers. The pupils’ mathematics teachers were present in the lesson and observed the pupils as well. The pupils were asked to write all their calculations down on the test sheet. They were not allowed to use calculators. The test took 20 to 40 minutes.

## Data analysis

Our study is of a mixed methodology design, consisting of quantitative and qualitative parts. For the quantitative analysis, the pupils' written solutions were analysed by trained helpers and the authors. The scoring was as follows: 0 points (no solution, incorrect solution or partially correct solution), 1 point (correct problem model (Kintsch \& Greeno, 1985) with a numerical mistake or correct solution). To analyse the parameters of problems, we used independent samples t-test in SPSS software. All the samples are large enough so that the assumption on normality of average success rate is justified.
Further, a qualitative analysis of the data was made. We carefully analysed the pupils' written solutions for their mistakes and solving strategies, using both our assumptions of mistakes and strategies which might be expected for the problems from our analysis a priori and techniques of grounded theory to find any phenomena unforeseen by us. In this case, we created a spreadsheet in which for each pupil's solution, a line was filled with the phenomena seen in it.

## RESULTS

## Quantitative analysis

|  | 3B |  | 4A (gr 4) |  | 4A (gr 5) |  | 7A |  | 7C |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Version | CZK | ZED | CZK | ZED | CZK | ZED | CZK | ZED | CZK | ZED |
| Solved (\%) | 69.7 | 60.7 | 50.3 | 44.2 | 61.5 | 59.4 | 20.8 | 17.5 | 68.5 | 49.6 |
| $N$ | 89 | 84 | 181 | 181 | 169 | 170 | 125 | 130 | 130 | 125 |
| Diff | $8.9 \%$ | $6.1 \%$ | $2.1 \%$ | $3.3 \%$ | $18.9 \%$ |  |  |  |  |  |
| $\boldsymbol{p}$-value | 0.220 |  | 0.208 | 0.690 |  | 0.437 | 0.002 |  |  |  |

Table 1: Success rates of pupils in selected tasks
Table 1 shows average success rates for both versions of the tasks. The $p$-values for the hypothesis that the average success rates are the same for both versions are shown in the last row. We can see that the difference is only significant for task 7C. Task 7A was complementary to this task (pupils who solved 7C in zeds, solved 7A in crowns and vice versa). But the difference for 7A turned out to be insignificant. Task 7A also proved to be much more difficult compared to all other tasks. Although the results are not significantly different in four of the tasks, the simple average success rate is higher for the version 1 (in crowns) in
all of them. The sample in case of task $3 B$ was relatively small and the difference may turn out as significant if a larger sample was used. Moreover, the pupils in approximately half of classes asked what the name "zed" means. It was not obvious for many of them that it is a currency name. Some influence seems to be apparent.
When looking for an explanation of the difference in 7 C , we also double checked the ability of pupils in both groups. For this we used Item Response Theory (or IRT) in IRTpro 3 software. A two-parameter logistic model was used (Lord 1980). The pupils' ability ( $\Theta$ ) was calculated for each pupil based on initial testing and the first two rounds of testing which they had undergone. Average $\Theta$ were very close to 0 for all groups (data are not shown here due to the space constraints) and the differences between groups were not significant. Further, we split the pupils into three groups based on their ability $(\Theta)$ - lower third (L), middle third (M) and upper third (U). The difference in performance appears in all three groups but the largest difference in performance was in the upper third (Table 2). More thorough analysis of strategies and mistakes follows in the qualitative section.

|  | CZK |  |  |  | ZED |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\boldsymbol{N}$ | Correct | in \% | No s. | $\boldsymbol{N}$ | Correct | in \% | No s |
| Low | 47 | 22 | $47 \%$ | 5 | 37 | 12 | $32 \%$ | 5 |
| Med | 39 | 26 | $67 \%$ | 4 | 44 | 21 | $48 \%$ | 6 |
| High | 44 | 41 | $93 \%$ | 0 | 44 | 29 | $66 \%$ | 3 |
| Total | $\mathbf{1 3 0}$ | $\mathbf{8 9}$ | $\mathbf{6 8 \%}$ | $\mathbf{9}$ | $\mathbf{1 2 5}$ | $\mathbf{6 2}$ | $\mathbf{5 0 \%}$ | $\mathbf{1 4}$ |

Table 2: Differences in success in task 7C per ability groups (No s. $=$ No solution)
Qualitative analysis - strategies and mistakes
Due to the space constraints, we will describe the most frequent solving strategies only for tasks 7A, 7C and 3B. In task 3B, we saw mainly numerical solving strategies consisting in trying various combinations of numbers 2 and 5 until the required sum was reached. This strategy was used by 59 pupils out of 173; only 9 of these were not successful. Other 69 pupils just wrote down their result without mentioning any calculation, 14 were not successful. Those strategies were expected since pupils in this grade do not possess other tools to solve this task. The majority of pupils with the numerical strategy used repeated addition in their calculations, only 20 expressed the computation as two multiplications - the number of those pupils was the same in both groups. 31 pupils also used pictorial representations of the coins which probably helped them in coordinating the units.
Only 11 pupils did not try to solve the task - more in the zed group (8), but the numbers are too low to consider the significance of the difference. There were only a few coordination mistakes (9) in total - pupils, for example, added the total value of coins to the number of coins. The frequency of this type of error
was approximately the same in both groups. More than $18 \%$ of pupils (32) used mathematical operations which did not correspond to the situation. Nineteen of them belong to the zed group, however, the difference is not significant. Ignoring at least one condition (either the total number of coins or the total value of coins) was observed in 17 cases, while 12 of them are in the crown group ( $13 \%$ of those who tried to solve the task compared to $7 \%$ in zed group).

Task 7A proved to be much more difficult for the pupils - only 52 pupils ( $20 \%$ ) solved it correctly. The most effective solving strategy was the trial and error method ( 36 pupils). Most often the pupils calculated the total value of coins for each boy for the corresponding number of coins (i.e., $n$ and $n+10$ ) and were looking for the difference of 40 between the values. Some of the pupils listed multiples of 2 and 5 and were looking for a difference between these pairs of values ( 22 pupils). Often, they focused their attention on multiples of 10 (probably because the difference in values is also a multiple of ten). This strategy did not always lead to a correct solution since the rows of multiples were not properly coordinated. There was no other successful strategy that we could recognise from the written solutions (some of the pupils wrote only the results).

More than $25 \%$ of pupils did not try to solve the task. Here we can see a significant difference between variants -43 pupils did not try to solve the task with zeds, compared to 23 for the task with crowns. One source of difficulty might be that values given in the text are in the role of operators (expressing how much more or less one of the boys has) and there is no fixed value to be used as a starting point. We can illustrate that by the fact that the "operator error" (i.e. working with "N. has 10 coins" instead of "N. has 10 coins more") was quite frequent in this task: it is evident in the solution of 20 pupils. Interestingly, the error appeared more often in the crown group ( $12 \%$ of those who try to solve the task) than in the zed group (8\%). Another source of the difficulty lies in the coordination between the value of coin, the number of coins and the total value of coins. It became a problem for 34 pupils - most often they mixed the number of coins with their total value. This error did not occur in case of task 7C when the coordination required was between price per item, the number of items, the price for all items. The frequency of this error was the same in both groups.

For task 7C, a lot of pupils (118 out of 155) used some kind of "numeric" strategy. Some of them split the number 17 (total amount paid) into two parts like: $9+8$ or $7+10$ or $6+11$ and tried to fit 2 , resp. 3 , numbers within each of the parts. Some of the pupils did not write down anything as they did all the corresponding calculations in their heads at this stage - only the correct price per pen and pencil is noted. This numerical method often overlapped with the trial and error method, when pupils took a number as a price of a pen/pencil, calculated the price for the other item $(+$ or -1$)$ and then calculated the price for

2 pencils and 3 pens to see whether it comes to 17 . Quite often ( $\mathrm{N}=35$ ), division in some form was used - e.g., $17: 5=3$ (rem. 2) and the prices of a pen and a pencil were derived from it. We do not know what was behind this strategy - 5 was used in the division probably because the price (17) was paid for 5 items. This strategy might not work if the item prices differed more. Anyway, in this case it was a very good estimate. Some of the pupils even used calculation in the form $(17-2): 5=3$, which means that they probably realised that what you paid for 2 pens was 2 crowns more than for 2 pencils. One of the pupils wrote the correct linear equation with single unknown and solve it. The solutions in the last two categories (i.e., division and equation) show the pupils' ability to coordinate units on all three levels. Only 35 pupils out of 255 tested ( $14 \%$ ) worked on this level, on the other hand, the task did not require this approach. The numerical solution and the trial and error methods work well here and are simpler from the coordination point of view - the pupils only coordinate two values at a time (like price per pen and price per pencil) and try if total result comes out as required.
As for the most frequent mistakes: 23 pupils did not even try to solve the task. Out of these 14 were in the zed group. Forty-one pupils used some operation that did not reflect the relationships in the task situation. A very frequent mistake in this category consisted in dividing 17 by 2 and then by 3 and considering the results as price for pen and pencil respectively. Sometimes the pupils repeated the division twice: $17: 2=8.5$ and $8.5: 2=4.25$ which they interpreted as the price of a pen. The mathematization of the relationships is incorrect and the pupils interpret the results incorrectly if at all. This type of mistake occurred in both groups approximately in $16 \%$ of cases. Six pupils made "operator error" - i.e., considered number 1 in the role of operator ("it costs 1 more"), or as the actual price - i.e. as "it costs 1 ", this error occurred only in the zed group.

## DISCUSSION AND CONCLUSION

The evidence about the effect of using an unknown currency name is not straightforward. In all the tasks, the average success rate was lower for zeds but the difference was significant only in one case - task 7C. An important difference between the variants of the tasks consists in the number of pupils who did not write anything in the solution, which, presumably means that they gave up on the task. There are more pupils providing no solution in all the zed variants of the tasks, but the difference was significant only for the most difficult task 7A. ${ }^{2}$ The unknown word might have contributed to the perceived difficulty of the task and the pupils were more likely to skip it. In other grades, the context of the task helped the pupils to get meaning of "zed" as a currency name.

[^1]One of the motivations of our study were "zed" tasks used in international testing. Task 7C in the zed variant was used in TIMSS testing of $8^{\text {th }}$ graders in 2007. The success rate of Czech pupils in TIMSS was much lower (25\%) than in our study (50\%) where the pupils were even younger. While in TIMSS, 29\% of pupils did not attempt the task, in our study it was $9 \%$. At least two factors contributed to this difference. First, in TIMSS, the correct answer without a calculation was considered as incorrect (nearly $4 \%$ of pupils). In contrast, we awarded full points even if there was only the correct answer without any explanation (which occurred for $20 \%$ of pupils) or if there was a numerical mistake in an otherwise correct strategy. ${ }^{3}$ The second factor is the choice of the sample. Our sample comes from Prague and results for pupils in the capital city tend to be higher in mathematics testing (e.g., Palečková, Tomášek \& Blažek, 2014). The significantly better result of the variant with crowns in our research might indicate that the low results of Czech pupils in TIMSS might have also been influenced by the use of the unknown currency.

In terms of prevailing solution strategies that pupils used, we can classify them as heuristic methods (trial and error, experimentation with input variables' values). Only about $14 \%$ of pupils in grade 7 used some more advanced method (some form of linear equations) to solve 7 C . We could not identify any significant differences in strategy use between crowns and zed groups for any of the tasks.

When we looked at the source of difficulties for the most difficult task (7A), the role of numbers in the word problem came out as a possible explanation. Hejný (2014) talks about three types of quantities: value, operator and frequency. While value denotes the number of pieces or the number of other units, the operator describes relation between two values. Operators in the task 7A are additive - i.e., expressed as "more than" or "less than". The role of numbers in the task strongly influences the task difficulty. If all the numbers given are values, the task is easier. An operator is always connecting two values - these might not be necessary for the solution of the task but the pupils might feel that they cannot understand the operator (like " 6 more") if they do not have at least one of the values that are being connected. In their meta-analysis of research, Nunes et al. (2016) found out that problems involving "comparisons have been the most difficult of all; they have been shown to be particularly difficult if the unknown in the problem has been the reference set" (p. 17). Our findings corroborate this conclusion.

To conclude: We found out that if the use of an unknown currency has any effect on the task difficulty, it is probably not very strong nor consistent across various type of tasks and grades. There might be an influence of the presence of

[^2]the unknown currency on the pupil's willingness to start solving the task, however, bigger samples of pupils are needed to show this. There were no significant differences among groups as far as the strategy use or unit coordination.

Our study has its limitations. Interviews would be needed if we wanted to account for written solutions whose strategies and mistakes remained unexplained. A large sample of pupils might render our conclusions more robust.

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# SELECTED ASPECTS OF WORKING IN GROUPS WHILE SOLVING A CERTAIN TASK IN A FOREIGN LANGUAGE 

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Bilingual education is becoming more and more popular in Poland. In the paper, I present a report from a small-scale study conducted at the mathematics class during which Grade 2 high school students (17-18 years old), solved a task in French while working in small groups. I study the communication processes in terms of the occurrence of metacognitive and discursive activities related to control and reflection concerning Mathematics and the foreign language.

## INTRODUCTION

Content-language integrated learning (CLIL), often referred to as bilingual education, is the simultaneous teaching of a subject and a foreign language, in other words, teaching subject content in a foreign language. The students of bilingual schools in Poland take their Polish 'matura' mathematics examination as well as a mathematics exam in French. One of the aims of mathematics teaching is to help students develop skills of the proper mathematical language usage. This is directly linked to developing skills of student's proper understanding (Krygowska, 1979). There are various opinions on mathematical language. Sierpińska (2005) gives three theoretical approaches to language: "language as a code (e.g., Laborde, 1982), language as representation (e.g., Duval, 1995; Janvier, 1986), and language as discourse (e.g., Kieran, Forman \& Sfard, 2001), (p. 250)". Considering language as discourse, we can refer to the 'mathematics register', defined by Halliday (1978) "in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes (p. 195)."
In case of Polish and French mathematical register there are certain differences in the symbols, mathematical terminology, the applied algorithms (Adamczak, 2014) and in stressing different mathematical meanings. As highlighted by Schleppegrell (2007): "As with all language development, students need opportunities to use the mathematic register in interactive activities in which they construct meaningful discourse about mathematics (p. 147)", so that they can, among others, express their ideas discuss and justify them. In bilingual education, cooperative task solving, e.g., in small groups can be a chance for developing the students' ability to use the mathematical register. It is closely connected to the skill of communication - not only in mother tongue (L1) but
also in a foreign language (L2). As it is emphasized by Cohors-Fresenborg and Kaune (2003) (as cited in Kaune \& Nowińska, 2012, p. 75) "discourse is the central element of lesson culture, which is to support the development of student's metacognitive activities". Fresenborg and Kaune, 2003 (as cited in Kaune \& Nowińska, 2012) classify precise reading, precise listening, following the line of argument, the evaluation of the correctness of used argument that is linked to the skill of explanation and reasoning as discursive skills. Whereas metacognitive activities in the process of task solving include:

- planning the next steps of its solving along with a choice of proper tools;
- monitoring which involves checking if the choice and usage of the tools is correct, if it leads to the intended aim and if the aim that has been achieved is consistent with the intended one;
- reflection - a mental activity that is directed to the results that have already been achieved. The formulated problems and understanding of the terms might be the subject-matter.

These activities can be observed through the interpretation of certain students' behaviours, including the elements of communication that occur during the lesson. A vital part of the process of communication in bilingual teaching is played by code switching "when an individual (more or less deliberately) alternates between two or more languages" (Baker, 1993 as cited in Setati, 1998, p. 35), e.g. for reformulation (to clarify instructions and to show the appropriate mathematical language use), for content or activity (to explain, inform and regulate) and to translate (e.g., Merritt, 1992, as cited in Setati, 1998). Code switching is a popular and valuable practice in bilingual teaching of various subjects including mathematics (e.g., Duverger, 2007, Gumperz et al., 1999).

## METHODOLOGY

The study was carried out in a Grade 2 high school mathematics class (17-18 years old) during the mathematics class in French. It was a practical lesson on geometric and arithmetic sequences. The students had 5 hours of math classes in Polish per week and 2 hours of them in French with another teacher. Most of the students demonstrated a good level of both French and mathematics. The most often used technique during the lessons was a group discussion. During this discussion the teacher or the students asked questions, evaluated their own and their fellow students' utterances, which were properly argued and justified. The code switching usually took place during the class interaction, e.g., when the teacher or the students encountered the language barrier, in order to provide more complete information and maintain teacher-student or student-student communication. The students had hardly any problems with using the mathematical register in L1. However, there were some difficulties connected with, e.g., understanding and proper usage of terminology and fluency in L2. The group work planned by the teacher seldom appeared during math lessons
conducted in French, except for spontaneous communication in pairs during task solving and which took place mostly in L1.
Eleven students took part in the study. The students divided themselves into groups of three and one group of two, in order to solve two tasks. Discussions in the group during the solving of tasks were recorded and after finishing work each group turned in a written solution to the task. The teachers asked the students to try to communicate in French. The teacher intervened only in the case of explicit request of the students. The task solution time was not limited.

In the article I analyse the work on the first task, which was a typical application of the theory (Krygowska, 1979) containing a realistic context. The realistic context could cause difficulties in understanding the problem statement, because few such tasks are solved in mathematics classes in Polish in the chapter on sequences (most of the times tasks in placing appear in this section of Polish exercise books). The solution to the task required basic knowledge of the geometric sequence. The formulation of the task in French and some terminological differences could cause an additional difficulty. In French, the set of natural numbers is often considered a domain of a sequence, while in Polish language a set of positive natural numbers is a domain of sequence. This is a reason for the need to modify the formulas, among others for the general term and the sum of the first n terms. The task has been taken from matura exam set for 2011 which was prepared for Secondary School of Hotel Administration ("Matura exam" 2011).

In summer, at a temperature of almost $35^{\circ} \mathrm{C}$ and at 2 o'clock a piece of meat $^{\prime}$ containing 100 bacteria was placed on a counter. Under such conditions, the number of bacteria doubles every 15 minutes. Let's mark the number of bacteria at 2 o'clock as $u_{0}$, the number of bacteria at $2: 15$ as $u_{1}$, the number of bacteria at $2: 30$ as $\mathrm{u}_{2}$ and $\mathrm{u}_{\mathrm{n}}$ the number of bacteria of n quarters after 2 o'clock where n is a natural number.

1. Rewrite and complete the following table (we assume that the conditions do not change during the entire experiment):

| Hour | $14: 00$ | $14: 15$ | $14: 30$ | $14: 45$ | $15: 00$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Index: n | 0 | 1 | 2 | 3 | 4 |
| Number of bacteria $\mathrm{u}_{\mathrm{n}}$ | 100 |  |  |  |  |

2. If $u_{n}$ is the number of bacteria at the set time, then $u_{n+1}$ corresponds to the number of bacteria 15 minutes later. What is the relation between $u_{n}$ and $u_{n+1}$ ?
3. Specify the type of the sequence previously defined and its 'raison' ${ }^{1}$ [common difference or common ratio].

[^3]4. Present $u_{n}$ as a function dependent on $n$.
5. Calculate the number of bacteria at 17:00.
6. It is believed that if there are more than 150000 bacteria present in food, then it is not suitable for human consumption. Until what time rounded up to a quarter, can a man eat safely a piece of meat?
For all teachers who teach in a bilingual way, it is important to best understand the cognitive processes of students who study in a foreign language and to adjust their teaching practice to what students have problems with. During a lesson, we observe the communication process also in understanding mutual reactions and interactions. Metacognitive and discourse activities play an important part in this process. Their analysis may reveal to what extent the language difficulties imply problems with understanding the language and to what extent with understanding mathematics. In this regard the following question was posed:
To what extent does communication that takes place between students who are taught in a bilingual way during solving of a task in French in small groups contain the elements of control and reflection about foreign language and to what extent about mathematical questions?

To answer this question, I will try to analyse the students' metacognitive (monitoring and reflection) and discursive activities with regard to whether they more concern the French language or rather mathematical problems. In order to do it, I will classify communication in terms of above-mentioned activities by interpreting text of utterance or students' behaviour. The interpretation will be based on the category system (an interpretative, transcript-based analysis of metacognitive and discursive activities in class discussions) adapted for this purpose from Cohors-Fresenborg and Kaune (2007) and which example can be found in Kaune and Nowińska (2012). Moreover, due to the fact that students who are under examination are bilingual I expect that they will use code switching in conducting control and expressing reflection. All the more, this practice has often been used by the teacher.

In this article, I analyse the work of three groups of students based on the collected documents: two audio recordings (group 2 and 3), one video recording (group 1) and a solution in a written form as a result of group work.

## DESCRIPTION OF THE INTERACTIONS

In the following transcripts, I distinguish between the fragments of the text, which represent students' metacognitive activities (monitoring, reflection) and discursive ones; in some cases, at the end I make a comment concerning the marked activities.

## Group 1: Judyta and Bartek

Bartek is a student with high grades in mathematics, but Judyta sometimes has difficulties in understanding some mathematical problems. Here is a fragment of the initial dialogue:

Bartek: We'll solve the task in French. Let's start. [Bartek read the instruction] (L2)
Judyta: And what do you propose? (L2) (incentive to discussion)
Bartek: It's easy. [Bartek completes the table (figure 1)] We have 200, 400, 800, 1600. [He writes down in the table and asks Judyta] Hm? [expecting confirmation] (L2) (relating the given utterance to the others to make sure that what has been said is understood).
Judyta: Are_you multiplying by 2? (L2)
Bartek: By 2, q is equal 2. [Judyta writes down the solution on her piece of paper while glancing at Bartek] (L2)
Judyta: 200, 400, 800, 1600. (L2)
Bartek: Yes. The next instruction, I'll be writing... (L2)
Judyta: If $u_{n}$ is the number of bacteria... [Judyta reads it] (L2)
Bartek: I think that $\underline{u}_{\underline{n}}$ multiplies by 2 is equal to $\underline{u}_{n+1}$. (L2)
Judyta: What's it? [Judyta looks and points to the indistinct record] Is it equal to? But multiplied by 2? (L2)
Bartek: Yes, because $u_{n}$ is 100 times $q$ to the power of $n, u_{n+1}$ is 100 times $q_{-1}^{n+1}$ q is $2 .[\mathrm{He}$ writes it on the right side of the piece of paper] (L2)
Judyta: Oh yes. So it's like in this table. (L2)
Bartek: We see then that 100 times $2^{n+1}$ is equal to 100 times $2^{n}$ times 2 . This is the same. Hm? [Bartek looks at Judyta and waits for confirmation] (L2)
Judyta: [She wonders for a moment] So the answer? Write it down. (L2)
Bartek: Here it is. [He shows on the piece of paper] (L2)
Judyta: I can write it. [She writes down the word: the answer] (L2)
Bartek: Specify the type of the sequence previously defined and its 'raison'. [Bartek read the third instruction, Judyta thinks about the previous one, crossing off what Bartek wrote so indistinctly and she writes down again: $2 \mathrm{u}_{\mathrm{n}}=\mathrm{u}_{\mathrm{n}+1}$ ] Yes. (L2) [Bartek confirms and as if he wanted to ask Judyta but immediately he turns around to the group behind him]. And 'la nature' is it a formula like this? (L1)
Magda: So, whether arithmetic or geometric. (L1)
Bartek: Uh, okay. (L1)
[Judyta reads the instruction loudly] (L2)
Bartek: It is a geometric sequence. (L2)
Judyta: Geometric, yes, yes. (L2)
(...)

Bartek and Judyta did well in communicating in French. They also used the correct terminology. The only time when Bartek used code-switching was when he asked a person from another group to explain the term he did not know (see transcript).

## Group 2: Antek, Filip and Michal

Antek, Filip and Michał are students who do not have problems in mathematics and it sometimes happens that they come up with interesting ideas when solving problems (especially Michał). When starting the work on the task concerning the sequence, Michał and Antek read aloud the first instruction; after a moment of silence Michał reads it again. There is no trace on the recording when they solved the first subpoint. Filip immediately speaks Polish and he is followed by Antek and Michał speaking in French.

Filip: E guys, I have it too. (L1)
Antek, Michał: $\underline{u_{n}} \underline{n}^{\text {is equal to } u_{n+1}}$ (L2)
Antek: Specify the type of the sequence previously defined and its 'raison'? This is a geometric sequence. [5 seconds of silence] (L2)
Michał: What is 'précédemment'? (L2)
Antek: I do not know. (L1)
Michał: What is 'précédemment'? [Question to the teacher] (L1)
Teacher: Is this important? (L2) (incentive to reflection, according to the teacher, not knowing the abovementioned term should not be obstacle to the task solving)
Antek: A little bit. (L2)
Teacher: The previous year is 2014. (L2)
Filip: Previous. (L1) (making sure to what has been said by the teacher)
Michał: I see, it's geometric and it's about q=2. (L2)
(...)

Michat: $\mathrm{u}_{\mathrm{n}}=\mathrm{a} \ldots$, [what is immediately crossed off by him (figure 3)] and speaks loudly: 100 multiplied by 2 to n . (L2) (self-control)
Antek: To n, yes, yes. (L2)
Michał: To n-1, not to n. (L2)
Antek: No, no, no. (L2)
Michał: It's- $\underline{u}_{0,0}$ (L2)
(...)

They were the first ones to solve the problem. However, they made a calculation error in point 6, therefore they gave an incorrect answer as to the time when man can safely consume meat (Figure 1). Only after the teacher's intervention with a request to re-check the solution (the teacher's control and students'
encouragement to reflection and control), did they find the error. Filip quickly noticed that Michał crossed off an extra zero (Figure 2).


Figure 1


Figure 2

## Group 3: Magda, Maria and Zosia

Magda and Zosia are hardworking students who often achieve high marks in mathematics, however Maria tends to have difficulties with some tasks. There was a long silence at the beginning of the recording as if each of the girls was thinking on their own and was trying to solve the task on the separate pages. Only Zosia and Maria participated in the discussion when solving the first and second subpoints. When a question about the term 'nature' from another group was asked, Magda decided to take a part in the conversation (see transcript of group 1). The girls decided together that the sequence was geometric, and Maria gave her justification in Polish: "With power, it's probably geometric". Together, they considered the meaning of the term 'raison'. Zosia replied: "La raison is the common ratio of the geometric sequence". Magda and Zosia together determined that the common ratio was 2 . Maria committed herself to writing down the solution on the piece of paper which was going to be given to the teacher. On a sheet of paper, in points 1 and 6 ' $u_{n}$ ' is marked as ' $a_{n}$ ', perhaps due to the habit of marking such a sequence in Polish. Most often, Zosia and Magda were the ones to show the activity of control. After returning the piece of paper to the teacher, a person from another group asks a question about the fourth sub-point. Zosia explains that the answer is $\mathrm{u}_{\mathrm{n}}=100 \cdot 2 \mathrm{n}$, not the power of $\mathrm{n}-1$, because it is a zero-th term (reflection and control). Maria also asks for an explanation, despite the fact that it would seem that they have already completed the first task and that Maria also participated in the process. Before that, the conversation took place partly in French and partly in Polish. From now on, the dialogue takes place only in Polish.

Maria: Explain it to me, because I do not understand this.

Zosia: What is... what is that $u_{0}$ ? Because if it was not $u_{0}$ then it should be " $n$ " at theend. Can I write it down? There would be something like $\mathrm{u}_{\mathrm{n}}=\mathrm{u}_{1}$ multiplied by $q$ to $n 1$, and because of $\mathrm{u}_{02}$ it must be something like q to n without this one. [Sub - point 4]
Maria: I see , without this number 1 .
Magda: It is because $u_{0}$ is the first position and the second position is $u_{1}$.
Zosia: Because it does nothing. They take $u_{0}$ instead of $u_{1} \ldots$ and there is no such thing here.
Maria: And this one would be reduced, because it's like going forward by one ...
Zosia: Yes, exactly like that.
(...)

Zosia: The result will not be precise... Will it be accurate according to 16:30. [Sub - point 6]

Maria: How did you calculate it?
Zosia: We just watched when it would not exceed it.
Maria: You just raise to the power and there is no other way?
Zosia: There is not.
Referring to Zosia's statement from the dialogue with Maria that there is no other way to find the answer to sub point 6 except raising to the power. It is worth adding that one of the groups noticed that you can arrive at a solution using the logarithmic function: $\log _{2} 1500=n$.

## RESULTS

On the basis of observation of the course of communication, many displays of reflection could be noticed. These reflections were both connected to the language and mathematics terminology. The reflection that concentrated on the understanding of a given problem was usually linked to the unfamiliarity of the undermentioned terms and their explanation - most often in Polish:

- nature (a polysemic term, a mathematical term explained by a person from another group in L1),
- precédemment (a word from everyday language, explained by the teacher in L2),
- raison (a polysemic term, a mathematical term explained by a person within the group in L1),
- arrondie (a mathematical term explained by a person within the group in L1).

As recorded in solutions you could also notice the habit of using Polish notations: q- common ratio of sequence (all groups), $a_{n}$ for marking a sequence (group 3). Unfortunately, there was a lack of reflection about the problem with mixing up the notations (group 3). In addition, in the second and third group,
there were doubts (reflection) about writing the formula for a general word of the sequence (point 4), as indexing of terms started from zero. However, they came up with the correct answer after negotiating their standpoints (reflection and monitoring). An interesting fact is that the reflection in the third group occurred after the work seemed to be over and the question asked by a person from another group regarding the fourth point could be an inspiration. So, the reflection of one student caused the other student's reflection in the form of a question. At that moment Maria was asked to explain the other remaining points as well, initiating Zosia's reflection and partly Magda's reflection about the solved task. It was also an example of mutual learning, Zosia took on the role of a teacher, clarifying any points that were still unclear, having control and showing a good understanding of the basic issues related to the sequences. The above-mentioned control and reflection took place only in Polish due to the fact that at given moment the task understanding was the most important. The foreign language could be an obstacle because of not sufficient language skills (L2) of discussion participants.
In all groups, the control and self-control were accompanied by displays of reflection. It can be shown in the analysis of transcripts of dialogues and written assignments (some attempts of task solving were crossed out and the calculations were redone). Still the reflection and control concerned more the mathematical questions than the language, sometimes both - as it was in the case of choosing the formula for a general word of the sequence.

While analysing the presence of discursive activities it can be noticed that students from group 2 were the best at listening to each other and following the line of argument despite the fact that not many moments of explanation and reasoning were observed. It might be so due to the fact that there was no such need. Explanation and reasoning often happened in the group 3, e.g. during the reflection after having been presented with the solution. It seems that out of all groups the group 1 had the fewest moments of deeper reflection despite the fact that the outer form of communication in foreign language was very good. The work of both students was not synchronized: Judyta was often one step behind Bartek and perhaps Bartek did not feel the need for discussion or any help from Judyta, because he was able to find a solution on his own. The displays of negative discourse (see Kaune \& Nowińska, 2012) connected with, among others, language problems e.g. concerned the reading of large numbers are worth noticing. In this case, what the students did was to read either the subsequent digits one-by-one or groups of numbers. Especially, the boys had problems with pronunciation and expressing their thoughts in French. It could be noticed that it was more important for boys to solve the task than to communicate properly in French.
As it could be expected, all groups used code switching to a greater or lesser degree. In group 1, the code switching was used only once so that the fast
explanation of the meaning of the French term could be done. In the remaining groups this practice was used to explain, inform, regulate and translate.

## CONCLUSIONS

The presented small-scale study makes one think about the possibility to improve the effectiveness of mathematics bilingual education. Special attention should be paid to the communication aspect of this process. The teacher should organize the activity of students which is directed in greater degree to the usage of mathematical register in foreign language. Additionally, the teacher should also make students aware that not knowing a term is not always an obstacle in understanding the essence of the task and its solving. While observing the groups, it could be noticed that students preferred switching to Polish because the communication did not always run smoothly and some pronunciation and grammar errors appeared. Despite the fact that using code switching seems to be a good practice in bilingual education, I think that a large part of 'awareness' discourse should take place in a foreign language because it happens along with the development of language skills (L2) that are also connected with mathematical register. The teacher should keep a close eye on students whether the foreign language is not an obstacle in understanding terminology as it was in Maria's case. In such a situation, the teacher can give, for example some language help before solving the problem. It should also be noticed that the correct outer form of utterance does not guarantee by itself the mastering of metacognitive and discursive skills when there is no precise listening, following the line of argument and deeper reflection (Judyta's example). It would be also desired to expand the research question to the problem of communication in class, which takes place in various forms during lessons. Moreover, not only interaction between students but also the ones between students and teacher should be further examined. The teacher should serve as a model for didactically desired behaviours of students and create situations which are in favour of them, e.g., to involve metacognitive and discursive activities, also when communicating in foreign language. Therefore, the observation of teacher and students' behaviour should be accompanied by reflections about the conditions that are in favour of constructing the knowledge in the process of communication. This process is far more complex in bilingual education, thus the need of further examination in this field is required.

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# Issues <br> in Teaching and Learning Mathematics 

# HOW CAN WE USE MATHEMATICS EDUCATION RESEARCH TO UNCOVER, UNDERSTAND AND COUNTERACT MATHEMATICS SPECIFIC LEARNING DIFFICULTIES? 

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#### Abstract

Mathematics education research from the last four decades has helped us to understand more and more about the nature and processes of mathematical learning. This has further helped us to uncover and understand characteristic obstacles that most learners of mathematics - and not only those with general learning difficulties - encounter during their attempts to learn mathematics, some even to a detrimental degree. Mathematics specific learning difficulties seem to be of a rather universal nature across cultures, countries and students. In my lecture I shall highlight some of these difficulties with a focus on recent work and findings. I shall further present a research based in-service education programme for upper secondary school teachers enabling them to detect and diagnose upper secondary students with mathematics specific learning difficulties and eventually to remedy or reduce these difficulties.


## INTRODUCTION AND BACKGROUND: FROM ELITE TO MASS EDUCATION

In Denmark, like in many other countries, education in general, and mathematics education in particular, have become a mass enterprise during the last four decades, not only at the primary and lower secondary levels, but at upper secondary and tertiary levels as well. This has given rise to major changes at the higher levels, where the transition from the selective and elitist study programmes of the past to programmes addressing, today, large segments of society has not at all been smooth. As the development of educational demographics has been gradual and somewhat slow, the changes of the study programmes, too, have been gradual and slow. One implication of this is that instead of making more radical changes in curricula, in teaching and learning materials, and in assessment, corresponding to the changes in the audiences, authorities have attempted to preserve the goals and the ethos of mathematics education of the past, at least in spirit, while making series of piecewise adjustments so as to avoid too drastic discontinuities in the transition from the past to the present. Bluntly put, instead of curricula, materials and modes of assessment designed from scratch, to accommodate the new boundary conditions, we have witnessed a sequence of gradual dilutions of these components of mathematics education, which has given rise to increasing incoherence and inconsistencies within the programmes. Another implication of the development just outlined is that the upper secondary school levels - which
will be my focus in this paper - have received lots of new categories of students, quite a few of whom experience severe difficulties in coming to grips with mathematics, especially because the transition from lower secondary to upper secondary mathematics education still represents quite a gap despite the development described. This means that in every upper secondary mathematics classroom one finds a marked and increasing number of students who simply "do not get it", and that includes students who work hard, but unsuccessfully, to come to grips with mathematics. Generally speaking, such students still form a minority in their classrooms, albeit a growing minority, a minority which, in some classrooms and schools, will eventually become a majority.

In this paper, the students in focus of interest are upper secondary students like the ones just described, but with the additional characteristic that they do not display learning difficulties in other subjects, except in the ones that rely heavily on mathematics, such as physics and chemistry. In other words, these students appear to have mathematics specific learning difficulties, neither general learning difficulties nor insufficient working morale. On the contrary, they are motivated to learn mathematics, not necessarily because of an intrinsic interest in the subject, but if not for other reasons then because it is a compulsory part of all upper secondary programmes and hence contributes to the end-of-school high stakes marks and exams that determine students' further path though the education system. It is exactly because of this motivation that the students at issue work hard to come to grips with mathematics, even though they remain unsuccessful in their endeavours. In the past, such students would not have been able to enter upper secondary programmes in which mathematics was part of the curriculum, but they can today, because, as mentioned, all programmes include mathematics, and admission to upper secondary school is possible also for students with weak backgrounds and low marks in mathematics.

It goes without saying that the learning difficulties of these students do not come out of the blue. They were founded in primary and lower secondary school and hence are often deeply rooted, and hence cannot be expected to be remedied by quick fixes. The upper secondary programmes (grades 10-12), even those with the lowest demand level in mathematics, are not designed to include remedial activities, nor do they leave room for such activities. Moreover, mathematics teachers for the upper secondary level were never trained to deal with students with mathematics specific learning difficulties, especially difficulties grounded in much earlier stages of schooling. Upper secondary school teachers are educated in universities and must have a master's degree in mathematics and one other subject (or an equivalent background), whereas primary and lower secondary school teachers are educated as generalists in separate non-university teacher training institutions, in recent years with a very limited background in mathematics.

## GROWING RESEARCH INSIGHTS INTO THE NATURE OF AND OBSTACLES TO MATHEMATICAL LEARNING

During roughly the last half a century mathematics education research has made immense progress in understanding the conditions, phenomena and processes involved in mathematical learning as well as obstacles to its unfolding. This progress is undoubtedly due to certain commonalities in human cognition and human affect across individuals, societies, countries, and cultures, even though there are also, of course, lots of differences in these respects. Unfortunately, as Celia Hoyles has pointed out in a private conversation, this growth of insight into learning has not been accompanied by a similar growth of insight into what kinds of teaching will ensure successful learning under general conditions. This is undoubtedly due to the fact that the cultural, economic, political, structural, educational and organisational conditions for teaching vary dramatically across continents, regions, countries, states, provinces, counties, municipalities, schools and classrooms. This is one of the important reasons why insight about the learning of mathematics have not had too much of a general impact on teaching.
It was against the dual background just outlined that I and a colleague, Morten Blomhøj, at Roskilde University, Denmark, a few years into the $21^{\text {st }}$ century discussed the possibilities of activating research insights concerning the learning of mathematics education to seriously attempt to counteract mathematics specific learning difficulties with students who work hard, but to no significant avail, to master, at least to an acceptable level, the mathematics they are taught. However, as no funds were available to devise and implement a programme based on our ideas, we had, at first, to give up and shelve our ideas. But the problems we had observed in the beginning of the century continued to grow. Most fortunately, a few years later a funding possibility emerged. Two other colleagues at Roskilde University, Jesper Larsen and the late Bent C. Jørgensen, succeeded in obtaining funding from the European Social Foundation for a larger and somewhat more general project, "STAR", within which it turned out to be possible to build a programme designed to counteract mathematics specific learning difficulties with upper secondary students. The design of this programme was made by myself and another colleague, Uffe T. Jankvist (now at Aarhus University, DPU), for whom a temporary postdoctoral position was secured by STAR-project funds. So, we were able to establish a special research-based professional-development educational programme for in-service upper secondary mathematics teachers, aiming at enabling participants to help their own and colleagues' target students in their schools to remove or at least markedly reduce their learning difficulties. It was a crucial part of the idea that teachers educated from the programme not only should become better teachers themselves but also should be qualified to undertake functions at their respective schools as specialised so-called mathematics counsellors, assisting both teacher colleagues and students in their respective schools. This programme started in

2012 and continues to this day. The content, structure and logistics of the programme will be presented in the next section (see also Jankvist \& Niss, 2015, 2016, 2017, 2018).

## THE MATHEMATICS COUNSELLOR PROGRAMME AT ROSKILDE UNIVERSITY, DENMARK

Based on experiential knowledge of the curricula and the general state of affairs in Danish upper secondary schools, including teachers' working conditions, we decided to focus the programme on three themes that we found particularly significant, even though several other themes would certainly have been pertinent as well.
These themes are concepts and concept formation in mathematics, mathematical reasoning, proof and proving, and mathematical models and modelling. The first theme was chosen because it lies at the heart of all mathematical work and activity, but also because experience told us that several upper secondary students have a very frail grasp of mathematical concepts, even the most basic ones such as 'number'. It may come as a surprise to some that we included the second and the third theme in the programme. Is it not the case that reasoning, proof and proving are higher order activities primarily relevant for students with or beyond a minimum level of mathematical competence, understanding, and knowledge, i.e. students who might not be supposed to belong to the target group of students in focus of this programme? And, furthermore, is it not the case that models and modelling, too, are for students who have already learnt mathematics to such a degree that they can put it to use in extra-mathematical contexts and situations, i.e., once again, students outside the target group? In other words, do the latter two themes not point to luxury problems in relation to the overall point of focusing on students with mathematics specific learning difficulties at a much more elementary level? The answer to both of these questions is 'no'. As to reasoning, proof and proving, these are certainly key mathematical activities because they deal with one of the most fundamental components of mathematical work: the justification of mathematical claims and statements. This is not restricted to justifying mathematical theorems and formulae by way of formal proof but pertains to all sorts of mathematical claims and statements, such as the results of numerical and symbolic calculations, solutions to problems, and inferences concerning the use of mathematics in extra-mathematical contexts and situations, to mention just a few. Moreover, experience and research show that many students' mathematics specific learning difficulties are to do with their perception of mathematics as a huge set of disconnected and incoherent facts, procedures and rules that have to be learned by rote, and of mathematical processes and methods as being meaningless and illogical inventions by strange people with weird minds. In other words, problems with reasoning, proof and proving are part of the foundation of students' learning difficulties, and these learning difficulties are exacerbated if
reasoning, proof and proving are left out in favour of a one-sided emphasis on rote learning, memorisation and drill of facts and procedures. When it comes to mathematical models and modelling, similar considerations apply. Many students with mathematics specific learning difficulties have severe problems with the relationship(s) between mathematics and the extra-mathematical world, especially as regards making sense and meaning of mathematics as having something to do with the world in which they live. They perceive mathematics as an abstract and inconsequential set of games that few (other) people can learn - and love - to play, but how, so the argument goes, can school allow itself to give such a high priority to abstract games instead of to something that really matters in people's lives? Of course, quite a few of such students are victims of the so-called relevance paradox (Niss, 1994, p 371). These students know that mathematics is important in society and in many attractive careers and jobs, but they do not understand why. Unfortunately, however, mathematics is not relevant to them with respect to their own perspectives on their current and future lives. Since mathematical models and mathematical modelling constitute the bridge by which mathematics and the extra-mathematical world are connected, an insufficient representation and grasp of these aspects of mathematics in mathematics education are co-responsible for students' learning difficulties. In addition to all this, models and modelling are in fact explicitly emphasised in Danish upper secondary curricula and hence generate obstacles to students who cannot come to grips with the relationship between mathematics and the extra-mathematical world.

So, the three themes mentioned were the ones we selected for inclusion in the programme. Other themes of significance to upper secondary school mathematics and students' learning difficulties in it were under consideration as well, namely 'mathematical problem solving', 'the role of technology in mathematics', and 'language and mathematics', and we would easily have been able to expand the programme to deal with these themes as well. However, for pragmatic and financial reasons it was considered too ambitious to double the size of the programme and we didn't want to dilute the treatment of the themes by having the time devoted to them be cut down to half.
For each of the three themes, Uffe Jankvist and I selected quite a number of research articles and book chapters that deal with issues and topics pertaining to theme and its "entourage", 18 publications for the first theme, 23 for the second theme, and 21 for the third theme. These publications were certainly not meant to represent anything like an exhaustive coverage of the respective themes but to provide a sufficiently broad, yet specific, avenue into the literature on the themes, and the more general environment in which they are imbedded, on the basis of which further literature could be sought by participants with guidance by the programme directors.

