Inquiry based mathematical education

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INTRODUCTION

Inquiry based learning is among the popular trends in teaching in the recent years. In many school subjects students are encouraged to get involved in projects, study many relations, search for solutions, discover properties, put and verify hypotheses. It is widely accepted that such an approach relies on independent solving of a given problem by students or researchers. It is also assumed that students in the course of solving a given problem can gain knowledge about the subject and learn about its properties. The role of the teacher is not to demonstrate ready-made solutions, but to rather suggest ways to search for them. By independently searching for solutions the students are able to learn much more and that active learning supports the establishment of deeper knowledge. In the course of exploration, the students are able to reach higher scientific thinking and at the same time develop intellectually.

Teaching and learning mathematics poses some unique challenges to researchers and educators. Not every solution tested in other fields can be adopted to mathematics classes. Some years ago the results of many studies have shown that the 'transfer' model of teaching brings disappointing and short-term effects in mathematics education. Thus, the constructivist approach was developed as an opposite pole. The main assumption of constructivism is the support to the student in an individual building of knowledge. So, how are the assumptions of "Inquiry based mathematical education" related to the general principles of constructivism and in which points does the organization of classes differ in the two approaches? What is the role and the place of the teacher within the new organization of the learning process? How far does the deductive nature of mathematics enable the acquisition of knowledge by inquiring? How to implement these principles of teaching at different educational levels? How to prepare teachers for the new challenges that this approach creates?

The authors of the studies included in this publication attempt to find answers for these as well as other research problems. We invite the reader to study these texts thoroughly and we hope that the proposed solutions and examples can be a starting point for the reader's own research in the field.

Rzeszów, June 2016

The Editors

Characteristics of inquiry based mathematics education

Part 1

BASING ON AN INQUIRY APPROACH TO PROMOTE MATHEMATICAL THINKING IN THE CLASSROOM

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In mathematics the art of proposing a question
must be held of higher value than solving it.

(G. Cantor)

One of the most delicate issues in the teaching/learning of mathematics is ensuring that students acquire the mind-set for grasping the mathematical sense of the teaching situations they face: researchers like A. Schoenfeld speak of mathematical sense making (1992). He argues that each discipline has its own version of "sense making": historians have their own way of looking at the world, as well as anthropologists and physicists. What makes each of these distinct areas are the instruments, norms and habits of mind. The same happens for mathematics: it is the exact opposite of the image that many people have of mathematics as a set of rules and algorithms to be learned by heart to answer questions sometimes far from any real sense.

How can we point out the main features of mathematical sense-making? How can we make concretely alive in the classroom? My intervention will try to answer these two questions.

MATHEMATICAL SENSE MAKING: EPISTEMOLOGICAL AND DIDACTICAL ISSUES

According to Schoenfeld (*ibid*.)

> Mathematics is a social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics").[…] The tools of mathematics are: abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman.

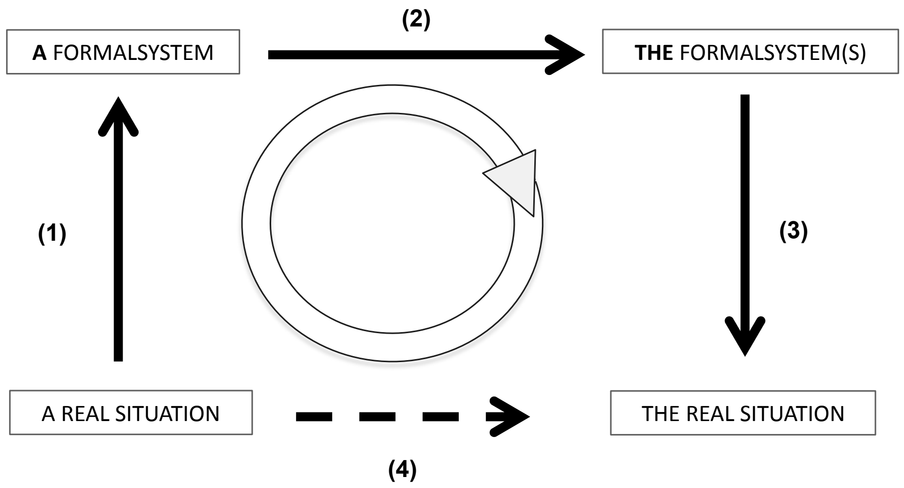
Learning mathematics means learning to think, act and communicate mathematically. More precisely:

(a) developing a mathematical point of view — valuing the processes of mathematization and abstraction and having the predilection to apply them according to precise goals;

(b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure

(c) developing competencies in communicating, first to herself and then to others, the results of one's mathematical thinking and doing, that is "the mathematician's way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution". (Rav, 1999, p. 13) The word 'problem' is used here in a generic sense for any open question, posed to the students or which can emerge during the solution process, as it will be discussed below.

In fact, the classes are cultural environments in which the activities and daily practices define and give a meaning to the topics that are taught: hence students develop, more or less consciously and more or less consistently, but inexorably, (a protocol of) rules to follow, for example to succeed or at least to "survive" to the questions of the teachers. This is the way they develop their own sense for mathematics. The trouble is that there can be a big difference between the teachers' intended meaning for mathematics she/he is transmitting to the students and the sense of mathematics for students, as a result of their experiences and practices in this domain inside and outside the school, and this may generate misunderstandings.



(1) Aspects of the real situation represented in the formal system

(2) Treatments within a formal system/ Conversions between formal systems

(3) Interpretation of the results of the formal system in the real situation

(4) Interpretation/theorization of the real situation with a theoretical lens

Figure 1. A model of a virtuous dynamics real-formal

Many times these misunderstandings can be very subtle and based on the complex interactions that practices in the classroom produce in the mathematical sense given by students to the discipline. A remarkable example for this is given in a research of L. Healy and C. Hoyles (2000) about the sense that high-attaining 14- and 15-year-old students attach to proof in algebra. The survey was made in UK and concerned almost 2500 students. They found that students simultaneously held 2 different conceptions of proof: those about arguments they considered would receive the best mark and those about arguments they would adopt for themselves. In the former category, algebraic arguments were popular. In the latter, students preferred arguments that they could evaluate and that they found convincing and explanatory, preferences that excluded algebra. A sort of double moral, hence of a double sense that students attached to mathematics: so to say, an epistemic one, in which they believed, but was rooted on an empirical basis, and a pragmatic one, the one that they thought was loved the most by their teacher, which was more formal.

Unfortunately, in many cases in our classrooms we find that in students' mathematical reasoning the formal and informal aspects are deeply separated: instead of having a virtuous circle between the two (see Fig. 1) a barrier seems to inhibit any communication between the two levels. A big didactical and cognitive problem arises when the cycle is interrupted: possibly a "suspension of sense" for mathematical statements is generated. Often in classroom practices there are two separated worlds (like in the students examined by Healy and Hoyles), one represented by the components in the top part of the picture in Fig. 1 (the formal side), and the other instantiated by the components in the bottom part (the empirical side).

The sense suspension can be one of the main roots of many mistakes of our students: it causes real comedies of errors in their interpretations of problems and of mathematical statements. Giving them the right definitions seems not enough!

THE METHOD OF VARIED INQUIRY: AN EXAMPLE

In what follows I will illustrate how one can design suitable learning situations and pursue classroom practices, which can generate a genuine mathematical sense basing on an inquiring approach, where problem solving is deeply and dynamically intertwined with problem posing because of the didactical situation, which is designed for the classroom. The goal is to propose a figure of the teacher, who is not (any longer) a transmitter of rules, but a promoter of mathematical sense making for students.

The proposed teaching-learning method is called Method of Varied Inquiry (MVI): it is based both on the Method of Variation and on the Logic of Scientific Inquiry, which dates back to Galileo, and nowadays can also make use of ICT.

I will introduce it through an example and after that I will draw some general comments.

An example

Let me simulate with you a class situation, adapted from Brown & Walter (2005). The didactical plan is divided in two main steps.

Step 1: "Looking for patterns"

1	3	3
2	4	8
3	5	15
4	6	24
5	7	35

Figure 2a

1 • 5	5
2 • 6	12
3 • 7	21
4 • 8	32
5 • 9	45

Figure 2b

1 • 5	5	4 + 1	9 – 4
2 • 6	12	9 + 3	16 - 4
3 • 7	21	16 + 5	25 - 4
4 • 8	32	25 + 7	36 - 4
5 • 9	45	36 + 9	49 - 4

Figure 2c

Question1: what do you observe?

Examples of students' answers:

O1. There are always two factors, which reproduce the succession of natural numbers, starting from 1 and from 3.

O2. In every line there is a difference of two between the two first numbers.

O3. The third number is the product of the first two.

O4. The products (3, 8, 15, 24, 35) are almost perfect squares: to get them you must add 1.

O4. These perfect squares are the same as the squares of the numbers between the two factors: 3-4-5 → 16; 5-6-7 → 36; …

O5. Also the differences between the products show an interesting pattern: 8-3 = 5; 15-8 = 7; 24-15 = 9; 35-24 = 11; …

…

One can also use an excel spreadsheet to continue the sequences of numbers.

Step 2: "What if…?"

Starting from the observations O_i made by the students, the teacher can suggest to change some property, for example O_2: "now suppose that the factors differ by 4. We get the table of Figure 2b.

Remembering the observation O_4 one can ask (Q2):

what about squares?

are still they there?

can we discover them again?

Students can investigate the situation and hunt for squares. They generally find two types of solutions, illustrated in figure 2c. Then the teacher can ask what of the two types of found squares is more "equal" to O_4: those in the third or in the fourth column? Why?

The situation can go further on looking for other situations, where the difference between the factors is a generic number *d* and see that there is a difference between the case when *d* is even or odd. Algebraic language can enter the scene and explain better the situation.

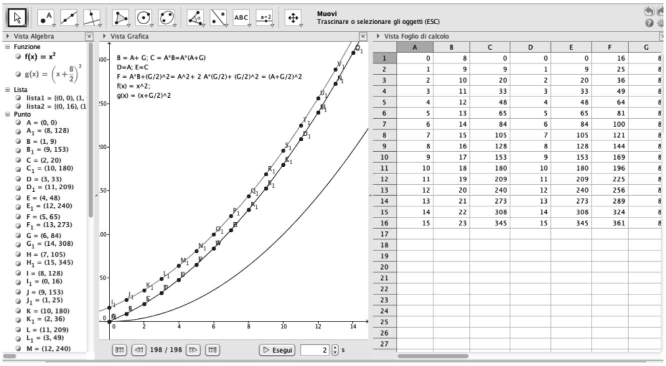


Figure 3

The section can end with a mathematical discussion in the classroom, where the students can write down some mathematical sentence of the type "if…then…", e.g. "if the difference *d* between the factors is even, then adding $(d/2)^2$ to the product the result is a perfect square" (namely the square of the smallest factor increased of $d/2$). The algebraic language can explain all these regularities, and possibly excel can help students to arrive to the formulas:

$n(n+ d)+ (d/2)^2 = (n + d/2)^2$, and $(a+b)^2 = a^2 + 2ab + b^2$.

Using GeoGebra allows investigations through a "game of frameworks" (Douady, 1986) interacting each other (Fig. 3).

It is interesting to discuss the didactical and cognitive difference between a sequence of problems as that generated by our method and problems that directly ask to prove the algebraic formulas behind our tables.

Using the framework of A. Sfard (2008, p. 120) we can summarize what we have done from a higher point of view as a dynamic expansion and compression:

Step 1. A situation: observe (O_i), ask questions (Q_j), give answers (R_k): these processes correspond to the question "Why is it so?".

Step 2. Modify one or more O_i changing the situation → (not O_i). Fresh observations (O_i)*, further questions (Q_j)* and answers (R_k)* are so generated: these processes correspond to the question "What happens if it is not so?".

The full process can be so summarized through different levels (*Method of the Varied Inquiry*, **MVI**):

Level 0. Choosing a starting point

Level 1. Listing the observations (Oi); asking questions (Dj)?

Level 2. What if it is not so? (~Oi)

Level 3. Posing consequent fresh problems/questions (Dj)*?

Level 4. Analysing the different (Dj)*

Level 5. Metareflection:

- the question (Q_j) allows to produce deductions of the type If (O_i) then (R_k);
- the question (Q_j)* allows to produce deductions of the type If (not O_i) then (R_k)*

THE METHOD OF VARIED INQUIRY: GENERAL COMMENTS

The main feature of MVI is that it pushes students to varying a didactical situation exploiting its different possibilities, formulating conjectures, looking why these are true or false and formulating such reasons in a suitable mathematical language.

MVI is relevant from an epistemological, didactical, and cognitive point of view, with consequences for the teaching practices. In fact its pedagogical basis is the so-called Theory of Variation.

According to the Theory of Variation, a key feature of learning involves experiencing a phenomenon in a new light (Marton, F., Runesson, U., & Tsui, 2003). In other words, "learning amounts to being able to discern certain aspects of the phenomenon that one previously did not focus on or which one took for granted, and simultaneously bring them into one's focal awareness" (Lo, Chik & Pang, 2006, p.3). Thus, teaching with variation helps students to actively try things out, and then to construct mathematical concepts that meet specified constraints, with related components richly interconnected (Watson & Mason, 2005).

Watson and Chick (2011) highlight the importance of teachers selecting mathematical tasks and examples with adequate variation to ensure that the critical features of the intended concept(s) are exemplified without unintentional irrelevant features. Thus, a crucial point in the use of variation is that it should be controlled and systematic in every case.

Gu (1981) identified two major types of variations in mathematics teaching: *conceptual* variation and *procedural variation*. Conceptual variation aims at providing students with multiple perspectives and experiences of mathematical concepts. Procedural variation aims to provide a process for formation of concepts stage by stage, in which students' experience in solving problems is manifested by the richness of varying problems and the variety of transferring strategies (Gu et al., 2004). According to Gu et al. (2004), procedural variation is derived from three forms of problem solving:

1. varying a problem: extending the original problem by varying the conditions, changing the results and generalization;
2. multiple methods of solving a problem by varying the different processes of solving a problem and associating different methods of solving a problem;
3. multiple applications of a method by applying the same method to a group of similar problems.

The Method of conceptual variation consists in varying the didactical situations according the following four principles:

CONTRAST "... In order to experience something, a person must experience something else to compare it with..."

GENERALISATION. "... In order to fully understand what "three" is, we must also experience varying appearances of three..."

SEPARATION "... In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant."

FUSION "if there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously." (Marton, F., Runesson, U., & Tsui, 2004, p. 16)

MVI promotes hypothetical thinking: in fact, it triggers and supports discourses that allow students:

- to go back on what has been done, seen (etc.), producing interpretations, explanations, answers to questions like "why is it so?"

- to anticipate events, situations, etc., producing forecasts, talks about hypothetical worlds, answers to questions ”how will it be like?", "how could it be?"

MVI is an engine for reasoning and arguing: in fact, it supports the production of abductions by students. Abductions are a cognitive mechanism studied the first time by C.S. Peirce, which produce discovering in mathematics and sciences.

A typical example of abduction is explained by Pierce as follows: suppose that we know that that sack is full of white beans. Consider the statements:

A) these beans are white (a RESULT: A) ;

B) if these beans came from that sack they would be white (a RULE: C → A);

C) probably these beans come from that sack (a CASE: C).

An abduction is different from a deduction, which would be: if C and if C → A, then A.

The general form of an abduction (Peirce, 1960, p. 372) is:

- One observes a fact A;

- were C true, certainly A would be true;

- hence it is reasonable to assume that C is true.

According to Pierce abductive processes are the unique logical operations which introduce new ideas .

Another basis for the MVI is its logical frame, namely that of the so called Logic of Inquiry. It has been developed by Jaako Hintikka, an important Finnish logician.

> This idea [of the Logic of Inquiry] is as old as Socrates, and hence older than most of our familiar epistemology and logic. It is the idea of knowledge-seeking by questioning or, more accurately, of all rational knowledge-seeking as implicit or explicit questioning. I am using the phrase "inquiry as inquiry" to express the idea. For what my leading idea is precisely an assimilation of all rational inquiry in the generic sense of searching for information or knowledge to inquiry in the etymological sense, that is, to a process of querying, or interrogation. (Hintikka, 1999, p. ix)

The basic idea on which this model is based […] can be expressed most easily in the jargon of game theory. We define a certain two-persons semantical game played within an “environment” M (= model for the language of the theory).

> The players are called Myself (the initial verifier) and Nature (the initial falsifier). When the game starts, Myself has the role of verifier and Nature that of falsifier. At each stage of the game, intuitively speaking, the player who all that time is the

> verifier is trying to show that the sentence considered then is true and the falsifier is trying to show that it is false. (Hintikka & Sandu, p.416)

This model is pushed forward by the MVI. Its methodology is near to that of a controlled experiment in science: in fact, it main idea is that the scientist can vary, within certain limits, one variable (the controlled variable) and observe how another variable (the observed variable) changes accordingly. The outcome of a controlled experiment is hence a statement of the dependence between the two variables. It will say that for each value of the controlled variable x there corresponds such-and-such a value of the observed variable y. An answer will specify a function, say *f* which states how *y* depends on *x.* This happens every time there is a sentence of the form "for all *x* there is a *y* such that...".

This approach can be nicely implemented in DGS environments, particularly in multitouch devices, where you can have two players simultaneously moving different points, each with her own aim, which conflicts the other one. In this case the logic of inquiry can be the result of an instrumentation process with the computer (Arzarello, Bairral, Soldano, 2014).

The logic of inquiry and through it the MVI is a producer of abductions. A typical example is discussed in Arzarello et al. (2012) in DGS environments. Here I give an example taken from the Sherlock Holmes Novels, which is one of the best examples of the intertwining of the logic of inquiry with abductive moves. It is taken from the Novel Silver Blaze.

<u>Background</u>: the famous racing-horse Silver Blaze has been stolen from the stables in the middle of the night, and in the morning its trainer, the stable master, is found dead out in the heath, killed by a mighty blow.

> Inspector: «Is there any point to which you would wish to draw my attention?»
>
> Sherlock Holmes: « To the curius incident of the dog in the night-time.»
>
> Inspector: «The dog did nothing in the night-time»
>
> Sherlock Holmes: « That was the curious incident.»

Sherlock Holmes is in effect asking three questions:

- Was there a watchdog in the stables when the horse disappeared?

  Yes

- Did the dog bark when the horse was stolen?

  No, it did not even wake the stable boys in the loft

- Now who is it that a trained watchdog does not bark at in the middle of the night?

  His owner, the stable master, of course [Abduction]

The following deductive argument is the transposition of the three questions above:

There was a watchdog in the stables

The dog did not bark when the horse was stolen

A trained watchdog does not bark only at its owner

Hence the thief was the owner.

The abduction marks the transition from an inquiring to a deductive approach.

MVI has important didactical consequences: the teacher can promote and support it in her/his teaching practices, fostering in this way the transition of its use from everyday life to the mathematical context. Doing so, she/he allows the construction of mathematical skills, in which knowledge is intertwined with argumentative skills of the students in situations where they pose and solve problems.

In short, MVI induces an open attitude to the inquiry, in which the students:

- pose and solve problems;
- produce hypotheses, definitions, arguments;
- are not embalmed in the typical closed scheme: given situation → solve/show;
- technology can strongly support and fasten such processes.

The variations are generated by the students themselves (with the support of the teacher, stronger at the beginning, more and more attenuated when the method is progressing in the class): hence the control in posing the problem passes from the "others" to themselves: it so generated a broader conception of what is a problem and a greater emotional sharing.

Variations deal with the same subject under different points of view: hence they generate a deeper and wider understanding.

The result can be a shared and positive sense for mathematics.

CONCLUSIONS

In my paper I have discussed the following points:

1. The sense of students for mathematics and the danger of the “suspension of sense” induced by some educational practices.

2. The suggestion of MVI as a method against the suspension of sense, since it involves students as actors in the process.

3. MVI helps students to consider a topic from different points of view, and to understand it in a deeper way (according to the method of variation).

4. MVI as a method based on the Logic of inquiry.

5. MVI as an engine to generate arguments through problem solving/posing and to support the transition from forms of "natural" arguments (abductions) to forms of mathematical reasoning (deductions).

These comments give reason of the didactical, cognitive, and epistemic sense of MVI, whose main ideas is also expressed by the quotation by Cantor at the beginning.

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Preparing teachers for an inquiry based mathematics instruction

Part 2

INQUIRY-BASED LEARNING APPROACHES AND THE DEVELOPMENT OF THEORETICAL THINKING IN THE MATHEMATICS EDUCATION OF FUTURE ELEMENTARY SCHOOL TEACHERS

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In the years 2013-2015, together with my research assistants, Georgeana Bobos and Ildiko Pelczer, we conducted a design experiment (Cobb, diSessa, Lehrer, & Schauble, 2003) on teaching fractions with three cohorts of undergraduate students taking a "Teaching Mathematics II" course offered as part of an Elementary Education program at a North-American university. Our approach to teaching fractions to these groups of pre-service elementary teachers was inspired by V.V. Davydov's "Measurement approach" to teaching fractions to children. In the first year of the experiment, our pedagogy had several features of an inquiry-based learning approach. This appeared to be a rational choice for us, because our design of the Measurement Approach was still very sketchy at the time and we needed to be open to modifying it based on the future teachers' response to it. Thus, we treated the future teachers more as fellow learners and teachers than as students. We tried to establish a "community of inquiry" where we would learn from them and they would learn from us, as we all solved and posed mathematical problems to each other. Assessment in the course was not based on quizzes, tests or exams but mostly on the products of the students' creative activity as teachers: the math problems they posed and the activities they planned and simulated with their peers. At the end, the future teachers were quite satisfied with the course, some telling us that they have learned "a lot." We, too, have learned from interacting with them, and a much more mature form of the Measurement Approach was born as a result. But we were not satisfied with the students' understanding of fractions; their understanding at the end of the course hardly differed from their initial understanding of fractions as, overwhelmingly, "parts of a whole." How did it happen? What aspects of the pedagogy (the inquiry-based learning approach), of the didactics of fractions (the Measurement Approach), and of the future teachers' ways of thinking could be responsible for such results? In the paper, I will reflect on these questions while telling the story of how our initial approach to fractions morphed, under the influence of future teachers' response, into the more mature Measurement Approach that we taught in the two subsequent rounds of the experiment.

INTRODUCTION

After having observed my colleagues teaching courses aimed at preparing pre-service elementary teachers to teach mathematics (Sierpinska & Osana, 2012), I took courage to teach such course myself. I applied and obtained the dean's permission to do it. This permission was necessary because the course was in the department of education while I belonged to the department of mathematics and statistics. The name of the course was "Teaching Mathematics II". Its curriculum included fractions, ratio and proportion, and geometry. I was not interested in just teaching the course; I wanted to teach it my way. I had a special interest in fractions, because I did not like the way fractions were presented in school textbooks and in books addressed to pre-service elementary teachers. Those presentations were usually close to what Davydov and Tsvetkovich (1991) called the "Visual Approach", while I leaned more towards the "Measurement Approach" (ibid.) which made more sense, epistemologically (in history, fractions arose in the practice of measurement), and mathematically. But the approach they proposed in the cited paper was a description of teaching fractions to children, not to future teachers. Developing a Measurement Approach for pre-service teachers required much preparatory theoretical work, design and experimentation – in brief, a design based experiment (Cobb, diSessa, Lehrer, & Schauble, 2003) – and so this is what I embarked on with the help of my doctoral students at the time, Georgeana Bobos and Ildiko Pelczer. Of great help in the first year of experimentation was also Edyta Nowinska, a mathematics education researcher from Poland, who was visiting Concordia University at the time[1].

After several months of preparations, I stood, in January 2013, in front of a class of 34 university students, future elementary teachers, with Georgeana and Ildiko as my research and teaching assistants, and Edyta as an observer. This paper will focus on the first year of the experimentation: the first version of the Measurement Approach to fractions for teachers, the difficulties we encountered in implementing the approach and what we learned from them.

I will start by presenting some features of the Measurement Approach to fractions for pre-service teachers, describing mainly the first version but also mentioning how it developed in subsequent years[2]. I thought that the approach will be clearer if I explained how it departs from the more commonly used Visual Approach as presented in a textbook destined for elementary pre- or in-service teachers (Parker & Baldridge, 2003).

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[1] When, in this paper, I use the plural "we" in speaking about observations or findings from the first year of experimentation, I am referring to Georgeana, Ildiko, Edyta and myself. When talking about further developments in the research, "we" refers to Georgeana, Ildiko and myself.

[2] Details of the most recent version of the approach can be gleaned from the "Class Notes" for students in the 2015 class of the Teaching Mathematics II course (Sierpinska & Bobos, 2015).

Next, I will describe our pedagogical approach in the course. It had, I shall argue, some features of the *inquiry-based approaches*.

In section 4, I will present an overview of the future teachers' conceptions of fractions as they were revealed through the activities on fractions for children that they designed – the actual products of the inquiry based learning.

I will conclude by discussing the changes that we decided to introduce in *the content* (Measurement Approach) and *the pedagogy* as a result of this first year experience.

MEASUREMENT APPROACH TO FRACTIONS FOR ELEMENTARY TEACHERS

I will explain the measurement approach (MA) to fractions by contrasting it with the treatment of fractions in a textbook for elementary teachers by Parker & Baldridge (2003). (I will use the abbreviation P&B in the references below to the book). I have chosen this textbook, because it is quite close to what Davydov and Tsvetkovich call the Visual Approach to teaching fractions and they define Measurement Approach by its contrasting it with the Visual Approach. Moreover, P&B is considered to be a good book by American mathematics educators[3].

The textbook starts with whole numbers and, for the authors, "Mathematics begins with counting." (P&B, p. 1). They stress that "A number is an abstract idea. We can speak of '3 apples' or '3 people', but when we say simply '3' or '3 + 3 = 6' we mean something more abstract."(P&B, p. 2). In MA, we also stress the difference between "3 apples" and "3". But we do not suggest that saying "3 apples" might be a symptom of a misconception about numbers. We go on to saying that "3 apples" is a "denominate number" which represents a quantity, while "3" represents an "abstract number": an answer to the question "How many?" For example, "3 feet" is a denominate number. It represents a length. Three is an abstract number; it answers the question: How many times this given length is larger than the unit called 'foot'?"

Already on the first pages of the textbook, P&B do not restrict "number" to the "counting" (or "whole") numbers – MA does not, either – but still P&B treats whole numbers and other numbers as separate entities, which MA does not:

> "Numbers arise in one of two broad settings, usually called the set model and the measurement model.
>
> - Set model – Here one counts discrete objects. The answer must be a whole number….

[3] A review of the textbook says that it has been "among the highest rated textbooks in NCTQ's textbook evaluations" (http://www.nctq.org/teacherPrep/review2014/resources/elementary/mathContent.jsp), where NCTQ refers to "National Council on Teacher Quality."

> - Measurement model – Here one thinks of a scale along which one measures a quantity such as distance, time or weight. The answer is often not a whole number….
>
> Thus the question 'How many people are in this class?" asks us to use a set model while the question 'How high is the ceiling?' asks us to use a measurement model. Sometimes both models can be used." (P&B, p. 3)
>
> "The simplest version of a measurement model is the number line." (P&B, p. 3)

In MA we do not distinguish between "set model" and "measurement model." Number is an answer to the question, *How many?* Quantity Q is **how many** units U? The number can be thought of as the ratio of the quantity Q to the unit in which it has been measured. So it is always an abstraction from measurement. Counting is also measurement. When the answer to, "Q is how many units U?" can be obtained by counting objects, the unit is an abstraction from the specific individual characteristics of the objects and the way they are ordered. If we ask, *This class is how many students?,* Q is the "number of students in the class" and U is "a student" (not John or Mary, but just an individual assigned to the institutional position "student").

Chapter 6 in P&B is devoted to fractions. The first paragraph of the section 6.1 "Fraction basics" says:

> When counting or measuring we always have some unit in mind. When measuring distances we may use inches, feet, or miles; when counting students we may count individuals, classes or schools. The unit is sometimes very explicit for clarity (as in "3 cups of flour"), while at other times one has to think a moment to recognize the presence of a unit (as in "the table sits six"). But the count makes no sense without a unit ("I have 3 water")." (P&B, p. 131)

In MA, we could say the same thing. In the first year of the experimentation, we went on to ask how does one measure a given quantity Q using a given unit U? Our answer was Euclid's Algorithm in its generalized form as a method of finding if two quantities are commensurable; that is – if there is a common unit that fits a whole number of times into both, and therefore their ratio is equal to a ratio of two whole numbers.

We said that the answer to "Q is how many U's?" can be found starting by repeatedly subtracting the unit U from the quantity Q (imagine measuring the volume of flour in a bag using a cup). If there is no remainder, Q measures a whole number of units U. The answer to "Q is how many U's?" is that whole number. If the remainder is non-null (call it "r"), we repeat the procedure and measure the unit U with r. Again, either there is no remainder or the remainder is non-null (r').

If there is no remainder, the unit U measures a whole number of the remainders r and therefore also the quantity Q measures a whole number of remainders r: $U = n \times r$ and $Q = m \times r$; r is a common unit that measures both U and Q in whole numbers. Q to U is as m is to n. We decide to say that the answer to "Q is how many U's?" is $\frac{m}{n}$.

It may happen that the second remainder (r') is non-null. In this case we repeat the procedure in search for a yet smaller common unit for both Q and U. Sometimes we arrive at a situation where there is no remainder, but it may also happen that no remainder is ever null.

In the latter case, Q is not a fraction of U.

In the former case, we would have found a common unit (call it "u") such that $Q = m \times u$ and $U = n \times u$. Then we can say that Q is a fraction of U.

And we decide that this is what we will mean by the sentence, "quantity A is the *fraction* $\frac{a}{b}$ of quantity B", where a and b are whole numbers and b is not 0: that there is a common unit such that A measures a such units and B measures b such units.

In the first year of our experimentation, we only introduced this definition towards the end of the course. We used it to explain equivalence of fractions (as a change of unit of measurement) and to justify why the fraction $\frac{a}{b}$ of the fraction $\frac{c}{d}$ of a quantity is necessarily $\frac{a\times c}{b\times d}$ of that quantity[4].

In the 2014 and 2015 versions of the MA, we would start the course not from the general notion of number but directly from the above definition of fraction of a quantity.[5] This definition assumes only the existence and knowledge of natural numbers, so these versions of MA can be called *bottom-up-MA*, while the first version was *top-down-MA*. A substantial part of those courses was then devoted to problems of combining fractions of quantities in various ways, thus building a whole arithmetic of fractions of quantities. Only once this arithmetic was well developed, would we start the process of abstraction of the notion of fraction as a "stand-alone" number. In the first, top-down approach, fractions were directly introduced as abstract numbers: we did not stop at stating that both Q and U are whole multiples of a common unit, but went on to speak about the *ratio* of Q to U as being equal to a ratio of whole numbers. The ratio was a number: it answered the question, Q is how many U's?

In P&B, there seems to be an intention to directly define or introduce fraction as an abstract number. Fractions of quantities (with the quantities often only

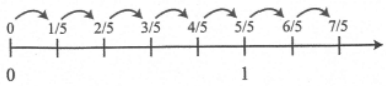
[4] We did not impose our way of reasoning (using the definition of fraction of quantity quoted above) on students – the future teachers. They could use their own ways and those who had their own ways did choose them to explain concrete cases of equivalence of fractions and of the fraction of a fraction of a quantity operation. They did not feel the need to prove general statements. Some strongly defended their own ways of reasoning.

[5] We decided to do that because we observed in the first year of experimentation that many of the future teachers completely missed the point of the generalized version of Euclid's Algorithm and their notions of number remained unchanged.

implicitly suggested) play the role of illustrations (or models) of those abstract fractions.

In Chapter 6 "Fractions", there are various explanations and representations of fractions, none of which is titled as a definition, although one of the explanations, given in the introductory paragraph of the chapter, is referred to as a definition:

> A fraction is a point on the number line. For example, to locate 7/5 we start at 0, find the step size so that 5 equal steps gets us to 1, and then take 7 such steps, landing at the points called 1/5, 2/5, 3/5,… until we get to 7/5. By this definition the fraction 3/1 is the same point on the number line as the whole number 3.



(P&B, p. 131)

The definition suggested in this paragraph is that a fraction *m/n* is the point on the number line whose distance from 0 is m times the distance u such that $n \times u$ is the distance from 0 to 1. In MA (both top-down and bottom-up) the construction described in the paragraph could be a solution to the problem: "On a number line, construct a point P whose distance from 0 is 7/5 of the distance from 0 to 1." The construction would be derived from the definition of fraction of a quantity; it would not be treated as *the* definition. The possibility of dividing a segment into 5 equal parts would be discussed.

In section 6.1 "Fraction basics" of P&B, after the paragraph about units quoted before, there is another explanation of fraction:

> We use fractions when there is a given unit, called the whole unit, but we want to measure using a smaller unit, called the fractional unit. (P&B, p. 131)

In MA, we would not formulate the problem leading to using fractions this way. Why would we *want* to measure using a smaller unit? Do we need to? What is the problem? What we would say is this:

We use fractions when we want to know how many times one quantity (call it A) is bigger or smaller than another quantity (call it B) of the same kind and it turns out that B does not fit into A a whole number of times, but one can find a common unit u *that fits a whole number of times into both A and* B.

In this statement of the problem, the quantity B corresponds to the "whole unit" of P&B and the common unit "u" corresponds to their "fractional unit". Terms such as "whole" or "whole unit" and "fractional unit" are not used in MA, because the focus is on a multiplicative relationship between two quantities and the possibility to express this relationship using a pair of whole numbers; if A is a fraction of B then B is also a fraction of A, so there is a kind of symmetry between the two quantities. If necessary, one quantity is called the "reference quantity": in "A is $\frac{a}{b}$ of B", B is the reference quantity.

P&B text continues with something that the Authors call “an example”. It is an explanation, on an example, of the words “whole unit” and “fractional unit”, and of the meaning, in those terms, of the numerator and the denominator of the fraction:

> For example, if 4 laps around a track is 1 mile, then mile is the whole unit and lap is the fractional unit. We write: 1 lap = $\frac{1}{4}$ mile, 3 laps = $\frac{3}{4}$ mile, 5 laps = $\frac{5}{4}$ mile with the numerator counting the number of fractional units and the denominator specifying the fractional units. (P&B, p. 132)

I find this text ambiguous and confusing. The “whole unit” is a quantity – length – and its measure is given as a denominate number: 1 mile. “Lap” sometimes refers to the run around a track but we are told that 4 laps = 1 mile so we understand that “lap” is intended here as a unit of length. Therefore, when we read “lap is the fractional unit”, we conclude that “fractional unit” refers to a quantity. But then we read that the denominator “specif[ies] the fractional units” and we are confused because the denominator is an abstract number. Is there some hidden meaning in the verb “specifying” that we are missing?

In MA, this example would be formulated as a problem of application of the definition of fraction of a quantity:

Given the conversion equation 4 laps = 1 mile, how many miles is (a) 1 lap; (b) 3 laps; (c) 5 laps?

The expected correct response to, e.g., (b), would be: 3 laps is ¾ of 1 mile because if we use 1 lap as the common unit, then 3 laps is 3 times the common unit and 1 mile is 4 times the common unit.

So far the text of the section 6.1 could be addressed to any learner of fractions. But a couple of paragraphs later, the Authors speak to teachers, warning them about children’s difficulties with the “standard notation” and related misconceptions about fractions:

> The standard notation for fractions is confusing when first encountered. Teachers should be on the lookout for two common student misconceptions:
>
> 1) thinking of $\frac{3}{4}$ as a pair of numbers (rather than a single number), and
>
> 2) thinking that larger fractions have larger denominators (thereby concluding that $\frac{1}{5} > \frac{1}{3}$). (P&B, p. 132)

I agree that thinking of 3/4 as a pair of numbers is common, but mentioning it as a “misconception” at this point in the book is surprising because, after the descriptions so far of the meaning of the word “fraction”, there is no reason to think that this expression refers to a single abstract number. So far, only the notion of fraction of a quantity has been introduced (or hinted at). Fraction as a single number has not been abstracted from the context of quantities.

In the first year of our experiment, fractions were introduced as numbers from the outset, as answers to the "how many?" question (as in the example above: 3 laps is how many miles? Answer: ¾). But this did not help our students, the future teachers, to get over the "misconception 1". This was one of the reasons why, in the subsequent years, we decided to work on the process of abstraction of fraction as a number much later in the course, after a thorough study of the operations on fractions of quantities and the problem of comparison of fractions of quantities. In fact, it was in the context of comparison of fractions of quantities that the process of abstraction would begin. Fraction as a single number was derived as a measure of a multiplicative difference between two quantities, e.g., a measure of how-much-ness of a substance in some other substance or in a container: fullness (How full is this bottle of water? Is it fuller than this other one?), sweetness (How sweet is this drink?...), salinity (What is the salinity of the Dead Sea?...). For example, fraction as a single abstract number is an answer to questions such as,

If a 750 mL bottle contains 300 mL of water, how full is it?

If suburb A counts 2574 households and 2/3 of them compost their organic waste, while suburb B counts 3878 households and 1/2 of them compost their organic waste, which one is more environmentally conscious?

With regard to the second misconception, in MA, in the theory of fractions of quantities, a statement such as $\frac{1}{5} > \frac{1}{3}$ would not make sense. One would ask, "$\frac{1}{5}$ of what quantity? $\frac{1}{3}$ of what quantity? It depends." Only after the passage to abstract fractions as measures of *how-much-ness* would the question of deciding which number is bigger, $\frac{1}{5}$ or $\frac{1}{3}$, arise and make sense: "If one bottle is $\frac{1}{5}$ full and another is $\frac{1}{3}$ full, which one is fuller?"

Section 6.1 continues with a presentation of the "Curriculum sequence" for fractions in elementary school. "Stage 1" is "Introducing fractions". The Authors start by describing how young schoolchildren think about fractions:

> Curriculum sequence Stage 1 – Introducing fractions. Most children start school already informally knowing the simplest fractions: halves and quarters. They understand and use these as adjectives, always speaking of fractions of something, as in 'my glass is half full'. (P&B, p. 132)

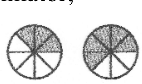
We agree that children speak mostly of "fractions of something" rather than of fractions as abstract numbers, and so do adults in everyday contexts. Ironically, however, the word "half" in "my glass is half full" is not an example of a fraction of something, but an abstract number, a measure of how-full-ness. But in many other places P&B speak of fractions of something even when they write expressions involving only abstract fractions. They consider not thinking of fractions as abstract numbers a misconception, yet they constantly amalgamate statements about abstract fractions with statements, expressions, and graphical representations of fractions of quantities. Let us consider, for example, the representations in Figure 1.

Figure 1. Illustrations of the fraction 3/4 in P&B, page 134.

The double arrow in the diagram on the left suggests looking at the distance from 0, not at the point labeled “3/4”. So “3/4” may be interpreted as representing not an abstract number but a fraction of a quantity: three quarters of the distance between 0 and 1. In the diagram on the right, “3/4” indicates three quarters of the area or length of the whole strip. In MA we would not write “3/4” if we meant three quarters of a given length or distance.

In P&B, the diagrams representing fractions of quantities are referred to as “illustrations” (p. 134). In MA, we do not “illustrate” statements on abstract fractions with pictures representing fractions of quantities. We do draw diagrams representing fractions of quantities but we treat them as fractions of quantities and not abstract fractions. We treat fractions of quantities as theoretical objects in their own right. We develop a whole theory of fractions of quantities, where statements are justified using concepts internal to the theory, not by “illustrations” or analogies with objects external to the theory.

But this explicit status of “illustrations” of statements about abstract fractions in P&B may be misleading, because the discourse is often about fractions of quantities. Let us look at a fragment of the text about comparing fractions. The intention is to teach about comparison of abstract fractions, but the illustrations suggest that fractions of quantities are being compared, except that the quantities are not specified.

> [E]xercises asking children to compare fractions point out the different roles of numerator and denominator. Those roles are evident when comparing fractions with either
>
> -the same denominator,
>
> $\frac{3}{8} < \frac{5}{8}$ 
>
> … [Here] we are comparing the number of fractional units…
>
> … Of course, when comparing fractional amounts it is important to use the same whole unit…. (P&B, p. 133-134)

In the above excerpt, inequality between abstract fractions is accompanied by a diagram hinting at fractions of areas. The circle diagram suggests that the reference areas are the same, but this condition is not made explicit at this point (it is, but later in the text), and so this may give the impression that 3/8 of any quantity is less than 5/8 of any quantity. In the top-down-MA, in the first year of experimentation, the difference between comparing fractions of quantities and comparing abstract fractions was emphasized and there was clear evidence that it made a lasting impression on the future teachers. In the first class of the course, students were asked to respond to a questionnaire, which contained, among others, these two questions

Question 8.

Mary used $\frac{3}{4}$ of a 500 mL bottle of water. Jane used $\frac{1}{2}$ of a 750 mL bottle of water. Which one used more water? Justify your answer.

Question 9.

Mary ate $\frac{3}{4}$ of an 8-inch pizza. Jane ate $\frac{1}{2}$ of a 10-inch pizza with the same filling. Which one had less pizza? Justify your answer.

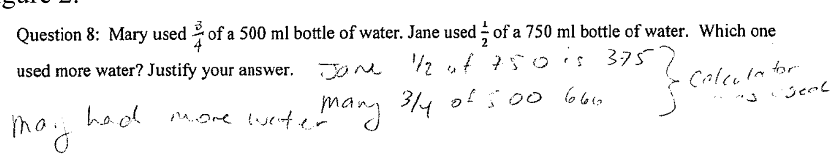
Of the 35 students who responded to these questions, 27 (77%) obtained the correct answer to Question 8 (Mary and Jane used the same amount of water). The solution strategy used in all responses with correct answers was:

Mary: $500 \div 4 = 125 \quad 125 \times 3 = 375$

Jane: $750 \div 2 = 375$

So Mary used the same amount as Jane.

This strategy – dividing the numerical value of the quantity by the denominator, multiplying the result by the numerator, ignoring the units – was used also by students who produced incorrect answers. For example, see the solution in Figure 2.


Question 8: Mary used $\frac{3}{4}$ of a 500 ml bottle of water. Jane used $\frac{1}{2}$ of a 750 ml bottle of water. Which one used more water? Justify your answer.
Jane 1/2 of 750 is 375
Mary 3/4 of 500 666
Calculator was used
Mary had more water

Transcript:
Jane ½ of 750 is 375
Mary ¾ of 500 is 666
(calculator was used)
Mary had more water.

Figure 2. Example of a future teacher's solution to Question 8 in a pre-course questionnaire.

Students then used the same strategy to solve Question 9, and the success rate was predictably very much lower: only 4 (11%) of the students produced the correct answer (Mary had less pizza). Ignoring the units did not pay off in this case. The commonly used strategy was:

Mary: $8 \div 4 = 2 \qquad 2 \div 3 = 6$

Jane: $10 \div 2 = 5 \qquad 5 \times 1 = 5$

So Jane had less pizza.

The same student who wrote the above quoted response to Question 8, answered Question 9 as reproduced in Figure 3.

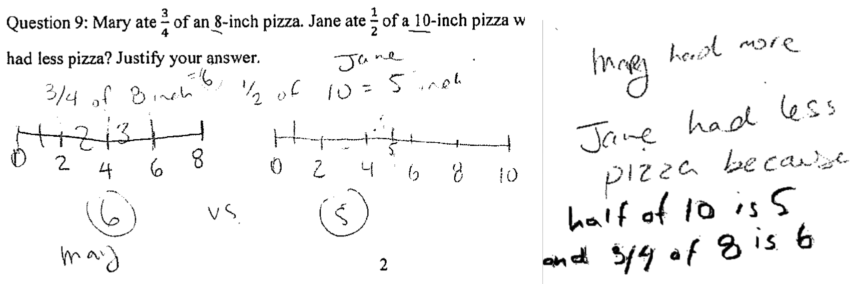


Figure 3. Answer to Question 9 of the pre-course questionnaire, by the same student whose answer was reproduced in the previous figure.

These results were then thoroughly discussed in class and certainly the massive failure on Question 9 made a lasting impression on the future teachers. They then became interested in inventing problems that would show that the inequality relations between abstract fractions are not always preserved in the context of quantities.

The question of equivalent fractions is an important topic. In the theory of fractions of quantities, *equivalence of fractions* appears as a result of the operation of change of unit. What fraction one quantity is of another depends on the choice of the common unit used to measure the two. This is already assumed in the definition of the expression, "quantity A is the fraction a/b of the quantity B." But if only the unit of measurement is changed, the quantities do not change, so the multiplicative relationship between the two quantities that these fractions represent stays the same. So it makes sense to treat these fractions as equivalent. For example, we can say that an amount of 350 g of flour is $\frac{350}{1000}$ of 1 kg of flour if we use 1 gram as the common unit, but we can also say that it is $\frac{7}{20}$ of 1 kg if 50 g is used as a common unit. We consider the two fractions $\frac{350}{1000}$ and $\frac{7}{20}$ as equivalent.

The question of equivalent fractions is tackled in P&B as part of Stage 3 of the curriculum, under the heading "renaming the fractions".

> Stage 3 – Renaming fractions. Each fraction can be represented in many ways: 2/3, 4/6 and 8/12 all represent the same fraction. Students who miss this fundamental fact will be mistified by the arithmetic of fractions. Fortunately, the idea can be

easily ILLUSTRATED *(my emphasis)* and understood at the beginning of students' exposure to fractions.

- Fractions Strips (a concrete approach) (P&B, p. 134)

This text is accompanied by a diagram, reproduced in Figure 4.

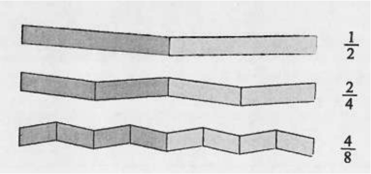


Figure 4. Fraction strips, P&B, p. 134.