It is now time to consider the structure and organisation of the professional development programme. Each year a cohort of up to 12-16 practising high school teachers are admitted to the programme. Typically they register in close agreement with their respective schools, who then also contribute towards the costs of the programme, since schools are often eager to have expert maths counsellors in their staff, as schools are more than well aware of the problems caused by students' learning difficulties with mathematics. However, there are also quite a few cases of teachers who have taken part in the programme entirely on their own initiative and at their own expense.

The programme is designed as a three-semester part time in-service programme, during which participants work as usual in their schools, though normally but not always with some reduction in their teaching load. The magnitude of the full programme is normed to 30 ECTS points, but we have to admit that this hardly matches participants' work load during the three semesters. Each cohort "travels" together for the three semesters, which allows for our conscious utilisation of synergy effects, in that the participants within a cohort collaborate in various ways throughout the course.
Each cohort begins its studies in early September year $n$ and completes them with a final exam at the end of January year $n+2$. Twice in each semester, in the beginning and at the end, the members of the cohort gather at residential seminars at a special conference facility owned by Roskilde University. At the very first residential seminar in the course the cohort is divided into 2-3 person project groups who stay together throughout the three semesters. Between the residential seminars of the semester each project group works, under supervision by the programme directors, on their own projects (see below) concerning the semester's theme. At the end of the semester each group submits a written report on its work. The report presents the empirical investigation undertaken by the group, with particular regard to the questions the group sought to answer, and also contains an account of what research literature was used and of how it was used. This report is presented orally to the entire cohort and the supervisors at the semester's second residential seminar, where one of the other groups has the responsibility of offering constructive commentary and criticism. So, all groups both have to present their own work and to assess another group's work. This is meant to further strengthen participants' educational outcome of the course. Project groups also receive feedback on their work from the programme directors.

After successful completion of the third semester each project group edits and integrates its three semester reports into one final comprehensive report, which forms the basis of a final oral exam at which participants receive individual marks. A passing mark results in a diploma for successful completion of the programme.

Now, what are participants' projects like and how do they come about? Well, first of all for each semester's theme, Uffe Jankvist and I have developed a socalled detection test (Jankvist \& Niss, 2017) with the purpose of assisting participants in detecting students with potential mathematics specific learning difficulties with respect to the theme. A detection test consists of questions that are supposed to be as elementary as possible in technical terms without referring to any particular section of students' current or past curricula. Most of the test questions are unusual in relation to the test questions students typically encounter as part of their everyday mathematics teaching. This means that the test is neither meant to uncover students' general mathematical competencies, knowledge and skills corresponding to the school level they are at, nor to screen the population of students at large. Nor is it designed to uncover the nature of students' learning difficulties, let alone their causes. For this, other instruments are required. Metaphorically, a detection test can be likened with a metal detector, which is able to locate metal objects in the earth but is unable to tell you anything specific about the nature of the objects, whether they are old rusty cans or gold treasures from ancient times. The construction of the detection tests was informed by the research literature but was also a result of our own ideas and analytical considerations.

At the beginning of the semester, the prospective mathematics counsellors administer the detection test to one or more of their own mathematics classes and sometimes also to some of their colleagues' classes. The detection tests are not supposed to stand alone in detecting students with mathematical learning difficulties. They have to go hand in hand with teachers' professional and personal knowledge of the students in their classes. It is interesting to observe that the outcomes of the tests almost always give rise to some surprises to the teachers, typically by detecting students who hadn't previously been identified by them as having potential learning difficulties, but in a few cases also by not detecting students that teachers had considered weak in mathematics. Here, it should be kept in mind that the detection tests are geared towards the special theme of the semester and not towards mathematics teaching and learning at large.
Once the outcomes of the detection test have been analysed by prospective mathematics counsellors and put together with their prior knowledge of students, a small number, 2-5, of students are identified for an invitation to participate in the project. In addition to being candidates for possessing mathematics specific learning difficulties (which implies that their learning difficulties are not seen to be of a general nature, pertaining to all the other school subjects as well), those invited also are supposed to be willing and able to spend extra time and effort on activities that can counteract their learning difficulties. Usually the students detected already know themselves that they have difficulties in mathematics, and most of them are eager to get help so as to
have their problems reduced. The students who have been identified and subsequently accept the invitation to participate in the project are called the focus students of the project.

The next step of the project is to diagnose the nature of the focus students' mathematical learning difficulties with particular regard to the theme of the semester. In this process, which is assisted by the research literature made available to participants, the prospective mathematics counsellors combine an a priori analysis of the pattern of each student's answers to the detection test questions with an a posteriori analysis of the outcome of special tasks and diagnostic interview sessions with the focus students, alone or in small groups of focus students. In these sessions the mathematics counsellors interact directly with the student(s), seeking to uncover the character of the learning difficulties at play, and if possible, some of their possible causes as well. Here, the research literature and its findings serve as lenses through which the nature of the learning difficulties might be understood and articulated. It is not unusual, though, that teachers are able to also discover novel learning difficulties that haven't been satisfactorily dealt with in the research literature.
It is also not unusual that focus students are found to have several different mathematical learning difficulties at the same time. Unless these are closely linked, mathematics counsellors decide to zoom in on one or two of them for the next stage of the project, the intervention stage. Whilst the diagnosis stage is largely informed and guided by the research literature, this is less true of the intervention stage. The reason for this is parallel to what was mentioned above about the non-universal character of successful teaching. A wide variety of structural, organisational, institutional and personal conditions and circumstances frame what interventional measures are possible. Moreover, the mathematical, cognitive and emotive specifics of the individual focus student, too, have to be taken into account in mathematics counsellors' design of an intervention scheme for the student. Even though it is certainly possible to get specific or general inspiration from the research literature, counsellors have to display a non-negligible amount of independent inventiveness and creativity in their intervention designs, which, by the way, is also one of the programmes' professional attractions for teachers.
Basically, three sorts of intervention schemes are at the counsellors' disposal. The first scheme is targeted on activities for the individual focus student. The second scheme involves a small group of focus students who are asked to collaborate on the tasks and activities designed by the counsellors. The final scheme involves the whole class to which the focus students belong, where all students engage in the tasks and activities, typically in small groups, but where the focus students, whether they work in a group composed of focus students only, or each of them in a group otherwise composed of non-focus students. Of course it is also possible to make combinations of the three basic intervention
schemes. In the past, all sorts of options have been explored and exploited by the prospective mathematics counsellors as part of their studies within the programme. It goes without saying that a key issue for the intervention, for the project work and for the report written at the end of the semester is the degree to which the intervention worked, i.e. the degree to which it was possible for the prospective mathematics counsellors to significantly reduce - perhaps even remove - the learning difficulties uncovered at the beginning of the semester for the focus students. Here, given the depth of these students' difficulties, it is encouraging to note that for the far majority of the focus students it was indeed possible to considerably reduce their learning difficulties during what amounts to a relatively short intervention period. Although we have limited knowledge about the long term-effects of the interventions, the mid-term effects seem positive in the sense that most of the focus students subsequently took their final high school exam in mathematics with very respectable marks, sometimes even with flying colours. We need further research to delve more deeply into longterm effects of the interventions undertaken by the prospective mathematics counsellors. At this stage the most interesting observation is that even deeply rooted and resilient learning difficulties in mathematics are not immune to remedial endeavours. Extensive, systematic and lengthy efforts to uncover, diagnose and intervene against mathematics specific learning difficulties with students who are willing to invest time and efforts into their learning of mathematics are likely to yield even better results.

There are several reasons why this programme seems to be pretty effective, but two linked reasons are of particular importance. The first reason is that the programme is research based, the second that it is linked up with and dependent on the prospective mathematics counsellors' everyday teaching practice, and especially that participants learn by working with authentic students - usually their own students - with real learning difficulties, and that the effect of their undertaking is immediately visible before their eyes. As specifically regards the research component of the programme, it is of outmost significance that it is not shaped as isolated theoretical "dry swimming" but is put to direct use in the projects participants make.
Let me finish this section by mentioning that the first two cohorts of the mathematics counsellors have written the chapters of two books edited by Uffe Jankvist and myself (Niss \& Jankvist 2016; Niss \& Jankvist 2017). These chapters present selected aspect of the projects done by the teachers. A third book in this series is in the pipeline.

## FINDINGS CONCERNING MATHEMATICS SPECIFIC LEARNING DIFFICULTIES

In this part of the paper we shall consider a number of selected findings concerning students' mathematics specific learning difficulties. Some of these findings are known from the research literature but were confirmed in our and
the mathematics counsellors' work on and within the programme. Other findings are new and corroborated by research conducted by Uffe Jankvist and myself on the basis of the data collected from students' work during the six years the programme has been in existence so far.
However, before going into details let me first point to two overarching factors that turned out to be intimately linked to most students' learning difficulties across all the themes of the programme. These factors are captured by corresponding theoretical constructs, well-known from the research literature: mathematics-related beliefs and the didactical contract.
Beliefs about "something" are a person's relatively stable, permanently held convictions about what is true concerning the "something" at issue (Philipp, 2007). It is not important for the existence of a belief whether these convictions are actually true or not. When it comes to mathematics, a person's mathematicsrelated beliefs can address at least four different aspects of mathematics as well as the interplay between these aspects. Beliefs about the nature and characteristics of mathematics as a discipline; beliefs about the place and role of mathematics in the world, i.e. in nature, society, culture, technology and science; beliefs about mathematics education, i.e. the teaching and learning of mathematics, and the place of mathematics in the curriculum at large; beliefs about the self's relationships with mathematics in the three manifestations just mentioned. The belief targets are sometimes depicted as a tetrahedron (see also Jankvist, 2015), in which the triangle with the vertices 'maths as a discipline', 'maths education', and 'maths in the world' are placed in the "ground floor", whereas the vertex 'self' is a point in 3 -space, placed above the ground floor. The six edges of the tetrahedron represent the connections between the four aspects taken into consideration.

The "person" at issue can be a student but also a teacher. Oftentimes people's mathematics-related beliefs do not manifest themselves directly and explicitly. They have to be uncovered. A multitude of studies show that students' and teachers' work and behaviour in, and attitudes, to mathematics and mathematical activity are markedly influenced by their beliefs about mathematics.

Indeed, the focus students involved in the mathematics counsellor programme were all greatly influenced by their mathematics related beliefs. They often saw mathematics as a swarm of disconnected concepts, facts and rules, not governed by any natural logic, a field which only rarely makes sense in the real world and which by ordinary people can only be learnt by rote memorisation. As one focus student explained, in the first year of upper secondary school it was still possible to learn everything by heart, but in the second and third year concepts, rules, procedures and theorems piled up to such an extent that it became out of reach to learn it all by heart, so the only option was to give up while at the same time wondering why some students evidently were able to catch up with the demands. "They must have extraordinary memories!". Most students
experienced great difficulty in reconciling theoretical and empirical considerations. Basically, they were inclined to perceive mathematics as an empirical discipline, but they knew that this was not the way the teachers and their textbooks looked at things. So, many students were utterly confused as to when empirical work was acceptable and when not. The far majority of the focus students held what Harel and Sowder (2007) call external or empirical proof schemes, i.e. for them to be convinced of the truth of a mathematical claim either some external authority, such as a teacher, a textbook or a high performing peer, was needed to testify to the truth of the claim, or the students needed sufficiently many (or rather insufficiently few!) empirical examples, and nothing else, to become convinced of the truth of the claim. Whilst these students were normally aware of the "system's" requirements of some formal proofs and proving they, themselves, couldn't see the need for it. As particularly regards mathematical modelling, it is crucial whether a student (or a teacher) holds views of mathematics as a discipline - or of mathematics as an education subject - that excludes or includes the relationships between mathematics and the extra-mathematical world, including mathematical modelling. Several students felt that extra-mathematical considerations and activities have nothing to do with mathematics, so they were reluctant to engage in such considerations and activities. Ironically, the very same category of students oftentimes complained about the lack of sense and meaning of mathematics for the world outside the classroom.

The notion of didactical contract is a key construct in Guy Brousseau's theory of didactical situations (Brousseau, 1997) which he has developed over several decades. The didactical contract amongst a teacher and his or her students in the classroom specifies the division of labour between the teacher and the students. What can the students expect that the teacher will do, and will not do, in- and outside class? What kinds of tasks will the teacher give to the students to work on, and within what time frame? What sort of help will the teacher provide to students, prompted or unprompted? Dually, what can the teacher expect the students to do, and not to do, inside and outside class. What kinds of tasks can the teacher expect the students to (accept to) undertake, individually, in small groups or in whole class settings? How does the teacher expect the students to behave, individually, and vis-à-vis peers, in relation to their mathematical work inside and outside class? And so on and so forth.

Typically, the contract is tacit, shaped by years of habit and experience on the part of both the teacher and the students, and not only the actual ones but also previous teachers and previous students. It is of course asymmetrical due to the general inequality in the power balance between teacher and students, even though we know that students may sometimes rebel against a teacher with dramatic consequences. According to Brousseau, learning can only really take place when the usual didactical contract is broken, because only then will
students make learning their own project, which is a necessary condition for learning to occur. Breaking the didactical contract implies that it goes from being tacit to being explicit. Such a breach requires re-negotiation of the contract, although not necessarily a shared agreement on a new contract. Nevertheless, if the teacher wants students to undertake mathematical work e.g. mathematical modelling, stating conjectures, proposing and proving theorems and so on and so forth - that are new in relation to the existing didactical contract, this contract has to be broken, re-negotiated and replaced by a new one.

All students in the mathematics counsellor programme were strongly influenced by a rather narrow didactical contract according to which the teacher "on a normal day" would present new material to the class - usually with close reference to textbook sections - showcase the solution of what would constitute a standard exercise with well-defined steps and procedures, and then ask students to solve a number of more or less similar exercises in class, either individually or in small groups. Typically, similar tasks are given to students to work on at home, and every now and then they are required to submit individual written solutions for the teacher to correct and mark, work that contributes to their highly significant seasonal marks. The role of technology, including CASsystems and graphing calculators, in all these activities is manifest. Probably no other country attributes a more positive role to technology in education than does Denmark. When students work on their tasks in class, they frequently call on the teacher, asking for confirmation of what they have done so far or for help if they've got stuck. The tasks students are asked to work on are typically closed, relatively short, and require short answers only (only seldom multiplechoice responses are an option), and there is normally only one correct answer. So, it does not take much novel activity to break the didactical contract, which both happened with the detection tests students were given, and with intervention schemes concerning the three semester themes, especially reasoning, proof and proving, and models and modelling. Most students were not used to such tasks, usually with several reasonable solutions, and often got confused about how to deal with them. In some cases they even rejected the tasks altogether, because they deviated too much from what they were used to. It was therefore necessary for the mathematics counsellors to articulate a change of the didactical contract (without using that term, though) and explicitly negotiate a new one with the students.
Let us now consider a set of more specific observations and findings. Needless to say, it is only possible to deal with a small set of observations and findings in this paper.

Firstly, we shall look at selected findings concerning concepts and concept formation. It is well known that many students experience difficulties at really capturing the concept of number, ranging from place value representations of
natural numbers, over the problems involved in seeing the differences as well as the links between fractions and rational numbers, understanding the nature of negative numbers and operations with them, through to the definition and meaning of irrational numbers and decimal representations of any real number. The position of numbers on the number line, including the denseness of the rationals (and the reals), give rise to learning challenges to most students. However, what was surprising to us was the extent to which focus students rejected 0 as a number. Several focus students explicitly stated that " 0 is not a number". Surprisingly many students - and not only focus students - claimed that $a^{5} / a^{5}=0$, because nothing is left when all the $a$ 's have cancelled out, which means, so the students reasoned, that the result is 0 . Some of these students also claimed that $(2 / \sqrt{3}) \cdot(\sqrt{3} / 2)=0$, but found $(2 / 3) \cdot(3 / 2)=6$, because in the latter case you can compute the product explicitly, which you can't in the former case. Furthermore, focus students often wrote ' $0 \cdot x=x$ ', and explained that since 0 stands for nothing it has no effect on $x$ in multiplication. Other students, in contrast, claimed that ' $0+x=0$ ', because "nothing" annihilates what is next it. These students' common perception seems to be that 0 is not a number, but "a marker of absence". How come that several focus students found that the solution to the equation $38 x+72=38 x$ is 0 ? Based on interviews with these students, the explanation seems to be that when you have done everything you can to this equation you obtain $72=0$, and as we "know" that the solution to an equation is what you end up with on the right-hand side of the equal sign when your manipulations have been completed, 0 must be the answer. Upon closer scrutiny, none of these students thought that the number 72 is actually 0 , suggesting, once again, that 0 is not a number. This was exacerbated by the fact that it was usual to get the answer 'six' from focus students, when they were asked to tell how many integers there are in the interval $[-2,5)$, namely $-2,-1,1$, $2,3,4$, leaving 0 out of the count.

Another well-known finding from research and practice over several decades is that the equal sign gives rise to many obstacles to students. The main reason for this is that the equal sign plays a lot of rather different roles in mathematics, although these roles are closely related, which is why we use the same symbol for all the roles. One such role is in definitions: In a right-angled triangle with sides $a$ and $b$, we define sine of the acute angle $A$ between $a$ and the hypotenuse $c$ as $\sin A=|b| /|c|$. Also we define what is called the discriminant of the quadratic equation $a x^{2}+b x+c=0$ as the number $D=b^{2}-4 a c$. We define the derivative $f^{\prime}(x)$ of the function $f$ at $x$ as $\lim _{h \rightarrow 0}[f(x+h)-f(x)] / h$ (provided the limit exists). Another role, close to that of definition but used in different contexts, is assignment. For example, given the numbers $a_{1}, a_{2}, \ldots, a_{n}$ we let $s_{n}=a_{1}+a_{2}$ $+\ldots+a_{n}$. A very important role is that of identity. In contrast to the previous uses of the equal sign, identity represents a statement, which may or may not be true, within some domain. We always have $a=a$. In the classical number domains we also have $a b=b a$ and $(a+b)(a-b)=a^{2}-b^{2}$ for all $a$ and $b$ (but not
if $a$ and $b$ are square matrices). For plane right-angled triangles (but for no other plane triangle) with side lengths $a$ and $b$ and hypotenuse length $c$, it is always true that $a^{2}+b^{2}=c^{2}$. Also $1=0.999 \ldots$ (infinitely many 9 's). A more general version of the notion of identity is that of equivalence within a mathematical domain. We write $m=n(\bmod p)$ for integers $m, n$, and $p$ if and only if $p$ divides $m-n$. Two different fractions $m / n$ and $p / q$ are equivalent, and we write $m / n=p / q$, exactly if $m q=p n$ (here ' $=$ 'stands for identity). One of the crucial roles of the equal sign is that of a query. A query is not a definite statement, but two combined questions: If $U(x)$ and $V(x)$ are variable expressions depending on $x$ varying over a certain domain, does there exist a value of $x$ such that $U(x)=V(x)$ ? If so, what is/are the $x /$ 's that satisfy this predicate? This role of the equal sign is the one we find in all sorts of equations. The final role of the equal sign is that of a prompt or a command, like in: $\sin (\pi / 4)=$, and $1+2+\ldots+n=$.
Grasping these different roles of the very same omnipresent equal sign is complex enough for ordinary students, but even more so for students with mathematics specific learning difficulties. The focus students in our programme seem to adhere in large part to the prompt interpretation. They perceive the sign as a procedure indicator meaning 'gives' or 'yields' or simply 'moving on', which is why they can easily use it several times, one after the other, in long chains of calculations, even though the resulting statements become mathematically meaningless. This interpretation probably also explains the highly surprising fact that several focus students answer 'no' when asked whether one can infer from $a=b$ that $b=a$. If $b$ is perceived as resulting from processes performed on $a$, there may be no reason to believe that $a$ will result from a process performed on $b$. When faced with an algebraic equation, focus students often believe that they have to perform operations on the left-hand side so as to obtain the right-hand side. This become meaningless to them when there are unknowns on both sides. How is it possible to make manipulations of the left-hand side that yield an unknown right-hand side? In other words to many students with mathematics specific learning difficulties the equal sign does not stand for a relation, let alone a symmetric one.
Solving algebraic equations is known to be notoriously difficult to many students, and the difficulty is often explained by insufficient operational skills including lack of knowledge of the manipulative rules and tricks - or lack of effective solutions strategies: what are we aiming at, what to do first, and next, and so on, till the equation has been solved? This is certainly also part of the problem. But there are some deeper problems, which are particularly manifest for student with learning difficulties.

In the same way as the equal sign in general is interpreted by focus students as a prompt to carry out certain procedures, an equation is interpreted as a prompt to undertake certain more or less well-defined steps. Students do not perceive an equation as a mathematical object such as a predicate, or a query. Similarly, to
them a solution to an equation is what comes up when the sequence of procedural steps has come to a stop. We already know this from research (Bodin, 1993) concerning younger students, where a large proportion of the (grade 7) students who were able to successfully come up with the unique solution to the equation $7 x-3=13 x+15$ were unable to subsequently tell whether $x=10$ is a solution to the equation they had just solved. It is also reflected in the focus students who propose 0 (usually not $x=0$ ) as the solution to the equation mentioned above, $38 x+72=38 x$. Embarking head over heels on the procedures prompted by the equation, they never reflect on the nature of the object put in from of them. Also focus students faced with the equation $3 x-x=2 x$, rush off to perform the procedural steps they know, arriving at $0=0$, and then finish with several question marks. When interviewed about their response they say that they didn't really know what to do, because suddenly $x$ disappeared from the equation, which has never happened before. So, once again, as the solution to an equation is what you obtain at the right-hand side, when nothing more can be done, several students propose 0 as the (only) solution. This was also found when students were asked to solve the equation $x=1$, where some wrote ' $x-1=0$, so $x=0$ '. It has to be admitted that it was never part of the standard didactical contract for these students to ask them to solve an equation that already has a trivial solution put in front of their eyes, equations with no solutions at all, or equations for which any number is a solution.

With the focus students we also encountered the standard finding that for something to be recognised as an equation it has to contain $x$ 's. When students were asked to answer the question 'does it ever happen that $a^{2}=2 a$ ?' a typical answer is 'no, for $a^{2}=a \cdot a$ and $2 a=a+a$ '. If asked whether it ever happens that $x^{2}=2 x$, the focus students are transferred to "equation mode" and give one of the following answers: 'I don't know, because I don't know how to solve such an equation' or 'yes, if $x=2$ '. Focus students almost never provide ' $x=0$ ' as an answer, which is probably yet another reflection of their conviction that 0 is not a number.

The final examples of findings concerning concepts and concept formation are special cases of the general finding that mathematical symbolism at large is a massive stumbling block for many students, and in particular symbolism involving letters. Many focus students were convinced that letter symbols are subjected to a game or a calculus of its own, with rules and procedures invented by some game constructors, and having no bearing whatsoever on dealings with numbers. As one of them stated when faced with the equation $38 x+72=38 x$, 'I don't understand, for you cannot add $x$ 's to numbers'. They were resiliently unaware of the fact that letter symbols are stand-ins for numbers in different roles, be they unspecified constants or parameters, be they variables or be they unknowns sought to satisfy equations or inequalities. We also saw several
students who believe that letters used in applied contexts are labels of entities or objects not numbers, as in variants of the famous Student/Professor-problem, 'In a certain school there are six times as many students as teachers. If $S$ is the number of students and $T$ is a number of teachers, write an equation that expresses the relationship between $S$ and $T$ ', where lots of students made the socalled "reversal error" and wrote ' $6 S=T$ ' giving reasons like 'for every six students there is a teacher', thus mixing the two sorts of entities 'students' and 'teachers', respectively, up with their numbers.
We also saw several focus students who believe that letter symbols have a personal identity of their own, like in the Pythagorean Theorem, $a^{2}+b^{2}=c^{2}$, where $c$ is always the (length of the) hypotenuse, and $a$ and $b$ the (lengths of the) two other sides, so that it is simply illegal to interchange them. More generally, the fact that we are allowed to introduce names of variables as we wish, as long as some basic conventions and consistency rules are being observed, is very alien to most of the focus students. This also plays out when they are asked to engage in modelling tasks. One of the conventions some focus students have difficulty in coming to grips with is that an $x$ in an equation is the same number in every occurrence. In an equation such as $7 x-2=12 x+2$, why can't we say that the $x$ on the left-hand side is 4 and the $x$ on the right-hand side is 2 , since $x$ is after all unknown in all occurrences?

I have spent quite some space on findings concerning concepts and concept formation because these permeate all mathematical work. It is now time to address the second theme, reasoning, proof and proving, which for space reasons will be dealt with in a less detailed manner. Our, and the mathematics counsellors', point of departure is the notion, already mentioned, of proof scheme, introduced by Harel and Sowder (2007). Based on empirical studies of university students, Harel and Sowder wanted to investigate what for students constitutes their conviction of the truth of mathematical statements. They identified three basic proof schemes, which might also be called conviction schemes: external proof schemes, where conviction is established by external features (e.g. the exterior appearance of a proof) or external authorities (such as teachers, textbooks, peers); empirical proof schemes, where conviction is established by empirical probes into the truth of a claim, primarily by way of confirmative examples of special cases of the claim, but also overall plausibility and compatibility with existing experiences; and deductive proof schemes, where conviction is established by logico-mathematical deduction from a set of premises. Focus students almost never possess a deductive proof scheme, although they sometimes know that this is what teachers have, especially versions, so the students believe, of deductive proof schemes involving formal calculations. Many focus students adhere to external proof schemes, which conforms to the fact that these students do not see mathematics as something that makes much sense, let alone something that can be understood. So, these
students are happy to "take an authority's word for it". Many other focus students have an empirical proof scheme, predominantly based on examples. They often tend to discard deductive reasoning as something that constitutes truth, because truth for them is an empirical matter. Hence deductive reasoning and proof are simply useless and superfluous.
When focus students were presented with an erroneous proof (based on division by 0 in disguise) that every number is 0 , and then were asked whether it is true that every number is 0 and whether the proof presented was correct, surprisingly many answered that the proof seems correct (an instance of an external proof scheme), so it must also be true that every number is 0 . In interviews some of them were then asked whether they really believe that every number is 0 , and replied 'in reality, of course not, but it is evidently true in mathematics'. Other focus students found no flaw in the proof but disagreed that every number is 0 , and when asked how these two statements were compatible, they answered 'that is because in mathematics you can prove whatever you like'.
Although several focus students did have problems with fundamental mathematical logic, especially implications and their reverse, quantifiers, and non-formal negation of implications and quantified statements, only a few of them suffered from serious deficiencies in everyday logic, i.e. by and large they were able to perform ordinary reasoning to a fair extent. The problems seem to arise when logic is combined with mathematics, in which case there is no experiential corrective to prevent flawed reasoning from occurring. When asked to decide whether the following two statements are the same, the far majority of the focus students answered 'yes': 'if the sum of two integers is even, then their product is odd' and 'if the product of two integers is odd, then their sum is even'. When faced with the question 'A square is a rectangle in which all sides have the same length. Is every square a rectangle?', the majority of the focus student answered 'no', because they discarded the premise stated in the question and activated their everyday "knowledge" that a rectangle is a right-angled quadrilateral where the two pairs of sides are of different lengths. This was confirmed when the same students said 'no' when subsequently asked whether a rectangle is ever a square.
Let us finally turn to observations and findings concerning mathematical models and modelling. The first significant observation is that almost a fourth of the students actually reject, or give up on, modelling tasks. In interviews the focus students typically say that they gave up, because they had no idea what to do and how to get started. They also tend not to pay attention to the setting, information and questions provided in the task but either rush off to make somewhat meaningless mathematical calculations or activate irrelevant (or wrong) everyday conceptions or ideas, e.g. when they respond to two models of the long term exploitation of an oil field by saying that the models are both wrong,
because the oil will be regenerated in the earth, or because mankind will always be able to find and exploit new oil fields.

For quite some time it has been established knowledge from research that students at large tend to over-generalise proportionality in word problem and modelling contexts (De Bock et al., 2011; Van Dooren et al., 2008). This is certainly the case with our focus students as well. We see this when students are asked to solve the following PISA problem: Two pizzas of the same kind and thickness cost DKK 30 and 40, respectively. The diameter of the cheaper one is 30 cm , and that of the more expensive one is 40 cm . Which pizza is the best value for money?' Virtually all focus students respond that it does not matter, because in both cases you get 1 cm pizza per DKK. In other words, they have adopted a proportionality model based on the diameter (or, equivalently, the circumference) of the pizzas. When interviewed about their responses, it typically takes quite some time to make students realise that the model should be based on the area of the pizzas, which scales with the square of the diameters. Some focus students even believe that 'diameter' is yet another unit along with centimetre or metre. Once the students have realised that the area is the relevant magnitude for the model, they typically have no problem in finding the areas involved, as they usually know the formula for the area of a circle. Another instance of over-generalised proportionality is present in students' answer to the question: 'A (massive) wooden cube with all edge lengths 2 cm weighs 4.8 g . How much does a massive cube made of the same wood, but with edge lengths 4 cm weigh?' Practically speaking all focus students answer 9.6 g , and once again, this answer turned out to be highly robust towards attempts to make them realise that the scaling factor is 8 and not 2 . Upon further probing, most of the students were convinced that also the volume - and not only the weight - scales by a factor of 2 , and not 8 . By inviting students to make several 2D and 3D drawings or to make use of building blocks, the mathematics counsellors were able to eventually generate the insight with students that the correct scaling factor is indeed 8 . A few focus students were able to correctly scale the volume but not the weight. It turned out that these students were missing the notion of density of a homogeneous material as a fixed ratio between weight and volume.
Over-generalisation is a special case of a general phenomenon: overgeneralisation of the models students are familiar with, above all function models. When invited to model more complex situations, students picked one of the models on their shelves of experience -multiplicative models, linear models, power function models, exponential or logarithmic models - without paying due attention to the specifics of the situation. For example, when asked to model the above-mentioned student/professor-problem, several focus students proposed meaningless linear models such as $y=6 S+T$, or power function models such as $T=E^{6}$, or no model at all, such as $6 S / T$. A focus student asked to forecast the body temperature of a person whose temperature, due to a fever, rose by $1.1^{0}$ per
hour proposed an exponential model, because 'this is what we usually have when something increases by a fixed amount per time unit, like with bank savings or populations growth'. When prompted by her mathematics counsellors to make a table and then a graph of the temperature growth, she realised that the growth was in fact linear, whilst maintaining that this clashed with her perception of what kind of model describes constant growth per time unit, without realizing that the nature of the growth, additive or multiplicative, is a key factor to consider.
Space does not allow me to go any further in describing the multitude of findings obtained by working with upper secondary school students with mathematics specific learning difficulties. However, what I have presented hopefully suffices to show that students are faced with a very large number of such difficulties and that these difficulties are far from lying on the surface of students' learning but are in fact deeply rooted in it. Furthermore, the learning difficulties are underpinned - if not co-created - by overarching misconceptions resulting from misguided mathematics-related beliefs and too narrow and less than constructive didactical contracts. Therefore it takes much more than correcting, telling and showing or other quick fixes to counteract these difficulties. This takes us to the final section of this paper.

## CONCLUSION

In view of the severity and deep roots of students' mathematics specific learning difficulties, shaped and consolidated throughout a decade of schooling, it seems as a mission impossible to dream of effectively counteracting these difficulties without beginning again from scratch. However, our programme, as well as our follow-up research, has demonstrated that engaged and competent mathematics counsellors trained in a professional development programme that makes explicit use of research in dealing with real students encountered in everyday teaching practice can in fact make a significant difference. We do not know yet how large this difference is. What we do know is that it is large enough for most (albeit not all) of the students taking part in the programme to make visible progress to a point where they obtain respectable and in some cases fantastic success in the remainder of their upper secondary mathematics education, while changing their mathematics-related beliefs and attitudes to mathematics and mathematical work to much more constructive ones.
Further research and development is needed to study long-term effects of the interventions implemented by the mathematics counsellors. Further research and development is also needed to investigate how the everyday role of mathematics counsellors in their schools can be optimised so as to not only remedy mathematics specific learning difficulties when detected and diagnosed but, much more importantly, how these can be prevented from occurring.

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# DOES SCHOOL EDUCATION ENHANCE THE DEVELOPMENT OF CREATIVITY? 

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#### Abstract

Among the aims of mathematics education in primary school, inter alia, the development of creativity, creative and critical thinking are mentioned. It seems that the higher the school level, the greater should be the openness and creativity of students. The paper presents the results of a research carried out among students in the third and seventh grade of primary school. Both research groups received the same mathematical task: in a presented sequence of numbers they had to select the one that in their opinion did not match the others. The justification provided by the students was the focus of our study. The obtained results turned out to be quite surprising and provoking the question whether school reality fosters the development of creativity.


## INTRODUCTION

Today, teaching mathematics is understood in a specific way and mathematics is perceived as a social activity (Schoenfeld, 1992). The focus of mathematics education has also changed (da Ponte, 2008, Krygowska, 1985, 1986). Learning and teaching mathematics is primarily understood as learning to think, act and communicate mathematically (Arzarello, 2016). It is expected that a student who graduates from primary school will not only demonstrate knowledge of relevant mathematical facts. In addition to substantive knowledge she or he should also demonstrate a whole range of mathematical skills. These include, above all, the ability to analyse and make hypotheses, argument and justification ability, and creative and critical thinking. Especially critical and creative thinking is particularly important (Oldridge, 2015). Changes in the curriculum that have recently taken place in Polish education put the main emphasis in the teaching of mathematics focused on the development of thinking. The idea is to educate in such a way that the student will be a self-thinking person (MEN, 2008).

Developing students' mathematical thinking is at the heart of mathematics education, also according to the Polish curriculum. However, the concept of mathematical thinking is not clearly defined by researchers. As Schoenfeld (1992) writes, in order to study mathematical thinking, one should recognize the following aspects: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and practices. So in other words student's "activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking" (Viitala, 2015a, p. 138).

## THEORETICAL FRAMEWORK

Research on creativity and creative thinking has been going on for some time. Researchers have given different definitions of creativity. Most often, mathematical creativity is defined as the students' ability to create original solutions in problem solving. Creativity of students during solving this kind of tasks is understood as the ability of finding non-standard ways of solution as well as the ability to find more than one method of solving a problem (Bures \& Novakova, 2015). Mathematical creativity can be seen as the ability of students to create useful and original solutions in authentic problem-solving situations (Chamberlin \& Moon, 2005). We can find references distinguishing the following basic features of mathematical creativity (Silver 1997): a) fluency, referring to the number of correct responses that the student produces, b) flexibility, referring to the number of different mathematical concepts and ideas that the student discovers, usually breaking away from stereotypes, c) elaboration, indicating the complexity of mathematical thinking, as the student integrates different pieces of mathematical knowledge, and d) originality, illuminating the extent that the student's ideas are insightful, new and lead to unexpected and unconventional solutions.

Studies show that on the first stage of education many students show their talents towards mathematics. There is a belief that every child is gifted, meaning that each child can work creatively (Brandl 2011, Clements \& Sarama, 2007, Gruszczyk-Kolczyńska, 2009, Munz 2013,). Therefore, it is necessary to raise the efforts to find and develop mathematical talents within pupils of the lower grades of primary school (Tirosh, Tsamir, Levenson \& Tabach, 2011). Mindful that students at different levels of education feel the satisfaction from creative activity, you need to create the conditions for them to present their achievements. That possibility leads to classes in which the student has the opportunity to meet with unusual tasks that do not impose a single method to solve (Ramani \& Siegler, 2007).
As the research shows, of great importance in the field of developing mathematical capabilities and mathematical creativity is the selection of tasks. The tasks called open-ended are particularly helpful (Klavir \& Herskovitz 2008). The open-ended problems are more cognitively challenging, because they allow for multiple interpretations and solutions and offer students the opportunity to solve problems using their actual skills.

Solving these types of tasks allows students to take their first steps towards developing mathematical creativity (Mann, 2006). It is believed that solving carefully selected problems may help to develop and cultivate students' creativity.

## METHODOLOGY OF RESEARCH

The research described in this paper has been carried out over 5 years. Two research groups participated in it. The first group consisted of students from the third grade of primary school ( 20 children in the age $9-10$ years old), and the second one consisted of students from the seventh grade ( 33 people aged 14). The research groups were not specifically selected. All of them were regular school classes. There were both mathematically talented students and those who had difficulty in learning mathematics. The idea was to test students in their natural environment. The third graders were examined in 2013. At that time, they participated in additional classes aimed at arousing their interest in mathematics. It took place once a week and lasted 45 minutes. Two different topics were discussed during the classes: one of them was related to geometry (mainly concerned with 3D geometry and the relationship between 2D and 3D geometry), and second one - to the arithmetic. The results obtained during these meetings have already been discussed and presented (Pytlak, 2013, 2015). Conclusions from the obtained results indicated a large potential of students in terms of creativity and critical thinking. The solving of mathematical puzzles, discovering rules and dependencies gave the third grade students great satisfaction (Pytlak, 2014, 2016). Then the question arose: is this a typical situation, a general case? Do students gain knowledge in this field during school education or maybe rather school reality is breaking their natural abilities? An attempt to answer these questions was to conduct a research among students of the seventh grade of primary school. Students received exactly the same research task as their younger colleagues. It was an intentional treatment. In this way I wanted to compare the results of both research groups. The use of the same research tool gave such possibility.
I wanted to find the answers to the following questions:

- Will students be able to see relationships and dependences in a given sequence of numbers?
- Will the students be able to properly justify their choice?
- Will students be creative?
- What criteria will guide their choices?
- Will the solutions presented by the seventh grade students be similar to those received by the third grade students?

The research task was as follows:
Among the given numbers, select the one that does not match the others. Justify your choice. Is another solution possible? For each choice, justify why this number should be deleted according to your opinion (selected number - justification).

| 9 | 15 | 24 | 16 |
| :--- | :--- | :--- | :--- |
|     <br> 21 42 41 14 <br>     <br> 12 16 18 20 |  |  |  |


| 25 | 16 | 34 | 18 |
| :--- | :--- | :--- | :--- |
|     <br> 33 15 25 30 <br>     | 15 | 12 | 6 |

This is an open-ended task. The numbers have been selected in such a way so that there is more than one choice, depending on the used criterion. The idea was to check what criterion students would apply. Will they be creative and constructive, how many different choices will they discover in individual examples? What features will be discovered first, what will they usually take into consideration while solving the task? Thus, this task developed the ability to: analyse, perceive similarities and differences between objects (here: numbers), put and verify hypotheses; also develop the skill of critical thinking.
The students worked independently for the one school hour. Before solving the task, they received the worksheet. Students were also informed that each task may have different solutions. They should give this one which is the most appropriate in their opinion. They could also make more than one choice, but every time they must give reasons for this selection.

At the beginning all students' worksheets were coded and after that they were analysed. A qualitative and quantitative analysis was carried out. All answers given by the students were collected and accordingly grouped. After analysing all students' work, the phenomena occurring and repetitions in their solutions were distinguished and highlighted. Then the successive analysis of the works was made and each of the answers was classified in the previously listed phenomena. Due to differences in the level of knowledge of both research groups, there were also differences in the used strategies. In addition, a comparative analysis of the results obtained by the third and the seventh grade students of primary school was made.
Due to limited space, the results obtained by the seventh grade students will be briefly discussed here. A comparative analysis of both research groups will also be presented.

## RESEARCH AMONG STUDENTS OF THE SEVENTH GRADE

All students participating in the study solved the task. Expectations regarding the results were such that each student will give at least two different solutions to each of the presented puzzles. In addition, I expected a large variety in the used criteria of selection.

The initial analysis of the work of the seventh grade students led to quite unexpected results. It turned out that everyone gave only one solution to each of the examples (in the third grade half of the respondents gave two or more options to solve a given task). Interestingly, only eight respondents indicated
that other solutions are possible, but it was limited only to the recording of this statement, without giving specific examples. Two students pointed out in their work that there is definitely only one solution in each case. The next two were not entirely sure and they were marking with some examples that another solution is possible, while at others - that there is only one solution. As many as two-thirds of the students solved the task by providing only one example and did not respond in any way to the question of possibility of other solutions. These results are summarised as follows:

1. Reference to digits (digit, digit of tens, the lack of specific digit in the remaining numbers): $8,1 \%$
2. The number of single-, two-digit: 4,6 \%
3. Odd, even number: $2,5 \%$
4. Small, big number: $4,6 \%$
5. Sum/difference of digits: $2 \%$
6. divisibility/multiplication: $57 \%$
7. the relationship between the numbers in the sequence: $8,2 \%$
8. prime number: $0,5 \%$
9. others: $11,8 \%$

As can be seen from the above, the most frequently chosen criterion was related to the divisibility of numbers. More than half of all responses were related to this criterion. Most often this criterion was used in the case of task No. 1, 2, 4 and 5. Students were choosing a specific number and as justification wrote: "because it is not divisible by ...", "because it does not divide by ..." or "because it is not a multiple of...". Such argumentation can be observed in the following works:


Each number is divided by 5 except the number 18

Besides the verbal justification, there appeared some calculations. They played the role of additional justification and they have to authenticate the choice.
The multiple criterion was applied in a quite original way by the students in the
sixth task (sequence of numbers $33,15,12,6$ ). Here students noticed that only 33 cannot be represented as multiples of 60 , because from each remaining number in the sequence, 60 can be obtained by appropriate multiplication, e.g. $15 \times 4=60,12 \times 5=60,6 \times 10=60$.


A lot of students justifying their choice focused on the visual aspect of the number. Here they referred both to how the number was built (i.e., what specific numbers were used to write this number), or paid attention to the specific digit standing on the position of tens or unities. This type of argument referred mainly to examples 2 and 3 (the sequences of numbers 21, 41, 42, 14 and 12, 16, 18, 20). This situation we can observe in the following works:


The criteria related to the construction of a number (one or two-digit number) and its size were equally chosen often. Usually, this way of solution concerned tasks No. 1, 3 or 6 . The criterion related to the parity of the number was also low. It was applied only to task 4 (sequence of numbers $25,16,34,18$ ). Perhaps it was related to the fact that more often in this situation, the students referred to the divisibility of the numbers by 2 .


Few students studied the relationship between the sum of digits in individual numbers. This occurred only in the task 6 (sequence of numbers $33,15,12,6$ ), although it also could be used in other examples (e.g. in tasks $2,3,4$ ). Only one person indicated the prime number as the selection criterion. It has not been said explicitly. This student noticed that the number 41 is not a multiple of any number. Perhaps this student has forgotten how such numbers are called. It was in the task 2:

| 21 | 42 | $2 x$ | 14 |
| :--- | :--- | :--- | :--- | 47 - inepent wielductrocicin 41 - is not a multiple

Particularly noteworthy is the criterion related to the dependence between numbers in a given sequence. We can see two different approaches while applying this criterion. One of them is as follows: the students noticed that three of the four numbers form a sequence change by a constant value (i.e., it is a certain arithmetic sequence with a constant difference). This is particularly evident in the task 3 (number sequence $12,16,18,20$ ), where two different arithmetic progression were distinguished: $12,16,20$ or $16,18,20$. Perhaps the manner of recording the numbers in the task helped the students to discover such dependence. An example of using this criterion is the following work:


The second approach was related to connecting three numbers with each other using arithmetic relationships. This was particularly used for the task No. 1 (sequence of numbers $9,15,24,16$ ) and No. 4 (sequence of numbers 25, 16, 34, 18). Here the students noticed that $9+15=24$ and $16+18=34$.