Here again, relations between abstract fractions are "illustrated" by graphical representations of fractions of quantities, without developing any theoretical discourse about these entities: the operation of change of unit represented in the diagram above is not named or discussed. Writing symbols of abstract fractions beside representations of fractions of quantities in textbooks may be one of the factors which encourage students to ignore units in problems about quantities and lead to errors such as those we observed in future teachers' responses to Question 9 of the pre-course questionnaire.

Further stages of the fractions curriculum are devoted to operations on abstract fractions in P&B, "illustrated", as always, with fractions of implicit quantities:

> Stage 4 – Addition and subtraction with same denominators…. Key Principle of Fraction Addition. Once we agree on the fractional unit, we count, add, and subtract just like whole numbers. (P&B,(p. 135)

"Fractional unit" refers to a quantity, so here again, an operation on fractions of quantities is being disguised as an operation on abstract numbers. In the first year of our experimentation, we did not teach an arithmetic of fractions of quantities based on the common unit definition. We left the problem of designing teaching of operations on fractions to the future teachers. A couple of them did take up the challenge, but they did not follow the MA approach. There will be some details of the work of one of these students in section 4.

In the bottom-up-MA presented to future teachers in the subsequent years of the experiment, regarding the operation of addition, for example, we would start by posing the problem of extending the operation of addition of whole numbers of a quantity to addition of fractions of a quantity. Then we would define the problem of addition of fractions of a quantity in precise terms:

> We define this problem and its solution more precisely below.
>
> **The problem of adding two fractions of the same quantity**
>
> If A is a fraction $\frac{a}{b}$ of some quantity Q, and B is a fraction $\frac{c}{d}$ of the same quantity Q, what fraction of the quantity Q are the quantities A and B taken together?

> We can formalize this problem as:
>
> $\frac{a}{b}$ of Q + $\frac{c}{d}$ of Q = $\frac{?}{?}$ of Q
>
> We start with a couple of examples, which we will then generalize to provide an answer to this question. (Sierpinska & Bobos, 2015, p. Chapter 4)

The main element of the solution of this problem is the operation of change of unit, which is necessary whenever $b \neq d$. If we let A denote the fraction $\frac{a}{b}\ of\ Q$ and B – the fraction $\frac{c}{d}\ of\ Q$, then $A = a \times u$ for a unit u such that $Q = b \times u$, and $B = c \times w$ for a unit w such that $Q = d \times w$. We cannot say how much the quantities A and B measure together unless we express them in the same units. But since we know that $Q = b \times u = d \times w$, we have a conversion equation between the units u and w. In particular cases it may be easy to deduce from it how many u's is w or vice versa (e.g., if $Q = 2 \times u = 6 \times w$, then, proportionally, $u = 3 \times w$, and both A and B can be expressed in units u). But in general, we can take a unit v such that $u = d \times v$ and then deduce from $b \times d \times v = d \times w$ that $w = b \times v$. This allows us to represent both A and B in the same unit v and add the two quantities together: $A + B = a \times d \times v + c \times b \times v = (a \times d + c \times b) \times v$. In terms of the unit v, Q measures $d \times b \times v$. So, by the common unit definition of fraction of a quantity, $A + B$ is the fraction $\frac{a \times d + c \times b}{d \times b}$ of Q.

In the second and third years of the experiment, the above type of reasoning - conducted on general situations such as above, or on generic examples[6] - is representative of the way of thinking that was not only presented in lectures but also required of the future teachers. In those courses, the future teachers were being treated as university students following a mathematical course (Bobos-Kristof, 2015). They were expected to learn this theory and use it in solving problems about fractions and in designing activities for children.

But in the first year of the experiment, the future teachers were treated as partners in the construction of sensible activities for elementary children. The discourse of the teacher in the course (myself) and her teaching assistants assumed that students already know fractions. The task was to reflect on how fractions can be conceptualized and realize how they arise in the context of measuring one quantity using another as a unit. The teacher proposed a certain conceptualization, but did not require the students to learn it and nor to use it in solving and posing problems. They were free to use any other conceptualizations they liked. We discuss our pedagogy in more detail in the next section.

[6] "Generic examples" in the sense of (Yopp, Ely, & Johnson-Leung, 2015).

THE PEDAGOGY: AN INQUIRY-BASED APPROACH

In the first year of the experiment, our pedagogy had several features of an inquiry-based learning approach. This appeared to be a rational choice for us, because our design of the Measurement Approach was still very sketchy at the time and we needed to be open to modifying it based on the future teachers' response to it. Thus, we treated the future teachers more as fellow learners and teachers than as students. We tried to establish a "community of inquiry" (Goodchild, Fuglestad, & Jaworski, 2013), where we would learn from them and they would learn from us, as we all solved and posed mathematical problems to each other.

The claim that the pedagogical approach was inquiry-based can be justified by the sense given to this term by Wells (2000). Wells derives his notion of inquiry learning from ideas proposed by Dewey in "The School and Society" (Dewey, 1900) and "Experience and Education" (Dewey, 1938).

> As is well known, Dewey proposed starting with 'ordinary experience', emphasizing the importance of involving students in 'the formation of the purposes which direct [their] activities' (Dewey, 1938, p. 67) and in selecting 'the kind of present experiences that live fruitfully and creatively in future experiences' (Dewey, 1938, p. 28). (Wells, 2000, p. 9)

In our pre-service mathematics teaching course, the "ordinary experience" consisted in ordinary teacher's tasks: inventing mathematical tasks for students, designing classroom activities, implementing them, and reflecting on the outcomes of the implementation. These are, we believe, "the kind of present experiences that live fruitfully and creatively in future experiences" of a teacher. Every week, the future teachers were asked to invent a task on a given mathematical topic. In the middle of the semester, they had to prepare, run and write a report on a simulation of a classroom activity with their peers in the role of students. The content of the activity had to be related to the mathematical curriculum of the course – fractions, ratios, percents or geometry – but otherwise the future teachers were free to choose the specific purposes of the activity and the instructional means to achieve their purposes. As a final assignment in the course, they were asked to write a Problem Book, addressed to teachers, with descriptions of 12 classroom activities for elementary school, on fractions, ratio and proportion and geometry. They could include a revised version of their workshop activity in the Problem Book.

An important feature of the course which encouraged independent inquiry was the emphasis on noticing mathematics in the world and solving situational problems, that is, economically, socially or culturally relevant questions that could be solved using mathematics. Every class started with the future teachers being asked to report if they had come across some interesting mathematics in their daily lives in the past week. Often, these situations were related to shopping "deals" that would not be as advantageous as the sellers wanted the customers to believe. Some were transformed into exercises and included in the textbook written for the next iteration of the experiment. Here is an example:

> Scotch tape is sometimes sold in packages of three. The package advertises that you get "33% more" (Figure 5). On the package, "7.62 m" is crossed out and replaced by "10.1 m"; also "22.86 m" is crossed out and replaced by "30.3 m."
> To what does the "33% more" refer? Justify your answer.
> Do you think that this is an honest advertisement? Justify.
> (Sierpinska, 2015, p. Problem III.5)

Figure 5. Scotch tape package advertised as "33 % more".

Hands-on experience in and of itself is not the end goal of inquiry learning. For Wells, an important characteristic of inquiry learning is an "interplay between theory and practice, involving different complementary modes of knowing" (Wells, 2000, p. 15). Problems of practice should be chosen so that there be "a perceived need for theoretical constructs that provide a principled basis for understanding those problems and making solutions to them" and, vice versa, "there should be opportunities to put the knowledge constructed to use in some situation of significance to the students, so that, through bringing it to bear on some further problem, they may deepen their understanding." (ibid.)

Wells speaks here of teaching and learning in general, but Dewey expressed the same idea in the context of teacher education:

> On one hand, we may carry on the practical work with the object of giving teachers in training working command of the necessary tools of their profession; control of the technique of class instruction and management; skill and proficiency in the work of teaching. With this aim in view, practice work is, as far as it goes, of the nature of apprenticeship. On the other hand, we may propose to use practice work as an instrument in making real and vital **theoretical instruction**: the knowledge of subject-matter and of principles of education. This is the laboratory point of view. The contrast between the two points of view is obvious […] From one point of view, the aim is to form and equip the actual teacher; the aim is immediately as well as ultimately practical. From the other point of view, the immediate aim, the way of getting at the ultimate aim, is **to supply the intellectual method and material of good workmanship**, instead of making on the spot, as it were, an efficient workman. Practice work thus considered is administered primarily with reference to the intellectual reactions it incites, giving the student a better hold upon the

> **educational significance of the subject-matter** he is acquiring, and of the science, philosophy, and history of education. (Dewey, 1904, pp. 9-10)

In our experiment, we did supply the pre-service teachers with a theoretical conceptualization of fractions – the top-down-MA described in the previous section. This approach builds the concept of fraction as an epistemologically necessary consequence of a measurement situation: "when one wants to measure a quantity and the existing units of measurement do not represent it precisely enough, a change of unit into finer ones may be necessary" (Bobos-Kristof, 2015, p. 25). But this situation may create an epistemological need for fractions only for children who have never seen fractions before. The future teachers in our course have already encountered fractions, mostly through the Visual Approach in the earlier grades of elementary school, and a Formal Approach, i.e., a calculus on fractions in the upper elementary grades, often without any systemic connection between the two. The Measurement Approach could be a response to a perceived pedagogical need to teach children better than they have been taught themselves, for example, by connecting the visual representations of fractions (such as partly shaded shapes such as circles or rectangles) with operations on fractions as expressions of the form *a/b*.

But there was no obligation in the course for the future teachers to learn the Measurement Approach – we considered not imposing a definite didactical approach to teaching a topic an important feature of an inquiry-based approach to teacher education – and so many chose to look for inspiration in designing their activities elsewhere than in the lectures (internet resources for teachers; textbooks), seeking something resembling more closely the Visual and Formal approaches they were used to in their own education. Some did, however, become interested in the Measurement Approach and adapted elements of it in their activities.

But nobody was interested or felt the need for constructing the connections between the Visual and the Formal approaches to fractions as a theoretically coherent and consistent conceptual system. They were satisfied with "illustrations" such as in Parker & Baldrige, or analogies, metaphors, or chains of associations.

Theory is never an epistemological necessity. Theoretical thinking, in the sense it is practiced in mathematics (Sierpinska, 2005), is perhaps best described as a *cultural need*, similar to a person's need to occasionally visit an art exhibition or go to a concert of a symphony orchestra. Cultural needs require nurturing. Theoretical thinking in mathematics rarely (if at all) develops naturally, without the individual having encountered examples of such thinking in their education and engaged in it. This was not the case in our experiment, as no mathematical courses beyond basic level secondary school mathematics are required for admission into the Bachelor of Education program. Examples of theoretical thinking about fractions demonstrated in class by the teacher were met mostly with future teachers' incomprehension or downright rejection on the part of some of them.

In the next section, I describe how the future teachers in the first year of our experiment chose to conceptualize fractions, illustrating my hypotheses by elements of the activities they designed and their responses to more straightforward questions in the assignments.

RESULTS OF THE FUTURE TEACHERS' INQUIRIES INTO FRACTIONS

Given the freedom afforded by the inquiry pedagogy, knowing that they will not be tested on the conceptualization of fractions presented in the lectures, many future teachers (FTs, for short) chose to stay within the comfort of their old conceptions of fractions, perhaps adding a few new elements but without changing the structure. The result was often something like a curio cabinet, filled with catchy phrases (e.g., "a fraction is a part of a whole"), pretty pictures, fun activities, standard examples used to explain what a fraction is, symbols of some fractions ($\frac{1}{2}$, $\frac{3}{4}$, maybe $\frac{2}{5}$, but not $\frac{11}{12}$), rules for reducing fractions (curiously expressed as, "divide the fraction by the common factor"), rules for the four arithmetic operations (such as, "invert and multiply" for division), and even songs[7]. Some elements were connected but the connections would not constitute a coherent and consistent system; there could be gaps and contradictions. Most FTs' conceptions appeared to belong to the Vygotskian categories of "syncretic heaps" or "complexes", rather than to "concepts" (Vygotski, 1987, pp. 134-166).

In fact, the lectures were not perceived as a presentation of a certain theoretical conceptualization of fractions but as a marketplace with more curios to add to one's collection, no different from things one could find on the internet. Euclid's Algorithm was one such curiosity; some FTs interpreted it as belonging to the "measuring with non-standard units" type of application problems on fractions previously introduced as "parts of a whole." It was rarely, if at all, understood as a method of deciding whether or not one quantity is a fraction of another quantity, and if yes, constructing this fraction. Problems, discussed in class, highlighting the fact that order between abstract fractions is not always preserved if the same fractions are taken as fractions of different quantities were another curiosity. This type of problems was considered a particularly precious item in the collection, and many FTs included them in their Problem Books. But this did not necessarily remove the FTs' habit of neglecting to mention the reference quantity or ignoring the units of measurements when solving or posing problems about fractions of quantities.

[7] One popular song is "Fractions, fractions", available at https://www.youtube.com/watch?v=DnFrOetuUKg .

Incongruity as a characteristic of FTs' conceptions of fractions – Inquiry or shopping at a flea market?

In the chapter on fractions of her Problem Book, one of the FTs (coded as S-I.39) included, side by side, activities on fractions of quantities where the quantities and measurement units were explicitly mentioned, modelled after problems posed in class, and activities, copied from the internet, based on the Visual Approach to fractions, where the reference quantity was implicit, units not mentioned, and abstract numbers were equated with drawings of shapes. On the one hand, there was an activity on the discrepancy between order in abstract fractions and fractions of quantities. On the other, there was an activity copied from the website of the "Synergy Learning" organization[8], where children were asked to "Circle the figures that represent one half" with a worksheet full of partly shaded squares. And there was an activity, titled "Understanding fractions using pattern blocks" with a worksheet, taken from the same website, where a shape of a hexagon was equated with the number 1 (Figure 6).

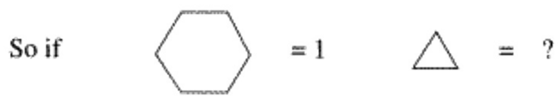


Figure 6. Part of an activity on fractions found on the internet by one of the FTs and reproduced in her Problem Book.
The diagram equates the drawing of a shape with an abstract number.

In a questionnaire passed after the presentation of the top-down-MA in the first two lectures in the first year of the experiment, S-I.39 claimed that "fraction is a part of a whole" is a true statement, justifying it by the authority of a popular website for elementary teachers[9]. (Figure 7)

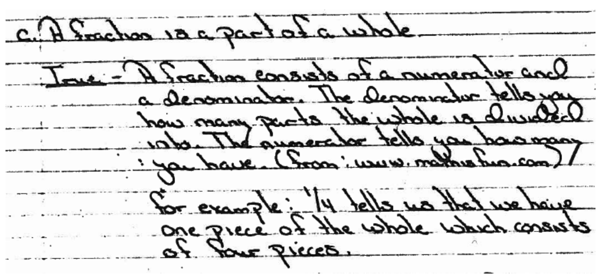

c. A fraction is a part of a whole
True - A fraction consists of a numerator and a denominator. The denominator tells you how many parts the whole is divided into. The numerator tells you how many you have. (From: www.mathisfun.com)
For example: 1/4 tells us that we have one piece of the whole which consists of four pieces.

Figure 7. An FT's justification of her claim that the statement "Fraction is a part of a whole" is true.

S-I.39 was perhaps not too imaginative, but at least she was attentive to the mathematical content of the sources she was borrowing from and appeared to

[8] The page is no longer available.

[9] "Math is fun", at www.mathisfun.com

understand well the instructional objectives of the activities she proposed in her Problem Book.

Careless "inquiry"

Other FTs were not always as accurate in the reproduction of the results of their "inquiries." A particularly disturbing example of careless reading are S-I.10's responses to the already mentioned questionnaire passed after the presentation of the top-down-MA in the first two lectures. S-I.10 referred to Parker & Baldridge in her responses, but she had obviously not read the chapter on fractions very attentively. Her responses also illustrate the frequently disconnected and internally inconsistent aspects of FTs' conceptions of fractions.

One of the items in the questionnaire was: "True or false? Justify. 'A number that is greater than 1 is not a fraction.'" S-I.10 wrote:

> False because **irrational fractions** are plausible. For example, 7 halves is represented as $\frac{7}{2}$, as in Parker & Baldridge, which is an irrational fraction. It is irrational because the numerator is larger than the denominator. Usually this type of fraction is written as a mixed number, $3\frac{1}{2}$. (S-I.10)

In response to "True or false? Justify. 'A fraction is a part of a whole.'", S-I.10 wrote (we divide her response into numbered paragraphs for ease of reference):

> [1] False because fractions represent a subunit or fractional unit of a whole unit.
>
> [2] For example, we are talking about 8 donuts as a whole unit and 5 have been eaten. The fraction of the whole that we are discussing is $\frac{5}{8}$.
>
> [3] A fraction can also be viewed as a division, for example, what fraction of 16 litres of water will 5 people receive; $\frac{16}{5}$, $16 \div 5$, will provide the answer.
>
> [4] A fraction can also represent a ratio. For every 1 pair of shoes I have 2 pairs of socks, $1 : 2$ or $\frac{1}{2}$.
>
> [5] Therefore a fraction can be seen as many things but it is not seen as part of a whole. It is a fractional unit of a whole unit, a division or a ratio. (S-I.10)

Further terminological and conceptual errors appear in her justification of the statement 'A fraction is a number that can be represented as a ratio of two whole numbers' which she considered true:

> [6] True because this is what a fraction is at the base. The numbers used to write a fraction are rational numbers and rational numbers also include whole numbers. If I were to say I ate 12 of 20 cupcakes and ask **what ratio I ate of them** it would be 12:20 or 6:10 or 3:5. This means for every 5 of the 20 cupcakes I ate 3 or in the 20 cupcakes I ate 12.

If we take the term "fractional unit" in the sense that Parker and Baldridge use it, the statement in [1] implies that only unit fractions are fractions. But the example in [2] implies that the FT is not restricting fractions to unit fractions. Hence she does not interpret "fractional unit" in the same sense as the authors of the book from which she drew her "definition." This example also suggests a "part of a whole" conception of fraction, which she denies in her answer, "False." But maybe "false" does not require a logical or conceptual contradiction for her. It may mean only *differently worded*.

She mentions other "constructs" of fractions in her answers ([4], [5], [6]), based on her readings, but does not try to connect them to the "definition" she cited at the beginning.

The example in [3] implies that, if 16 liters of water are shared (presumably equally, although she does not mention this) among 5 people, then each person receives $\frac{16}{5}$ of 16 liters. She makes the common error of not paying attention to the reference quantity (or the "whole unit") in a fractional relation.

In the shoes and socks example in [4], she identifies ratio $1 : 2$ with fraction $\frac{1}{2}$ in a formal way. She does not stop to think which quantity is the fraction one-half of which quantity in her example. In her donuts example, she seems to think of fraction as a relation between things: 5 donuts were the fraction $\frac{5}{8}$ of 8 donuts. In the shoes and socks example, is she now thinking of her shoes being the fraction $\frac{1}{2}$ of her socks? To be able to say that one quantity is a fraction of another quantity, both quantities must be of the same kind. What kind of quantity is it in this case? Number of pairs of footwear, perhaps?

She does not ask herself such questions, however. Fractions are formal expressions that can be vaguely illustrated by situations in which appear the same numbers as in the numerator and the denominator. Ratios are also formal expressions as evidenced in the incorrect grammar of the phrase with "ratio" highlighted in [6]. The ratio $a : b$ is merely an alternative notation for the fraction $\frac{a}{b}$. There is **a disconnection between the material representations of fractions and the formal representations**.

Looking for a structure in FT's conceptions

When we tried to identify some order in the FTs' conceptions of fractions, we realized that the "items of their curio cabinets" could be grouped into one or more of the following "elementary" conceptions:

- expressions of the form $\frac{a}{b}$ with rules of arithmetic operations on them ("formal conception");
- things such as slices of a pizza ("material conception");
- quantities such as $\frac{3}{4}$ of 1 kg ("quantitative conception");

- measures of qualities such as “fullness”, “discrepancy”, etc. (“numerical measure” conception).

A conscious effort to connect these sub-conceptions was rare. The most common combination was the formal-material-quantitative/disconnected conception - we identified it in 10 (29% of the 34 FTs in the class) (Table 1). S-I.39’s conception belonged to this group. The second most popular combination was the formal-material/disconnected conception (7 FTs, 21%). S-I.10, described in section 4.2, was classified into this group.

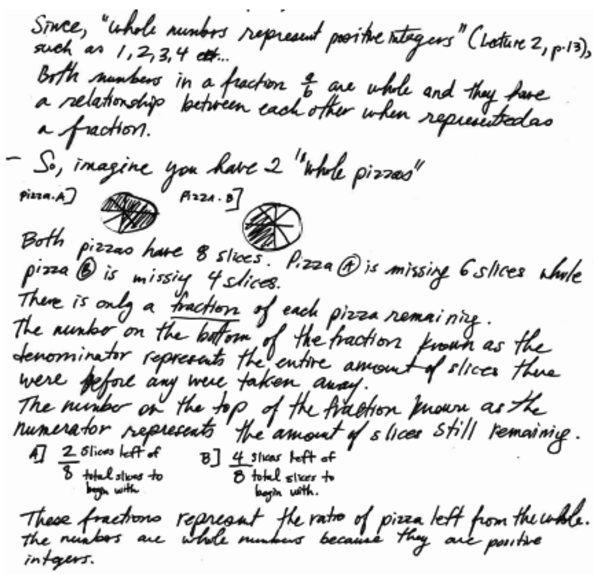
Conception	Number and percent (N = 34) of FTs who showed signs of the conception
formal-material-quantitative/disconnected	10 (29%)
formal-material/disconnected	7 (21%)
quantitative	6 (17%)
formal-material/connected	5 (15%)
formal-quantitative/connected	2 (6%)
quantitative-numerical measure/connected	2 (3%)
formal-material-quantitative/connected	1 (3%)
numerical measure	1 (3%)

Table 1. Distribution of conceptions of fractions among the FTs in the first year of the experiment.

Example of a formal-material/disconnected conception

The examples we gave so far were both based on borrowed texts and activities. But effort of inventing original activities was manifest in the work of some FTs. One outstanding example was an activity titled “Fractions are fun with music” by S-I.38.

S-I.38’s conception was classified as formal-material/disconnected based on evidence in her productions throughout the course. Evidence of the “material conception” component was most clear in her response to a question posed in the questionnaire already mentioned in section 4.3.1: “True or false? Justify. ‘A fraction is a number that can be represented as a ratio of two whole numbers’” reproduced in Figure 8. Here, a fraction is a piece of a pizza.



Since, "whole numbers represent positive integers" (Lecture 2, p.13), such as 1,2,3,4 etc...
Both numbers in a fraction $\frac{a}{b}$ are whole and they have a relationship between each other when represented as a fraction.
– So, imagine you have 2 "whole pizzas"

Both pizzas have 8 slices. Pizza (A) is missing 6 slices while pizza (B) is missing 4 slices.
There is only a fraction of each pizza remaining.
The number on the bottom of the fraction known as the denominator represents the entire amount of slices there were before any were taken away.
The number on the top of the fraction known as the numerator represents the amount of slices still remaining.

A] $\frac{2}{8}$ 2 slices left of 8 total slices to begin with
B] $\frac{4}{8}$ 4 slices left of 8 total slices to begin with.

These fractions represent the ratio of pizza left from the whole. the numbers are whole numbers because they are positive integers.

Figure 8. S-I.38's response to "True or false? Justify.
'A fraction is a number that can be represented as a ratio of two whole numbers.'

Evidence for the "formal conception" was found in her "Fractions are fun with music" activity, designed for her Workshop and then included, after revisions, in her Problem Book. In this activity, drawing on her guitar playing experience, S-I.38 aimed at making children realize that fractions are not just a dry school subject but that they are used in something as exciting as music composition – musical notes represent fractions of a whole note. The connections she made between fractions and musical notes remained, however, on the formal level. She asked children to perform formal operations on fractions disguised as symbols of musical notes without regard to the musical meaning of the operations (Figure 9). What would subtraction of a note mean in music? She also expected children to "reduce" the notes. But two eights notes are not equivalent to a quarter note: this is not the same music. So the reduction is a formal operation.

> Worksheet #1: (Teacher needs to remove answers before handing them out to students).
>
> Question- Add and/or subtract the notes according to what mathematical operation is given in each question. (…)

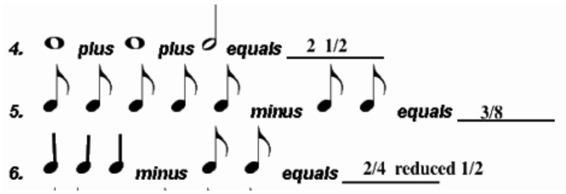


Figure 9. Formal operations on fractions represented as musical notes.

We also considered her conception as disconnected because there is no attempt in her work to explain the formal operations of addition and subtraction of fractions in terms of the material "piece of a pizza" conception of fraction.

A unique case: the exclusively numerical-measure conception

Conceptions made of a single "elementary conception" were not frequently detected but there were a few. An interesting singular case was S-I.5 who consistently displayed the "numerical measure" conception. This student closely associated expressions of the form *a/b* with the result of the operation *a:b* or even *(a:b)* x 100. He frequently immediately converted fractions into percents and used the obtained values to represent relations between quantities and compare them. His mathematical knowledge seemed to be "situational", organized by problem situations, not theoretical concepts. In his Problem book, titles of chapters were not "Fractions", "Ratio and proportion", "Percents" and "Geometry" as in most other FTs', but "Let's measure time and space", "Let's distribute" and "Let's talk money."

In the "Let's measure…" chapter, one of the tasks is to measure the lengths of one's "body parts" and find relations between pairs of measurements in the form of fractions and percents, and then relate the size of the "discrepancy" between the measurements of two body parts and the size of the number represented by the fraction or the percent:

> The students will be asked to bring in a measurement tape from home, which they will use for the activity. In pairs, they will record measurements of their various body parts. This data will be collected on a sheet provided by the instructor. The students are required to fill out the sheet by working together (i.e., one doing the measurement for the other). Once the sheet has been filled out, they will be asked to compare their various body parts measurements as fractions and percentages of themselves. (…)
>
> The handout sheets will be collected and analyzed by the instructor to ensure that the task has been performed adequately and that the objectives have been met. Along with the answers provided in the handout, I will test whether or not the students have learned what was intended by asking them:
>
> 1-Which body part yielded the largest measurement?
>
> 2-Which two body parts had the largest discrepancies in length?

3-Which two body parts yielded the smallest fraction when comparing their measurement (smallest/biggest)?

4-Which two body parts yielded the largest percentage when comparing their measurement (smallest/biggest)?

There was no chapter on geometry and geometry was not integrated in any of S-I.5's activities. This does not mean that S-I.5 was not interested in geometry. He just avoided it in the Problem Book assignment. His workshop was focused on geometry; it was based on an activity about the sum of angles in a polygon. It was an interesting activity but the formulation of the problem and its expected solution left much to desire and his grade for the workshop report was only 64%. The teacher wrote a 5-page feedback to the student, discussing the various aspects needing improvement and suggesting ways of improving the activity. S-I.5 chose, however, not to work on improving the activity and decided not to include it in his Problem Book. The feedback was ignored.

An advanced formal-material/connected conception, and the dangers of becoming excessively fond of one's creation

Inquiry-based pedagogy affords students equal opportunity to do the bare minimum to pass the course and to surpass themselves and the teacher's expectations. FTs in the course presented a full spectrum of behaviors between these two extremes. Those who engaged less creatively in the construction of their activities reacted less emotionally to critical comments and suggestions of change. But those comments and suggestions might have been lost on them. Emotions sometimes flared in those who did engage, but the effect of comments and suggestions on their conceptions was no higher. It seems that the more effort and thought one invests in the creation of something, the more one gets attached to it - as Pygmalion became attached to the sculpture he created - and the more difficult it becomes to accept criticism and agree to modify it. This is probably true for anybody and any type of creation but if the engaged creator is a student in a course and the critic is the teacher, the situation becomes difficult. The teacher may think she is not doing her job if she only praises the students' work, but the student in an inquiry-based approach has the right to think that the ultimate judgement of the quality of her work does not belong to the teacher of the course, but to instances external to it, more objective and impartial. For example, the student may say, "I know this is a good way to teach fractions; I have tutored this way for years and my students have always succeeded on ministerial exams. So I am not going to change it, despite your criticism."

I am not making this up. Such words were indeed addressed to me by one of the FTs in the course, the highly engaged and ambitious S-I.16. How do you think I felt, especially when, and after saying this, the student left the classroom?

In an e-mail exchange with my research assistants after this memorable event in class, I wrote:

> The obstacle that you are discussing, Georgeana – future teachers' negative attitudes to learning in the sense of changing their ways of thinking – was something that was deeply disturbing for me as their teacher. I had the impression of hitting a wall of resistance. Especially in the last class, I had the feeling that if I don't bend my teaching to their ways of thinking about fractions, they will all just get up and walk out of the class.

In this exchange, we were trying to identify obstacles to the measurement approach to fractions we were faced with in the experiment.

S-I.16's way of thinking about fractions can be classified as formal-material/connected and it was very similar to Parker & Baldridge's. But the FT did not copy and paste activities from that textbook for her workshop or Problem Book. She invented her own. The aim of one of her activities was the introduction of division of fractions to children. Already the fact that she was inventing an activity to *introduce* an operation on fraction distinguished her from her peers, most of whom assumed children have already been introduced to fractions and only had to "use it" or "practice" it in their activities.

Another distinctive feature of her conception was that she was trying to connect its components, the material and the formal conception, if not into a conceptual system, then at least in some systematic way, building a meaning for operations on fractions up from the meaning of operations on whole numbers.

Her conception can be described as follows. Fractions are expressions $\frac{a}{b}$, with a and b being whole numbers, and b nonzero. This is their formal meaning. There exists a formal calculus on these expressions (arithmetic operations are defined by standard algorithms). The meaning of these expressions and operations on them can be conveyed to children by interpreting the expression $\frac{a}{b}$ as referring to a things, b of which constitute a whole (the material meaning). The whole can be represented by a rectangle, cut out from paper (Figure 10).

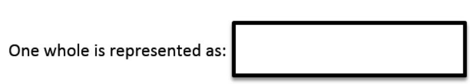


Figure 10. Element from S-I.16's Problem Book - Chapter I - Activity 4.

One can fold "one whole" into two equal parts. Each part is then labelled $\frac{1}{2}$.

The meaning of the operations on the expressions $\frac{a}{b}$ can be explained and understood by manipulating the rectangles. For example, S-I.16 expected children to "easily solve the problem $4 \times \frac{1}{2}$ mentally, with the standard algorithm, or with the assistance of manipulatives as follows: [Figure 11]"

Figure 11. S-I.16's expected solution "by using manipulatives" of the "problem" 4 x ½.

She offered no examples of material interpretation of multiplication of a fraction by a fraction. But this would be a challenge. The focus of her activity for the workshop (then reproduced in the Problem Book) was division.

S-I.16 interpreted the operation $x \div y$ as the problem "how many times y fits into x"? For example, 1: ½ =2 was interpreted as "one-half fits into 1 twice." (Figure 12)

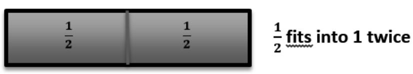


Figure 12. "Material representation of the formal operation 1 : 1/2 = 2 in S-I.16's activity on division of fractions.

After several such easy examples, S-I.16 boldly attacked operations that could not be easily or convincingly represented using the paper models. She started with division of whole numbers where the result was a fraction: 32:6. She expected students to have some difficulty with it, and here is how she planned to guide them:

> The answer to this problem [32 ÷ 6] may not be immediately evident. I suspect the students may give the answer of 5 with a remainder of 2. If this is the case, I will ask them what the 5 in the answer represents. The answer to this should come out that there are 5 groups of 6, to which I will respond that if a complete group has 6 in it, how much of a complete group does the remaining 2 represent. The students should be able to arrive at an answer of $\frac{2}{6}$ or $\frac{1}{3}$." (S-I.16)

How would children tackle a similar problem without guidance? After the remainder of 2 is obtained, it is not clear why one should seek the answer to "2 is what fraction of 6" rather than "2 is what fraction of 32"? Also, it is not clear why the interpretation of the problem, implicit in the reasoning is, *How many groups of 6 things fit in a collection of 32 things?* and not *If a collection of 32 things is divided into 6 equal groups, how many things are in each group?* Yet, she expected children to sufficiently generalize her example with division of whole numbers to be able to apply the "same method" in division of proper fractions: $\frac{1}{2} \div \frac{1}{3} = 1\frac{1}{2}$.

> This problem may be more difficult for the students to understand. It is very likely that students could give an answer of 1 as a third can only fit into a half once in its entirety. Therefore, if need be, they will be reminded that we can represent the

> remainder by a fraction as was done in the case of 32:6. The use of the manipulatives can help the students conceptualize the answer to this problem. (S-I.16)

The text was followed by the figure reproduced in Figure 13.

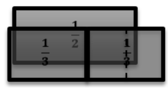


Figure 13. Material representation of the operation 1/2 : 1/3 = 1 1/2 in S-I.16's Problem Book.

But if, indeed, a similar reasoning was used, the first question would be, How many times one-third fits into one-half? The answer would be: 1 whole time and there would be a remainder. How would the children calculate this remainder? Subtraction of fractions would be necessary to evaluate the fraction exactly. The operation is non-trivial; the answer is one-sixth. The next question would be: one-sixth is what fraction of one-third? This is, in fact, a problem of division: $\frac{1}{6} \div \frac{1}{3}$. This problem is not much easier than the initial one. The answer may be not evident from the manipulatives if they are not very precisely done and doing them precisely would require the knowledge that they are supposed to make the child to understand. If a representation of one-sixth is obtained by folding the strip of paper then it will certainly not fit very precisely into one-third twice. Getting an approximate answer is contrary to the precision of measurement that fractions have been invented for. The transfer of the concept of "remainder" to the domain of fractions is not obvious.

S-I.16's material representation of the expression $\frac{a}{b}$, as referring to a things, b of which constitute a whole, is not far from the notion of fraction of a quantity; it is, at least, " $\frac{a}{b}$ of *something*." In the expression "quantity A is the fraction $\frac{a}{b}$ of quantity B", a refers to the number of units A measures when B measures b such units." The distance is, however, sufficiently significant to explain S-I.16's refusal to switch from the one to the other.

In S-I.16's material conception, this *something* that $\frac{a}{b}$ is a fraction of is not a specified quantity (length, area, or weight) but an object, a thing that may have several quantitative aspects, but none is identified, or considered specifically. The quantities do not matter, really. What matters are the operations on formal expressions and getting the answers right. The manipulatives give some sense to the operations – how many times a given piece fits into another piece – and they can help in getting the right answer or check an answer obtained by a formal standard algorithm.

As I see it, the problem with this approach is that it is trying to justify statements of the theory it aims to teach – the arithmetic of abstract fractions – using statements from another theory – the theory of fractions of quantities – which is not taught but is assumed to be intuitive and not requiring precise formulation and explicit teaching.

Is it indeed as intuitive as that? It may be intuitive – for some – but making it explicit shows that it is non-trivial. I think that the theory of fractions of quantities deserves more attention as it is the missing link between the material and the formal conceptions of fractions.

To show how non-trivial this theory is, let us look at how the problem of division of $\frac{1}{2}$ by $\frac{1}{3}$ would be tackled in the MA approach, in the more mature, bottom-up version that we have introduced in the second year of the experiment. Before being formulated as $\frac{1}{2} \div \frac{1}{3}$ in the theory of abstract fractions, it would appear in the theory of fractions of quantities, in two types of division word problems. That these problems can be interpreted as division problems is a theoretical question to which special attention and time would be devoted.

Type I – Division of a quantity by a quantity: A is WHAT FRACTION of B?

Example:

I have $\frac{1}{2}$ L of honey and I will be putting it into $\frac{1}{3}$ L jars. How many jars shall I fill?

Reasoning:

The problem is:

$\frac{1}{2}$ L is how many $\frac{1}{3}$ L?

To see this problem as, A is WHAT FRACTION of B? one needs to understand a fractional multiple of a quantity as the fraction of the quantity. In MA, the expression $A = \frac{a}{b} \times B$ is defined as meaning, A is the fraction $\frac{a}{b}$ of B.

Let A be the amount of honey I have and B the capacity of the jar.

A is one-half of a litre: this means that A measures one unit u such that

$1\,L = 2 \times u$

B is one-third of a litre: this means that B measures one unit w such that

$1\,L = 3 \times w$

Obviously, u and w are different units. But to answer the question, A is what fraction of B?, A and B must be converted to the same unit. The unit must be such that the equality $3 \times w = 2 \times u$ holds; both quantities are equal to one litre.

If we take a unit v such that $u = 3 \times v$ – such a unit exists; any amount of honey can be divided into 3 equal parts – then $3 \times w = 6 \times v$. By proportionality, one w is 3 times less; so $w = 2 \times v$.

Since $A = 3 \times v$ and $B = 2 \times v$, then, $A = B + 1 \times v$, and, by definition, $1 \times v$ is $\frac{1}{2}$ of B. So A is $1\frac{1}{2}$ of B.

So I shall fill one and a half jars.

Type II – Division of a quantity by a fraction: A is $\frac{a}{b}$ of WHAT QUANTITY?

Example:

$\frac{1}{2}\ L$ of water fills $\frac{1}{3}$ of a container. What is the capacity of the container, in litres?

Reasoning:

The problem is: $\frac{1}{2}\ L$ is $\frac{1}{3}$ of how many litres?

The problem can be solved directly from the common unit definition of fraction of a quantity, but if the operation of fraction of a fraction of a quantity ($\frac{a}{b}$ of $\frac{c}{d}$ of Q is what fraction of Q?), or of addition of fractions of a quantity have already been studied and generalizations were obtained then the reasoning can be simplified. For example, using addition:

Let $A = \frac{1}{2}$ of $1\ L$ and let C be the capacity of the container.

Given that A is $\frac{1}{3}$ of C, by definition of fraction of quantity, $C = 3 \times A$.

Multiplication of a quantity by a whole number means repeated addition:

$C = A + A + A$.

So $C = \frac{1}{2}$ of $1\ L + \frac{1}{2}$ of $1\ L + \frac{1}{2}$ of $1\ L = 1\ L + \frac{1}{2}\ L$

The capacity of the container is one and a half liters.

CONCLUSIONS

We knew it was not going to be easy. It wasn't. Perhaps the biggest disappointment for us was that some students came close to theoretical thinking about fractions yet refused or did not feel the need to make that step. But there were other difficulties, too. So after the first trial, it was back to the drawing board for us. We had to re-think our conceptualization of fractions, our assumptions about students - the future teachers – and our ways of interacting with them.

The structure of the future teachers' knowledge

The conceptualization of fractions proposed in the lectures had a vertical structure (Bernstein, 1999). This did not fit well with FTs' conceptions which were often made of disconnected pieces. They were not concerned about coherence or consistency. In fact, few of them even noticed that the content of the first two lectures was a presentation of a vertically structured conceptualization of fractions. In particular, they did not interpret Euclid's Algorithm as a theoretical means of distinguishing between pairs of quantities where one of the quantities is a fraction of the other, and those that are not in such relation. Many saw it as an application problem for fractions already introduced otherwise; a “fun measurement activity with non-standard units”.

In revising our design for the next rounds of experimentation, we did not, however, give up structuring knowledge about fractions in a vertical way. We only stopped counting on the FTs to notice it on their own and use it when they wanted to. We decided to be much more explicit in our presentation of this knowledge. We presented it as a mathematical theory and we required that the FTs learn it and use it to validate statements about fractions on a class test and a final examination.

The Measurement Approach

We did, however, make concessions to the FTs’ ways of thinking about fractions. We modified our conceptualization of fractions from the top-down measurement approach to a bottom-up measurement approach. The top-down MA starts with the construction of number as measure. But unlike in children following Davydov’s curriculum, FTs’ first encounter with number was counting objects, not measuring quantities and comparing them (Schmittau & Morris, 2004). Re-conceptualizing number as measure at an adult age would be very difficult. So we decided to build a bottom-up version of the measurement approach, assuming only knowledge of whole numbers and operations on whole numbers and whole numbers of quantities (e.g., 5 times 3 cups of flour = 15 cups of flour).

We adjusted the conceptualization to act as the missing link in the FTs’ disconnected conceptions of fractions. FTs’ conceptions were made of mainly three components: the material, the quantitative and the formal conceptions. The theory of fractions of quantities that we developed was meant to give the quantitative conception a vertical organization. With this vertical organization, we explicitly and systematically constructed the connection between the material and the formal conceptions.

How did it go? An analysis of the second experimentation, in winter 2014, can be found in Georgeana’s doctoral thesis (Bobos-Kristof, 2015). An account of the third experimentation (winter 2015) is in progress.

The inquiry-based pedagogy

The inquiry-based pedagogy, or the way we practiced it, was good, but mainly for us, the researchers. Allowing the future teachers to do what they wanted opened wide the windows to their minds for us and we could see what and how they spontaneously think about fractions.

But we were not very satisfied with the knowledge about fractions that the future teachers developed through the course. Perhaps, in general, inquiry-based pedagogy benefits the teachers more than the students?

For students to learn, they have to be receptive to their peers' and the teacher's critique and open to changing their conceptions. This, as we learned the hard way, they were not.

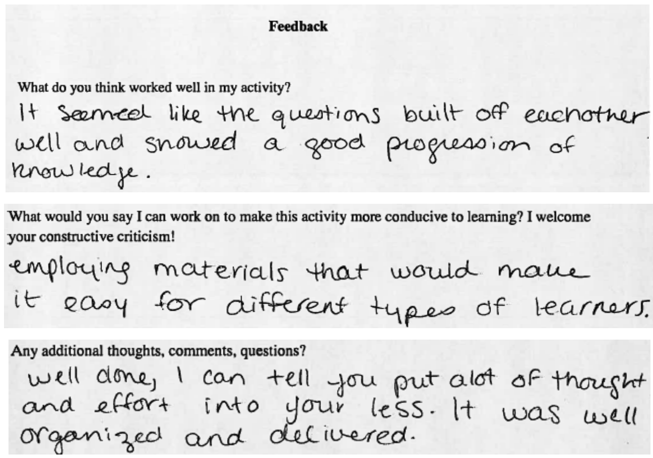
In mathematics courses offered by the department of mathematics, it is common practice for the teacher to discuss, in class, the more important mistakes made by students on a test. Students' names are not mentioned, not because the teacher does not want to hurt anyone but because what is discussed is the mathematical truth of a statement and not a person being "right" or "wrong." Learning to detach from one's knowledge is part of the process of learning to think theoretically about it. In the Teaching Mathematics course in the department of education, discussing common mistakes in class, even without mentioning students' names, can be perceived by some students as a serious offence.

Here is an example. As I was discussing the problematic character of an approach to teaching geometric transformations[10] after several FTs used it in their workshops, one of the students interrupted me and said, "Don't you think it is a bit rude to criticize us in public like this?" To my reply, "But I am not speaking about anybody; I am talking about mathematical concepts. Rotation in 3d is not the same thing as rotation in 2d", the student answered, "You are not naming names but everybody here knows whom you are talking about." After which this student and two others left the classroom. Knowledge was definitely a personal matter for them. This did not augur well for the development of theoretical thinking about mathematics in this class. Theoretical issues are not resolved by taking offense and walking out on the interlocutor anytime a contradiction is found.

To benefit from inquiry pedagogy, students should also take seriously the work of giving feedback to their peers. Saying "Good job!" or "Well done!" and offering vague advice is socially nice and makes the addressee feel good but not very useful as feedback. (Figure 14)

Just as they were uncritical towards their peers' work, FTs were indiscriminating in their choices of activities they would find on the internet. It is a good thing that there are so many resources available on the internet, but one needs to already have some mathematical knowledge to separate the wheat from the chaff. Having learned fractions in elementary school oneself is not enough, yet some FTs seemed to think that it is.

[10] In their workshops, some FTs were introducing transformations of the plane by means of "hands-on" activities involving three-dimensional objects and actions, such as looking at oneself in a mirror, step dancing, walking, turning around, etc.


Feedback

What do you think worked well in my activity?

It seemed like the questions built off eachother well and showed a good progression of knowledge.

What would you say I can work on to make this activity more conducive to learning? I welcome your constructive criticism!

employing materials that would make it easy for different types of learners.

Any additional thoughts, comments, questions?

well done, I can tell you put alot of thought and effort into your less. It was well organized and delivered.

Figure 14. Feedback an FT received after her workshop on equivalent fractions.

Learning without having to change one's previous conceptions is merely assimilation of new information. But changing one's conceptions requires effort and may be stressful. For many FTs, the ideal teacher makes learning effortless for her students. Stress must be eliminated; activities must first of all be "fun". Some applied the same criteria to their own learning for the course. The pedagogy allowed it. The teacher provided detailed feedback, but one could just ignore it, without any institutional sanctions. Others, more ambitious, invested much effort and creativity in the design of their activities. Sometimes they became emotionally attached to their creations; their knowledge became personal knowledge, making it all the more difficult for them to counter criticisms with theoretical arguments. In any case, it was difficult to engage the future teachers in a theoretical debate.

In some publications, *inquiry based pedagogy* is presented as the opposite of *transmission pedagogy*:

Although the notion of inquiry varies across contexts, common to all variants is an emphasis on active student participation, engaged thought and the investigation of mathematical and pedagogical ideas; it signals a departure from the belief that teachers can 'deliver' or transmit conceptual ideas and recognizes the learner's active part in making sense of experience. (Klein, 2004, p. 36)

I find this alternative questionable, because a "conceptual idea" cannot be transmitted by a "sender" without the "receiver" actively engaging in interpreting the message: the teacher who believes that his or her task is to "transmit conceptual ideas" (and not just information) necessarily also "recognizes the learner's active part in making sense of experience." So if the alternative makes sense to Klein and other authors she quotes, then she must associate "transmission" with something else than "any educational incident which sets the learning of knowledge previously planned or defined by the teacher as the basic objective" (Goodson, 2005). In fact, it seems that she associates "transmission" in mathematics teaching with "mathematical knowledge… equated with remembered facts, skills and procedures" (Klein, 2004, p. 40), and the teacher in the position of power as an authority and ultimate judge of students' competence or incompetence (Klein, 2004, p. 38), defined in terms of "getting the 'right' or 'wrong' answer" (Klein, 2004, p. 41).