Applying this criterion could testify the analytical approach of students. Such dependence was not obvious. It required analysing the sequence and making several calculations.

## COMPARATIVE ANALYSIS

Comparing the results obtained by the third and the seventh grade students of primary school, we can notice a lot of differences. Many of them resulted mainly from the level of knowledge available to students at a given educational level. Some of results, however, were quite surprising.

The table below provides a summary of the choices made by students in both research groups. In order to unify the results, the criteria used by the students were coded as follows:

- A - criterion related to divisibility (multiples, divisibility by a given number)
- B - criterion related to the construction of the number (one- or two-digit number, use of specific digits to record the number)
- C - criterion of dependence (creating sequences by numbers, arithmetic relationships between numbers)
- D - criterion of sum / difference of digits of a given number
- E - criterion of the size of the number (large, small number)
- F - parity criterion (even number, odd number)
- G - other criterion (all other criteria that cannot be unambiguously defined)

The results (in \%) are presented in the Table 1:

| $\begin{aligned} & \text { No. } \\ & \text { of } \end{aligned}$task | Number sequence | Selected number | III grade |  |  |  |  |  |  |  | VII grade |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | A | B | C | D | E |  | F | G | A | B | C | D | E | F | G |
| 1. | $\begin{aligned} & 9,15 \\ & 24,16 \end{aligned}$ | 9 |  | 65 |  | 3 |  |  | 3 |  |  | 12 |  |  | 12 |  |  |
|  |  | 15 |  | 3 |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  | 24 |  |  |  |  | 8,5 |  | 8,5 |  | 64 |  | 6 |  |  |  | 6 |
|  |  | 16 |  |  |  | 8,5 |  |  |  |  |  |  |  |  |  |  |  |
| 2. | $\begin{aligned} & 21,41, \\ & 42,14 \end{aligned}$ | 21 |  | 70 |  |  |  |  |  |  |  | 21 |  |  |  |  |  |
|  |  | 41 | 8 |  |  |  |  |  |  |  | 67 |  | 3 |  |  |  |  |
|  |  | 42 |  | 19 |  |  |  |  |  |  |  | 3 |  |  |  |  | 3 |
|  |  | 14 |  | 3 |  |  |  |  |  |  |  |  |  |  | 3 |  |  |
| 3. | $\begin{gathered} 12,16, \\ 18,20 \end{gathered}$ | 12 |  |  |  | 12 | 20 |  |  |  |  |  | 16 |  | 3 |  |  |
|  |  | 16 |  |  | 4 |  |  |  |  | 4 |  |  |  |  |  |  |  |
|  |  | 18 | 12 |  |  |  |  |  |  |  | 45 |  | 9 |  |  |  | 6 |
|  |  | 20 |  | 44 |  |  | 4 |  |  |  |  | 9 |  |  | 6 |  | 6 |
| 4. | $\begin{gathered} 25,16 \\ 34,18 \end{gathered}$ | 25 |  |  |  |  |  |  | 50 |  | 57 |  | 6 |  |  | 16 | 6 |
|  |  | 16 |  |  |  |  | 4 |  |  |  |  |  |  |  |  |  | 3 |
|  |  | 34 |  |  |  |  | 12 |  |  |  | 3 |  |  |  |  |  |  |
|  |  | 18 |  |  |  | 30 |  |  |  | 4 |  |  |  |  |  |  | 9 |
| 5. | $\begin{aligned} & 18,15 \\ & 25,30 \end{aligned}$ | 18 | 12 |  |  |  |  |  |  | 16 | 85 |  |  |  |  |  | 6 |
|  |  | 15 |  |  |  |  | 3 |  | 3 | 3 |  |  |  |  |  |  |  |
|  |  | 25 | 6 | 6 |  |  |  |  |  |  | 3 |  |  |  |  |  | 3 |
|  |  | 30 | 3 | 22 |  |  | 10 |  | 3 |  |  | 3 |  |  |  |  |  |
| 6. | $\begin{gathered} 33,15, \\ 12,6 \end{gathered}$ | 33 |  | 4 |  |  | 12 |  |  |  | 12 | 9 |  |  | 3 |  | 22 |
|  |  | 15 |  |  |  |  |  |  | 4 |  | 12 |  |  |  |  |  |  |
|  |  | 12 | 4 |  |  | 12 |  |  |  |  | 3 |  |  | 12 |  |  | 3 |
|  |  | 6 |  | 56 |  |  | 8 |  |  |  |  | 12 |  |  | 6 |  | 6 |

Table 1. Students' choices of criteria
As can be seen from the presented list, the younger research group more often found justification for selection of each number in the sequence than the older
one. The seventh-grade students in four cases did not find justification for choosing particular number. In addition, making choices significantly differed between these two research groups. It was related to the type of used criterion. As we can observe in the task No.1, the third grade students definitely chose the number 9 (using the criterion related to the construction of the number), while the most of seventh grade ones chose the number 16 using the criterion of divisibility.

We can observe similar situation in the task No. 2 - the majority of the third grade based on a visual criterion associated with a particular digit and chose the number 21 or 42 , while students from the seventh grade using the criterion of divisibility chose the number 41 . In the third task the older group most often chose 18 based on the criterion of divisibility. As much as one third of the choices was guided by the number sequence criterion ( 12 or 18 were selected). In the younger group, students chose 20 , referring primarily to the construction of the number. As far as the sequence of numbers is concerned, they were only applied to the number 12 - perhaps there was a suggestion here of the numbers in the table, and the students did not attempt to investigate other dependencies.
In the task No. 4 both groups most often chose the number 25, however, they followed a different criterion: for the third grade it was connected with number parity, and for the seventh one - to divide the number by 2. For the task No. 5 the most commonly used criterion by the oldest students was the reference to divisibility (choice of 18). Unfortunately, it is difficult to distinguish the criterion used by the younger students in this sentence. Although the most frequently chosen number was 30 , different justifications and different criteria were used. As we can see the results for the task No. 6, in the first group the number 6 was most often chosen, arguing that it is single-digit (criterion related to the construction of the number). In the second group, the number 33 were the most popular, however giving various justifications (and none of the justifications related to the construction of this number).
A criterion seldom used by the older research group was this one with reference to the sum or the difference of digits. Equally not very popular was the criterion related to the parity of numbers and the size of numbers. Definitely the seventh grade students referred most frequently to the divisibility or multiplicity of numbers. With regard to the younger research group, we note that the most frequently used criterion was the reference to the construction of the number. The criterion of dependence was the least frequently used (that is, the relationship between numbers was not examined).

## SUMMARY

The task the students received was available at both school levels. In both research groups, students had the appropriate knowledge and skills to solve this task. Expectations regarding the obtained results were such that students from
the higher level will apply much more diverse criteria. As the analysis of the collected research material showed, the younger research group presented a wider range of possibilities. In addition, it was the younger students who were more creative when solving the task. Besides, they tried to give more than one answer to each puzzle. They did not limit themselves only to indicating a number that does not match the others, but in each case they gave justification for their choice.

Meanwhile, the seventh grade students did not even attempt to give more than one answer to each task. This may be due to the fact that students are expected to give one specific answer at school, during math lessons. The tasks they meet in the classroom are unambiguous and require a precise solution. Therefore, they do not have enough opportunity to develop their creativity, to be creative. Typical tasks require the use of specific mathematical knowledge. Usually this is related to the subject currently being processed and students are not encouraged to look for alternative solutions that go beyond the currently processed material.

Students have great potential associated with creativity and creative thinking in the field of mathematics, as evidenced by the results obtained by the younger research group. It is the role of the mathematics teacher to discover and develop this potential. It should not be wasted. It is necessary to undertake appropriate activities conducive to the development of creative thinking, mathematical thinking. Above all, putting open problems in front of the students and encouraging them to look for different, non-standard solutions.

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# "IS THIS AN ACCEPTABLE MATHEMATICAL PROOF?" A SYSTEMIC INVESTIGATION OF HIGH SCHOOL STUDENTS' PROOF BELIEFS AND EVALUATIONS 

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In this study, we introduce a systemic approach to investigate the high-school students' proof beliefs and evaluations, as they emerge through their experiencing the perceived 'official' proof reality (the teacher and the textbook). A mixed methodology was adopted to identify the convergences and divergences in a school class amongst the students' proof belief systems and evaluating criteria, the teaching practices and beliefs about proof and proving, the appearances of proof and proving in the school textbook and answer book, and the audience (self, peers, teacher) of the proof. Complex, otherwise conflated or over-simplified, relationships were revealed, supporting the chosen approach.

## PROOFS AND PROVINGS IN SCHOOL: BELIEFS AND PRACTICES

Mathematics education researchers have investigated the students' experiencing the various proof functions (Balacheff, 1988; Hanna, Jahnke \& Pulte, 2010; Hanna \& de Villiers, 2012), including "verification (concerned with the truth of a statement) [...] explanation (providing insight into why it is true) [...] systematisation (the organization of various results into a deductive system of axioms, major concepts and theorems) [...] intellectual challenge (the selfrealization/fulfilment derived from constructing a proof)" (De Villiers, 1999, p. 5). The students' difficulties have been linked with their having limited opportunities to be engaged with proof and proving in their everyday school experience (Thompson, Senk, \& Johnson, 2012). In this study, we investigate the students' proof beliefs and evaluations, theorising that their daily experience of the official proof reality emerges at the interactions of diverse sources of authority, immediately experienced and are recognised as such: the official texts (the textbook and the answer book) and the teaching and assessment practices.

## A SYSTEMIC PRAGMATIC APPROACH TO INVESTIGATING PROOF BELIEFS AND EVALUATIONS IN HIGH SCHOOL

Bieda (2010) notes that the teachers and the textbooks serve as signposts in the students' efforts to give meaning to what a valid proof is; what is rendered acceptable to the given educational setting and its perceived constitutional principles. Hence, the students construct what Harel and Sowder (1998) term as a proof scheme ("what constitutes ascertaining and persuading for that person", p. 244), with ascertaining referring to "the process an individual employs to
remove his or her own doubt about the truth of an observation", whilst persuading referring to "the process an individual employs to remove others' doubts about the truth of an observation" (p. 241). Following these and drawing upon the pedagogical differentiation "convince yourself, convince a friend, convince an enemy" (Mason, Burton \& Stacey, 1982, p. 95) and Segal's (1999) ideas of private and public aspects of proof, as well as Harel and Sowder's (1998) notion of proof scheme, Moutsios-Rentzos (2009) suggested that there are qualitatively differences in the students' conceptions of 'prove to yourself', 'prove to a friend' and 'prove to an enemy', being evident in their proving strategies (including constructing, presenting or evaluating a proof). Furthermore, he differentiates proof constructing from proof evaluating as qualitatively distinct processes, since the criteria developed for evaluating a proof, which may or may not be in line with their actual proving.

Various classifications of the arguments that may be employed as proving arguments have been suggested, including Harel and Sowder's (1998) finegrained categorisation and Balacheff's (1988) hierarchy: naive empiricism, the crucial experiment, the generic example, and the thought experiment. However, the links between non-deductive and deductive arguments are not clear for the students, as, for example, the use of specific examples may help in accepting an argument as mathematically valid (Bieda \& Lepak, 2014), while a deductive argument for the general may not be considered to be valid in the specific (Duffin \& Simpson, 1993). Moreover, authority is crucial to the formation of the proof evaluation criteria, including undergraduates and even research-active mathematicians who seem to be affected by the authority status of the person who utters the argument (Inglis \& Mejia-Ramos, 2009). Furthermore, the students' evaluations are affected (even determined) by whether or not the presented proof follows the established communicational norms; for example, mathematical symbolism (Harel \& Sowder, 1998; Pfeiffer 2009).
The accumulated effect of the students' experiences with proof and proving is evident in their beliefs, that are their "multiply-encoded cognitive/affective configurations, usually including (but not limited to) prepositional encoding, to which the holder attributes some kind of truth value" (Goldin, 2002, p. 64). Conversely, their beliefs are linked with their proving strategies and their proving evaluating criteria (Bieda, 2010; Moutsios-Rentzos, 2009).

In this study, we adopt a systemic perspective to investigate the convergences and divergences between the students' beliefs about proof and proving (the notion of proof and their proof evaluation criteria) and the official experienced school class reality (including the teaching practices, the textbooks and answer books). In specific, in our investigation about proof beliefs, we agree with the view that beliefs form relatively isolated, internally structured clusters to constitute a construct sometimes referred to as a belief system (Green, 1971). Belief systems have been viewed as being complex, bearing properties and
characteristics that transcend their constituting beliefs-elements (Beswick, 2012). These echo Bertalanffy's (1968) ideas of the system as being a complex whole with clear purpose and boundary, consisting of elements and subsystems, having a structure, with its constituting parts being interlinked in ways that allow for properties to emerge that cannot be immediately attributed to the system parts. Consequently, in this study, we focus on belief systems within the school class conceptualised as a subsystem of the school unit system (cf. Moutsios-Rentzos \& Kalavasis, 2015; Moutsios-Rentzos \& Leontiou, 2016) to identify the students' proof belief systems and evaluating criteria, as they are formed at the interactions of: a) the teaching practices about proof and proving and the underlying teacher's related beliefs, b) the appearances of proof and proving in the school textbook and answer book, and c) the perceived as acceptable proof and proving evaluating criteria of the audience (the students themselves, their peers or their teacher).

## METHODS AND PROCEDURES

The study was conducted with 16-year old high-school students ( $\mathrm{N}=63$ ) and Amelia (their Algebra teacher), after they finished the first grade in high-school in Greece. Amelia is a mathematician with 25 years of teaching experience (15 years teaching Algebra in this grade).

## If $\mathbf{a}, \mathrm{b}$ are signed numberswith different sign then $|\mathbf{a}+\mathrm{b}|<|\mathrm{a}|+|\mathrm{b}|$.

Naive empiricism
For $\mathrm{a}=2$ and $\mathrm{b}=-5$ then $\mathrm{a}+\mathrm{b}=-3,|\mathrm{a}|=2,|\mathrm{~b}|=5$ and $|\mathrm{a}+\mathrm{b}|=3$ consequently $|\mathrm{a}+\mathrm{b}|<|a|+|\mathrm{b}|$.
For $a=-6$ and $b=9, a+b=3,|a|=6,|b|=9$ and $|a+b|=3$ consequently $|a+b|<|a|+|b|$.
For $\mathrm{a}=15$ and $\mathrm{b}=-1, \mathrm{a}+\mathrm{b}=14,|\mathrm{a}|=15,|\mathrm{~b}|=1$ and $|\mathrm{a}+\mathrm{b}|=14$ consequently $|\mathrm{a}+\mathrm{b}|<|\mathrm{a}|+|\mathrm{b}|$.
For $\mathrm{a}=-10$ and $\mathrm{b}=10, \mathrm{a}+\mathrm{b}=0,|\mathrm{a}|=10,|\mathrm{~b}|=10$ and $|\mathrm{a}+\mathrm{b}|=0$ consequently $|\mathrm{a}+\mathrm{b}|<|\mathrm{a}|+|\mathrm{b}|$.
For $\mathrm{a}=32$ and $\mathrm{b}=-30, \mathrm{a}+\mathrm{b}=2,|\mathrm{a}|=32,|\mathrm{~b}|=30$ and $|\mathrm{a}+\mathrm{b}|=2$ consequently $|\mathrm{a}+\mathrm{b}|<|\mathrm{a}|+|\mathrm{b}|$.
It follows that this statement is true for these five pairs of signed numbers with different sign. Consequently the statement is proved.

## Crucial experiment

Let a random pair of signed numbers with different sign $\mathrm{a}=3, \mathrm{~b}=-5$. Then $\mathrm{a}+\mathrm{b}=-2$ and thus $|\mathrm{a}+\mathrm{b}|<|\mathrm{a}|+|\mathrm{b}|$. Consequently, the statement is true and since the pair was taken at random, the statement is proved.
Generic example
Let a random pair of signed numbers with different sign $\mathrm{a}=-8, \mathrm{~b}=7$. Then $|-8+7|^{2}=(-8+7)^{2}=(-8)^{2}+2(-8)(7)+(7)^{2}=1$.
Also $(|-8|+|7|)^{2}=|-8|^{2}+2|-8| 7|7|+|7|^{2}=225$. Consequently, $|-8+7|^{2}<(|-8|+|7|)^{2}$ and since the bases of the squares are positive it will be $|-8+7|<|-8|+|7|$. And since the chosen pair was taken at random, the statement is proved.

## Thought experiment (direct proof)

Let $|a+b|<|a|+|b|$. Since both the two members of the inequality are positive numbers: $|a+b|^{2}<(|a|+|b|)^{2}$, so $(a+b)^{2}<$ $|a|^{2}+2|a||b|+|b|^{2}$, so $a^{2}+2 a b+b^{2}<a^{2}+2|a b|+b^{2}$. Finally $a b<|a b|$ which is true since $a b<0$ and $|a b|>0$. Consequently the statement is proved.

## Thought experiment (reductio ad absurdum)

Let the statement is not true, which means that if $a, b$ are signed numbers with different sign then $|a+b| \geq|a|+|b|$. Since both the two members of the inequality are positive numbers: $|a+b|^{2} \geq(|a|+|b|)^{2}$, consequently $(a+b)^{2} \geq|a|^{2}+2|a||b|+|b|^{2}$, consequently $a^{2}+2 a b+b^{2}-a^{2}+2|a b|+b^{2}$. Finally $a b \geq|a b|$ which is not true (contradiction) since $a b<0$ and $|a b|>0$. Having started from something that I assumed that it was true and ending up with something that is not true, then my assumption wasn't true. So the 'opposite' of what I started should be true. Consequently the statement is proved.

Figure 1: 'Statement 3' (theorem proved in the textbook) and the five arguments.
The research instruments included:
a) Textbook analysis. Drawing upon Moutsios-Rentzos and Pitsili-Chatzi (2014), the school textbook and answer book were analysed to identify
functions of proof (only systemisation, explanation and verification), proving practices, employed language (symbolic or mixed) and proving methods.
b) Structured interview, to investigate Amelia's teaching practices and beliefs about proof and proving.
c) Questionnaire, to investigate the students' beliefs about proof (29 five-point Likert type items, drawing upon Almeida, 2000; Hemmi, Lepik \& Viholainen, 2010; Kögce \& Yıldız, 2011) and their evaluation criteria about proof and proving: 3 statements (a theorem not proved in the textbook, a textbook exercise, a theorem proved in the textbook) with 5 arguments for each statement in line with Balacheff's (1988) classification including two thought experiments (direct proof and reductio ad absurdum; see Figure 1). The students' evaluations concerned three audience types (self, peers, teacher; Moutsios-Rentzos, 2009): on a 7-point scale for self and peers and as a grade (out of 20) for the teacher. The questionnaire was administered to a broader sample to identify its structure (see Table 1): Validity (offered by a proof, based on verification and systemisation). Proof for some (not for all students). Intellectual challenge-Understanding (affective, personal gains and understanding). Certainty (assigned to the idea by a proof). Explanation (gives meaning to the idea).

## STUDENTS' BELIEFS AND EVALUATIONS

The students' proof beliefs are outlined with respect to their agreement for each of the items constituting a factor in Table 1. The students agree or are neutral with respect to the items of Validity, Certainty and Explanation, neutral to Intellectual challenge-Understanding and negative-neutral to Proof for some. Thus, the students seem to believe that proof is a tool validating with certainty a given statement and that this tool is accessible to all students and not only for some. Furthermore, they do not seem to have a positive or negative opinion with respect to the less functional aspects of proof, such as the affective or personal gains and understanding that may be linked with a proof (Almeida, 2000).

|  | F1 ${ }^{1,2}$ | F2 | F3 | F4 | F5 | $M d{ }^{3,4}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Proofs in mathematics both verify and explain. | 0.265 |  |  |  |  | 4.0 |
| If a result in mathematics is proved, I can be certain that it is true. |  |  |  | 0.458 |  | 3.0 |
| Examples illustrating a result do not always help me understand why the result is true. |  |  |  |  | -0.259 | 3.0 |
| Proofs are essential in pure mathematics. | 0.347 |  |  |  |  | 4.0 |
| I can't see the point of doing proofs: all the results in the course have |  | 0.626 |  |  |  | 2.0 |

already been proved beyond doubt by famous mathematicians.
Proofs are necessary in mathematics. $[-0.278] 0.512 \quad 1.0$
I like doing proofs in mathematics.
Working through a proof of a result helps me to understand why it is true.
Reading through a proof of a result in $\quad 0.608 \quad 3.0$ a textbook helps me to understand why it is true.
Different proofs of a theorem help me 0.316 3.0 to understand it better.
$\begin{array}{lll}\text { Proofs in mathematics depend on } & 0.378 & 4.0\end{array}$
other mathematical results.
$\begin{array}{llll}\text { Proofs show how everything is } & 0.422 & {[0.340]} & 3.0\end{array}$
connected in mathematics.
$\begin{array}{ll}\text { Proofs help seeing that "evident } 0.378 & 3.0\end{array}$
statements" are not necessarily true before they are proved.
Proofs show the beauty of 0.6563.0 mathematics.
Not all students can cope with proof; $\quad 0.457 \quad 2.0$
only those who are good at mathematics
$\begin{array}{lll}\text { Proofs are logical structures in } & 0.447 & 4.0\end{array}$
reasoning where the various steps are motivated with known theorems and definitions.
$\begin{array}{lll}\text { I can't believe any math statements } & 0.319 & 3.0\end{array}$
without proof.
$\begin{array}{lll}\text { Proofs are sequences of logical } & 0.520 & 4.0\end{array}$
statements that imply each other, a logical derivation of results.
Proofs strengthen logical reasoning. $0.560 \quad 4.0$
$\begin{array}{lll}\text { Proving sometimes reveals the } & 0.610 & 4.0\end{array}$ inaccuracy of a theorem which seems correct.
$\begin{array}{lll}\text { Proving is made for our convincing } & 0.477 & 3.0\end{array}$
someone for our claims.
$\begin{array}{lll}\text { Proving can lead us to new } & 0.305 & 3.0\end{array}$
discoveries in mathematics.
I think there is no need for proving, $0.654 \quad 2.0$ because it is confusing.
$\begin{array}{lll}\text { Proofs show where mathematical } & {[0.310] 0.654} & 4.0\end{array}$ relationships come from.
$\begin{array}{lll}\text { Proofs answer the question why } & 0.573 & 4.0\end{array}$
$\left.\begin{array}{llll}\begin{array}{l}\text { theorems and statements hold } \\ \text { undoubtedly true. }\end{array} & & \\ \begin{array}{l}\text { Proofs explain mathematical } \\ \text { expressions and relationships } \\ \text { through already known facts. }\end{array} & 0.289\end{array}\right)$

[^4]Table 1: The questionnaire of students' beliefs.

|  | Not proved theorem | Exercise | Proved theorem |
| :---: | :---: | :---: | :---: |
| Naive empiricism |  |  |  |
| Self ${ }^{1}$ | 4.0 | 4.0 | 4.0 |
| Peers ${ }^{1}$ | 4.0 | 4.0 | 4.0 |
| Teacher ${ }^{2}$ | 14.2 | 14.4 | 14.5 |
| Crucial experiment |  |  |  |
| Self | 3.0 | 4.0 | 3.0 |
| Peers | 4.0 | 4.0 | 4.0 |
| Teacher | 12.6 | 13.5 | 13.7 |
| Generic example |  |  |  |
| Self | 4.0 | 3.0 | 4.0 |
| Peers | 4.0 | 3.0 | 4.0 |
| Teacher | 13.0 | 12.2 | 13.9 |
| Thought experiment |  |  |  |
| Direct Proof |  |  |  |
| Self | 4.5 | 5.0 | 5.0 |
| Peers | 5.0 | 5.0 | 5.0 |
| Teacher | 15.1 | 15.5 | 16.2 |
| Reductio ad absurdum |  |  |  |
| Self | 5.0 | 5.0 | 5.0 |
| Peers | 4.0 | 4.0 | 5.0 |
| Teacher | 15.0 | 15.3 | 15.8 |

Table 2: Students' evaluating criteria about proof and proving.

The students' evaluating criteria are outlined in Table 2. In general, the students appeared to be reluctant to provide a high score to any of the provided 'proofs', regardless the audience. This may be linked with their mathematics grading experiences. Nevertheless, there seems to be a correspondence amongst the expected grade (teacher), the level of ascertaining (self) and persuading (peers): they correspond roughly to the same percentage. Moreover, it is clear that the thought experiment (being a direct proof or reductio ad absurdum) are the more acceptable arguments, followed by the naïve empiricism, with the crucial experiment and the generic example receiving similar evaluations. The direct proofs are the only ones on the positive-neutral side of the evaluation scale (over ' 4 ' on the 7 -point scale) or grade (usually 15 out of 20 is considered as a grade to denote the average attaining student in high school mathematics in Greece). Interestingly, almost all the arguments were acceptable: neutral (' 4 ' on the 7 point scale) or below the 'average' (15/20), but not a failing grade ( $<10 / 20$ ). Moreover, the students seem not to differentiate the crucial experiment from the generic example, which combined with the higher evaluations of the naive empiricism argument may imply that the students focussed on the number of cases given, rather on the line of thinking being communicated.

## MAPPING THE OFFICIAL PROOF AND PROVING CLASS REALITY

## Proof and proving: the textbook and the answer book

As outlined in Table 3, the students predominantly encounter in both textbook and school book direct proofs communicated in mixed language, whilst half of the identified functions of proof are linked with systemisation, about one third with explanation and the remaining one sixth with verification.

| Appearances of proof <br> and proving | Textbook <br> Theory | Applications Total | Answer <br> book | Total |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Functions |  |  |  |  |  |
| Systemisation | $26(47.3 \%)$ | $10(62.5 \%)$ | $36(50.7 \%)$ | $44(48.4 \%)$ | $80(49.4 \%)$ |
| Explanation | $22(40.0 \%)$ | $3(18.8 \%)$ | $25(35.2 \%)$ | $32(35.2 \%)$ | $57(35.2 \%)$ |
| Verification | $7(12.7 \%)$ | $3(18.8 \%)$ | $10(14.1 \%)$ | $15(16.5 \%)$ | $25(15.4 \%)$ |
| Practices |  |  |  |  |  |
| Application after proof | $15(45.5 \%)$ | $1(7.7 \%)$ | $16(34.8 \%)$ | $4(3.7 \%)$ | $20(13.1 \%)$ |
| Example before proof | $2(6.1 \%)$ | $0(0.0 \%)$ | $2(4.3 \%)$ | $24(22.4 \%)$ | $26(17.0 \%)$ |
| Identification as 'proof' | $15(45.5 \%)$ | $10(76.9 \%)$ | $25(54.3 \%)$ | $78(72.9 \%)$ | $103(67.3 \%)$ |
| Figure | $1(3.0 \%)$ | $2(15.4 \%)$ | $3(6.5 \%)$ | $1(0.9 \%)$ | $4(2.6 \%)$ |
| Language |  |  |  |  |  |
| Only symbolic | $3(10.0 \%)$ | $9(90.0 \%)$ | $12(30.0 \%)$ | $26(33.3 \%)$ | $38(32.2 \%)$ |
| Mixed | $27(90.0 \%)$ | $1(10 \%)$ | $28(70.0 \%)$ | $52(66.7 \%)$ | $80(67.8 \%)$ |
| Methods |  |  |  |  |  |
| Direct | $22(73.3 \%)$ | $6(60.0 \%)$ | $28(70.0 \%)$ | $58(74.4 \%)$ | $86(72.9 \%)$ |
| Reductio ad absurdum | $1(3.3 \%)$ | $1(10.0 \%)$ | $2(5.0 \%)$ | $2(2.6 \%)$ | $4(3.4 \%)$ |
| Analysis | $4(13.3 \%)$ | $3(30.0 \%)$ | $7(17.5 \%)$ | $17(21.8 \%)$ | $24(20.3 \%)$ |
| Counterexample | $3(10.0 \%)$ | $0(0.0 \%)$ | $3(7.5 \%)$ | $0(0.0 \%)$ | $3(2.5 \%)$ |

Find the fault $\quad 0(0.0 \%) \quad 0(0.0 \%) \quad 0(0.0 \%) \quad 1(1.3 \%) \quad 1(0.8 \%)$

Table 3: Appearances of proof and proving in the textbook and the answer book.
Though the majority of proofs are explicitly identified as such, when comparing the two books, only half of them are in this category in the textbook, contrasting the almost three quarters of those in the answer book. Furthermore, one third of the textbook proofs are accompanied with a subsequent application, an almost non-existent practice in the answer book. In contrast, one quarter of the answer book proofs are introduced with an example, which is not evident in the textbooks. Consequently, the students regardless if they read only the textbook or they refer to the answer book as well, they seem to be experiencing a relatively coherent representation of what an acceptable proof usually is: a direct proof, written in mixed language, referring to other proven or accepted statements. Interestingly, the links of example and application with a proof are qualitatively different, thus affecting the experienced official reality between the students who read only the textbook and the students who read both, with the former constructing a deductive mathematical world (with proofs of the general, followed by its application the specific), whilst the latter also experience an inductive approach (with the specific introducing the general).

## Amelia's proof and proving beliefs and practices

Considering beliefs about proof and proving, Amelia stressed the role of proof in mathematics in appropriately grounding the mathematical ideas, in gaining deeper understanding about the rationale and line of thinking of the proving arguments, as well as in providing the bedrock for other mathematicians to conduct further investigations: "First you ground it, second you can see the way the one who did it the way he [sic] thought it and third you are given the opportunity to search for something else yourself' (lines 27-28). 'Appropriately grounded', "the right way", seems to be especially important for her: "I believe that proof just offers satisfaction and pleasure to the one who manages it and proved and proved it the right way, based on the right line [of thinking]" (lines 51-53) [...] "for me the proof is something that grounds something that intuitively or empirically was conceived by someone and then it [the proof] grounds it" (lines 129-131). Amelia stresses that such a process strengthens reasoning and intuition: "[...] it is more about admiration for the person who set the proof and secondly it gives you more ideas. This broadens more thinking, intuition" (lines 35-37). The affective aspects of proof are evident in these quotes (admiration for the prover and the pleasure in proving), mentioned by Amelia several times, suggesting the importance she assigns to these qualities.
Considering proving practices, Amelia employs an inductive, mediating approach, as she introduces a context (an example or a specific problematic situation) that may give meaning to the theorem and its proof for the students: "I consider this wrong. To tell them the theorem and now we'll prove it. I teach
them more indirectly and try to let them be involved in its [proof] construction and more on seeing the end result of this process" (lines 204-206). Such an approach is linked with her belief that the examples help the students in gaining deeper understanding about mathematics through explicit linking mathematics with their applications: "so that the value of proof will be known to the students and to link it [mathematics] with something more practical and to interest them [the students] a little bit more (lines 489-491). Her broader belief that proof and proving strengthens the students' reasoning is evident in her trying to present more than one ways of proving a statement, as she believes that this practice helps in broadening the students' thinking, imagination and creativity: "ehh it is beneficial not directly for the specific exercise, it [multiple proofs] gives them [...] experience that for the child I think would be multiplied, to be applied to something completely different later on, to broaden [...] his [sic] thinking and imagination" (lines 395-398).

Amelia crucially differentiates proof reading from proof constructing as teaching practices with respect to the opportunities for learning for students of different mathematics attainment. Proof constructing is valuable for all students (regardless their mathematical attainment) as it has multiple advantages, including pleasure and links to everyday applications of mathematics: "even for the average student to try this thing [proof constructing] it really gives him pleasure" (lines 453-456) [...] "if they construct it would be better, it is more interesting in the times that we live" (lines 509-511). Nevertheless, Amelia considers proof reading to be more appropriate for high-attaining students as it helps in broadening their thinking. This belief implicitly aligns qualities of the high-attaining students with those of the mathematicians, as similar benefits were mentioned by Amelia in her view of proof for mathematicians. Furthermore, it reveals that the gains of proof constructing are linked with the fact that the students are actively involved in doing mathematics, which has the broader benefits of being engaged with an activity, which for Amelia is also in line with her view of everyday living. Thus, Amelia considers proof reading appropriate only for students who are already successful in mathematics and not for the low-attaining or even 'average' students. Importantly, Amelia does not mention whether such proof constructions may help 'average' students to develop the characteristics that would make proof reading suitable for them.

## APPROACHING THE COMPLEXITY: CONCLUDING REMARKS

Considering the experienced by the students' official reality, it was revealed that with respect to the school books, proof is usually identified as such predominantly and is communicated as being a direct proof, written in mixed language (Pfeiffer, 2009), explicitly linked with already proven or accepted arguments. Importantly, the role of the specific as means of introducing and identifying the general is only evident in the answer book, whilst in the textbook the specific is only employed as a special case of the general. Regarding

Amelia's teaching practices, it was revealed that they were in line with the answer book, relying on the specific to introduce and give meaning to the general. Furthermore, Amelia emphasises systemisation, as she is especially interested in the appropriate foundation and construction of the proof, with the latter being at the heart of her teaching practices.
Which of these appearances of the official or their interactions may be linked with the students' beliefs and evaluations? The results of the conducted analyses revealed that the students of the specific class have formed a proof belief system, identifying proof predominantly as functional means for securing with certainty the validity of a statement and that is a mean that may be employed by all students regardless of their attainment. These may be partially linked with the systemisation function of proof strongly communicated by both Amelia and the school books. On the other hand, the pro-'proof for all' belief may be only linked with Amelia's efforts to engage all the students with proof constructions, in the text book proving exercises (to be solved by the students) are included only in the set of exercises characterised as difficult (Moutsios-Rentzos \& Pitsili-Chatzi, 2014). Furthermore, the students seem to be neutral with the affective qualities that Amelia identified, which may be linked with her not engaging the students with proof reading (that she considers may help a few, high-attaining students to experience aspects of proof and proving linked with mathematicians), as well as with the lack of communication of such aspects of proof in the school books.

Considering the students' evaluations, it seems that the students' experiences with both the school books and Amelia's practices are evident in their considering direct proofs as the most acceptable arguments. Nevertheless, the fact that they also consider reductio ad absurdum as equally acceptable, a method that is scarcely evident in either school books or Amelia's practices, may imply that the students may have over-evaluated the importance of mixed language in a proof as communicated by the school books. Or, that a proof is different from the specific, since the example or the application are usually differentiated from the proof in the school books and Amelia. This may be further investigated in future studies by asking the students of a school class system to also evaluate a proof by counter example. However, the fact that none of the arguments was evaluated as unacceptable by the students, suggests that the students conceptualise an acceptable mathematical argument differently from what Amelia considers as a proof or what the textbook identifies as proof (in an introductory section entitled as such). This might be linked with the fact that many proofs included in the textbook (and fewer in the answer book) are not explicitly identified as such, as well as with the Amelia's preferred way of teaching that effectively legitimises the use of examples in the proving process (though not in the proof itself), maybe without clearly differentiating it from the proof, thus maybe assigning a proof degree to the specific, rather than absolutely
not being a proof. Hence, it is posited that the students seem to form a hierarchy of acceptable proving arguments (one specific case, several specific cases, the general; cf. Balacheff, 1988), which may be linked with aspects of the daily experienced official reality, rather than on Amelia's underlying beliefs or the introductory section of the textbook.
Consequently, it is argued that the employed approach helped in meaningfully approach the complexity of the real world classroom, regarding the links amongst the official reality daily experienced by the students and their proof and proving beliefs and evaluating criteria. It was revealed that the role of the teacher and the school books are crucially linked with the students' beliefs and evaluation, in less than expected or obvious ways, with the same experiences diversely and seemingly incompatibly affecting the students' constructions, thus effectively approaching the complexity of the lived daily in-class reality. Finally, a current research project draws upon this approach to include in-class observations and longitudinal data, in order to gain deeper understanding of the aforementioned complex phenomena.

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# RECOGNITION OF BASIC SHAPES BY $4{ }^{\text {TH }}$ GRADERS 

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#### Abstract

Relevant research data and research analyses related directly to Slovak children of younger school age concerning how children think about planar geometric shapes and how they describe them are presently missing. We were interested whether the characteristics of van Hiele levels of geometric thinking are generally applicable, for example when also taking into account socio-cultural and linguistic aspects. The article briefly informs about the results of an extensive research conducted in the fourth grade of primary school in a form of a non-standardised test of knowledge. The aim was to identify pupils' conceptions and misconceptions of basic planar shapes - disk, triangle, square and rectangle.


## INTRODUCTION

It is incredibly difficult to find out how children think because from their answer to a question we usually get to know only the result of the child's thinking. We do not know how the process of thinking itself was going on. GruszczykKolczyńska (2009) states that it is important to identify the nature of children's intellectual activities because they affect the impact of mathematical education. Children desire to understand the world around them and this desire, or inner motivation, must be utilized at the right time and supported adequately to the child's age.

Children's cognition is spontaneous, predominantly experiential and it is strongly affected by emotions. This is also a reason why children's conceptions are extraordinarily firm and resistant to correction attempts. Children's conceptions, so called naive theories of children, are initially imperfect because a child has less experience, it looks onto the world from its point of view, it cannot take into account more aspects at once, it creates illogical explanations and it involves its fantasy. They differ qualitatively from the adults' understanding of the world and often are not in line with scientifically presented interpretations (Gavora, 2007). According to Jirotková (2010), the suggestive power of such knowledge often even subdues the need to formalise the subjected knowledge.

## THEORETICAL BASES

From the above-stated it is apparent that the problematic of research of children's conceptions, or the identification of children's misconceptions, is a complex process that is of an interdisciplinary character. We base our research
on the work of Jean Piaget (1896-1980), Lev Semjonovič Vygotskij (18961934), Jerome Seymour Bruner (1915-2016) and Piere van Hiele (1909 2010). Among the theories of Piaget, Vygotskij and Bruner we may find certain parallels but also some differences. Instead of (separate) stages, which are exactly sequentially differentiated by age and which may be found in Piaget's work, the representations of Bruner are integrated and intertwined, the division into sections is very loose and individual types of representations may exist concurrently. Progression through different types of representations increases the flexibility of thinking and the ability to solve problems. Individual types of representations are not bound by age. Based on these representations new mental representations to understand mathematic terms, principles and rules are created. Advancement to a higher level of cognition may be initiated by suitable educational activities and in suitable environment. Bruner, similarly to Vygotskij, accentuated the importance of cultural and social environment. This procedure is similar to the conception of Vygotskij's zone of proximal development. Bruner's theory states that the creation of children's conceptions is conditioned by education and influence of adults, whereas it is necessary to use the correct type of representation.
Milan Hejný (born 1936), contemporary Slovak and Czech mathematician has been dealing with didactics of mathematics his whole life. Drawing on the ideas of his father Vít Hejný (1904-1977) he prepared and systematically examined the mechanism of obtaining mathematic knowledge. The supporting structure of this process according to Hejný (1990) is formed by the sequence motivation $\rightarrow$ experience $\rightarrow$ knowledge. Nowadays, after several amendments, the taxonomy of cognitive process according to Hejný (2014) contains 5 stages (levels) motivation, isolated models, generic model, abstract knowledge and crystallisation and 2 cognitive shifts - generalisation $(1 \rightarrow)$, and abstraction $(2 \rightarrow)$ (Figure 1). He excluded the stage of automatization from the cognitive process; this has primarily psychological grounds because it does not concern new cognition, but merely a practice of the known. Some pedagogic theories tend to think that the knowledge is really learned only when it is automatized to such level that one can automatically apply it in the given situation.


Figure 1: Model of cognitive process according to Hejný (2014)
Hejnýs theory of the generic model not only describes the substance of the cognition process, but it also enables the diagnostic of incorrect mathematical conceptions of children/pupils, the change of mechanic knowledge into knowledge with understanding and it provides impulses for the creation of reeducation procedures.

The theoretical base of the research of cognitive process in geometry is (besides others) the theory of geometric thinking (1957), of which the authors are Dutch teachers of mathematics Dina van Hiele-Geldof and Pierre van Hiele (for more information see van Hiele, 1986). There are many supporters of the van Hiele model of geometric thought worldwide, but also some disapproving opinions occurred. Van Hiele model of geometric thought with detailed characteristics of individual levels in cooperation with other models of cognitive process provides in general a theoretical framework for the research on geometric ideas of pupils of younger school age.
We consider the revision of the mechanism of cognitive process by Hejný to be a successful outcome of considerations that is also in line with our pedagogic persuasion. Whereas the original division of stages reflected Piaget's developmental stages from the sequential point of view and taking biological age into consideration, the revised version corresponds with the considerations and conception of Bruner, who accentuated the need of spiral arrangement of curriculum and thus implied that the cognition stages may not be strictly separated. Hejný (in Hejný, Novotná \& Stehlíková, 2004) states that "the sequence of individual levels corresponds up to certain extent with the timeframe of cognition process". At the same time, he states that new experience or knowledge affects several levels at the same time and therefore he does not understand them as disjunctive. As far as time duration of individual stages is concerned, Hejný (2014) states that in case of some subject matters of cognition the individual levels may last a short time (maybe several seconds/ minutes/weeks), in other cases they last longer (several months, even years). Duration of cognition within individual levels varies. There are differences among individuals, but these are also related to differences in the quality of each individual's understanding of the subject. These statements and conclusions are important for our research, on one hand for the comparison with the theoretical bases of the van Hiele model of geometrical thought, but also for the results of our research.

## RESEARCH

The aim of our research on geometric conceptions and misconceptions of primary school pupils of the fourth grade was to find out what are their conceptions about planar geometric shapes and their elementary properties. We assumed that the pupils of the fourth grade will think of the geometric shapes on the level of analysis according to the van Hiele theory and that they will be able to recognise significant elements of shapes and describe basic properties of triangles, squares, rectangles and disks. We wanted to find out the differences in the difficulties of identification of models and non-models of planar geometric shapes. We also examined which significant elements and properties, and to what extent, are the pupils of the fourth grade aware of when thinking about geometric shapes.

## Research tool

In order to examine conceptions of primary school pupils of the $4^{\text {th }}$ grade of elementary school planar shapes (triangle, square, rectangle and disk) and their properties, we used a non-standardised test of knowledge. The content of the test was designed so that it would correspond with the current content and performance standards of State educational programme for the $1^{\text {st }}$ stage of primary school (2015) in the educational field Mathematics and work with information. At the same time, we reflected the theoretical bases, in particular the van Hiele model of cognitive process, therefore tasks in the test corresponded predominantly with the first two van Hiele levels - visualisation and analysis, with some tasks borderlining also with the abstraction level as they contain elements checking the ability of abstraction.
Individual tests were aimed to check the following abilities of primary school pupils of the fourth grade:

- to name a planar geometrical shape by the image template (task 1);
- to identify the model/non-model of a planar geometric shape by its name on the basis of image pattern (tasks 2-5);
- to know significant elements and properties of a planar geometrical shape (tasks 6-9);
- to create a model of a planar geometric shape in square grid (task 10).