But isn't it possible to set, as the basic instructional objective, a previously planned mathematical knowledge that would not be equated with "remembered facts, skills and procedures"? Theoretical mathematical knowledge, which is necessarily based on a good balance between theoretical and practical mathematical thinking, cannot be learned by memorizing facts and applying procedures (Sierpinska, 2005). When a theoretical mathematical system is gradually developed in a mathematical course from scratch, from the first definitions and explicit assumptions, the teacher is no longer in the position of authority over the knowledge he or she teaches. Students are as well-equipped to prove the teacher wrong as the teacher is to prove the students wrong. And this is not even how the teacher and the students have to look at it, because the discussions are not about *a student or the teacher being right or wrong* but about *statements being true or false* within a theoretical system. It is not personal beliefs, personal "theories", or personal identity as being good or bad at mathematics that are being discussed but just a system of hypothetical statements: a theory. Students and the teacher may emotionally detach from the matters they discuss. Humiliation is replaced by humbleness in the face of mathematical necessity.

This is what we aimed at in revising our pedagogy and content of the course for the next rounds of experimentation in 2014 and 2015. On the surface, the 2013 classes may have looked like inquiry-based and the 2014-15 classes more like the transmission model but inquiry was present in both. The difference was that the 2014 and 2015 students – future teachers – had to engage in a much more advanced mathematical inquiry than any of the 2013 students. The 2014 and 2015 students' spontaneous conceptions of fractions could be just as disconnected and material or formal as those of the 2013 students, but I was able to discuss them with the class calmly without anybody feeling offended.

It makes sense to assume that learning – mathematics or mathematics teaching – is either inquiry-based learning or it is not learning at all. But for inquiry to occur, classes do not have to become laboratories where students behave as researchers, passionately discussing their own projects, with the teacher and other students in the role of advisors or reviewers. Inquiry-based learning can occur also in lecture-based courses, with the teacher presenting and explaining mathematical problems, techniques, methods and theories, and students working on assigned exercises and problems, and projects. This was the pedagogy adopted in the next two years of experimentation. You can call it a "transmission model teaching", but it did not turn the FTs into docile memorizers of procedures. There were still many ways to think about and interpret any problem in the theory of fractions of quantities, and the debate was lively. FTs discussed in small groups in class and out of it, and the discussions were focused on the mathematical content. The conceptualization proposed to them was not easy but they had no choice but to *reflect* on it, identify the difficulty and try to resolve it and accommodate their ways of thinking to overcome the difficulty. This, for me, is also *inquiry-based learning*.[11]

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[11] According to Dewey (1910), only "reflective thought is truly educative in value" (p. 2) and all reflective thinking is an inquiry: it is triggered by a realization of a difficulty, its formulation as a question and is powered by the will to resolve the difficulty (p.72).

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ON THE WAY TO IMPROVE PRIMARY TEACHERS´ PROFESSIONALISM: THE CASE OF INQUIRY AND CONCEPT CARTOONS

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The paper focuses on inquiry-based education as a possible way to development and refinement of primary teachers' professional competence in pre-service teacher training, in an optional university seminar on didactics of mathematics. We deal with issues related to argumentation; in particular we focus on the development of the awareness of the need and reason of proper argumentation in mathematics classroom. As tools for this purpose we use Concept Cartoons and substantial learning environments. We outline how students assessed benefit of inquiry-based mathematics lessons. In particular, we present students' responses, e.g. their reflections on inquiry-based mathematics education, and Concept Cartoons posed by them.

INTRODUCTION

For many years we have been involved in research in the area of development and improvement of teachers' professional competence, i.e. the set of skills and qualifications needed for successful performance in the classroom. We pay special attention to pre-service primary teacher training. Our aim is systematic development of teachers' knowledge base, improvement of didactic knowledge of the subject, and development of subject didactic competence.

This paper reports an ongoing investigation which is a part of a project named *Enhancing mathematics content knowledge of future primary teachers via inquiry based education*. In the project we aim to implement inquiry-based methods into several university courses on mathematics and didactics of mathematics for future primary school teachers, and observe what impact the implementation has on students' knowledge and beliefs.

In this particular investigation we implemented inquiry-based methods into an optional university seminar on didactics of mathematics. Our research question is: Whether and how can this course form participants' views of inquiry-based mathematics education?

As an additional issue in our study we introduced to students a new educational tool called Concept Cartoons, and used it to diagnose misconceptions in their argumentation, in mathematics content knowledge.

BACKGROUND

Inquiry-based education

Recently a lot has been said and written about inquiry-based education as one of the important means of gaining new knowledge. That is why inquiry-based education and its implementation in pre-service teacher training are often seen as one of the key ways to development and improvement of teachers' professional competence. We share I. Stuchlíková's (2010) view that the trend is not brand new. We can for example recall characteristics formulated by J. S. Bruner, who emphasises *learning by discovery* and claims that to educate means to teach a person to take an active part in the process of gaining, organizing, and storing knowledge (1965). However, the term *inquiry* is rather new in the area of mathematics education. Formerly, math educators were considering application of the ideas of *genetic education* and talking of exploring and discovering in this respect. According to H. Freudenthal, genetic approach is characterised by *guided rediscovery* as a step in the process of learning (1973). According to E. Wittmann, genetic approach is based on natural cognitive processes in construction and use of mathematics (Wittmann, 1974; Hošpesová & Tichá, 2013). In the Czech Republic, genetic approach and guided discovery were discussed by J. Vyšín in his lectures and articles (Vyšín, 1976).

Let us also remind the fact emphasised by a number of authors that the core and background of inquiry are based on formulation of a problem or a question. And that problem solving is the most efficient and the most elaborate heuristic teaching strategy (Gagné, 1975). At the same time we must bear in mind that during inquiry, the learning subject must have something to think about. This implies that the learning subject must have some basic knowledge to build on.

Our way to inquiry-based education in mathematics

We have always been trying to develop pre- and in-service teachers' ability to see mathematics in the world that surrounds us (we speak of developing contacts of school mathematics with reality). That is why we were involved in studying the process of grasping situations (Koman & Tichá, 1998). It is a process that results in posing problems stemming from mathematical and non-mathematical (real-world) situations. We showed that this process could be defined as one of the forms of inquiry-based education (Hošpesová & Tichá, 2013).

In our previous research we showed the motivational, educational and diagnostic potential of inquiry-based activities. The process of inquiry asks for conjectures, hypothetical answers and their verification. That is why we have recently focused also on the development of the awareness of the need to know how justify and prove, and why mathematical reasoning and justification should be included in mathematical lessons. We try to develop Semadeni's idea of *action proof* (1984) and Wittmann's *operative proof* (2014). With pre-service primary teachers we try transition from empirical reasoning to non-empirical reasoning, to formal proof (Samková & Tichá, 2016; Stylianides & Stylianides, 2009).

One of very beneficial tools – Concept Cartoons

As one of the unusual ways of assigning the tasks during our experiment we used pictures called Concept Cartoons (see Fig. 1). We met with these pictures during our participation in a European project focusing on inquiry-based education, and in primary school classroom they appeared to have successful motivational and educational function as well as they appeared to support argumentation (for details on research see Naylor & Keogh, 2013). Following this path, we conducted a study which showed that Concept Cartoons may play positive motivational, educational and diagnostics role also in future teachers' education (the diagnostic part is referred in Samková & Hošpesová, 2015).

Concept Cartoons are a special tool that helps us to record process of solving of a certain problem by various respondents. Each Concept Cartoon shows a group of several children in a bubble-dialogue. Texts in the bubbles present alternative views on the pictured situation or alternative solutions of the problem growing from the pictured situation. One bubble could be blank – it is place for respondent's alternative.

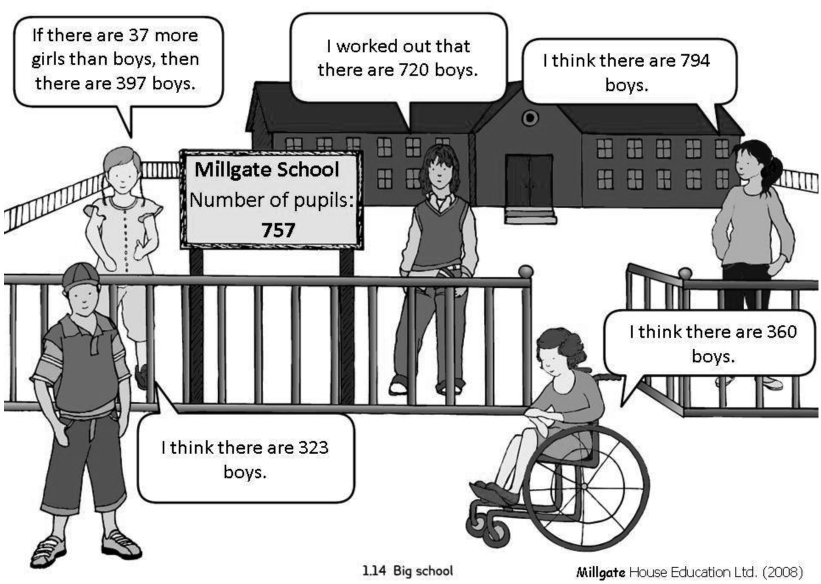

Figure 1. Example of Concept Cartoon (taken from Samková & Tichá, 2015)

This is a "reverse form" to problem solving – the students have to assess the correctness of answers. When working with Concept Cartoons, the students are expected to find bubbles with incorrect solutions, and to justify where the mistake is and what could be its source. In other words, the students are asked to assess the solutions, to look for the pupil's solving procedure, to look for possible sources of mistakes, wrong formulations, etc. All this is expected to develop their ability to discover, to reason and present arguments, to prove. Work with Concept Cartoons not only enables this development, it actually demands it and is built on it.

Another way how to work with Concept Cartoons is to ask students to write their own text into the blank bubble. Then the students' statements show whether they really understand the issue but also makes them aware of the importance and significance of the ability to formulate arguments and reason – both in teaching and when planning the lesson.

On one very substantial learning environment

One of the environments that seem to have a rich potential is the environment called *Rechendreiecke / addition triangles* (Krauthausen & Scherer, 2010; Wittmann, 2001). It allows us to pose problems of different difficulty (see Fig. 2).

Let us present some questions that have proved to be very stimulating over the past year:

> Find and describe the rule specifying how the numbers in the triangle are filled in.
>
> How do the numbers change if we use 11 instead of 10?
>
> What if there are numbers 6, 13 and 14 in the outer boxes?
>
> How can we fill in the numbers inside the triangle if only the numbers in the outer boxes are known?

We can also see that in this environment it is possible to start with algebra.

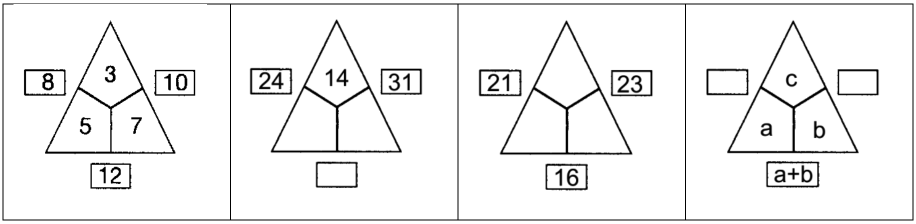

Figure 2. Examples of addition triangles

For the work in this environment we also created a Concept Cartoon (see Fig. 3). The students' task was to decide which statements in bubbles are right, and to justify their decision.

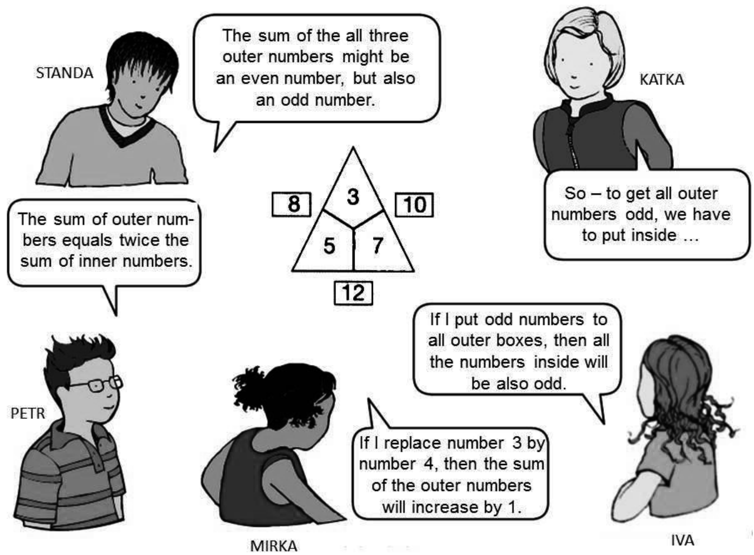

Figure 3. Concept Cartoon in environment of addition triangles; template of children with empty bubbles taken from (Dabell, Keogh & Naylor, 2008)

Future teachers' education

If we want to implement inquiry-based education into school mathematics, teachers have to be trained for it, as for many of them it is a step to *terra incognita.* In their teacher training they should have the chance to get hands-on experience with inquiries. Activities characteristic for inquiry-based education that are perceived as very important are argumentation, verification, justification, proving.

That is why we try to look for and assign to our students problems in substantial learning environments, and allow the students to gain the needed knowledge and skills through their solving. When solving a problem, it is essential to think it over and to verify the gained result.

THE STUDY

On one short survey

We conducted a short survey with two groups of 4th and 5th year teacher students (one in regular and one in distance form of studies) within the optional university course called Didactical Situations in Teaching Mathematics.

In the beginning of the course the students were asked to write what they understood by inquiry-based education in mathematics, and to list activities characteristic for inquiry-based education in mathematics.

During the course we tried to assign such tasks, whose solution required inquiry-based activities and approaches, e.g. searching, predicting, discovery, reasoning, verification, reflection etc.

It the end of the course the questions from the beginning of the course were asked again: *What do you understand by inquiry-based education in mathematics? How has your view of it changed during the course?* We tried to see if our teaching experiment led to a change in attitudes and opinions, and whether the participants of the course would try to use inquiry-based approach in planning their own lessons (Stuchlíková, 2010).

RESULTS AND DISCUSSION

We tried to classify students' characteristics of inquiry-based mathematics education. Only a few respondents stated that their view of inquiry-based education in mathematics had changed. Most often they claimed that their opinions had deepened, broadened, had been enriched by other aspects, had become more complex, etc.:

> My opinion is now clearer. I understand better now what the term inquiry-based education in mathematics represents.
>
> My opinion about inquiry-based education has not changed but got clearer contours. I had had a similar idea in my mind before. … I think everybody understands something slightly different by inquiry-based education.
>
> I do not think my views have changed completely but they have certainly grown more complex. I have learned that inquiry-based education in mathematics can have various forms.
>
> I do not think my view has changed but I have learned more about the issue and am more informed now. It is true that originally I thought this was a method typical for science education but I was not far from truth.

They stressed that inquiry-based education in mathematics requires careful planning on the teacher's part and clear stating of the goal, and that it also requires broader knowledge and deeper "general culture":

> My view of inquiry-based education has not changed but I have got a new perspective on how to plan the work in advance and how to think it over. Because we will never achieve good results unless we formulate the goal clearly. What do I want to teach and why? These questions are usually related to problems in textbook. But here more is needed – the teacher must know why.

Some students connected inquiry-based mathematics education particularly with science, especially with biology:

> Before the course I could only imagine inquiry-based education in biology.
>
> I do not think my view has changed but I have learned more about the issue and am more informed now. It is true that originally I thought this was a method typical for science education but I was not far from truth.

Some of them associated inquiry-based mathematics education with gifted pupils:

> Before this course I was convinced that inquiry-based education was exclusively for gifted pupils in mathematics and that they try to pose different types of mathematical problems. Having read literature and having finished the seminar work I found out that thus conceived teaching is for everybody. Inquiry, discovering and considering different variants of a solution helps to reinforce positive attitude to mathematics, develop logical thinking and develop general study skills. I will try to use this method in my lessons of mathematics as much as possible.

Some respondents appreciated that inquiry-based approach led to deepening of their knowledge of mathematics:

> Thanks to the literature I studied, my knowledge of mathematical concept and types of problems has broadened.

> The seminars made me astonished when I realized how many different types of problems existed in inquiry-based education in mathematics.

Several students appreciated inquiry-based methods but expressed concerns about weak level of their own mathematical knowledge:

> For me, inquiry-based approach brings calmness into mathematics. It showed me how beautiful mathematics can be. However, I am afraid my mathematical thinking and knowledge are not sufficient.

There appears also gratifying opinion about Concept Cartoons:

> Concept Cartoons are interesting and I will try to use them.

Certain statements appear as empty proclamation, "learned allegations”, as precepts:

> I understand inquiry-based education as a process of discovery that leads to asking many questions related to the problem, that encourages thinking and “stimulates” mind. It makes learners experiment and look for answers. At the end of such activities learners verify and discuss their conclusions.

> What are the demands of this kind of teaching? Careful planning from the teacher … and more knowledge from different fields and disciplines and its application.

Posing Concept Cartoons

During the university course we also assigned the students a task asking them to pose their own problem in the form of Concept Cartoon. The students wrote an essay describing the process of creation, and how they tried the Concept Cartoon with children.

One of the essays was really interesting from the didactical point of view (see Fig. 4).

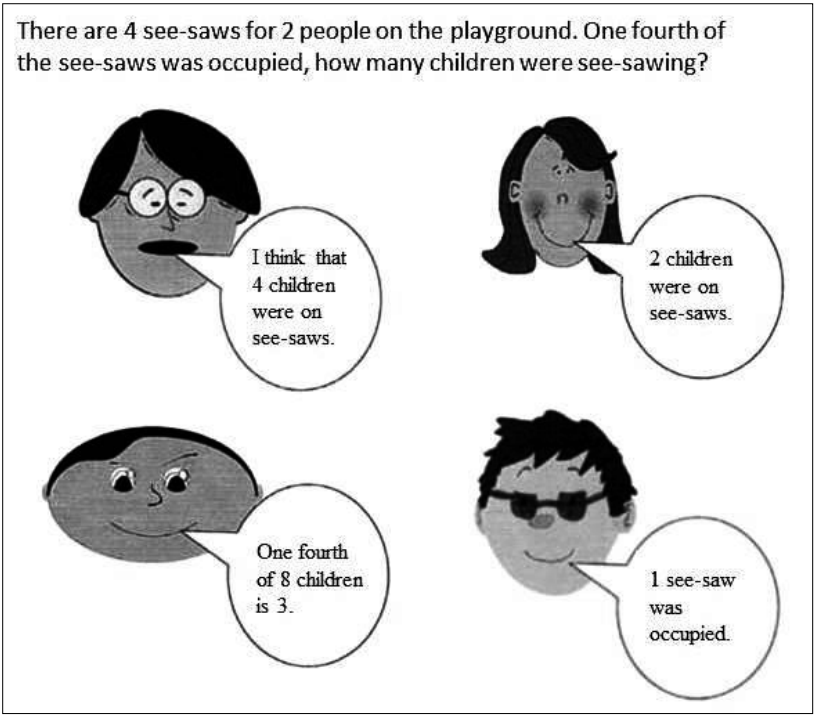

Figure 4. Concept Cartoon posed by one of the students.

The author supplemented the posed problem with a comment that only revealed further misconceptions in her approach:

> In my work I have realized how important it is to formulate questions accurately with respect to conjunctions, prepositions or phrases as these can completely change the question. .... In the second grade I wrote: **There are 4 see-saws for two people on the playground.** In another class I reformulated the assignment: **There are 4 see-saws on the playground, each for two people.** Younger children always circled only one of the two correct answers. There was too much information in the problem. If I were assigning the same problem in the second grade, I would simplify it for example in this way: **There were four see-saws in the playground. One quarter of them was used. How many children were there on the see-saws?**
>
> If I were working systematically in inquiry-based manner, I would start by easier questions with only one possible answer. And then I would gradually progress to more difficult problems.

The author also summarized her observations:

> Fourth and fifth graders were better at presenting arguments and defending their truth, even in case their answers were not correct.

> I was surprised by different points of view and their reasoning. These were really enriching for me and I started to perceive individual pupils' thinking in a new way. … some of the children used pictures that helped them solve the problem.

> I think the most seminal was the joint analysis of problems in which children who had made a mistake found the correct solution and felt the pleasure of achieving this. I realized children really want to discover. And it is up to us to enable them to do this.

CONCLUSIONS

We can say that our study shows two important results related to implementation of inquiry-based mathematics education into university courses for future primary school teachers. First, that experienced mathematical inquiry can cause shifts in students' views of inquiry-based education, and also in their beliefs and attitudes towards mathematics. Second, the study confirmed the significant role of Concept Cartoons in future teachers' education – they can be successfully used not only as an educational tool but also as a diagnostic one.

Once again (e.g. as in Samková & Tichá, 2015) we confirmed our belief that while working with Concept Cartoons, the students started to realize how important is to get in touch with various methods of argumentation (Stylianides, 2007). That is why we see Concept Cartoons as really beneficial, and we shall continue in using them in future teachers' education.

On the other hand, our experience shows that judging the statements in bubbles is very difficult for the students, maybe also because this activity is unusual for them. Some students even expressed the opinion that wrong statements should not be present in the bubbles, that they should be replaced by right ones.

Acknowledgement

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PROBLEMS POSED BY PRESERVICE TEACHERS – HOW DO THEY FIT IN AN INQUIRY MATHEMATICS CLASSROOM?

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According to the inquiry oriented mathematics education, the students are expected to engage in investigative and creative activities, similar to those of mathematicians. This requires a particular way of teaching by the teachers, which in turn requires specific training during their studies. Two basic elements of this training are problem solving and problem posing. Following the literature, we believe that these activities can and should be linked, in order for the teacher to be able to establish an inquiry tradition in the classroom. In the present study we engaged preservice mathematics teachers in activities that involved posing, solving and evaluating mathematical problems. We were interested in the characteristics of these problems in relation to an inquiry oriented didactical approach. Our results have shown the preservice teachers' ability to identify and utilise fruitful contexts for problem solving.

INTRODUCTION

Mathematical problems constitute a core element of mathematics education, especially in inquiry-based approaches. According to these approaches, the students are expected to engage themselves in activities similar to those of mathematicians: solving problems, formulating hypotheses, posing new questions based on an existing situation, etc. Thus, the processes of problem posing and problem solving are central to these approaches and, moreover, they are – or should be – related in a visible way in the mathematics classroom. The rationale for such an approach is that in a conventional mathematics classroom the teacher is posing problems which the students are expected to solve. Even when the problem leads to another problem (e.g. by generalising) the process is guided by the teacher in order for the students to reach the 'desired' results. On the contrary, in an inquiry mathematics classroom, students' participation becomes more active: they can investigate problems, they can pose new questions and, generally, they can propose their own problems. However, these activities are not easy to be established in a classroom; a continuous negotiation of the relevant norms is required (Yackel & Cobb, 1996). In other words, the students have to become accustomed to the fact that they can participate actively in the common construction of knowledge. This in turn requires a careful design by the teacher who wishes to create an inquiry tradition in the classroom. Thus, the need for preparing teachers for that approach is evident and has been researched (Crockett, 2002).

In line with this, in one of our previous studies (Tatsis & Maj-Tatsis, 2014) we have chosen problem posing as one of the main elements of an inquiry-oriented approach to teaching mathematics. Particularly, we engaged preservice mathematics teachers in problem posing as a means to improve their own knowledge on what constitutes a 'good' mathematical problem – and on how the notion of appropriateness of a problem may affect its formulation (Kontorovich et al., 2012). In the research presented here, we build upon the results of that study with a more recent one, that aimed to explore in greater depth the connections between solving and posing problems. Particularly, our research focus was the preservice teachers' ability to pose problems adhering to an inquiry-oriented didactical approach. Our research questions were the following:

a) Were the preservice teachers able to discover and utilise fruitful contexts for challenging mathematical problems?

b) Were the preservice teachers able to perceive the mathematical potential of their posed problems?

Before we proceed with the description of our study, we will present our theoretical background.

THEORETICAL FRAMEWORK

The inquiry-oriented mathematics education is an approach to teaching and learning mathematics that places the students at the centre of the learning process. According to this approach, the students are expected to actively participate in mathematical discussions, pose problems or conjectures and solve problems (Richards, 1991). An important function of inquiry is that it empowers "learners to see themselves as capable of reinventing mathematics and to see mathematics itself as a human activity" (Rasmussen & Kwon, 2007, p. 190). In other words, the students are engaged in doing mathematics in the way mathematicians do. This has significant implications on the way that classroom instruction is organized: of great importance are some mathematical activities which should be entailed in students' engagement with mathematical problems: conjecturing, questioning, and generalizing are some of them (Johnson, 2013). To these we can add reasoning and mathematisation (or modelling), which according to Krygowska (1986) are higher-level mathematical activities.

In an inquiry-oriented mathematics classroom the teacher and the students' roles differ from a conventional classroom. The teacher instead of being the sole authority in the classroom, leaves room for the students to experiment with the mathematical situations, to make conjectures and to try to confirm or refute them. Problem solving and the connected problem posing (Kilpatrick, 1987) provide a fertile ground for such explorations.

Problem solving in mathematics has received a huge amount of attention during the last decades and appears in numerous contemporary curricula and research reports. Within problem solving lies problem posing, since the students, while solving a problem, may develop new problems by changing the initial questions, adding new data, etc. (NCTM, 2000). Even in Polya's (1957) classic work *How to solve it?* we read about a heuristic that suggests to the solver to initially solve a problem which is related or analogous to the proposed one. In other words, the solver is asked to *pose* a different problem. Generally, problem posing can be defined as the creation of a new problem or as the reformulation of a given problem (Silver, 1994), where the initial conditions may vary. Stoyanova & Ellerton (1996) refer to three problem posing situations:

a) free, where there are no restrictions for the students; they can pose any problem they wish;
b) semi-structured, where the students are provided with an initial situation and they are asked to create a problem based on it;
c) structured, where the students are provided with an initial problem and they are asked to reformulate it.

There are many aspects of problem posing that make it a very useful tool in mathematics education and also in (mathematics) teacher education. The basic aspect is the close connection of problem posing with creativity (Silver, 1997). Additionally, by problem posing students are put at the centre of the learning process, which is usually not the case in traditional teaching/lecturing or even in problem solving, where the problem is posed by somebody else, e.g. the teacher or the author of the textbook (Kilpatrick, 1987). Moreover, asking students to pose problems within realistic contexts (English, 1997) provides them a great opportunity for critical thinking, since they have to interpret the given data, discriminate between significant and insignificant information and "investigate if the numerical data involved are numerically and/or contextually coherent" (Bonotto, 2013, p. 40).

The process of problem posing itself has been investigated in various studies (e.g. Bonotto, 2013; Koichu & Kontorovich, 2013), while some of them extend their analyses to possible relationships between the posed problems and the posers' mathematical knowledge (Leung & Silver, 1997), and more particularly problem solving (English, 1998) or creativity (Silver, 1997). In one of our previous studies (Tatsis & Maj-Tatsis, 2014), we analysed preservice mathematics teachers' problem posing activities from three different perspectives: the problems' level of difficulty, the homogeneity of the problems' solutions by the posers and their peers and the effect of the initial context which was provided. Our results have shown the importance of engaging the students in such activities: our participants saw in their own eyes how different solvers can interpret differently the givens of a problem and thus solve it in a different way. The issue of language in the sense of avoiding ambiguities came also to the fore. The experience we gained by this study as well as our other previous studies (e.g. Tatsis, 2014) led us to design a research that had many similarities with these, but a rather different research focus.

CONTEXT OF THE STUDY AND METHODOLOGY

The participants of our study were 14 students, at the third year of their studies at the Department of Mathematics and Natural Sciences of University of Rzeszow in Poland; they had all chosen the teacher specialisation, which means that by the completion of their studies they would be able to teach Mathematics in the primary school (grades 4-6). The students were attending the third semester of the “Didactics of Mathematics” course and they had also attended a relevant course entitled “Problem solving seminar”. During the previous two semesters of “Didactics of Mathematics” the students discussed and solved some mathematical problems, talked about the different types of mathematical tasks and also discussed on the features of a ‘good’ problem. During their problem solving experience, they were engaged in prolonging tasks, formulating new questions, finding new strategies, generalising and transferring of a method (Klakla, 2002).

That group of students can be characterized as a mixed-ability group, since it contained students with high as well as average or even low marks in their studies. The authors of the paper were both present in the two sessions. All materials were presented to the students in Polish, with the exception of some initial theoretical comments which were presented in English and translated in Polish. Part of that presentation contained a description and examples of the revised Bloom’s taxonomy (Anderson & Krathwohl, 2001), with a focus on the higher-order thinking activities: analysing, evaluating and creating. That taxonomy was presented as an aid for posing challenging problems and for evaluating the problems posed by their colleagues. The first session lasted two hours; the students were initially introduced to a warm-up problem posing activity taken from Silver & Cai (1996). Then they were given a worksheet that contained two different contexts (Tatsis & Maj-Tatsis, 2014) and asked to work in pairs in order to formulate six challenging problems. We clarified that the solution of a challenging problem should contain some higher-order mathematical activities. Then, we collected the worksheets and handed them randomly to other pairs, the “evaluators”. After the problems were solved for a second time, we handed the worksheets with the second solutions back to the initial posers, and asked them to answer the questions: Is your solution of your problem the same with the solution of the other pair? If not why? Would you change something in the initial formulation of your problem? If yes, what would that be? The students were also asked to identify the strategy they had used to formulate their problems. Then, as a homework, we asked them to search in various grade 6 textbooks for contexts that could be used as starting points for problem posing by the pupils. They were also asked to justify why they had chosen the particular contexts. Finally, they were asked to formulate and solve three problems based on the chosen context, at least one of them being an open-ended one; all problems were expected to be challenging but within the reach of a sixth-grader.

Our research data consisted of mainly two parts: the first part was collected during the two sessions, which lasted five hours (two plus three) and the second part contained the students' homework, which was assigned to them after the end of the second session. The first part of data is similar to that contained and analysed in (Tatsis & Maj-Tatsis, 2014), since the students were given the same worksheet, initial contexts and peer evaluation scheme. For the purpose of the present paper we will focus on students' homework, which consisted of 42 problems produced by them, as well as their solutions.

According to our research aim and our particular research questions we have decided to focus on the following features of the students' work:

- the contexts chosen and the justifications given by the students on their choice;
- the problems posed and their solutions: particularly, we compared:
    - the mathematical activities that were present in the solutions provided by the students;
    - the mathematical activities that were identified by us as needed for the solution.

We were generally interested to see whether the students' work would contain elements of mathematical inquiry, since we have been trying to engage students in such activities during their courses. Particularly, the chosen contexts would provide us with information on the students' ability to identify the mathematical features of the textbook situations. Then, we analysed the students' problems and searched for advanced mathematical activities, according to the way we perceived the problems' solutions. Following our theoretical framework, we have deployed the following categories of advanced mathematical activities: multi-step calculations, reasoning, modelling, systematising, decision making, identifying and/or performing geometrical transformations. Finally, we examined the students' solutions and compared them to their initial justifications and our own solutions; this process was expected to give us a clear indication on our students' ability to utilise the textbook contexts in a way that would be compatible with an inquiry-oriented didactical approach.

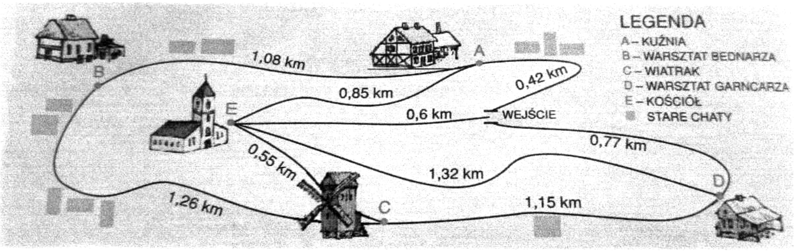
RESULTS

Our students have chosen a variety of contexts, taken from six different textbooks for grade 6. We have to note that in Poland there is no one textbook for all schools, but each school decides on the textbooks of all subjects. The contexts chosen by the 14 students can be categorised as follows:

Context type	Number of works
textbook task	4
geometrical figures	4
picture showing prices	4
picture showing maps	2

Table 1. Categories of contexts.

The justifications provided varied from “close to the pupils’ everyday life” to “it connects different mathematical concepts”. Below are two characteristic examples referring to the works 9 and 14 (from now on signified as W9 and W14, see Figures 1 and 2 below):



Wybrałam wyżej przedstawiony obrazek, bo uważam że przedstawia on ciekawy plan. Plan do którego można wymyslić sporą ilość zadań w tym problemowych. Rysunek nie zawera zbyt dużo informacji. Odległości zaznaczone na planie mogą wystąpić w zadanich, ale można je również pominąć.

Zadanie 1. (zadanie otwarte)
Wyobraż sobie, że odwiedziałeś właśnie warsztat garncarza w którym odbyły się zajęcia pokazowe. Kolejne zajęcia na które się zapisałeś mają odbyć się w warsztacie bednarza. Którą drogę wybierzesz, aby się tam dostać?

Zadanie 2.
Zwiedzając skanes pokonujesz 1 km w czasie 15 minut, 10 min zwiedzasz/oglądasz obiekt. Ile czasu zajmie nam oglądnięcie kuźni, kościołu i warsztatu garncarza.

Zadanie 3.
Masz przed sobą plan skansenu.Wchodząc do skansenu, chcesz zwiedzić jak największą ilość miejsc, pokonując przy tym jak najmniej kilometrów. Jak wyglądała by trasa Twojej wycieczki?

Figure 1. Context, justification and tasks of W9.

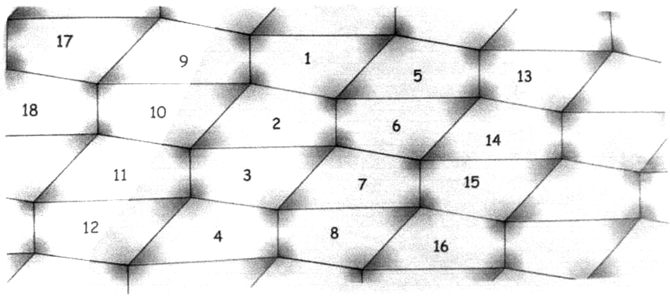


Figure 2. Context of W14.

I chose that picture because I think it represents an interesting plan. It is possible to think up many tasks, also problems. The picture does not contain too much information. The distances at the plan can be put in the tasks but you can also delete them. (W9)

I think that the quadrangles presented as a floor will stimulate the imagination of the pupils and give motivation for developing the topic. You can notice here many relations such as: translations, rotations or congruent figures. The topic is also connected to the sum of the angles in a quadrangle. During classes with that picture you can expand the topic and discuss other polygons and motivate the pupils to create their own floors. (W14)

Most contexts were what we may call typical textbook situations: an image taken from a textbook task or a textbook word problem. However, there were few exceptions which were considerably different, for example some geometrical figures, which belong to a pure mathematics context (Figures 2, 3 and 4).

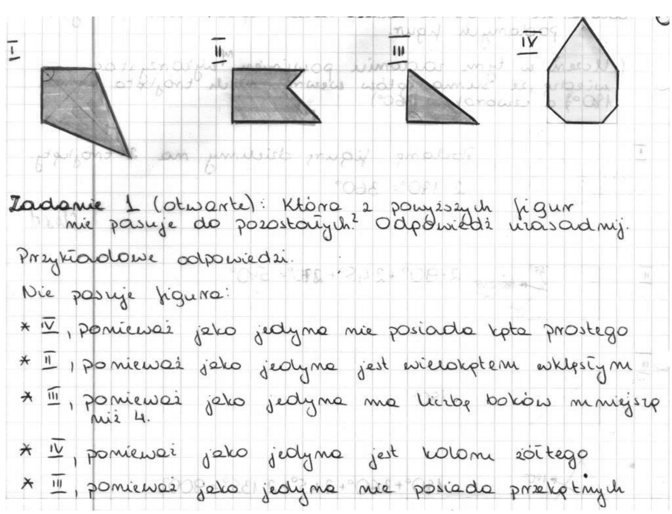


Figure 3. Context, Task 1 and its solutions from W1.

In Figure 3 we read:

Task 1 (open): Which figure from the above does not fit to the others? Justify your answer.

Solution:

Examples of answers:

IV as the only one which does not have a right angle.

II as the only one which is a concave polygon.

III as the only one which has less than 4 sides.

IV as the only which one is yellow.

III as the only one which does not have any diagonals.

Task1 described above includes the mathematical activities of discovering commonalities among geometrical figures and reasoning. Thus, generally, the problems posed by the students were mainly categorised according to their solutions as provided by the posers and as identified by us. Table 2 below presents only the more advanced of the mathematical activities contained in each problem's solution. For instance, in the case of Task 1 from W1 we chose to keep only reasoning as the more advanced mathematical activity. The students are referred to as "Posers" and we as "Researchers". We have to note that there were 20 problems which contained only one- or two-step simple calculations, thus they were not considered as challenging and are not included in Table 2.

	number of problems according to	
activity type	**posers**	**researchers**
multi-step calculations	5	8
reasoning	1	3
modeling	4	4
systematising	2	5
decision making	0	1
proving a theorem	1	1
identifying and/or performing geometrical transformations	2	3

Table 2. Mathematical activities in problems' solutions.

Table 2 shows that there are significant differences in the ways that the students and we have perceived the mathematical potential of the problems.

A characteristic example of such a disparity appeared in W9 (Figure 1). The problems contained in W9 were the following:

> Task 1 (open task): Imagine that you just visited the pottery workshop (D at the picture), where you attended classes and your next classes are in the barrel workshop (B). Which way will you choose to get there?
>
> Task 2: While visiting an antique building museum you make 1 km during 15 min, it takes 10 min to visit one place. How much time would it take you to see the forge (A), church (E) and the pottery workshop (D)?
>
> Task 3: On the map you can see the plan of an antique building museum. You want to visit as many places possible by walking the less kilometres possible. How would your trace look like?

For Task 1 the student gave the following solution, which does not contain any mathematical justification:

> I will choose the way through church and the forge in order to see a couple of places.

For Task 2 the student's solution contained multi-step calculations, which is in line with our analysis. For Task 3 she gave the following solution:

> 0.42+0.85+0.55+1.15+0.77 = 3.74 (km)
>
> My plan of the excursion is Forge – Church – Windmill – Pottery workshop

In Task 3 the student offers only one alternative, without any justification; according to our analysis, this task calls for systematising and reasoning.

Another characteristic example of a discrepancy between the student's and our views of a problem's potential comes from W11 (Figures 4, 5).

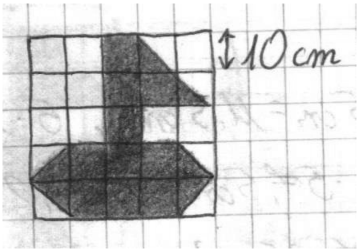


Figure 4. Context of W11.

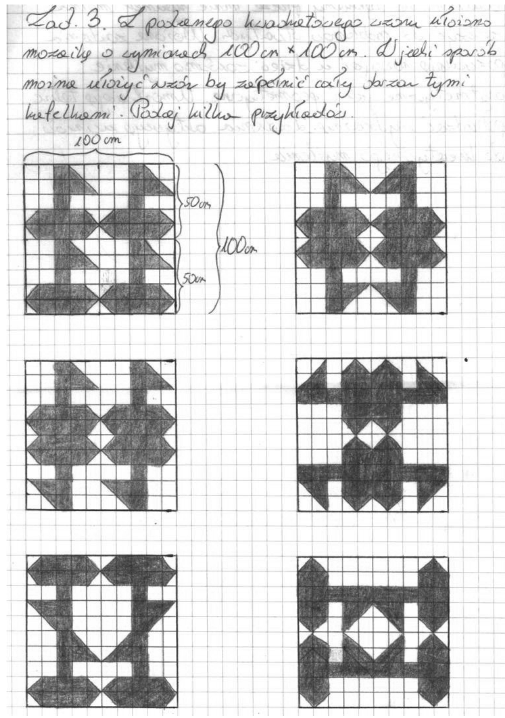


Figure 5. Task 3 and its solution from W11.

In Figure 4 we read Task 3's description:

> From the given square pattern a mosaic of the dimensions 100 cm × 100 cm was made. In what ways can you put a pattern in order to fill that area by the tails? Give some examples.

This task has a significant teaching potential: instead of merely asking the pupil to draw the different mosaics, the teacher could ask the pupil to identify the geometrical transformations which are needed in order to acquire these mosaics. Thus, the particular task as formulated by the student fails to include the advanced mathematical activities of identifying and/or performing geometric transformations.

CONCLUSIONS

Our study was designed in order to investigate preservice teachers' ability to pose problems adhering to an inquiry-oriented didactical approach. In order to investigate this we focused on the contexts the students used and on the solutions of their own problems. Our data came from the homework that was assigned to them after two problem posing sessions. Although it was stressed to them that they will not be assessed for it, they all put a great amount of effort, judging by the quality of their problems. Half of these problems were characterised as challenging by us, since they contained at least one higher-order mathematical activity in their solution.

Thus, coming to our research questions, our students were able to discover and utilise fruitful contexts from textbooks, in order to formulate challenging mathematical problems. Most contexts were typical, but few had distinctive characteristics, which made them even more appropriate for problem posing. At the same time, our analysis has shown that our students were only partially able to grasp the potential of their own problems; in some cases they provided simplified solutions to promising problems. This can be attributed to their limited experience in problem posing combined with problem solving.

Summing up, we believe that only when a student puts oneself to the position of the poser *and* the solver, can grasp the totality of mathematical activities which are contained (or sometimes even hidden) in mathematically challenging problems. Our students have had that experience and acknowledged the richness and the potential of such activities in an inquiry-based mathematics classroom.

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THE ATTITUDE OF PRE-SERVICE MATHEMATICS TEACHERS TOWARDS INQUIRY-BASED EDUCATION

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This paper presents a selection of the results of a questionnaire conducted among Polish pre-service primary and secondary school teachers of mathematics. The aim of the study was to recognize the theoretical attitude of the pre-service teachers of mathematics towards using the Inquiry-Based Education method in their practice. On the basis of the participants' opinions, the positive and negative factors affecting the implementation of Inquiry-Based Education are presented. Additionally, some conclusions are drawn of the process of teacher training in regards to the issue.

INTRODUCTION

In the 60s of the last century, thanks to well-known psychologists, among whom such names as Piaget, Dewey (1997), and Vygotsky (1961) should at least be mentioned, the idea of constructivism bloomed. It lasts in teaching to this day. Constructivism as a theory of knowledge, according to Dylak, has two major approaches:

> It is, on the one hand, a neurobiological theory of brain functions, and on the other hand, pedagogical conceptions which indicate the effectiveness of rules of pedagogical action (…) (Dylak, 2016).

The so-called *Inquiry-Based Education (IBE)*, based on the idea of constructivism, is currently a popular instructional method in many countries and in the context of many subjects. Although some communities of psychologists, such as Kirschner, Sweller & Clark (2006), are strong opponents of such methods, the IBE is intently promoted by the majority of researchers, educators, and by the European Union.

International projects are financed by the European Union in order to promote this idea in teaching. To illustrate the scientific interest, we can mention, among others, such projects of natural sciences as:

- PROFILES - "Professional Reflection-Oriented Focus on Inquiry-based Learning and Education through Science",
- SAILS - "Strategies for Assessment of Inquiry Learning in Science",
- ESTABLISH – "European Science and Technology in Action Building Links with Industry, Schools and Home".

Or, combined with mathematics:

- PRIMAS – "Promoting Inquiry in Mathematics And Science Education Across Europe", and FIBONACCI PROJECT (Artigue et al., 2012).

Some of them address their aim explicitely to teachers, for example: “to prepare teachers not only to be able to teach through Inquiry-Based Science methods, but also to be confident and competent in the assessment of their students’ learning” (SAILS).

INQUIRY-BASED EDUCATION VERSUS THE CORE CURRICULUM IN POLAND

A new Polish school curriculum was launched in 2008. The new general objectives of education and the students’ key competences to be developed during class were defined in this document. Currently, in the school year 2015/2016, it is being implemented into first year classes of middle schools.

Bernard et al. (2012) prove that the competences in science education formulated in the new Polish science core curriculum are strictly in line with the competences that might be developed by *Inquiry-Based Science Education* (IBSE). Included was the answer of the Ministry of National Education to some of the questions formulated by the Department of Chemical Education at Jagiellonian University. The following formal excerpt of the correspondence expresses the issue.

> The core curriculum, despite the empirical education, recommends the use of active teaching methods in the teaching-learning process. In particular, (…) it is recommended that students perform experiments on their own (under the teacher supervision), conducting and recording observations, followed by critical analysis and public presentation of the results. Active methods promoting direct understanding, such as educational trips, educational projects, debates, etc. are also indicated. Students should have a possibility to observe, study, explore laws and relationships, achieving satisfaction and enjoyment from gaining knowledge on their own in lessons. (…) The scope of the teaching content provides many opportunities to use project methods (especially of research type), practical chemical experiments or other active methods, that enable students to gain information from a variety of sources and to process them in various ways. A self-contained student’s observation is the foundation of the experience, reasoning, analysis and generalisation of phenomena and so experiments play an important role in the accomplishment of the above content. (Bernard et al., 2012, pp. 50-51)

The use of a similar approach in mathematics education is confirmed not only by the above general interpretation of the core curriculum provided by the Ministry of National Education but also by the explicitly formulated core curriculum in mathematics. The following excerpt concerns the middle and secondary school level:

> “The goals of teaching – general requirements,” particularly:
>
> III. Mathematical modelling: The students build a mathematical model of a given situation, taking into account the restrictions and remarks.

IV. Using and creating strategies: The students creates strategies for solving a problem.

V. Reasoning and justification: The student creates a string of arguments and justifies their correctness.

(Podstawa programowa z komentarzami - Core curriculum with comments, p. 41)

BACKGROUND OF THE RESEARCH

The research presented in this paper is designed on the basis of the definitions and descriptions of *Inquiry-Based Science Education* (IBSE). The definition of the IBSE method of teaching and learning is attributed to Schwab and Brandwein (1962), which was formulated in the context of science education. It is regrettable that there is no specific Polish translation of this method, neither in the community of science, nor of mathematics teachers, educators, and researchers. This can cause slight difficulties in the process of teacher training in Poland.

The emphasis in this method is on organizing the activities for a classroom in such a way that the students themselves ask questions in order to solve a problem, as well as discover, pose, and verify hypotheses with minimal guidance from the teacher. A quite similar attitude is presented in the context of mathematics education (Jaworski, 1994).

The definition of IBSE is agreed upon by science educators according to Linn, Davis, Bell, (2004), where scientific inquiry in a classroom is defined as:

> the intentional process of diagnosing problems, critiquing experiments, and distinguishing alternatives, planning investigations, researching conjectures, searching for information, constructing models, debating with peers, and forming coherent arguments. (Linn, Davis, Bell, 2004, p. xvi)

Bernard et al (2012) tried to establish how Polish in-service science teachers establish methods based on inquiry. Six areas, which, according to them, may affect the application of IBSE in Polish schools, were examined. The issues in question were the nature of IBSE, as well as: IBSE and teachers, students, school curricula, assessment, and IBSE and the attitude of the public. The participants of the study were 33 in-service science teachers of middle or secondary schools. The study took place just after a theoretical workshop regarding the IBSE method, which means they had knowledge of the topic. This sample of teachers consisted of those who were interested in the method, since they volunteered to participate in the training. The results of the questionnaire (Bernard et al, 2012) are brought up for the discussion of the results and mentioned in the summary of the paper.

AIM OF THE RESEARCH

The aim of the empirical research was to acquire the general opinions of pre-service mathematics teachers of primary, middle and secondary schools regarding the idea of *Inquiry-Based Learning* and investigate how the method is perceived by them in the context of teaching mathematics.

In particular, the purpose was to discover their attitude towards the method proposed theoretically, and what they think are the strengths and limitations of this method.

Additionally, a comparison is made with the results of the research prepared by Bernard et al. (2012) on the sample of in-service science teachers of primary and secondary schools.

METHODOLOGY OF THE RESEARCH

General remark

The research was intentionally designed on the basis of science education by the use of the description and definition of *Inquiry-Based Science Education* (IBSE). The purpose of such an approach was to gain a **real, natural answer from the respondents on the possibility of implementing and using this method in mathematics**. At the same time, it ensured a possibility of disclosing the respondents' natural attitude towards the method.

It is worth mentioning that the researcher was at the same time the academic teacher of a "didactics of mathematics" course. That is why pre-service teachers were asked about their opinions **as experts** in mathematics education in the context of science education. Designing the research only in the context of mathematics would not cause the disclosure of their natural attitude and spontaneous opinions. The knowledge that the method exists would influence them to adapt their responses to the academic teacher's expectations, being treated not as experts, but as students taking a didactics of mathematics exam.

Participants

The participants were pre-service teachers in mathematics of two groups, hereinafter referred to, in order to simplify the presentation: PRIMARY and SECONDARY.

PRIMARY was a group of 29 pre-service primary school teachers of mathematics. They were finishing their 3rd year of first cycle degree studies (Bachelor's) in mathematics, with a teaching specialization, at Pedagogical University of Cracow.

The SECONDARY group also consisted of 29 participants, who were formally prepared to teach mathematics at primary schools and were also continuing their education at the Pedagogical University of Cracow to acquire permits to teach mathematics at middle and secondary schools. They were finishing their 1st year of second cycle degree studies (Master's) in mathematics, with a teaching specialization, at the Pedagogical University of Cracow.

During their studies, both groups attended many courses on theory of mathematics education as well as practical courses at schools. They already had knowledge of many methods and approaches to teaching. The general idea of constructivism was also introduced and promoted. However, during their studies they have not explicitly heard the name of the *Inquiry Based Education method*.

At the beginning of the meeting, the participants (the students) were told that they are taking part in a research conducted only in order to obtain their opinions. They were assured that their answers would not have any influence on any of their graded assignments. They were also informed that their answers would not be assessed from the point of view of their didactical knowledge, but would be the starting point for further discussions concerning the issue.

Questionnaire

The questionnaire consisted of three informal parts. The first part was introductory. The second part was a presentation of the IBSE method and involved obtaining the participants' opinions on the method in general. The third part concerned using the method in mathematics education.

The participants were to answer 7 questions in a written questionnaire. It took them approximately 45 minutes. The time was not limited. In this paper, due to content limits, I will focus only on some of the questions regarding the participants' opinions on IBSE and the possibility of using the method during mathematics lessons.

The presentation of the IBSE method started with the following statement: "Look intently at the features of the two science classes presented in the table. Both lessons are related to practical experiments on a pendulum."

Directly following the presentation of the table (Figure 1), **Question 3** was posed: "Which of the lessons do you think is more powerful? Justify your answer."

After providing some time for the justification, the following information was provided: "Mr. H carried out his lesson in accordance with the *Inquiry-Based Education* method. Have you ever heard of this method?" The answer to **Question 4** consisted of choosing either "Yes" or "No".

Mr Shaw's lesson	Mr Hammond's lesson
The teacher poses the questions that are to be explored.	The teacher introduces a stimulus to the class and invites students to observe, describe and pose questions. He awakens curiosity.
The teacher gives each pair of students the equipment they will need.	The students are allowed to select the equipment they need.
There is no room for predicting and testing. Possible mistakes and misconceptions are avoided.	Predictions are discussed and tested. For example, students assume that the relationship between length of pendulum and time is linear and test this.
The task is completely structured by the textbook. Students make very few decisions. They mainly follow instructions.	Students are allowed to tackle the problem in any way they wish. For example, they are allowed to use trial and error. They make decisions for themselves.
The teacher tells students to check their work for accuracy.	The students check each others' work for accuracy.
The teacher mainly instructs and gives information and evaluates work.	The teacher challenges, questions and provokes students to think for themselves. Students present and evaluate each others' work.

Figure 1. What is Inquiry-Based Learning? PRIMAS: „INQUIRY-BASED LEARNING in maths and science classes" (2013, p. 17)

After Question 4, the definition of IBSE according to Linn, Davis & Bell (2004), cited above, together with the Constructivist Inquiry Cycle (Llewellyn, 2002, p. 47) in the context of science education was provided (see Figure 2).

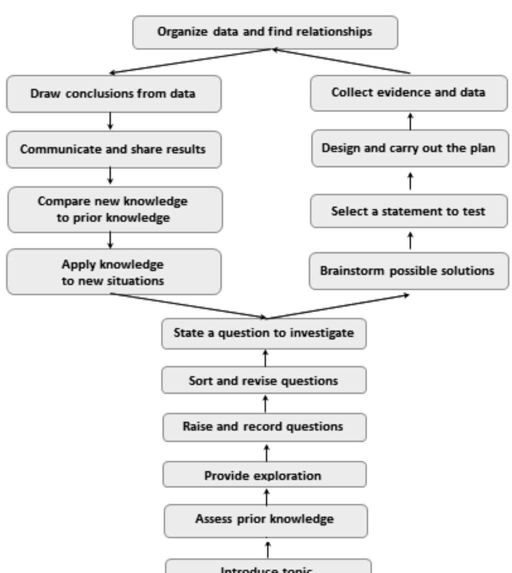

Figure 2. Constructivist Inquiry Cycle (based on Llewellyn, 2002, p. 47)

Question 5 was followed by the graph presented above. I used the same questions as presented by Bernard et al. (2012) referring to the teachers' attitude towards the nature of IBSE, presented below. The last, tenth question, on the equipment of science laboratories, was omitted in my research on mathematical education as it was not relevant. The question was:

"Do you agree with the following statements? Indicate on the scale your response."

The scale was five-point, bipolar scale: strongly disagree, disagree, not sure, agree, strongly agree. The statements were the following:

5.1 IBSE requires more thinking than traditional methods.

5.2 IBSE is more suitable for foundation level courses.

5.3 IBSE is more suitable for higher level courses.

5.4 IBSE favours the better students.

5.5 IBSE favours the weaker students.

5.6 IBSE requires more time than traditional methods.

5.7 IBSE requires discussion but there is insufficient time for this in school.

5.8 Inquiry methods require longer blocks of time than are not normally available in the school timetable.

5.9 Step by step exercise instructions defeat the purpose of IBSE.

The formulation of **Question 6** was the following:

Can you see the possibility of using the IBE method in mathematics classes? Select your answer: "YES", "NO", "IT IS DIFFICULT TO DECIDE".

If you think that this method can be used during **mathematics** lessons **write 3 examples of topics/issues it could be used for**. If you think it is impossible – **explain why**.

Because of content limits, Question 7, as well as Questions 1 and 2, are not the focus of the paper, so their presentation is omitted.

RESULTS

Ad Question 3

The answers of 100% of both groups (58 participants) of respondents to Question 3 stated that the lesson of Mr. H was more powerful. Below, the most popular statements used by pre-service teachers in order to justify their answers are provided:

"The students retain more of the knowledge obtained during lessons, as they discover correlations and form the knowledge on their own, which means that it will be more effective. Thanks to this method, the students can not only become scientists, but also learn about independence, make mistakes, notice them, learn from them, use any testing method including trial and error, learn in their own way, and decide how they want to do it. This method enforces creativity, activity, and

> involvement (whether the students like it or not), attracts interest and motivation for learning, as well as the feeling that what is being done makes sense. It showcases the point of the subject and makes the students think. They will definitely remember such a lesson as well as their findings for an extended period of time."

All of the respondents were enthusiastic about the method. However **one man** from the SECONDARY group of pre-service teachers included his doubts concerning using this method in Polish schools:

> "Mr. H's lesson is conducted in a more interesting way and will definitely pique the interest of students, as it motivates them to make and verify hypotheses on their own. However, I do think that in Polish conditions, time restrictions would not allow for such a lesson to take place, and that it is more suited for private lessons or school clubs."

Ad Question 4

17 people from the SECONDARY group and 21 from the PRIMARY group answered that they knew of the method beforehand. However, they have orally specified that they had not known the name of the method nor its specific instructions, but only the general idea of constructivism.

Ad Question 5

The differences in the teachers' opinions on the statements provided by Question 5 were measured using a methodology used by Bernard et al. (2012) in order to compare the results. The aforementioned scale: strongly disagree, disagree, not sure, agree, strongly agree was coded respectively by the numbers: –2, –1, 0, +1, +2. The length of the bars of agreement and disagreement corresponds respectively to the absolute value of the positive and negative numbers, summed independently. The answer 'not sure' did not influence the length of the bars of agreement and disagreement. Figure 3 shows the following results concerning the nature of IBE.

The number of "not sure" answers for the all statements in Question 5 was 68 for the PRIMARY group, and 41 for the SECONDARY group, which is shown in detail in Table 1.

Group	Number of answers of "not sure" to Questions 5.1 - 5.9								
	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
PRIMARY	1	10	10	9	8	2	4	10	14
SECONDARY	0	6	6	3	2	5	2	10	7

Table 1. Number of answers of "not sure" to Questions 5.1 - 5.9 of 58 pre-service teachers - PRIMARY and SECONDARY groups.

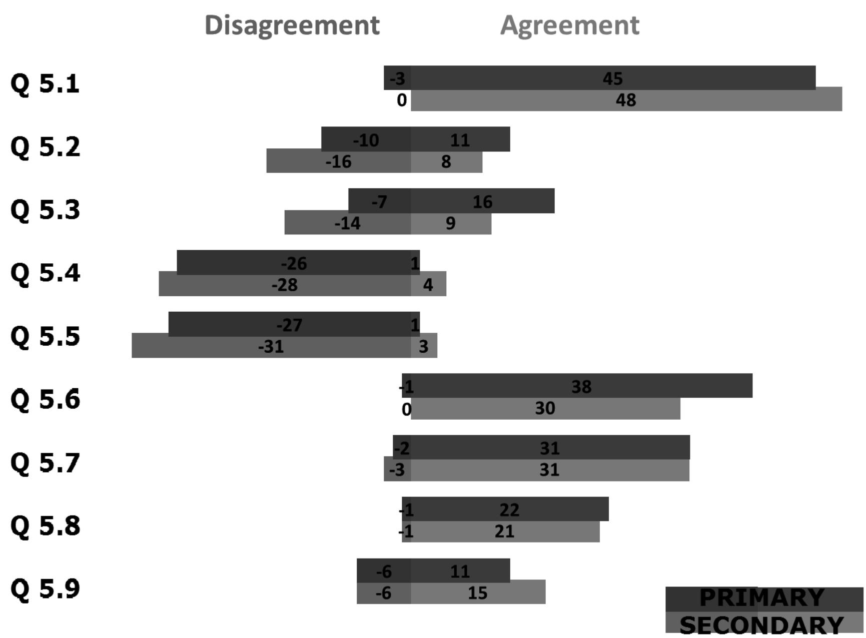

Figure 3. Answers to Questions 5.1 - 5.9 on the nature of IBE of 58 pre-service mathematics teachers in two groups: PRIMARY and SECONDARY.

Ad Question 6

The opinions on the possibility of using the method in mathematics education varied between "yes" and "not sure" with **only one exception -** it was the same man who expressed his doubts in the context of Question 4. The distribution of the answers is illustrated in Figure 4.

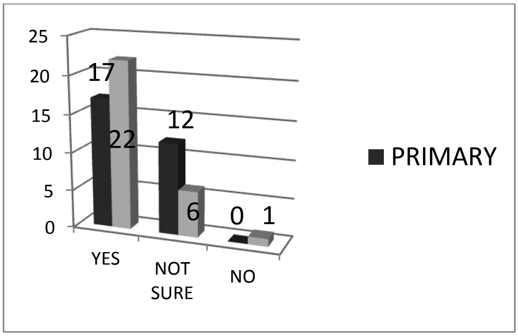

Figure 4. The possibility of using the IBE method **in mathematics classes** in the opinion of the PRIMARY and SECONDARY groups of pre-service teachers.

DISCUSSION OF THE RESULTS

The reactions of the pre-service teachers to the questionnaire was very positive and matter-of-fact. The participants unanimously judged that Mr. H's lesson was more valuable. In regards to the theory, they were able to appreciate and point out the advantages of the *Inquiry Based Education* method. Their justifications were focused on constructivism. It can be concluded that this is also the result of their successful training in didactics of mathematics during their studies. Very strong agreement (+93, only one 0, and -3 of disagreement) was given to the statement that the IBE required more thinking than traditional methods. The strengths of this method were identified by the future teachers.

In their opinion, the IBE does not favour weaker students, but is also not regarded as favouring the very good ones. In the pre-service teachers' responses, no preferences for applying IBSE at higher or lower levels of teaching can be observed.

However, despite such enthusiastic opinions on the method, **a large percentage of the respondents had doubts regarding the application of this method in teaching mathematics** (PRIMARY: 41%, SECONDARY: 18%, ALL 33%). Generally, SECONDARY teachers were more positive: 22 (yes), 6 (not sure) and 1 (against) than PRIMARY: 17 (yes) and 12 (not sure).

The justification for the doubts regarding the approach was provided by the respondents in various parts of the questionnaire. The prospective teachers strongly agreed (+68) that the IBSE required more time than traditional methods and that it required discussion, but there was insufficient time for this during class (+62). Similarly, the respondents stated that the inquiry methods required longer blocks of time than are normally available in a school's timetable (+43). It was a powerful factor which weakened the usefulness of this method in the opinion of the respondents.

The respondents, when explaining their doubts, mentioned the following reasons:

> Lack of time, poor self-preparation for making use of a very demanding method, overflow of content in the core curriculum, and the necessity of studying for external exams.

It is also worth quoting the arguments of the only person who was generally opposed to the use of this method in teaching mathematics. More so due to the fact that he firstly admitted the positive aspects of this method in teaching physics. It is worth mentioning that the student did not attend first cycle studies at the Pedagogical University of Cracow and his experience and preparation in didactics of mathematics was different than the majority of others. He was so opposed to this method that he even numbered his arguments, writing everywhere he could, despite the space limitations:

1) In many Polish schools, the curricula are overloaded and there is simply no time to make use of such methods.

2) Multiple students, especially the gifted ones, will prefer traditional learning methods, as it is easier to have a readymade formula provided.

3) There are also students which will not be interested in the lesson, no matter how interesting it is (this could also be result of their fatigue, when e.g. physics is their last lesson of the day). Such a lesson will not be interesting to them.

4) This method creates a lot of commotion and noise in the classroom, which could disturb and strain the more sensitive students.

5) Those who would like to show off will check the textbook and find the appropriate formulas, and then "miraculously" discover the dependencies.

6) Such lessons usually involve at most 2-3 students who dominate the lesson, with the rest feeling exempt from actively participating.

7) Many physical values are related to particular constants (e.g. g, G, k – coefficient of resilience) – determining these constants can be troublesome in a classroom environment.

8) The students may remember an interesting lesson, but not necessarily the formulas they discovered themselves.

Many of the barriers mentioned by the student can be eliminated through an adequate and detailed preparation of activities and by conducting them properly (e.g. 5, 6, 7). The fact of raising them may confirm a hypothetical lack of ability to mitigate these difficulties. Argument (3) seems to be irrelevant to the subject, and argument (1) was raised by multiple respondents. Many of these statements should not be ignored, but analysed and tested. For example, it should be noted with regard to argument (8) that it was also brought up by psychologists, as already mentioned in this paper (Kirschner, Sweller & Clark, 2006). One can also see some of the arguments as disputable, for example statement (2).

Unfortunately, not all the respondents understood the idea of the method. The quite strong disagreement (-12) with the statement "Step by step exercise instructions defeat the purpose of IBSE" confirms the occurrence of a misunderstanding in both groups.

Moreover, a misunderstanding was observed through some of direct comments of several pre-service teachers from the PRIMARY group, e. g.:

> I do, however, think that it is a fantastic method, as learning during a teacher-student conversation is an important aspect of a lesson. The teacher should guide the students towards the proper reasoning, instead of providing them with solving methods (…)

In the group, there were also students who revealed another misunderstanding in their reasoning. For example, they equated the possibility of applying this method with the need of introducing each new topic in accordance with the method.

(…) it would be strenuous to conduct every introductory lesson this way.

Such lack of understanding was not apparent in the SECONDARY group. Concerning a further comparison between the two groups, it was visible that, much more often, pre-service teachers from the PRIMARY group lacked opinions or were unsure. The percentage of "0" responses to the statements of Question 5 was 26% in this group, whereas in the SECONDARY group, it was 16%.

Similarly, they were not sure about the possible use of this method for teaching mathematics - as much as 41% of the PRIMARY group had no opinion or mixed feelings about this.

The SECONDARY group was also more creative. Fifteen people tried to provide examples of topics recommended by them for the application of this method into school practice, but, in the PRIMARY group, only 10 people tried to do so.

It would be interesting to investigate the reasons for such differences. There is a difference of only one year of study between these groups, which is not much. Although second cycle studies with a specialization in teaching bring together people who do plan their future in this profession.

SUMMARY

The analysis of the research will be continued, as the respondents' answers were very interesting and worth of further exploration.

Both the examined pre-service mathematics teachers as well as in-service science teachers (based on Bernard et al. 2012) shared the same opinions on the nature of inquiry-based education, despite the differences both in the subjects, as well as the specialization, experience, and decision-making methods of the participants of the study. All of them were able to perceive and appreciate the strengths of the method.

However, many of them claimed that there is no time in Polish schools for *Inquiry-Based Education*, even though this statement is contrary to the core curriculum as well as its interpretation and the expectations of the Ministry of Education.

The results show the possible ways of continuing teacher training involving the method and its relevant issues. Pre-service teachers strongly agreed that they would like to learn about the method. It is a good indication that they appreciate the method theoretically and despite having doubts "if" and "how" to use it, they truly want to learn about it.

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APPROPRIATE SELECTION OF MATHEMATICS TASKS AS A SUPPORT OF EFFECTIVE TEACHER'S WORK

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The article contains a list of basic information about the project "Narzędzia w Działaniu", which was conducted in Polish lower- secondary schools in the years 2015-2016 by the Instytut Badań Edukacyjnych. The aim of the project was, among other things enrichment conducted by teachers of mathematics classes with new, interesting and diversifed content and methods for their implementation to see whether these changes have provoked an increase in the efficiency of the work of teachers. The author of the article as a mentor and co-author of the final report of the project "Narzędzia w Działaniu" shares in this publication fragment conclusions of the project, on the impact of selected types of tasks and methods of their implementation on the work style of the teacher.

INTRODUCTION AND THEORETICAL FRAMEWORK

A competent teacher of mathematics is, in the teachers' opinion, the one who is a very good mathematician and has, at the same time, the teaching, communication and pedagogical skills due to which the mathematical knowledge transmitted to the learners becomes operative and durable (NCBiR, 2014).

Stefan Turnau in his book "Lectures on teaching mathematics" points to four prerequisites for a successful lesson that is the one which is provided in such a way that a learner is encouraged to active and conscious participation in the teaching-learning process. The prerequisites comprise: problem posing, availability, motivation and activity. In the author's opinion "…teaching mathematics should always involve problem posing; a problem already solved by the learners, which is sometimes equivalent to a fundamental problem, should be a starting point and an assistant in this process. Problem solving must create difficulty whose overcoming accelerates progress in the learning process. Such a problem, however, should be available to each learner, that is each learner should have knowledge, skills and intellectual maturity which suffice to deal with difficulty arising out of the problem. Problem posing and availability generate motivation in the learners which corresponds with understanding and acceptance of the sense of the effort as well as persistence necessary to solve the problem. Finally, an available problem and motivation which arises out of it trigger learner's activity without which teaching mathematics is not possible" (Turnau, 1990; p.91).

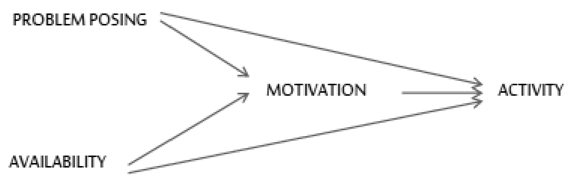

Figure 1. Prerequisites for a successful lesson; source: S. Turnau

Fulfilling each of these prerequisites is related to teacher's special skills, such as communication skills and questioning skills, as well as knowledge of, among others, methods and forms of classwork, conception of teaching mathematics, the role of heuristics in solving mathematical problems, using different visual aids with regard to particular time and purpose, proper reaction to learner's mistake and proper selection of mathematical problems which will be helpful in achieving short-term and long-term educational goals during a lesson.

In the publication of IBE (the Educational Research Institute): “Report on the State of Education in 2013, Teachers matter” a profile of a teacher of mathematics was constructed. The authors write: “The vast majority of teachers participating in the survey on teachers' needs declared that preparing a lesson they use textbooks (over 97%), teacher's methodological manuals (90%) and the publishers' websites (86%). Therefore one may presume that the direction a teacher takes and the way she/he implements the recommendations provided in the core curriculum (if she/he focuses only on detailed requirements or perceives also the general ones) depend, to a large extent, on publishers' propositions. The selection of materials for teaching mathematics offered by publishers influences the types of problems the learners will deal with, the knowledge they will acquire and the skills they will develop” (Czajkowska, Orzechowska; 2014, p.185).

In the process of teacher development, it is important to observe the effects on their own implementation of theory into practice, the effects of implementing new didactics solutions. It is important to discuss them with other teachers as well as specialists in the field of teaching mathematics.

One of the forms of professional development occurring in the Poland, and in all over the world (Meyer, 2011, Murata, 2011, Jones, 2013) is lesson study.

Lesson study places teachers at the center of the professional activity with their interests and a desire to better understand student learning based on their own teaching experiences (Murata, 2011; p.2).

In Japan, which in recent years has achieved high results in the process of teaching-learning of mathematics at all levels of education, teachers perceive their success is in wide use of lesson study (Murata, 2011; Shimizu, 1999).

Lesson study is used there to examine and better understand new educational approaches, it is for connect theory and practice. In the United States and in countries all over the world lesson study is known as a small, school-based collaboration (Murata, 2011).

Lesson study provides an opportunity to present an example of a new educational idea and/or approach for teachers to discuss, to ask question about, and to a shared understanding of the new idea (Murata, 2011; p. 4).

Lesson study usually consists of the following steps[1]:

- Consider goals for student learning and development;
- Plan a "research lesson" based on these goals;
- Observe the research lesson and collect data on student learning and development;
- Use these data to reflect on the lesson and on instruction more broadly;
- If desired, revise and re-teach the research lesson to a new group of students.

GOALS, ORGANIZATION, METHODOLOGY AND TOOLS

In order to enrich the classes conducted by teachers with new, interesting and diversified content and ways of passing it on in a form which contributes to initiation or enrichment of the process of stimulating activities specific for mathematicians and available for learners on a particular level of their education, a project "Narzędzia w działaniu" (Tools in Action) was carried out. In the project there were 9 sets of tasks containing 30 implementation units, called Baza Dobrych Praktyk (Database of Good Practice)[2], developed by the experts of Mathematics Section of the Educational Research Institute and proposed to teachers to be used during classes. The project was run in lower-secondary schools from October, 2014 to February, 2015 with participation of 210 teachers in the whole country. The majority of them were:

- certified teachers,
- living in the country,
- 24-38 years old,
- with 12-20 years of experience.

The author of this article carried out a function of mentor in the project "Narzędzia w Działaniu" (Tools in Action). According to the author of this article was inter alia, the professional support of teachers and the preparation of the final report. Theme of the project, according to the author, was very

[1]Based on the schema from the article Murata, 2011, p.2.
[2] See: http://bnd.ibe.edu.pl/subject-page/6

important for teachers to realize importance of implementation of the general requirements for math class. Unfortunately, the final report has not been published, prompting the author of the article to share bits and pieces of experience working in the research on its pages.

As it has been mentioned before, 9 sets of tasks were implemented[3]. Each group had to persuade teachers to try to work towards different goals specific in teaching mathematics.

1. **Team work[4]** – it is a group of tasks designed to be handled in groups of several students. The set "Groups" comprises the following three implementation units: "Area of a circle", "Pythagoras' theorem" and "Statistics".
2. **Tasks for summative classes[5] –** aimed to make students perceive the questions in wider context. The set "Relations" comprises the following three implementation units: "Percents", "Algebraic expressions, Pythagoras' theorem" and "Proportions".
3. **To trigger self-reliance – tasks with numerous correct answers[6]** – aimed to encourage students to experiment by means of exposing them to atypical situations where one task may have numerous different solutions and answers. The set "Superopen" comprises the following tasks: "Arithmetic, numbers", "Arithmetic, numerals" and "Geometry".
4. **Reasoning and argument – tasks which give opportunity to practise mathematical reasoning[7] –** aimed to activate the processes of reasoning and inference. The set "Explain" comprises the following tasks: "Algebraic expressions, divisibility", "Area of a triangle" and "Congruence, angles in a triangle".
5. **Tasks serving as good warm-up activities[8].** This set contains the tasks that are a reference to the previous lesson (short task similar to those that have been resolved in the previous lesson) or preparation for a new theme (a reminder of previously known messages needed to introduce a new topic or preparing to introduce a new issue). The system "warm-up" There are four units of implementation, "Introduction to the exponential notation", "Introduction to linear equations", "Introduction to solving systems of linear equations by substitution" and "Preparing for calculating the sum of the interior angles of the polygon."

[3] See: http://bnd.ibe.edu.pl/search?subject_id=6
[4] See :http://bnd.ibe.edu.pl/practice-page/68
[5] See: http://bnd.ibe.edu.pl/practice-page/66
[6] See: http://bnd.ibe.edu.pl/practice-page/64
[7] See: http://bnd.ibe.edu.pl/practice-page/62
[8] See: http://bnd.ibe.edu.pl/practice-page/60

6. **Individuation – complex tasks solvable in numerous different ways depending on individual student's capacities[9].** The set „Steps" comprises the following tasks: "Powers", "Areas of figures drawn in coordinate system" and "Inequality of a triangle".
7. **Tasks motivating students to focus attention on the lesson[10].** This set has been designed with the concept of "task-clamp": the teacher begins placing interesting questions to which answers with students at the end of the class.The set "Clamp" comprises the following tasks: "Transformation of algebraic expressions", "Application of Pythagoras' theorem", "Volume units".
8. **Tasks whose presentation is important[11].** Unlike other sets, the task presented here are characterized by unconventional presentation of content. The set „Differently" comprises five implementation units: "Ratio", "Similarity of rectangles", "Perimeter and area of a circle", "Evaluating algebraic expressions", "Volume of a cylinder, density".
9. **Missing element – tasks which require careful content analysis and indicating additional necessary information from students to be solved[12]**. These tasks called tasks with lack of data allow to practice the process of logical reasoning and accurate formulation of questions. The set „What is needed" comprises three tasks: "Surface area of solids", "Volume of a cylinder, density", "Speed, distance, time".

Content tasks mentioned above can be found on the website of Best Practice on the cards with the description of each of the 9 sets (See: http://bnd.ibe.edu.pl/subject-page/6).

The teachers taking part in the project received a ready tool together with a proposition of its implementation and information about general goals which should be achieved during the implementation of each unit. As in the research described in the IBE's report, the working style and the way the teachers operated depended on the type of tool (the sort of task) used during the lesson and the way she/he was prepared or she/he prepared for the implementation of new tasks, i.e. what goals she/he decided to pursue and what methods and forms of work she/he was going to employ. The teacher's working style also depends on the group she/he works with – if the youth are willing to cooperate when working in groups, if they are able to express their opinions involving other students in the process of problem solving, and finally it is affected by the students' level of knowledge and their mathematical skills as well as grade statistics with regard to the course or branches of mathematics.

[9] See: http://bnd.ibe.edu.pl/practice-page/58
[10] See: http://bnd.ibe.edu.pl/practice-page/54
[11] See: http://bnd.ibe.edu.pl/practice-page/52
[12] See: http://bnd.ibe.edu.pl/practice-page/50

Coming back to the Report on the State of Education in 2013, we obtain confirmation of the above mentioned theses: “Teachers declare that their working style depends on two basic factors: the topic discussed during the lesson and the type of the group – students intellectual potential, interpersonal relationships amongst themselves, activity level, and their willingness to cooperate with teachers and with classmates. According to teachers' opinions, expository methods (a talk or a lecture) prove the best in the process of the introduction of new mathematical content. On the other hand, problem-focused methods or practical methods may be used during the lessons aimed at systematising and strengthening the knowledge, practising an acquired skill or using new tools. The majority of teachers of mathematics declare that the active training brings about the best effects and the growth of students' involvement. Some of them claim, however, that applying these methods is possible only in groups of students who are willing to cooperate and developed good interpersonal relationships amongst themselves. They emphasize that in case of groups in which the majority of students have lower mathematical abilities, are not involved in the process of acquiring knowledge, do not like one another or are involved in conflicts, introduction of expository methods is indispensable” (Czajkowska, Orzechowska, 2014, p. 188).

EXAMPLE

Mentors of the project did not participate in the study as observers lessons, did not receive the scenario analysis tasks or filled by students work cards. Put up their requests based on the contents of the reports of teachers completed tasks.

After each completed task from the database design teacher filled a report in which he was asked, among other things:

- General requirements.
- Particular requirements.
- Educational goals use implementation unit.
- At what point lesson has been used Implementation Unit.
- How to unit has been used Implementation.
- Tasks of the students and teachers accompanying solving tasks.
- Strengths and weaknesses and difficulties of using implementation unit.
- Errors committed by students.

Below are presented several statements of teachers, which most often appear in reports. They concerned work on a task No. 2 from set of tasks called “task-clamp”.

The content of the task:

On the football field between the posts opposite goals stretched and tensioned rope length 100m. Then it was extended by 1 meter and one of the players standing on the center line of the pitch raised middle rope to the top. At the greatest height which can raise the center of the rope?[13]

Teachers participating in the project suggested that the implementation unit Application Pythagoras task that begin implementation of lessons allow students to first analyze the content of the task and guess the outcome, then perform some proper lessons, which concerned the Pythagorean theorem, and only at the end of the lesson to get back to the task to look at them again.

This task mobilized teachers involved in the project activity, and to do something more than just a drawing to the task on the board. The activity, which rarely occurs in the classroom mathematics is formulating hypotheses, verify them and experimentation. The occurrence of these activities in the implementation of the duties of the teachers declared very often.

Reports show that most often the task was carried out in the introduction to a lesson or part of the relevant lessons, which most teachers modified the idea of the authors of the project for the task.

Despite the difficulties that this task is made disciples as much as 93% of teachers believe that they do not have to modify the content of the task and up to 97% of them believe that certainly will use the Unit in the future in his professional career.

Teachers of students' motivation to work on this task wrote:

- It is much easier to solve tasks in the "backyard".
- Students most attention focused on the result, which appeared to them to be too large compared with their expectations. After reaching the result returned to the calculations to see if it made a mistake.
- The situation described in the problem is that everyone knows the field and willingly took for drawing the auxiliary and solve the task, the students predicted results. They replied that 0.5m, very surprised result.

Teachers of students' activity most often write that:

- All students have worked with full commitment, to invent different ways to solve.

Teachers of work organization wrote:

- After the topic, lesson objectives, gave students the task in the form of key questions for the lesson. (It was written on the external wing of the

[13] See: http://bnd.ibe.edu.pl/files/upload/3-klamra%20gimnazjum_337f.pdf

board). I asked students to read content and an explanation of how the situation comes. Using brainstorming, jointly we established as the rope is hooked on the pitch. I asked the students to answer the proposal. The number fell 20 cm; 50 cm; 50 m; 1 m. I told the students that come back to the task at the end of the lesson. We took care of solving other tasks. At the end of the lesson we returned to the task.

- Unit I introduced right at the beginning of classes (triangle properties have been discussed in previous classes). Task flashed on the blackboard and asked students if they understand the content of the job. Since it confirmed that the content of the task is clear asked if know how to present the situation in the drawing. Reported only a few people. I asked one of the students to present the situation on board. A student drew a sketch pitch as seen from the air. I asked if the figure presented in the form you can see all the essential elements of the situation in question. Students according to confirmed that they do not. Therefore, I asked another willing person. Schoolgirl drew a model situation (as the height of the triangle drawn a 'puppet' and she added that this is an isosceles triangle, and that you can apply the Pythagorean theorem. I asked the students if they can guess what will be the result of the calculation. Some students said that probably 1 meter. Other after my the question whether there are other proposals found that probably 2 meters. After performing the calculations by one of the students were surprised that it was so many meters.
- The plaque was jointly made a schematic drawing of the pitch. Then the teacher posed questions: How was stretched rope on the pitch? - Draw; As it has been stretched? - Draw; Where to apply the Pythagorean theorem? Please apply the dimensions?
- Some students have difficulty understanding instructions. after the demonstration, tasks using a piece of string, joint forces found the correct solution.

Teachers of the difficulties the students wrote:

- Students have a problem with the analysis of the task of imagining a situation referred to in the task. The task proved to be difficult for my class, only one student knew that it must be used the Pythagorean theorem. Not everyone understands what it means, that the line was extended by 1 meter, assumed that the base of an isosceles triangle is 101 cm., Or that the triangle is equilateral.
- Incorrect analysis tasks and incorrectly drawn figure to the task.

- Students not fully aware of what to solve a given task with which the use of property. They tried to draw, to invent a logical solution. Some combined with a triangle.
- Some students have difficulty understanding instructions. After the demonstration, the task using a piece of string, joint forces found the right solution.
- Students have not noticed that we are talking about the 2 ropes. Part could not imagine the situation described in the task. Part misapplied tw. Pythagoras.

RESULTS AND CONCLUSIONS

Summarising the description of teacher of mathematics' working style, the authors of the Report write: “There is a possible hypothesis that teachers of mathematics teach perfectly the skills specified in the detailed requirements but fail to sufficiently teach those which constitute the general requirements of the core curriculum. Maybe they underestimate the meaning of general requirements in the educational process or do not believe in their students' potential” (Czajkowska, Orzechowska, 2014, p. 192).

Taking this opinion into consideration one should notice that many teachers participating in the project “Narzędzia w Działaniu” (Tools in Action) tried to pursue these difficult, long-term goals which are specific for mathematician's work and referred to as general requirements in the core curriculum. We learn from the Report about numerous attempts to work towards these goals and the attempts proved effective in the opinions of teachers implementing the units. Therefore including new, interesting and atypical tasks in the work of teachers of mathematics in lower-secondary schools prompted the change in working style or partially affected the selection of methods and forms of classwork making lessons more attractive and stimulating the students, which was acknowledged by teachers and this information is mentioned in the summary report.

The summary report shows that tools proposed by IBE:

- made lessons more attractive and diversified (41%),
- enriched and enhanced teacher's skills and tools (37%),
- stimulated the students (12%),
- influenced the team work (9%),
- affected the development of students' competences (out of school curriculum) (7%),
- facilitated teacher's work (2%),

- supported the process of revising and strengthening the students' knowledge (2%),
- affected preparations for the exam (1%).

80% of teachers claimed that implementing the units affected their working style. According to their declarations presented in 13 using the educational tools of IBE most often moderately affected the working style (45%). Almost one in four teachers claimed that using units from the proposed set of tasks affected their working style to a great extent (23%). A lower proportion of teachers acknowledged that the educational tools did not affect at all the methods of classwork (20%).

Teachers reported in their reports that part of the task was more difficult than those with which their students met every day. Hence, in addition to the formation and consolidation of mathematical concepts, the goal set themselves:

- make students aware of the task that at first glance seem difficult, turn out to be easier if we earlier steps that will facilitate the discovery of the road solution
- arouse the curiosity of students other types of tasks than those that appear every day on the math class,
- familiarize students with the types of tasks that do not have all the data needed to solve the task, as a rule, students rarely have to deal with such tasks in the classroom,
- learning plan the next steps, thinking about what will be needed to solve the problem,
- to develop the students' ability to reason and draw conclusions and arguments
- encourage students to experiment.

In the opinion of the majority of novice teachers it is the internships and not studies which prepared them to work at school and let them familiarise themselves with the profession (Walczak 2012). One of the respondents admitted that several weeks of internship taught him more than three years of studies. In case of some people the period of internships influenced their final decision concerning their career path.

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Inquiry based learning of mathematics in the early school years

Part 3

PATTERNS FOR KINDERGARTNERS: A DEVELOPMENTAL FRAMEWORK

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The importance of patterning for children's mathematical development is increasingly highlighted in recent years. For this reason there are considerable research results in that domain proposing that patterning should be included in the mathematics curriculum from the early years. However, teaching practice is not giving the necessary space and time to patterning, for several reasons. Among these reasons is the difficulty of teachers to design and realise pattern tasks. The aim of this paper is to propose an initial patterning developmental framework, in order to be used for designing pattern tasks for the early schooling years.

INTRODUCTION

Patterns is a mathematical construct that is very important for children's mathematical development (Clements & Sarama, 2009; Mulligan & Michelmore, 2009), for significant gains in numeracy (Hendricks, Trueblood, Malabonga, Willson-Quayle, Ciancio & Pasnak, 1999; Herman, 1973), for cultivating students' understanding of generalizations (Billings, Tiedt, Slater, & Langrall, 2007/2008; English, 2004; Waters, 2004), for developing their algebraic thinking (Stalo, Elia, Gagatsis, Theoklitou, & Savva, 2006; Threlfall, 2005), as well as for developing multiplicative reasoning (Papic, Mulligan & Mitchelmore, 2011) and analogical and inductive reasoning (English, 2004).

Young children have informal knowledge and various experiences on patterning (Synoś & Swoboda, 2007) when they collect, organize, discuss and graph data (Enright, 1998), as well as when they make constructions using various materials with different attributes, such as shapes, sizes, orientations, colors, positions (Garrick, Threlfall & Orton, 2005) etc. Additionally, research results show that young children have the ability to identify, complement, reproduce, extend, describe and create patterns (Klein & Starkley, 2004; Lannin, 2005). They mention, however, that their performance is dependent and influenced both by the pattern's structure, type and material (Skoumpourdi, 2013), as well as by teaching (Hendricks, Trueblood & Pasnak, 2006). Most of the above researches come to the conclusion that, to further improve the performance of children on patterning, teaching interventions are necessary from the early years of schooling. For these reasons patterning is (suggested to be) included in the mathematics curriculum from the early years (ACARA, 2010; Depps, 2001; DfEE, 1999; NCTM, 2000).

Though, patterning is underestimated in teaching practice. Experiences with patterns usually lack in the kindergarten (Skoumpourdi, 2013) and the elementary school (Warren & Cooper, 2008), and if they exist are related to very simple tasks (Papic, Mulligan & Mitchelmore, 2011). This may come from teachers' lack of awareness of the importance of patterning in mathematical reasoning, their low confidence in teaching patterning (Papic & Mulligan, 2005; Waters, 2004) as well as their difficulty to design creative and developmentally appropriate pattern tasks. In this paper, a patterning developmental framework is suggested, to be used for designing pattern tasks for the early schooling years.

RESEARCH RESULTS ABOUT PATTERNING

Research on patterning in early years is focused on investigating young children's (5-6 years old) ability to identify, complement, reproduce, extend, describe and create patterns in different structures, constructed with a variety of materials.

Klein and Starkey (2004) in their study found that pre-kindergartners can learn to duplicate simple concrete patterns and that kindergartners can learn to extend and create patterns. They mention that many 4-year-old children at the beginning of the kindergarten find it difficult to identify and to analyze the core unit of a repeating pattern.

Hendricks, Trueblood and Pasnak (2006) investigated the effects of teaching patterning to 1st-graders. They found that at the end of the school year, the patterning group significantly outperformed the control group on measures of patterning and academic achievement.

Papic's, Mulligan's and Michelmore's (2011) findings show a link between patterning and multiplicative reasoning. In their 6-month intervention focusing on AB repeating patterns found, that the intervention group of pre-schoolers outperformed the comparison group across a wide range of patterning tasks. The intervention group revealed greater understanding of the repeating unit and the spatial structuring, whereas most of the comparison group treated repeating patterns as alternating items and rarely recognized simple geometrical patterns (Papic & Mulligan, 2005).

In another study (Skoumpourdi, 2013) kindergartners' (5-6 years of age) performance on extending and reproducing different types of patterns using a variety of materials, before formal teaching, was investigated. The results of the study showed that kindergarten children have the ability to extend and reproduce different patterns constructed with a variety of materials before teaching. Their performance on patterning was strongly influenced by the pattern's type and structure and relatively less influenced by the material's type. More than half of the participants were successful in extending repeating patterns with AB structure and about half of them, when the pattern structure was ABC. When the core unit becomes more complicated (like ABBC) the children's successful actions got fewer. It was found that the core unit AAB was more difficult for young children to handle than the ABC. The use of physical materials led the children to more successful actions than the use of printed[1] ones. The reproduction of repeating patterns was more difficult for children than the extension of repeating patterns, especially with the use of physical materials. In this pattern type, children were more successful when they used the printed materials independently from the pattern structure. The extension of a growing pattern was very difficult for kindergartners and led most of the children to modify it.

The ability of kindergartners to identify, recognize, describe and complement patterns in iconic representations of cultural devices was investigated in a study (Skoumpourdi, 2014). The results showed that kindergartners have the ability to identify, recognize and describe those patterns but not to complement them.

METHODOLOGY

In order to create a patterning developmental framework we collect and analyse pattern tasks from several contexts, such as research, textbooks and mathematical books for kindergarten from the Greek market, in order to distinguish their functional characteristics. The above data will be related to the research results, in regard with young children's capabilities on patterning and will form the proposed patterning developmental framework.

RESULTS

Pattern tasks from research

Klein and Starkey (2004) used in their study two types of tasks: 1. The pattern duplication task that assessed children's ability to copy a linear repeating pattern and 2. The pattern extension task that assessed children's ability to complete a linear repeating pattern. In both tasks, patterns were constructed with small colored blocks. The structure of the patterns was AB.

In the 6-month intervention about patterning (Papic & Mulligan, 2005; Papic, Mulligan & Michelmore, 2011) children had to copy AB patterns with materials, to draw or to make them from memory, identify a screened element, or extend the patterns. The materials used were colored blocks, triangular dot patterns with counters, grid patterns, cut-out tiles and hopscotch templates.

Hendricks', Trueblood's and Pasnak's (2006) learning set, for the four months instruction, contained 480 problems in recognizing, comprehending and reproducing both logical and arbitrary patterns involving numbers, letters, shapes, colors, orientations, causes and effects, as well as temporal events. The means they used to present the patterns, varied from colored plastic beads in wooden frames to animal stickers, cartoon sequences, flash cards, memory

[1] Printed material: The iconic representation of the manipulatives

games, activity sheets and computer-generated graphic patterns. The tasks ranged from simple linear orderings on one dimension to multidimensional sequences presented as matrices.

Skoumpourdi (2013) used in her study ten pattern tasks. The pattern tasks were constructed with physical and printed materials. There were three different pattern types incorporated in the interviews: 1. Five tasks for the extension of repeating patterns: two of them with AB structure, constructed with white and red cube blocks as well as with white and red printed squares, two with ABC structure, constructed with blue, yellow and green marbles as well as with blue, yellow and green printed circles and one task with ABBC structure, constructed with green, yellow and red 'connected people', 2. Four tasks for the reproduction of repeating patterns: two of them with AB structure and the other two with AAB structure, constructed with white and red cube blocks along with white and red printed squares, and 3. One task for the extension of a growing pattern with A AB ABA … structure, constructed with yellow and red cube blocks.

In another study (Skoumpourdi, 2014) seven tasks for pattern's identification and completion were realized. The structure of the patterns were A and AB and were presented through photographs and iconic representations of Rhodes town sights.