## Research group

The research group consisted of 345 pupils of the $4^{\text {th }}$ grade from 26 primary schools, mainly from regions located in the North of and North-east of Slovakia. The research subjects were selected based on their availability; the research group consisted of 181 ( $52.5 \%$ ) boys and 164 ( $47.5 \%$ ) girls. The test was administered in the period from April to June 2015 in the traditional pencilpaper form. Fourth-graders either circled the answers or they filled in short answers.

## Course of Research

The main focus of the test was finding out the conceptions and misconceptions of fourth-graders about planar geometric shapes. We distinguished pupils’ ability to name a shape, identify it (distinguish it among other geometrical shapes), to know its elementary properties and to create its model. Despite the fact that the test is not standardised, after the statistical analysis it was confirmed that it shows signs of a quality test with diagnostic potential.
The aim of the analysis of the acquired data, in relation to the research task being solved, was to find out whether it is possible to create a graded sequence of models and non-models of shapes according to the difficulty of their identification by pupils for each of the planar shapes (square, triangle, rectangle
and disk). We regard this as important input data for the creation of quality teaching materials, or tools of intervention for the removal of misconceptions.

The next aim of the analysis was to put the models of planar shapes (square, rectangle, triangle and disk) in order according to the difficulty of their correct identification. This way we will obtain the information on which of the mentioned shapes are identifiable more easily for fourth-graders in terms of their geometric thinking, or identify shapes that may prove to be a potential obstacle in the children's further geometric abstraction advancement.
The conception of the test of knowledge was structured into two parts. The first part of the test (tasks 1-5) was focused on examination of fourth-graders' ability to correctly identify planar shapes (triangle, square, rectangle and disk) according to the graphic template. Graphic templates contained models and nonmodels of the shapes in question. The aim of this part of the test was the diagnostic of geometric thinking of fourth-graders at the visualisation and analysis levels.


Figure 2: Task 1 - To each shape write its name.

| clusters | shape |
| :---: | :---: |
| 1 | D |
| 2 | F, G, B , C, E |
| 3 | A |

Table 1: Clusters of shapes from the 1st task.

By means of the task 1, we observed the ability of pupils to name the shape according to the picture (Figure 2). From among all the items, pupils achieved the highest success rate in case of the item $\mathrm{D}(97.1 \%)$ and the lowest success rate in case of the item $\mathrm{A}(54.8 \%)$. Based on the values of the difficulty parameter, we divided the items into three clusters by means of cluster analysis. The first cluster contains the shapes least difficult for the pupils to name while the objects in the third cluster proved to be the most difficult (Table 1).

In tasks 2 to 5 we observed the ability of pupils to distinguish models and nonmodels of the given shape, the tasks were assigned using pictures (e.g. Figure 3) with models and non-models of individual shapes depicted. In task 2 we verified the fourth-graders' visual conception of squares, in task 3 of triangles, in task 4 we examined the ability to distinguish and select shapes that are not rectangles and in the last task of this part of the test we verified the fourth-graders' visual conceptions of disks. In two tasks the pupils should select and mark models of the shape (tasks 2 and 3), in two tasks the pupils should select and mark nonmodels of the shape (tasks 4 and 5). We have evaluated the success rate and
used cluster analysis for models and non-models of the shape separately for each task. To illustrate, we present the values for non-models of a square (Table 2).

Limited space does not allow us to describe the results of individual tasks in details, therefore we will state the final conclusions of the test results in the research results part of this paper.


Figure 3: Task 2 - Is the shape on the picture a square?

| clusters | shape |
| :---: | :---: |
| 1 | C, D, F |
| 2 | G |
| 3 | H, J, I, B |
| 4 | E |
| 5 | L |

Table 2: Clusters of non-models of a square from the task 2.

The tasks in the second part of the test aimed to test the selected properties of shapes corresponding to the second and third van Hiele level, respectively the analysis level and abstraction level. Task 6 focused on the ability to distinguish the significant elements of triangles (points belonging or not belonging to a triangle, vertices and sides of a triangle). In task 7 we observed significant elements of a disk and a circle, the ability to distinguish them (points belonging to the disk, point belonging to the circle, radius and diameter of a circle). In task 8 we verified selected properties of a square among fourth-graders (adjacent and opposite side of a square, diagonals of a square, length of sides in a square etc.). In task 9 a non-convex hexagon was pictured and we tested whether the fourthgraders are able to determine the number of its sides and vertices.
In task 10 we verified the higher cognitive processes and tested whether the fourth-graders are able to draw a triangle, square, rectangle and pentagon in a square dotted grid, when one of the shape's sides is already pre-drawn.
Thus, the aim of the second part of the test was to determine the level of geometrical thinking of primary school pupils of the fourth grade from the point of view of van Hiele levels and their basic characteristics and conceptions and misconceptions of significant elements and properties of planar geometric shapes at the level of standards defined in the national curriculum (triangle, disk, circle, square, polygon). We tested which significant properties of planar shapes stated in the national curriculum cause difficulties for the pupils of the fourth grade and whether their perception is affected by misconceptions.

## Research results

Ability of pupils to name a shape according to a picture. The task of naming the shapes was relatively easy for the fourth-graders. Even so there were differences in the difficulty of naming of individual shapes. To name a disk was the easiest task for the pupils ( $97 \%$ ) and to name the shape of a square rotated by $45^{\circ}$ was the most difficult task ( $55 \%$ ). Some fourth-graders still named the shapes on the basis of their overall look and the position of the shape played an important role for them. This was reflected in a lower success rate of identification of a rectangle with a great difference in the length ratio of its sides.

Pupils' conceptions of squares. The square in a standard position was the easiest to identify from among all of the shapes depicted in the template (95\%). The identification of a rotated square was more difficult for the fourth-graders. At the same time, we may state that the degree of rotation of models also influenced the result. The fourth-graders experienced the greatest difficulties with a square, whose diagonals are in horizontal and vertical position ( $80 \%$ ). We have observed this problem already among younger children. The fourth-graders also had tendency to identify the shapes on the basis of holistic perception. They marked the shapes, which in fact were not models of squares but had a square shape, as squares. They experienced the greatest difficulties with the shapes of square shape with rounded sides and vertices. Another problematic shape in terms of identification of models and non-models of squares was a rhombus, which was marked as a model of a square by as many as $49 \%$ of the fourthgraders. We suppose that the congruence of sides fulfilled an important role as a significant property characterising a square during the decision making. We found out that more than $80 \%$ of the fourth-graders from the researched group know the significant elements and some selected properties of squares. They managed to identify the side of a square, the diagonal, they distinguished the adjacent sides (less successfully the opposite sides) and they could decide on the lengths of adjacent and opposite sides. Despite the fact that during the identification of squares in the pictures the pupils manifested the characteristics of the visualisation level, the results of the task concerning the significant elements and properties of squares indicated that they were able to give elementary descriptions of the basic characteristics of squares, which is typical for the van Hiele analysis level.

Pupils' conceptions of triangles. The results of the identification of models and non-models of triangles by primary school pupils of the fourth grade in the graphic template showed that the conceptions of triangles are of a slightly higher quality than the conceptions of squares. The success rate of the identification of models of triangles ranged from $88.7 \%$ to $98 \%$ and the success rate of the identification of non-models of triangles ranged from $67 \%$ to $89.6 \%$. This result may be interpreted in a way that the pupils recognise the models of triangles
more easily and better than the non-models of triangles. The lower success rate of the identification of non-models of triangles was caused mainly by the shapes that holistically resembled a triangle, so the thinking about triangles of at least one third of the pupils of the fourth grade from the chosen group showed the characteristics of the van Hiele visualisation level. Besides the holistic perception of triangles, we expected the fourth graders to be able to verify at least elementary significant characteristics of triangles (e.g. 3 sides, 3 vertices). These anticipations were not fulfilled. A relatively big part of the fourth-graders was not able to perceive the details of triangles. That means that they do not have correct conceptions of the facts that the side of a triangle must be a line segment and that the vertices of a triangle are points from the geometrical point of view. They also are not able to perceive the triangle as part of a plane. Under the term triangle they understand only its border. All of these indicators unambiguously show the fact that in the field of triangles the research group of fourth-graders manifested the characteristics of geometric thinking only on the visualisation level.
Pupils' conceptions of rectangles. We assumed that primary school pupils of the fourth grade should be able to relatively reliably determine the models and non-models of rectangles from the image template. The task focused on the identification of rectangles was formulated in the form of negation, as opposed to the previous tasks. The research results from the field of the identification of models and non-models of rectangles by the fourth-graders showed that the easiest task for the pupils was to determine those non-models of rectangles, whose shape did not resemble a rectangle. For example, they managed to mark a triangle, square, trapezoid or square-shaped figure as a shape, which is not a rectangle. More complicated models of shapes for the fourth-graders were those that holistically resembled a rectangle. Interesting discovery was that considering difficulty the rhomboid was the most complicated to identify for the fourth-graders. More than one half of the them regarded the rhombus as rectangle, similarly to many younger pupils. We may state that the research group of the fourth-graders showed the characteristics of the van Hiele visualisation level. At least $20 \%$ of the pupils did not take significant elements and most important properties of rectangles into consideration during the classification of the shapes. They did not pay attention to any details, not even to the roundness of the shapes, which is out of the question within the identification of a rectangle. They distinguished shapes not using deeper analytical thinking, but merely by visual holistic appearance. That means that the tested pupils do not have enough experience with rectangles or with rectangle-shaped figures that are not really rectangles. Once again it turned out that the position of a rectangle proved to be a significant attribute for the pupils, by which they identified the rectangle.

Pupils' conceptions of disks. The identification of planar shapes that are models of disks by their graphic representations was not a difficult task for the pupils and, in principle they managed to reliably distinguish these shapes from the shapes that are not disks. Even though the position of the shape does not play any role during the identification of a disk, it was shown that the size of a disk was not negligible for the primary school pupils of the $4^{\text {th }}$ grade. They experienced more difficulties when identifying a significantly smaller disk. Two regular polygons, which visually resembled disks, belonged to the most difficult items to correctly identify. The easiest thing was to exclude the shapes, whose shapes were "too angular"; the fourth-graders did not consider these shapes to be disks. Empirical data gathered during the identification of non-models of a disk showed that the division of shapes into groups corresponded with the holistic perception of these shapes very accurately. Visual thinking was confirmed also by the results of the task concerning significant elements of disks. At least half of the fourth-graders in the research group did not manifest the ability to determine the points of a disk, the radius and diameter of a disk, thus they were not able to provide elementary description of the shape. Despite the fact that disks turned out to be the most easily recognizable planar geometric shapes to identify by picture, their identification showed only the characteristics of the visualisation level.

Our measurements proved that the identification of models and division of nonmodels into groups for each individual shape (disk, square, rectangle, triangle) very accurately corresponds to the holistic perception of these shapes, and this in turn corresponds to the visualisation level of the van Hiele theory of the cognitive process in geometry. Primary school pupils of the $4^{\text {th }}$ grade still preferred the similarity of a shape with a visual prototype during its identification. In case of triangles this was shown by preferring triangles with one side in a horizontal position, in case of rectangles by the incorrect identification of rhombus and rhomboid in the horizontal position, and, at the same time, the inability to identify a square, whose diagonals are in horizontal and vertical position. Our assumption that these pupils will achieve the analysis level (of description) with signs of visualisation level were not confirmed. Only within the subject of squares we discovered some signs of the level of description related to the significant and elementary properties of squares. We may rather state that fourth-graders remain at the visualisation level and that they manifested signs of the higher van Hiele analysis level only partially and only with certain shapes.

## CONCLUSION

The results of our research confirmed the typical basic attributes of visualisation level and analysis level according to the van Hiele theory. At the same time we have observed such children's statements and demonstration, that were seemingly on the border between the levels. The original van Hiele theory does
not describe these observations. According to our results, we may not think of the van Hiele levels as separated and disjunctive stages. Even though the differences between the characteristics of the individual levels are apparent, we understand the border between them to be very thin. We tend to think that the individual levels overlap within the cognitive process. The signs and demonstrations that are beyond the extent of one level, but are still not sufficient for the higher level could be regarded as such overlaps.
Our data provided the evidence that children's conceptions of geometric shapes may be at various levels. For example, a child's geometric conceptions of triangles may be at a different van Hiele level than its conceptions of squares. The level of geometric thinking is usually determined by qualitative and quantitative individual experience of a child, but also by the diversity of shape properties (position, size, form).
The stability of geometric preconception of children is high. For children the prototypes mean certain models, which represent geometric shapes and children create their own mental representations or figural schemes based on them. These representations and schemes are so fixed, meaningful and understandable for children that they are not willing to change their conceptions and even adapt their argumentation to the already existing mental structures (Žilková, 2013).

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# LET'S EXPLORE THE SOLUTION: LOOK FOR A PATTERN! 

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In this paper we analyse some aspects of students' cognitive factors in problembased learning. The problem we chose is closely related to the mathematical concept of sequence and offers also multiple solution strategies, multiple representations of the subject and possibility for mathematical communication. We report results of a study in the age group of Grades 5th and 6th, focusing on their problem solving strategies and the characteristics of their inductive reasoning.

## INTRODUCTION

"Make the subject problematic!" - it is a conceivable way the teachers approach the curriculum. Hiebert et al. argue that "... instruction should be based on allowing students to problematize the subject. Rather than mastering skills and applying them, students should be engaged in resolving problems" (1996, p. 12). In the subject's research, professional and ethical issues are constantly emerging about problem-based learning. Is it possible to expect independent (or directed) discovery from every learner? Is the role of examples and counter-examples understandable to everyone? May the problem-based learning lead to meaningless learning in some cases? These dilemmas can only be resolved if the problems raised are examined with scientific certainty. The more we understand the problem-solving thinking of students at different ages, and the more thoroughly we analyse the effectiveness of problem-based mathematics teaching in classroom environments, the more secure we can apply this method. In this paper we analyse some aspects of students' cognitive factors in problem-based learning (including complex thinking and reasoning strategies, e.g. conjecturing or justifying), in order to understand students' problem solving thinking better.

## CONCEPTUAL BACKGROUND

The key concept of our paper is problem-based learning, which has a continuously enriching conceptual structure in the literature of mathematics education; therefore, we first clarify why we use this concept. In a problembased learning environment, a problem drives the learning material (Roh, 2003). The problem or task should be an activity that focuses students' attention on a particular mathematical concept that matches the goals of the curriculum. Students can also make connections between mathematical concepts and processes that are familiar to them. Good problems for problem-based learning offer also multiple solution strategies, multiple representations of the subject and
possibility for mathematical communication that includes proof-based activities or justification (Erickson, 1999). Good problem solving skills are prerequisites of problem-based learning; additionally problem-based learning in mathematics classes would provide students more opportunities to think critically. In our opinion, in the mathematics field, problem-based learning means that a learner must analyse a mathematical problem situation; he or she must approach critically the thinking of their own and their classmates. Furthermore, students explain and justify their thinking (Csíkos, 2010). The problem solving process we are studying in this paper is characterized by all the three elements of the above definition, thus providing a suitable conceptual framework for describing our research.

The purpose of a problem appearing in the classroom is focusing students' attention on a particular mathematical concept, idea or skill. The model by Stein, Grover and Henningsen (1996) based on the fact that mathematical tasks pass through three phases in the classroom: as written by curriculum developers, as set up by the teacher in the classroom, and as implemented by students during the lesson. Teachers' goals, knowledge of subject matter and knowledge of students influence the setup of the mathematical task as represented in the curricular materials. Factors influencing student's implementation are classroom norms, task conditions, teachers' instructional dispositions and students' learning dispositions.
Mason, Burton \& Stacy argue that "The process of conjecturing hinges on being able to recognize pattern or an analogy, in other words on being able to make generalizations" (2010, p. 73). More generally, this cognitive process is involved in the inductive reasoning activity. Haverty, Koedinger, Klahr and Alibali (2000) argue that fundamental areas of inductive reasoning are data gathering, pattern finding and hypothesis generation. Within the process of inductive reasoning Polya (1954) distinguishes stages, such as observation of particular cases, formulating a conjecture (generalization), testing the conjecture with other particular cases. Following these sources, we use a five-levels model for describing the inductive reasoning process (Kónya \& Kovács, 2017).
(1) Observation of particular cases including looking for possible patterns as well.
(2) Following the observed pattern, i.e. applying it for other cases. It often happens without formulation of a general statement.
(3) Formulating a conjecture.
(4) Testing it by other particular cases.
(5) The result is a general statement at this stage, but the mathematical problem solving process requires the deductive closure. The form of deductive closure could be either a rigorous proof or justification using the underlying mathematical structure.

Patterns in school mathematics often are represented either numerically or figurally (Rivera, 2013). In this study we use a figurally given pattern. The underlying mathematical structure can be represented numerically by a sequence. Students are expected to continue the pattern figurally, and they are also expected to formulate generalizations concerning this sequence, e.g. to determine a "near", "far" or "arbitrary" element of the sequence. We also look for mathematically valid explanations or non-proof arguments, i.e. empirical arguments or rationales in the sense of Stylianides (2009).
For describing the background of our research, we outline the Hungarian traditions of problem-based learning. Problem-based learning is an essential element of the Hungarian mathematics-teaching traditions, which is closely related to heuristics, inductive reasoning or to Polya's principle of active learning (Polya, 1965, pp. 102-106). Problem-based learning is one of the fundamental principles of the "Complex Mathematics Teaching Experience" conducted by Tamás Varga in the Sixties and Seventies in Hungary (Varga, 1988). One of the important effects of the Complex Mathematics Teaching Experiment is that this principle has always been present in the everyday practice of Hungarian mathematics teaching and learning. In this place, we emphasize C. Neményi Eszter's pedagogical work (C. Neményi, 1999), where one of the focal points is the pattern recognition in a sequence which is uniquely defined by some activity, drawing, or procedure. C. Neményi argues that pattern recognition activities support

- recognizing the modelling function of sequences (i.e. a sequence is the mathematical model of a problem),
- identifying functional relationships between quantities,
- understanding mathematical concepts, and ideas.


## METHODOLOGY

We conducted a cross-sectional study, where we used a textbook-problem for fifth-graders, but we used five different setups of this problem for various ages. Table 1 presents the number of students in each grade who took part in the investigation.

| Grade | $5-6$ | $7-8$ | $9-10$ | $11-12$ | $12+$ | Total |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N | 47 | 21 | 60 | 33 | 71 | 238 |

Table 1: Number of students in the sample of investigations (12+ refers on teacher trainees.)

## The sample

In this paper we report results of the study in the age group of Grades 5th and 6th. We implemented the task in the classroom using pair work method. 47 pupils (24 pairs altogether) involved in the classroom observation, which took place in 2016 in an urban school in Hungary. The sixth-graders had results in the
lower part of the top tierce of the National Assessment of Students' Mathematical Competences (similar to the international PISA test) in the year of our research; and we suppose that this result is exhibitive also for fifth-graders. (The National Assessment is conducted only for 6-, 8 - and 10 -graders.) It means that based on their achievement in mathematics, they are average or slightly higher than average students.

## The "House of cards" problem

In a Hungarian textbook for fifth graders the following problem appears: "Build a house of cards shown in the picture. Discuss how many cards you need to make 1, 2, 3, ... level house!" (Gedeon, Korom, Számadó, Tóthné Szalontay, \& Wintsche, 2016)


Figure 1: The picture in the textbook
We used this problem in the cross-sectional survey, but we have changed the text by age. There was a notable change in the fact that in the different age groups we asked students about different storeys: about "low" house (e.g. 5storey), "high house" (30-storey), or generally about an $n$-storey building. Appendix 1 contains the worksheet prepared for $5^{\text {th }}$ and $6^{\text {th }}$ graders. Tasks 4 , and 5 contain questions about "low" houses, i.e. "near" elements of the sequence. In Task 6 there is the option of "far" element of the sequence. In fact most of the groups built the problem for "high" houses. (We consider a house high when it is difficult to draw it accurately and counting the cards; e.g. a 30 -storey house is a "high" house.)

We think that this problem has all the features of a "good problem" and gives the possibility of a problem-based learning activity. It proved oneself to be a real problem situation in all grades. It points to curriculum material connected with sequences, but in different depth in different grades. For the $5^{\text {th }}$ and $6^{\text {th }}$ graders the focus is on recognition of functional relationship. It gives the possibility for different representations, i.e. enactive, iconic and symbolic in the sense of Bruner (1971). Accordingly, in the survey we made $5^{\text {th }}$ and $6^{\text {th }}$ graders build the house. The problem is also suited for deep mathematical communication and reasoning: the pupils should formulate a generalization, and he or she is expected to explain it. Also, critical thinking is relevant in this problem, because it contains possibilities of typical misconceptions. For example, while the number of cards grows as the house becomes higher; many students thought that the number of cards is linearly proportional to levels. Another misconception is that the function in question is additive.

Moreover, several approaches are possible, because the problem can be modelled by different sequences:
A. number of cards in the sequence of houses: $2,7,15,26,40, \ldots$
B. number of new cards one needs to complete the previous house in the sequence: $5,8,11,14, \ldots$
C. number of slanted cards in rows in a particular house (from up to down): $2,4,6,8, \ldots$
D. number of horizontal cards in rows in a particular house (from up to down): $1,2,3,4, \ldots$
E. number of triangles in rows in a particular house (from up to down): $3,6,9,12, \ldots$ (excluding the last row, i.e. the "basement").
The textbook proposes that group work should be implemented for this problem. We agreed with the cooperative method, because the problem-based learning style requires students' critical and active attitude to the problem and to their own and their classmates' thoughts. Furthermore communication is an essential part of this learning approach. Taking all of this into consideration, we implemented the task in pair work in our survey.

## RESEARCH QUESTION

1. What kinds of problem solving strategy are used by the $5^{\text {th }}$ and $6^{\text {th }}$ Graders?
2. What are the characteristics of their patterning?

## RESULTS AND DISCUSSION

In order to answer to the questions we investigated the written works. We analysed the works of 9 pairs of $5^{\text {th }}$ Graders and 15 pairs of $6^{\text {th }}$ Graders. They worked on the "House of cards" problem during the class together and were asked to complete the tasks on their worksheet (see Appendix 1). First they built the 3 -storey house from cards (enactive representation) then completed the figural sequence (iconic representation) with the next two elements (3- and 4storey house). In the third task they counted the number of cards and wrote it under the figures of the houses (symbolic representation). With the exception of 1 pair everybody solved the Task 1-3 correctly. This means that 23 pairs understood the problem itself and were able to identify and use its iconic representation form. The correct figure of the 4 -storey house shows us that they saw the structure of the card-building too.
Special attention was paid to the solution of the Task 4-6. Concerning the first research question, we examine the strategies occurring in the solutions.
We developed our coding system for problem solving strategies inductively. The authors performed a pilot coding and gave a coding system for coders. Every written work was coded by two different coders, and in the last step we
consolidated the corpus. Task 4, 5 , and 6 in the worksheet were the coding units. The coding system for problem solving strategies as follows:

- Counting. The students draw the house and count the cards without any sign of looking for patterns. (Figure 2)


Figure 2: Example of the Counting strategy

- Patterning. The students refer to sequences A-E in problem solving. (Figure 3)

$$
\begin{aligned}
& 10 \text { shintes (eqy) }=\$ 55 \quad 30 \text { mint }=1365 \\
& 2<7<15<26<40<57<7 \nexists<100<126<195<187 / \text {. } \\
& \begin{array}{llllllllllll}
5 & 8 & 11 & 14 & 17 & 20 & 23 & 26 & 29 & 32
\end{array}
\end{aligned}
$$

$$
\begin{aligned}
& 495 \angle 551 \angle 610 \angle 672<737<805 \angle 876<950
\end{aligned}
$$

Figure 4: Example of the Patterning strategy (Sequence A and B; 10- and 30storey house)

- Recursion. The students recall the one storey lower house while counting the cards of a particular house. (Figure 4)



Figure 4: Example of Recursion (40, because we added 14 cards to the 4 -storey house.)

- False scheme. The students refer to linear proportionality (Figure 5) or additivity (Figure 6).


Figure 5: Example of False scheme, proportionality (It has 6 storeys. We can build it from 57 cards. $\rightarrow 12$ storeys: $57 \cdot 2=114 \rightarrow 24$ storeys: $114 \cdot 2=228 \rightarrow 48$ )


Figure 6: Example of False scheme, additivity (13-storey house; we can build it from 140 cards. 5 -storey +8 -storey $=40+100=140$ )

- No answer

Table 2 gives an overview about the distribution of the applied strategies in the three tasks.

| Number of works | 5-storey house | 8-storey-house | arbitrary house |
| :---: | :---: | :---: | :---: |
| 5 | Counting | Counting | Counting |
| 1 | Counting | Counting | Proportional scheme |
| 1 | Counting | Counting | Additive scheme |
| 1 | Counting | Counting | Patterning |
| 2 | Counting | Patterning | Patterning |
| 1 | Counting | Patterning | Additive scheme |
| 7 | Patterning | Patterning | Patterning |
| 2 | Recursion | Recursion | Recursion |
| 2 | Recursion | Recursion | Patterning |
| 1 | Recursion | Additive scheme | Recursion |
| 1 | No answer | No answer | No answer |

Table 2: Strategies applied in the tasks
We can conclude that 14 pairs use the same strategy during their work and the most popular was the Patterning ( 7 works) then the Counting ( 5 works). 9 pairs used different strategies in the three tasks. 6 of them started with Counting in the
case of 5-storey house, and then half of them recognized a pattern in the 8 -storey house, while the others continued with Counting again. In the last task, where the drawing was difficult, False scheme appeared besides of the Patterning.

Figure 7 summarizes the applied strategies by tasks. We can see clearly, that the number of Counting strategy decreases, while the number of Patterning increases as the house gets higher. False scheme appears only in the case of high houses, when the Counting strategy does not work.


Figure 7: Distribution of the strategies by tasks
Concerning our second research question we investigate students' inductive reasoning, so we focus on those solutions which applied the Patterning strategy. The first phase of the inductive reasoning process i.e. Observation of particular cases was obvious in 23 works, because of the completing the Task 1-3. The next phase, namely Following the observed pattern, appeared in those works, where the Patterning strategy was applied. We detected all of the five sequences. Sequences A and B was used in 14 solutions (see Figure 4 as an example), sequences C and D, similarly in 14 solutions (Figure 8). Sequence E also appeared in 1 solution (Figure 9).


Figure 9: Example of the sequence C and D


Figure 10: Using the sequence E ( 10 -storey house, 172 cards. One row is 3 times more as the first one. Every row is divisible by 3 , except the last one.)
We can conclude that in 14 works from the 24 Patterning strategy i.e. Following the observed pattern was detected at least in one task. Furthermore, the third phase of inductive reasoning Formulating a conjecture was observed only in some cases. We should make a difference between describing the way of counting they use and formulating a conjecture. The formulated conjecture contains typical phrases, like "always" we found it in 9 works, for example: "As much as the previous one has increased, you have to add 3 more to it." (Sequence A-B) or "Going from the top there is always 2 more cards in the rows; the cards that separate the rows always increase by 1." (Sequence C-D)
The control, namely testing the conjecture by other particular cases, didn't appear in this form. However, we observed in 9 works that the students drew the figure of the house because of the control of the patterning activity. Another way of the control appeared in one work only: they checked their result gained by using the Sequences C-D with the help of Sequences A-B, which was also recognized.

We couldn't find any clue of any kind for the argument for the discovered rule, except one work (Figure 10), where it was explained by marking the triangle on the top with a circle.

## CONCLUSION

The simple Counting strategy was the most frequent one, especially in the case of 3-, 4-, 5- and 8 -storey houses. The Patterning strategy also occurred in many cases, thanks to the possibility of using sequences in this problem's situation. The Recursion, i.e. the recursive thinking is closely related to the patterning activity. The lack of the generalization ability causes the appearance of the False schemes. The linear proportionality and the additive thinking are very common in the mathematics classrooms; the students use them automatically without any doubt about their compliance.

Following and observing a pattern is a well-known and often used strategy in the investigated age group. However, the further phases of the inductive thinking process are not realised at all. After the teacher requested it, the students were able to formulate and explain a "rule" or argue for it, using the real context that defined the pattern, but they didn't feel the need for such an explanation.

Our problem is closely related to the mathematical concept sequence. The problem solving activity contributed to the better understanding of that concept and to practice the flexible transition between the iconic and numeric representations.

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## APPENDIX 1

## Cards (Grade 5)

Name:
Mark in mathematics after grade 4:
Name:
Mark in mathematics after grade 4:

1. This is a three-story house. Build the house after the photo!

2. We drew the one-storey and two-story house. Continue drawing with the three-storey and four-storey housel

## 1-storey 2-storey 3-storey

3. Calculate the number of cards needed to build the houses and write in the rectangle under the figure.
4. How many cards are needed for the five-story house? Explain your answer
5. How many cards are needed for the eight-story house? Explain your answer!
6. Think of another house! Tell us how many levels it has! How many cards are need to build it?

# OLD AND NEW METHODOLOGIES FOR FACTORING QUADRATIC EQUATIONS 

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#### Abstract

This study investigated the effect of using graphing tools for factoring trinomials to students in introductory algebra classes. Students in the study were taught to graph parabolas to establish factorability of quadratic trinomials, to estimate solutions, and to verify their answers by using algebraic methods. The study found that students using graphing tools significantly outperformed the control group that used computational methods only in understanding and solving quadratic equation.


## INTRODUCTION

Various advancements of science and technology, along with constantly changing the socio-economical world in the 21st century, called for the revision of the goals of teaching mathematics. Current curricula for high school or college level mathematic assume that students master algebra by the age of eighteen. The expectations include, among others, the use of the proper mathematical language and structures to represent, analyse, generalize, model and solve problems of various complexity in various contexts (RAND \& Ball, 2003). An important part of the algebra curriculum is the ability to factor polynomials and use it to solve algebraic equations and graphing of functions. However, it turns out to be one of the most challenging topics to teach. The difficulties come from the complexity of the tasks and the amount of different methods of factoring that students are exposed to.

## DIFFERENT METHODS OF SOLVING QUADRATIC EQUATIONS

Each of the four main historical periods in the development of algebraic methodology (geometric, static equation-solving, dynamic function, and abstract) added new views on factoring and solving quadratic equations (Katz \& Barton, 2007). At first, they were interpreted as geometric problems (Babylonian algebra and ancient Greece), then algorithms started to be used, still justified by geometry (al Khwarizmi), and after that, new non-geometric methods were developed (Islamic mathematics). The improvement of the mathematical notation in the seventeenth century allowed algebraic equations to interpret science underlying their importance.

Currently, we recognize three basic methods of solving quadratic equations: the quadratic formula, completing the square, and using zero property of multiplication (factoring). All these techniques are part of the United States Common Core Mathematics Standards. Out of those three methods, factoring
polynomials is the most challenging to teach (Kotsopoulos, 2007), particularly in introductory algebra courses. Heavily based on previous mathematical knowledge, factorization requires students to learn and incorporate many formulas like: the square of a binomial, the difference of two squares, the greatest common factor of monomials, the "guess and check" method, "grouping," and more. To make a situation even more complex, teachers and educators have been creating new procedures (Table 1), hoping to produce a better and easier way for students to understand the concept. Therefore, we can find a variety of algebraic techniques at schools, for example: Moskol's method, Autrey's and Austin's method, Baker's method, Savage's method, slip-slide method, and X method. Teachers also use geometric interpretations, physical models, puzzles, and manipulatives (algebra tiles). Even though, physical and geometric models have limitations, they could be viewed as "the opportunity to bridge the gap to algebraic thinking" (Patterson, 1997, p. 240).


Physical
Models
Factor $x^{2}+3 x+2$.


Therefore, $x^{2}+3 x+2=(x+2)(x+1)$.
Hirsch (1982, p. 388)
Factoring Puzzles Factor $10 x^{2}+23 x+12$.
The first puzzle corresponds to factorization of the trinomial.
Therefore, $10 x^{2}+23 x+12=(2 x+3)(5 x+4)$.


Hollingsworth \& Dean (1975, p. 428)
Factor $8 x^{2}+10 x+3$.
Autrey's\& Austin's
First write:
$(8 x)(8 x)$ leaving space to write other numbers after each $8 x$.
Consider $8 \times 3$, where 8 is the coefficient of $x^{2}$ and 3 is the constant. We want to find two integers whose product is 24 and whose sum is 10 , such as 4 and 6 .
Write these integers after $8 x$ to get
$(8 x+4)(8 x+6)$ (It does not matter whether 3 or 6 is first). Now, divide each parenthesis by their greatest common factor. Here 8 and 4 are divided by 4 , and 8 and 6 are divided by 2 .
Therefore, $8 x^{2}+10 x+3=(2 x+1)(4 x+3)$.
Autrey \& Austin (1979, p. 127)

| Slip-Slide Method | Factor $6 x^{2}-19 x+10$. <br> (Using integers only) <br> Let's multiply the leading coefficient by the <br> constant. <br> We will get: $x^{2}-19 x+60$ |
| :--- | :--- |
|  | Now, we can factor this trinomial. |
| We are looking for two numbers |  |
| with a sum of -19 and a product of 60. |  |
| Both numbers must be negative: -15 and $-4:$ |  |
| $(x-15)(x-4)$. |  |$\quad$| Next, "slide" back the leading coefficient: |
| :--- |
|  |
| $\left(x-\frac{15}{6}\right)\left(x-\frac{4}{6}\right)$ |

Then, let's reduce it:
$\left(x-\frac{5}{2}\right)\left(x-\frac{2}{3}\right)$
Next, "slide" back the denominators
in front of each $x:(2 x-5)(3 x-2)$.
Therefore, $6 x^{2}-19 x+10=(2 x-5)(3 x-2)$.
Steckroth (2015)

Factor $6 x^{2}-38 x-80$.
X-Method $\quad 6 x^{2}-38 x-80=2\left(3 x^{2}-19 x-40\right)$
or
Thus, $a=3, b=-19, c=-40$
Diamond Method


Therefore, $2(3 x+5)(3 x-24)$
Divide each factor by its GCF:
$2\left(\frac{3}{1} x+\frac{5}{1}\right)\left(\frac{3}{3} x-\frac{24}{3}\right)=2(3 x+5)(x-8)$
Therefore, $6 x^{2}-38 x-80=2(3 x+5)(x-8)$.
Lemmon (2004, p. 35)

Factor $6 x^{2}-19 x+10$.
Moskol's Method
Draw $3 \times 3$ box:

| $6 x^{2}$ | 10 | $60 x^{2}$ |
| :--- | :--- | :--- |
|  |  | $-4 x$ |
|  |  | $-15 x$ |

Because
$(-4 x) \times(-15 x)=60 x^{2}$
$(-4 x)+(-15 x)=-19 x$
Find a factor of $6 x^{2}$ and a factor of 10 whose product is $-4 x$. Put those numbers into second row.
By convention, the linear factor must always be positive.

| $6 x^{2}$ | 10 | $60 x^{2}$ |
| :--- | :--- | :--- |
| $2 x$ | -2 | $-4 x$ |
|  |  | $-15 x$ |

Find a factor of $6 x^{2}$ and a factor of 10
whose product is $-15 x$.
Put those numbers into second row.

| $2 x$ | -2 | $-4 x$ |
| :--- | :--- | :--- |
| $3 x$ | -5 | $-15 x$ |

Therefore, $6 x^{2}-19 x+10=(2 x-5)(3 x-2)$.
Moskol (1979, p. 676)

Factor $6 x^{2}-41 x+63$
Baker's
Method
Solution:
$6 x^{2}-41 x+63=6 x^{2}+a x+b x+63$
Where:

1. $-41=a+b$
2. $\frac{6}{b}=\frac{a}{63}$

Therefore,

$$
\begin{aligned}
& a b=6 \cdot 63=378 \\
& (a-b)^{2}=(a+b)^{2}-4 a b
\end{aligned}
$$

So,

$$
(a-b)^{2}=(-41)^{2}-4 \cdot 378=1681-1512=169
$$

Therefore, $a-b= \pm 13$

$$
\begin{aligned}
& \text { If }\left\{\begin{aligned}
a-b & =13 \\
a+b & =-41
\end{aligned}\right. \\
& \text { then } \begin{aligned}
2 a & =-28 \quad \text { and } \quad 2 b \\
a & =-54 \\
a & =-14
\end{aligned} \\
& b=-27
\end{aligned}
$$

$$
\begin{aligned}
& 6 x^{2}-14 x-27 x+63=2 x(3 x-7)-9(3 x-7)= \\
& (3 x-7)(2 x-9) \\
& \text { Therefore, } 6 x^{2}-41 x+63=(3 x-7)(2 x-9)
\end{aligned}
$$

Baker (1969, p. 631)

## Factor $10 x^{2}+31 x+15$ <br> Box <br> Method <br> 1. Factor GCF <br> 2. Draw $2 \times 2$ box

3. Put the ax2 term in the top left box and the c term in the bottom right box
4. Multiply this diagonal. The other blank diagonal has to multiply to be the same product
5. Find the positive or negative factors of this product to get the bx term. Those two factors will go in the blank boxes. Don't forget the variable!
6. Find the CCF from the top row and "solve" the box.
7. Write these factors using parenthesis. Don't forget the GCF from step 1!

| $10 x^{2}$ |  |
| :--- | :--- |
|  | 15 |

$$
\begin{aligned}
10 x^{2} \times 15 & =150 x^{2} \\
150 & =1 \times 150 \\
150 & =2 \times 75 \\
150 & =3 \times 50 \\
150 & =5 \times 30 \\
150 & =6 \times 25
\end{aligned}
$$

Therefore:

|  | $5 x$ | 3 |
| :--- | :--- | :--- |
| $2 x$ | $10 x^{2}$ | 6 x |
| 5 | $25 x$ | 15 |

Therefore, $10 x^{2}+31 x+15=(5 x+3)(2 x+5)$.
Harbin, D. (n.d.)
Table 1: Different methods of factoring quadratic functions.

Challenges with teaching factoring quadratic functions inspired us to search for solutions. Vygotsky (1978) stressed the importance of introducing different interpretations of a concept when constructing students' knowledge: "if one changes the tools of thinking available to a child, his mind will have a radically different structure" (p. 126). Many educators support using multiple
representations to enhance students' learning (Ainsworth, 2006; Cabahug, 2012; Ogbonnaya, Mogari \& Machisi, 2013). According to Ainsworth (1999) "a common justification for using more than one representation is that this is more likely to capture a learner's interest and, in so doing, play an important role in promoting conditions for effective learning" (p. 131).
All the functions of multiple representations apply to graphing technology, the modern tool in mathematics classrooms. In particular, incorporating graphing calculators to teach factoring quadratic functions as early as possible to introductory algebra students could play an essential role in teaching-learning processes. Graphs of parabolas could be used to establish factorability of quadratic trinomials (existence of real solutions), to estimate possible solutions, and to verify answers calculated by algebraic or computational methods.

## USING TECHNOLOGY

In modern times, it seems natural to expect technology to be included in mathematics classrooms. Almost twenty years ago, (Pan, 1998) emphasized: "the robots of the future are waiting", "speedy computer algorithms offer new answers." And in fact, the technological era has opened completely new options for mathematics educators. For instance, computer-assisted instructions (CAI) allow students to learn at their own pace (access to computers and Internet is necessary). A more affordable option and suitable for algebra classrooms are smaller graphing tools (such as iPads or graphing calculators). The National Council of Teachers of Mathematics and National Research Council encourage using graphing calculators as a supplementary tool in mathematics classroom (Dreiling, 2007; NCTM, 1989). Graphing tools are often seen as powerful teaching enhancements - not only for teachers, but also for students, as a teaching and investigative tool (Laughbaum, 1998).

## GRAPHING/GUESS AND CHECK METHOD

We advocate for an enhancement of the existing algebraic methods by early introduction of graphs of polynomials related to the equations. This study investigated the effect of using graphing calculators as a basic tool in guess and check method of factoring trinomials. After a comparative analysis of different methods of factoring quadratic equations, we proposed the new approach. In our study, students were taught, how to use calculators to graph polynomials to establish factorability of quadratic trinomials (existence of real solutions), to estimate possible solutions, and to verify their answers by algebraic or computational methods.

## METHODS

The experimental group participants were chosen to represent different ages and levels in math. We used one-on-one tutoring sessions and regular classrooms settings. Some students were diagnosed with learning difficulties. We
administered a four-question pre-test concentrating of factoring different types of quadratic equations.

The study had three parts. In the first session, students were trained to make graphs on graphing calculators and track points to get their coordinates. They learned to interpret graphs, especially identify zeros of a function. The level of mastering the first part was evaluated by a teacher through the individual oral assessment. In the second session, students learned factoring quadratic equations using guess and check methods. They were asked to use graphing calculators to check factorability (they established the number of solutions based on the number of x-intercepts). Students used graphing calculators to track x-intercept points and estimate their x coordinates. Then, they used those numbers in their guess and check method. Students used graphing calculators and their graphs to confirm the final solutions. In the third session students learned the difference of squares, the quadratic formula and the completion of the squares method.

The control group had also three sessions following the lectures and examples based on a standard textbook. We started by administering the same pre-test to this group. At the end of the study all students took a post-test and a survey was administered to participants and teachers.

## RESULTS

This study showed that using graphing technology as a supplementary tool for factoring quadratic equations significantly improved students' abilities to solve the problems, and the study group significantly outperform the control group on the post-test (Table 2).

| $(\mathrm{p}<0.05)$ |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- |
|  | Control Group (N=16) |  | Study Group (N=21) |  |
|  | Mean | SD | Mean | SD |
| Pre-test | 0.25 | 0.45 | 0.14 | 0.36 |
| Post-test | 4.50 | 1.51 | 5.33 | 0.97 |

Table 2: Pre-test and Post-test Mean and Standard Deviation for Both Groups
The main goal of using graphing tools was to help students with guess and check method of factoring quadratic functions and indeed it improved their abilities to solve problems like $x^{2}+6 x+8$ and $x^{2}-4 x+3$. Especially students with learning differences (for example dyslexia) really appreciated an option to be directed by graphs to find or check their solutions since they experience difficulties with basic operations on integers. In addition, the study showed that using graphs of parabolas helped differentiate between two formulas: the difference of the squares and the perfect square formulas. Students with dyslexia have tendencies to confuse $x^{2}-4=(x+2)(x-2)$ with $x^{2}-4 x+4=(x-2)(x-2)$ and $x^{2}+4 x+4=(x+2)(x+2)$. The students using graphing tools were able to catch the mistakes if they mixed up two opposite
solutions with a double solution. However, graphing calculator did not help them with double roots, such as $x^{2}-4 x+4=(x-2)(x-2)$ and $x^{2}+4 x+4=(x+2)(x+2)$. Note that teachers' responses on the survey evaluating the method were positive. They underlined the visual aspect of solving equations via graphing and pointed out that the method could be beneficial for higher degree polynomials. Some teachers pointed out that the incorporation of graphing calculators in the mathematics curriculum should be a part of their professional development.
Students taking introductory algebra at the middle school level liked to use graphing calculators, because it was a piece of technology that made them look smart ("cool"). Struggling high school students liked the predictability of this method. Some of them stressed that at least they knew where to start, because with guess and check methods they were often simply 'stuck.' Based on the students' self-assessment they felt more confident solving quadratic equations having graphing calculators for visualization.

## CONCLUSION

There is a general belief about the necessity of incorporating technology into mathematics instruction. Many research studies have been done to demonstrate the significance of graphing tools in higher levels math classes, such as precalculus and calculus. However, there are very few studies about using graphing tools in pre-algebra and algebra courses. The study shows that graphing tools are effective if introduced early as a supplementary visualization for solving quadratic equations in introductory algebra.

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# Emerging Mathematics through Realistic Situations 

# HOW DO STUDENTS CONSIDER REALISTIC CONTEXTS IN MATHEMATICAL PROBLEMS? - A CASE STUDY 

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In the present study, we investigate the influence of realistic contexts on students' problem solving. We adapted three mathematical tasks from relevant studies, which were based on a realistic context and we asked seven students of grade 6 to solve them. The students demonstrated different levels of consideration of the realistic constraints of the given tasks. Our analysis shows that they performed considerably better in the task which was assumed to be closer to their lives. Thus, the distance of the context of the tasks from the students' realities together with the researcher's interventions in the whole class discussions proved to be important factors in the contextualised problem solving of these students.

## INTRODUCTION

Mathematics is a core subject in school curriculum; however, for most students it is a source of anxiety or even fear. Negative connotations can be also traced in popular culture, such as films and books (Darragh, 2018). For many students, mathematics is seen as "'hard', 'logical', 'certain' and 'ultra-rational'" and "mathematicians as eccentric, even insane, and [...] highly emotional" (Epstein, Mendick \& Moreau, 2010, p. 49). By acknowledging these facts, several movements in mathematics education within the last decades have attempted to increase the popularity of mathematics among students at all educational levels. One of the ways to achieve this was by enhancing students' ability to transfer their learning of school mathematics to different contexts, including their everyday lives (Boaler, 1993). In the same line, for the last decades, 'Realistic Mathematics Education' (RME) (e.g., Freudenthal, 1973; 1991) has been putting the focus on the role of context in mathematical problems. Nowadays, several countries have adopted the views of RME, whose influence is clearly visible in international mathematical surveys, such as O.E.C.D.'s Programme for International Student Assessment (PISA). At the same time, as we will present in the next section, the effect of context in mathematical problems is continuously being investigated with sometimes conflicting results.
Our study stems from our interest in the ways context affects mathematical problem solving. Particularly, we aim to investigate whether grade 6 students consider the realistic contexts of the tasks given to them. Additionally, we were
interested to see whether students' familiarity with the contexts would affect their solutions. Our research questions are:

- Do students consider the realistic constraints of contextualised mathematical problems?
- How do different contexts affect the students' solutions? Particularly, what is the role of the distance of the context from the students' lives?
- Can the teacher's actions affect the degree of students' consideration of the realistic constraints of contextualised mathematical problems?


## THEORETICAL FRAMEWORK

Mathematics is a discipline usually linked to logic and unambiguousness; however, as Hersh (1991) points out, there is a 'frontside' and a 'backside': the frontside is based on the widely spread beliefs about the unity, objectivity, universality and certainty of mathematics. These beliefs (which are called myths by Hersh) "need not be true; they need to be useful" (p. 132); a mathematician though, needs to make a transition to the backside and "develop a less naive, more sophisticated attitude toward the myths of the profession" (p. 132). According to this attitude, mathematics is not an isolated discipline, but it takes part in the interpretation of interdisciplinary phenomena from various aspects of reality. Such an approach allows for the appreciation of the aesthetic dimensions of mathematics, engaging in critical thinking and generally avoiding a dogmatic and sterile certainty, especially in economy and education (Ambrose, 2017).
At the same time, mathematics education also seems to suffer from similar dichotomies; for example, rote memorisation vs. active, participatory learning. The last decades we have seen movements that have put the student(s) at the crux of teaching and learning (for example, constructivism or socio-cultural approaches). In line with these, concerns are expressed about the content of school mathematics. One of the main issues is about the way to improve the students' ability to apply their school mathematics knowledge to out-of-school situations. The design of contextualised mathematical tasks has been proposed as a fruitful way to achieve the 'transfer' aim. Apart from enabling students to transfer their mathematical knowledge to real life (and vice-versa), contextualised tasks increase students' motivation, thus they can combat the negative feelings associated with mathematics:

Students must understand that the mathematics instruction they receive is useful, both in immediate terms and in preparing them to learn more in the fields of mathematics and in areas in which mathematics can be applied (e.g., physics, business, etc.). Use of ill-structured, real-life problem situations in which the use of mathematics facilitates uncovering important and interesting knowledge promotes this understanding. (Middleton \& Spanias, 1999, p. 81)

With respect to contextualised task design, one of the main issues that needs to be decided is what is a real (or an authentic) context, or, from a mathematics educator's point of view, how to design such a context. Niss (1992) claimed that an authentic context is "one which is embedded in a true existing practice or subject area outside mathematics, and which deals with objects, phenomena, issues, or problems that are genuine to that area and are recognised as such by people working in it" (p. 353). At the same time, contextualised tasks are expected to be compatible with assessment demands, thus they also have to be meaningful and informative (Van den Heuvel-Panhuizen, 2005).
The Realistic Mathematics Education (RME) movement, which was grounded on the work of Freudenthal (1973; 1991) has taken the above considerations seriously, and this has resulted in a large amount of studies. In most of these studies, the issue of how to design a 'good' realistic task is of central importance. A first potential concern was identified by Boaler (1993), who cautioned mathematics educators not to design tasks that "have little in common with those faced in real life" (p.343), because they are merely school problems 'dressed'/disguised with a thin coating of 'real world' elements. Palm (2009) addresses the same issue, proposing a framework that "comprises a set of aspects of real-life situations that are reasoned to be important to consider in the simulation of real-world situations" (p. 8). The aspects of real-life situations mentioned are:

- event: "the event described in the school task has taken place or has a fair chance of taking place" (p. 9);
- question: "being one that actually might be posed in the real-world event" (p. 9);
- information/data: it should match the one in the real situation;
- presentation: the language used, together with other forms of visual representations;
- solution strategies: "the match in the strategies experienced as plausible for solving the task in the school situation and those experienced as plausible in the simulated situation" (p. 11);
- circumstances: available tools that may assist the solver, time restrictions, as well as possibilities for solving the task alone or in collaboration;
- solution requirements: should be consistent with the real life situation;
- purpose in the figurative context: it has to be clear enough to the solver.

The above considerations constitute a comprehensive framework for realistic task design. However, we should not overlook the fact that sometimes there is a discrepancy between the aims of the task as perceived by the designer, the
actual presentation/implementation of the task by the teacher and the task solution by the students (Clarke \& Roche, in press). Thus, in order to fully grasp the phenomena that may intervene from the task design to the students' solutions, one has to consider a multitude of factors, including among others, the classroom social and sociomathematical norms (Yackel \& Cobb, 1996), as well as the teacher's subject matter and pedagogical knowledge (Shulman, 1986).

In the present paper, we choose to focus only on the context of the task. Particularly, and according to our research questions, we were interested in whether some of the aspects of real-life situations have affected our students' solutions/solving. Furthermore, we focussed on the distance of the context to the students' lives: with the closest being the one related to the students' daily life, followed by school, sports and work, then by the local community and with furthest being the scientific contexts (De Lange, 1999). Moreover, the relevant research has not led to conclusive results on the relationship between the context's distance to the students' lives and their performance, although there is the assumption that the underperforming students prefer contexts closer to their daily life, because these contexts are more supportive in the students' comprehending of the proposed task (De Lange, 1999).

## CONTEXT AND METHODOLOGY

The participants of our study were seven students - four girls and three boys attending a primary school in an urban part of Rzeszow, Poland. All students were at grade 6 (12-13 years old), who volunteered to participate in a series of additional mathematics classes (taught by the second author of the paper), including the one which is the focus of the paper. Their mathematical abilities varied considerably from a girl who was characterised as mathematically gifted by her teacher to a boy who was characterised as inactive by the teacher, although he was eventually offering some fruitful ideas. The research session lasted 45 minutes. The students were given three tasks, each one in a separate worksheet. After working on every task, a discussion was initiated with the researcher (the second author of the paper) about the solutions. At the end all students were asked about the task they preferred and why. The whole process was video recorded and then the discussions that took place were transcribed.
Our choice of tasks was based on our research aims, given the time restrictions. Thus, we had to choose a limited number of tasks; we concluded that three tasks would suffice for the given time, including the planned discussions. Then, based on relevant studies we firstly decided to choose tasks that would contain varying contexts, i.e., coming from different aspects of everyday life. Following Palm's (2008) framework, the tasks were differentiated in terms of the possibility of the events described, while at the same time, we were careful that the questions posed matched those that could be posed in real life. As we will show below, the contexts chosen were: work, scientific/school context and everyday life - these were chosen in order to ensure the variation in their distance from the students'
lives. Each context posed different realistic constraints to the potential solvers, but the form of presentation was the same in all tasks. In one case (Task 2) the information/data given did not match those that one might expect in reality. Additionally, the type of context affected the nature of mathematical activities needed in order to reach a solution.

Work Task 1: The elevator.
This is the sign in a lift in an office block:


In the morning rush, 269 people want to go up in this lift. How many times must it go up?

Task 1 initially appeared in the English secondary testing programme (Schools Examinations and Assessment Council, 1992) and according to the proposed marking scheme, only the answer "20 times" was considered correct. In order to reach that answer the students had not only to successfully perform the division 269:14=19.21, but also consider some realistic (and pseudo-realistic) factors: "lifts go up in whole numbers, the lift never has fewer, where possible, than 14 passengers or ever more than this, and no-one use the stairs (Cooper \& Harries, 2009, p. 94).

School Task 2: Traffic.
The children in Year 6 of a school conduct a traffic survey outside of the school for 1 hour.

| Type | Number that passed in one hour |
| :---: | :---: |
| car | 13 |
| bus | 8 |
| lorry | 50 |
| van | 10 |

When waiting outside the school they try to decide on the likelihood that a lorry will go by in the next minute.

Put a ring round how likely it is that a lorry will go by the next minute.

| certain | very likely | likely | unlikely | impossible |
| :--- | :--- | :--- | :--- | :--- |

They also try to decide on the likelihood that a car will go by in the next minute.
Put a ring round how likely it is that a car will go by the next minute.

| certain | very likely | likely | unlikely | impossible |
| :--- | :--- | :--- | :--- | :--- |

Task 2 is a revised task, taken from the English primary tests (Schools Examinations and Assessment Council, 1993). In the original version the numbers of vehicles recorded in one hour were: 75 cars, 8 buses, 13 lorries and 26 vans. What was noticed was that in order to correctly respond to that task, the students did not have to consider the given data at all; they merely had to draw "on their knowledge of typical frequencies of cars, lorries, etc., in their everyday worlds" (Cooper \& Harries, 2009, p. 95). In order to avoid the "false positives" produced by that task (i.e. correct responses not based on the given data), Cooper and Harries (2009) revised it in the form shown above. Another reason for the particular choice was that students were not taught probabilities at school, but they were familiar with the term.

Everyday Task 3: The airport.
You need to arrive at Krakow-Balice international airport at 18:00 to pick up a friend. At 16:00, you left for the airport that is 180 km away. You drove the first 90 km in an hour. Will you be on time?


A similar version of Task 3 was included in a study on whether the task's expressed goals have led the students to considering realistic factors in their solutions (Inoue, 2008). The consideration of realistic factors (such as the traffic at the particular time and place) may affect remarkably the response given.
As soon as the solving process of each task was completed, in which students worked individually, a discussion was initiated by the researcher. Among the questions asked were (Inoue, 2008):

- What was your answer in the task?
- How did you reach your answer?
- Imagine yourself being in the situation described in the task. What would you do?
- Is the given task realistic according to you?
- Which of the given tasks did you like the most and why?

According to our research questions, our analysis was based on students' written solutions and their contributions to the discussion that took place. In order to analyse this rather disparate set of data we deployed methods from the studies mentioned in the theoretical section of the paper. Particularly, following Inoue (2008), we firstly categorised the solutions into calculational (CL) and realistic (R). A solution was considered calculational if the student merely performed one or more operations and provided the result of the operations as the answer to the task. A solution was considered realistic if the student considered some realistic factors before providing the answer. Then, based on the discussion that followed, we expanded the solution categories according to the following scheme, adapted from Inoue (2008); the adaptation was necessary, since the original categories were proposed for a clinical interview:

- calculational (CL): refers to the students who provided a calculational answer and while they admitted that they ignored realistic factors, they did not justify their initial answer; we also assigned that code to the students who did not participate in the discussion;
- reflecting a shared understanding of reality (SR): refers to the students who provided a realistic answer;
- reflecting a personal understanding of reality (PR): refers to the students who provided a calculational answer and during the discussion they provided a justification based on their personal understanding of the context.
- recognition of realistic constraints ( RR ): refers to the students who although were aware of the realistic constraints of the problem, they chose to ignore them since they are not explicitly stated.
At the same time, we considered particular aspects of the tasks, in order to examine any relations between them and students' consideration of the contexts. The results of that process are presented in the next section.


## RESULTS

As mentioned in the previous section, we firstly analysed students' written solutions in the tasks and categorised them into calculational (CL) and realistic (R). The results are summarised in Table 1.

| Students | Work Task 1 |  | School Task 2 |  | Everyday Task 3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | answer | cat. | answer | cat. | answer | cat. |
| S1 | 18 | CL | very likely | CL | yes | CL |
|  |  |  | likely |  |  |  |
| S2 | 20 | CL | likely | CL | no | R |
|  |  |  | likely |  |  |  |
| S3 | $\sim 20$ | CL | unlikely | CL | yes | CL |
|  |  |  | likely |  |  |  |
| S4 | 10 | CL | very likely | CL | yes | CL |
|  |  |  | likely |  |  |  |
| S5 | 22415 | CL | unlikely | R | yes | CL |
|  |  |  | certain |  |  |  |
| S6 | 20 | CL | very likely | CL | depends | CL |
|  |  |  | unlikely |  |  |  |
| S7 | 16 | CL | likely | R | depends | R |
|  |  |  | likely |  |  |  |

Table 1: Students' written solutions
We may firstly note that the vast majority of written solutions (17 out of 21) is calculational. Especially in Task 1 all solutions were calculational, based on the division 269:14. It is noteworthy that only three were mathematically correct;
among the mistaken ones the most unanticipated answer was 22415. In Task 2 we encountered two types of calculational solutions: the first type contained a calculation of vehicles per minute and the second was based on merely comparing the numbers of vehicles in an hour. The variety of calculational answers is remarkable. Here are two examples of these types of solutions:

S6: A car passed only 13 times during 1 h , which gives us 1 car per 4.5 min . That is why it is rather impossible for a car to pass in the next minute.
S1: It is very likely, because the number of passing lorries is the largest.
Concerning the two realistic responses in Task 2 below we can see a student's responses on the two posed questions:

S5: It is unlikely, because next to the school there are no lorries passing.
S5: It is certain, because next to the school there are many cars.
In Task 3 the calculational responses were mostly based on calculating time by using the distance and the speed of the car, which was considered constant throughout the journey by all students except one. Figure 1 shows a characteristic example of such solution (S1):


Figure 1. A calculational solution of Task 3
The realistic solutions in Task 3 contained some considerations about traffic and expected traffic jams on the way, as well as in the city of Krakow. A characteristic example is the following:

S7: Yes, however, traffic jams can appear on the way, so we are not $100 \%$ sure that we'll be on time. Even if we are late, our friend will wait.
After the analysis of the written responses was completed, we performed the analysis of the transcribed discussion that took place after each task's solution. Table 2 summarises students' consideration of the tasks' contexts in their written responses (already presented in Table 1) as well as in their participation in the classroom discussions.

| Students | Task 1 - Elevator |  | Task 2 - Traffic |  | Task 3- Airport |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | written | discussion | written | discussion | written | discussion |
| S1 | CL | SR | CL | PR | CL | PR |
| S2 | CL | PR | CL | PR | R | SR |
| S3 | CL | SR | CL | PR | CL | RR |
| S4 | CL | SR | CL | RR | CL | RR |
| S5 | CL | - | R | CL | CL | CL |
| S6 | CL | - | CL | - | CL | SR |
| S7 | CL | - | R | SR | R | SR |

Table 2: Students' consideration of realistic constraints before and during discussions
As seen in Table 2, some students did not participate in the discussion for particular tasks, thus we could not assign any category to their participation. What is quite clear though, is that in all but two cases (which both refer to S5) the students switched their approach and considered the tasks' context, either in a personal or in a shared way of understanding.
After completing the individual solving of Task 1, a discussion was initiated. The students were firstly asked to present their answers. Student S2 clarified that her response (20) represented the minimum number of times that the elevator should go up:

S2: Dividing by 14 doesn't mean that 14 people have to go every time, once can be 6 people going up.

This response was categorised as reflecting a personal understanding of reality, since S2 provided a justification of her response based on her understanding of the situation. The interesting thing though, was that this utterance triggered a reconsideration of the context of the task by other students; this was enhanced by the researcher, as shown in the exchange that follows:

R: Imagine that you are one of these 269 people. You are entering the building, there is the elevator and there are 268 other people. What do you do?
[No reply by the students for 5 seconds]
S3: We are waiting for everybody to go up, until it will be our turn.
S4: We are pushing ourselves to the front!
S1: We are going by the stairs.
After this exchange, all four students who participated in the discussion realised that the problem's solution is not as straightforward as initially thought and that 20 is just an indicative number of times; the actual number can be bigger or smaller than that.

During the discussion for Task 2, the students initially presented their solutions. Then the following discussion took place:

R: Okay, so when you look at the task, at the table... Does it make sense?

## [All students look at the table]

S1: Usually there are more cars than lorries.
R: So, why didn't you consider it in your answers?
S1: I did it because the number of cars is the second biggest number.
S3: It also depends on time because lorries like to go more during the day and rest during the night, although sometimes the opposite. It depends also where... If we go to the motorway, you need to see if there is traffic jam or...
R: But it's right out of the school.
All students immediately reacted to the last comment of the researcher, showing that they realised a significant factor of the problem. Then the following exchange took place:

R: Hasn't anybody thought that these are quite a lot of lorries?
S4: No, because I am used to.
R: To what?
S4: That in mathematical tasks a person buys, for example, 50 watermelons.
The last utterance by S4 demonstrates not only his attitude towards school mathematics, but also his reluctance to consider the real life factors in the given context - although he was aware of them. The same attitude was expressed by student S4 in Task 3 as well, thus was categorised as a recognition of realistic constraints.
Another interesting case in Task 2 was student S5, who switched from his realistic written solution mentioned before to a calculational approach, by referring merely to the table provided:

S5: It is very likely, because there were many lorries.
S5: It is likely, because there were few cars.
In Task 3 we have encountered the most active participation by the students; it is noteworthy that the researcher did not have to give any prompt to the students to imagine themselves in the given situation. This is because this context, as we anticipated, proved to be the closest to the students' everyday lives. This fact led them to easily identify realistic factors that might affect the answer to the task:

S6: Depends on the speed.
S2: The same with S6.
S3: If there will be traffic jams. But we should be on time because there's nothing written about it. [in the task]
Additional factors mentioned were: higher speed (even if it is against the law), accidents, broken car, broken tire and busy city. There were still cases though, like student S 1 , whose participation reflected a personal understanding of reality. Despite the fact that she mentioned possible traffic jams or accidents on
the way, she claimed that these happen rarely; so, if she maintained a constant speed of $90 \mathrm{~km} / \mathrm{h}$ she would be on time at the airport.

## DISCUSSION

Our study aimed to shed light on phenomena related to contextualised mathematical tasks. Bearing in mind that the students while solving such tasks demonstrate a "suspension of sense-making" (Schoenfeld, 1991), we provided three contextualised tasks to seven students and examined the effect of varying contexts to the solutions. We have seen that students without the teacher's guidance (thus in their written responses) were eager to provide calculational solutions, without any consideration of the realistic constraints of the situation. Thus, the "suspension of sense-making" has been evident throughout the tasks, and students seem bounded by the prevailing notions of mathematical truth and objectivity.
Another finding (although not among our research aims) is the variety of solutions in Task 2; this is probably due to students' informal knowledge of probability. We have also encountered a considerable amount of mathematical mistakes in Task 1, although the required operation could not be considered demanding.
As the results from the whole class discussions have demonstrated, many students were capable of considering the realistic constraints of contextualised tasks, as soon as these were highlighted by the researcher. Thus, the role of the teacher becomes prominent in two aspects: contextualised mathematical problems should be carefully designed, and continuous prompts should be made for realistic interpretations of the given situations.

Seeing students’ performance among tasks and especially their eagerness to (re)consider the realistic aspects of the given situations, we may claim that the students performed better in Task 3 - The Airport. Particularly, they were able to identify more realistic factors that might affect the task solution. Our assumption is that this was due to the context's closeness to the students' everyday experiences. The nature of our case study does not allow for safe generalisations; a larger amount of data is needed for that. Of particular interest would be a study in which varying tasks with varying contexts would be provided to a bigger group of students. A larger group of students would also allow for the examination of other factors, such as gender and socioeconomical background.

However, and in line with relevant research (e.g., Inoue, 2008), we believe that open-ended, contextualised problems given by an informed teacher can encourage students to interpret the given data, reconstruct the described situations and try to reach a meaningful and sensible solution. Moreover, if we accept as an aim of mathematics education the appreciation of creativity,
aesthetics, risk taking and non-typical thinking (Sriraman, 2005), we need to provide ample such opportunities to our students from the early school years. Appreciating these dimensions of mathematics can strengthen its links to the real world and help contest mathematical anxiety as well as views that mathematics has little to do with the real world.

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# NON-STANDARD AND STANDARD UNITS AND TOOLS FOR EARLY LINEAR MEASUREMENT 

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#### Abstract

Taking into consideration that research results on kindergarten children's capabilities in linear measurement are not always in agreement, we assumed that the auxiliary means used for early linear measurement may play a crucial role. To investigate kindergarten children's actions when using non-standard and standard units and tools for linear measurement, we conducted an inquirybased classroom experiment in which the designed task created an environment for children to investigate linear measurement in teams. The results showed that through children's measurement actions specific parameters arose that varied according to the different characteristics of units and tools.


## INTRODUCTION-THEORETICAL FRAMEWORK

Length is one of the main magnitudes in the content area of measurement in early childhood mathematics curriculums (Smith, Tan-Sisman, Figueras, Lee, Dietiker \& Lehrer, 2008; Smith, van den Heuvel-Panhuizen, \& Teppo, 2011). Its importance is highlighted by the fact that it is the simplest form of measurement (quantification of continuous quantities) and thus is considered an accessible and understandable magnitude even by young children (Tan-Sisman \& Aksu, 2012). It is also fundamental for perceiving other magnitudes, such as perimeter, area and volume; for connecting mathematical content areas for example number and geometry; as well as for linking mathematics to the real world that children live in.

The perception, comparison, and measurement of length as a magnitude, as a length or width of two-dimensional shapes, as a height of a three-dimensional shape, as a distance or as a movement between two points, is a slow and evolving process and develops through several stages (Sarama, Clements, Barrett, Van Dine \& McDonel, 2011). According to Clements and Sarama (2007) there are eight main concepts that are fundamental for children's understanding of length measurement, 1 . Understanding of the attribute of length. 2. Conservation of length. 3. Transitivity. 4. Equal partitioning of the object to be measured. 5. Iteration of the unit; the placing of the unit end to end alongside the object and the counting of these iterations. 6. Accumulation of distance; the number words of the counted iterations signify the space covered by the units up to that point. 7. Origin; any point on a ratio scale can be used as the origin, 8 . Relation between number and measurement. The sequence that these concepts are developed in is not commonly accepted yet, since it is influenced by age, experience and instruction. Different pedagogical approaches
do not seem to influence children's performance on linear measurement (Kotsopoulos, Makosz, Zambrzycka \& McCarthy, 2015) whereas the complexity of measurement tasks does (van den Heuvel-Panhuizen \& Elia, 2011).

Curriculums suggest starting to teach length with the qualitative perception of the concept using relevant words such as big-small, long-short, as well as the ability to make direct comparisons, such as length-based ordering of objects. After that they suggest continuing with estimations, with indirect comparisons and with the ability to quantify length, giving it a numerical value. Indirect comparisons can be made both by placing multiple units or by iterating a unit. Initially, non-standard units are used and then standard units. The final stage of teaching is the cultivation of the ability to use measurement tools, such as rulers (Ministry of Education 2010; NCTM, 2006; $\Pi \Sigma N, 2011$ ). This sequence of instruction which is also proposed by many researchers (Barrett, Cullen, Sarama, Clements, Klanderman, Miller, et al. 2011), is based on Piaget's theory of measurement. However, there is also research that suggests beginning the instruction with standard units and rulers, for an initial understanding of measurement, and a later introduction of non-standard units. This suggestion comes from the fact that young children show a preference for rulers and are able to use them before they fully understand the unit represented on rulers (Clements, 1999; Mac-Donald \& Lowrie, 2011; van den Heuvel-Panhuizen \& Elia, 2011).

Research results on kindergarten children's capabilities in linear measurement are not always in agreement. Most of the research suggests that young children have an intuitive understanding of length (Clements \& Sarama, 2007) and are able to make direct comparisons and classifications of objects according to their length (Barrett, Jones, Thornton \& Dickson, 2003; Clarke, Cheeseman, McDonoug \& Clarke, 2007). They perform length estimations by activating the cognitive processes of holistic visual recognition, classification and unification (Van den Heuvel-Panhuizen \& Elia, 2011). They can measure the objects’ length by following the necessary procedures such as placing the units from one end to the next, without gaps and overlays, measuring the number of units and communicating the result of the measurement (Sarama et al., 2011).
However, there is also research, suggesting that young children use units in a non-systematic way and are not able to determine the length of an object (Barrett et al., 2003; Castle \& Needham, 2007; Clarke et al., 2007). Nevertheless, even if they measure length using an appropriate method, not all of them give the right numerical value. This is affected by the measuring material and the object to be measured, which could lead them to meaningless measurement results (Skoumpourdi, 2015). Children's main strategies in measuring length are the linear, the perimetrical and the spatial placement
strategy. Additionally, research results indicate that young children, although they show a preference for the use of rulers (Kotsopoulos et al., 2015), they find difficulties in using them methodically, despite their repeated use during teaching experiments (Sarama et al., 2011).
The most reported students' errors during length measurement are (Tan-Sisman \& Aksu, 2012): units overlapping, mixing length units with other measurement units, confusing the concept of perimeter with area, incorrect alignment with a ruler, starting from 1 rather than 0 , counting hash marks or numbers on a ruler/scale instead of intervals and focusing on end point while measuring with a ruler. Problems arise also when children have to iterate units-blocks to measure a length when blocks are fewer than the necessary (Kotsopoulos et al., 2015).

From the above mentioned we made the assumption that the role of the auxiliary means used in a linear measurement may be crucial. The type of the magnitude to be measured, as well as the units and tools that are used for the measurement influence children's ability to measure accurately. Thus, the purpose of this paper is to investigate kindergarten children's actions when using non-standard and standard units and tools for linear measurement. The research questions posed were the following:

1. How do kindergarten children use anglegs ${ }^{1}$ and Cuisenaire rods ${ }^{2}$ as nonstandard units for linear measurement?
2. How do kindergarten children use a ribbon as a non-standard tool for linear measurement?
3. How do kindergarten children use snap cubes ${ }^{3}$ as standard units for linear measurement?
4. How do kindergarten children use a ruler as a standardized measurement tool for linear measurement?
5. What characteristics of the units and the tools used seemed to influence early linear measurement?

## METHOD

To investigate kindergarten children's actions when using non-standard and standard units and tools for linear measurement, an inquiry-based classroom

[^5]experiment took place. A pre-service kindergarten teacher, through a designed linear measurement task, created an environment for the children to explore, experiment with and investigate linear measurement in teams.
The task was implemented in a public kindergarten ${ }^{4}$, with 18 students (6 girls and 12 boys) divided in five teams of three or four persons. In the designed linear measurement task, children had to measure the length of the four sides of a field, for ordering a fence to protect the planted carrots. Common non-standard and standard units and tools for linear measurement, different for each team, were used to investigate children's actions. Anglegs and Cuisenaire rods were used as non-standard measurement units, because of their multiple sizes and colours, but also because anglegs could be snapped together, whereas Cuisenaire rods could not. A roll of 3 meters ribbon was used as a non-standard measurement tool that, due to its continuousness, covers a length easily. Snap cubes were used as standard measurement units because of their consistent size and their multiple colours. Also, a ruler was used as a standardized measurement tool. All the units and tools were familiar to the children with no specific knowledge of their used required, except for the ruler.

## RESULTS AND DISCUSSION

At the beginning of the process, and before children's separation into teams, the scenario and the carrot field were presented to the students, who were asked both to show what they should measure and to estimate the length of the fence. Children seemed to understand what they should measure, and a child showed how to do it by moving his hands and saying "this, all around". They seemed to be willing to estimate, but offered answers at random without much consideration. Their estimations were numbers with a measurement unit, such as " 3 meters", " 2 meters", " 10 meters", " 20 meters", etc. Because of the variety of the estimations, the need for a more accurate measurement came up. To the teacher's question about how they should measure in order to have an accurate result, all of them answered "with a ruler".

After that episode, children in teams had to measure the length of the fence and write the result of their measurement on a piece of paper. The first team had to measure with a ribbon, the second team with the anglegs, the third team with the cubes, the fourth team with the Cuisenaire rods and the fifth team with a 50 cm ruler.

[^6]
## Measuring with a ribbon

The first team, 2 boys and 2 girls, had to measure with the use of a ribbon. One of the boys, who had the ribbon in his hands, asked the teacher some clarifying questions about how they should measure, while one of the girls started to count the carrots. Counting objects is a common activity in their class. The teacher reminded to the girl that they had to measure the carrot field to order the fence and not count the carrots and at the same time she gave the initiative to the boy to decide with his team what to measure and how to measure it. The boy showed with his hands where to measure saying "here all around", and then, with the help of the other two team members he started measuring with the ribbon. They placed the ribbon around the carrots, holding the ribbon with their fingers firmly on the corners and saying, "approximately this much" (photo 1). To the teacher's question about what the result of their measurement was and what they were going to write on the paper, their answers varied from 1 to 15 meters. Finally, the team members came up with the result "4 meters". They did not determine the ribbon's actual length nor did they attempt to cut the ribbon to match the perimeter of the carrot field.

## Measuring with the anglegs

The second team, 3 boys and 1 girl, had to measure the carrot field with the use of a set of anglegs. The measurement started with a boy who placed 2 red pieces. Then he picked purple pieces, which were shorter, intending to place them beside the red ones but the girl preceded and placed a blue one beside the red ones. At the same time the other two boys placed multiple sized pieces along two other sides of the field. The former placed first 1 yellow, then 1 blue, then 1 yellow and finally 1 purple piece. The latter placed 1 blue, 1 blue and 1 yellow piece. The last piece to be placed was a matter of concern, because in the meantime the first boy had already completed the fourth side with 5 purple pieces but there was a small gap left. After several attempts, two of the boys filled the gap by placing 2 orange pieces (photo 2 ). To the teacher's question about what the result of their measurement was and what they were going to write on the paper, their answer was 17 , the number of pieces they had placed around the field, without regard to the different sizes.


## Measuring with cubes

The third team, 2 boys and 2 girls, had to measure the field with the use of cubes. Three of the children ( 1 boy and 2 girls) connected some cubes and placed them along one side of the field. The length of these cubes, however, was longer than the side of the field and a girl removed the excess number and continued placing cubes on another side. The two girls seemed to be concerned with the accuracy of the placement and perhaps for this reason they were careful to place the cubes exactly along the perimeter. Or perhaps they just wanted to use as many cubes from the box as they could (Photo 3). The second boy did not place any cubes, but he tried to count all the arranged cubes. To the teacher's question about what the result of their measurement was and what they were going to write on the paper, their answer was 56 , a number not corresponding to the actual number of cubes (90).

## Measuring with Cuisenaire rods

The fourth team, 2 boys and 1 girl, had to measure with the use of a set of Cuisenaire rods. One boy started the placement with the orange rods, which were the longest rods. He used them all (4) and he added 1 purple rod. Then he continued along the next side with 1 black, 1 blue, 1 brown, 1 dark green, 1 yellow and 2 white rods. At the same time the other two sides had already been covered by the rest of the team members: the boy had covered one side with 2 blue, 1 black and 2 yellow rods, while the girl had placed, 1 purple, 1 blue, 1 black, 2 dark green, 1 yellow, 1 light green and 1 purple rod along the other side (Photo 4). To the teacher's question about what the result of their measurement was and what they were going to write on the paper, their answer was 25 , the number of rods they had placed around the field, without distinguishing the different sizes.

## Measuring with ruler

The fifth team, 3 boys, had to measure with the ruler. The first boy placed the ruler along one side of the field in such a way that the side of the field matched the middle section of the ruler (the ruler was 50 cm and the side 40 cm ). The second boy pushed the ruler so that the its edge matches the side of the field's edge. Then, the first boy started counting imaginary units with his index finger ignoring the units on the ruler (Photo 5). The second student interrupted him and told him "No need to count, its 40", indicating the ruler's units. However, the first boy continued to use the same strategy on the next side starting from 40 and ending at 54 . To measure the 3 rd side he placed the ruler as he did in the beginning, counting his imaginary units along the length, 10 in total and continued along the next side in the same way, announcing " 20 " as the measurement's result. Essentially, the student acted on his own. The other children of the group, apart from the original correction and the placement of the ruler, did not interfere.

After the measurements with the units and tools the class had to decide on the final length of the fence, so the carpenter would know how much material would need. But the answers varied since each group reported a different number as the result of their measurement without any justification. To the teacher's question about which of these auxiliary means they considered to be the best for a measurement they replied that the best for measuring was the ruler because it had numbers on it.

|  | Non- <br> standard tool | Non-standard units |  | Standard units | Standard tool |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Ribbon | Anglegs | Cuisenaire rods | Cubes | Ruler |
| Strategy | Perimetrical along sides | Linear placement | Linear placement | Linear placement | Linear placement |
| Team work | Placement along sides/ holding on corners | Place pieces along sides/ separately | Place rods along sides/ separately | Place <br> cubes <br> along <br> sides/ <br> separately | Placement along sides by a student |
| Answers | $4$ <br> meters | 17 | 25 | 56 | $54+20$ |
| Measurement result | A length of the ribbon | Pieces ${ }^{5}$ : $2 \mathrm{r}, 2 \mathrm{o}, 3 \mathrm{y},$ $4 \mathrm{~b}, 6 \mathrm{p}$ | Rods ${ }^{6}$ : <br> $1 \mathrm{lg}, 1 \mathrm{~b}$, 2w, 3p, 3b, $3 \mathrm{dg}, 4 \mathrm{~b}$, $4 y, 40$ | 90 cubes | $\sim 170 \mathrm{~cm}$ |
| Time | 1.5 minutes | 3 <br> minutes | 2.5 <br> minutes | 9.5 minutes | 1.5 <br> minutes |

Table 1: Parameters that vary according to the auxiliary means and its use.
Children's measurement actions set parameters that seemed to vary according to the non-standard and standard units and tools specific characteristics and influenced the measurement results (table 1). Regarding the anglegs and the Cuisenaire rods, as non-standard units, it seemed that their multiple sizes did not seem to trouble the children who used them but counted them as if they were of the same size. The time they spent measuring with these materials was not long (2.5-3 minutes) and the numerical value they gave when quantifying their result

[^7]was meaningless because it was equal to the non-standard units' total number of pieces. The number of the pieces allowed all the children of the group to use them independently, without cooperating with each other. In the case of the ribbon, a non-standard tool, it was clear that although it made the measurement a quick and easy process, it was not helpful for quantifying the measurement's result. As for the cubes, standard units, children used them in the same way as the non-standard units, although the time they spent to complete the measurement differed significantly. The uniformity of their size could have led to a correct quantification of the measurement's result, but that did not happen, because of their large number. The ruler, a standard tool, also led to incorrect measurement results.

From the above results many questions arise: The method of measurement with the cubes is considered to be a correct one but should the method with the anglegs and the Cuisenaire rods are considered to be wrong? Since in both cases we do not know if children have any understanding of (non) standard units. If we choose to give to the children units of the same size, such as cubes and the children fit them correctly, measure them accurately and quantify their measurement, should we consider that children have understood linear measurement? Do they realize transitivity, equal partitioning of the object to be measured, iteration of the unit, accumulation of distance? And if the above does occur with the children of this age, should this be the starting point for teaching measurement in the kindergarten? Should also emphasis be given on the accuracy on the quality data of the measurement rather than solely to the numerical result?

## CONCLUSIONS

The classroom experiment showed that the children effectively used nonstandard and standard units, but not tools, to perform a linear measurement yielding a result that was logical for them. However, this result cannot be accepted as an actual result of measurement.
Through children's measurement actions specific parameters arose that varied according to the different characteristics of the units and tools used as auxiliary means for measurement. These were the strategy used, the type of the cooperation, the children's answers and the time spent for the measurement in relation to the measurement result.

As it is often suggested, children have to be educated in the use of the ruler, as a standardized tool. From this experiment it became clear that it is necessary to educate children also in the use of any auxiliary means used for measuring, and mostly in the announcement of an accurate measurement result with qualitative data related to the unit/tool used. Thus, we can add to the dilemma, about how to start teaching linear measurement from non-standard or from standard units and tools, the necessity to educate children how to use units and tools for linear
measurement, as well as how to quantify their results giving the qualitative data that come from the means they used.

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# USING YOUNG CHILDREN'S REAL WORLD TO SOLVE MULTIPLICATIVE REASONING PROBLEMS 

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This study focuses on kindergarten children's multiplicative reasoning. The participants were 12 children (5-6-year-olds) from Viseu, Portugal. Pre- and post-tests were used to assess the effect of an intervention program focused on multiplicative reasoning. The intervention program comprised 12 multiplicative reasoning problems and was carried on in four sessions, during three weeks. Children's performance and arguments were analyzed when solving selective problems of multiplication, partitive and quotitive division. The results suggest that children can succeed in some multiplicative reasoning problems, presenting valid or partially valid arguments, and that their multiplicative reasoning can be improved relying on their informal knowledge.

## INTRODUCTION

Children possess informal knowledge relevant for the learning of mathematical concepts. The mathematical ideas children acquire in kindergarten constitute the basis of future mathematical learning. Thus, the development of the mathematical skills in early age is crucial to the success for future learning (NCTM, 2008). In Portugal, the Curricular Guidelines for Pre-School Education (Silva, Marques, Mata \& Rosa, 2016) emphasize the importance of mathematics, in everyday life as in the structuring of the child's thinking, with a special focus on problem solving. In practice, it can be said that solving problems enables the development of thinking skills and stimulates a creative search for solutions to everyday problems. Children involvement in resolution of tasks and problem solving that allow different strategies, improve their mathematical reasoning (NCTM, 2017).
Concerning quantitative reasoning, literature reveals children's difficulty establishing a multiplicative reasoning, and the long period of time that is necessary to develop the ideas involved on it (see Vergnaud, 1983; Clark \& Kamii, 1996; Sullivan, Clarke, Cheeseman \& Mulligan, 2001; Siemon, Breed \& Virgona, 2005), contrasting with the relatively short time that is required to develop additive reasoning. However, there is evidence that many children have already an informal knowledge that allows them to solve some multiplicative reasoning problems (see Becker, 1993; Frydman \& Bryant, 1994; Nunes et al., 2007). Children can use their informal knowledge to analyse and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes \& Bryant, 1996). But they can also know quite a lot about multiplicative reasoning when they start school (Nunes \&

Bryant, 2010). Here, some research results are presented from a study focused on kindergarten children's multiplicative reasoning, in Portugal.

## THEORETICAL FRAMEWORK

Numbers are used to represent quantities and to represent relations. Nunes and Bryant (2010) argue that when numbers are used to represent quantities they are the result of a measurement operation from which a quantity can be represented by a number of conventional units (e.g., 3 children, 4 chairs). When a number is used to represent relations, the number does not refer to a quantity but to a relation between two quantities, expressing how many more or fewer (e.g., there is 1 more chair than children). In mathematics children are expected to be able to attribute a number to a quantity, which is measuring (Nunes \& Bryant, 2010), but they also are expected to be able to quantify relations. When quantities are measured, they have a numerical value, but it is possible to reason about the quantities without measuring them. In agreement with Nunes, Bryant and Watson (2010), it is crucial for children to learn to make both connections and distinctions between number and quantity. Quantitative reasoning results from quantifying relations and manipulating them (Nunes \& Bryant, 2010). Quoting Nunes and Bryant (2010), "[...] quantifying relations can be done by additive or multiplicative reasoning. Additive reasoning tells us about the difference between quantities; multiplicative reasoning tells us about the ratio between quantities." (p.8). In literature additive reasoning is associated to addition and subtraction and multiplicative reasoning is associated to multiplication and division problems (see Nunes \& Bryant, 1996; Vergnaud, 1983).

The fact that children learn about addition and subtraction before multiplication and division maintains the idea that multiplicative reasoning is accessible to children only when they already master additive thinking. This idea supports the notion of an additive phase predictive of multiplicative reasoning (Hart, 1981; Karplus, Pulos \& Stage, 1983; Piaget \& Inhelder, 1975). Piaget and Inhelder (1975) argued that there should be any superior qualitative transformation in children's thinking to understand and perform such complex operations as multiplication and division. Moreover, because some multiplicative problems can be solved with additive strategies such as repeated addition, it has preserved the idea that multiplicative reasoning depends totally on the additive reasoning, so, this should be consolidated first. However, understanding multiplication as a complicated form of addition is a very reductive way of realizing multiplicative reasoning.
In spite of his undoubted contribution to research, more recently research has been giving evidence of a different position. Thompson (1994), Vergnaud (1983) and Nunes and Bryant (2010) support the idea that additive and multiplicative reasoning have different origins. Vergnaud (1983), in his theory
of conceptual fields, distinguishes the field of additive structures and the field of multiplicative structures, considering them as sets of problems involving operations of the additive or the multiplicative type. Vergnaud (1983) argues that "multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects" (p.128). Nunes and Bryant (2010) also consider that additive and multiplicative reasoning have different origins, arguing that "Additive reasoning stems from the actions of joining, separating and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant." (p.11).
Multiplicative reasoning involves two (or more) variables in a fixed ratio. Thus, problems such as: "Joe bought 5 sweets. Each sweet costs 3p. How much did he spent?" Or "Joe bought some sweets; each sweet costs 3p. He spent 30p. How many sweets did he buy?" are examples of problems involving multiplicative reasoning. The former can be solved by a multiplication to determine the unknown total cost; the later would be solved by means of a division to determine an unknown quantity, the number of sweets (Nunes \& Bryant, 2010).