Pattern tasks from mathematical textbooks

Given that in Greece there is no textbook for the kindergarten, we recorded the tasks of the 1st grade textbook, as the tasks that children from kindergarten will have to deal with, in their next schooling year. The tasks were seventeen in total.

In the 1st and 2nd task children have to identify and verbally describe patterns with ABB (boy-girl-girl) and AAB (green carriage - green carriage - red carriage) structure. In the 3rd task children have to extend an AB pattern (white - green parts on a snake). In the 4th task students have to reproduce a pattern in a grid. In the 5th, 6th, 7th and 8th tasks students have to extend repeated patterns with AB (blue marble - yellow marble and orange ball - white ball) and ABB (red square - green square - green square and fish with bubbles - fish without bubbles - fish without bubbles) structures. In the 9th and 10th tasks students have to extend patterns in grids. In the 11th and 12th tasks students have to extend a pattern ABBB (red marble - yellow marble - yellow marble - yellow marble) and a pattern in a grid. In the 13th task they have to reproduce a given AB (triangle-square) pattern. In the 14th and 15th tasks they have to extend repeated AB patterns (pink line - blue line and angle with short sides - angle with long sides). In the 16th and 17th tasks they have to create their own patterns in the given blank marbles and squares.

The core unit, in the majority of the tasks, is repeated twice at least. But there are two tasks in which the core unit is presented just once. In the extension tasks students have to extend the pattern in spaces that are given.

Pattern tasks from mathematical books for kindergarten from the market

In a total of ten books there were forty five activities for patterns. The activities are related to the identification of patterns (5 tasks), to the extension of repeated patterns (31 tasks), to the identification of the mistake of the patterns (6 tasks) and to the completion of a part of the core unit of the patterns (2 tasks).

For the identification of patterns, AB (2 tasks), ABC (2 tasks) and ABCD (1 task) patterns are presented, with the core unit being repeated from two to six times. In the majority of the tasks (4 tasks) one dimensional simple linear orderings are presented.

For the extension of repeated patterns, AB (18 tasks), AAB (1 task), AABB (1 task), ABC (10 tasks) and ABCD (1 task) patterns are presented, with the core unit being repeated in most of the tasks (18 tasks) once (and an item more in some instances) and in the rest of them (13 tasks) twice or three times. Children are asked to extend the pattern by creating (17 tasks) or by painting (14 tasks) the core unit. In seven of the creating tasks children are asked to create the next item that is to create just one of the items of the core unit. In three of the painting tasks the number of the empty spaces is not a multiple of the number of elements of the core unit (e.g. for an AB pattern there were three empty spaces).

In the majority of the tasks (17 tasks), one dimensional simple linear orderings are presented. In the others (14 tasks), two dimensional orderings are presented. All the tasks have horizontal arrangement except of four having vertical (2 tasks), circular (1 task), and path arrangement (1 task).

For the identification of the mistake of the patterns, ABC (5 tasks) and ABCD (1 task) patterns are presented, with the core unit being repeated twice (1 task), three (4 tasks) and four times (1 task). Children are asked to identify the mistake in the patterns. Mistakes are either in the core unit (2^{nd} or 3^{rd}) or between the core units (between 2^{nd} and 3^{rd} or between 3^{rd} and 4^{th} core units). All the tasks are one dimensional simple linear orderings in horizontal arrangement.

For the completion of a part of the patterns' core unit, ABC (1 task) and ABCD (1 task) patterns are presented, with the core unit being repeated seven times. Children are asked to complete the pattern by pasting the corresponding stickers in the empty spaces (3^{rd}, 4^{th}, 5^{th}, 6^{th}, 7^{th}). The tasks are two dimensional simple linear orderings in horizontal arrangement.

All the patterns were presented in iconic representations of various types: shapes (such as circles, squares, triangles and rectangles), animals (such as owls, frogs, centipedes), as well as other objects and symbols (such as strawberry, apple, trefoil, flowers, water lilies, trees, pail, shovel, ball, carriages, star, moon and abstract symbols for decorating carpets etc.).

Functional characteristics of pattern tasks

Common activities that occur both in the books and the research are related to the exploration of simple repeating patterns using shapes and objects. Young students are asked to copy and continue these patterns, identify the repeating part and find missing elements. Analyzing the above activities we recorded and categorized four functional characteristics of the pattern tasks: the type, the structure, the material and the arrangement of the pattern. These functional characteristics influence both the difficulty of the tasks and the children's patterning performance.

Type of the pattern: Four main types of patterns were recorded from the pattern tasks, for early schooling years, indicating the action the activity requires. These are: 1. Identification of a pattern, 2. Reproduction of a pattern, 3. Completion of a pattern and 4. Creation of a pattern. In each type, patterns were spatial structures or repeated patterns and in one instance, growing patterns.

Identification of a pattern includes identifying the core unit of a pattern that is being repeated or growing and identifying a mistake in the pattern. In the former, children have to point or to verbalize what they identify and in the latter, to correct the mistake after recognizing it. The mistakes on the recorded tasks concern an extra item in the core unit or/and between the core units. Children have to identify and delete this extra item. In that type of pattern the difficulty lies on the complexity of the core unit, as well as in its repetitions.

Reproduction of a pattern demands an ability to copy or duplicate a pattern. These actions, copy and duplication, can take place either with the same, like the initial pattern materials, or with other ones. In the former, which is more difficult, manipulatives have to be placed or paintings have to be made in the right order. In the latter, the core unit recognition is demanded in order to be translated with the other materials.

Completion of a pattern, in the recorded tasks, includes both the filling of the empty spaces in-between a pattern, as well as its extension. The former, requires the filling of the empty spaces of the pattern either by creating/drawing the items, or by painting them, or by putting stickers, or other items. The latter, requests the ability to expand the pattern, by repeating the core unit in a repeating pattern, or by growing the core unit in a growing pattern. Growing patterns are more difficult for young children. The extension of the patterns has to be performed in predefined or not spaces. In the tasks that the spaces are predefined, an inferior amount of them appears. The problem is that the available spaces are not multiple to the number of the items of the core unit and therefore not sufficient for the pattern to be completed. In the tasks that the spaces are not predefined, the creation of the core unit of the pattern (once or more times) has to be done in a way that extends the pattern. In that type of pattern the difficulty lies on the complexity of the core unit as well as on the material used. Creation of the pattern in the recorded tasks includes both the creation by memory of a pattern that has been presented and the creation of a new pattern.

<u>Structure of the pattern</u>: The structure of the pattern is describing the core unit of the pattern. The range of the core units, of the recorded tasks, was large. The core units used were: A, AB, ABA, AAB, ABB, ABC, AABB, ABBB, ABBC, ABCD, A AB ABA, in which A, B, C and D differed between them in one, two or three features (color, type, size etc). For example, the pattern red circle-orange circle has AB structure like the pattern red circle-orange triangle but in the former differed in one variable (color) and in the latter in two variables (color and shape).

<u>Material of the pattern</u>: The materials of the patterns are related to the kind of the means used for constructing it and the way it is presented. The material that has been recorded was: 1. Manipulatives and 2. Iconic representations. Specifically the materials that are used for patterns construction are objects from everyday life (e.g. buttons, marbles, counters, stickers), the human body (e.g. steps), cultural tools (e.g. clock, ruler) and educational materials (e.g. colored blocks, number line, cubes, connected people, flat shapes representations, printed , iconic representations of objects, grids, hopscotch templates). Topics of folk tradition and everyday life of children have also been used in the studies (e.g. photos of sights).

<u>Arrangement of the pattern</u>: The arrangement of the pattern is the way the items of the core unit of the pattern are arranged. The arrangements that were recorded are: Linear/horizontal, circular, path arrangement and vertical. The vast majority of the patterns have linear/horizontal arrangement.

Patterning developmental framework

Considering the research results about the patterning ability of kindergartners in relation with the functional characteristics of the recorded pattern tasks, a patterning developmental framework was shaped.

According to the type of the pattern the identification and completion of a pattern task are easier for young children than the reproduction and creation of a pattern task. The reproduction of repeated patterns is more difficult for young children compared to the extension of repeated patterns, but easier than the creation of a pattern. The growing patterns are very difficult for children in all the pattern types.

Based on the research data, the structures A and AB are the easiest. When the structure of the core unit becomes more complicated, such as ABBC, the kindergartners' actions are less successful, with the exception of the completion of a composite structure AABBC, where children's actions were successful. The AAB structure is more difficult to handle than the ABC structure. Children's success significantly reduces when the pattern has to be translated with other materials. When the pattern must be reproduced by different materials (translation), children have more success when using printed shapes than manipulatives, regardless of the structure of the core unit. In the other structures the use of manipulatives leads children in more successful actions than the use of printed materials.

Some general data to be taken into account is that linear arrangements are more familiar to children than the other types of placements. Pattern is the ability to recognize an ordering of numbers, letters, shapes, symbols, objects, or events according to some rule of progression. Therefore, the core unit has to be presented at least twice, so that the regularity is perceived by the children. Yet, specifically for the identification and the completion of a pattern, more than two repetitions of the core unit are required.

From the above it seems that patterns' functional characteristics are interdependent and that the different combinations of these characteristics influence children's performance. According to the data so far and based on what is easier for children, the developmental framework is suggested to be the following: 1. For the type of the pattern: observe, recognize, identify, describe, complement, copy, extend, translate and create patterns. 2. For the structure of the pattern: A, AB, ABC, ABA, AAB, ABB, ABCD, ABBC, AABB, ABBB, A AB ABA, in which A, B, C and D differed in one feature. The more the features are, the bigger the difficulty of the pattern is. 3. For the material of the pattern: Manipulatives and then iconic representations. 4. For the arrangement of the pattern: linear arrangement, vertical arrangement, path arrangement, circular arrangement.

CONCLUSIONS

In order to help teachers overcome their difficulty to design creative and developmentally appropriate pattern tasks, a patterning developmental framework is suggested in this paper to be used for designing pattern tasks for the early schooling years. The framework is just an indicator of what a child can manage in patterning. The patterning developmental framework together with children's informal knowledge, experiences, interests and mathematical abilities can be used to create interesting and creative pattern tasks which can engage children in challenging patterning experiences. Taking into consideration that the pattern' functional characteristics - type, structure, material and arrangement - are interdepended, we come to the conclusion that the design of pattern tasks is a complex situation. A different combination of these characteristics leads to different pattern tasks with a varying degree of difficulty, which requires different actions from children.

Further research is needed to determine the framework more accurately. More activities must be analysed in order to record more functional characteristics of patterns and more research must be done to record the ability of children in other types of patterning activities

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TWO – STEP PROBLEM SOLUTION BY 7-8 YEAR-OLD STUDENTS

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The topic of our research is the comparative portrayal of the performance of 7-8 year-old students in the ability of selecting and conducting the operations for the solution of a two-step problem with and without the presence of an informational representation. The research which was carried out on 94 pupils aged 7-8 demonstrated the difficulty that students encounter in the selection of the appropriate operations for the solution of a two-step problem with an informational representation, whereas on the other hand it facilitates the execution of algorithms.

THEORETICAL FRAMEWORK

The domain of the mathematical problem solution is especially wide and does not focus on one parameter only. A big number of researches has been involved with the individual features which determine the degree of success in the solution of a mathematical problem: size of work memory (Tronsky & Royer, 2002), basic computation (Fiori & Zuccheri, 2005), tracing of the type of the problem and its association with its shape (Nesher & Hershkovitz, 2003), exploitation ability of the various forms of representation for the more facile comprehension of the fundamental elements of the problem (Kafoussi, Skoumpourdi & Kalavassis, 2003 ; Chrisostomou, 2013).

Many researchers acknowledge the fact that there is an increased amount of difficulty in the two-step problem (Shalin & Bee, 1985 ; Kafoussi & Ntziachristos, 1989 ; Nesher & Hershkovitz, 2003 ; Geary, 2006 ; Castro – Martinez & Frias – Zorilla, 2013). The "scheme approach" (Shalin & Bee,1985 ; Nesher & Hershkovitz, 2003), the "decrease – increase relationship" (Rico et all 1994) and the approach of "problems with two nodes" (Frias & Castro, 2007) attempt to decode the structure of two-step problems so that they can be integrated in the pupils' cognitive schemata.

Nesher and Hershkovitz (2003) by utilizing the approach of schema in pupils of the third, fourth, fifth and sixth grade of Primary School in Israel observe that the type of shape (of hierarchy, common sum and common part) drastically influence the extent of difficulty of the two-step problem. The schema of "hierarchy" is proved to be the easiest of the three, whereas there follows the "common sum" (sharing the whole) and the most difficult one is the "common part" (sharing a part).

The research by Shalin and Bee (1985) in students of third, fourth and fifth grade of Primary School traces an increased rate of success of students with the two-step problems who belong to the “hierarchy” schema.

The approach of decrease-increase (decrease – increase relationship) distinguishes four types of two-step problems: Increase-increase (it corresponds to the hierarchy schema), decrease-decrease (it corresponds to the hierarchy schema), increase-decrease (it corresponds to the common sum) and decrease-increase (it corresponds to the common part). The research conducted by the team of Arithmetical Thought of the University of Granada in students of the fourth, fifth and sixth grades of the Primary Schools of Granada comparatively determined the degree of difficulty that the four types of problems created for pupils (Rico et all 1994). The results of the research highlighted with an increasing degree of difficulty the following types of problems: increase-increase, increase-decrease, decrease-increase, decrease-decrease. The approach of the connection (problems with two nodes) approaches the two-step problems as if they were two simple problems, which have a linking element.

Castro and Frias (2013) carried out a research in 172 pupils of the fifth grade of a primary school in Spain on the one hand in order to compare the pupils’ performance in the simple and two-step problems and on the other to analyze the pupils’ errors in the two-step problem. The error typology in the two-step problem as it was identified by the researchers contained the following types of error: a) execution of only one operation (omission of the first or the second relationship), b) solution of the problem with the order the data appear in the text, c) double repetition of the same piece of information at the solution of the problem and d) other errors. The majority of errors were spotted in the first error type and specifically in the omission of the second relationship of the two-step problem.

Vilenius – Tuohimaa, Aunola and Nurmi (2008) attempted in a research of 225 pupils aged 9-10 in Finland to associate the ability of text comprehension with the performance in the solution of the mathematical problem. The pupils were motivated to solve two-step, combination, comparison and focus problems. According to the results of the research the ability of text comprehension is mostly associated with focus problems and less with the two-step problems. The compare and combine problems are exclusively connected with the mathematical ability. Moreover the ability of technical reading is associated with the solution of the mathematical problem as well as with the reading comprehension.

Nortvedt (2009) in a case of a 12-year-old pupil who attempted to solve a two-step problem studied the pupil’s ability to comprehend the data of the problem he has been reading. Nortvedt points out that the pupils have been practising from the Curricula to look for information within the problems. However, they encounter difficulty in acting on the problem and attempting to interpret it, a fact which explains the low performance of the pupils in the solution of the problem.

The inspection of the bibliography underpins the fact that the exploitation of the informational representations, one of the four representation categories of Elia, Chrysanthou and Filippou (2003) who exploited the classification of images in the process of a literary text by Carney and Levin (2002), can improve the pupil's reading comprehension ability in the solution of the problem (Chrisostomou, 2013).

We decided to choose the topic of our research because of the increased difficulty two-step problems present to pupils aged 7-8, as this is demonstrated by the bibliography. Our study uses as a variable the presence of informational representations for the solution of a two-step problem, due to the open discussion going on as regards the question whether the existence of representations contributes to or hinders pupils to proceed with problem solution. The research is conducted among students aged 7-8 because of the critical stage this age exhibits in relation to the respective class in order to assimilate addition and subtraction. The current research is aiming at looking into the ability of pupils aged 7-8 to select and execute the operations for the solution of a two-step problem with and without the aid of the informational representations. More specifically, the research queries posed were the following:

- What operations do children choose for the solution of the two-step problem and how do they put it into practice?
- How does the presence of the informational representation affect the pupils' performance in the two-step problem?
- Which kind of informational representation do pupils prefer in order to be supported with the solution of a two-step problem among the sketch, the table and the realistic one?

METHODOLOGY

Sample

The research, which is part of a wider study pertaining to the investigation of the influence of representations in problem solution, took place in three public Primary Schools at a mid-urban area of Athens, at the 2nd Grade during the school year 2014-15 in a three-day period. The schools participating in the research will be referred to as A, B and C during our work. 94 pupils of various learning levels participated in the research. School A has 70% immigrant students, school B has 20% students with special needs and school C is a typical school.

Research tools

To attain the goals of the research two two-step problems were used, which had a common phrasing and could be solved with the same operations and the same order. The first problem said: "Miltiadis wants to buy a bicycle which costs 100€. In his money-box he has 69€. He has also got 15€ of pocket money by his grandmother. How much money does he still need in order to buy the bicycle?" The first two-step problem was not accompanied by a representation and belongs according to the schema approach to the category of "hierarchy", which is thought to be the easiest one. Respectively, in the "decrease-increase" approach the problem falls into the category of increase-decrease, which is thought to be the second easiest one. The second two-step problem was accompanied by three kinds of informational representation[1]: a sketch, a table and a realistic representation from which each child had to pick one. The sum claimed by the problem was to find the remainder from an initial sum after the purchase of two products. At the sketch and the realistic representation we had the depiction of the two products together with their prices, whereas in the rubric of the problem the initial sum was mentioned. The pupils were told not to write the executions of the algorithms on the desks, but on the paper of the tests. They were given no further explanation.

RESULTS

In the *two-step problem* with the help of representations we pin-pointed specific types of errors encountered by the pupils per school unit.

ERROR TYPES IN THE TWO-STEP PROBLEM				
	A	**B**	**C**	**TOTAL**
Half-finished	07	05	09	21
Wrong Selection of Operation	01		03	04
Wrong Execution of Algorithm	09	07	06	22
Didn't attempt to solve	10	01	4	15

Table 1: "Error types in the solution of problem without representations"

[11] The informational representation gives information which is necessary for the solution of the problem. The remaining three types of representation, according to the categorization of Ilia, Chrysanthou and Filippou (2003) are: a) decorative b) organizational c) representational

✓ *Half-finished* s*olution*: the pupils who delivered a half-finished solution of the two-step problem did only the addition without proceeding with the subtraction from one hundred.

✓ *Wrong* selection of operation: the pupils did not comprehend the data of the problem.

✓ *Wrong execution of operation:* Many pupils encountered difficulty in the execution of subtracting from one hundred.

It is worth mentioning the strategy used by the pupils who solved the two-step problem correctly. In the *two-step problem* certain strategies were spotted that were used in order to solve the problem correctly.

- *"Subtraction by addition"*
- *One addition and one subtraction*
- *Analytical verbal phrasing without the demonstration of operations*

	STRATEGIES OF TWO-STEP PROBLEM			
	A	**B**	**C**	**TOTAL**
Subtraction by addition	02	04	02	**08**
Addition & subtraction	04	12	06	**22**
Verbal phrasing only	1		1	**02**

Table 2: "Ways of solution of the problem without representations"

The pupils of B and C Primary School appeared to encounter a lesser degree of difficulty in relation to their fellow pupils of A Primary School who attempted to solve the two-step problem with horizontal operations. We can therefore speculate that the pupils of A Primary School because they execute more easily the addition rather than the subtraction attempt to solve the problem through "subtraction by addition".

A week after the solution of the two-step problem without representations the pupils participating in the research were given a two-step problem *with the aid of an informational representation*. The pupils were asked to select among different kinds of informational representation: a sketch, a realistic representation and a table and seemed to opt for the table for the solution of the two-step problem in their majority (Tab. 3).

REPRESENTATION PREFERENCES				
	A	**B**	**C**	**TOTAL**
REALISTIC	10	08	06	24
SKETCH	02	02	01	05
TABLE	22	18	25	65

Table 3: "Pupils' representation preferences"

Table 3 demonstrates the explicit preference of pupils to select the table in relation to the two other types of informational representation. Moreover, the pupils' attitude to the sketch is remarkable, despite the fact that there is a strong presence of sketches in the pupils' school books. On the contrary, we observe that the pupils aren't accustomed to the realistic representations in school books they appeared to be interested in being helped by that type of representation. The actual responses of the pupils to our question "*why did you choose this representation instead of the other one?" were:* the pupils who chose the table said: *"the table shows the data more briefly"* or *"I have all the data in front of me";* the pupils who chose the realistic approach told us : *" I have the same ball as that",* or *"I liked this doll"*; the pupils who showed their preference for the sketch said: *"I'd like these toys"*, or *"I like this sketch".* In the two-step problem with the aid of informational representations the pupils came up with the same error types with the one without representations. The errors that occurred were the following:

- ✓ *Half-finished solution*: The pupils who handed in a half-finished solution of the two-step problem did only the addition without proceeding with the subtraction from one hundred and fifty.
- ✓ *Wrong selection of execution*: The pupils did not comprehend the data of the problem and as a result they subtracted the prices of the two toys from one another.
- ✓ *Wrong execution of operation*: Many pupils encountered difficulty in the execution of subtraction from one hundred and fifty.

On the one hand Table 4 depicts the difference in the performance of the pupils in the two-step problem from school to school and on the other hand the fact that the table, even if it attracted the majority of the pupils' preferences, it also recorded the majority of the errors. It should also be highlighted that an equal number of students coincides with the selection of the wrong solution strategy, of the wrong execution of the algorithm and the solution of the two-step problem only as regards its first part.

ERROR TYPES IN THE TWO-STEP PROBLEM										
	SKETCH			**REALISTIC**			**TABLE**			
	A	**B**	**C**	**A**	**B**	**C**	**A**	**B**	**C**	**Total**
Half-finished		**1**		2		**1**		**3**	**2**	**09**
Wrong Choice of Operation					**3**			7	**9**	**19**
Wrong execution of Algorithm	**1**			1	**2**	**3**	**11**	**1**	**5**	**24**
Did not attempt to solve	**1**			5		**3**	**6**		**5**	**20**

Table 4: "Error types in the two-step problem"

The pupils do not seem to have managed a better performance in the solution of the two-step problem with the aid of representations than in the solution of the two-step problem without the aid (Tab. 1). In particular, it is noted that the minority of the pupils solved correctly the two-step problem which is accompanied by representations, as the pupils who selected wrong operations for the solution of the problem were many. On the contrary, it is noted that in the two-step problem with representations less students left the half-finished problem by adding the value of the two toys.

It is worth pointing out the strategy used by the pupils who solved the two-step problem correctly. In the *two-step problem* there were traced specific strategies that the pupils used in order to solve the problem correctly.

- ✓ *Two subtractions*
- ✓ *An addition and a subtraction (vertical algorithms)*
- ✓ *An addition and a subtraction (horizontal algorithms)*

TWO-STEP PROBLEM STRATEGIES										
	Realistic			**Sketch**			**Table**			
	A	**B**	**C**	**A**	**B**	**C**	**A**	**B**	**C**	**TOTAL**
Two subtractions		**2**			**1**			**1**		**04**
Addition & subtraction vertically		**3**	**1**	**2**				**3**	**4**	**13**
Addition & subtraction horizontally	**2**						**2**	**1**		**05**

Table 5: "Two-step problem solution strategies"

It is inferred from Tab. 5 that the solution of the two-step problem with representations in relation to the solution of the problem without the aid of representations exhibits a smaller variety of strategies, since we did not trace any solution approach of the problem subtraction by addition or only verbal phrasing. Also, it seems that the pupils prefer to solve the problem mainly through vertical algorithms and with a horizontal solution way.

Tab. 6 presents the total amount of errors by pupils in the two-step problem without representations and in the two-step problem with representations.

<table>
<tr><th colspan="6">ERRORS - SOLUTION STRATEGIES IN THE TWO-STEP PROBLEMS</th></tr>
<tr><td>ERRORS</td><td>WITH-OUT REPRES.</td><td>WITH REPR.</td><td>SOLUTION STRATEGIES</td><td>WITH-OUT REPR.</td><td>WITH REPR.</td></tr>
<tr><td>Half-finished</td><td>21</td><td>09</td><td>Subtraction by addition</td><td>08</td><td>-</td></tr>
<tr><td rowspan="2">Wrong selection of operation</td><td rowspan="2">04</td><td rowspan="2">19</td><td>Addition & subtraction vertically</td><td>22</td><td>13</td></tr>
<tr><td>Addition & subtraction horizontally</td><td>-</td><td>05</td></tr>
<tr><td rowspan="2">Wrong execution of algorithm</td><td rowspan="2">22</td><td rowspan="2">24</td><td>Verbal phrasing only</td><td>2</td><td>-</td></tr>
<tr><td>Two subtractions</td><td>-</td><td>04</td></tr>
</table>

Table 6: “Condensed table of pupils’ errors and strategies in the problem”

In the solution of a two-step problem without representations the pupils made use of the subtraction by addition vertically and horizontally, the vertical algorithms of addition and subtraction and the verbal phrasing only. On the contrary, in the solution of the two-step problem with representations the pupils made use of the strategies of the two subtractions, of the vertical algorithms of addition and subtraction and the horizontal operations of addition and subtraction.

DISCUSSION-CONCLUSIONS

The solution of a two-step problem proved to be a difficult activity for the pupils. Although the two-step problems we posed to the pupils are included in the hierarchy schema, and are considered to be the easiest schematic type (Nesher & Hershkovitz, 2003) during the schema approach they were very difficult for the pupils. The decrease-increase approach (Rico, et al, 1994) attributes to the problems of increase-decrease the second in difficulty degree in a row. However, the pupils’ performance in the two problems they were given proves that apart from the type of schema a very important factor is the numbers which participate in the problem. The pupils’ errors in the two-step problem makes it clear that they had not probed in the connection approach (Nesher & Hershkovitz, 2003) resulting in not fully comprehending the data of the problem. The pupils' errors during the solution of the two-step problems had to do with the half-finished solution of the problem, the selection of inappropriate solution strategies and the wrong execution of algorithms.

The error categories we spotted in the two-step problem are not in absolute accordance with those of Castro and Frias (2013). The category of errors which addresses the solution of the problem with the order of appearance of the data in the text cannot be applied in our survey since the existence of informational representation does not coincide with the linear reading of the data. Moreover, this error category is not found in the problems without representations either. As regards the category of double repetition in the solution of the problem of the same piece of information it was not traced in the results of our survey an error of this category. Finally, as regards the execution of only one operation, of the first or of the second relationship, in our results we did not trace an omission error of the first relationship, and for this reason we named this error category "half-finished solution".

The solution of a mathematical problem that pupils are invited to solve from a very young age is not always more easily accessed with the aid of imagistic representations. The speculation for the representations is continued with the positive (Kafoussi, Skoumpourdi & Kalavassis, 2003; Chrisostomou, 2013) and the negative (Verschaffel & Van Dooren, 2014) results to which researches have reached. Many factors can affect the result of the performance of the pupils in the conquest of the mathematical concept, such as the schema of the mathematical problem, the numbers participating in the problem, the kind of the mathematical problem and the level of performance of the participants in the pupils' research.

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AN ARITHMETIC TASK DEVELOPING THIRD GRADE STUDENTS' CREATIVE THINKING

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One of the most important issues in mathematics education is to teach creative and critical thinking. The typical school activities do not always give this opportunity. The article presents an example of classes in the third grade of primary school, whose aim was to develop a creative mathematical activity of students. The described activities were primarily related to the identification of differences and similarities between the numbers in a sequence.

INTRODUCTION AND THEORETICAL FRAMEWORK

Mathematical education of children and youth has become the most important subject raised by various bodies. Researchers, teachers and politicians take up the discussion on improving the education system, especially mathematics education. In the latest trends in the early education a large emphasis is placed on developing the skills needed for a child to explore and understand the world and to cope with different situations of everyday life. The skills that are particularly useful in various situations include analyzing, critical thinking, putting and verifying hypotheses. The tasks of the school according the new curriculum include the care that a child could acquire the knowledge and skills needed to understand the world and equipment the child needs in math skills in real-life and school situations and for solving the problems (Podstawa Programowa, 2011).

The most important skills acquired by the student in the course of general education in elementary school should be, inter alia, mathematical thinking, comprehension as the ability to use the basic tools of mathematics in his or her daily life and carrying out the elementary mathematical understandings (.

In recent years, Polish school has undergone several reforms. They mainly focused on changes in the curriculum. Referring to the current trends it formulated new tasks and objectives that should be realized by school. An important task of school is to prepare students for life in the information society. It is associated with the ability to search and organize information, verification of hypotheses and critical thinking. These noble intentions are not always reflected in the school reality. A too rigid adherence to the curriculum and the inflexible approach the teacher are not helpful to realizing these goals. In addition, the curriculum is designed for a student with average abilities. Meanwhile there is a belief that every child is gifted, meaning that each can work creatively (Gruszczyk-Kolczyńska, 2009, Brandl 2009, Munz 2013, Clements, Sarama, 2007). Studies show that on the first stage of education many students show their talents towards mathematics. Therefore, it is necessary to raise the efforts to find and develop mathematical talents within pupils of the lower grades of primary school (Tirosh, Tsamir, Levenson, Tabach. M. 2011).

Mindful that children feel the satisfaction from creative activity, you need to create the conditions for them to present their achievements. That possibility leads to classes in which the student has the opportunity to meet with unusual tasks that do not impose a single method to solve (Ramani, Siegler, 2007;).

To meet these expectations a series of classes with students of the third grade of primary school were carried out. The classes were prepared by the class teacher and the author of this paper. We were leading a series of lessons with students from the 3rd grade of primary school (8-9 years old) from the October 2012 untill the January 2013. Lessons took place once a week and lasted one lesson hour (45 minutes). Their aim was primarily to develop the students' interest in mathematics. The subjects of the classes were very diverse and dealt with both issues of geometry and arithmetic. One of the ongoing topics was “fun with numbers”. In these classes students had not only to improve skills, but also shape their mathematical language, they were able to demonstrate the ability of making hypotheses and verifying them.

METHODOLOGY OF A SERIES OF ACTIVITIES RELATED TO NATURAL NUMBERS

Classes related to arithmetic consisted of six meetings of 45 minutes. The subject was related to natural numbers. The topics discussed during the course included: knowledge of the natural numbers, their construction, positional decimal system, basic arithmetic operations (addition, subtraction, multiplication and division). It was assumed that students were familiar with the following concepts: digit, number, sum, difference, divisibility, multiplicity, unit digit, tens digit, hundreds digit. The aim of this series of activities was mainly to develop creativity and critical thinking among students, argumentation skills, perceiving dependencies and relationships between the sequences of numbers. I was looking for answers to the following questions:

- Will students be able to see the dependencies and relationships in certain sequences of numbers?
- Will students be able to adequately justify their choice?
- Will they be creative?
- Will they be able to think critically with regard to their actions?
- In which ways will they develop their mathematical language?

The classes described were conducted among pupils from the third grade of elementary school. They took place once a week and 20-25 students participated in them. The classes were recorded and after that the relevant protocol was prepared. The research material consisted of students' work (a worksheet fulfilled by students) and the video from the classes. At the beginning students were informed that their work will not be assessed. Pupils could work in any way that they considered as appropriate.

First stage of research

All series of arithmetic activities were divided into three parts. The first of them concerned the introduction of students to the subject, to familiarize them with the appropriate vocabulary which will be used in the next stage. These classes started from a common play called "What number are you?". Students picked out some cards with numbers, and their task was to present the drawn number in an interesting way. The purpose of this task was to draw the students' attention to the various features and aspects of the number and lead to the appropriate mathematical vocabulary (pay attention to the meaning of the words: number, digit, even, odd).

The next step was to draw up pairs: students looked for "a partner" for their number, and they had to justify their choice. The aim of this task was to draw attention to the common features of numbers.

Second stage of research

The second stage of the study concerned the analysis and argumentation skills. During these classes, students had to solve the following task:

Point to the one that does not match the other the numbers. Justify your choice.

9	15	24	16
21	42	41	14
12	16	18	20

25	16	34	18
18	15	25	30
33	15	12	6

The numbers in particular examples have been chosen in such a way that there were more than one possibility of choice, depending on the adopted criterion. The idea was to see what criteria the students will apply. Will they be creative and productive, how many different choices they will discover in the particular examples. What features will be soon discovered, what mostly they will take into account. Therefore, this task was planned to develop the ability to: analyse, see similarities and differences between the objects (here: the numbers), make and verify hypotheses; and also develop critical thinking skills.

The students worked single-handed. Each of students received a card, on which they can write all their thoughts and discovered dependences. The students worked during one hour of classes; they were informed that each task can have different solutions, and they should include those that are the most appropriate according to them.

Third stage of research

The third stage of the study required from the students to demonstrate their ingenuity and creativity. This time they had to create their own tasks. At the beginning their received a worksheet like previous but without numbers. There were only six rectangles (1x4) divided into four squares (1x1). Based on their previous experience the pupils had to enter in the blank spaces four numbers in that way so that one of them did not fit with the others. The criterion for selection was arbitrary. For each of the puzzles they had to write the rule, which was used.

The task posed in this way placed them in a slightly different role than in the previous situation, when they discovered the existing relationships and dependences between the numbers. This time the students first chose the criterion according to which they were fitting the numbers. The aim of this stage was to check if the students will be creative and constructive; if they will be able to prepare different task-puzzles or rather they will create very similar ones, by using the same rule.

Similarly, they worked single-handed. After finishing their work they presented their puzzles to the others.

SOME RESULTS

First stage of research

At this stage, the argument most commonly used by the students concerned the parity odd parity/data numbers. This task was repeated on subsequent activities, and the arguments used by students were far more diverse. The students were trying to present different characteristics of the number – less than half of them invoked the parity this time, and more than one-third of them highlighted the number of tens and the number of units. This may indicate a more analytical approach of the students to the task.

Second stage of research

This task was a challenge for the students, however they willingly joined the work. Each of the 20 students pointed out at least one number in each of the given examples, which according to him or her did not fit with the others. Half of the pupils reported more than one example of a number that does not match the others.

The analysis of all students' work helped to distinguish the following types of arguments on which the students appealed in their responses:

1. Reference to digits (digit, digit of tens, the lack of specific digit in the remaining numbers)
2. The number of single-, two-digit
3. Odd, even number

4. Small, big number
5. Sum/difference of digits
6. divisibility/ multiplication
7. the relationship between the numbers in the sequence
8. others –

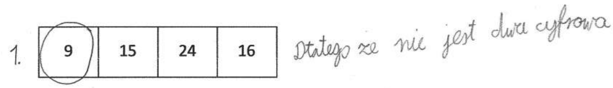
The results obtained by students during this stage of research are as following:

The type of argumentation	1	2	3	4	5	6	7	8
Number of answers (total: 197)	56	40	24	24	16	16	13	8
%	28%	20%	12%	12%	9%	9%	6%	4%

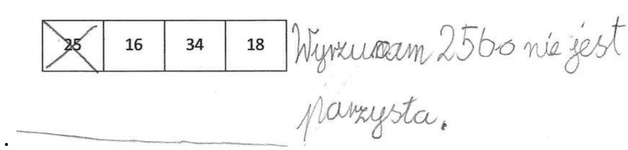
Table 1. The results from the first stage of research

As can be seen from the above statement, the most frequently used criterion associates with the visual aspect. So it was, inter alia, in example 2, where the students chose number "21", arguing that "in the rest of the numbers there is digit "4", and in this one-is not" (other numbers in this sequence is 41, 42, 14).

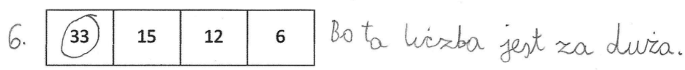
The visual aspect was also associated with the amount of digits in the number. A lot of students chose number that had a different number of digits than the others, for example "because it is not a two-digit number":



The next criterion used by students was connected with odd-even numbers. We can see it in following example: a student crossed out 25 and wrote "I throw 25 because it is not even".

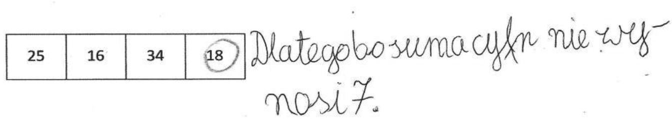


Then they made an analysis of the numbers contained in the series and, above all, compared to the basic characteristics of the number, which is the parity and odd parity. An equally frequent argument related to the size of the numbers was mentioned, for example. "I throw out 33 because it is the biggest", "6-because it is the smallest".

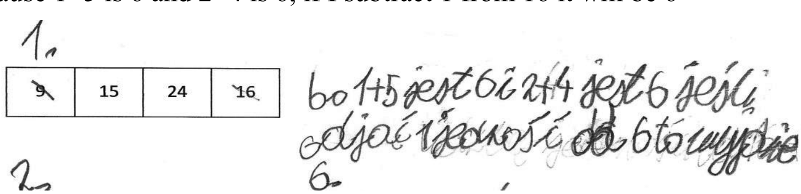


The application of this criterion was associated with the arrangement of the numbers in the specified sequence, referred to the aspect of the ordinal numbers. Although this is a very simple criterion, it required the student to carry out at least a brief analysis of the figures given in the statement.

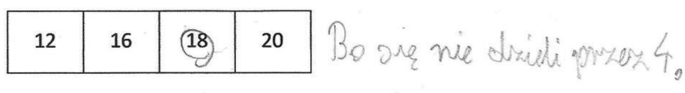
Another criterion used by students was referring to the sum or difference of digits in different numbers. This argument was usually used in example 6. Students choosing 12, argued, "because the sum of the digits is 6". Likewise, in example 4: deleting the number 18 was associated with the justification "because the sum of the digits is not 7".



This criterion was applied quite originally by a student in example 1. He deleted "9" claiming, that the sum and difference of the other numbers is 6. He wrote "because 1+5 is 6 and 2+4 is 6, if I subtract 1 from 16 it will be 6"



Indeed, the sum of the digits in the numbers 15 and 24 is 6. The situation is different in the case of 16. It seems that the number to which the student applied the "difference" but understood not as a subtraction, but just deletion. Thanks to duelist of the digits "1" in the number "16" you can get the desired result. In example 5. when choosing 18 the students wrote "it does not divide by 5" or "because it does not compose of fives." Similarly in example 3 - the deletion of 18 is justified by the fact that "it is not divisible by 4". The use of this type of argument provides a more analytical approach, looking at the numbers from another point of view. It is very interesting that at this stage of education students don't learn more about characteristics of severability and numbers and they solve any tasks relating to this issue.



Among others, a quite original justification was the following: "because adding the remaining numbers we receive a full of dozens". Such reasoning was applied in the third example (in the sequence 12, 16, 18, 20 they deleted 16) and the fifth (among the numbers 18, 15, 25, 30 they deleted 18).

These students performed a more complex analysis, since they studied the relationships between various combinations of the given numbers. They demonstrated the ability to complex analysis. It seems that students were able to go beyond the traditional scope of school they began to invent new things. And it is the most important activity during such classes.

Third stage of research

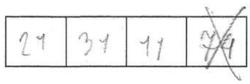
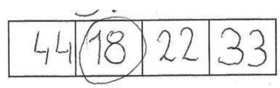
In this level twenty pupils took part and seventeen of them undertake the task. More than half of them created six puzzles, each time trying to apply different criteria. Three people reported to their puzzles more than one justification (presented various options for the same configuration of numbers). In total the students prepared 92 puzzles.

The analysis of all students' work helped to distinguish the similar types of arguments like in the previous stage, on which the students appealed in their creation of puzzles:

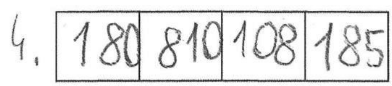
The type of argumentation	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Number of answers (total: 92)	13	17	10	12	19	8	7	6
%	14%	18%	10%	13%	21%	9%	8%	7%

Table 2. The results from the second stage of research

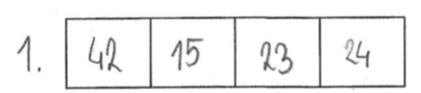
As can be seen from the above statement, still the most frequently used criterion is associated with the visual aspect. This was related both with a focus on the size of the number and the number of digits. This time, however, there were less by half of the arguments concerning the presence (or absence) of specific digits. Mostly there were questions like "all the numbers with the same digits, one of the various" or write three different numbers using the same digits for each of them. We can see it in the following examples of student work:

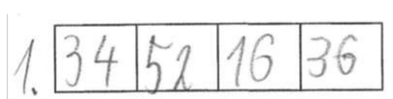



A similar criterion which appealed to the digits was used by the student who wrote: "because all numbers have the same digits, and 185 did not." In her work she used combinations of three digits to write three different numbers. In order to write the fourth number she used two digits from the previous and added another one.

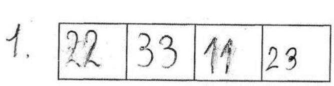


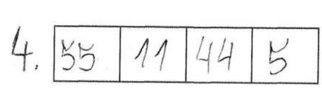
On this stage of research much more students applied the criteria related to the sum or the difference of digits. Very often the students justified their choice as follows: “I crossed out 23 because the sum of those numbers [others] is 6 and in the 23 – not” (S5) or “because the sum is not 7” (S6).





The criterion of "difference digits" was applied by one student in a rather unusual way. In her puzzles she used three numbers written by the same digits and one number with different digits. As a justification of her choice she wrote: “because the difference of numbers is not zero”. It means that the difference between the digits in the first three numbers is zero, and in the fourth one – not. After preparing the first puzzle the girls asked the teacher if it is correct. After the confirmation she thought it will be safe to use the proven criterion, and therefore she applied it again.





There were also some quite original puzzles. One of pupils gave the following sequences of numbers: 33, 10, 5, 8. As the incorrect number he show “33”. His justification was: “this number does not exist in the calendar as a day of month”.

Pupils had some problems with the precise expression of their criteria. In some situations it was very hard to understand what they were trying to say. Examples of such situations are presented in the following table:

The expression given by the student	**Example of puzzle**	**Meaning**
Because there are not tens	**7,** 16, 61, 84	It is a single-digit number
Because it is not sum of 8	44, 53, 35, **66**	The sum of digits is not 8
Because it if full and even	18, 70, 15, **20**	It has even decimal digit
Because it does not increase by 2	8, 10, 12, **20**	The other numbers consist a sequence increased by 2
Because 8 has not pair.	14, **8,** 15, 33	It is a single-digit number
Because all numbers in digits are 10	64, 82, 28, **52**	The sum of digits is 10
Because it does not increase by 6	6, 0, 12, **14**	It does not divide by 6
Because it does not equal 7	7, 16, 61, **84**	The sum of digits is not 7

Table 3: Students' expressions and their meaning

The examples presented above show how it was difficult for students to express their ideas by using correct mathematical language. Although they met earlier concepts such as number, digit, sum of the digits, one-digit number, two-digit number, they had difficulty in applying them correctly. Students writing their puzzles preferred in most cases the language used in everyday life. They used their own "mental shortcuts" that could always be understood by the other students.

SUMMARY AND CONCLUSION

Students very eagerly take new challenges, especially if they are presented to them in a fun way. The classes in which our students participated were for them a new challenge. Until that moment they had not met such kind of tasks. Nobody expected to provide a particular result, but to give the appropriate arguments to justify their choice. By observing students' work we could see a great joy for them during the discovery of the next relationship, and by being creative. The element of competition (who will discover more possibilities) meant that the task was more attractive. They eagerly attempted to solve each of the examples, some students tried to enter more than one solution. By creating their own puzzles, they tried to use as much different criteria as possible. They showed a great creativity and critical thinking.

The need to justify each of their choices forced the students to work on their own mathematical language. Initially, they used natural language, since it seemed to appeal to their own understanding of individual concepts. The common discussion after their work and the need to present the results of this work to the class showed that such a way of expressing their thoughts was not fully adequate. In order to all class talk about the same thing, it is necessary to apply a uniform language. And the most appropriate in this situation was mathematical language. So, it was a very good opportunity to show the students how important it is to talk correctly and accurately.

Expressing an argument, making a classification, identifying the differences and similarities between objects are very desirable skills not only in lessons of mathematics, but also in everyday life. Classes which are propely prepared may allow students to develop these skills. It seems that the series of "fun with numbers" gave the students this opportunity. However, the role of teacher who should inspire and encourage learning through play it is very important.

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DISCOVERING GEOMETRICAL CONCEPTS IN THE COUNTRYSIDE AND IN MATHEMATICS EDUCATION

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In our contribution we present one of the many possible ways of using activities of primary school pupils motivated by elements of "outdoor mathematics" to discover relations and links between geometrical concepts.

INTRODUCTION

Exploring space and related geometrical concepts are typical examples of how mathematics education provides an opportunity to reflect and develop children's real-world experience. Geometry as a world of shapes, sizes and movements in school mathematics sometimes becomes theoretical world full of incomprehensible abstract concepts and terms that do not have much in common with a multifaceted children's world.

In our contribution we present one of the many possible ways of using activities of primary school pupils motivated by elements of "outdoor mathematics" to discover relations and links between geometrical concepts.

THEORETICAL AND METHODOLOGICAL BASIS

Our contribution is based on two basic theoretical concepts: outdoor or fieldwork education and inquiry-based education (IBE).

Outdoor education as a form of experiential education is meant to provide pupils with opportunities for their complex development. As a method of education it is based on direct experience and takes place outdoors, focuses not only on the cognitive area, but appeals to all the senses, is interdisciplinary and includes relationships between people and natural resources (Payne, 2015).

As summarized by Dofková (2016), teachers should pay particular attention to:

• Diverse activities, some of which will focus directly on mathematical activities, while others will use activities in the nature and problems and activities following from everyday life, which are imaginative, entertaining and promote the development of children's communication skills,

• Balance between teaching in the classroom (school) and outdoors – thus to devise activities that share both classroom and outdoor education aspects,

• Use of "new" ideas, concepts and language in the pupils' own activities and encouraging children to discover problems, creating and providing opportunities for observation, forecasting and planning further stages in pupils' learning.