Research has been giving evidence that children can solve multiplication and division problems of these kinds even before receiving formal instruction about multiplication and division in school. For that they use the schema of one-tomany correspondence. Carpenter, Ansell, Franke, Fennema and Weisbeck (1993), reported high percentages of success when observing kindergarten children solving multiplicative reasoning problems involving correspondence $2: 1,3: 1$ and $4: 1$. Nunes et al. (2005) analysed primary Brazilian school children performance when solving multiplicative reasoning problems. When children were shown a picture with 4 houses and then were asked to solve the problem: "In each house are living 3 puppies. How many puppies are living in the 4 houses altogether?", $60 \%$ of the $1^{\text {st }}$-graders and above $80 \%$ of the children of the other grades succeeded. When children were asked to solve a division problem, such as: "There are 27 sweets to share among three children. The children want to get all the same amount of sweets. How many sweets will each one get?", the levels of success for $1^{\text {st }}$-graders was $80 \%$ and above that for the other graders ( $2^{\text {nd }}$ to $4^{\text {th }}$-graders).
In Portugal, there is still not much information about kindergarten children understanding of multiplicative reasoning, relying on their informal knowledge. This study focuses on children's ideas when solving multiplicative reasoning problems. It tries to address three questions: 1) How do children perform when solving multiplication, partitive and quotitive division problems? 2) What arguments children present to justify their resolutions?

## METHODS

An intervention program was conducted with 12 kindergarten children (5-yearsold, $\mathrm{n}=6$; 6-years-old, $\mathrm{n}=6$ ), from a public supported kindergarten in Viseu, Portugal. These children belong to an economic middle class group. Pre- and Post-tests were used to identify changes on children's understanding during the intervention. The study integrates a wider research program conducted by Soutinho (2016).

Individual interviews were used in the Pre- and Post-tests, and were conducted in a separate room in the Kindergarten, prepared for it. In each interview children solved 28 problems ( 18 additive structure problems; 6 multiplicative structure problems; 4 control problems). The problems presented in the interview followed an established order, and was the same for all children. Due to the higher number of problems, each child was interviewed in two different moments, during two straight days. The same procedure was use with all the children.

The problems presented to the children were selected and adapted from Vergnaud's classification (see Vergnaud, 1982, 1983). The problems of both tests were similar. The additive structure problems presented to children in the tests comprised: i) composition of two measures; ii) transformation liking two measures, with the starting and element of transformation omitted, ( 2 for addition, 2 for subtraction); iii) static relation linking two measures, ( 2 involving "more than", 2 for "less than"). The multiplicative structure problems in the tests comprised: iv) Isomorphism of Measures, selecting the problems of Multiplication, Partitive Division, and Quotitive Division. The control problems included only geometry tasks (geometric regularities, shape with tangram). The problems presented to the children in each test comprised two problems of each type. Tables 1 and 2 give, respectively, some examples of problems of additive and multiplicative structures presented to the children in the Pre- and Post-tests.

| Type of problem | Examples of problems of additive reasoning structures |
| :--- | :--- |
| Composition of | Mary has 8 dolls but only 2 are in the box. How many dolls are |
| two measures | outside the box? |
| Transformation | Bill had 7 marbles. He gave some to Paul and now Bill has only <br> liking two <br> measures |
|  | 4. How many marbles did Bill give to Paul? |
| There are 5 frogs in the lake. Some more join the group. Now <br> there are 8 frogs. How many frogs came to join the group? |  |
| linking two <br> measures | Anna has 4 puppies. John has 2 more than Anna. How many <br> puppies does John have? |
|  | Mary has 5 bananas and 2 strawberries. How many strawberries <br> are there less than bananas? |

Table 1: Examples of problems presented to the children in Pre- and Post-tests.

| Type of problem | Examples of problems of multiplicative reasoning structures |
| :--- | :--- |
| Partitive division | Sara has 10 candies to give to 5 children. She is doing it <br> fairly. How many candies is each child receiving? |
| Multiplication | Bill has 3 boxes with pencils. Each box has 4 pencils. How <br> many pencils does Bill have in total? |
| Quotitive division | The teacher Anna has 12 children in her group. She wants to <br> seat the children in groups in the tables. Each group must <br> have 4 children. How many tables does teacher Anna need? |

Table 2: Examples of problems presented to the children in Pre- and Post-tests.
All the problems were presented to the children by the means of a story, and materials were available to represent the problems. After each resolution, each child was asked "Why do you think so?" in order to reach a better understanding of his/her reasoning. All the information was registered in video. A quantitative analysis of Pre- and Post-tests results was conducted using the Statistical Package for Social Science (SPSS 20.0).
In the intervention, the participants were divided into three groups of four children each, having each the same age and Pre-test results conditions. The intervention took place in the pre-test following week and lasted for 3 weeks. Four sessions were planned, organized by level of difficulty, equal to all the groups. In each session children solved 3 problems, and the same kind of problems was explored twice a week. Each group had the opportunity to discuss and solve the same type of problem 4 times, in a total of 12 problems. The tasks presented to the children, during the intervention comprised 4 partitive division problems, 4 multiplication problems, and 4 quotitive division problems. The problems presented to the children in the intervention program were similar to those of the multiplicative structure problems given in the tests (see Table 2).
The interviewer presented the problems to the children orally by the means of a story. In each session, the interviewer presented each problem to the group and the material related to the context of the problem was available for representation. The children were challenged to solve the problem individually and present his/her response to the group. After each resolution, the interviewer asked questions related to their resolutions in order to gain an insight of children's reasoning and stimulate their discussion. All the information was video and audio recorded. Qualitative methods were used to analyse children's interviews when solving the problems.

## RESULTS

## Children's performance in solving problems

One point was awarded to each child's correct response. Children's performance in solving Pre- and Post-tests problems was analysed to understand the effects of
the intervention on the children's performance. Table 3 presents the mean of proportions (and standard deviation) of correct responses for Pre- and Post-tests, according to each type of problem.

|  | Mean (s.d.) |  |  |
| :--- | :---: | :---: | :---: |
| Type of Problem | Pre-test | Post-test |  |
| Additive Structure | $.45(.22)$ | $.55(.21)$ |  |
| Multiplicative Structure | $.40(.31)$ | $.61(.31)$ |  |
| Control | $.77(.23)$ | $.77(.25)$ |  |

Table 3: Mean of proportions (standard deviation) of correct responses, in Pre- and Post-tests.

The Wilcoxon's Test reveals that children's performances improved significantly from pre- to post-tests in both the additive structure problems ( $\mathrm{W}=50.000 ; \mathrm{p}<.05$ ), and in the multiplicative structure problems $(\mathrm{W}=42.000$; $\mathrm{p}<.05)$. No significant improvements on children's performances were observed regarding problems of control, despite the higher level of success in these kinds of problems. This indicates that the intervention on multiplicative structures problems was effective.
By focusing the attention on multiplicative reasoning problems, it becomes relevant to analyse children's performance when solving multiplication, partitive and quotitive division problems. Figure 1 presents the distribution of percentage of children's correct responses when solving these problems, in Pre- and Posttests.


Figure 1: Distribution of percentage of correct answers when solving problems.
The intervention focused on multiplicative reasoning problems seemed to improve children's understanding of multiplication, but also partitive and
quotitive division. Regarding the multiplicative structure problems, the Multiplication problems seemed to be easier for children to understand than division problems. Quotitive division problems revealed to be the most difficult ones for children. Nevertheless, some improvements were observed with the intervention. According to Friedman's test, in post-tests there are significant differences between Multiplication and Quotitive division problems ( $\chi_{\mathrm{F}}^{2}(2)=$ 7.786; $\mathrm{p}<.05$ ). Friedman's test also revealed that differences between children's performances in Pre- and Post-test are only significant in Multiplication problems, ( $\mathrm{W}=28.000 ; \mathrm{p}<.05$ ). Thus, this intervention program seemed to be effective for children understanding of multiplicative reasoning problems.
In order to clarify that children's performance was not reached by chance when solving the multiplicative reasoning problems, their arguments were analyzed as they were always challenged to explain their answers.
Children's arguments after solving the problems
After solving each problem in Pre- and Post-tests, children's verbal explanations were required when asked "Why do you think so?". An analysis of children's arguments was conducted among those who solved the problems correctly, in order to have an insight of their reasoning when solving the tasks. Four categories of arguments were distinguished when solving the multiplicative structure problems: valid argument (V), comprising an explanation that articulates correctly all the quantities involved in the problem; partially valid argument (PV), comprising an explanation in which a child attends only to part of the quantities of the problem, producing an incomplete argument; no argument (NA), comprising expressions such "I don't know", and the absence of an argument; and invalid argument (I), comprising an explanation that could not be understood or is decontextualized from the problem. Table 4 summarizes the frequency of type of argument given by the children when solving multiplicative reasoning tasks correctly.

| Type of argument | Pre-test (\%) | Post-test (\%) |
| :--- | :---: | :---: |
| Valid | 48.3 | 50 |
| Partially Valid | 20.7 | 13.6 |
| Invalid | 27.6 | 25 |
| No argument | 3.4 | 11.4 |

Table 4: Type of arguments presented for correct responses, in Pre- and Post-test.
Many children presented valid arguments in the explanations of their correct resolutions, revealing an understanding of the problem. In many cases, the partially valid arguments were presented using material, representing the situation correctly, in spite of the difficulty in the verbal communication. To present an explanation is not a simple task for young children; some children
were able to solve the problems correctly providing no explanation at all. This can be explained by the difficulty that some children have in putting into words their reasoning. This difficulty was expected among children of ages 5 and 6 as, according to Piaget (1977), they have to reflect upon their action. With the intervention, children seemed to become more aware of process and explanation, being more prone to be quieter, giving no explanation, than to give an answer that was not compatible with their procedure. The decrease of invalid arguments and the increase of valid arguments after the intervention discards the possibility of the success in problem solving be achieved randomly.
Also children's explanations in the intervention sessions when solving the multiplicative structure problems revealed some improvement on their way of thinking. The following Transcript gives evidence of children arguing solving the multiplication problem "Bill has 3 bicycles without wheels in his garage. Each bicycle must have 2 wheels. How many wheels does Bill need to fix all the bicycles?". Figure 2 shows children presenting their arguments solving the multiplication problem.

Child 1, 3: Two.
Child 2: $\quad$ Six.
Child 4: [Remain in silence.]
Child 1: It's 2 ! [Argues while getting 2 wheels to represent it.]
Researcher: There are 2 wheels in each bicycle... Show me why do you think so?
Child 2: I got 2 [shows it with paper material] and it is only for 1 , got more and it is for 2 [takes more 2 paper wheels], got 2 more and is for 3 [takes 2 more paper wheels].
Researcher: So, how many wheels do you need?
Child 1,2,3: Six!
Child 4: It's $4 \ldots$ No... Two are for 1, and more wheels are for another [put it below the previous ones], and these are for bicycle 3 [put them below the last ones].


Figure 2: Children presenting their arguments when solving a multiplication problem. In many problems, the material was mostly used by children not to solve the situations but to explain their resolutions.

## FINAL REMARKS

This study explores the effects of a short intervention program focused on multiplicative reasoning on young children solving problems of additive and multiplicative structure. The intervention was effective as children improved their understanding of multiplicative reasoning problems. Multiplication problems revealed to be easier for children than division ones. Also children's arguments revealed improvements. Young children provided arguments and explanations that sustain the idea that their successful resolutions were not obtained randomly.
Previous research carried out with kindergarten children solving multiplicative reasoning problems (see Carpenter et al., 1993) reports levels of success, but does not refer to children's explanations or arguments to give a better insight of children's way of thinking. Also Nunes et al. (2005) report remarkable success levels when $1^{\text {st }}$-graders solve multiplication and division problems, but give no reference to their explanations. The study reported here gives evidence that young children can reach success levels when solving multiplication and division problems, relying on their informal knowledge, presenting arguments that show that they are able to establish the correct reasoning when solving the tasks, articulating properly all the quantities involved in the given problems.

This study suggests that children's multiplicative reasoning can be enhanced when they can experience problem solving being able to interact with peers and discuss their ideas, after receiving some prompts from teacher, and the problems are presented by means of stories. It also suggests that both additive and multiplicative reasoning, in their simplistic forms, seem to be simultaneously reachable to kindergarten children, making sense to them. Thus, perhaps kindergarten mathematics should include more of these experiences in order to develop children's mathematical reasoning. When problems are presented to young children through a story connecting them to the children's real world, the mathematics make sense for children.

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# YOUNG CHILDREN CAN LEARN TO REASON AND TO NAME FRACTIONS 

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This study investigates the effects of a teaching intervention on children's reasoning and labelling of fractions in Quotient, Part-whole and Operator situations. A Pre-test, Intervention and Post-test design was used with 37 six- to seven-year-olds from Primary schools in Braga, Portugal. The children had not been taught about fractions in school. Reasoning and labelling questions were presented in the three situations in the Pre- and Post-test. During teaching, each intervention group learned about fractions in only one of the three situation. Children who were taught in the Quotient situation made significant progress in the reasoning and naming fractions; Children taught in the Part-whole or in the Operator situations only learned how to label fractions.

## INTRODUCTION

Fractions can be used to represent quantities in different types of situation. The aim of this study was to investigate the impact of the situation in which fractions are taught on children's learning. Three types of situation were included: Quotient, Part-whole and Operator. In quotient situations, a/b represents the relation between a number of items shared equally among $b$ number of recipients (e.g., $2 / 3$ represents 2 chocolate bars shared fairly by 3 children); $\mathrm{a} / \mathrm{b}$ also represents the quantity received by each recipient (e.g., $2 / 3$ represents the amount of chocolate received by each child). In part-whole situations, $\mathrm{a} / \mathrm{b}$ represents the relation between $b$, the number of equal parts in which the whole is divided, and $a$, the number of these parts taken (e.g., $2 / 3$ of a chocolate bar means that the bar was divided into 3 equal parts and 2 of these parts were taken). In operator situations, which involve a set of discrete items taken as a whole, $b$ indicates the number of equal groups into which the set was divided and $a$ is the number of groups taken (Nunes \& Bryant, 2008).

Quotient situations involve sharing (Streefland, 1997; Mamede, Nunes \& Bryant, 2005), where the denominator and the numerator of a fraction involves variables of distinct nature, recipients and items being shared, respectively (Nunes et al, 2007). Part-whole situations involve dividing continuous quantities into equal parts, and the denominator and the numerator involve variables of the same nature (Nunes et al, 2007), respectively the number of equal parts into which the whole was cut and the number of those parts taken. Fractions in operator situations also involve variables of same nature, the denominator and the numerator refer to the number of equal groups initially made and the number of groups taken, requiring the child to ignore the number of elements of each
group. Although quotient, part-whole and operator situations may seem very similar to an adult, they may be perceived as quite different by children.

## FRAMEWORK

Previous work (Correa, Nunes \& Bryant, 1998; Kornilaki \& Nunes, 2005) on children's understanding of division has shown that children aged 6 and 7 understand that, the larger the number of recipients, the smaller the part that each one receives. So in sharing situations, they display some informal knowledge and are able to order the values of the quotient. It should be noted that these studies were carried out with divisions where the dividend was larger than the divisor. In the present study, all situations involve dividends that are smaller than the divisor. So it is necessary to see whether the children will still understand the inverse relation between the divisor and the quotient when the result of the division would be a fraction. The equivalent insight in part-whole situations - the larger the number of parts into which a whole was cut, the smaller the size of the parts (Behr et al., 1984), has not been documented in children of this age. Research is needed to know more about how do young children understand this inverse relation in situations where the divisor is larger than the dividend, when they do not have to deal with it numerically, but only make a judgement, similar to those required by Correa et al. and Kornilaki and Nunes in quotient situations.

There is little information regarding equivalence in quotient situations but Empson (1999) found some evidence for children's use of ratios with concrete materials when children aged 6 and 7 years solved equivalence problems. Concerning part-whole situations, Piaget, Inhelder and Szeminska (1960) found that children in this age level understand the equivalence between the sum of all the parts and the whole and some of the slightly older children could understand the equivalence between parts $-1 / 2$ and $2 / 4$ - if $2 / 4$ was obtained by subdividing $1 / 2$. Different informal strategies have been identified (drawing and shading, using knowledge from money situations) by other researchers but these were observed at later ages, after children had already received instruction on fractions.

Previous research on children's informal knowledge (Empson, 1999) shows that children aged 6 and 7 found it difficult to understand the operator concept, but this difficulty is reduced after receiving instruction. Research with older children, who received instruction on fractions (Behr et al., 1984; Post et al., 1985), shows that for some children the operator concept is still difficult. However, these studies were not focusing on children's informal knowledge and do not compare children's reactions across situations.
Thus, one still needs to investigate children's understanding of equivalence and ordering of quantities represented by fractions in distinct situations, before being taught about it in school. Although there are some studies on informal
knowledge, systematic and controlled comparisons between the quotient, partwhole and operator situations have not been carried out. These situations may seem very similar to an adult, but it is hypothesized that they are perceived as quite different by children as the meaning of numerator and denominator varies across situations. Thus it is predicted that, if children learn about fractions in one type of situation, they will not transfer easily what they have learned to the other two types of situation.

Literature presents some studies on the effects of situations in which fractions are used on children's understanding. Previous research shows that children perform differently in parallel items presented in the context of quotient and part-whole situations. For example, 8- and 9-year-old British children answered items about fraction equivalence in quotient and part-whole situations; when comparing $1 / 2$ and $2 / 4$, the rate of correct responses was $35 \%$ in part-whole and $66 \%$ in quotient situations (Nunes et al., 2007). Similar results were found amongst Portuguese children aged 6-7 years: when ordering fractional quantities, $42 \%$ of the answers were correct in part-whole and $61 \%$ in quotient situation; in equivalence items, $14 \%$ correct answers were observed in partwhole and $22 \%$ correct answers in quotient situations (Mamede, Nunes \& Bryant, 2005). In another survey Nunes and Bryant (2008) asked to 318 Year 4 and 5 pupils to judge whether the fractions $1 / 3$ and $2 / 6$ were equivalent, or not. The items were presented simply as numbers, without a context, in the context of part-whole situations, and in the context of quotient situations. Pupils were most successful in quotient situations ( $68 \%$ correct), followed by part-whole situations ( $41 \%$ correct) followed by numerical problems without context ( $39 \%$ correct). Similar results were obtained in a study with 8 - and 9 -year-olds in England, who had been taught about fractions in part-whole situations and attained $40 \%$ ( 8 -year-olds) and $74 \%$ ( 9 -year-olds) correct responses in partwhole problems; their rates of correct responses to the quotient questions were $71 \%$ and $83 \%$ (Nunes \& Bryant, 2011).

In Brazil, Campos, Magina, Canova and Silva (2012) compared the impact of intervention sessions focused on fractions in quotient, part-whole, operator and intensive quantities on 138 Brazilian $3^{\text {rd }}$ and $4^{\text {th }}$-graders. The authors refer that students of the quotient situation intervention group registered the higher improvement. More recently, Canova (2013) analysed the effect of a teaching experiment, comprising reasoning and naming fractions tasks with part-whole and quotient intervention groups, involving 378 fourth- to sixth-graders from Brazilian primary schools. The quotient intervention group performed better on the reasoning fractions problems, and the part-whole intervention group performed better in the naming of fractions.

These results strongly support the significance of the distinction between quotient and part-whole situations for educational practices. However, previous
studies did not investigate the consequences of teaching and learning about fractions in these different situations; teaching had been done in schools without the researchers' interference. The present study analyses the effects of teaching children about fractions in each of these types of situation in comparison to the others. It is hypothesized that what children learn about fractions is at first connected to the situation in which they were taught. If the situations are truly distinct from the children's perspective, their newly acquired knowledge will be situated rather than generalized. Thus further teaching and experiences with fractions would be required to allow for a more general understanding of fractions that can be used in a variety of situations.

## METHODS

Participants were 37 six and seven-year-olds (mean age 6.6 years) from two state supported primary schools, in Braga, Portugal. According to the information given by the teachers, the children had not received formal instruction on fractions at school. This study was carried out with un-instructed children, otherwise the results would be influenced by the type of instructions that they had received. In Portugal, the children contact with equal sharing activities in the $2^{\text {nd }}$ grade ( $7-$ to 8 -years-old) and were formally introduced to fractions in the $3^{\text {rd }}$ grade, and part-whole and operator situations were the most common ones to explore fractions in the $3^{\text {rd }}$ and $4^{\text {th }}$ grades.
Pre- and Post-tests, administered individually, were used to assess whether there was progress after the intervention. These tests comprised 12 reasoning items, involving equivalence or ordering fractions, presented in each type of situation quotient ( Qt ), part-whole ( Pw ) and operator ( Op ) - without the use of fraction labels. Figure 1 gives an example of an equivalence problem presented in the Pre- and Post-tests.

| Type of situation | Example |
| :---: | :--- |
| Quotient | Three boys are going to share 1 chocolate bar fairly. Six <br> girls are going to share 2 chocolate bars fairly. Does <br> each boy eat more chocolate than each girl? Does each <br> girl eat more chocolate than each boy? Or do the boys <br> and girls eat the same amount of chocolate? Circle the <br> one that you think that ate more or both if they ate the <br> same. Explain your answer. Write in the box a number <br> to show how much chocolate each girl (each boy) is <br> going to eat. |

Part-whole Betty and Ruth have each a chocolate bar. But as they are not very hungry, they decide not to eat all the chocolate bar at once. Betty divides hers into 3 equal

parts and eats 1 part; Ruth divides hers into 6 equal parts and eats 2 parts. Does Betty eat more chocolate than Ruth? Does Ruth eat more chocolate than Betty, or are they eating the same amount of chocolate? Circle the one that you think that ate more or both if they ate the same. Explain your answer. Write in the box a number to show how much chocolate each girl is going to eat.

Operator


1. Anna and Phil have each 12 sweets (first slide).
2. Anna decided to share hers into 3 equal bags, with the same number of sweets in each; Phil shares his into 6 equal bags, all with the same number of sweets (second slide).
3. Anna eats 1 bag of sweets and Phil eats 2 bags (third slide). Does Anna eat more sweets than Phil, does Phil eat more sweets than Anna, or do they eat the same number of sweets? Circle the one that you think that ate more or both if they ate the same. Explain your answer. Write in the box a number to show how much chocolate each one is going to eat.

Figure 1: Examples of an equivalence problem of the Pre- and Post-tests.
After solving the reasoning questions, the children were also asked to name the 12 pairs of fractions in each of these situations. Fractional language is relatively rare in Portuguese culture in everyday life. The most common fraction in everyday language is "metade" (half), but is often used to refer to a division in two parts without rigor in the equality of parts. So in order to examine whether children can adopt fractions signs in writing and oral language more easily in one type of situation than another, the children received a brief explanation of
how to use fractional representation and then were assessed on their ability to use these representations.

The same fractions were used across the different situations making it possible to compare the children's performance on reasoning and naming problems in each situation. Table 1 presents the pairs of fractions used in the problems of equivalence and ordering of quantities represented by fractions in the Pre- and Post-tests.

| Pre-Test |  | Post-Test |  |
| :---: | :---: | :---: | :---: |
| Equivalence | Ordering | Equivalence | Ordering |
| $1 / 3 ; 2 / 6$ | $1 / 3 ; 1 / 4$ | $1 / 3 ; 2 / 6$ | $1 / 3 ; 1 / 4$ |
| $1 / 2 ; 2 / 4$ | $1 / 3 ; 1 / 6$ | $1 / 2 ; 2 / 4$ | $1 / 3 ; 1 / 6$ |
| $3 / 5 ; 6 / 10$ | $2 / 3 ; 2 / 9$ | $3 / 5 ; 6 / 10$ | $2 / 3 ; 2 / 9$ |
| $2 / 3 ; 4 / 6$ | $2 / 5 ; 2 / 10$ | $2 / 3 ; 4 / 6$ | $2 / 5 ; 2 / 3$ |
| $1 / 2 ; 3 / 6$ | $3 / 4 ; 3 / 6$ | $1 / 4 ; 2 / 8$ | $3 / 4 ; 3 / 8$ |
| $3 / 6 ; 6 / 12$ | $4 / 5 ; 4 / 10$ | $3 / 4,6 / 8$ | $4 / 5 ; 4 / 10$ |

Table 1: Fractions used in the problems of equivalence and ordering of fractions in each condition of study for Pre- and Post-tests.
Children were randomly assigned to learning in one of the three situations Quotient (Qt), Part-whole (Pw), or Operator (Op) intervention - or to a Comparison group, who solved multiplication and division problems with whole numbers.

Eight groups of 5 children (one of them with 3 children) participated in two teaching sessions of about 35 minutes each. These teaching sessions took place outside the classroom, in a small room of their school. Thus, no changes on the curriculum were provided due to this intervention.
In the first session, the children received instruction on how to label fraction in their working situation, and then they had to solve 2 problems of labelling and 2 of ordering of fractions. In the instruction sessions on how to label fractions, the unitary fractions up to $1 / 5$ and the non-unitary fractions $2 / 3$ were used. After being taught to label the fractions, the children were asked to name the fractions in the subsequent labelling and ordering problems, and their answers were discussed in the group by the researcher. In the second session, the children had to solve 2 problems of equivalence of fractions. Table 2 summarizes the fractions involved in the intervention sessions when solving reasoning and naming problems.

| Naming | Ordering | Equivalence |
| :---: | :---: | :---: |
| $3 / 7$ | $1 / 2 ; 1 / 3$ | $2 / 3 ; 4 / 6$ |
| $5 / 8$ | $2 / 3 ; 2 / 4$ | $3 / 4 ; 6 / 8$ |

Table 2: The fractions involved in the problems used in the intervention sessions.

All problems were presented using an approach similar to the test items exemplified in Figure 1, in which the researcher showed the children an illustration while presenting the problem orally, and the children had a booklet with the same illustration in which they could write or draw as they wish.
The researcher presented the problem and then explained the question; each child answered in their own booklet. For the problems of labelling, each child had to write down the answer; for the problems of reasoning, they had to judge about the equivalence and ordering of fractions individually, drawing a circle around those that they considered to be having/eating more. When all the children had finished and all the answers were written dawn, each child had to say why they answered so. Finally, the researcher discussed their answers with the children of the group.

No judgements were made by the researcher whose role was to pose the questions, create opportunities for the children to present their individual responses to the group, and manage the group discussion.

## RESULTS

One point was awarded for each child's correct response, the maximum score on reasoning problems of fractions is 12 . Table 3 presents the means and standard deviations for accuracy on reasoning items in each situation by testing occasion. The means are separated by intervention group. At Pre-test (Table 3), all children performed better on reasoning problems presented in quotient situations, irrespective of the group to which they were later assigned. There were almost no correct responses to reasoning problems presented in part-whole or operator situations. At Post-test, children in the Quotient Intervention Group improved in accuracy in the quotient reasoning items, but no other improvement in reasoning is noticeable.

|  | Reasoning problems (Maximum score $=12)$ |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :---: |
|  | Pre-test |  | Post-test |  |  |  |  |
|  | Qt | Pw | Op | Qt | Pw | Op |  |
| $\mathrm{Qt}(\mathrm{n}=10)$ | $5.6(3.3)$ | 0 | 0 | $8.6(3.1)$ | 0 | 0 |  |
| $\mathrm{Pw}(\mathrm{n}=10)$ | $2.7(3.4)$ | $0.1(0.3)$ | 0 | $3.0(3.7)$ | $0.6(1.9)$ | 0 |  |
| $\mathrm{Op}(\mathrm{n}=10)$ | $2.5(2.6)$ | 0 | 0 | $3.8(3.7)$ | 0 | 0 |  |
| Control <br> $(\mathrm{n}=7)$ | $3.0(3.9)$ | 0.29 | $0.43(1.1)$ | $3.0(4.5)$ | $1.57(4.2)$ | $1.71(4.5)$ |  |

Table 3: Mean accuracy (standard deviations in brackets) by Testing Occasion on Reasoning Items in Each Situation by Intervention Group.
In the naming problems, one point was awarded to each fraction correctly named. The total score of naming problems ranged from 0 (minimum) to 24
(maximum). At Pre-test, no child was able to label a fraction correctly but there are improvements in the children's accuracy in labelling items in the post-test (Table 4).

|  | Labelling problems (Maximum score = 24) |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | Post-test |  |  |  |
|  | Qt | Pw | Op |  |
| $\mathrm{Qt}(\mathrm{n}=10)$ | $8.6(3.1)$ | 0 | 0 |  |
| $\mathrm{Pw}(\mathrm{n}=10)$ | $3.0(3.7)$ | $0.6(1.9)$ | 0 |  |
| $\mathrm{Op}(\mathrm{n}=10)$ | $3.8(3.7)$ | 0 | 0 |  |
| Control $(\mathrm{n}=7)$ | $3.0(4.5)$ | $1.57(4.2)$ | $1.71(4.5)$ |  |

Table 4: Mean accuracy (standard deviations in brackets) by Testing Occasion on Labelling Items in Each Situation by Intervention Group.

The improvements are selective: children in the Quotient Intervention Group improve their performance in naming fractions in Quotient situations being able to name more than half of the fractions presented in Part-whole and Operator situations. The children in the Part-whole and Operator intervention groups improve their accuracy in naming fractions in both types of situation and are able to transfer these learning to name fractions among Part-whole and Operator situations. Nevertheless, the Part-whole and Operator intervention groups find more difficult to name fractions presented in Quotient situations.
In view of the floor effects in pre- and post-test accuracy scores in reasoning items in Part-whole and Operator situations, it was only possible to analyse the effect of the intervention on reasoning items in Quotient situations. In order to analyse whether one type of intervention led to greater improvement than the other on Quotient reasoning items, an ANCOVA was carried out, controlling for the Pre-test. The score for the pre-test Quotient reasoning problems was a factor and Type of Intervention session (Quotient, Part-whole, Operator, Control) was a between-participants factor. The dependent variable was the score for post-test Quotient reasoning problems. The results showed that the covariate predicts significantly the children's performance in solving the Quotient reasoning items $(\mathrm{F}(1,32)=86.74, \mathrm{p}<.001)$. There was also an interaction of Quotient reasoning items by Session Intervention Group $(\mathrm{F}(3,32)=4.48$, $\mathrm{p}<.05)$. Post-hoc (Bonferroni) tests revealed that the Intervention Sessions on Quotient Situations significantly increased children's performance compared to both the Part-whole Intervention Session Group, $\mathrm{t}(32)=-3.15, \mathrm{p}<.05)$, and the Control Intervention Sessions Group ( $\mathrm{t}(32)=-3.19, \mathrm{p}<.05$ ), but not with the Operator Intervention Sessions Group ( $\mathrm{t}(32$ )=-2.07, n.s).

As there was no variation in the children's accuracy in naming fractions in the pre-test, only post-test performance can be analysed. A repeated Measures ANOVA was carried out, with naming problems as a repeated measure in Posttest and the Type of Intervention Group as a between participants factor. There is an interaction between the type of Group Intervention and the situation to name fractions, $(\mathrm{F}(6,66)=36.37, \mathrm{p}<.001)$; Children in the Quotient Intervention Group performed better on naming problems presented in Quotient situations than those of the Part-whole or Operator Intervention Groups ( $\mathrm{p}<.001$ ), but weaker on problems presented in Part-whole or in Operator situations; on naming problems in Part-whole situations, the children of both Part-whole Intervention Group ( $\mathrm{p}<.001$ ) and Operator Intervention Group ( $\mathrm{p}<.001$ ) performed better than the Control and Quotient Intervention Groups; on naming problems on Operator situations, children of both Part-whole ( $\mathrm{p}<.001$ ) and Operator Intervention Groups ( $\mathrm{p}<.001$ ) also performed similarly and better than Control and Quotient Intervention Groups.

Thus, the type of situation in which fractions are used to present the problems to children affects differently children's reasoning and naming of fractions.

## FINAL REMARKS

The findings of this study show that some changes occurred with the teaching experiment in which the children were introduced to fractions, in each type of situation analysed. The children who were introduced to fractions in Quotient situations improved their performance on reasoning problems, involving equivalence and ordering, revealing some understanding of the inverse divisorquotient relation. This understanding was also found previously in the literature (see Mamede, Nunes \& Bryant, 2005), when fractions were introduced to young children, but also when comparing fractions problems were solved by older children in Quotient situations (see Nunes \& Bryant, 2008; Canova, 2013). Contrasting with these findings, the children who were introduced to fractions either in Part-whole or Operator situations did not show improvement with the instruction sessions when solving reasoning problems. These findings suggest that Part-whole and Operator situations are very difficult situations for the children to attend to all the dimensions involved in the problem.

It is concluded that learning in Quotient situations was more effective, as the children progressed both in reasoning and naming items, but it was situated: there was no transfer. In contrast, learning in Part-whole and Operator situations was limited, as there was no progress in reasoning, but the use of fraction labels was generalized between the two situations.
Teaching about fractions in many countries is often done in part-whole and operator situations, with emphasis on learning to name fractions. Children easily learn to name fractions in specific situations, so it is easy to believe that they understand the reasoning underlying this new numerical form. This study
underscores the limitations of teaching in these situations and the need to combine different situations in teaching fractions, as each of them has strengths and weaknesses.

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# READINESS OF PRIMARY SCHOOL STUDENTS TO SOLVE MATHEMATICAL TASKS REQUIRING THE USE OF FORMAL OPERATIONS (PART OF RESEARCH) 

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#### Abstract

The article contains information about the preliminary results of pilot studies carried out by the author on 127 students with the age 14 in the period 20172018. The aim of the research was to check students' readiness to use formal operations while solving mathematical problems, as well as to check the correctness of the tool construction by means of which the author attempted to search for answers to the questions posed.


## INTRODUCTION AND THEORETICAL FRAMEWORK

Mathematics is "a science using the method of deduction, dealing mainly with the study of sets of numbers, points and other abstract elements" so modestly about mathematics is written in one of the dictionaries of the Polish language (Dubisz, 2003, p. 583). School Encyclopaedia from the 80s of the last century describes the motto of Mathematics up to 8 pages in small print, showing its development and showing the richness of mathematical concepts, their use and activity and skills in which a professional mathematician should be equipped to be able to handle tools flexibly and simultaneously this scientific discipline (Pańkowska, 1988, p.140). Finally, about the mathematics in the book Lectures on Mathematics Teaching, (Turnau, 1990) writes the famous Polish didactician of mathematics Stefan Turnau. He looks at mathematics - the school subject and, answering the question of his tasks in the process of developing a potential student, makes the reader aware of how different this object is from other taught at school. Well, here and on other objects, we develop by mastering a certain fragment of knowledge defined in the core curriculum, and even often transcending it. The knowledge gained on each subject should contribute to supporting us in action, based on its application. In mathematics, we say that the knowledge gained should help us in solving tasks and performing certain operations, it should be assimilated operatively, it should be flexible and functional. Are the skills developed in maths lessons only useful for solving school math problems? Well, no, good and operatively assimilated mathematical knowledge will certainly pay off not only in maths classes, but also in everyday life. After all, in the process of solving mathematical problems, you could develop the following skills:

- perceiving and using analogies;
- schematization;
- defining, interpreting a given definition and its rational application;
- deduction and reduction;
- coding, constructing and rational application of mathematical language;
- algorithmisation and rational use of algorithms (Krygowska, 1986).

Today it is difficult to consciously function in everyday life without having the above mathematics activities, although the vast majority of them are specific only to the work of a professional - a mathematician.

We are not born with the abovementioned skills, we acquire them and shape them on the path of intellectual development. This process, in accordance with the theory of the Swiss psychologist Jean Piaget, is a constant struggle with what we know and new information on the path of adaptation. It consists of two processes: assimilation and accommodation. The first one is adapting new information of external origin to what the person already knows and knows, the second is adapting his knowledge to new information. Man builds new patterns, his intelligence develops. There is a kind of competition between accommodation and assimilation, which is based on comparison, a process that is an important basis for the development of mental operations. According to Piaget, the child's development depends primarily on him, the very actions he undertakes, which underlie thinking, or a continuous cognitive process. It consists of intellectual operations, mental operations as an internalized action, which is internalized and therefore runs in the mind (Piaget, 1963, 1966, 1972, 2012).

According to Piaget, this internalization takes place in four separate stages of development of intelligence, linearly following each other:

- Sensorimotor (from birth to about 2 years of age), the child learns the world by perceiving directly through the senses and acting in space.
- Pre-operational (from 2 to about 6 years of age), then symbolic thinking is shaped, but intellectual possibilities are still dominated by observations and not thinking and using concepts.
- Concrete operations (from 7 to about 12 years old), the child uses operational reasoning, when he can manipulate particulars, tries to solve problems logically, loses self-centeredness.
- Formal operations (over 12 years of age) the child acquires the ability to reason with abstract thinking, is able to solve problems in the mind, thinking becomes more and more similar to the thinking of an adult man.

According to Piaget, everyone, regardless of their place of residence and the environment in which they grow up, goes through all the above-mentioned stages of development, in which reasoning changes from simple forms, strongly
related to perception and performed activities, to forms implemented in the mind, abstract and hypothetical. These are qualitative changes, not quantitative changes, new behaviours are built on previous ones and they do not cause their disappearance only they complement and correct.
Piaget's research shows that not all people, regardless of where they live and what they do, reach the level of formal operations, it is also true that many of us do not use formal operations for many aspects of their lives (PrzetacznikGierowska, Tyszkowa, 2000).
A person who thinks at the level of formal operations is characterized by:

- abstract thinking, i.e. the ability to logically use symbols in relation to abstract concepts (without the need to link them with reality), hypothetical-deductive reasoning and development of your thinking about abstract objects.
- metacognition, or the skill of parallel reasoning and its monitoring, it is the ability to constantly reflect on one's own cognitive process.
- the ability to logically and methodically solve problems, in particular those of mathematics.
The theory of Jean Piaget is embedded in development-cognitive constructivism. The child's / student's knowledge is actively created, not passively drawn from the environment. It is not the environment that gives shape to the child, but it actively pursues its understanding. He studies, manipulates and analyses objects and people in his own environment (Dylak, 2013, Gofron, 2013).


## GOALS, ORGANIZATION, METHODOLOGY AND TOOLS

In his intellectual development model, Piaget determined that over 12 years of age most students are already thinking at an early level of formal operations. However, from British research conducted in the seventies, on a sample of around 10,000 students aged 14 , it turned out that the vast majority of them, about $80 \%$ did not reach this level yet (Shayer, Kuchemann, Wylam, 1976).
There is a need to check the current state and level of reasoning used by Polish students aged around 14. Professor Edyta Gruszczyk-Kolczyńska, who for many years has been studying mathematically gifted children and those with specific difficulties in learning mathematics, writes in her books that one of the potential causes of Polish children's difficulties in the creative development of mathematics may be the shift in the age in which we begin to think with the use of formal operations, the author of the article is not familiar with research in which in recent years a discussion on this subject would have been undertaken in Poland (Gruszczyk-Kolczyńska, 1992, Gruszczyk-Kolczyńska, 2012). The author of this work has undertaken to examine the manifestations of formal reasoning that characterize Polish students.

The aim of the research was, among others, to check whether in the process of solving mathematical problems students:

- logically use symbols in relation to abstract concepts,
- they reason hypothetically and deductively,
- think about abstract objects, or do they perceive and do things?

The aim of the research was, among others, to check whether in the process of solving mathematical problems students:

- logically are using symbols in relation to abstract concepts,
- they reason hypothetically and deductively,
- think about abstract objects, or do they perceive and do things?

The article will present the results of some pilot studies. They were conducted in two rounds in 2017 on a group of 100 students from one of the Poznań gymnasiums (these studies were part of the research carried out for the diploma thesis of Mrs. Monika Drgas, prepared under the direction of the author of the article). The second part of the research was carried out in 2018 on a group of 27 students from two classes 7 from one of the primary schools in Poznań.

The research tool in the first round of tests consisted of three tasks, all of them come from a textbook for class 1, Gdańskie Wydawnictwo Oświatowe. The goal of each of these tasks was to run students' reasoning to check or justify a certain mathematical regularity or fact.

The tool used in the second stage of the study consisted of 6 tasks, all geometric ones, two of the tasks from the first stage were used in the second study and they will be analysed in this article. The students had exactly 45 minutes to solve the tasks in both the first and the second stage.

## Examples

## Task 1

Points A, B and C divide the circle with centre O into three equal parts. Justify that the triangles ABO and BCO are congruent.


## Task 2

Check what part of the rectangle's field is shaded field of the figure.


The first task was to, among other things, check whether students know the concept of congruent triangles. For the correctness of the solution, it was important to choose the appropriate congruence feature and justification for the choice made. The second task forced the student to analyse the conditions of the task, and then by reasoning in accordance with the state of knowledge available to answer the question.
As you can see in both tasks, it was necessary to use a symbolic language of mathematics for rational use of knowledge and launching hypothetical-deductive reasoning in order to obtain a full and correct solution.

## Analysis of task 1 solutions

## Round 1

During the first round of the pilot study, the content of the task was devoid of a drawing, which was attached in the second part. Over 60 works contained only a drawing, which was not always executed correctly, did not reflect the data conditions in the task. In 20 works there were no solutions. 15 students did not remember what triangles we call congruent, and 35 did not have an idea how to show it. Two students did not see the point of showing something that is obvious.

Below is a table with a quantitative description of the features that support launching formal reasoning.
Data in the table concerns 18 works, only those attempts have been made to solve the task.

|  | Feature |  |  |
| :---: | :---: | :---: | :---: |
|  | logically <br> use <br> symbols | they reason <br> hypothetically and <br> deductively | think about abstract objects / <br> perceive / perform activities |
| Amount <br> of <br> students | $13-$ yes <br> $5-$ no | $8-$ no <br> $5-$ manifestations <br> $5-$ yes | $9-$ abstraction |
| $6-$ perceive |  |  |  |
| $3-$ perform activities |  |  |  |

Table 1: Features of formal reasoning_task 1_junior high school students

## Example 1



Figure 1
This reasoning is correct. It is true that there is no commentary supplementing the full process of deduction, no indication of the moment of using the assumptions of the task, but the student showed that she has knowledge and can use it correctly. The mathematical symbols were used in the right way, the record shows the deductive path of reasoning. What is more, a person concludes on the basis of abstract knowledge about mathematical concepts, perhaps it is supported by insights, although the drawing is made in such a way that it is difficult to deduce specific regularities from it.

## Example 2



Figure 2
The student mistakenly equated the equality of the triangle fields with their congruention. He decided that the radius is equal to the height of the triangles analysed in the task. The student did not need to make an auxiliary drawing. Reasoning detached from the concrete, but incorrect and the knowledge used in a wrong way, the lack of manifestations of reflection on their own work and the belief that this is a well thought-out deduction.

## Round 2

10 out of 27 works do not contain a solution to this task. The remaining solutions are only an attempt to indicate in the figure some perceived regularities, the lack of a full and correct solution, which is confirmed by the data contained in the table below.

|  | Feature |  |  |
| :---: | :---: | :---: | :---: |
|  | logically <br> use <br> symbols | they reason <br> hypothetically and <br> deductively | think about abstract objects / <br> perceive / perform activities |
| Amount <br> of <br> students | $6-$ yes <br> $11-$ no | $12-$ no <br> $5-$ manifestations | $2-$ abstraction <br> $10-$ perceive <br> $5-$ perform activities |

Table 2: Features of formal reasoning_task 1_pupils of grade 7.

## Example 3

Opisz sposób rozwiqzania:



Figure $3^{1}$
The student drew two triangles, described them incorrectly using the uppercase letters used to mark the points indicated on the circle to indicate the length of the sides. The student marked it manually, without indicating the angle of the simple height of the data of the triangles, and then after measuring all the lengths necessary to calculate the triangles, he obtained the obtained quantities to the formula. The reasoning is not detached from the concrete, moreover, the student does not use the regularity and facts observed in the drawing, but only performs the measure of length. Congruation justifies incorrectly because of equality of fields.

[^8]
## Analysis of the task number 1

From the analysed solutions, it appears that students:

- they conclude, but most often based on some insights (from the figure),
- they conclude, but the facts based on which they carry out their reasoning are not always correct mathematical knowledge, they are not always knowledge about the analysed situation,
- students do not need to proof the regularities observed in the drawing,
- students incorrectly conclude,
- they mostly do not reason with deduction.


## Analysis of task solutions 2

## Round 1

Junior high school students correctly commented on their solutions. More than half of the students noticed that the area of the shaded figure is half of the area of the rectangle, but they could not justify it.

The data from the table 3 confirm the above thesis.

|  | Feature |  |  |
| :---: | :---: | :---: | :---: |
|  | logically <br> use <br> symbols | they reason <br> hypothetically and <br> deductively | think about abstract objects / <br> perceive / perform activities |
| Amount <br> of <br> students | $5-$ yes <br> $7-$ no | $4-$ no <br> $6-$ manifestations <br> $2-$-yes | $3-$ abstraction <br> $9-$ perceive |

Table 3: Features of formal reasoning_task 2_junior high school students.

## Example 4



Pole zacieniowaney figury jest $\frac{1}{2}$ prostolusta
Figure $4^{2}$
The student introduces additional divisions in the drawing, thus obtaining four rectangles divided by their diagonals. It introduces markings that are surely

[^9]related to the fields of the resulting triangles. The idea is correct, the lack of a comment showing the applied reasoning allows to presume that the lack of detachment from the concrete given in the drawing, enforces inference based on observations, although the student may draw from knowledge that he cannot write neither verbatim nor symbolic.

## Example 5



Figure $5^{3}$
The student using the notation similar to the film tries to show the "shifting" of the hatched triangle in such a way that the initial rectangle can be transformed into a parallelogram. There are no essential assumptions for the entire reasoning process that authorize such conduct. The attempt to prove the fact is based on specific activities and observations, and this certainly does not testify to formal reasoning.

## Round 2

Out of 27 participating in the study, only 11 students attempted to solve the task, not every one of them answered the question posed in the task, and those who gave it correctly. Most of the students did not carry out correct and full reasoning, certainly they were not formal reasoning, students were not able to use symbolic language, the conclusions were based on intuition, often correct but not supported by knowledge, and about specific activities, ie measuring the length of line segments and indicating certain facts or approximate values.

|  | Feature |  |  |
| :---: | :---: | :---: | :---: |
|  | logically <br> use <br> symbols | they reason <br> hypothetically and <br> deductively | think about abstract objects / <br> perceive / perform activities |
| Amount <br> of <br> students | $9-$ yes <br> $2-$ no | $7-$ no <br> $4-$ manifestations | $1-$ abstraction <br> $1-$ perceive <br> $10-$ perform activities |

Table 4: Features of formal reasoning_task 2_ pupils of grade 7.

[^10]
## Example 6



Figure $6^{4}$
The student introduces his own markings, which are to be field sizes, but it is not obvious why this is the case, the student introduces some assumptions himself and does not explain this step. In addition, the facts observed are completely false. The student combines a symbolic record with the text, tying it with the sign of equality. He puts the correct hypothesis, but based on false premises.

## Analysis of the task number 2

From the analysed solutions, it appears that students:

- most of them gave the correct answer,
- the answer in the vast majority was not the result of formal / abstract reasoning,
- their response was due to the perception of certain regularities that they saw in the drawing,
- their response was often a consequence of specific actions performed with the use of a ruler, the sides of figure were measured and although the results were often different, the correct answer was given,
- pupils do not need to justify their statements,
- students incorrectly conclude,
- students do not apply the rules of deductive reasoning,

[^11]- students cannot tear their reasoning away from the concrete given in the drawing,
- their intuitions are often correct, but students cannot tear themselves away from them to launch formal reasoning.


## SUMMARY

The article presents preliminary results and conclusions from a pilot study aimed at describing the readiness of Polish students at the age of 14 to use in the process of solving mathematical tasks of formal operations, which according to the theory of Jean Piaget's intellectual development after the age of 12 should already have their place in the pupil's mathematical thinking. Initial conclusions are unambiguous, the vast majority of students are not ready to reason with deduction in detachment from the facts, in isolation from imaginary operations and concrete actions. Pilot studies ${ }^{5}$ were divided into two stages, in the first there were 3 tasks of type of evidence, the solution of which turned out to be an easy task for most students, but the correction of the tools and equipping them with a set of tasks to level the difficulty of knowledge and skills needed to solve the task did not bring no positive changes in the test results. Before the author of the work, the proper research and an attempt to answer a series of additional questions about the reasons for this state of affairs in Polish education.

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# GENERALIZING ALGEBRAIC MODELS THROUGH INTERACTIVE LEARNING ACTIVITIES 

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#### Abstract

We report on hands-on interactive activities that require building algebraic models and their generalizations. We analyse the performance by three groups of learners: teenagers, pre-service teachers and in-service teachers and change in their attitudes towards the unfamiliar situations and modelling uncertainty to routine lecture based teaching.


## INTRODUCTION

This study is a part of a larger project focused on designing and assessing learning activities leading from basic to advanced levels of critical and analytical thinking that promote mathematical modeling, engagement, excitement, discussions and students' creativity. The three activities described here targeted early algebra curricular experiences including modeling, predictions, development of strategies, analysis of patterns and generalizations to other contexts. Initial engaging problems for each session included simple algebra based tricks or games, that through explorations, discussions and predictions, lead to formalization of the models and their further generalizations. Participants discussed the process of creating new structures and ideas, focused on making connections and attempting different solutions that were evaluated for their creative approach. Following Savic and associates (2017), we defined creativity as a process of offering new solutions or insights that are unexpected for the learner, with respect to his/her mathematics background or the problems $\mathrm{s} /$ he has seen before, as well as discoveries made within a specific reference group that creates something new (Vygotsky, 2016).

Problem solving and creative thinking are necessary for a professional success in a fast-passed technology intensive global setting of 21st century. At every level of mathematics education, there have been criticisms about the excessive amount of structure imposed on learners, especially at the $\mathrm{K}-12$ level, where students are rarely encouraged to solve open-ended problems, think creatively or pose their own questions. Already in 1989, the National Council of Teachers of Mathematics addressed the need for standards that include modeling, creativity and independent thinking, but nearly two decades later situation in American schools is not much better, as mathematics education still concentrates on basic skills and traditional problem solving (Schoenfeld, 1992). Additionally, the worldwide emphasis on high-stakes testing brought basic skills back to the center of attention (Lesh \& Sriraman, 2005). While for a long time Polya style problem-solving strategies (draw a picture, identify the givens, work backwards,
solve similar problems) have been advocated as important abilities for students to develop their mathematical maturity (Polya, 1957), they are not leading pedagogy in our schools (Chazan, 2008; Drew, 2011).

Contemporary students live through many stimuli in their lives, and they prefer innovative rather than traditional pedagogy (Star et al, 2008), learning with multiple representations (Ainsworth, 2006), through hands-on activities (Cruse, 2012) that are related to their interests (Whaley, 2012), in an engaging, playful environment (Kuh, 2003). Hence, to make mathematics learning more attractive and interactive our motivation was to get learners involved in unpredictable realistic situations to promote concept development and understanding. The activities that we developed promote curiosity, explorations and creation of algebraic models, logical thinking through pattern recognition and proposing definitions for underlying rules, development of various representations, extensions and modifications as well as verbalizations of the thinking. Algebraic concepts are now introduced early in the curriculum (Stephens et al, 2015), but test results show that even high school seniors have problems understanding algebraic ideas (Kuh, 2003). Learning hands-on activities (like puzzles, games, art projects, poems) bring fun back to the classroom and provide a new way of teaching that mixes context, explorations, and applications and brings new interdisciplinary connections to the abstract curriculum (Jones, 2016, Kurz, 2017, Stylianou et al. 2005, Whaley, 2012). Recently, there have been some efforts in various states of systematically implementing new pedagogical strategies (such as inquiry-based learning or problem-based learning) to improve students' skills that are related to mathematical modeling and creativity.
However, such efforts are generally not included at the university level, even in mathematics education for future teachers, (Karakok et al., 2015).

## METHODOLOGY

Our research goal was to evaluate participants' ability to generate new problems, to recognize patterns and to build related mathematical models, which required defining many variables. Additionally, we administered pre- and post- survey asking about preferred learning activities to assess the change of attitudes towards unfamiliar problems and generalizations for three different groups: 12 summer school pupils aged 13-14 (we denote this group P), 20 pre-service high school mathematics teachers who were university students (group S), 24 inservice mathematics teachers (T). The groups were very different in terms of preparation and maturity, however they turned out to be similar in terms of their expectations about of teaching methods in a mathematics classroom. The presurvey administered before the activities had the two questions stated below, while the post-survey had the question 2 only and requested comments.

1. Evaluate your knowledge of high school level algebra.

Learner (pre-algera) Proficient (passed algebra) Expert (applied/taught algebra)
2. Circle all teaching activities that you prefer in mathematics classroom.

| Lecture | Reading Math Text | Watching Math Videos |
| :--- | :---: | :---: |
| Solving of Problems on Board | Graphing | Individual Problem Solving |
| Group Work | Discussions | Asking Questions |
| Using Manipulatives | Guessing Answers | Games | Using Technology

Note that the activities above are grouped by level of engagement and creativity. We assigned scores 1-5 for each level (i.e. Lecture- score one, Modeling-score 5) and analyzed answers for each group (see Results).

During the study each one-hour activity was repeated with the three groups with reported different preparedness levels (see Table 1) with the same lead instructor and 2-3 assisting instructors. Each session started with a magic trick or a simple game, which was discussed to uncover underlying patterns, rules and/or optimal strategies to be modeled in algebraic language. Small teams implemented modifications and/or generalizations to the model and presented them to the others. Then participants designed their own patterns creatively; hence variety and complexity were added to individualized patterns, which were later modeled. Further discussions and explorations led to even more generalized problems, which often were formalized as formulas that included several different variables. Most of the hands-on tasks were done individually or in small teams/pairs and shared with the group for comparison and discussions. Some of the participants were openly frustrated at first when asked to work in unfamiliar contexts and with initially undefined variables. While the activities proved to be quite challenging, learners were fruitfully engaged at trying to design the models through logic and reasoning.

## Description of the activities

Magic tricks with dice activity starts by building various towers consisting of two dice. The instructor performs magic by guessing the sum of the hidden faces on each tower. By discussion, participants figure out how the trick works and model the situation using the equation, 14-x, where the variable $x$ represents the number at the top of the tower. Next, they build towers with three dice and try to figure out how to guess the sum, 21-x. Then they use four and five dice, described by $28-x$ and $35-x$. Now they study taller towers and the arising the patterns to come up with the linear algebraic formula depending on the number of dice and the number on the top of the tower modeling the situation by the equation with two variables, $7 n-x$, where $n$ is the number of dice.

Next, students put two, then three, then four dice in a row touching each other by one side (a tower lying down), where more sides are invisible than in
previous situation. At each level, they try to figure out the formula for guessing the sides that cannot be seen. After discussing, they come up with the general formula depending on the number of dice $n$ and the sum of the visible faces, which is unexpected, as it introduces several variables, sequence summations and generalizations.
In the exploration part, participants create their own designs using increased number of dice and trying to relate the geometric and numerical patterns. Then they are expected to choose their own variables and generate algebraic models for the sums of the hidden faces or for other generalized questions.
Guess my number activity starts with each student picking its own secret natural number and then following a set of algebraic operations given by the instructor. Students share the results of their final calculations, and the instructor guesses their individual secret numbers (using the simplified formula). Through discussions, students try to figure out the underlying rules and use algebra to make computational shortcuts. The activity can be repeated several times with different instructions. Once the group understands the underlying algebraic models, they design their own guessing games by creating new sets of rules and calculating the answers algebraically. They play out their scenarios in small groups. Note that the underlying equations could be linear, quadratic or of any complexity.

Guess my graph game involves graphing activities and is similar to the traditional battlefield game played in pairs. To start, on the 4 by 4 square in the first quadrant of the coordinate system each player secretly draws a line of his or her choice that has equation with integer coefficients only. Taking turns, they try to 'hit' each other's lines by 'throwing' points with integer coordinates. In each round, they get the information about the point being 'above', 'below' or 'on' the line. To win, a player needs to give the equation of his or her opponent's line. Parabolas (or other algebraic curves) can be used for more advanced students. Discussion that follows introduces various strategies and the minimal number of points on the linear or quadratic graph needed to uniquely identify it. At the end of the activity instructor's Guess my Parabola trick contradicts the usual belief that you always need three points to define a parabola. The instructor-magician can guess your parabola if you provide one point that lies on it. The underlying concepts involve a bit more advanced, but accessible mathematics.
Note the above activities promote curiosity and participation, address students' different learning styles, by providing visual, auditory, kinesthetic and languagebased tasks. They promote discussions, conjecturing and collaborations as well as creativity, problem posing and solving. Students can be divided into groups of various sizes, depending on the number and the level of participants.

## RESULTS

We collected all participants work and structured instructors observations for data analysis from all three groups: pupils (P), pre-service teachers (S), inservice teachers (T), as well as data coming from surveys and evaluation of participants' performance on tasks. Note that knowledge and competency in introductory algebra (Algebra Levels in Table 1) were self-reported by participants on the survey question 1 and the answers turned out to be quite homogenous across each group as expected (see Table 1). Engagement in activities for each group was ranked based on reports from instructors. During each modelling activity participants worked in pairs or teams of three. The generalized more complex models and their creativity were evaluated by the instructors based on the following scale:

Model is a direct generalization of the introductory model (Low)
Model introduces some new ideas into the generalized model (Moderate)
Model introduces creative/unexpected ideas to generalized model (High)
For example, in dice activity, the generalized model involving building another simple dice tower was considered as low creativity, designing a simple 2D or 3D pattern with dice and working out the algebraic formula was marked as moderate, while proposing interesting geometric 2D or 3D patterns generating interesting formulas (possibly with parameters) was considered as highly creative.

| Modelling <br> Dice | Sample Size | Algebra Level | Engagement | Creativity |
| :---: | :---: | :--- | :---: | :--- |
| $\mathrm{P}=$ Pupils | 12 | Learners | High | Moderate |
| S - students | 20 | Proficient | High | High |
| T- Teachers | 24 | Expert | Moderate | High |
| Total | 56 |  |  |  |

Table 1: Participants engagement and creativity by groups.
It is worth noticing that all groups were engaged in the activity at least moderately, each group provided generalized models, and participants more proficient in algebra provided more sophisticated and creative models.
The following results for each activity are based on participants' individual written work and coded observations of the instructors. In dice activity, the 2dice model was generalized to the 3 -dice model and to the $n$-dice model by teamwork and the instructor lead discussions.

| Modelling <br> Dice | 2-dice model | n-dice model | Generalized <br> Model | Mastered <br> Gen. |
| :---: | :---: | :---: | :---: | :---: |
|  | $100 \%$ | $92 \%$ | $75 \%$ | $50 \%$ |
| S - students | $100 \%$ | $100 \%$ | $90 \%$ | $75 \%$ |
| T- Teachers | $100 \%$ | $100 \%$ | $92 \%$ | $80 \%$ |

Table 2: Dice activity.
Almost all participants mastered and were able to use the n-dice model for different numbers of dice. Large percentage in each group was able to generalize the model in some way and solve the related specific problem (note that pupils' models were less sophisticated than models for the other two groups). Over three quarters of students and teachers were able to provide the accurate algebraic formulas (Mastered the Model) and discuss the parameters involved.

The initial Guess My Number activity involved the entire group and the underlying linear model was uncovered by discussions. Then small teams designed their own guessing tricks. Table 3 shows the complexity of the models, where underlying formula such as $(2 x+6) / 2-2$ was considered simple, $(x+1)^{2}-2 x$ was classified as the use of quadratic functions. Mastering the model meant that students were able to simplify the algebra of their formulas to create a quick answer for guessing the original number $x$. Some of the models used more than 5 steps.

| Modelling | Simple algebra <br> Quadratic |  | Created model <br> Mastered |  |
| :---: | :--- | :--- | :--- | :--- |
| P = Pupils | $92 \%$ | $50 \%$ | $92 \%$ | $33 \%$ |
| S - students | $90 \%$ | $90 \%$ | $100 \%$ | $80 \%$ |
| T- Teachers | $100 \%$ | $92 \%$ | $100 \%$ | $92 \%$ |

Table 3: Algebra use in Guess My Number activity.
During the initial graphing activity participants worked in pairs trying to figure out each other's line. Some of them asked about random points, while others were able to describe their strategy (such as finding two points on the line to figure out the equation). Then they were designing new games involving lines and parabolas (at least 3 points to define the graph). If they were able to explain their games and related strategies for finding the secret graphs, they mastered the methodology of finding equations for lines and parabolas.

| Strategies | Linear strategy <br> Linear | General | Quadratic Model <br> Mastered |  |
| :---: | :---: | :---: | :---: | :---: |
| P = Pupils | $75 \%$ | $50 \%$ | $50 \%$ | $50 \%$ |
| S - students | $90 \%$ | $80 \%$ | $80 \%$ | $75 \%$ |
| T- Teachers | $90 \%$ | $90 \%$ | $90 \%$ | $75 \%$ |

Table 4: Strategies in Guess my Graph activity.
$50 \%$ of the pupils had problems finding lines (and parabolas) using coordinates of two (three respectively) given points. $25 \%$ of students and teachers had similar problems, even though the vast majority of them understood the strategies for quadratic models. The most common technical problem in this case was setting up and solving appropriate systems of linear equations.

The above data shows that all participants were able to generalize and then model unfamiliar problems with some support from the instructors. To evaluate participants' attitudes toward various learning activities we evaluated data collected on question 2 from pre- and post- surveys preferences as stated in Methodology. Every activity chosen on any level was assigned appropriate score between 1 and 5 and the scores were than averaged for each participant and then each group. Interestingly, the scores on the pre-survey did not differ much per group (the mean score for P was 2.1 , the mean for S was 2.3 and the mean for T was 2.5 ) as level 4 and 5 activities were rarely selected (and no one selected Guessing, Manipulatives or Creating Problems). The corresponding scores on the post-survey were significantly higher for each group (the mean score for P was 3.9 , the mean for S was 3.8 and the mean for T was 3.7 ) and majority of participants included as preferred the level 5 unfamiliar situations and modelling uncertainty ( $86 \%$ ) and only $52 \%$ included routine lecture based teaching in their preferred group (and that included $88 \%$ of teachers, and only one student). In comments several participants suggested including songs, poems and kinaesthetic activities in question 2.

Some interesting comments. Below we quote some of the comments on the activities and the learning process from post- surveys. Teenagers were engaged and came up with creative generalized models, but were generally were worried that their math skills are not good enough to analyse them.

Pupil 1: I liked how algebra is magic. I designed my own trick that is hard to solve.
Pupil 2: My design is complicated. I had to use $n, m, l, k$ to make the correct formula to find the totals. But it worked.

Pupil 3: How do I know we always need $n+1$ points to find a graph of degree $n$ ? Maybe some of them need more or less? Like $x^{n}+c=y$.

The comment of Pupil 3 refers to finding coefficients of the general equation of a polynomial of degree $n$. This is an interesting abstract question, shows his higher level thinking skills.

In general, pre-service teachers were more confident about the algebraic models, and enjoyed the analysis of patterns. Many comments referred to the satisfaction of being able to handle a complicated situation they have created.

Student S1: The magic part was really fun and I was trying to figure out something all the time. I never though I can come up an interesting complicated formulas by myself, but I did.
Student S2: I want to continue on more complicated dice patterns in 3D. Then I will find a pattern for patterns. Generalizing is fun! I wish my math courses were taught that way, more interactive.
Student S3: Talking about strategies was interesting. I liked debating the ideas freely. For the first time I looked at math as a game. But staff can get too complicated.

The teachers were the most reflective group, commented about the teaching process, and pointed out some problems with classroom implementations.

Teacher T1: These activities were involving and innovative. They taught me how to generalize problems. We can get students to make connections and think algebraically on their own. Some of the tricks were hard and require some preparation.
Teacher T2: I can do these simple dice models with my students. I'm worried about their generalizations, as they may come up with something too difficult for their level.

Teacher T3: I never thought to ask my students to be creative. I want to try these activities with my algebra students, probably the simpler cases only. I don't want to confuse them with complex equations.

Over all comments regarding the activities, discussions and the learning process were positive across the groups. The initial tasks inspiring curiosity were liked the most, as was the task of designing generalized models. Participants showed perseverance analysing these models and expressed concerns about their own ability to formalize them in algebraic language.

While the activities proved to be quite challenging, learners were fruitfully engaged at trying to design the models through logic and reasoning. The activities supported teamwork, discussions and mathematical perseverance when challenged. Participants found them interesting, rewarding and supporting their growth and confidence. The group of pupils presented some of the activities as magic tricks to their teachers and parents, who gave them enthusiastic reviews.

## CONCLUSION

In order to support student thinking in algebra, it is important for students to experience critical thinking and original model building in various contexts. It is beneficial to them to struggle a bit and discuss the possible solutions before coming out with correct models. They should have an opportunity to make sense of algebra as a tool for predictions and modeling patterns. Our study provided all these opportunities to learners as well as teachers and our assessment shows that all three groups at different developmental levels of introductory algebra engaged fully into the proposed activities, discussions, modeling processes and generalizations. They stayed focused throughout the sessions, and came up with creative solutions that required perseverance and advanced thinking to analyze. Almost all participants showed their potential for generalizations using multiple representations. They performed well when exposed to the uncertainty and the difficulty of creating mathematics. Teacher's comments indicate the suitability of the activities for regular classrooms (with appropriate preparation) and suggestions that may change their teaching styles to include more open-ended and creative problems. Hence, there is a need for further training of teachers applying levels 4 and 5 activities in their classrooms, as well as further development and testing of interactive learning activities for other topics in mathematics that use variety of tools (such as manipulatives, technology, games, art and science concepts, etc.). Hopefully, this pedagogy supporting students' engagement and creativity will become more common and schools will educate more creative, open minded and thoughtful students that can meet future demands of the society.
Interestingly, majority of two older groups tried to come up with 'nice' formulas, paying attentions to aesthetic, i.e. beauty of the mathematical models. Some of the participants' comments suggest that they would like to learn more about their own cognition and the regulation of the creative processes. These suggest that we should study not only learners' creative actions, but also their meta-cognitive skills.

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# Professional Approaches to Constructing Mathematics 

# IS DIMENSION A SIZE, A SURFACE OR A SPACE? PRE-SERVICE TEACHERS' PERCEPTIONS OF THE CONCEPT 

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This paper presents a mixed methods study whose objective was to learn how pre-service elementary school teachers understand and interpret the concept 'dimension’. The research sample consisted of 132 pre-service mathematics teachers who enrolled in an asynchronous online course entitled "When Mathematics Meets Art". The preliminary results suggest that research participants have insufficient prior knowledge about dimension; the majority of them have misconceptions about dimension, which have at least two origins: prior concept image and the linguistic meaning of the word. In the case of dimension, artwork did not contribute to the understanding of the concept.

## THEORETICAL FRAMEWORK

The world we live in consists of complex shapes and forms. To understand it and live in it, we must be able to characterize it and to be able to describe different properties of natural phenomena correctly. One such characteristic is 'dimension', a concept that we use practically on a daily basis. The concept of dimension is so ingrained in geometry that it is often used in schools and even in teacher training institutions without it being given a proper definition (Vitsas \& Koleza, 2000). In textbooks, the word 'dimension' appears as a characteristic of a geometric object alongside examples of objects with different dimensions such as points (zero-dimension), lines (one-dimension), rectangles (two-dimension), and cubes (three-dimension).

What is the definition of dimension and why it is not presented to learners? Selkirk (1990, p.170) presented a mathematical dictionary definition of dimension: "The number of measures needed to give the place of any point in a given space, the number of coordinates needed to define a point in it". Morgan (2005) points out that one of the reasons that the definition of dimension is not presented to learners is its ambiguity: the nature of the 'given space' is left open. However, she claims that "this is not a weakness in the definition but a characteristic of the mathematical concept itself" (p. 104). This conclusion contradicts the notion that mathematical language must always attempt to be defined precisely so that there is no "possibility of more than one interpretation for a mathematical expression arising from sloppy use of language rather than any uncertainty in the mathematical ideas" (Barwell, 2005, p. 118). Having this in mind, for the purpose of this paper, I based on the topological definition of
dimension as the number of directions in which an object can expand (Skordoulis, Vitsas, Dafermos, \& Koleza, 2009).

Several studies focused on the way students, pre-service teachers, and in-service teachers understand dimension in the context of Euclidean geometry. It was found that not all teachers could correctly determine the dimensions of known objects, and about half of them named a criterion according to which they determined the dimension, but were usually inconsistent in applying it (Ural, 2014; Vitsas \& Koleza, 2000). Skordoulis and his colleagues (2009) suggested that the difficulty of determining the object's dimension stems from both a lack of clarity regarding the use of the concept in school and the confusion created when the geometrical object is placed in a Cartesian coordinate system.
However, none of these studies examines what is the concept image of dimension that researches' participants have. The concept image can uncover learners' misconceptions. Tall and Vinner (1981, p. 151) define concept image as:

The total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

The literature review indicates that dimension is important concept and helps us understand the nature, yet it is not taught in an organized manner on any level, neither in pre-academic programs nor in teacher training colleges. Studies that examined the knowledge of pre-service and in-service teachers indicate that most of teachers have insufficient mathematical knowledge about dimension in Euclidian geometry. Therefore, it was decided to submit activities that deal with dimension as one of the topics in the asynchronous online course "When Mathematics Meets Art", offered by the Gordon Academic College of Education in Israel. The course was designed to implement the principles of STEAM (Science, Technology, Engineering, Art and Mathematics) education, and in this case, artwork that emphasizes mathematical concepts was included in the course curriculum. This approach was chosen to enable pre-service teachers to experience interdisciplinary learning, which is considered more suitable for the 21st century (e.g. Okbay, 2013; Thuneberg, Salmi, \& Fenyvesi, 2017).

## CONTEXT OF THE STUDY AND METHODOLOGY

The preliminary results of the student learning outcomes from the course "When Mathematics Meets Art" suggest a positive and significant partial overlap between mathematics and art regarding topics such as zero and infinity (in the context of calculating area and perimeters), the golden section, and spatial vision (focusing on impossible shapes), which is innovative, intriguing, fun, and inspiring (Nutov, 2018). However, in the case of dimension, there were no correlations between the artwork and the above-mentioned mathematical
concepts and so I decided to analyse pre-service teachers' learning outcomes regarding dimension as a case study. The purpose of the study was to learn how pre-service teachers understand and interpret dimension.

The research questions were:
(a) How do pre-service teachers understand and interpret dimension?
(b) Does artwork contribute to the pre-service teachers' understanding of dimension?

## The research method

Of all the mixed methods research strategies, the explanatory design research was deemed most suitable (Creswell \& Plano Clark, 2007). The first phase of the research consisted of the collection and analysis of quantitative data. The second phase included the interpretation of the quantitative data using the qualitative data. In this way, the advantages of both research paradigms are exploited: A quantitative method enables to examine the relationship between variables, while the qualitative method provides the participants' interpretation of the quantitative findings.

## The research environment

The course "When Mathematics Meets Art" was offered for the first time in the fall of 2017 by the Gordon Academic Educational College, Israel for pre-service elementary school teachers majoring in mathematics. A total of 132 pre-service teachers participated in the semester-long (14 weeks), asynchronous online course. The objectives of the course were to expand the pre-service teachers' mathematical knowledge, to create a community of learners, and to apply mathematical concepts to art, daily life and natural occurrences. The course covered six mathematical topics presented in the following order: tessellations, zero and infinity (in calculating area and perimeter), the golden section, spatial vision (focusing on impossible figures), dimension, and self-similarity. These six topics were selected using three criteria that relate to the elementary school curriculum: topics included in the curriculum (zero and infinity and spatial vision); concepts mentioned in the curriculum but not studied in depth (dimension and self-similarity); and concepts that are not part of the curriculum but are tightly coupled with both mathematics and art (tessellations and the golden section).

The course was designed on the Moodle platform and consists of six units, one for each mathematical concept as indicated above. The following are the student requirements for each course unit: (1) Pre-service teachers must take part in a forum discussion or answer a survey. These tasks are designed to check the pre-service teachers' prior knowledge ( $15 \%$ of final grade; pre-service teachers are graded for participation, not knowledge); (2) Pre-service teachers study the
mathematical concept theoretically using a specially prepared video, an article, or a PowerPoint presentation. They then apply their acquired knowledge by solving exercises or carrying out an inquiry task. Finally they check their knowledge by taking a short online test (test score is $20 \%$ of final grade); (3) Pre-service teachers contribute an artwork (which need not be original) to an online cooperative gallery ( $9 \%$ of final grade); (4) A final exam ( $56 \%$ of final grade).
The mathematical content of the Dimension unit consists of a paper that explains the topological definition of dimension and presents the Hausdorff formula used to calculate it. The unit includes exercises that implement these definitions. Preservice teachers were encouraged to watch movies such as "La Linea" that was created by Osvaldo Cavandoli and "Flatland" that is based on the book of Edwin Abbot.

## The research tools

Prior knowledge online survey - consisted of open-ended questions: How would you explain to your students what dimension is? What is the dimension of a circle, a circle perimeter, a pyramid apex, a pyramid edge, the surface of a sphere, and a room? (Table 1). What are origins of your knowledge?
Online tests - consisted of multiple choice questions which applied the definition of dimension to Euclidean objects (point, curve, surface, and space), the Hausdorff formula to calculating the dimensions of Euclidean objects and to estimating the dimensions of fractals (for example, the dimension of the snowflake curve that is between 1 and 2 ). The pre-service teachers had two attempts to answer the questions. The tests were graded automatically based on pre-determined criteria and the final test grade for each pre-service teacher was the higher of the two.
Collaborative gallery - the gallery consisted of the pre-service teachers' art contributions. They could contribute an artwork, original or not original, with a short explanation of how dimension is expressed in it. Students could have commented on the contributions of their colleagues and indicate 'like'. The Padlet (https://padlet.com) website was selected for the collaborative art gallery. All contributions were graded according to a published rubric.

## Data analysis

The quantitative data were analysed statistically and an attempt was made to find a connection between the online mathematics test grades and the artwork grades. The qualitative data (prior knowledge) were analysed according to the Strauss-Corbin method (1990): first, each of the survey responses was analysed to identify primary and secondary themes. Next, links were identified between the different categories, and finally, an attempt was made to correlate the quantitative and qualitative results.

## RESULTS AND DISCUSSION

The data analyses offered the following preliminary results: (a) pre-service teachers have insufficient prior knowledge regarding the concept of dimension; (b) the majority of research participants have misconceptions about the concept of dimension, which have at least two origins: prior concept image and the linguistic meaning of the word; (c) contribution of art to understanding the concept of dimension was not observed. Each of these results is discussed in what follows.

## Insufficient prior knowledge of pre-service teachers

Table 1 presents analyses of the prior knowledge online survey. These results suggest that pre-service elementary school mathematics teachers have unsatisfactory knowledge of dimension. Even a simple question like "What is the dimension of a room?" produced only $55 \%$ of correct answers.

| Question <br> No. | What is a dimension <br> of... | Correct <br> answer | Wrong <br> answer | Didn't <br> answer | 'Creative' <br> answers |
| :---: | :--- | :--- | :--- | :--- | :--- |
| 1 | a circle | $51.81 \%$ | $27.27 \%$ | $10 \%$ | $10.9 \%$ |
| 2 | the perimeter of a <br> circle | $30.9 \%$ | $45.45 \%$ | $10.9 \%$ | $12.72 \%$ |
| 3 | the apex of a pyramid | $40.9 \%$ | $40 \%$ | $18.81 \%$ | $7.2 \%$ |
| 4 | the edge of a pyramid | $46.36 \%$ | $27.27 \%$ | 142.72 | $13.63 \%$ |
|  |  |  |  | $\%$ |  |
| 5 | the surface of a sphere | $22.72 \%$ | $53.63 \%$ | $12.72 \%$ | $10.9 \%$ |
| 6 | a room | $55.45 \%$ | $13.63 \%$ | $10.9 \%$ | $20 \%$ |

Table 1: Prior knowledge survey results $(\mathrm{N}=110)$
Some of the pre-service teachers 'creative' answers to the question "How would you explain to your pre-service teachers what 'dimension' is?" demonstrated a total confusion regarding the frequently used concept. Here are some examples: "Dimension is a region that is defined as a specific area, sometimes abstract and sometimes real that can be seen and felt". "Dimension is a 'wide' place that we can enter". "A more sophisticated painting, which can be seen from different angles". "Dimension is a space that occupies some form or an object, there are several indices and each of them sees it differently."

## Misconceptions about the concept 'dimension'

The data analysis reviles two possible roots of pre-service teachers' misconceptions: the concept image of dimension and the linguistic meaning of the word 'dimension'.

Concept image of dimension: Some pre-service teachers stated that they would explain that: (a) Dimension is a measurement (length, width, height) ( 20 preservice teachers); (b) Dimension is a space or a surface (18 pre-service teachers).
In the case of dimension, the concept image presented by the pre-service teachers, i.e. dimension as a measurement, is understandable. The pre-service teachers had never learned the proper definition and their knowledge is built upon examples of different objects. Objects that has only one dimension such as a line or a curve have only length; objects that have two dimensions -such as rectangles, that have both length and width and are related to a plane; and objects that, have three dimensions -such as cubes that have length, width, and height and are related to a space. In addition, in the case of the State of Israel, where this study was conducted, the online geometry dictionary available on the official site of the Ministry of Education contains no definition of dimension, yet offers the following definitions: "Length - a dimension connected with a straight line"; "Space - a dimension connected to a plane". These definitions are in line with the misconceptions of pre-service teachers observed. It seems that the lack of a clear and coherent definition contributes to pre-service teachers' confusion in the case of the concept image of dimension.

Here are some examples of pre-service teachers' answers. Dimension as a measurement: "Dimension is a numeric or a quantitative value that defines measurements of an object or a shape, such as length, width, height, or the quantity that this object can contain, such as volume or capacity."

Dimension as a space or a surface: "Dimension is a number that describes a particular space".
Linguistic meaning of the word "dimension": Although "mathematical language appears to be identified with its vocabulary" (Morgan, 2005, p. 103), many mathematical concepts are, at the same time, used in everyday language in which the meaning of the word is different from its mathematical meaning. This situation creates ambiguity and can be a source of misconception. Data analysis reveals four different kinds of misconceptions related to the linguistic meaning of the word 'dimension' in Hebrew: (a) Dimension as size or weight (4 preservice teachers); (b) Dimension as an axis in space (5 pre-service teachers); (c) Dimension as a point of view (7 pre-service teachers); (d) Dimension as a protected space ( 2 students).
These explanations of the concept 'dimension' given by pre-service teachers can be based on the different meanings of the word 'dimension' found in a Hebrew dictionary. For example, to describe a large project, one can say "a big dimensional project".
Here are few examples for each misconception: (a) "Dimension is a concept that helps us to notice a certain size. For example, an elephant is huge - it is large
and heavy. Its size and weight are enormous." (b) "A dimension is an axis in space that describes a volume. For example, when a person is said to be large in size." (c) "Dimension describes my point of view, which directions I looked at".

The most surprising misconception is the last one, (d), since it comprises two different misconceptions - one is connected to mathematics and another to Hebrew as a language. Protected space and dimension are written identically but are pronounced differently: protected space is pronounced "mamad" (and is in fact an abbreviation) while dimension is pronounced "meimad". Here is an illustrative example: "I will explain that dimension is a place where we hid during the war".

## Contribution of art to understanding dimension

The preliminary research results indicate that in the case of dimension there is no connection between the pre-service teachers' online math test grades and their artworks ( $\mathrm{r}=0.061$, sing. 1 -tailed $=0.247$ ). However, a more in-depth examination of the data showed that pre-service teachers who uploaded original artwork to the art gallery did very well on the test. Unfortunately, only six, out of the 101 submitted artworks, were original (four of them are presented in Figure 1). Based on this data any conclusions cannot be make regarding the contribution of original artwork to the understanding of dimension.


Figure 1: Some of original student art on dimension
It is interesting to note that all the artworks (original or not) which were uploaded to the art gallery, represented three-dimensional objects. However, there is not even one example of one-dimensional or zero-dimensional object, or fractals (which have a fractal dimension). That was although pre-service teachers had such examples in the theoretical material (films like La Linea and Flatland).

## CONCLUSIONS

This research is part of a larger study whose objectives were to examine the possible contribution of art to pre-service teachers' understanding of mathematical concepts and to identify the challenges of learning mathematics in combination with art via an online course. The preliminary results of the main study confirmed the hypothesis that there is a possible overlap between
mathematics and art, inspired by a mathematical concept. This overlap, or the math-art connection, can help pre-service teachers enhance their mathematical ability to solve a given assignment or to perform an inquiry task; it can also help them develop mathematical intuition, since art enables expression that is beyond words and numbers. However, the preliminary results of the present research do not support this claim; on the contrary, they indicate that, in the case of dimension, there is no overlap between mathematics and art. This contradiction warrants an explanation and future research to explore the question: Do artworks contribute to the understanding of any mathematical concept or only to those mathematical concepts that are easy to draw or are not particularly abstract?

The study results show that dimension is an abstract mathematical concept with an ambiguous definition (Morgan, 2005) that, in most cases, is not previously taught in any curriculum (Skordoulis et al., 2009). There are at least two possible explanations for the almost zero correlation between the math test results and the artworks in the case of dimension.

One explanation can be the challenge to understand the abstract nature of this concept: in the real-physical world, we can model only two types of objects -two-dimensional and three-dimensional; there are no zero-dimensional, onedimensional or fractal objects. When pre-service teachers were ask to present an artwork, their choice to represent three-dimensional objects seems natural because we live in a three-dimensional world. A representation of threedimensional object on a two-dimensional surface requires a deep understanding of the mathematical concept of dimension and good art skills (as opposed to concepts such as zero and infinity or tessellations).
The second explanation can be the nature of the course. The course presented here was designed as an asynchronous online course, which means, that each student can complete the course tasks at a time that is convenient for her or him. This freedom made it impossible to identify the pre-service teachers' misconceptions while they were studying the unit and to provide immediate, real-time response. It is very reasonable to suggest that when the mathematical concept is unclear the ability to express the knowledge through artwork is limited.

I believe that the math-art connection holds great potential for math education, particularly at the elementary school level, although in the case of dimension the results did not support this claim. As a lecturer at an education college, I see it as my responsibility to expose pre-service teachers to STEAM as well as to online learning. In that spirit, my future plans are to complete the entire data analysis and to verify my preliminary results. In addition, I plan to organize an exhibition of the pre-service teachers' original artwork and to study its impact on the entire Mathematics Department, including both pre-service teachers and staff.

The course "When Mathematics Meets Art" is a new online course and the research results indicate several directions for improving the course so that it meets student needs more precisely. The questions I will consider are: (a) What previous knowledge do pre-service teachers need so that no knowledge gaps exists and they feel confident of their knowledge and what are the best ways to provide it? (b) Preparing original student artwork versus analyzing professional artwork: which is more beneficial for student learning? (c) How can pre-service teachers be provided with richer experiences so that they are able to form a more coherent concept of dimension? (d) The inclusion of a synchronic lesson in each unit and an option of addressing questions pre-service teachers have after studying the theory and before taking the online test.

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# PRE-SERVICE TEACHERS' KNOWLEDGE ABOUT SHIFTING BETWEEN FUNCTION REPRESENTATIONS 

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The current study examines the knowledge of pre-service teachers regarding the relations between different function representations as reflected in the way they solve various mathematical tasks requiring fluent transition between different function representations. By documenting their solution process and explanations, we examined their developing knowledge about function representations. The research participants comprised students at colleges of education studying to become high school mathematics teachers. The findings indicate that pre-service teachers lack the knowledge required to understand relationships between a function's changing values, its graphical representation and changes in its symbolic representation. By solving mathematics tasks, students expanded and enhanced their knowledge on this topic.

## THEORETICAL BACKGROUND

## Required knowledge for mathematics teachers

Sullivan (2003) noted that as a result of the sharp increase in mathematical complexity in the transition from high school math to mathematics in academic institutions, some beginning teachers lack a profound understanding of concepts in the mathematics curriculum. On the other hand, teachers are expected to be aware of in-depth mathematical concepts required for teaching already at the initial stages of their career. In recent decades, mathematics education researchers have attempted to investigate and characterize the knowledge needed by mathematics teachers. Based on the different types of knowledge defined by Shulman (1986), Ball and Bass (2003) noted that the knowledge required for teaching mathematics is unique and constitutes a decisive factor in effective and high quality teaching. Hence, they defined the concept of mathematical knowledge for teaching (MKT) as knowledge that, among other things, traverses all areas and levels of school mathematics. This knowledge supports teachers' pedagogy knowledge and the mathematical knowledge required to solve, integrate and manage appropriate assignments in class. One of the components of MKT is defined as specialized content knowledge (SCK), which among other things includes mathematical knowledge and skills for teaching as well as the ability to use a variety of representations of mathematical
concepts and to connect between them (Ball, Thames \& Phelps, 2008; Ball \& Bass, 2003).

## Difficulties in learning different function representations

There is a broad consensus among researchers in the field of mathematics education that the learning process entails solving mathematical tasks requiring flexibility and adaptation as well as the use of a variety of representations and shifting fluently between the various representations of the concepts involved in the task (Kilpatrick, Swafford \& Findell, 2001; Heinze, Star \& Verschaffel, 2009). The ability to shift flexibly and continuously between different representations of a mathematical concept is a critical component of solving mathematics problems and provides learners with in-depth and broad knowledge of the concept (Duval, 2002; Kaput, 1989). Bieda and Nathan (2009) defined the concept of representational fluency as "the ability to work within and translate among representations" (p. 637). This central mathematical skill should be integrated into mathematical operations in the learning process in order to construct knowledge and understanding. Students' difficulties in shifting between representations are liable to emerge while describing or mapping the connections between an abstract or unfamiliar concept and a concrete or known concept (Glenberg, De Vega \& Graesser, 2008; Nathan, 2008). These difficulties can be manifested in a lack of continuity in using different representations, and in the transition between representations, and are likely to fall into one of the three following categories:
a) physically grounded - when learners make limited use of graphs, and the information they represent, and view the graph as an invariable physical object.
b) spatially grounded - when learners see the graph as a limited display. They can make physical changes in the graph by using a new criterion or extension, but they are not able to translate the information displayed in the graph into any sort of generalization.
c) interpretatively grounded - when learners see continuity between the graphical representations supported by limited explanations, while preserving the link to the original form. (Bieda \& Nathan, 2009)
Mathematics teachers are required to have mathematical and pedagogical knowledge and skills, including the ability to process and translate transitions between different representations of mathematical concepts, and aware to students' difficulties, in order to initiate, organize and manage mathematical operations and tasks in the classroom environment with students.