The term inquiry-based learning is going to be used in the sense of Samková (2014), i.e. as learning which includes activities focused on exploration and discovery. Obviously, research requires knowledge and specific learning environment. For details concerning inquiry-based learning, see e.g. Dostal (2014) and their relevant non-Czech sources such as Jaworski (2006), Dorier, Maas (2014).

Like research in science, research in mathematics also begins with posing a question or wording a problem to be solved through observation and exploration; through realizing mental, physical or virtual experiments; through looking for already previously addressed and solved similar questions and problems. We use and adapt, if necessary, known mathematical techniques. The process is led by or leads to hypothetical assumptions - answers that need to be verified (Samková et al., 2015).

Basic features of inquiry-based education include:

• Tasks and questions which can be interpreted in various ways - problems are usually open, can be solved in more than one way, have more correct answers. Pupils solve the problem on their own, to a greater or lesser degree with individual help of the teacher, where the teacher acts as a facilitator, using questions that direct pupils' activity and regulates their learning process and in the end also summarizes the situation and leads pupils to understanding key aspects of the problem,

• Discovering and rediscovering (as addition to the deductive approach),

• Learning from mistakes (especially one's own, but also of others; mistake is seen as an integral part of the learning process),

• Sufficiently dense network of basic knowledge (which has a potential to be enhanced),

• Cumulative learning style (linking new knowledge to already previously acquired knowledge)

• Interdisciplinary connections with other branches of study (incl. unusual ones, eg. Czech language or historiography) (Artigue, Baptist, 2015).

Methodological approach of our research is inspired by the methodology of critical didactic incidents - CDI), which falls within the area of qualitative methodology. It is based on direct observation of the work of pupils, which is appropriately supplemented by subsequent reflection (Amade-Escot, 2005). In our research, we directly observed a 4th grade mathematics class and subsequently coordinated joint collegial reflection with the teacher. The emphasis on microanalysis of teaching situations accompanied by focus on specific learning content - design based research (Trna, 2011) is based on the constructivist perspective of pupil's own knowledge acquisition and on the crucial requirement to achieve the highest possible level of understanding of the content and cognitive activation of pupils. At the same time, one can find out what specific issues and epistemological obstacles appear in the process of teaching and learning the specific learning content (Lech, Ametler & Scott, 2010, p. 8), which, in our case, is the relationship between elementary geometric concepts.

THE ACTIVITY

Basic features

The experiment was composed as a one-day project activity of both outdoor and classroom education performed during four consequential lessons of science and math, in a class of the 4th grade of primary school of 16 pupils. For the outdoor walk, no formal groups were set up, teams for classroom part were formed based on individual preferences.

Motivation

During a natural walk through a park near the school we focused on, besides botanical knowledge (type of tree, features of leaves, characteristics of fruits, etc.), estimating the age of selected trees: when doing so we used a rough calculation based on the idea that 2.5 cm of the trunk circumference corresponds to one year of the tree age.

Tree	Trunk Circumference	Approximate age
Pedunculate oak	210	84
Small-leaved lime	180	72
Common beech	140	56
White poplar	115	46

Table 1. Overview of date from walk

Using tape measure the pupils measured the circumferences of trunks of selected trees and calculated their approximate ages:

During our walk we came across a fallen tree. On a stump in the shape of a disc, we noticed annual rings and discussed another way of acquiring data for our calculation: measuring the diameter of the stump.

When back at school pupils found out the exact names of the measured trees and filled them into the table. Teacher suggested to discover the way of determining the diameter and radius of the tree trunk from a known circumference. An experiment was prepared in the classroom environment.

Goals of the experiment, tools:

The experiment aims at exploring the relationship between the diameter of the circle (disc) and its circumference, i.e. propedeutics of the relation $o = \pi.d$, or $o = 2\pi.r$, which is introduced in later grades of school. Discovering the relation is based on comparing lengths of strings, which represent the circumference, and the diameter. This may also lead to a preliminary understanding of the idea of direct proportionality - circumference of a circle is a function of its diameter (or radius).

Strings of different colors, scissors and various measuring instruments such as band, tape measure, school wooden meter, folding ruler, triangle ruler were used. We also needed some papers for taking notes and performing calculations, pens and calculators. We also used data written in the prepared table. There the pupils realize the lengths of the circumferences.

Assumed knowledge of the pupils:

The following knowledge and skills were assumed: basic skills in length measurement and use of available tools and instruments, knowledge of concepts of circle and discs (and their differences), radius, diameter, circumference of a circle and of a planar shape.

Tasks for students:

1. Cut strings of different colors of same lengths as the circumferences of the respective tree trunks, and then compare lengths of the strings, which represent circumferences of the given trees.

2. Find a way to determine the approximate diameter of the tree trunk.

3. How will you determine the radius of the tree trunk?

The experiment performed and recorded:

ad 1) First, pupils chose “their” trees. When cutting strings of the same length as the circumference of the tree trunks, pupils had first of all to determine the appropriate length of the string. The circumferences of the tree trunks in the table were given in centimeters. Pupils had no instructions on how to deal with this issue, so they had to develop their own strategies of measuring the length of the string. The usual classroom equipment and measuring tools that pupils had previously worked with were available. The selection of a suitable tool was influenced mainly by the circumference of the respective tree (210 cm, 180 cm). Pupils searched for instruments which allowed them to measure the entire length at once. First group has therefore chosen a tape measure. Instruments chosen by other groups were (in this order):: tape measure/folding ruler, school wooden meter, paper ruler. The group which had to select their instrument after the removal of tape measure considered their task unsolvable without the above mentioned instrument (not available). Sylva commented on this: "This is unfair, we also want the tape measure or a folding ruler. How are we supposed to measure a string of 210 cm using a 100 cm long ruler string? Thus the need to re-use measuring instrument in case it did not have the desired length (circumferences of the respective trees) appeared. Pupils chose their "auxiliary units" to enable them to perform the measurement, and they applied it gradually on the string. They did not employ the approach usual especially when buying fancy goods (ribbons, rubber, ...), where the shop assistant holds the point of n-multiple of meter, but marked the point on the string with a marker. The unit used was 1 m (Fig. 1).

Figure 1. Designation of the measured interim length with a marker

A similar procedure was applied by a group with paper rulers - to re-use the unit of 1 meter. They stuck together several rulers to get the desired length. At first, the group made a mistake when sticking the rulers together not exactly at the points of 100 cm of one ruler and of 0 cm of the other. Only after teachers' note the error was detected and corrected (Fig. 2).

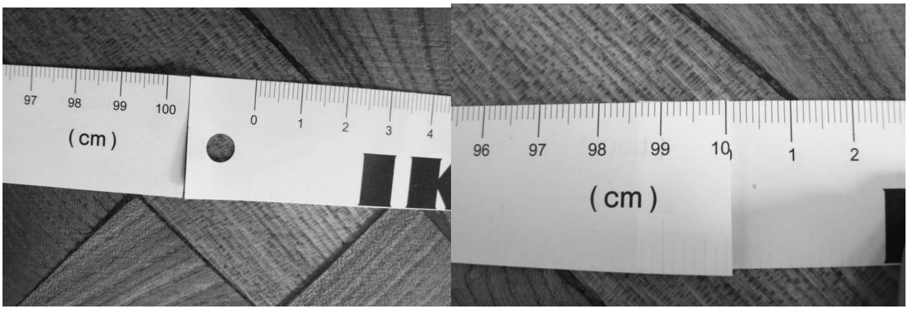

Figure 2. Wrong and correct "extension" of the ruler(s)

Ad 2) Pupils correctly understood it necessary to start from the circumference when determining the diameter of the tree. Sometimes, in their group discussions, somewhat radical views of determining the diameter appeared.

Luke: "The best thing would be to cut down the tree and to measure the diameter of the stump, as we saw in the park”.

After cutting the string of desired lengths corresponding to the circumferences of the tree trunks listed in the table, pupils tried to create circles on the floor and to shape the circumference of the trunks. Circumferences represented by strings were far from perfect despite close cooperation of all members of the groups. One group created an almost perfect ellipse (Figure 3). The pupils therefore accepted the proposal of their teacher, which reflected the idea of drawing a circle on paper, to mark the center of the circle on the floor (stick) first and to estimate the approximate radius. And subsequently to "draw" the circle. The group managed this quite well. This process seemed to solidify the idea of radius as a line segment, with endpoints in the center of the circle and at any point on the circle.

Figure 3. Attempt to create a circle using the strings

In the next step pupils could already make use of their knowledge about the concept of the diameter of the circle (disc). The starting point of their reasoning was the idea of the diameter as a line segment both ends of which lie on the circle. Such a line segment needs to be depicted using a string. In all groups several attempts were needed to draw such a line segment which passed through the center of the circle on the floor, i.e. the longest string circle, the actual diameter. The lengths were measured (with a certain degree of inaccuracy, resulting from the approximate shape of circles) and pupils recorded their findings in tables prepared by the teacher. All groups received the same table where they filled-in not only their measured data but also data of the other groups.

	1. group (Oak)	2. group (Lime)	3. group (Beech)	4.group (Poplar)
Circumference measured (park)	210	180	140	115
Diameter measured (classroom)	73	58	48	36
Radius as calculated	36,5	29	24	18
Circumference divided by the diameter (rounded)	3	3	3	3
Circumference divided by the radius (rounded)	6	6	6	6

Table 2: Circumference, diameter and radius of the tree trunks

Ad 3) The answer to the last question was relatively easy as it was based on the fact that radius of a circle is one half of its diameter. The value of diameter was divided by two, and the result was filled into the table.

Exploring the relationship between circumference, diameter and radius. The last part of the activity consisted of working with the above tables, focusing on the numerical calculations using a calculator. Pupils were asked to divide the particular tree trunks circumferences by the values of diameters, subsequently by the values of radius. However, it was important that the teacher steered the activity- the same table as drawn by the pupils was projected by a projector on the board and the results were filled into the workbook and on the board simultaneously. After the calculations were performed and results were rounded, an interesting fact could be seen: the diameter of the trunk was in all cases approximately three times smaller than the circumference, the radius of the trunk about six times smaller than the circumference - regardless of the different size of the trunk circumference.

It should be noted that at this stage pupils started losing their attention. Some of them were surprised by the result (three, six) but with no sign of thinking about the underlying rule.

CONCLUSION

Our research dealt with specific aspects of CDI methodology which focus on key events that constitute substantial parts of the work of teacher in cooperation with pupils, interpreting them as crucial for the success of teaching.

We believe that the activity discussed in this contribution uses elements of inquiry based learning. Activities of pupils and the teacher has been documented by numerous photographs and video recording. It turned out that modeling trunk circumference of given value and diameter or radius of the circle is within the capabilities of the pupils who at this stage worked spontaneously enthusiastically. Attempts to generalize, to find a functional relationship between the diameter, radius and circumference, however, had to be closely and consistently encouraged by the teacher.

Our experience has shown also some difficulties. In our opinion, from the perspective of pupils, this includes appropriate motivation, suitable timing as well as problematic background of necessary mathematical knowledge and skills. From the perspective of the teacher - besides the already mentioned necessary professional competencies - considerably demanding preparation and organization of the activities, material resources and relating limited possibilities of implementation in common educational reality. Although teachers can prepare elaborated scenario in advance, there still exists obvious risk of missing set expectations or of not achieving the objective. From the above the emphasis on creativity and flexibility of the teacher is paramount as he /she must be able to manage and take advantage of such educational situations.

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HOW TO COMPARE SURFACES?

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The paper presents the analysis of an experimental activity, based on the inquiry model, realised with 9-10 years old pupils. The experiment consists of three main phases: 1) submission of a worksheet, 2) individual interview, 3) communication in class. The worksheet introduces the problem of the equivalence of figures with respect to their surfaces, in a context of 'fences and sheep'. The pupil is asked to verify and to check his answers in a personal interview. The final discussion in the classroom plays a fundamental role in the experiment. The research reports a qualitative analysis of the results, with particular attention given to difficulties and misconceptions that emerged on the topic.

THE EXPERIMENT AND ITS ORIGIN

The work originated from a research (Marchetti & al., 2005) intended as a preliminary investigation into the spontaneous procedures adopted by pupils in problems concerning perimeter and area. During the experiment, the researchers observed particular difficulties in comparing the surfaces of the figures. This highlighted the need to organize new activities to examine this aspect in depth.

In the first experiment, the researchers prepared two worksheets, one on the comparison of perimeters, another on comparison of areas. The first presents the following problem: a shepherd must choose a fence for his sheep; his choice is among nine figures. He wishes to put a barbed wire around the fence spending the least amount of money. The second worksheet uses the same figures to submit a problem on the concept of surface: the shepherd wants the sheep to have as much grass to graze as possible. The worksheet finishes with this task: "Now put all the fences in order, from the smallest to the biggest, according to how much ground they enclose". The topic of the present paper is the analysis of pupils' behaviours faced with this problem of surfaces comparison. It gives important information on the strategies employed by the children, on the misconceptions and errors that emerged, but especially on the role of mathematical investigation and discussion in the promotion of new knowledge.

THEORETICAL FRAMEWORK

The "inquiry model" in mathematical education (Borasi, 1992) aims to include both the pupils and teacher with interest and responsibility. Its main assumptions are the following:

- Knowledge is a dynamic process of inquiry. Doubt, uncertainty, cognitive conflict motivate pupils to investigate and to construct new knowledge. This process occurs inside a research community.
- Learning is a process of construction of meaning by social interaction. It also involves personal and individual engagement by motivating situations.
- Teaching must provide environments that stimulate inquiry. It organizes the class as a community of learners, who must construct their own knowledge.

This model promotes reasoning and communication in the classroom through difficult and complicated problems. Some typical aspects of the inquiry model are the complexity in the production of knowledge and the uncertainty of the result.

Geometry requires a cognitive activity that stimulates sight, language and gesture. Duval (2005) identifies the origin of the difficulties in geometry in the intuition that relies on the perception, which plays a fundamental role in the visualisation process:

> […] by the perception the visual thought organizes itself as starting point of the insight and the reflection, mental activities which contribute to the formation of the concepts (Marchini et al., 2009).

In the specific case of the equivalence of figures with respect to their surface, perception can influence the evaluation. The equivalence of surfaces is based on problems of conservation of magnitudes, studied in depth by Piaget and Inhelder (1975) and Bang and Lunzer (1965). As written by Chamorro (2002, p. 63):

> […], the dominant component of the notion of surface is, in fact, the notion of shape: it is impossible to conceive a surface without shape.

In other words, the recognition of a surface passes through the shape, if the change of the shape is relevant, the pupil does not recognize the conservation of the surface. Moreover, the terminological confusion between 'shape', 'surface' and 'area', often present in the scholastic practices, creates problems of understanding. For these reasons, I decided to propose a work based on the comparison of surfaces of figures having different shapes, without measuring them.

In Italian primary schools, the concepts of 'surface' and 'area' (as a measure of the surface) usually appear in the fourth grade (pupils 9-10 years old). It is important to work on surfaces without measure them, maybe using their equi-decomposability (division in equal parts) to find their equivalence in respect of the surfaces. The early introduction of area presents the risk of confusing the concepts of 'surface' and 'area'; furthermore, it promotes too fast a transition from the geometrical to the arithmetical field:

> It's important to distinguish the two stages since the competences, necessary for the established aims, are different, each using a distinct level of abstraction (Marchini, 1999).

In conclusion, it is necessary to communicate that the surface is a property of a figure, that an object has a surface even if we do not measure it. An early introduction of measurement and formulae can create obstacles (Vighi & Marchini, 2011).

METHODOLOGY

The experiment took place at the end of the academic year in three classes of two schools[1]. It engaged 75 pupils of fourth grade (9-10 years old). The topic 'figures and their surfaces' was completely new in two of these classes (class A and B), while in the third class (class C) the students knew the formulae for the areas of squares and rectangles. These two different kinds of background allow the comparison of the behaviours of pupils with different initial knowledge. In particular, in class C it would be possible verify if the concept of surface is present and if the practice with the formulae modifies the answers.

The experiment consisted of three phases:

<u>Phase 1: submission of a worksheet</u>. In the classroom, the teacher submits a problem of comparing surfaces, using the same worksheet employed in the first experiment (Marchetti & al., 2005). Only the last question is changed. The figures presented in the worksheet as possible fences are the following[2]:

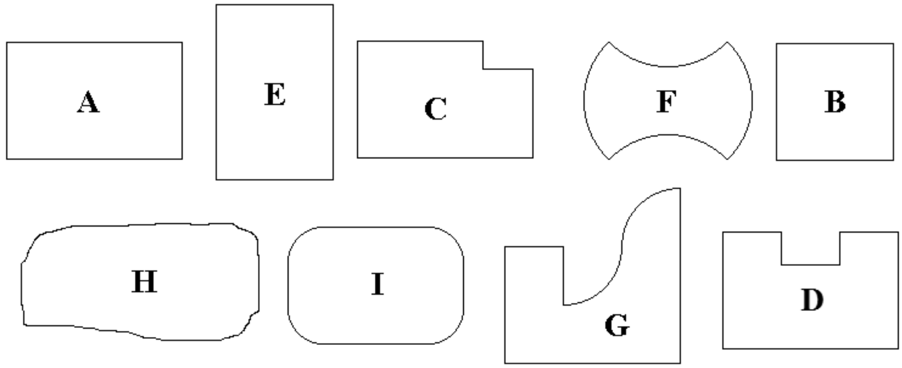

Figure 1: The figures in the worksheet

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[1] Specifically, the Primary Schools of Vicofertile and Vigatto, two small towns near to Parma (Italy). We acknowledge the teachers G. Barantani, L. Ferrarini, A. Tomasini for the hospitality in their classes and for their collaboration during the experiment.

[2] In the worksheet, the 'real measures' of figure A are 3.5 cm e 2.8 cm; obviously, the dimensions of the other are proportional to these.

Phase 2: individual interviews. Some weeks later, the researcher (the author of the present paper) carries out individual interviews submitting her/his worksheet to each child, asking the pupil to read, comment and check her/his own answers. Lastly, the researcher places on the desk a template of each figure present in the worksheet, obtained by cutting out them with scissors (Figure 2). The use of templates changes the task, moving the attention from figures to objects: the first can only be observed by the eyes, while the second are objects that can be manipulated. The templates are bigger than the figures drawn on the worksheet, to avoid problems of manipulation. The task is "Put them in order, from the smallest to the biggest, according to how much grass they enclose". The researcher observes, without comment or evaluate the execution of the task by the pupil.

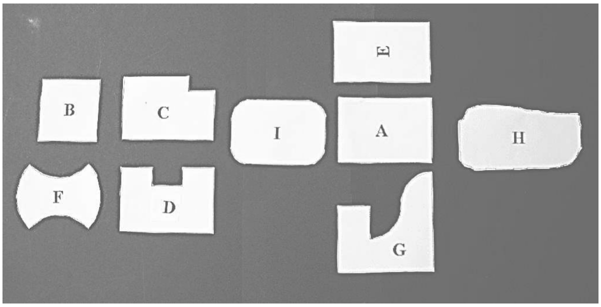

Figure 2: The templates and their correct order.

Phase 3: communication in classroom and final solution to the problem. After the individual interviews, on the same day the researcher goes into the classroom, she presents enlarged copies of the nine figures, each of them reproduced on an A4 sheet of paper. This increases their dimensions and highlights the difference among their surfaces, simplifying the task in part. The task is to put the figures in order again, and to obtain a final sequence that all the pupils agree on. During the discussion, the teacher has the role of mediator, the researcher only of observer.

The researcher collected and analysed the worksheets completed by the pupils and video-recorded the individual interviews and the discussion in classroom. Each interview took ten-fifteen minutes, the discussion one hour.

THE EXPERIMENT AND ITS A-PRIORI ANALYSIS

Analysis of 'Phase 1'. The context chosen, a problem of fences for sheep, supplies the motivation of observing and comparing figures and their shapes. The choice of comparing many figures aims to stimulate a process of investigation that, starting with open questions, through doubts and difficulties, encourages the student to organize his mathematical activity. It is a typical situation of inquiry: the pupils must explore the problem and assume personal initiatives, reaching an ultimate and common solution by negotiation.

Some shapes are usual in scholastic activities (A, B, E), but the most not. There are polygons concave or convex (A, B, C, D, E) or figures with curved outlines (F, G, I); figure H is particular, its boundary is indented, it is 'irregular' but very similar to a real fence. Figures A and E are isometric; they may seem different since they are drawn in different ways. Their congruence implies that they are 'equal' concerning the shape and equivalent in respect of the surfaces. Figure G, obtained by a rotation of a 'piece' from figure A, has the same surface, but its strange shape, very different from this of A, can hinder the recognition of their surfaces equality. Furthermore, the figures C and D have different shapes, but equal areas. Both can be obtained from A by cutting out a small rectangle, so their surface is smaller than that of A. The figure I could be obtained by a modification of A, 'cutting out and curving its vertices'. The figure F is obtained from B by operations of 'cutting out and gluing', their surfaces are equal. To simplify the task, the worksheet asks to compare sets of three figures (A-I-B, E-C-F, D-G-H), in increasing order of difficulty.

The worksheet finishes with the question: "Are there any fences which enclose the same quantity of grass? If so, write them down and explain why". The discovery of this kind of figures leads the children again to observe and to compare the fences. The aim is to promote the idea that different figures can have the same surface, though they have very different shapes. In particular, the experiment allows the investigation of the concept of surface, remaining in the geometrical field.

Analysis of 'Phase 2': The researcher suggests to solve the task of putting the fences in order, not only using sight and perception as in the first experiment, but also manipulation of templates. This problem and its solution is the core of the individual interviews. The individual interviews take place outside of the classroom. At the start, the researcher presents each child her/his protocol, asking her/him to read the worksheet again and check the answers given to the questions. In this experiment, checking has a fundamental role, while in traditional mathematical activities it is often neglected or omitted.

The second part of the interview is on the ordering the templates with respect to the extension of their surfaces. The choice of using templates is a strategy that moves the attention from the visual and perceptive level to that of manipulation. It is a fundament transition, aimed at stimulating and improving investigation using suitable tools. This new variable promotes the transition from a static to a dynamic observation of the figures. In this way, the comparison of figures completely changes, for instance, it is possible to superimpose the shapes, to cut and glue some parts to show their equivalence and so on.

The activity of comparison is not usual in school. Traditionally, work in geometry is only on single figures, while here there are many shapes to compare. Mathematically, the comparison can be reduced to five figures (A, B, C, I, H), since some of them are equivalent with reference to their surfaces, but from the didactical point of view it is opportune that pupils compare all the figures. The high number of comparisons can be a cause of problems in pupils' performance. The request is intentionally ambiguous. The correct sequence of figures, "from the smallest to the biggest" is the following: B-F; C-D; I; A-E-G; H (the symbol "-" connects figures with same surfaces).

Analysis of 'Phase 3': this phase is fundamental in an inquiry-based activity. It is motivated by the need to reach a common solution to the problem. The result must be negotiated by communication in the classroom, starting from the personal ideas of each member of the community, students and teacher.

RESULTS

Phase 1 results: The following table contains the percentages of correct answers and of recognized equivalences:

Comparison A-I-B	Comparison E-C-F	Comparison D-G-H	Equivalence A-E	Equivalence C-D
93%	73%	55%	39%	25%

Table 1: Percentages of correct answers.

It is evident that the percentages decrease with the increasing difficulty in comparing the figures. As regard the recognition of figures 'with the same grass', only pairs A-E and C-D are meaningfully recognised. In particular, in the initial investigation only four pupils found the equivalence of A-E-G immediately, and one found the equivalence between B and F.

Phase 2 results: The individuation of figure B as 'the smallest' is easy, while the recognition of its equivalence with F seems to be very difficult. The observation of F comes late, after the collocation of the majority of the other templates. Its shape like a "bone" leads the child to observe and to privilege its breadth, while the 'empty spaces' above and below suggest an idea of 'thinness'. The square B, on the contrary, appears 'robust' and bigger than F. This sometimes induces pupils to put F before B in the ordered sequence: the perception prevails. The possibility of 'cutting and gluing' some pieces of F to obtain B or vice versa is not evident: only 20% of those interviewed discover this strategy by themselves.

The equality between figures A and E appears evident by manipulation, also for the pupils (19%) who consider the areas different in the execution of the worksheet tasks (phase 1), possibly conditioned of the dimensions, the width or the height. An important aspect that emerges is the influence of the traditional work of the classification of figures in geometry with the relative terminology.

For instance, some pupils affirm that "A and E are equal since they are both rectangles", without noting their dimensions. This kind of reasoning hinders the recognition of the equivalence between the figures A-E and G, since the shape of G 'is not rectangular'. Figure G can become a rectangle by a suitable operation of 'cutting out and gluing a piece' or by a rotation of a piece. This idea spontaneously emerges from the child in few cases; it could be caused from a lack of work on geometrical transformations. The need to place G in the ordered sequence induces the child to investigate its surface in comparison to the others. The films made during the interviews document that the discovery of the equivalence between A-E and G is the result of a long and laborious work. Sometimes the prevalence of a 'hole in the rectangle' leads child to put G before A-E in the sequence, with the conviction that 'it is smaller'. In other cases, G is placed after A-E 'since it is higher'.

The comparison between figures C and D sometimes is influenced by the perception: the "rectangle cut out" in C appears smaller than the one in D. A suitable superimposition of the templates is sufficient to confirm that their cavities fit. The comparison between C-D and I is a real problem: how to organize and quantify their omitted (or cut) parts in respect of the rectangle A? During the individual interviews, this problem is approximately solved using perceptive suggestions. Some children draw the rectangles circumscribed to C and I, to improve the 'consistence' of the "cut out parts": this strategy improves the performances and results but, in fact, it remains an open question to solve in phase 3. The estimation of the surface of figure H presents the same problem: H is "high as A, but wider than it" or "it is long and thin": these argumentations lead the 30% of the pupils to place H incorrectly before other figures in the sequence of templates. This mistake will be another opportunity for discussion during the following phase of work.

<u>Phase 3 results</u>: When the individual interview is over, the child returns to the classroom. Even if he gave a correct final order of the templates, often he forgets the sequence constructed, he only remembers his argumentations and reasoning. The discussion in the classroom starts from this point.

At the beginning, the pupils solve the easiest cases, for instance the comparison between A and E or C and D. The role of the geometrical terminology and the 'weight' of the traditional nomenclature used for some figures clearly emerged. The presence in the worksheet of "figures without a name" (no rectangles or squares or …) causes trouble and sometimes it forces the child to say, for instance: "All the figures A, C, D, E are rectangles". In class B, this reasoning promotes a methodology of comparison of figures based on the concept of 'equi-complementarity' (instead of superimposition): some pupils draw the rectangles that contain C and D, showing that "to have the same rectangle, starting from C or D, it needs to add the same small rectangle". In other cases, some children say: "C and D are equal, and also the piece omitted is the same small rectangle".

The use of the word 'equal' requires a deeper analysis. Sometimes, during the discussion, this kind of dialogue takes place: "C and D are equal" referring to their surfaces, with the prompt answer: "Not, their shapes are different!". This exchange of opinions is important, it promotes the distinction between the shapes and the surfaces. The conditioning induced by the shapes clearly emerges in the discussion. For instance, about figures A and G, some pupils continued to affirm that "A and G have different shapes and, as a consequence, different surfaces". In class A, the equivalence between G and A is accepted when a girl, using her hands and gestures, shows the rotation that transforms the G in A. In classes B and C, the discussion was focussed on this initial affirmation: "G is higher, so it is bigger than A"; at the end, the use of the scissors solves the problem.

Other comparisons between figures, which are less evident, require long and interesting negotiations. Interesting beliefs and misconceptions emerge. The following transcript of a dialogue, made during the comparison of figures C and I, documents this aspect.

1 Marco: I think that the grass in figure C and this in figure I are the same.

2 Luca: This was my first idea, but it was wrong since in figure I the four pieces omitted create a square, while in figure C the part omitted is a rectangle.

3 Marco: But, we must observe the quantity of grass!

This dialogue is an example of the prevalence of the shapes, square and rectangle, and their names, which hides the initial problem, highlighting a misconception. The discussion in classroom on these argumentations leads the rejection of 'Luca's thesis'. The conversation continues following new inputs. A girl suggests a possible solution of the problem, using scissors (Figure 3).

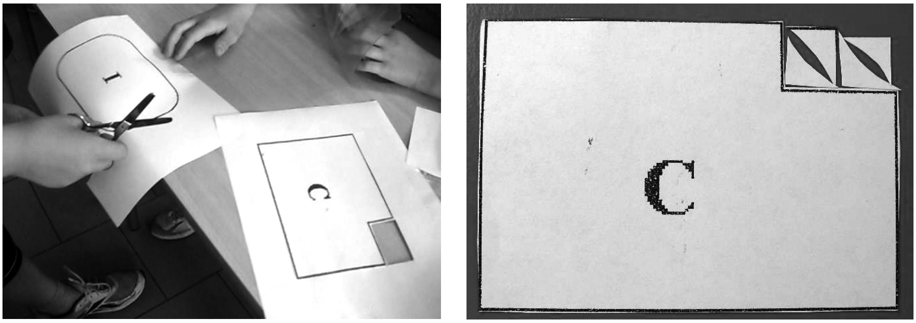

Figure 3: A problem solved with the use of the scissors

At the end, the problem of the evaluation of H-surface required a long and articulated discussion. I noticed different behaviours in the three classes. In class A, pupils observe that the figure H is not 'regular' (not a polygon), it has the same height of A but it is wider; the superimposition of the templates A and H solves the problem, since “a big part of H stick out from A”. In class B, the teacher suggested this idea: to use grains of rice covering the figure and counting them, or better the weigh the rice. Unfortunately, in classroom there was not balance. In class C, a boy proposes to use a ruler and to measure the width and the height of the shapes. It documents the attempt to repeat the strategies used by the teacher for squares and rectangles. This also highlights the correct idea that the surface depends on both dimensions. In fact, pupils are unable to conclude their reasoning using the measurement made with a ruler. Finally, emerges this suggestion: to use squared paper superimposed to the figures and count the squares, following the same methodology used for squares and rectangles. This strategy prepares the concept of measurement of a surface in cm^2.

CONCLUSIONS

The activity proposed seems to be very rich and meaningful, but is also difficult, sometimes without the finding of a common solution. As written previously, the complexity of the situation is a choice, aimed at realising a real situation of an “inquiry-based lesson”. During the discussion, students presented their own answers, obtained in the individual interview, they explained their ideas using verbal language, but also non-verbal communication based on gestures and manipulation of the templates. The transition from the observation of drawn figures to the manipulation of their templates provoked a fundamental change in the reasoning on surfaces: this allows overcoming the didactical obstacle created by the use of drawings in the treatment of surfaces (Chamorro, 2002). The role of touching, cutting and transforming a figure into another clearly emerged during the discussion.

This paper is focused on the mathematical results of the activity. It could be interesting, using videos, to analyse the experiment, analysing the “sociomathematical norms” that support the ‘inquiry-based discussion’ (Yackel, Cobb, 1996) and the role of the gestures (Arzarello, 2006) and (Radford, 2003). The mathematical discussion among pupils highlighted the need to communicate in a proper way their own ideas and also to listen the opinions of the others. Often the pupils tried to engage the teacher and his authority, but in this case, he was only “a voice that represents the mathematical culture” (Bartolini Bussi, 1996, p. 17), he could only orchestrate the debate and, if possible, promote the acquisition of common shared conclusions.

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Inquiry based teaching of particular mathematical topics

Part 4

EXPLORING PATTERNS USING INQUIRY-BASED LESSONS

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This study focuses on the exploration of patterns by 6th-graders in their mathematics class. It addresses two questions: 1) How do students embrace tasks involving patterns? 2) What difficulties do they face when solving these tasks? An intervention of three inquiry-based mathematics lessons comprising 3 tasks of patterns (3 to continue a sequence and 2 to investigate the sequence) was carried out. Qualitative methods were used to analyse students' solving the tasks. Results show that students succeeded in the majority of the tasks. The inquiry-based lessons gave them an opportunity to solve problems and stimulate mathematical reasoning and students' oral mathematics communication, creating a challenging environment in the classroom.

PATTERNS IN THE TEACHING OF MATHEMATICS

In agreement with Orton (2005), patterns are difficult to define as the word has several meanings. In the literature patterns relates to expressions such as regularities, sequences, order and structure. Vale, Palhares, Cabrita and Borralho (2006) argue that the concept of pattern is used when one refers to a particular disposition of shapes, colours or sounds that involve regularities. Devlin (2002) points out that patterns are the essence of mathematics, and what mathematicians do is to examine abstract patterns – numeric patterns, patterns of shape, patterns of movement, among others.

Concerning the types of patterns, Orton (2005) distinguishes geometric patterns, in which the regularity relates to some idea of symmetry, and numeric patterns, involving numeric sequences. Vale and Pimentel (2009) distinguish patterns of repetition, that has an identified motive that repeats cyclically indefinitely, and patterns of growth, in which each term changes in a predictable way from the previous one. There are linear and non-linear patterns of growth, which the related algebraic expression can be, or not, represented by a 1st-grade polynomial expression. Zazkis and Liljedahl (2012) distinguish numeric patterns, geometric patterns, computational procedures patterns, linear and quadratic patterns, and repeated patterns. But more important than discuss the diversity of classifications of type of patterns given in the literature is to focus the attention in the context in which the patterns can emerge.

In this study the numeric and figurative contexts are used to present patterns to the students. To know more about students reactions solving tasks of patterns, to focus attention on students reasoning during their resolutions gives relevant information about how teachers can approach the tasks in the classroom.

To explore patterns in the mathematics classroom is fundamental and pertinent as it appeals to the development of creativity, makes possible the establishment of several connections between distinct topics, it develops the ability to classify and order information as well as to understand the connection between mathematics and the world (Vale & Pimentel, 2009). Patterns allow students to discover connections, establish generalizations and make predictions. In the classroom, patterns can be an excellent context to develop mathematical concepts, beyond being an opportunity to develop the ability to solve problems, reasoning and communicate mathematically (Vale, Palhares, Cabrita & Borralho, 2006; Vale & Pimentel, 2009). The development of algebraic thinking requires an approach to Algebra through problem solving of patterns, as the exploration of patterns is a powerful strategy of problem solving; to solve non-routine problems can become a way for students to explore and formalize patterns, allowing them establish conjectures, verbalize relations between distinct elements of a pattern and generalize (Vale & Pimentel, 2009). Vale and Fonseca (2011) point out that patterns can give the students an opportunity to develop mathematical knowledge as they allow students to relate different concepts and contents in distinct contexts. Thus, patterns are a relevant topic concerning the development of students' ability in mathematical processes such as problem solving, mathematical reasoning and mathematical communication. These ideas are also expressed in Portuguese official curricula documents (see DGIDC, 2007), and also international documents (see NCTM, 2007).

Frobisher and Threlfall (2005) referred that, in the first years of contact with patterns, students should develop abilities to describe, complete and create patterns, transform a written expression into a symbolic one, or vice-versa, extend a pattern to solve problems, explain a generalization of a pattern and also use patterns to establish relations. Also Garrick, Threlfall and Orton (2005) argue that to explore tasks involving regularities, in group or individually, challenge students to verbalize their ideas and present their knowledge, improving their ability to communicate mathematically. Regarding these issues, literature suggests that teachers should select tasks for their students that challenge them to find regularities, make conjectures, discuss ideas, improve their argumentation based on coherent and valid justifications, and also to improve their ability to reasoning and communicate mathematically (see Vale & Pimentel, 2009). Thus, it becomes essential to challenge students to describe patterns, by their own words, and justify how they can extend or create it.

Orton and Orton (2005) consider that the learning of patterns is influenced by the level of conceptual difficulty beyond students' motivation. For the authors, students' level of abstraction affects their understanding of mathematical concepts, as there are students that simply cannot make sense of essential aspects of mathematics. If the mathematics teaching do not comprise relevant experiences for students, it will be difficult for them to improve their level of abstraction. Thus, teachers assume a crucial role in promoting relevant learning for their students.

The tasks of patterns

Concerning the tasks of patterns, Vale, Palhares, Cabrita and Borralho (2006) underlie the idea that teachers should present their students with tasks relevant to students' algebraic thinking. These tasks should comprise the identification of patterns, the creation and continuation of patterns, and the exploration of different properties of relations. Vale and Pimentel (2009) point out that tasks of patterns should comprise the multiple representation of a pattern, the recognition of a pattern, the prediction of terms, the generalization and building of a pattern, and the oral and written description of a pattern.

Patterns can emerge from numeric, visual or mixt approaches (Orton, 2005). Since early school years, students should have the opportunity to observe patterns and represent them geometric and numerically, starting to establish connections between geometry and arithmetic. Vale (2012) argues that mathematics teaching should include challenging tasks that emphasize the understanding of a generalization, through numeric and figurative aspects, making the most of students' ability to think visually. To improve students' visual representations may comprise their use of different representations, such as describe patterns using tables and adequate numeric expressions (Vale, 2012). For Vale (2012) generalization involves higher order thinking such as reasoning, abstraction, or visualization. Thus, the selection of tasks is crucial if one intends to offer problem solving experiences that allow students to make generalizations.

This study tries to understand how sixth-graders explore patterns. It tries to address two questions: 1) How do students embrace tasks involving patterns of repetition on and growth in the mathematics class? 2) What difficulties do they face when solving these tasks?

METHODS

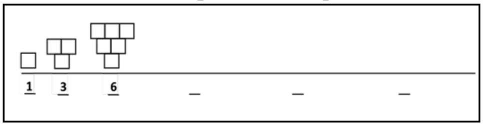
Qualitative research methods were adopted, and an interpretative approach was used. Bogdan and Bicklen (1994) argue that qualitative methods occurs in a natural environment that provides a natural source of data in which the researcher is the main tool. This research is strongly descriptive, keeping the focus on the processes rather than the products.

The participants were 28 students of a sixth-grade from a state supported school form Braga, Portugal. There were no students with special education needs.

An intervention was conducted in three mathematics lessons focused on patterns that took place during two consecutive weeks. Each session last 90 minutes long. The intervention was implemented in the class by one of the authors of this paper.

Tasks of problem solving that require students' ability to explore patterns, establishing conjectures, verbalizing relations between the elements of the pattern and accomplish generalization were selected. The tasks presented to the students comprised on: continuation of sequences; identification of the rule (or law) of formation of a sequence; types of sequences; determination of terms of a given sequence; and exploration of sequences. Figures 1 and 2 present examples of some tasks to continue a sequence presented to students in the intervention sessions.

Observe and complete the sequences.



25, 22, 19, 16, ____________

Figure 1. Examples of tasks involving figurative and numeric pattern.

Observe the following sequence:

47, 49, 51, 53, 55, …

Indicate the next four terms of the sequence. ____________

Indicate the rule to build the sequence. ____________

Is 86 a possible term of this sequence? Why? ____________

Figure 2. Example of a task to continue and determine the term of a sequence.

After each session, an analysis of the tasks implementation and of the students' difficulties solving them was carried out by the researcher, one of the authors of this paper.

The first session comprised three tasks; the first and the second tasks had four questions each, the third had three questions. The session involved repetition and growth patterns (numeric, geometric and figurative). The repetition patterns comprised two, three and four repetition terms; the students had to observe and continue the sequences, identify the rule of formation, and verify the existence of particular terms. The second session comprise two tasks, each with four questions. Students had to draw the following terms of a given sequence, find out and explain the rule of formation of the sequences, find the terms of particular positions in the sequence, analyse the possible existence of particular terms and their positions in the sequence. Both tasks comprise patterns of growth, an illustrative and a geometric one. The third session comprised three tasks, with three, one and four question each. Students were supposed to continue patterns, explain their reasoning, find out a particular term and its position in the sequence, find out the possible existence of others, and explore the growth of a sequence to establish a generalization. The three tasks involved patterns of growth, of numeric and geometric types.

An inquiry-based approach was adopted in the sessions during the implementation of the tasks in which students centred activities were developed. The students were challenged explore the patterns and the questions presented to them prompt their work. They were encouraged to regulate their own activity while exploring the questions. Nevertheless, this was the first time students contacted with tasks of sequences, and very seldom they were challenge to discuss the problems in small groups. Naturally they felt more comfortable in solving the tasks individually, and then share the resolutions in the whiteboard with the whole class.

Data collection was accomplished using videorecorder and photos, researcher field notes and also students written work. All the students names used in this work are fictitious.

RESULTS

Lesson 1

The lesson started with a whole class discussion about regularities and patterns. Students knew expressions from their informal settings, but struggle with the types of sequences. Soon students become familiar with the terminology and types of patterns. When analysing one of the example given to students, they were asked about the existence of a particular term, a debate emerged and the solution was found. Immediately after, one of the students asked about the rule of formation of a particular sequence. The teacher sent the question to the whole class and a debate of different arguments took place to reach the solution. At this moment, the motivation was high.

The first task was presented to the students; two illustrative sequences involving repetitive patterns were given and students were asked to extend them. Then students were challenged to complete two patterns of growth (see Figure 1). They start to develop work individually and then shared their resolutions with their pair, waiting for the whole group discussion. Only in the following sessions students were able to work together to reach a solution to the tasks. The first two questions, concerning patterns of repetition, were correctly solved by the students with no major difficulties; in the illustrative pattern of Figure 1 most of the students attended only to the numbers and, due to miscomputation, failed to extend the sequence. They extended the sequence only considering that the difference between consecutive terms was 3, presenting the sequence 1, 3, 6, 9, 12 e 15 as result, drawing the correspondent geometric picture, as given in Figure 3. When discussing their solving processes in whole group, they realize that should had attended to both, the numeric and the geometric sequence. The numeric sequence was easily accomplished by the students.

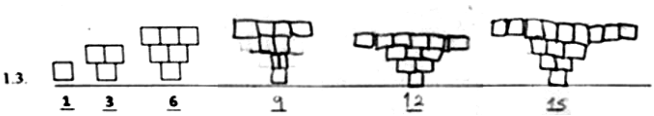


Figure 3. Incorrect resolution made by a student.

The second task (see Figure 2) was successful solve by students, presenting different justifications for the last question, arguing that 86 cannot belong to the sequence "because it ends in 6" or "because the sequence only has odd numbers and 86 is an even number". In the whole class discussion students start to discuss other numbers, such as 733, 200 or 2015. Students were enthusiastic about the tasks and when any resolution was not correct, they argue about it and correct it by their own.

The last task (see Figure 4) comprised three questions. The first two concern the extension of the sequence and was easily solved by students, in spite of some discussion about the rigour of the drawings of geometric shapes that emerged between them when sharing their answers.

Figure 4. Sequence of the last task presented in the first session.

In the second question, students were asked to identify the geometric shape of the fourteenth term. More than half of the student referred that have had used counting as a strategy to answer the question. Immediately the researcher asked "What about the 765th position?" provoking a general laugh in the class. Shortly Bill explained "We could use the multiplication table by 3 as the repetition group has 3 distinct pictures, and look for the number that multiplied by 3 is 765… or close to it.". In the last question, the students were asked "Which geometric figure is in the 19th position? And in the 88th position? Explain how do you know it?". This was the most difficult question of the task.

Students discussed a lot between them when solving this problem, presenting their arguments, trying to reach a solution. When sharing their resolutions with the whole class, the students presented their justifications. One student argued that "as the repetition group comprises 3 terms and the first is a triangle, I counted 1, 4, 7, 10, 13 and realised that these positions matched always a triangle, thus the picture of the 19th position must be a triangle.". Other argued that "thought of the multiplication table by 3 and realised that 3 times 6 is 18, and was a circle. As I wanted the 19th, had to consider one term more in the sequence, which is a triangle".

When finding the picture in the 88th position, another student shared his resolution with the whole class, and argued "the sequence has 3 terms and I know that 3 times 10 is 30 and that the circle was the last one. Thus, 3 times 20 is 60, 3 times 30 is 90 and, as I want the 88, I thought 3 times 29 that is 87. Then was just to add 1 term to get 88, which was a triangle.”.

Lesson 2

Two tasks were presented to students in this lesson. In the first task, the illustrative pattern of growth of Figure 5 was presented to students. The prompt question was “How many dots would have the 100th picture? How do you know it?”. Immediately students started to discuss ideas, sharing strategies, having to listen and understand each other ideas and argue about it.

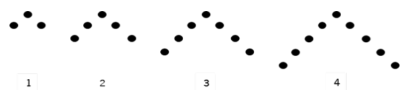


Figure 5. Sequence given to students in lesson 2.

Students realised that they need to find the law of formation to discover the 100th term, and several strategies emerged to achieve it. Explanations such as “the number of term is also the number of dots of the right side of the picture, ignoring the top one. The left side has 1 more dot than the right one.”, or “I multiplied the number of the term of the picture by two, added 1 unit and got the total number of dots of the picture. Thus, for 100 is 100 times 2 makes 200, and 200 plus 1 is 201.”, or even that “I saw that the number of a term plus the number of the next term equals the number of dots of that term. For instance, the number of dots of term 2 equals the number of that term, that is 2, plus the number of the next term, that is 3. That means that 2 + 3 = 5, which is the number of dots of the 2nd term. Thus, the number of dots of the 100th term is 100 + 101 = 201.”. One student with poor achievement in Maths explained also that “the 1st term has 2 dots on the left side, the 2nd has 3, the 3rd has 4, thus the 100th has 101; on the right side, the 1st has 1, the 2nd has 2, the 3rd has 3, so the 100 will have 100. All together will be 201”. Another student, based on the idea of symmetry of the pictures of each term, also explained “I realised that the term is the number of dots in each side of the picture ignoring the top one. So, the 100th term will have 100 dots in each side (100 + 100) plus the top one, making a total of 201 dots”. Figure 6 shows different students’ resolutions presented in the whiteboard to the whole class.

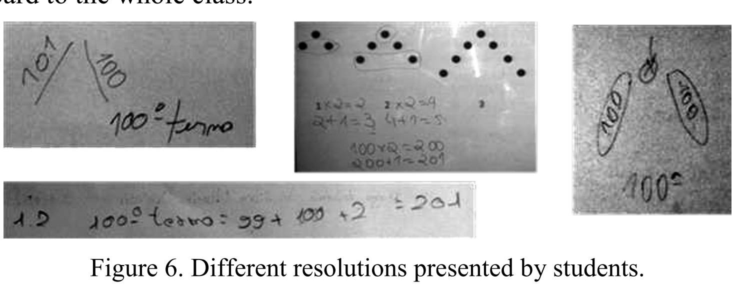


Figure 6. Different resolutions presented by students.