## RESEARCH METHOD

This study is based on the rationale that until a decade ago, the mathematics curriculum in both middle and secondary schools in our country did not include content and skills regarding shifting between symbolic and graphic function
representations. Ten years ago, the Ministry of Education issued a new middleschool curriculum that taught this content and these skills by means of linear and quadratic functions. In secondary school, the content and skills were extended to other types of functions (polynomial, trigonometric, logarithmic), and included in the matriculation exams through questions whose solutions required understanding the transition between symbolic and graphic representations of functions. As a result, we assumed that some students who were unfamiliar with the new mathematics curriculum lacked knowledge and skills related to shifting between representations of functions. Hence, we decided to examine the subject in depth through the current study.

The objective of the current research was to investigate the development of the knowledge of pre-service high school mathematics teachers about different representations of functions, particularly the transitions between symbolic and graphic representation as this topic emerges in problem solving.

## Research population

The research participants included two groups of pre-service teachers comprising 17 bachelor's degree students at colleges of education enrolled in a one-year course.

## Research questions

1. What knowledge do pre-service teachers possess regarding symbolic and graphic function representations and the transitions between them, as manifested in their problem solving at the beginning of the research intervention?
2. What characterizes the development of pre-service teachers' knowledge about symbolic and graphic function representations and the transitions between them?

In this framework, we will address only the first research question

## Research instruments

The research instruments comprised 13 mathematics problems whose solutions relied on familiarity with different function representations and the transitions between them. Some of the problems were taken from the underground mathematics website of Cambridge University in England. The tasks on this site were written for mathematics teachers to integrate in their teaching of advanced high school mathematics. We chose those tasks that were non-routine and unfamiliar, and whose solution is based on a meaningful understanding of the transition from graphic representation to symbolic representation of functions. The chosen tasks were formulated in accordance with the needs of the study. Some of the tasks were formulated and designed by the researchers. In cases
where we identified that students had a specific difficulty that needed to be overcome before they continued learning, we formulated a suitable task. The research tools included documentation of the discussions between the students and the whole-class discussions, observations of student work, and a researchers' diary that included documentation of the discussion between the students while solving the tasks and considerations that guided us in selecting, formulating and integrating the tasks.

In the following sections we describe two examples of tasks assigned to the preservice teachers. The first task was assigned prior to the intervention program with the purpose of examining the students' knowledge on the topic: e.g., what do they know about the transition between symbolic representations and graphic representations, and how do they understand the changes in the graph and in the symbol representation of the functions? The second task was assigned to the students at the beginning of the intervention program, which in the first stage entailed become familiar with various symbolic representations of quadratic functions, sliding the quadratic function graph and understanding the connection between the symbol and the graph representation. Subsequently the intervention program was extended to other types of functions.

## Task 1

Choose two of the transformations (transitions between symbolic representation and graphical representation) below and apply them in turn starting with the function $f(x)$ (Figure 1). Sketch the resulting graph after you have applied one transformation and then the other. Does it


Figure 1: Starting function matter in which order you apply the two transformations? Does the order matter for all pairs, some pairs or none of the pairs?
The transformations: 1. Horizontal sliding of -2 units. 2. Vertical sliding of -4 units. 3. Stretch by factor 3 parallel to $x$. 4. Contraction by factor $\frac{1}{2}$ parallel to $y$.

Task 2 (designed by the researchers)
The graph of the quadratic function $f(x)$ follows (Figure 2):
a. Find symbol representations that fit the graphical representation of $f(x)$.
b. The graph of $f(x)$ and other quadratic functions $g(x), h(x), t(x)$ and $k(x)$ are shown on the same Cartesian axis system (Figure 3). Based on the symbol representations you found for the quadratic function $f(x)$, find appropriate symbol representations for the quadratic functions $g(x)$,


Figure 2: Quadratic function


Figure 3: Quadratics functions
$h(x), t(x)$ and $k(x)$. Please explain your considerations in selecting the symbol representations for $g(x), h(x), t(x)$ and $k(x)$.

## DATA ANALYSIS

In the first stage, data were collected from the students' process of solving the mathematical problems, including solution methods, reasoning methods, sketches and written explanations. In addition, we collected data from the documentation of the class discourse in the whole-class discussion and among the students. In the second stage we analysed the collected data while attempting to formulate categories reflecting the students' knowledge as well as the difficulties that arose while solving the problem related to making a fluent transition between the representations. The categories referred to those outlined by Bidea and Nathan (2009). We documented and analysed the whole-class discussions and the discussions between the students during their work in class. We examined whether the categories matched the new data and made changes in the categories accordingly.

## FINDINGS

Here we present partial findings based on the above-mentioned resources.
While solving Task 1 the students were required to choose two of the four transformations that include transition between symbolic representations and graphic representation, apply them one after the other on the given function graph $f(x)$, and present the resulting graph and symbolic representation after application of the two transformations. Table 1 summarizes the findings from the students' operations in Task 1 in accordance with the chosen transformations. The left-most column describes the chosen transformations, where the notation $(x, y)$ indicates that transformation $x$ was applied first, followed by transformation $y$.

| Chosen <br> Trans | \# of Students | Graphical Graphic Representation |  |  | Symbolic Symbolic Representation |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | Trans 1 correct sketch | Trans $1 \& 2$ correct sketch | Incorrect sketch | Trans 1 correct symbol | Trans $1 \& 2$ correct symbol | Incorrect symbol |
| $(1,2)$ | 4 | 2 | 1 | 2 | 2 | 0 | 2 |
| $(1,3)$ | 2 | 1 | 0 | 1 | 0 | 0 | 2 |
| $(1,4)$ | 4 | 2 | 1 | 2 | 2 | 1 | 2 |
| $(2,3)$ | 1 | 1 | 1 | 0 | 0 | 0 | 1 |
| $(2,4)$ | 4 | 2 | 2 | 2 | 2 | 0 | 2 |
| $(3,4)$ | 2 | 1 | 0 | 1 | 1 | 0 | 1 |
| Total | 17 | 9 | 5 | 8 | 7 | 1 | 10 |

Table 1: Students' operations in Task 1 according to chosen transformations (transition between symbolic and graphic representation)

The data in Table 1 reflect the students' missing knowledge at the beginning of the intervention regarding the use of the symbolic and graphic representations and the continuity in shifting between them. Of the 17 students, nine sketched the correct graphical representation for the first transformation, while only five students sketched the correct final graphical representations after the second transformation was applied. Eight students were unable to sketch the correct graphical representation for either of the transformations. Moreover, the students also lacked knowledge regarding the symbolic representation of the function and regarding the shift from the graphical representation to the symbolic representation.
Through analysis of the data in Table 1, the students' explanations, our observations of the students as they worked and the whole-class discussions, we attempted to characterize the students' missing knowledge.

## Missing knowledge for understanding the relationship between changes in values and changes in the graph

Students in this category had difficulty in correctly translating the required changes in the graph for each of the axes. For example, one student chose to apply transformation (2,4): "vertical sliding of -4 units" followed by "contracting by a factor of half parallel to $y$ ". The graph the student drew to represent the results after applying the two transformations showed the vertical downward slide and was contracted by a factor of 2 relative to the $y$ axis and also relative to the $x$ axis. The student had difficulty seeing the contraction of the function relative to one axis only. This difficulty may stem from a misinterpretation of the concept of "contract by a factor of half" relative to the $y$ axis as "contract by a factor of 2 " for both the $x$ values and the $y$ values.
Another student chose to apply transformation (1,2): "horizontal sliding of -2 units" followed by "vertical sliding of -4 units". This student was able to explain the results of the transformations verbally. The shift to the graphical representation, however, included horizontal sliding of the graph to the left by 2 units, and then reflection of the graph relative to the $x$ axis: "In the horizontal sliding, the change is only in the values of the $y$ axis". Reflection relative to the $x$ axis, the change is only in the values of $y$.

## Missing knowledge for understanding the relationship between changes in values and changes in the structure of the symbolic representation

Only seven of the students were able to provide a correct symbolic representation for the first transformation. Of these, only one student provided a correct symbolic representation after applying two transformations. Ten students did not manage to provide a correct symbolic representation for any of the transformations they chose. The students' difficulties in switching to the symbolic representation emerged in their translation of "horizontal sliding of
two units to the left" to the symbolic representation $f(x-2)$ rather than the symbolic representation $f(x+2)$.

Students who had problems with symbolic representation of the transformation "stretch by factor 3 parallel to $x$ " suggested $\frac{1}{3} f(x)$ as the symbolic representation. The discussions led us to conclude that this answer was apparently based on the students' intuitive feeling that, some factor in the symbolic representation, should be divided by 3, but they lacked an understanding of which factor this should be. Similarly, for the "stretch by factor 3 parallel to $x "$ transformation, other students suggested $f(3 x)$ as the symbolic representation. This response was based on the students' intuitive understanding of the translation of "stretch by factor 3 parallel to $x$ ". "I noticed that point $(2,0)$ on the original graph corresponds to point $(6,0)$ on the new graph, that is, we multiplied 2 by 3 "; "I do not understand the difference between vertical contraction or stretching and horizontal stretching, because sometimes they seem to me to be the same thing, as for example in the function $f(x)=x^{2}$ ".

In addition, the students used partial knowledge from the graph, while they referred only to a limited number of points on the graph. For example, some students suggested the symbolic representation $f(x)-2$ as appropriate for transformation 4: "stretch by factor half parallel to $y$ ". They did so by referring only to the change in a limited number of points on the graph: $(0,0)$ and $(0,4)$.

Before assigning Task 2 to the students and based on their missing knowledge that emerged while they solved Task 1, we held a lesson dedicated to teaching the various representations of quadratic functions. The students were familiar with the standard symbolic representation of the quadratic function $y=a x^{2}+b x+c$ and the roles of the parameters including this symbol representation. During the lesson the students refreshed their knowledge about the symbolic representation of the quadratic function represented as a multiple of two linear factors: $y=a\left(x-x_{1}\right)\left(x-x_{2}\right)$, and for the first time they learned about the vertex symbolic representation $y=m(x-p)^{2}+k$ that results from sliding, stretching or contracting the function $y=x^{2}$. They also learned about the role of the parameters in this vertex symbol representation. Table 2 presents the representations selected by the students in the presentation of the various functions:

|  | $f(x)$ | $k(x)$ | $h(x)$ | $g(x)$ | $t(x)$ | $k(x)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $y=a x^{2}+b x+c$ | 8 | 2 | 1 | 1 | 1 | 2 |
| $y=a\left(x-x_{1}\right)\left(x-x_{2}\right)$ | 3 | 5 | 4 | 1 | 5 | 4 |
| $y=m(x-p)^{2}+k$ | 6 | 10 | 12 | 15 | 11 | 11 |

Table 2: Students' operations in Task 2 according to chosen representations

The students' processes in formulating and applying this new knowledge became apparent as they solved Task 2. Their new knowledge was reflected in the fluent transition between the different representations of functions, and in this case the quadratic function. The students used different symbolic representations, and fluent transitions between the graph representations, and symbol representations. For example, in solving the task, one student chose to represent the function $f$ through this symbolic representation: $f(x)=(x-1)(x-3)$. He then provided the symbolic representation of function k that resulted from horizontal sliding to the right relative to function $f: k(x)=(x-1-6)(x-3-6)$. He also adjusted the values to the location of the function on the Cartesian axis system. For representing function $g(x)$, he moved from a symbolic representation as a multiple of two linear factors two linear factors to representation by vertex. Each of the other representations of the quadratic function chosen by the students provided other examples.

## CONCLUSION

In this study we examined the knowledge and ability of pre-service teachers to shift between representations. The data indicate that even before the topic of shifting between representations was raised, the students' concept about functions was limited. Bardini, Pierce, Vincent and King (2014) found that students either did not create connections between the different types of representation of functions or at least were more likely to identify a function correctly from its graph. Moreover, more students appeared able to recognise a hybrid function when they were provided with its algebraic rule than when they were shown its graphic representation.
According to Vinner and Dreyfus (1989) and Sierpinska (1992), students need to develop a concept of functions through experience with many examples in which the several representations are connected. They then need to summarise and formalise this learning by applying the definition of function. It is critical that students be aware of which aspects of the definition of function they call on when asked to consider functionality and be able to make connections between the various function representations. Dreyfus (2002) contends that to be successful in mathematics, students need rich mental representations of concepts, that is, many connected aspects of the concept. He notes that "poor mental images of the function concept...are typical among beginning college students, who think only in terms of formulas when dealing with functions" (p.32). Connecting representations and gaining "representational fluency" through which students can interpret mathematical ideas in dissimilar representations and then move between them supports the development of strong conceptual schema. Oehrtman, Carlson and Thompson (2008) pointed out that a strong procedural emphasis in which students think about functions only in terms of symbolic manipulations and procedural techniques has not been effective for building a deep conceptual understanding of functions.

The students participating in this study lacked SCK regarding the conceptual image of functions, different representations of functions and the relationship between the representations, and the fluent transitions between them. Their difficulty in shifting to a graphical representation was related to their problems in seeing the changes in the graph for each of the axes separately as well as their difficulties in seeing the changes in the graph for all the points on the graph. This difficulty diminished as they solved additional problems but continued to find expression during the entire intervention process. Bieda and Nathan (2009) used the term "spatially grounded" to describe this type of difficulty that teachers encounter in the context of different representations of functions. In the current study, spatially grounded difficulties were manifested in partial transitions between representations along with only partial generalizations due to incomplete understanding of the relationship between changes in values and changes in the graph on the axis system. Interpretatively grounded difficulties are based on incomplete understanding of the relationship between changes in values and changes in the components of the symbolic representations. The students were unable to interpret the changes in the graphical representations correctly and apply the interpretation appropriately in the symbolic representation. The students' body of knowledge regarding the various representations of functions and the transitions between them increased during the problem-solving process, and typical difficulties at the beginning of the research gradually diminished. Mathematics Teacher educators who train preservice teachers to become high school mathematics teachers must be aware of the students' missing SCK and thus assign the students appropriate tasks to provide them with this knowledge.

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# THE ACTIVITY APPROACH AS THE MAIN MEANS OF TRAINING FUTURE TEACHERS OF MATHEMATICS 

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This article discusses the use of the activity approach as the main means of training future teachers of mathematics in the formation of their ability to selfdevelopment on the basis of reflexive self-organization, which is put forward among the actual requirements for graduates of higher education. Additionally, the formation of abilities for knowledge and the creative use of the received knowledge in any educational and vital situation as one of the leaders in structure of readiness of the teacher for professional activity is described. The purpose of the study: the use of the activity method in the practical lesson "methods of teaching mathematics". The relevance of the activity approach in the formation of students' ability to self-develop on the basis of reflexive selforganization is proved at teaching mathematics at physical and mathematical faculties of pedagogical higher educational institutions which provides acquisition of ability to organize at students search activity.

## INTRODUCTION

Currently, the higher school has the task to equip the graduate with knowledge, to form his ability and desire to learn all his life, to work in a team, the ability to self-development on the basis of reflexive self-organization. The activity method of training helps to accomplish this task. In this regard, the solution of the task in the training of future teachers of mathematics for professional activities is one of the most important goals of higher education.
The formation of the ability to self-development is possible through the involvement of students in search of active activity. The teacher who has the ability to implement social activities "may later be the steward of their own destiny, the continuation of education of his life" (Diesterweg, 2018, p. 25) However, the level of formation of the ability to self-development, selfdetermination, self-realization and reflection in the conditions of pedagogical University does not fully meet modern requirements, tasks of modernization of higher pedagogical education.

Activity method of training allows to carry out: 1) the formation of thinking through learning activities: the ability to adapt within a certain system with respect to its norms (self-determination), conscious building of their activities to achieve the goal (self-realization) and adequate evaluation of their activities and their results (reflection); 2) the formation of a system of cultural values and its manifestations in personal qualities; 3) the formation of a holistic picture of the world, adequate to the current level of scientific knowledge (Kholodenko, 2018, p. 5).

It is obvious that the traditional methods of training future teachers of mathematics, explanatory and illustrative methods, on the basis of which is based today in higher education, does not meet the objectives.

The modern transition to an innovative system of education poses a task for pedagogical science to form and develop a teacher who owns the ability of a person to implement socially significant activities, who owns the technology of development of students' abilities to cognition, creative use of knowledge in any educational and life situation. This technology is an activity-oriented learning method.
In this regard, the activity approach is used as the main means of training future teachers of mathematics. The unconditional requirement in the preparation of the future teacher is the formation of his activity abilities. Purposeful application of activity approach in training of the future teacher demands introduction of modern search methods and forms of training in educational process of higher education institutions. However, the current educational practice does not contribute to the maximum implementation of the activity method in the preparation of future teachers of mathematics and organization of their own educational and cognitive activity of students. There is a need to find ways to implement the activity method and develop new methods of formation of abilities to cognition, creative use of knowledge in any educational and life situations, as one of the leading in the structure of the teacher's readiness for professional activity.

## THE AIM OF THE RESEARCH

The activity method of teaching is theoretically presented in different fields of scientific knowledge and studied by many teachers and psychologists, but its application in practice in the conditions of pedagogical higher education is insufficiently studied that predetermined relevance of this subject.
Subject of research: the use of activity method in various forms of organization of the educational process.
The aim of the study is to implement the activity method in practical teaching. Tasks:

- to justify the relevance of the activity approach
- to show the application of the activity method in the practical lesson "methods of teaching mathematics".
The hypothesis of the study: if the activity approach is applied as the main means of training future teachers of mathematics, it will increase the readiness of the future teacher of mathematics for professional activities.

Research methods: study and theoretical analysis of domestic and foreign literature on the problem of research; observation, generalization.

## THEORETICAL BACKGROUND

The concept of "exercises through activity" was proposed by the American scientist Dewey (Yerokhin, 2006). The basic principles of its system: taking into account the interests of students; teaching through the teaching of thought and action; knowledge and knowledge as a consequence of overcoming difficulties; free creative work and cooperation. The main thing in the activity method is the activity itself, the activity of the students themselves. Getting into a problem situation, students are looking for a way out of it. The function of the teacher is only a guide and corrective in nature.
The activity approach is focused on mastering the ways of professional activity. Basics of personality-activity approach in psychology by the works of Vygotsky, Leontiev, Rubinstein, Davydov and others (Vygotsky, 1982). Personality in these works is considered as a subject of activity, which determines its personal development through activity and communication. The activity is characterized by common essential properties and a single structure, in which the presence of the necessary components (purpose, motive, content, methods, result) provides the result to which the student seeks.
The concept of the activity approach to learning is the situation: the assimilation of the learning content and the development of a student happen not by passing it some information, and in the process his own motivated and purposeful activities This is confirmed by Sukhomlinsky (1973): all our plans, searches and constructions turn into ashes, into a lifeless mummy, if there is no children's desire to learn. From this we can conclude that a necessary condition for the development of the individual student is its high cognitive activity, but not every activity develops the children's ability, and the only one that arouses interest.
According to the personal-activity approach, the purpose of training is formed in the language of activities, where the task is a situation in which it is necessary to achieve a certain goal; the activity itself is the process of achieving the goal; reception is a way of carrying out activities (Mamykina, 2009, p. 135). According to Elkonin (1995, p. 37) "the main difference of the educational task from any other tasks is that its purpose and result consist in change of the acting subject, but not in change of subjects with which the subject acts". Activity approach is carried out by us in the practical classes of the course "methods of teaching mathematics", when there is interaction of students, both with each other and with the teacher. Interaction is one of the integral and essential characteristics of learning in the context of the activity approach. The universality of this category is that it represents and describes the joint activities of students, their communication as a form of activity as a condition, means, goal, driving force. The mechanism of such interaction is seen in the
combination of the ability not only to act, but also to perceive the actions of others.

Interaction thus is a way of being - communication and a way of action - the solution of problems.

The teaching environment is an activity, diverse in content, motivated for the student, problematic in the way of mastering the activity, the necessary condition for this is relations in the educational environment, which are based on trust, cooperation, equal partnership, communication (Leont'ev, 1978, p. 21).
The basis of pedagogical activity is an action. It consists of a chain of interrelated actions, which form its structure. Markova (1994) identifies three main components in the structure of pedagogical activity: 1-motivational and indicative link (orientation in the situation, setting goals and objectives, the emergence of motives); 2-performing link (implementation) and 3-control and evaluation link (result).
At the first stage, the teacher formulates pedagogical goals and objectives (in any kind of activity), the second selects the necessary pedagogical tools for their implementation, not the third - analyzes and evaluates their own actions.
The implementation of the technology of activity method in practical teaching is provided by the following system of didactic principles: the principle of activity, the principle of continuity, the principle of integrity, the principle of minimax, the principle of psychological comfort, the principle of variability, the principle of creativity. The stated above didactic principles set the system of necessary and sufficient conditions of the organization-the activity method of training.

Activity method, we defined as a method of training in which the assimilation of the content of educational material by students and their development do not occur through the transfer of some information to him, and in the process of his self-development on the basis of reflexive self-organization and their own active activity.

## METHODOLOGICAL BACKGROUND

The methodological basis of the activity approach to learning mathematics consists of the following concepts:

- the concept of humanization and humanitarization of mathematical education: the training focuses on the personality of the student; mathematical knowledge is considered as the basis of intellectual development of students (Dorofeev, 1990; Mordkovich, 1985);
- the concept of student-centered learning: involves orientation to the student's personality, strengthening its independence and subjectivity (Yakimanskaya, 1979, etc.);
- the concept and strategy of modernization of General secondary education: the main goal is to prepare a diversified personality, oriented in the modern system of values, capable of active social adaptation in society and independent life choice, self-education and self-improvement (Galperin \& Talyzina, 1968)
- psychological and pedagogical concept of formation and development of cognitive interest of the student's personality in the learning process is considered as an electoral positive focus on the process of cognition that contribute to the internal motivation of the student, his self-activity (Markova,1994; Shchukina, 1979);
- the concept of developing learning mathematics determines the need for training, takes into account and uses the levels of knowledge and features of the student, aimed at the development of a set of qualities of the individual (Vygotsky,1982);
- the concept of the activity approach to learning is that the assimilation of the content of learning and the development of the student is not by transferring a certain amount of knowledge to the student, and in the process of his own activity.
- The concept of the level of assimilation - the ability of the student to perform targeted actions to solve a certain class of tasks associated with the use of the object of study.

Shamova (1982, p. 61) defines the following levels of assimilation of knowledge and methods of activity:

I level - willingness to play consciously perceived and recorded in memory knowledge;
II level-readiness to apply knowledge on a model and in a familiar situation;
III level-readiness on the basis of generalization and systematization to transfer knowledge and methods of activity in the situation of their application;

IV level-readiness for creative activity.
Psychological and pedagogical researches and experience of development and application of pedagogical technologies show that it is expedient to estimate knowledge and abilities of students at the same levels, namely:

Level I-understood, remembered, reproduced;
II level-acquired knowledge at the first level, applied them on the model and in modified conditions, where you can find a sample;

III level - have mastered the knowledge on the second level and learned how to transfer them into an unfamiliar situation without the presentation of ways of working;
IV level-creative activity - cannot be achieved by any of the students, it is the level of gifted (Sporova, 2014, p. 9).
In the field of didactics developed by active learning methods that stimulate cognitive activity and creative abilities of pupils the student is an active participant in the learning process, but he mainly interacts with the teacher.
One of the actual active methods is interactive learning, in which the interaction is carried out not only between the student and the teacher, in this case, all students contact each other. This is more conducive to the development of independence, self-education, solution of communicative problems.
A person becomes a person in the process of joint activities with other people. In other words, every person has the opportunity to become or not to become a person. And it largely depends on the teacher, on how he organizes the joint activities of his students.

An interactive (from "inter" - mutual, "act" - to act) learning model is carried out in conditions of constant, active interaction of students with each other, with the teacher, the environment and provides for certain joint activities of students. In this case, the student and the teacher are equal subjects of education. Note that in modern research interactivity is understood as interaction with the computer and through the computer. Interactive teaching methods "it is always interaction, cooperation, search, dialogue, game between people and information environment" (Kaskatayeva, 2015, p. 57).
Depending on the content studied, the method of the round table, practical work competitions with their discussion, trainings, problem method, modeling of production processes or situations, discussion of special videos, including recording of own actions, methods with the use of computer technology and a skilful combination of traditional and innovative means, forms and methods of training can also be used.
The choice of forms and methods of training used in the educational process depends primarily on the level of individual qualities and abilities of the participants of the group, the activity of the group, the specifics of a particular course, the content of the educational material.

The activity aspect of the content of learning in the activity model of learning is expressed in the fact that the content of learning is the activity in connection with solving the problem and the activity of communication as mastering the social norm, verbal activity and types of nonverbal self-expression, i.e. the educational process is: interaction, solution of communicative (problem) tasks.

Using active and interactive methods of teaching in the classroom, the teacher has the opportunity to form the ability to self-development. This means that interactive learning, as a reception of activity approach to learning mathematics, becomes particularly relevant in the formation of students' ability to selfdevelopment, to self-determination, to self-realization and to reflection.
For the organization of educational activities of the greatest interest are the tasks of intellectual and cognitive plan, which are realized by the students themselves as a thirst for knowledge, the need to learn this knowledge, as a desire to expand horizons, deepening, systematization of knowledge.

## ORGANISATION OF STUDENTS' ACTIVITY

Let us consider the problem of finding the Fermat point of intellectual and cognitive plan at the practical lesson of the course "MPM". The problem is offered to students of the third year of the specialty "mathematics". The theme of the lesson: "learning mathematics through problems". An interactive method is used, which is focused on wider interaction of students not only with the teacher, but also with each other. The teacher directs students to achieve the goals of the lesson. To solve this problem, including organizational and propaedeutic work, it takes two hours (academic).
Task: There are three deposits of oil. Engineers need to build only one refinery. Specify the nearest location of the refinery to visit the specified three oil fields?

This stage of the learning process involves the conscious entry of students into the space of learning activities in the classroom.
The teacher offers students a task that leads them to the self-discovery of a new one.

For the three specified oil fields, find the fourth point, such that if you draw straight lines from it to these points, the sum of the distances will be the smallest.

Decision:
There is a joint activity of students, their communication as a form of activity.
Students connect three points in segments. If the connection will turn out straight, the plant should be built at the point lying between two others (Figure 1).

If the connection of the three points lines you get a triangle, the angles are less than $120^{\circ}$, then search the location of the refinery -F , as follows:


Figure 1
If the connection of the three points lines you get a triangle, the angles are less than $120^{\circ}$, then search the location of the refinery -F , as follows:

1. We construct equilateral triangles $\mathrm{ABC}^{\prime}, \mathrm{BCA}^{\prime}, \mathrm{CAB}^{\prime}$ on the sides of an arbitrary triangle ABC .
2. The resulting triangles will describe a circle.
3. Lines $\mathrm{AA}^{\prime}, \mathrm{BB}^{\prime}, \mathrm{CC}^{\prime}$ intersect at a point F .
4. If all angles of the ABC triangle do not exceed $120^{\circ}$, then F lies in the ABC triangle and is the Fermat point. If one of the angles of the ABC triangle is greater than $120^{\circ}$, then F lies outside the ABC triangle, and the point F coincides with the vertex of the obtuse angle.
At this stage, the reflection of educational activities (outcome).
The new content studied at a lesson is fixed, and reflection and self-assessment by students of own educational activity is organized.
For the student who completed the task, his result is a subjectively new result, as it is new for him, who received it. But he was looking for ways and means to solve this problem, ways that he had never met in his practice and as a result acquired research skills.
At the end of the lesson, when students passed their assignments, the teacher informs them of E. Torricelli's theorem, which gives an algorithm for constructing a Fermat point using a compass and a ruler (Aksenova, 2001).

## CONCLUSIONS

Thus, as a result of observations and generalization of our experience, we come to the conclusion that the joint activities of students in the process of learning, learning material means that each makes a special individual contribution, there is an exchange of knowledge, ideas, ways of activity. This is done in an atmosphere of goodwill and mutual support that allows you to not only obtain new knowledge but also develops the research activities. This activity requires
students cognitive, intellectual ability, motivates to persistent and enthusiastic work on the educational task. So, there is a formation of cognitive, intellectual ability and research abilities of students and through them pupils of high schools.

To continue this study in the future we set the following task: to carry out experimental work to identify the effectiveness of the activity method in the study of higher mathematics courses.

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# A SYSTEMIC APPROACH TO THE IMAGE FOR MATHEMATICS: THE CASE OF SPECIAL EDUCATION 

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#### Abstract

In this paper, we present a theoretical methodological framework that has been developed as a part of a PhD study focusing on the image of mathematics in the special education in Greece. This systemic approach investigates three levels of educational discourse and their relevance to the affective domain of a learner's relationship with mathematics: the research literature; the official instructions, curricula and textbooks; and the various educational protagonists (students, teachers, parents, broader society) involved in both special and mainstream education. The interrelations, included in the proposed conceptualisation image, amongst the accumulated affective orientation of mathematics experience (beliefs, values, attitudes), the real time affective experience (emotions) and the affective pragmatic potential (expectations) construct a visual model of the relationships among the various subsystems of mathematics education.


## PAINTING AN IMAGE OF "IMAGE"

The crucial role of the affective domain in the teaching and learning phenomena has been widely acknowledged (Hannula, 2014; Oatley \& Jenkins, 1996), including the whole affective spectrum (for example, emotions, attitudes, beliefs, values). Ernest (1989, 1995, 2008a, 2008b) adopts a more holistic approach to the affective domain (cf. Hannula, 2011, 2012) to discuss the notion of image of mathematics, referring to a complex whole, consisting of a system of beliefs and views about mathematics. We posit that such a more holistic approach is in line with the affective complexity that the students (and other educational protagonists) experience with and about mathematics.

Moreover, we argue that a positive image for mathematics is being constructed, when a school unit fosters positive attitudes, beliefs, values, emotions and expectations towards mathematics, and that the more positive images for mathematics one student constructs, the better educational results are achieved. Furthermore, the image for mathematics is not constructed exclusively within the school experience, since, for example, the views about the importance or the usefulness of mathematics in everyday life and in the development of civilization are often independent of, or even contradictory to, the school mathematics experience. Thus, these broader constituting conditions crucially affect both the decisions about the role of mathematics in the school system and the family choices with respect to the educational paths of their children.
In this study, which is part of a PhD study, we build upon these ideas to broaden Ernest's conceptualisation, defining as "image for mathematics" the system of
both cooler and hotter aspects of the affective domain (beliefs, attitudes, values, expectations and emotions), thus allowing for a three-dimensional affective mapping of the mathematics experience, including: the accumulated affect, the real time affect and the affective potential. In specific, in this study we included:
a) The accumulated affective orientation towards mathematics experience, as represented by beliefs, values and attitudes. Beliefs are relatively stable across time and task subjective understandings and views (Philipp, 2007), values represent the importance assigned on a belief (Seah, Atweh, Clarkson \& Ellerton, 2008), whilst attitudes describe the positive or negative affective dispositions towards mathematics (Philipp, 2007).
b) The real time affective experience, which is represented in this study by emotions. Emotions describe short-term, spontaneous and volatile affective reflexes (Hannula, 2002; McLeod, 1992; Moutsios-Rentzos \& Spyrou, 2017).
c) The affective pragmatic potential of mathematical experience, which is represented in this study by expectations, defined as the perceived performance capabilities regarding to mathematics (Betz \& Hackett, 1983) that are related to appraisal of mathematical situations and results (Hannula, 2002).

We employ this broadened conceptualisation, in order to investigate the complex image of mathematics: a) both as a discipline and as a course, b) as appearing in the research literature, c) as appearing in the official educational documents, and d) as viewed by the various educational protagonists (students, parents, teachers etc). By adopting such a broadened perspective, it is posited that invisible aspects of the relationships amongst the affective experiences may be noticed, thus enabling a more pragmatic and effective educational planification. For example, these aspects may constitute a context for describing and explaining the genesis and development of otherwise invisible educational obstacles. Importantly, by investigating the images within and across both the mainstream and the special education school unit systems, the multidimensional cross-mappings of the existing educational interactions and networks is feasible (Moutsios-Rentzos \& Kalavasis, 2016), allowing for a pragmatic address of the call for inclusivity, which is at the crux of the contemporary educational programmes and curricula (Brusling \& Pepin, 2003; UNESCO, 2001, 2005; UNICEF, 2007; Vislie, 2003). It is posited that such a mapping, on the one hand, reveals the images of mathematics and their implicit interactions as painted in the different systems (mainstream and special education) and experienced by the educational protagonists. On the other hand, our approach crucially allows for the planification of the communication of those images, which constitutes an important first step for empathy and inclusivity.

Following these, in this paper, we discuss a study investigating the images for mathematics taught at special education school units from three aspects: a) the research literature, $b$ ) the official (and institutional) documents, and $c$ ) the views held by the educational protagonists (students, teachers, parents, broader society) of both special and mainstream education.

## A SYSTEMIC APPROACH TO THE SCHOOL UNIT

A system is defined as a whole, the parts of which are linked and interrelated in such complex ways that the constructed whole significantly differs from a simple adding of its parts (Moutsios-Rentzos \& Kalavasis, 2016; MoutsiosRentzos, Kalavasis \& Sofos, 2017). Every system may consist of parts or/and sub-systems which interact with respect to a specific goal (Moutsios-Rentzos \& Kalavasis, 2016). Systems are characterized by the level of interaction with their environment or/and their hyper-systems, and by the volume and speed of input and output they demonstrate (Moutsios-Rentzos \& Kalavasis, 2016).


Figure 1: An approach to the complexity of school mathematics education (MoutsiosRentzos \& Kalavasis, 2016, p.105)
In our approach, we view the special education school unit as an open learning system, consisting of subsystems and elements that interact with each other, and, at the same time, is a part of a broader interacting social and educational system, as well as with the family system (Kalavasis, 2007; Kalavasis \& Kazadi, 2015). In specific, we draw upon Moutsios-Rentzos \& Kalavasis (2016), who investigated the links of mathematics within and across the system of disciplines and the school unit system. Their approach was accompanied with a model, which helped visualising and quantifying the views that the different educational protagonists hold about mathematics (see Figure 1): a) as a discipline in comparison with other disciplines, and b) as a course in comparison other courses, including three foci (the symbolic/normative, the protagonists' perceived official regulations; the pragmatic, the actual lived school reality; the desired/intentioned actions, assuming the power to implement them).

## IMAGES FOR MATHEMATICS: A SYSTEMIC APPROACH

Consequently, in this study, we focus on the images for mathematics in special education school units, to investigate the complex relationships amongst the accumulated affective orientation of mathematics experience (beliefs, values, attitudes), the real time affective mathematics experience (emotions) and the affective pragmatic potential of mathematics (expectations). The image of mathematics lens is applied to three areas of experience: a) the documents of the research literature, b) the official documents, and c) the views held (the symbolic/normative, the pragmatic and the desired/intentioned actions) by the protagonists of special and mainstream education (students, teachers, parents, broader society).
In this section, we flesh out these ideas to propose a methodological theoretical framework paired with a methodological instrument, which allows us to consistently investigate the images for mathematics across all three areas of experience (research literature, official documents and protagonists). Thus, in each area, we investigate the constructed image for mathematics by investigating the five aforementioned elements of affect: beliefs, attitudes, values, emotions and expectations. Our approach broadens Ernest's conceptualisation (1989, 1995, 2008a, 2008b), as the included elements concern three different levels of affective experience: the accumulated affective experience, the real time affective experience, and the affective pragmatic potential of mathematical experience. As a result, the links amongst these five elements may be organised through a trigonal bipyramid (see Figure 2).


Figure 2: The trigonal bipyramid of image as conceptualised in this study.
Each element of image is possible to be investigated, as Moutsios-Rentzos and Kalavasis (2016) proposed, within two systems (also in line with Ernest, 1995): the system of all disciplines and the system of the courses taught in school (in this case, in special education; depicted as a dipole, see Figure 3 left). Though when investigating the documents the dipole model suffices, when investigating the protagonists' views requires a pyramid model, with the triangular base of the pyramid representing the aforementioned three foci of 'mathematics as school course' (Moutsios-Rentzos \& Kalavasis, 2016): the symbolic/normative ('s' on
the pyramid), the pragmatic ( p ' on the pyramid), the desired/intentioned actions ('d' on the pyramid). Hence, a triangular pyramid is constructed the base of which represents mathematics as a school course (through the three foci) and its apex represents mathematics as a discipline (see Figure 3, right). Consequently, the image for mathematics in the school unit with respect to the protagonists is a synthesis of the tri-focussed pyramid, applied on each of the element of the trigonal bipyramid of image (see Figure 4).


Figure 3: Mathematics as a discipline and as a school course (Documents, left; Protagonists, right).


Figure 4: The protagonists' image of mathematics as a discipline and as a school course.

When looking into the system of the School Unit, the images for mathematics can be investigated for each of the Protagonists (the students, the teachers, the parents, and the broader society; in line with Ernest's (2008a, 2008b) differentiation between social and personal images), as well as for the Documents (research literature and official documents). The students, the teachers and the parents are considered to have significant relevance in the case of mathematics and are explicitly elements of the school unit system (Galindo \&

Sheldon, 2012; Kafoussi, 2005, 2006; Kafoussi, Moutsios-Rentzos \& Chaviaris, 2017; Moutsios-Rentzos, Chaviaris \& Kafoussi, 2015).


Figure 5: The School Mathematics Image (Documents, left; Protagonists, right).
Thus, we posit that the School Mathematics Image emerges across and within the interactions of the images of the Protagonists and of the Documents. The 'triangular space' amongst the three poles (students, teachers and parents) represents the "school unit" system (mainstream or special education), which interacts with the broader society (see Figure 5, right). Each pole consists of the trigonal bipyramid of image. Each element of the image represents the five elements of affect (beliefs, attitudes, values, emotions and expectations). Each pyramid hints the investigation of image for mathematics as a discipline and as a school course at the special education school units in three foci (the symbolic, the pragmatic representations, the desired actions). Considering the written Documents (research literature and the legislation/ official instructions and documents), the pyramid of the protagonists is replaced by the documents dipole (Figure 5, left).
We posit that the proposed theoretical methodological framework explicitly includes and allows for the investigation of the possible hidden interactions, concerning the affective domain, amongst students, teachers, parents and the broader society, which would be otherwise be conflated and/or invisible. For example, the image for mathematics at special education, consisting of the five elements of affect (beliefs, attitudes, values, expectations, emotions) may be investigated in two systems: both as a discipline compared with other disciplines (system of disciplines) and as a course at special education in contrast to other subjects (system of subjects). It is argued the higher level of sensitivity, which the proposed framework has, is especially important for rendering inclusive education to be a pragmatic call, since the invisible obstacles pose barriers to providing equal educational opportunities to all children and, hence, equal citizenship opportunities. Through this perspective, we claim it is possible to
identify potential differences amongst, on the one hand, the images of mathematics in various sources of documents, including the curricula of special education and assessment (for example, Van de Rijt, Van Luit \& Pennings, 1999), as well as the related special education research literature (for example, Nunes, 2004), and, on the other hand, amongst the images of the educational protagonists about mathematics when referring to special education school units, including mathematical ability, content, methods and materials (for example, Beswick, 2008; DeSimone \& Parmar, 2006; Jungert \& Andersson, 2013).

## CONCLUDING REMARKS

In this paper, which is a part of a PhD study, we draw upon systems theory to conceptualise the school units as learning systems and sub-systems of the broader social and educational system, thus interacting with their sub-systems, their elements and the broader society. According to this theoretical framework, we propose a methodological approach to concurrently investigate the images for mathematics taught at special education settings and mainstream education settings in three areas of experience: official instructions and documents; research literature; views held by four educational protagonists (students, teachers, parents, broader society). We defined image to be a system constituting of five affective elements: beliefs, attitudes, values, emotions and expectations. In the investigation of the protagonists' image, we refined our approach by applying the lenses of three foci: the symbolic, the pragmatic and the desired actions. We posited that the proposed image construct would facilitate the realisation of inclusivity in education by revealing hidden aspects of the images (of both mainstream and special education school units) for mathematics both as a discipline and as a school course taught at special education school units, thus rendering possible to more pragmatically and effectively re-plan the teachinglearning process.

Overall, in this paper, we propose a systemic theoretical methodological approach that provides a mapping of the complex interactions between special education and mainstream education, as depicted in the images for mathematics in special education (both as a discipline and as a school course) of the educational protagonists and the documented realities of the research literature and the official instructions and documents. Importantly, the interactions, as well as the interactions of the aforementioned interactions may also be become visible to the educational policy makers. Crucially, being visible entails that some connections, which might affect the learning process, can be a part of the educational planning process; be predicted, corrected, and/or upgraded.

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[^0]:    ${ }^{1}$ In Czech language, the name of coins is one word.

[^1]:    ${ }^{2}$ Neither task 7A1 or 7A2 was at the end of the test in no version of the test and thus, time constraints were not to be blamed for this.

[^2]:    ${ }^{3}$ We wanted to see if the pupils were able to make a correct problem model, their ability to calculate was secondary.

[^3]:    ${ }^{1}$ In French there is one term used to describe common difference or common ratio.

[^4]:    Principal Axis Factoring ( $\mathrm{N}_{\text {Broader }}=185$ ). Varimax rotation with Kaiser Normalization. $44.2 \%$ variance explained. Cross-loadings $>0.240$ are reported with [ ].
    ${ }^{2}$ F1: Validity. F2: Proof for some. F3: Intellectual challenge-Understanding. F4: Certainty. F5: Explanation.
    ${ }^{3}$ Only for the students discussed in this paper ( $\mathrm{N}=63$ ).
    4 ' 1 '-' 5 ' scale. ' 1 ': Disagree a lot, ' 2 ': Disagree, ' 3 ': Neither disagree, nor agree, ' 4 ': Agree, ' 5 ': Agree a lot.

[^5]:    

[^6]:    ${ }^{4}$ This kindergarten (students from 3 years and 9 months to 6 years and 6 months old) was chosen a) because of the frequent cooperation we have with the teacher who likes integrating innovations in her teaching, b) because the children in that classroom were able to compare two objects directly and recognize their equality or inequality, c) because they were also able to place in order objects according to their length and d) because they knew to count and write numbers up to 100 .

[^7]:    ${ }^{5}$ r: red, o: orange, y: yellow, b: blue, p: purple
    ${ }^{6} \mathrm{lg}$ : light green, br: brown, w: white, p: purple, b: black, dg: dark green, b:blue, y: yellow, o: orange

[^8]:    ${ }^{1}$ Translation of the text from the solution: I drew two triangles, the first ABO , the Second BCO. After measuring, it can be stated that the triangles are congruent that they are the same.

[^9]:    ${ }^{2}$ Translation of the text from the solution: The area of the shaded figure is half of the rectangle.

[^10]:    ${ }^{3}$ Translation of the text from the solution: The area of the shaded figure is equal $\frac{2}{4}$ of field of the rectangle.

[^11]:    ${ }^{4}$ Translation of the text from the solution: $(x+x)+\left(y+\frac{1}{2} y+\frac{1}{2} y\right)=$ the whole rectangle. The shaded area is half of the rectangle, because 1 non-shaded triangle is divided into 2 parts after assembly are 2 shaded and 2 shaded.

[^12]:    ${ }^{5}$ Due to the limited volume of the article, the author does not put in full work sheets used in both stages of the pilot study. The interested in getting to know test cards author asks for direct contact.