This task gave students an opportunity to realise that there was several correct strategies to solve the problem. Then the researcher asked students: “Is it possible to have a picture with 125 dots in this sequence? Where it will be in the sequence?”. After a short discussion, students realised that was possible because “all the pictures of the sequence have an odd number of dots”. Trying to find the position of the term, one student argued that “if we take 1 dot of the top, and add the two diagonals, is 124 dots. If we divide the 124 by 2, we will have get the number of dots of each diagonal”, using the word ‘diagonal’ referring to the laterals of the inverted V shaped made by the terms of the sequence.

In the second task, a geometric growth pattern was given (see Figure 7), and some prompt questions were presented “How many dark grey squares are there in the 30rd picture? And light-grey squares? How do you know that?

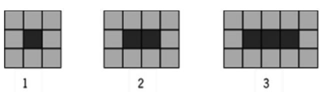


Figure 7. Geometric growth sequence presented to the students.

This was a more difficult task to students. They realised that was relevant to find out the law of formation, but found the job hard, and some difficulties concerning the mathematics communication become evident. The problem was solved by some students, but discussed only in lesson 3.

Lesson 3

This lesson started with last task of lesson 2. A debate on strategies used to find an answer emerged. One of the students explains that “In the first picture, there are 2 dark squares and above them there are a light square plus two more on the sides. Thus, if we imagine the 30rd picture, we would have 30 dark squares and above them we would add 2 more from the sides; that would make 32. We would do 32 times 2 because there are 32 squares on top and bottom, making 64. Then we would add 2 more for the sides, making a total of 66 squares.”. Students’ ability to communicate their ideas and arguments when solving the tasks improved since the first lesson, becoming more rigorous.

In another task, students were challenged to write the 15th term of the numeric sequence “1, 18, 3, 16, 5, 14, 7, …”, and also to find out if the 18th term was an odd or even number. The students realised that they would need to find the law of formation of this sequence, and soon some resolutions were presented by them to the whole class (see Figure 8).

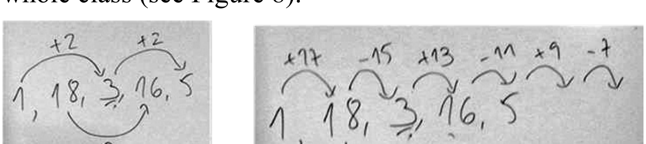


Figure 8. Students’ resolutions discussed with the whole class.

In the next task students were given a star with 5 vertices numbered from 1 to 5 following by the problem: "If we keep numbering the vertices using the same order, the number 6 will be in vertex 1, number 7 in vertex 2, number 8 in 3, and so on till 2007. In which vertex is the last number? Why do you think so?". After some discussion between them, some answers were given and arguments such as "I realised that 1 + 5 = 6, 2 + 5 = 7 and so on, and I found out that all numbers ended in 2 or 7 were in vertex 2. As 2007 end in 7, it must be also in vertex 2.". The last task of the lesson was the geometric growth pattern with equilateral triangles solved with no major difficulties by students, having presented their explanations.

FINAL REMARKS

Approaching patterns with freedom to explore relations and properties was possible due to the inquiry-based lessons implemented that allowed students to reach their solutions. The tasks revealed to be adequate to students knowledge and required the investigation of resolution processes and strategies. In the solving process, students had the opportunity to develop their ability to solve problems, their ability to reasoning and to communicate mathematically. Students established mathematical relations when dealing with the sequences, made generalizations, and presented and justified their resolutions. The inquiry-based lessons carried out gave them the opportunity to assume an investigative attitude, preserving the motivation, discussing their ideas among them, in small groups or whole class.

Regarding their difficulties, to have to search for terms of higher order in the sequences revealed to be a challenge for this students as they were not familiar with these tasks. Students also revealed difficulties concerning the mathematical communication, as they were not used to explain their reasoning or procedures because they were used to solve problems with only one resolution method and with only one possible solution.

Another remarkable point concerns students' attitudes when sharing their solutions. Initially, it was difficult for them to listen their colleagues and discuss their ideas; but by the end of the intervention, students easily embraced the tasks exploring patterns, trying to use different strategies, feeling proud in sharing them with the whole class.

This intervention allowed students to contact with relevant problems for the development of their algebraic thinking. Being able to explore pattern gave them the opportunity to make conjectures, formulate generalizations, establish mathematical relations and develop their ability to argue relying on valid justifications. This work offers the students the possibility to develop abilities related to their algebraic thinking that support their mathematical thinking, allowing them to go beyond their ability to compute.

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AREA MEASUREMENT TEACHING IN A GRADE 6 CLASSROOM

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In our paper we report on one part of a multi-stage teaching experiment conducted in 6th Grade, in which we dealt with the comparison of the areas of figures by visual estimation first, than by superimposing one onto the other, furthermore with the area measurement using a square grid. Our main conception is to apply general principles of measurement teaching for the area measurement.

INTRODUCTION

Examining the performance of Hungarian students of Grades 5-12 in connection with area measurement, we found many deficiencies and thinking failures (Herendiné-Kónya, 2014, 2015b). Students are often capable to solve only tasks which require the use of simple formulas, and we experienced the nonsense use of the learnt formulas. When determining for example the area of irregular 2-dimensional figures they are not able to use the additive property of the area, they do not see the parts in a compound figure, which area could be easily calculated. They have little knowledge and experience about the conservation of the area, that is, not only congruent shapes have the same area. In the light of this background, it seems reasonable to identify those teaching movements that trigger the explored problems and to design a teaching experiment that tries to avoid and exclude them.

In our paper we report on one part of a multi-stage teaching experiment conducted in 6th Grade, in which we dealt with the comparison of the areas of figures by visual estimation first, than by superimposing one onto the other, and with the area measurement using a square grid. The aim of this research is to analyse the teaching process, i.e. the teaching methods and cognitive characteristics of students.

TEORETHICAL FRAMEWORK

Research often reveals poor understanding of the processes used for area measurement of plane figures. Though it is generally accepted that mathematics should be taught through understanding but in the topic of area it would seem that children often rely on the use of formulae with little understanding of the mathematical concepts involved. Some studies point out that in order to determine the area, the measurement should be done indirectly: firstly measuring the lengths, secondly calculating the area (Nitabach & Lehrer, 1996; Tsamir, 2003; Murphy, 2010). In the initial phases of the concept formation it can be useful to consider area as a quantity independent from length. The study of Curry, Mitchelmore and Outhred (2006) analyses the measuring of length, area and volume independent from each other according to five measurement principles:

> … 1. the need for repeated units that do not change; 2. the appropriateness of a selected unit; 3. the need for the same unit to be used to compare two or more objects; 4. the relationship between the size of the unit and the number required to measure; 5. the structure of the repeated units. (Board of Studies NSW, as cited in Curry et al., 2006, p. 377).

Baturo and Nason (1996) describe the gist of measurement as continuous quantities being divided into equal discrete units and then counted. The measuring can be done in two ways: we take a unit and cover the whole quantity successively, or we take the required size unit and cover the whole quantity at once. Taking into consideration the nature of area measurement, only the latter approach can be used. The appropriate measurement tool is the transparent grid, especially the square grid. However we have to take into consideration that the investigation of different tessellation patterns using congruent tiles should precede the introduction of the grid as a measurement tool. In order to accelerate the process of tessellation the students are able to construct the grid from many square tiles spontaneously (Herendiné-Kónya, 2015a). What's more, Kamii and Kysh (2006) draw attention of the risk of counting squares in the process of area concept formation. Their experiment showed that "… for the 94% of the investigated students in Grade 8, squares were rigidly inviolable, discrete objects rather than objects that could be used as units that covered an area." (Kamii & Kish, 2006, p. 113).

The formulas of area calculation are introduced too early, long before a stable *concept image* (Tall & Vinner, 1981) would be formed in the students' minds. If the calculation rules are not linked to actual experiences, the knowledge of area is not effective (A. Baturo and R. Nason, 1996).

In understanding area measurement, area conservation has a crucial role, that is, the fact that the area of a figure doesn't change if the figure is cut up and a new figure is composed from the parts (Piaget, Inhelder & Seminska, 1960). According to Kordaki (2003), area conservation, area measurement and area formulas should be taught in an integrated way, in order to develop all three aspects. The study also shows that the type of the figures may have a role in recognising the area conservation.

Kospentaris, Spyrou and Lappas (2011) claimed that recognising area conservation could cause problems for secondary school students and even for first year university students. The idea that only the areas of congruent figures are equal is very strongly rooted. They investigated the positive and negative features of the visualization and discussed the role of the visualization in the area conservation and indirect comparison.

In Zacharos' paper (2006) we can read about the teaching practice of area measurement and the mistakes of concept formation. He saw the problem in the too early introduction of formulas, but also referred to the misunderstandings which roots from the wording. Unlike the word 'length', the 'area' is not used in the same way in our everyday life as in mathematics. Area is not only used to denote a measure, the quantity describing a plane figure, but very often by the word 'area' we mean the domain itself, and it can also occur that it means the multiplication of the width and length as it could be linked to the rectangular figure.

The cited studies confirm that for the formation of the area concept, measuring practice is needed independently from the length. So we applied the general steps of teaching measurement: the direct comparison of quantities without measuring; the need for repeated (standard or not standard) units; estimation; the relationship between the size of the unit and the number required to measure (reciprocity); choosing the appropriate unit for a concrete quantity. (Herendiné-Kónya, 2013).

RESEARCH QUESTION AND METHODOLOGY

The focus of our recent research is on the activities related to area estimation and conservation, furthermore area measurement using grid.

Research question

1. How is possible to realise certain activities related to the comparison, estimation and direct measurement of areas in a regular classroom environment?

2. What kind of typical mistakes make the students as well as the teacher in the observed teaching/learning process?

A teaching experiment was carried out among a group of 6th grade Hungarian students in December 2015. According to the discussion with the class teacher before the experimental teaching we considered the mathematical knowledge and skills of these students as average or a slightly below average. By the time, they have already learnt the methods of calculating the perimeter and area of rectangles and squares as well as the use of a few standard units like cm^2 and m^2. However the students haven't learnt the conversion between them and the area formulas connected to quadrilaterals or triangles. The teaching experiment was based on the curriculum framework topic *Measurement*. We designed the activities for regular classroom situation and paid special attention to being in line with the curriculum that is we kept to the required teaching time of the topic and didn't plan any extra lessons. In our opinion in this way the experimental activities could turn into an integral part of the teaching practice.

Four task were designed with different duration in time and were introduced in two consecutive 45 minutes classes. The experimental teaching material and the teaching aids were compiled by the author and the lessons were conducted by the class teacher in accordance with our guidelines. The tasks were done in groups, whereas setting the tasks and the discussion of the experience took place in the whole class. We applied cooperative teaching method, because it is suitable for these activity-based lessons: students have the opportunity to try, explain and control their ideas in a small group of their classmates. They were familiar with this way of learning; their teacher was expert in organization of students into groups and in supervision of the classroom work. The children divided themselves into 5 groups, one in which there were 2 members and in the rest there were 4. All the lessons were voice-recorded, notes and photos were taken.

DISCUSSION AND RESULT

On the first class we completed the first three tasks. On the second class we dealt with the fourth task then the teacher continued the teaching prescribed in her syllabus in a traditional way.

Task 1 – Recognizing that different shapes could have the same area by visual estimation at first and by equidecomposition later

We showed 8 figures with different shapes on the interactive whiteboard (Figure 1), and asked students to compare and order its area using visual estimation. The suggestions of the five groups were written on the blackboard.

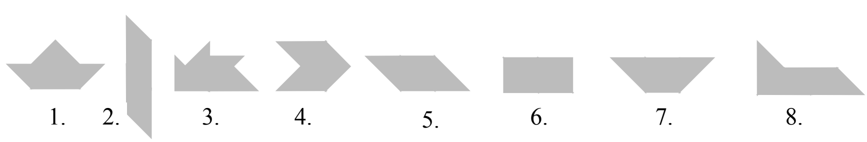


Figure 1. Figures which areas are equal.

As a next step we divided a square on the interactive whiteboard into 4 congruent triangles. The students came to the board and covered the figures using ‘drag and drop’ technique. After two attempts somebody already recognized that all of the figures can be covered with 4 triangles, but the classmates wanted to try the technological tool to control this statement (Figure 2). The interactive whiteboard seemed to be good tool for such kind of tangram-like activity, but we found that the emphasis was mainly on the use of technology and not on the essence of the activity itself. Nevertheless the students came up to the conclusion, that “It doesn’t matter that the figures have different shapes, their areas are equal.”

A class discussion was followed about the estimations of the order of the areas. It was clearly seen that the rectangle was at the beginning while the figures number 3 and 8 were at the end of the line.

1 T: Why did you think that the figures number 8 and 3 are the largest in area?

2 S1: Because they seem to be large.

3 S2: … because they have such interesting shape and we didn't know how large could they be.

The students listed several other reasons like "it's composed from many parts", "it has strange shape", "it hasn't a regular shape", "it's sharp", "it has more than 4 corners".

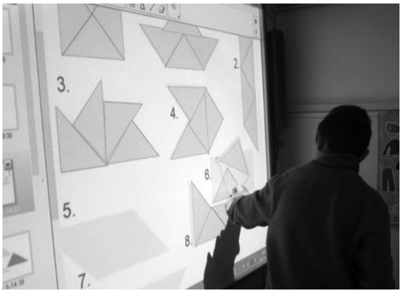
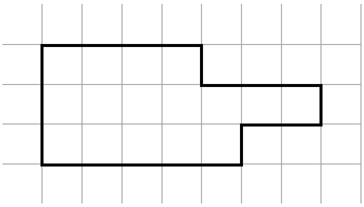


Figure 2. Covering the figures with congruent triangle.

Task 2 - Constructing different shapes of the same area on a square grid

A 4×4 square was given on a grid. Students had to draw 3 figures with different shapes but the same area as the square had. Some additional questions of pupils helped to understand the problem: "Should it be constructing from triangles?", "Is it allowed to use *half-cube*?" The students designed the figures together thereafter one of them explained their solutions to the whole class. Each group drew correct figures. All but one figure were rectilinear. We conclude from the way of students' justification that they are able to cut the figures into rectangles in their mind that is to say they applied the additive feature of the area unconsciously. For instance (Figure 3): "Here are … 4 times 4 … ehm…. 4 times 3 then a double vertically and the other horizontally." The multiplication rule according to the rectangle also came up (Figure 4): "Well, we made this long rectangle, here are 8 squares and here are 2." Only one group thought on the half-square and the right angled triangle as possible component.



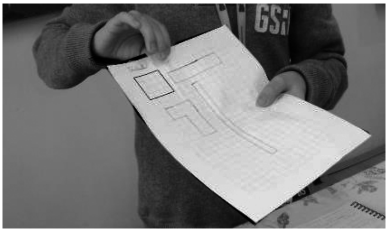


Figure 3. Rectilinear figure of area 16 □. Figure 4. Explanation of the constructions.

Task 3 – Comparing the areas of different shapes directly; then measuring these quantities using a centimetre grid as an instrument for measuring area

The figures were cut out from coloured paper and their areas, except of one, were whole numbers measured in the given unit squares (Figure 5).

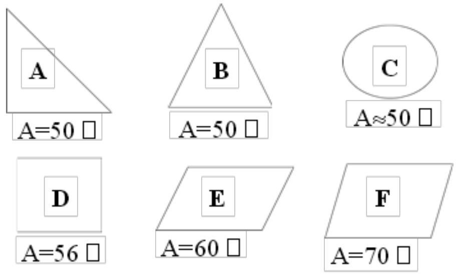


Figure 5. Six figures cut out from coloured paper.

The direct comparison of the areas of some pairs were easy (e.g. F>D), but some of them requires mental decomposition and visual comparison (e.g. E>B). One of the important experience gained from this activity is that most of the students didn't think that such kind of shapes like C has any area: "Does the *circle* have any area at all?!" or "We don't think so, … we don't believe that the *circle* has an area." One of the groups ignored the figure C when ordered them from the smallest to the largest area. When they started to measure the areas using grid, we observed the trouble with the ellipse again:

1	T:	Put the grid on the figure and count how many squares convert it.
2	S':	And what should we do with the *circle*?
3	T:	Figure out it approximately!

Table 1 shows the results of the measurement using a grid. We indicate the correct measure of the areas in the table's last row. Analysing the solutions of the groups it can be seen that

- the area of the rectangle (Figure D) is almost correct.
- the areas of the triangles and parallelograms are measured by counting the squares directly instead of converting to rectangles. They got for instance 48 or 59 instead of 50 or 60.
- the order of the areas after measuring was essentially correct in every case.
- the areas of every triangles and almost of every parallelograms were considered as smaller in area than the correct value.

	Figure A	Figure B	Figure C	Figure D	Figure E	Figure F
Group 1	49	50	53	56	59	69
Group 2	42	47	51	56	59	70
Group 3	48	48	42	56	55	60
Group 4	49	48		64	68	68
Group 5	45	43	43	56	53	64
Correct	50	50	≈50	56	60	70

Table 1: The values of the areas measured in unit squares.

We asked students how they counted the squares in the case of part-squares around the boundary. One of the answers: "I count in a way that I numerate first the whole squares after that the halves and if there are two halves, I count it as a whole. … and the *smaller half* with the larger one take together a whole." (She uses the word 'half' as synonym of the word 'part'). We recognised that all of the students thought in this way: mentally tried to combine part-squares to make whole squares.

Task 4 – Estimating relative areas of two irregular shapes after that measuring them using grid

By planning this task we took into consideration that the existence of the area of a shape with irregular outline caused trouble for many students. Our purpose was to give them further practice in the use of centimetre grid for measuring area, and in estimating relative areas. The students had to compare first, and then measure the area of two Hungarian counties on the map. (Figure 6). After that they searched the areas in km^2 on the internet and controlled their relative estimation.

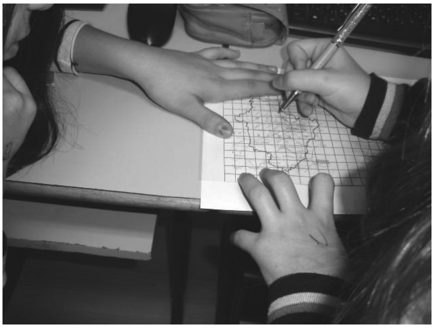


Figure 6: Practice in the use of the grid.

The results gained by counting the whole- and part-squares shown big differences, so a teacher-led discussion was initiated how to improve the accuracy in area measurement.

1	T:	How could we measure more precisely, ….., how could we determine even these small overhanging parts?
2	S1:	No way! It's not possible!
3	S2:	It's not possible with this (*he shows the grid*).
….		
4	S3:	Smaller grid!

After a relatively long time S3 gave the answer the teacher waited for, but the idea of the need for smaller measurement unit wasn't well elaborated in the class this time. (We should be noted that the teacher's question wasn't well thought-out.)

Traditional way of teaching

After the Task 4 the class teacher continued the 45 minutes class in a traditional way using the material prescribed in her syllabus. From this part of the lesson we pointed out two episodes: 1) making connection between the centimetre grid and the cm^2 as a standard unit; 2) calculation of the area of a rectangle by measuring the lengths of the sides.

1) After students established the size (1 cm) of one square on the grid the following discussion was detected:

1	T:	Draw a square, and denote its sides by *a*. We know that the length of the side *a* is 1 cm. How can we calculate the area of a square?
2	S':	*a* times *a* (*together*)
3	T:	Why is it possible that the area of a square is *a* times *a*?
4	S1:	Well, … because of the same size.
5	T:	The same size, good … What is the area of this square?
6	S1:	2.
7:	S2:	1 times 1.
8:	T:	And how many cm^2 is this? *(she ignores the wrong answer*)
9	S2:	1.
10	T:	1 cm^2.

The teacher emphasised the symbol *a* which often used in area calculation tasks and determined the area of the square applying the formula $a \times a$ and not the concept of cm^2 itself. The lines 3-5 show that the student S1 as well as the teacher thought only on the particular method and not on the concept of area in general.

2) The discussion below illustrates the conflict between the experimental and the traditional way of teaching:

1	T:	Measure width and length of your booklet using ruler, and then calculate the area. How can we calculate the area of a rectangle?
2	S1:	*a* times *b*

3	T:	And why is it *a* times *b*? Why isn't it else?
4	S1:	Because the pieces haven't the same size …
5	T:	Very well, because the two sides haven't the same size … ehm … and the areas are these small squares (*she shows on the board*)

It's clear that the teacher didn't have the intention of connecting the formula and the concept of area but she wanted only to choose the appropriate formula again ($a\times a$ or $a\times b$). Furthermore we can recognise the duality in the teacher's explanation as it is seen in line 5. It seemed that she understood the essential parts of the experimental activities and taught them, but she returned back very quickly to her traditional methods and tasks. So we can say the she actually 'put into brackets' i.e. neglected the experimental teaching material and confirmed 'the area is measured by ruler because it is length multiplied by width'.

CONCLUSION

Regarding our research question we summarize the experiences gained from the two lessons described above in the following points:

– The fact that a shape with a curved border also has an area was surprising most of the 6th graders.

– The students used the grid as a tool for measuring area in a right way and in the case of the part-squares around the boundary they combine the parts to make a whole squares.

– The statement that the accuracy in area measurement can be improved by using other grid with smaller squares requires more tasks. In this case the need for the introduction of different standard units would be better established.

– However the teacher understood the true concept of area as the number of unit squares which can be combined to cover the given figure, in particular teaching situations she reduced the area concept to a specific calculation method. We found that there are certain habits in teaching which are hard to give up or change.

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WORD PROBLEMS FOR THE PREPARATION OF THE FUNCTION CONCEPT[1]

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We present a teaching experiment that was conducted in the year before the function concept is introduced, namely in 6th grade, and that points towards the objectification of the function concept. For the objectification of the concept of covariant quantities and for the preparation of the function concept we use a model of sources of function, such as word problems and the table, the symbolic representation form in line with this. We discuss whether students are able to recognize covariant quantities presented in the form of real-life word problems and in what way can it be associated with the rule which describes the solution of a word problem if this rule is expressed with algebraic expressions.

INTRODUCTION

This paper reports findings from a stage of a longitudinal study. The purpose of this research is to study the preparation process of the function concept, to analyse its present teaching methods and cognitive characteristics of students as well as to plan and execution of teaching experiments in the interest of the effective foundation of the concept. The perspective taken here is that the study of functions should be treated longitudinally and in its full richness beginning in early elementary school (Keller & Hirsch, 1998).

In this study we will present a teaching experiment that was conducted in the year before the function concept is introduced, namely in 6th grade, and that points towards the objectification of the function concept. Here, we are focusing on the connection between covariant quantities presented in the form of real-life word problems as well as on the various representation forms of this connection, such as the table, the rule expressed in a narrative or algebraic form. The chosen educational environment, an Ukrainian (hereafter referred to as UA) 6th grade class. The students already met the idea of function machine and the table representation form of covariant quantities before the conduction of the teaching experiment which is being presented in this current paper. The aim of the three-hour long teaching experiment was to find an answer to the question whether students are able to recognize covariant quantities presented in the form of real-life word problems, and can use the representation forms highlighted above to define a connection between them. Besides these, we also wanted to find out in what way can it be associated with the rule that describes the solution of a word problem if this rule is expressed with algebraic symbols. That is to say, the UA curricula framework prescribes the process of familiarization of the expressions

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using literal symbols in 5^{th} grade, and lays the basis for the introduction of the function concept in 7^{th} grade. Results of our earlier research (Szanyi, 2015) suggest that having a practical understanding only of this form of representation is not sufficient when it comes to the foundation of the function concept.

THEORETICAL FRAMEWORK

In school, the introduction of a mathematical concept is preceded by long preparatory work. There are concepts that rely on students' spontaneously developing preliminary knowledge taken from everyday life. In cases like these, some particular examples are needed to facilitate the abstraction of the necessary mathematical concepts (Skemp, 1971). The concept of function is not like that. In this respect, there is no such informal basis and no analogical example on which we can build a new mathematical discourse (Nachlieli & Tabach, 2012). In agreement with the conclusions of the authors, we also believe that before introducing them to the function concept, students need to become familiar with the following forms of representation: graphs, tables of numbers, stories (word problems), rules, and the algebraic expressions (a rule described in a symbolic way). One of the conclusions of the paper quoted above is that before we point out that the function is one situation which could be represented in all of these five ways, students would need to gain some relevant experience of the acquaintance and practicing all of these representation forms. Therefore, it is obvious that – in a spiral manner – preparatory work should begin in the years that precede the introduction of the function concept, hence, typically before entering seventh grade.

Functions have many sources such as patterns, functions from real life phenomena, and function machines. These can be illustrated using concrete objects, diagrams, physical examples and solving word problems (Kwari, 2007). Some researchers such as Davindeko (1997) have experimented on the development of the function concept by starting with the modeling of real life situations. Van de Walle (2004) used meaningful contexts and their various representation forms in order to prepare the function concept. Findings of other related studies also confirm that discourses on real-world phenomena involving covariant quantities should be promoted in the concept preparation process, in conjunction with the discourses symbolic expressions and on tables. From each context the various aspects of the function concept (variables, relationships, rules) can be developed and from these a network of schemas will provide the basis for defining the function concept.

Rule recognition and rule following (hereafter referred to as RR and RF) skills (as express generalization) are closely related to the rule as one of the aspects of the function concept. The development of these skills can be started at a relatively early age, as early as elementary school age (Warren, Miller & Cooper, 2013). In agreement with the conclusions of Rivera (2013), "… the representational path toward the construction of generalizations proceeds from using concrete models, the numbers as quasi-variables, followed by words, and finally literal symbols and variables in algebra", we opted for models of real life situations, as one of the sources of function, that brought the recognition of covariant quantities into focus, in order of the preparation of the function concept and the development of RR and RF skills. Sierpinska write that "…the most fundamental conception of function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another" (as cited in Sierpinska, 1988, p. 572). Therefore the RR and RF skills are needed to figure out the relationship between quantities and recognize function-like relations. Finding rules that define the relationship between quantities can be used as an entry point to the development of the function concept.

A word problem is defined as any verbal description of problem situations wherein one or more questions are raised the answer to which can be obtained by application of mathematical operations to numerical data available … and wherein solver is required to give mathematical relationships between those quantities inferred from the text (Verschaffel et al, 2000). So, in the case if the rule lies in a word problem, the verbal description phase is also important and fundamental in the process of recognizing relationships between quantities and expressing the rule that define the relationship with algebraic expressions. Hence, it is recommended to start the development of generalization skills with the development of RR and RF skills as well as with the verbalization of rules. It provides us with the opportunity to facilitate "transitioning from a verbal expression to an algebraic rule" (as cited in Rivera, 2013, p. 76). To this we linked the table and symbolic representation forms of covariant quantities (the use of graphical representations has not been involved in this research material so far). Creating a table of covariant quantities involves generalizations "in terms of specific numbers and even to an example of any number before they can provide a generalization in language or symbols" or "thinking of numbers themselves as variables" (as cited in Rivera, 2013, p. 76.). Students produce a formula in order to express a general solution of a word problem, that is to say, to express a functional relation by recognizing a connection between covariant quantities arranged in a table.

TEACHING EXPERIMENT

Background

According to the above-mentioned studies, the concept of function can be developed as a co-variation relationship between variable quantities. The rules, patterns and laws are simply well defined relationships (Sierpinska, 1992). But analyzing the UA curriculum for the 5th and 6th grade, the RR and RF skills between development requirements are not mentioned. In our previous research we investigated the RR and RF skills of a group of 6th grade UA students

(Szanyi, 2015). Our result shows that at this stage of intellectual development the students are able to recognize and follow a rule in order to assign cohesive elements without any targeted development by the end of 6th grade. However, arguing in favor of a well-recognized rule either in a narrative form or with formulas remained a difficulty for them. According to these findings, we developed a teaching experiment. Our aim was to study the development potential of RR and RF skills.

The teaching experiment was conducted among another group of 6th grade UA students in December 2015. According to the results of an assessment that was conducted at the beginning of the school year, the mathematical knowledge of these students is average. By the time, they were already familiar with common and decimal fractions and had a clear understanding of the operations carried out with them. Already from first grade, students' attention was drawn to word problems that operate with concrete data and have only one solution. They are also aware of the process of solving word problems with the use of equations.

Current research was preceded by a teaching experiment, during which students had the opportunity of getting familiar with the idea of the function machine and the table, formula representation forms of covariant quantities. Followed by the teaching experiment, we used different word-problem contexts in order to facilitate the development of the recognition of covariant quantities, a connection between them, and also to facilitate the representation of this connection in several different ways. With this task we also aimed at having students recognize and express a connection in different ways not only between element matches expressed explicitly but also in the case of covariant quantities in word-problems pointing towards functions, which would let them solve the task by expressing the solution algebraically as well. The teaching experiment was based on the curriculum framework topic *Proportion, direct proportionately*. Three activities were designed with different duration in time and were introduced in 45 minutes classes. During the classes, students were either instructed by the teacher or had to work individually. In case of individual work, they recorded their responses in their exercise book. Notes, voice recording and photos were made about all lessons.

Descriptions of the teaching experiment and results

Lesson 1 (duration: 15 minutes). The aim of this lesson was the probable recognition of covariant quantities expressed in a narrative form, the connection between them, also the development of the various representation forms of a recognized rule defining this connection (table form, algebraic expression).

Each of the 20 students recognized the two quantities of the task (Figure 1). They managed to make correct calculations, i.e. computed the amount to be paid for the quantity of apples expressed explicitly; also they made an informal table of values (Figure 2). Creating a table of values based on the element pairs remained a difficulty for 8 students, only 12 succeeded. These 8 students either have skipped this part or created a table (2 students) where they assigned the price per kilogram to the given values (3, 5, 10 kilograms) (Figure 3).

Task: The price of 1kg of apples is 15 hryvnia. What is the price of a) 3 kg of apples, b) 5 kg of apples, c) 10 kg of apples? Answer the following questions based on your solutions: 1) What are the two quantities being compared? 2) Arrange quantities in a table based on the element pairs of the task! 3) Verbalize the steps of calculating how much a certain amount of apple costs?

Figure 1

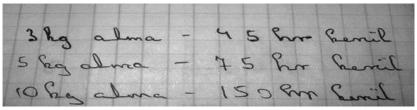

Figure 2

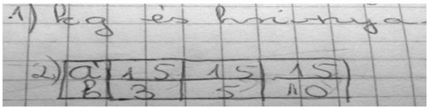

Figure 3

14 of the 20 students managed to answer Question 3 correctly („we have to multiply certain kilogram of apples by the price of 1 kilogram”). 2 of these students introduced a variable for indicating „certain kilogram” („we have to multiply 15 hryvnia by *a* kilogram that is the purchased amount of apples”). One student used the given data in order to provide an answer („we have to multiply 15 hryvnia by 3, 5, 10”). This student might have found it problematic to generalize the rule and express it algebraically; he could imagine the situation only by focusing on concrete data.

Individual work was followed by a discussion. Involving the students this time, we solved each task step by step on the board. Covariant quantities represented in a table form were put on the board as well. We filled the first row with values referring to the purchased amount of apples (3, 5, 10), and the second row with the price to be paid for them (15, 75, 150). After this, we tried to induct students to the generalization of the task, i.e. by using a formula, as described below:

1 T: What do the numbers show in the first row of the table?

2 S1: How many kilos of apples we have purchased.

3 S2: Or the purchased amount.

We extended the table with an extra column of which first row we filled in with the answers deriving from students.

4 T: And what do the numbers show in the second row of the table?

5 S3: They show how much we pay if we buy 3,5,10 kilograms of apples.

We filled the second row of the previously added column with “paid amount”.

6 T: How could we write something else instead of the “mass of apples”?

7 S4: Kg.

8 T: We have discussed earlier that mathematicians like to abbreviate things. Is there a way to mark it even shorter?

We received even more responses (a, x, k), then we agreed on indicating the purchased amount with x, and by adding another new column to our table, we filled its first row with "x". We asked students again:

9 T: Knowing that 1 kg of apples costs 15 hryvnia, if Peti buys *x* kg, how much does he have to pay?

We chose one of the students to reply that has provided an incorrect solution when working individually. The student was thinking, so we put the question again which was answered uncertainly: "$x \cdot 15$?" Therefore we explained the problem once again and what is the *x* appoint. Then the student's answer was entered in the second row of the new column. After this we discussed that this response is the amount to be paid and we marked it with letter "*y*". Then we wrote the general solution of the problem with an algebraic expression ("$y=x \cdot 15$").

10 T: Knowing these, shall we give it a try to verbalize what this abbreviation means?

We asked those students to respond who had skipped Question 3 of the task. With some guidance (we revived what letters *x* and *y* meant), they managed to verbalize a rule: "...if we buy some apples, we have to multiply the amount by 15 so we will know how much do we have to pay".

Lesson 2 (duration: 25 minutes). In contrast to the previously presented task, this one included more input data and thus more than one possible solutions as well that could be concluded only from the context. Hence, in order to solve this task, correct text interpretation was crucial (Figure 4).

Task: Peti was talking on the phone with one of his friends in a 5-minute break. The duration of their conversation is measured in whole minutes. 1 minute costs 0,5 hryvnia. How much could the conversation cost to Peti, if the connection fee was 1 hryvnia? 1) Arrange covariant quantities (element matches) in a table. 2) Express it a) with words, b) using the language of mathematics (with a mathematical expression), how much Peti had to pay if the duration of the conversation was *x* minutes?

Figure 4

Having regard to the fact that this is rather a complex task, we could forecast that solving this task could be problematic for many of the students. Thus we did not expect them to solve this task individually. We left some time for them (5 minutes) to read and interpret (everyone was trying to solve it). While observing students working, we noticed a student who has managed to complete this task correctly during this time. In case of the others, 2 solution categories could be defined: (1) calculated charges only for 1 and 5-minute phone call, ignoring connection fee; (2) calculated charges only for 1 and 5-minute phone call with connection fee included. Three of the students could answer each of the questions of Question 2 correctly. They probably had difficulties with the recognition of input data.

In order to initiate the recognition of keywords in context thus having students interpret every single word accurately, we enacted the situation. During the discussion, the role-play proved to be effective and students found out that Peti could have talked for 1, 2, 3, 4 or 5 minutes. As we received more and more responses, incoming quantities have been put on the board, serving as a preparation for representing covariant quantities in a table form. After this, we asked them which word they considered to be a reference to this from the text. They read the task once again and pointed out the expression „could cost" correctly as it can refer to more than one correct solution. Following this, we calculated together how much Peti paid for each and every minute and filled the appropriate column ("1·0,5+1") of the table with it. We added extra columns to the table as we did it earlier as well (Figure 1). We filled one of the columns with words, referring to what numbers in the first row mark („minutes") and what numbers in the second row mark („amount to be paid"). Firstly, we discussed how we used concrete data to calculate the amount to be paid for each and every minute („we multiply the minute in question by 0,5 followed by adding the connection fee"). After that, we asked students: „…knowing that we have to multiply the minute in question by 0,5 then add 1 hryvnia as the price of connection fee to it, how much did Peti pay for talking for x minutes long?" The table containing concrete data made it possible for the students to be able to generalize: "$x \cdot 0,5+1$". We discussed that with this act we used the language of mathematics to describe how much Peti paid for x minutes. Following this, we introduced a variable y in order to indicate the amount to be paid and also verbalized the general solution of the task („I multiplied the amount of minutes by 0,5 then added the connection fee, that's how much Peti had to pay"; we multiply the amount of minutes he uses for talking by 0,5 and +1 hryvnia").

Lesson 3 (duration: 35 minutes). Students were required to assign different algebraic expressions to given word problems of which solutions the formulas described. In the worksheet there were seven word-problems and four algebraic expressions included. The four algebraic expressions were applicable to describe six of the seven word-problems. Students were required to match the expressions to the tasks. Concrete input data were not included in the tasks, we used an x to indicate them in the text of word-problems, and we also used it in the case of algebraic expressions (Figure 5). In the followings, we present two tasks that were included in the worksheet.

Every one of the 20 students managed to match the task (Figure 5) to the correct formula describing it ($y=36 \cdot x$). However, there were only 11 students out of the 20 who noticed that this formula can be applicable to the other task as well (Figure 6). Students recognizing a rule more effectively in context as presented above can be explained by the fact that it is easier for them to interpret a word-problem if these tasks contain words of which meanings imply the necessary arithmetic operation to be done (e.g. 36 times as many) (same conclusion in the case of task Figure 1).

A fruit basket contains *x* apples, and *36* times as many pears as apples. How many pears are there in the basket?	There are *36* students in a class. Students get the same Christmas present. Each present contains *x* candies. Every one of the students gets a present. How many candies does the class have in total?

Figure 5 Figure 6

The two types of tasks demonstrated by Figure 7 and Figure 8 were introduced to students in the second experimental lesson (see Figure 4, Question 2). There, we used concrete input data to solve those tasks and also managed to build up an expression that described the general solution. In contrary to this, students had to choose one of 4 algebraic expressions that could be applicable for generalization.

An empty container is being filled with water from a tap at a rate of *2* liters per second. How many liters of water will the container hold after *x* seconds, if it contained *36* liters of water before opening the tap?	Zsuzsa bought a rose bouquet for her mother. There were *x* roses in the bouquet altogether. The price of 1 rose was *2* hryvnia; the price of packaging was *36* hryvnia. How much did the bouquet cost to Zsuzsa?

Figure 7 Figure 8

Both tasks required recognition of the same two-step rule occurring in different contexts. Regarding these different contexts, the recognition of the appropriate algebraic expression could be considered more effective in the case of the task demonstrated in Figure 7. There were 16 students who managed to match this task to the appropriate algebraic expression $y=2\cdot x+36$, while in the case of Figure 8, there were only 13 students who could accomplish. Considering our findings we can conclude that in order to be able to solve tasks like these, not only the context but the structure of the text is also an impact factor.

In order to enhance the effects of activities involved in the second experimental class, that is to say, facilitating the interpretation of the connection between context and a table by the context was represented. We prepared a table of data set of the task (Figure 9) in advance. Students were required to fill the blank cells of the table according to the various values of a variable (x). The purpose of this activity was to find out whether students are able to interpret the table prepared in accordance with the conditions of the task and also to see if they could fill the blank cells despite the varying values of the variable. We also requested students to make an attempt to solve this task based upon the value of a concrete variable ($x=11$) that was not a part of the table. Most of the students could fill in the second row of the table correctly, also could define the amount to be paid for a rose. However, defining the price in the case of other values remained a difficulty for them. These pupils summed the numbers in each column. By this time, they have probably neglected the previously defined condition and they summed these numbers automatically, paying no attention to what each row indicated. Three of the students performed less effectively when they made their calculations based on the table instead of concrete data that derived from the number of roses. Others, despite seemed to move in the right direction when attempted to solve the task and made their calculations with a concrete variable (11), computed only the price of roses ignoring the price of packaging.

Roses	1	2	3	4	5
Price of	2				
Price of	36	36	36	36	36
Amount					

Figure 9

CONCLUSION

The above presented tasks aiming for the preparation of the function concept can be well adjusted to the mentality of examined students at this age. As a source of function, word-problems facilitating the development of rule recognition and rule following skills can be considered to be in line with the level of mental maturity of students, along with the discussed representation forms such as tables and algebraic expressions. However, the detailed problem solving process and generalization of real-life word problems (as presented above) could serve as a satisfactory base only for the half of the students, to be able to abstract from the model and to recognize algebraic expressions that can serve as a general solution of similar tasks and also to interpret a table representation form of a certain real-life situation along with a condition of a task. Other impact factors are the presence or absence of words implying necessary arithmetic operations to be performed in order to generalize (or make a rule) of word problems, these types of words have an influence on rule recognition. The recognition of covariant quantities and the recognition of a connection between them present some difficulty as well, if these are referred only by “keywords” lying in context, that is to say, the success of the recognition depends on the context and the structure of the text as well.

task containing one input data item	→	more input data items	→	more input data items ”hiding” in text	→	arrangement of values in a table	→	generalization (introducing a variable)

Figure 10. Learning-teaching trajectory with the application of word-problems

Students' performance allows us to conclude that the table form of representation of covariant quantities presented as text and the generalization of a task by using algebraic expressions can be developed within a short time with the help of tasks that can be described with a one-step rule. However, in the case of two-step rule making process, it requires a longer, continuous development process. Hence, considering the findings of our experiment, it is recommended to carry out the learning-teaching trajectory with the application of different types of word-problems that can be described with one- and two-step rules and that aim at the preparation of the function concept, as shows the Figure 10.

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SCHOOL OLYMPIADS ON PROBABILITY THEORY

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The main goal of the paper is to represent intellectual competitions as an instrument for the creation of a functional core for teaching probability theory and statistics in the junior, middle and high school. In Russia, the probability theory and statistics were introduced into teaching in schools in 2004, as it became required by the Federal Educational Standards. Since then teaching of the subject meets usual difficulties that accompany the process, starting with lack of teachers' experience, absence of traditions for teaching of statistics and probability, and following with difficulties with understanding the content of the course from teachers' side, troubles with wrong interpretation of the objectives, and necessity to teach statistics in schools differently and not as a simplified university course. Here I consider one important additional aspect of the teaching of the probability theory and statistics in Russian schools: the Olympiad on the probabilities theory and statistics for students of 6–11 school years, held by the Moscow Center for Continuous Mathematical Education since 2008. In 2015, the Olympiad consisted of two rounds. In 2016, it was decided the first time that the Olympiad will have three rounds: an invitational round (school round), an individual round and the final intramural round. The Olympiad's final round had been held on February 21, 2016. Additionally to a usual contest in solving problems the participants of the final round were offered a possibility to write a plan for a statistical experiment on subjects given by organizers of the competition. We provide here an overview of selected Olympiad problems and tasks that in our opinion reflect interesting and important tendencies of school teaching for the statistics and probability theory and reveal perspective and promising methods for realization of the main objectives of the modern mathematical education.

INTRODUCTION

In 2004, elements of probability theory and statistics were introduced into the school mathematics curricula in Russia in accordance with the educational standards. Since 2012, problems on probability and data representation are included in the Unified state exam in mathematics. The Concept of Development for Mathematical Education in Russia, approved by the Government in December 2013, states that the probability theory and statistics are important and perspective sections of school mathematics. In 2015, the Federal Exemplary Curricula were developed, where the probability and statistics appear as complete sections that determine content for each educational level (see Appendix).

Meanwhile, the educational practice causes serious difficulties inescapable for the formation of a new school subject totally different from traditional courses that are being taught in universities. Difficulties in the preparation of teachers follow.

This paper focuses on only one of popularization dimensions for activity on formation of a public inquiry in the field of mathematics, namely, on methodology and practice of intellectual competitions for schoolchildren on probability theory and statistics by considering the example of the Olympiad held by the Moscow Center for Continuous Mathematical Education (MCCME) since 2008.

The Olympiad Organizing Committee is ready to cooperate with colleagues from all countries, in particular, we can provide English versions of all materials.

BACKGROUND

We often hear from mathematicians that the probability theory and statistics are too difficult, and, therefore, should not be taught in school. This opinion grows out of complications that follow studying probability theory in universities, which traditionally is deductive and based on combinatorics and wide knowledge of calculus. A 'combinatorial approach' to the probability theory is typical for many generations and grows from the Soviet period when the probability theory was torn out of statistics upon being announced as 'a social science'. In fact, combinatorics is not directly related to base ideas of statistics and the probability[1]. It's just a way to enumerate elements of vast probabilistic spaces and prove theorems. Experience and intuition are primary, and no combinatorial tricks are useful without them. One should meaningfully consider chances of events, especially unlikely events that play a significant role in daily life.

The problem is that events are less obvious than figures or numbers while concepts of chance and volatility are not as intuitive as length, area or volume. An event and its chances make special types of imaginary objects and their formalization into mathematical notions is much more complex than a formalization of a picture (transition to geometry) or a quantity (to arithmetic or algebra).

The second problem is that the majority of children until a certain age is alien to the concept of volatility and instability of events. At what age a child is ready to perceive changeable models and which models should they be, is still to discover. In the Soviet period the science of the laws of cognitive activity was

[1] Saying about the probability theory in school we often omit 'theory' to simplify the text and to follow the tradition that is being formed in Russian educational terminology.

destroyed (Petrovsky, 1991). Its place was taken by the paradigm of ‘a clear sheet of paper’.

But if in the early childhood the rejection of variability possibly serves as a defensive mechanism which simplifies adaptation to social and natural conditions and minimizes the number of necessary rules of behavior, then later an absence of general ideas of randomness and volatility start to hinder in life. Study of statistics in adulthood does not improve this situation (Kakihana, Watanabe, 2013).

The first thing to take care of is to make basic concepts of probability and statistics clear and familiar to math teachers who have difficulties in the transition from teaching abstract facts of math to the application of mathematical concepts and laws to solving practical problems.

In addition, one of the most important aspects of education is the popularization of knowledge. If algebra, geometry, and other sections of traditional school mathematics show no shortage of additional, scientific-popular and popular literature for adults and children of all ages, the analysis of the situation in the area of probability theory shows a distinct lack of a popular literature and other forms of popularization (Bunimovich, Bulychev, Vysotskiy et al., 2009; Bunimovich, Vysotskiy et al., 2009). Some number of popular books for school children was published in the Soviet Union, mainly in the 50s–80s of last century (see, for example, (Kolmogorov, Zhurbenko, Prokhorov, 1982; Mosteller, 1985). The number of new materials appearing in Russia is vanishingly small, even if we take into account translations of foreign publications on probability and statistics for students (eg, (Chjun, Ait-Sahlia, 2007)). At the same time, the amount of popularization literature on statistics and probability theory in the world increases. This is partly due to the increasing importance of teaching probability and statistic at school, partly because of the growth of the role of stochastic methods in different sectors of the global economy (Bunimovich, Bulychev, Vysotskiy et al., 2009). Popularization activities should be carried out in various forms. In addition to special literature and mathematics circles, Olympiads and other intellectual competitions of different levels for children and adults are of great value. Recent years show a spontaneous increase in number and quality of Internet sites dedicated to popular mathematics, in particular in the field of probability theory and statistics.

THE OLYMPIAD, ITS MAIN PRINCIPLES AND DESCRIPTION

The first Olympiad was held by MCCME on the initiative of Yuri Tyurin, Alexei Makarov, and Ivan Yaschenko. In different years, many mathematicians and educators (E.Bunimovich, V.Bulychev, P.Semenov et al) participated in selection and preparation of Olympiad problems. The rules of the Olympiad are simple and requirements for participation are common. The Olympiad is open to everyone and is held during a calendar month. Olympiad problems are designed for students of 6–11 grades.

Olympiad runs in two rounds. First invitational round goes in schools. In 2015, the number of participants was 2450, in 2016 – 2890. In 2016 an intramural final round was added, that consisted of two contests. The first was a competition on a statistical experiment, the second was solving problems.

The Olympiad contains problems recommended for students starting from a certain class. For example, a task that can be solved using only intuitive ideas, finite enumerations and classical probability definitions is recommended for students from 6th grade. If for a solution one needs to use simple transformations within the algebra of events, the task is recommended for students in grade 7 or older. If the problem involves the characteristics of random variables, such task we offer to students from 8th grade, etc. But the 'age of a problem' is not limited from above. Age differentiation occurs at the stage of grading solutions and awarding the winners.

Olympiad problems are placed on the website for free access for about a month. According to the rules of the Olympiad, participants can use any help, reference books, etc. Only 'an open hiring of adult labor' is considered non-sportive and ugly.

Materials of previous years are published on the Olympiad website in the Archive section on http://ptlab.mccme.ru/olympiad.

Olympiads of 2008 – 2011 years are published in the book (Vysotskiy, Zakharov, Nesterova, Yashchenko, 2011). Articles dedicated to this competition are published in 'Mathematics' journal (PH 'September First') (Vysotskiy, 2012; Vysotskiy, Borodkina et al., 2009).

ESSAY TASKS

A distinctive feature of the Olympiad is that it includes essay tasks besides problems. The essays are evaluated separately, regardless of the age. This turned out to be important.

Participants are required to analyze the offered situation and to write a short essay on a given topic. They are being immersed into an uncertain situation that requires imagination and activities on the estimation that takes into account real limitations and the nature of the data. Actions in an uncertain situation play a crucial role in the formation of a statistical and mathematical culture of students, because instead of performing the steps of the known or studied algorithm, an Olympiad participant enters the role of a researcher who plans an experiment by himself. A student has to determine important and unimportant factors of a random experiment, interpret the results, invent a method for describing the data and hypothesis. None of the proposed essays requires students to check the formulated hypothesis, as the mathematical tools available to school students are clearly insufficient for that.

In some situations, the creation of a hypothesis itself is a very complicated task that requires from a student a high culture of regulatory activity. Moreover, situations often arise where the number of possible plausible hypotheses is large. In these cases, the authors try to repose the situation in such way that either it contains a hypothesis formulated explicitly or the task implicitly directs the student actions (see two examples below). The most important part of the essay task for a student is to search for data aiming to find a pattern or to check the data on correspondence with certain assumptions. Some tasks require students to collect data independently through short surveys. The others demand an independent search on the Internet.

Some essay tasks are designed to generate critical scientific thinking in students. A student is being immersed in a situation where it is known that provided arguments contain an inaccuracy of some kind, an error or an unreasonable conclusion. Students are invited to understand the shortcomings of the study and make research steps. I must say that the tasks of this type appeared the most attractive to the participants. Among the submitted essays there are very bright and original works, which we placed on the Olympiad website in the "Solutions" section.

Below we provide several essay tasks from previous years.

All essays can be classified by the educational objectives and methods of doing.

1. *Checking of a statement* using either independently collected data or raw data provided in the essay task. In this case, usually, the essay offers to an Olympiad participant a partially proposed algorithm of actions.

2. *A study that may require a survey* among student's classmates, friends, parents, and a consequent data processing with a formulation of a plausible hypothesis or a refutation of an implausible one. The student's abilities to organize collected data, present it in the most appropriate way and produce hypotheses are important here.

3. *A search of a statistical method* for solving the problem (see essays 1, 2). That is aimed at developing constructive thinking and skills. An unbounded search appears very attractive to students. An essay about the estimation of a number of people in a crowd was the most frequently selected among participants of all age groups in 2013. None of them used a method that the authors of the problem considered the most natural, but we got many original ideas instead.

4. *A search of an error* in a complex and extensive reasoning presented in the formulation of the task (see, e.g., essay 6). Solving tasks of this type helps to develop a critically-destructive way of thinking, which is an integral part of intellectual culture. Speaking specifically on the implementation of an essay

about the relationship between air humidity and levels of snowfall, we note that many students succeeded to notice flaws in the arguments of the author, but the main problem – the unsuitability of the linear regression for the case – was not mentioned by any of them.

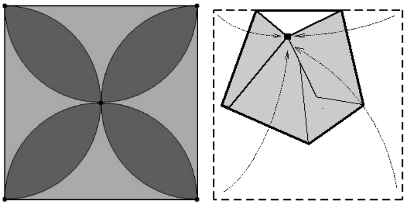
This shows that the basic ideas of the statistics stay on the border between intuitive and conscious.

Examples of essay tasks

1. A number of people in a crowd (2012). The photo shows a crowd of people. How one can estimate (approximately calculate) the number of people in this crowd? Try to develop an appropriate method, use it and make such estimation. Describe your method in all details, explain why it works correctly, how to use it and what did you get using it. We are looking forward to your answer – how many people are there in the photo?

2. Which way is faster (2014)? Is it true that an aircraft spends different time when it flies east or west? Is it always so? Go to a site of a large airport or a major airline, it is even better if you'll consider several airlines or airports. Select flights from east to west and vice versa. Collect and process the necessary information. How different are the durations of flights in one way and the other? Does it depend on the distance? You need to come up with a statistical measure that describes the difference. Is it stable? The difficulty is that a mere difference between the time there and the time back does not give us much: we have to take into account not only very long flights but also relatively short ones. If such difference does exist, by what can it be caused? Is it possible to estimate consistency and power of this amazing factor? Are east and west really guilty? Maybe a similar pattern can be observed with other flights, for example, from north to south and back? A lot of questions arise. Try to locate, describe, analyze data, and use your imagination.

3. Haga's problem (2011). Professor Kazuo Haga from the University of Tsukuba is the inventor of origamics (geometrical origami). Once he posed an interesting question. A paper square is divided into light and dark parts by four semicircles (the left figure). Obtained graceful ornament resembles a flower.

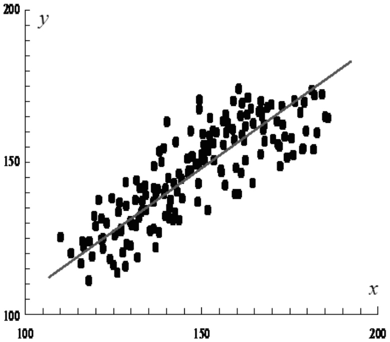


If we choose a point in a light part and then make folds consequently so that all vertices meet this point then we get a pentagon (right). So the union of light parts is ‘a domain of pentagons’ or simply ‘5-domain’. The dark parts make the 6-domain. The four vertices and the center of the square give quadrangles. Professor Haga writes: ‘I noticed that when asked to choose a random point the majority of people mark a point leading to a hexagon. Pentagons are much rarer. Very few choose points making quadrangles. The question is: if the number of those who choose a point in a certain domain is proportional to the domain’s area?

If sides of the square are 1 then each semicircle has radius 0.5 and its area is $\pi/8$. Therefore, the area of the petals (6-domain) equals to $\pi/2-1\approx 0.57$ whilst all the rest (5-domain) has the area some about 0.43. The difference isn’t too big. The ratio of those who choose points in the corresponding domains doesn’t fit the ratio of the areas. What is the reason for this – why points outside the petals are less attractive than points inside?

Conduct an experiment. You’ll need some tens of paper squares. Ask as many people as possible to mark a point on the clear square. If all points are put together on a new square we’ll get the distribution of the point. Maybe some properties of the distribution will help to explain why ‘hexagon admirers’ appear oftener than Professor Haga could anticipate having compared the domain areas.

4. Height correlation (2009). Once a teacher decided to show her students, that height of boys and girls are independent random values. For this, she made a research. In every class, she chose 10 girls and 10 boys at random, then composed random pairs ‘boy-girl’ and wrote down their heights x_k and y_k for every pair. When done and put all results on the scattering chart she found to her horror that all points are grouped around a slant line. This means that there is an obvious correlation between height of boys and height of girls! How could it be?



Write a short essay, in which try to explain whether the teacher made a mistake in her findings and if yes then what her likely misjudging is.

5. The insurance (2010). The insurable value of a car depends on its age. Agents of insurance company ABC estimate the write-down in very simple way – cars older than two years lose in price 10% yearly. Using http://www.auto.ru and other available sources, conduct a research on the topic whether the price policy of the company ABC corresponds to the practice that was formed in the market of used cars? When working take into account that amidst of cars there are tuned or exclusively equipped vehicles whose price doesn't meet the average for the cars of the same model and age.

PROBLEMS THAT REQUIRE A SOLUTION

In addition to three essays, the Olympiad traditionally includes 16 problems in statistics and probability theory. Some problems are easy and admit a very simple solution by brute force or a short reasoning. More complex problems in addition to looking for a key to a problem require from students the ability to perform operations on events and some knowledge of probability properties. Finally, there are some really complex problems that surrender only to those who devote enough time to thought and attempts, who study the literature and the problems from previous years. Authors deliberately include into the Olympiad special problems whose solving requires complex transformations. Such problems are few, but they must be in an online competition as the participation in such implies a scientific research made by a participant. A few examples are listed below.

Examples of problems

1. Defective coins (from 6th grade). For the anniversary of Saint-Petersburg mathematical Olympiads, the Mint produced three commemorative coins. One coin is made correctly, the second coin got two tails on both sides and the third coin has two heads on both sides. The Director of the Mint had chosen one of these coins without looking and tossed it. He got a tail. Find the probability that the second side of this coin also is a tail.

2. Three targets. A shooter fires on three targets as many times as he needs to hit all three. The probability to hit for one shot is p.

a) **(from 7th grade, 2 points).** Find the probability that the shooter will fire exactly five times.

b) **(from 8th grade, 2 points).** Find the expected value for the number of shots.

3. Intersecting diagonals (from 9th grade, 3 points). In a convex polygon with the odd number of vertices equal to $2n+1$, two random diagonals are chosen independently. Find the probability that these diagonals intersect inside the polygon.

4. Draws (from 9th grade, 6 points). Two hockey teams of equal strength have agreed that they will play until the total score will reach 10. Find the expected value for the number of the moments when a draw happened.

5. Stunning news. A conference is attended by 18 scientists, of whom 10 know some stunning news. During a coffee break, all scientists are randomly divided into pairs and in each pair a scientist who knows the news tells it to the other if the other did not know it yet.

a) **(from 9th grade, 1 points).** Find the probability that after the coffee break the number of scientists who know the news will be equal to 13.

b) **(from 10th grade, 4 points)**. Find the probability that after the coffee break the numbers of scientists who know the news will be equal to 14.

c) **(from 9th grade, 3 points).** Denote by X the number of scientists who will know stunning news after the coffee break. Find the expected value of X.

6. Mini-tetris. A tall rectangle of width 2 is open at the top, and randomly oriented L-shaped triminos fall into it.

a) **(from 9th grade, 3 points).** Let k trimino fall into the rectangle. Find the expected value of the height of the resulting polygon.

b) **(from 10th grade, 6 points).** Let 7 trimino fell. Find the probability that the resulting figure will have height 12.

7. A complicated case (from 10th grade, 4 points). While investigating a case, the police inspector Smart found that a key witness is the one of Malachowsky family who on that fateful day came home earlier than others. An investigation revealed the following facts.

1. A neighbor old pani Renata wanted to borrow salt, buzzed and knocked at Malachowsky's door, but no one was home. What time was it? Who knows? It was dark already…

2. Pani Iwona Malachowskaya came home in the evening and found both children in the kitchen and husband Adam on the sofa having a headache.

3. Pan Adam Malachowskiy stated that as he got home he immediately laid down on the sofa and fell asleep, saw no one, heard nothing, the neighbor did not come for sure, as a doorbell would wake him up.

4. The daughter Malgorzata witnessed that when she returned home she immediately went to her room, does not know anything about the father, but in the hallway, as always, tripped over Tomek's shoe.

5. The son Tomek does not remember when he came home, as soon as came he got to his room, had not seen father, but heard Malgorzata screaming about the shoe.

– Well, – thought the inspector. – That's so mysterious… What is the probability that Tomek returned home before his father?

THE STATISTICAL EXPERIMENT CONTEST

In 2016, the Olympiad first time had three rounds. On the final (intramural) round participants were offered a topic for the development of a statistical experiment. Organizers took a well-known scheme of degustation tests as a base for it. All participants were offered to design an experiment aimed to reveal the threshold sensitivity to sweetness (using a weak aqueous solution of sugar). All participants were asked to remember the following.

1. Hygiene. It is unacceptable to offer several people to try water from one cup.

2. Various effects that distort the result are possible. For example, a person may be less sensitive to a less saturated sugar solution after a stronger one. How to reduce this effect?

3. There may be many tiny factors that affect taste. Should we regard them all as non-significant? Maybe some of them can be taken into account?

4. How to process the collected data? We do not assume a deeply scientific approach, but we hope that a proposed procedure will be convincing.

Participants presented several plans of such experiment. After a discussion, the best-proposed plan was implemented.

Figure 1.On the photos: Left: the course of the experiment.
Right: the best plan author Alia (5th grade, Rep.of Bashkortostan)

During the discussion about this form of work, the organization committee came to the following conclusions.

1. The experiment may become the most dynamic and exciting part of the Olympiad.

2. The topic of the experiment should be chosen thoroughly so that learners should be able to replicate the experiment in the existing conditions within the announced time.

3. The form of the experiment should be chosen in such way that all participants could take part together with their parents, accompanying persons, etc., regardless of age.

4. The experiment should be planned so that each participant at any time would be busy, or at least, would have a chance to busy him/herself by performing tests, collecting statistics, processing collected data, etc. The best thing if we can organize work in groups with different duties for all members in a group.

CONCLUSION

The experience of the Olympiad on the probability theory shows that despite the fact that this branch of mathematics has traditionally been considered difficult and unusual for school learning, the interest to it is gradually increasing.

In 2015, the Olympiad school tour was held for the first time, and it was attended by 2465 students from 20 regions of Russia. While in the 2016 tour, 2859 students from 41 regions participated, as the Olympiad was joined by leading schools in many regions where such competition had not previously been known.

The growth is also indicated by the increasing number of requests from teachers and students to provide methodological support for teaching and learning. In 2014–2015 study year, the methodical site on teaching of probability and statistics http://ptlab.mccme.ru showed 876 queries. In 2015–2016 school year (up to 13th of May) the number of such inquiries was 1764. The number of queries related to the Olympiad only (applications, rules, results, appeals, etc.) reachs 20% of all requests (downloads).

One can conclude that the Olympiad plays an important role in the popularization of probability theory and statistics as a school subject.

Of course, the Olympiad should not be the only means of promoting it. Beside it, there must be math circle groups (extra-class activities), numerous publications on probability and statistics in teachers' and popular magazines (such as "Mathematics.1st of September", "Mathematics in School", "Kvant" and "Kvantik"). At same time, the number and variety of probabilistic problems in the national exam for primary and high school courses increase.

Unlike the problems from the regular school course, the Olympiad tasks are much more diverse in subject matter and level of difficulty. Taking advantage of this, the Olympiad developers are gradually expanding the range of tasks and invent new forms, some of which later will be included into school courses and methods.

APPENDIX

The curriculum of the course
“Probability theory and statistics” (grades 5–9)

1. Base level

Statistics. Tabular and graphical data representation; bar and pie charts; graphs. The use of charts and graphs to describe the real dependencies; extracting information from tables, charts and graphs. The descriptive statistics: the mean, median, maximum and minimum of numerical sets. Measures of scattering: range, variance and standard deviation.

Random changeability. The variability of the measurements. *Decision rules. Patterns in variable quantities*[2].

Random events. Random experiments, elementary events (outcomes). The probabilities of elementary events. Random events and favorable elementary events. The probabilities of random events. Experiments with equiprobable elementary events. Classical probability experiments with coins and dices. *Euler-Vienn diagrams. The opposite event, union and intersection of events. Addition of probabilities. Random choice. The use of trees for presenting random experiments. Independent events. A sequence of independent trials.* Iindependent events in real life.

Elements of combinatorics. *The rule of multiplication, permutations, factorial of a number. Combinations and formula for the number of combinations. Pascal's Triangle. Experiments with a large number of equiprobable elementary events. Calculation of probabilities in experiments using combinatorial formulas. Bernoulli trials. Success and failure. Probabilities of events in a Bernoulli test series.*

Random variables. *Introduction to random variables on examples of finite discrete random variables. The probability distribution. An expected value. Properties of the expectation. The concept of the law of large numbers. Measurement of probabilities. Applications of the law of large numbers in sociology, insurance, healthcare, public safety in emergency situations.*

2. Advanced level

Statistics. Tabular and graphical data representation, bar and pie charts, extraction of the necessary information. Diagrams of scattering. Descriptive statistics: the mean, median, maximum and minimum of numerical sets. Deviation. Random outliers. Measures of scattering: range, variance and standard deviation. The properties of the mean and the variance. Random variability. The variability of the measurements. Decision rules. Patterns in variable quantities.

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[2] Here and below the italic marks auxiliary topics included to curriculum but not to final assessments

Random experiments and random events. Random experiments, elementary events (outcomes). The probabilities of elementary events. Events in random experiments and favorable elementary events. The probabilities of random events. Experiments with equiprobable elementary events. Classical probability experiments with coins and dices. Euler-Vienn diagrams. Algebra of events. Addition of probabilities. Random choice. Independent events. A sequence of independent trials. The use of trees for presenting experiments. Trials until the first success. Conditional probability. The total probability formula.

Elements of combinatorics and Bernoulli trials. The multiplication rule, permutations, factorial. Combinations and the number of combinations. Pascal's triangle and the binomial theorem. Experiments with a large number of equiprobable elementary events. Calculation of probabilities in experiments using combinatorics. Success and failure. Probabilities of events in a Bernoulli test series.

Geometric probability. Random choice of a point from a figure on a plane, a segment or an arc of a circumference. Random choice of a number from a numeric interval.

Random variables. A discrete random variable and the probability distribution. Important discrete distributions (uniform, geometric, binomial). Independent random variables. Addition, multiplication of random variables. The mathematical expectation and its properties. The variance and standard deviation of a random variable. The variance of the number of successes in a Bernoulli tests series. The concept of the law of large numbers. Measurement of probability and measurement accuracy. Application of the law of large numbers in sociology, insurance, healthcare, public safety in emergency situations.

The curriculum of the course "Probability theory and statistics" for high school (grades 10-11)

1. The base level

Random variables. Operations with random variables (sum, multiplication). The distribution of a sum and a product of independent random variables. *The concept of correlation coefficient as a numerical measure of the linear relationship between two random variables.*

The concept of probability density. Examples of continuous distributions and densities. *The normal distribution as a distribution of measurement errors and variability in nature.*

2. The applied level

Random events and their probabilities. The conditional probability. The multiplication of probabilities. The total probability formula.

Random variables. Operations with random variables (sum, multiplication). Joint distributions. The distribution of a sum and a product of independent random variables. The expectation and variance of a sum of random variables. A geometric distribution. The expectation of the geometric distribution. *The variance of the geometric distribution. The Chebyshev inequality. Chebyshev's theorem. Bernoulli's Theorem. The law of large numbers. The sampling method. The covariance of two random variables. The concept of a correlation coefficient as a numerical measure of the linear relationship between two random variables.*

The concept of a probability density. Important continuous distributions (uniform, exponential, normal) and the parameters. Examples of random variables subject to the exponential and normal law. The standard normal distribution. Observing two random variables. *Concept of the correlation coefficient. The linear regression and LS method.*

The expansion of the applied level of the program that makes possible to continue the creative study of mathematics

Random events and their probabilities. A probability space. The axioms of probability theory.

Statistics. A statistical hypothesis. A statistics of a criterion and its significance level. Verification of simple hypotheses. Empirical distributions and their relationship with theoretical distributions. A selective correlation coefficient. The rank correlation.

Random variables. The Chebyshev inequality. Chebyshev's theorem. Bernoulli's Theorem. The law of large numbers. The sampling method. Sample variance. Estimating parameters of known distributions using samples. A covariance and correlation coefficient of two random variables. A binary random variable. The Bernoulli distribution. A calculation of characteristics of discrete random variables.

The distribution function of the random variable. The relationship between the distribution function and the density of a continuous random variable. The Laplace function.

The hypergeometric distribution. The Poisson distribution.

The relationship between the binomial and Poisson distributions. The central limit theorem.

3. The general compensating course

Statistics. A tabular and a graphical presentation of data, extraction of information from tables, bar and pie charts, graphs.

The statistical characteristics of numerical sets: the arithmetic mean, median, maximum and minimum values, range, variance and standard deviation. The errors of measurement. Simple decision rules.

Random events and their probabilities. Random experiments and their descriptions. The probabilities of random events. Experiments with equally possible elementary outcomes: throwing coins, symmetrical dices, random samples. A presentation of a random experiment as a tree. The ordered brute force approach, the multiplication rule. Calculation of probabilities using the brute force approach and the rules of multiplication. Success and failure. Bernoulli trials. *The concept of the law of large* *numbers: a proximity of frequency and probability.* An approximate calculation of probabilities.

Random variables. Introduction to random variables on finite examples of discrete random variables. The probability distribution tables. The expectation of a random variable. *The random variable "A number of trials until the first success." The random variable "A number of successes in a series of independent tests." The concept of the law of large numbers. Applications of the law of large numbers in sociology, insurance, healthcare, public safety in emergency situations.*

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Inquiry based learning of mathematics from various perspectives

Part 5

SYSTEMIC INVESTIGATIONS ABOUT PARENTAL INVOLVEMENT IN MATHEMATICS

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In this paper, we adopt a systemic perspective to investigate the convergences and the divergences about parental involvement in mathematics as perceived by high school students (age 13-15) and declared by their parents, investigating the role of mathematics reinforcements (mathematics books, private tutoring or tutoring schools) and of socio-developmental factors (gender, mathematics attainment, parents' education). The findings supported the chosen perspective confirming and refining existing results, revealing relatively concealed till now aspects; including structural convergences between the declared and perceived parental involvement and important divergences (notably with respect to the contrast between mother and father). The educational implications are briefly discussed.

FAMILY AND MATHEMATICS

Parental involvement has been identified as one of the factors affecting students' learning mathematics. The ways that the parents are engaged in their children's learning mathematics has been the focus of various research projects, gathering the interest of mathematics educators, of the protagonists of the educational systems, as well as of the educational policy makers (Campbell & Mandel, 1990; Cooper, Robinson & Patall, 2006). For example, the importance of the qualitative characteristics of the children's family –as manifested in mathematics involvement at home and the broader family practices– has been linked with their attitudes towards mathematics and their attainment (Cao, Bishop & Forgasz, 2006; Cobb & Yang, 1995; Moutsios-Rentzos, Chaviaris & Kafoussi, 2015).

Inquiry-based learning has also been at the crux of various research projects, especially with respect to the ways that such a perspective may be implemented in everyday teaching, where the several factors interact, often competitively (see, for example, Maaß & Artigue, 2013). Considering the role of parental involvement, mathematics educators investigate the ways that parents may help in promoting IBL (Mousoulides, 2013). Nevertheless, it is important to identify whether or not the parents' involvement in their children's learning mathematics is perceived by their children as such; that is, to identify the convergences and the divergences in the constructions that children and parents hold about the ways that the parents choose to get involved with their children's learning. In this study, we draw upon a systemic perspective, acknowledging the fact that in-school learning does not happen in vacuum, being at a continuous interaction with the broader socio-cultural environment. Following these, we investigated these links with the purpose to delineate the qualitative characteristics of these convergences and the divergences, as well as the links between the perceived (by the children) parental involvement and the declared (by their parents) parental involvement in high school mathematics (age 13-15) and socio-developmental factors (for example, gender and mathematics attainment).

SYSTEMS, PARENTAL INVOLVEMENT AND MATHEMATICS

Cai, Moyer and Wang (1997) differentiated between *direct parental involvement* and *indirect parental* involvement; the former focuses on immediate parents-children actions and interactions including the parents' helping their children to successfully deal with mathematics problems at home and the parents' participation in school activities and decision making, whilst the latter concentrates in less immediate actions and interactions including the family expectations with respect to the children's mathematical attainment, their encouragement, the parents' attitudes towards mathematics and the broader parental support for their children doing mathematics (for example, additional to the school textbook mathematics books). Cai et al (1997) found that direct parental involvement has weaker links with the children's mathematical attainment than the indirect parental involvement. From a different perspective, Epstein (1995) identified six types of parental involvement within the three-way interactions of parents-children-teachers (parenting, communicating, volunteering, learning at home, decision making, and collaborating with community), with the purpose to stress that different types are linked with different practices, challenges and results. Cao et al (2006), in a research project focusing on the children's perceived parental involvement in China and Australia with students aged 11, 13 and 15 years, combined top-down with bottom-up approaches to construct a questionnaire and to empirically identify the children's parental involvement aspects. Their analyses revealed four components of the children's perceived parental involvement components (p. 92-93): a) "perceived mother's and father's encouragement about mathematics learning" (*Parent Encouragement*), b) "perceived father's attitudes to mathematics and help given for mathematics learning" (*Father's Attitude and Help*), c) "perceived mother's attitudes to mathematics and help given for mathematics learning" (*Mother's Attitude and Help*), and d) "perceived mother's and father's expectations of their child's school achievement" (*Parent Achievement Expectation*). They found that aspects of the children's perceived parental involvement is actually conceptualised as 'parental', whilst others appear to be differentiated between mother's and father's parental involvement.

Parental involvement about mathematics seems to be linked with various factors, including: the socio-cultural environment, the parents' educational level, the frequency of the provided help at home, the way the additional mathematics material is employed (Cao et al, 2006; Hyde, Else-Quest, Alibali, Knuth &

Romberg, 2006). In specific, the parents with university level education hold higher expectations with respect to their children's attainment, crucially affecting their academic course, whilst the parents' higher socio-cultural level has been linked with their higher participation in the school activities as well as with more frequent helping their children with their mathematics homework (Ho & Willms, 1996; Young-Loveridge, 1989). Moreover, the children's perceived parental involvement has been found to decrease as the children get older, whilst the mothers have been found to be more active in supporting their children's self-confidence (Cao et al, 2006; Moutsios-Rentzos et al, 2015).

Consequently, the educators discuss different aspects of parental involvement notably differentiating that the role of the mother is not the same as the father's per se. Nevertheless, it seems that there is a lack of empirical research focussing on the potential divergences and convergences between the constructions that the parents and their children hold; respectively, between the children's perceived parental involvement and the parents' declared involvement. In this study, we draw upon a systems theory perspective to conduct such an investigation. Echoing existing systemic approaches to mathematics education (English, 2008; Davis & Simmt, 2003; Moutsios-Rentzos, da Costa, Prado & Kalavasis, 2015; Moutsios-Rentzos, Kalavasis & Sofos, in press), parental involvement can be viewed as a phenomenon occurring within and across interacting open systems. A system is a whole with a specific goal, the parts of which are related in ways that their inter-related emergent wholeness is more than the mere sum of its parts, whilst at the same time, the system is only one of the potential ways that its parts may be related, thus being in this sense less than its parts (Moutsios-Rentzos et al, in press). Following these, in this study, we conceptualise the school unit and the family as open social systems, in the sense that they are permeable to interactions amongst each other and the broader social environment. For example, the children function within and across at least three interacting systems: the family (sons, daughters, siblings etc), the school unit (students, peers etc) and the broader social system (in numerous diverse roles). Researchers suggest that the children's harmonious systemic functioning is strongly linked with their developing positive learning stance and self-image (Dowling & Osborne, 1994; Epstein, 1995).

Overall, in this study, we drew upon a systemic perspective to adopt and adapt the conceptualisation of Cao et al (2006), in order to investigate parental involvement about mathematics; both as perceived by the children and as declared by their parents, as well as with respect to their with socio-developmental factors, including grade, gender (children and parents), parents' educational background, the reinforcements provided to their children (for example, mathematics books or private tutoring) and the students' mathematical attainment.

METHODS AND PROCEDURES

This quantitative study was conducted in March 2015. The reported findings concern 108 children and one of their parents (N=216; 108 child-parent pairs) of a Greek Gymnasio (age 13-15; see Table 1). In Greece, high school includes six grades, divided in a three-grade Gymnasio (which signifies the end of the obligatory education in Greece) and a three-grade Lykeio (ages 16-18).

		N	Valid %	*M*	*SD*
Gender (children)	Boy	48	44.4		
	Girl	60	55.6		
Age (children) in years				13.6	1.1
Gymnasio Grade	A	36	33.3		
	B	40	37.0		
	C	32	29.6		
Gender (parents)	Father	36	33.3		
	Mother	72	66.7		
Age (parents) in years				44.2	7.2
Education Parent (participating)	Obligatory	13	13.0		
	Lykeio	52	52.0		
	Tertiary	35	35.0		
Education Parent (non-participating)	Obligatory	25	27.5		
	Lykeio	42	46.2		
	Tertiary	24	26.4		

Table 1: The participants of this study.

In order to identify declared and perceived parental involvement, we drew upon a Greek version of the Perceived Parental Influence scale (PPI; Cao et al, 2006) as translated and employed by Moutsios-Rentzos et al (Gr-PPI; 2015), to construct a corresponding scale for the parents' declared parental involvement (Gr-DPI), essentially by minor re-wordings of the children's scale (one of the methods employed in family research; for example, Edman, Cole & Howard, 1990). PPI is a sixteen-item scale (4-point Likert type items; 'Strongly Agree', 'Agree', 'Disagree', 'Strongly Disagree'): eight items measuring the students' perceived mother involvement and eight measuring the students' perceived father involvement. The four components of Gr-PPI were found to correspond well with the original PPI (suggesting its good cross-cultural validity and reliability; see Moutsios-Rentzos et al, 2015) with the addition of two levels of specificity: 'general' and 'maths', referring to whether or not the items loading to a component explicitly contained 'mathematics' in their wording. The four components of Gr-PPI are (see Table 2): PEC_g (Parent EnCouragement; general), FAH_m (Father's Attitude and Help; maths), MAH_m (Mother's Attitude and Help; maths), and PAE_g (Parent Achievement Expectation; general). The parents' declared involvement was identified through a scale constructed with appropriate rewording of the Gr-PPI items. The Declared Parental Influence scale (Gr-DPI; see Table 2) consisted of eight 4-point Likert type items. The Principal Component Analysis (Varimax rotation with Kaiser normalisation; 54.82% of the variance explained) of Gr-DPI revealed only two components (instead of expected three, based on the Gr-PPI analysis): the parents seem to conceptualise Parent Encouragement and Parent Achievement as one component. Hence, Gr-DPI consisted of two components: a) Parent's Attitude and Help (PAH_m; maths), and b) Parent Encouragement and Achievement Expectation ($PECAE_g$; general). Though both the identified components conceptually correspond to the children's components (and the assigned to them items), further research is conducted by the authors to gain deeper understanding in the parents' constructions. Gr-PPI and Gr-DPI constituted the core of the two versions of the questionnaire (children version and parent version) of this study, along with additional questions about the socio-developmental factors included in the study.

Perceived Parental Involvement (Gr-PPI) [a]	
My mother is good at maths.	MAH_m
My mother checks my maths homework frequently.	MAH_m
My mother asks me about my assessment results in maths.	MAH_m
My mother helps me with some difficult maths problems.	MAH_m
My mother makes me feel that I can do well in maths.	MAH_m
My mother tells me that a person must do something carefully in order to do it well.	PEC_g
My mother tells me a person must work hard in order to do something well.	PEC_g
My mother expects me to be the best student in maths and other subjects in my class.	PAE_g
My father is good at maths.	FAH_m
My father checks my maths homework frequently.	FAH_m
My father asks me about my assessment results in maths.	FAH_m
My father helps me with some difficult maths problems.	FAH_m
My father makes me feel that I can do well in maths.	FAH_m
My father tells me that a person must work hard in order to do something well.	PEC_g
My father tells me that a person must do something carefully in order to do it well.	PEC_g
My father expects me to be the best student in maths and other subjects in my class.	PAE_g
Declared Parental Involvement (Gr-DPI) [b]	
I am good at maths.	PAH_m

I check my child's maths homework frequently.	PAH_m
I ask about my child's assessment results in maths.	PAH_m
I help my child with some difficult maths problems.	PAH_m
I try to make my child feel that he/she can do well in maths.	PAH_m
I tell my child that a person must work hard in order to do something well.	$PECAE_g$
I tell my child that a person must do something carefully in order to do it well.	$PECAE_g$
I expect my child to be the best student in maths and other subjects in his/her class.	$PECAE_g$

[a]. PEC_g (Parent EnCouragement; general), FAH_m (Father's Attitude and Help; maths), MAH_m (Mother's Attitude and Help; maths), PAE_g (Parent Achievement Expectation; general). [b]. PAH_m (Parents' Attitude and Help; maths), $PECAE_g$ (Parent EnCouragement and Achievement Expectation; general).

Table 2: Measuring Perceived and Declared Parental Involvement (Gr-PPI & Gr-DPI).

RESULTS

The declared and perceived parental involvement across the Greek Gymnasio is outlined in Table 3. It is noted that although both the parents' and the children's constructions seem to follow a descending trend as the students progress through Gymnasio (which is in line with the literature), these trends were not found to be statistically significant (the overall Gr-DPI marginally not significant) and, thus, cannot be generalised. Furthermore, the overall declared Gr-DPI was higher than the overall Gr-PPI: the parents claim to be more involved with their children's learning mathematics than what the students claim to perceive.

	Grade A		Grade B		Grade C		K-W
	M	*SD*	*M*	*SD*	*M*	*SD*	*P*
Gr-PPI	2.9	0.4	2.9	0.4	2.7	0.6	0.376
MAH_m	2.8	0.7	2.6	0.7	2.5	0.7	0.280
FAH_m	2.7	0.7	2.8	0.8	2.6	0.9	0.601
PEC_g	3.3	0.5	3.3	0.6	3.1	0.7	0.357
PAE_g	3.0	0.5	2.9	0.8	2.6	0.8	0.093
Gr-DPI	3.2	0.5	3.2	0.5	3.0	0.5	0.052
PAH_m	3.1	0.6	3.0	0.7	2.9	0.6	0.115
$PECPAE_g$	3.4	0.6	3.5	0.5	3.3	0.4	0.173

Notes. *M*: Mean, *SD:* Standard Deviation. K-W: Kruskal-Wallis H test. Values range from '1' (strong disagreement) to '4' (strong agreement).

Table 3: Perceived and Declared Parental Involvement across a Greek Gymnasio.

Moreover, no statistically significant differences were found with respect to the students' gender (see Table 4). On the other hand, the students whose fathers participated in the present study were found to hold statistically significantly higher perceived Father Attitude and Help than the students whose mother participated in the study. It is noted that we did *not* find corresponding findings for the perceived Mother Attitude and Help. Furthermore, the fathers declared statistically significantly stronger overall parental involvement than the mothers.

		Gender (children)		Gender (parents)		Education Parent (participating)		Education Parent (non-participating)	
		Boy	Girl	Father	Mother	Lykeio	Tertiary	Lykeio	Tertiary
		Mdn	*Mdn*	*Mdn*	*Mdn*	*Mdn*	*Mdn*	*Mdn*	*Mdn*
Gr-PPI		2.9	2.9	3.0	2.8	2.9	3.0	3.0	2.9
	U	1223.5		982.0		719.0		431.0	
	P	0.299		0.051		0.098		0.225	
MAH_m		2.6	2.8	2.6	2.8	2.6	3.0	3.0	2.8
	U	1054.0		1172.5		679.0		463.5	
	P	0.034		0.484		0.044		0.427	
FAH_m		2.6	2.8	3.0	2.6	2.8	3.0	3.0	3.0
	U	1240.5		728.5		848.5		518.0	
	P	0.351		<0.001		0.593		0.930	
PEC_g		3.3	3.3	3.3	3.3	3.3	3.5	3.4	3.3
	U	1323.0		1241.5		684.0		450.0	
	P	0.683		0.808		0.048		0.330	
PAE_g		3.0	3.0	3.0	3.0	3.0	3.0	3.3	2.7
	U	1175.5		1212.0		881.5		366.5	
	P	0.175		0.660		0.803		0.037	
Gr-DPI		3.2	3.3	3.3	3.1	3.1	3.4	3.3	3.3
	U	1385.5		960.5		634.0		522.5	
	P	0.848		0.037		0.016		0.977	
PAH_m		3.0	3.0	3.2	3.0	3.0	3.3	3.1	3.2
	U	1406.0		897.5		613.5		497.5	
	P	0.950		0.012		0.010		0.724	
$PECPAE_g$		3.3	3.7	3.5	3.3	3.3	3.3	3.3	3.7
	U	1245.0		1240.0		875.0		502.5	
	P	0.274		0.801		0.757		0.769	

Notes. *Mdn* 'median'. *U* 'Mann-Whitney test'. Values range from '1' (strong disagreement) to '4' (strong agreement).

Table 4: Perceived and Declared Parental Involvement, gender and education.

Considering the parents' educational level, we contrasted the participating (in this study) parent with the non-participating parent, assuming that the participating parents would be more involved with their child education. The educational level of the participating parent was found to be statistically significantly positively linked with only with two components of the perceived parental involvement (in line with the literature) MAH_g and PEC_g, as well as with the overall declared Gr-DPI. Nevertheless, the non-participating parents with lower educational level were perceived to show higher PAE_g (in contrast with the literature).

The provided reinforcements appeared to have diverse links with parental involvement. Private tutoring and private tutoring schools were not found to be statistically significant factors with respect to parental involvement, except for the declared PAH_m only for private tutoring. On the other hand, the existence additional mathematics books was statistically significantly linked with the overall perceived parental involvement, the perceived MAH_m and FAH_m and the corresponding declared PAH_m. The intensity of these reinforcements was not statistically significantly correlated with parental involvement.

Finally, no statistically significant correlations were found between the students' grades (test or semester) and parental involvement (declared or perceived).

	Private Tutoring School		Private Tutor		Mathematics books		Test Grade	Semester Grade
	Mdn_{No}	Mdn_{Yes}	Mdn_{No}	Mdn_{Yes}	Mdn_{No}	Mdn_{Yes}	P^{τ}	P
Gr-PPI	2.9	2.8	2.9	2.8	2.8	3.0	0.548	0.892
U	676.0		466.0		920.0			
P	0.845		0.517		0.015			
Intensity$^{\tau}$	0.154 ^{P}		0.776		0.729			
MAH_m	2.8	3.0	2.8	2.7	2.4	3.0	0.794	0.856
U	627.0		445.5		863.5			
P	0.514		0.385		0.005			
Intensity	0.589		0.421		0.621			
FAH_m	2.8	2.6	2.8	2.6	2.6	3.0	0.481	0.407
U	683.5		476.0		977.5			
P	0.900		0.585		0.042			
Intensity	0.457		0.817		0.595			
PEC_g	3.3	3.3	3.3	3.3	3.3	3.3	0.591	0.364
U	670.0		527.		1159.5			
P	0.800		0.998		0.431			
Intensity	0.238		0.908		0.969			
PAE_g	3.0	3.0	3.0	3.2	2.7	3.0	0.313	0.331
U	607.5		462.5		1095.5			

P	0.402		0.489		0.220			
Intensity	0.300		0.386		0.723			
Gr-DPI	3.3	3.3	3.3	3.0	3.0	3.3	0.463	0.141
U	675.5		328.0		1098.5			
P	0.725		0.067		0.132			
Intensity	0.421		0.485		0.408			
PAH_m	3.1	3.0	3.2	2.8	3.0	3.2	0.375	0.160
U	697.5		296.5		987.0			
P	0.880		0.028		0.024			
Intensity	0.669		0.332		0.659			
$PECPAE_g$	3.3	3.7	3.3	3.3	3.4	3.3	0.576	0.741
U	604.0		457.5		1311.5			
P	0.307		0.675		0.924			
Intensity	0.619		0.699		0.492			

Notes. *Mdn*: Median. *U*: Mann-Whitney *U* test. Intensity: Hours per week. $^{\tau}$: Kendall's tau coefficient. ^{P}: Statistical significance. Values range from '1' (strong disagreement) to '4' (strong agreement).

Table 5: Perceived and Declared Parental Involvement, reinforcements and attainment.

CONCLUDING REMARKS

In this study, we employed a systemic approach in order to re-visit parental involvement, crucially by differentiating the perceived by the children's involvement from the declared by the parents' involvement. The quantitative analyses helped in gaining deeper insight about existing research findings and in identifying new conflated till now relationships. First, the parents declare higher involvement than the one perceived by their children. Furthermore, the higher educational level of the participating parents was as expected linked with the perceived Parent Encouragement and the Mother's Attitude and Help, as well as with the overall declared involvement. Contrasting the expectations, the lower educational level of the non-participating parent was linked with higher perceived Parent Achievement Expectation. Moreover, these are further complicated by the fact that the mothers' constructions appear to differ from the fathers'. For example, in line with existing research (Moutsios-Rentzos et al, 2015), the participating fathers declare higher involvement, which also seems to have a stronger effect on the perceived by their children involvement. Furthermore, no links were found in this study between parental involvement and attainment. Finally, the existence of additional mathematics books – essentially a family, within the home establishment reinforcement– was positively linked with parental involvement, which may be because through these books the parents actively interact with their children to improve their grades. This hypothesis is strengthened by the absence of such links with private tutoring, where the parents' role is minimal. Thus, considering the parents' key-role in inquiry-based learning in school (Mousoulides, 2013), it may be inferred that successful IBL practices may combine the parents' active participation with the parents' being convinced that these practices actually reinforce their children's grades. Overall, the diverse findings of this study reveal novel aspects of the complex parental involvement phenomenon, thus rendering crucial to conduct further systemic research –including, for example, both parents– with the purpose to delineate the complex interactions amongst parents (father & mother), children and school mathematics.

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"THERE ARE MORE PARTS"- LEARNING AND TEACHING THROUGH MONITORING PUPILS' PROBLEM SOLVING

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This paper presents a discussion of the problem-solving approaches of primary and secondary school pupils in relation to the following issues: developing strategies, communicating, and receiving guidance. The first issue deals with various strategies developed by pupils of different ages when involved in solving a particular problem. The second is connected to pupils' reasoning in written and oral form. Guiding is the role of the teacher who should be sensitive enough to support pupils' thinking, when necessary, but not direct it. A group of pupils (35 pupils between 10 and 19 years old) were given a geometrical problem that required them to define the number of parts created when a single plane was divided by straight lines. Each pupil tackled the problem individually, while primary teacher students from the Faculty of Education observed and guided them. After analyzing the pupils' work from various perspectives, we came to the following conclusions: the problem was a challenge for all pupils, almost all pupils needed hints during the problem-solving process, and only the oldest pupils were able to produce a general rule. Until presented with a problem that required a geometrical approach, the differences among the age groups in terms of successful problem solving were not that noteworthy. The difference among age groups was observed in examples of more complex problem solving where a shift towards an arithmetical approach was needed.

INTRODUCTION

Problem solving in the mathematics classroom is by no means new idea. Beside the fact that there are many researchers dealing with different ideas connected to problem solving (e.g. Pólya, 1945; Reid, 2002; Cañadas and Castro, 2007; Radford, 2008; Mason, Burton, and Stacey, 2010; Schoenfeld, 1985 ect.) who are in favour of encouraging problem solving among students, it is also the fact that problem solving is not accepted by the teachers and by those who develop teaching materials in the way as other topics in mathematics are (e. g. mastering written algorithms, solving equations...). Short analysis of the mathematics curriculums in most countries of the world proves that the students should solve problems and reason mathematically. It is very broad aim, in most cases not presented with examples and it is on the teachers to find their way to realize this aim. Some teachers help themselves with different sources of problems; some do the problems in their classroom only for the purpose of some kind of research. Why? "Problem solving is time consuming, it is not possible to carry it with the whole group of students and we do not know what the students actually learn", is the typical answer we get from our teachers. Even if they try to do some problem solving, they find it interesting but not enough to implement their teaching with it. Problems in the mathematics textbooks (there are not many) are in most cases planed for the gifted students and are very often solved at home or individually after finishing the 'obligatory' tasks. With our present research in the area of problem solving we do hope to show again that problem solving is beneficial for the students and for the teachers. Firstly, prospective primary teacher students were involved in the role of the teacher's researchers and secondly, the problem presented is interesting and appropriate for the students of different age.

LEARNING TO REASON THROUGH PROBLEM SOLVING

There are different types of problems. We are going to distinguish between procedural and conceptual problems in a similar way as we distinguish between procedural and conceptual knowledge. Considering the definition of procedural and conceptual knowledge by Haapsalo (2003) we are going to name procedural problems those which require mere procedural knowledge for their solving; in this case a problem solver is more focused on procedures, rules and algorithms. On the other hand, the conceptual problems are those which require the solver to be familiar with the specific mathematical concepts. We are also proposing that there are no disjunctive categories of problems in this manner: however, one of them (procedural or conceptual knowledge) prevails over the other one at problem solving; some kind of relation between procedural and conceptual knowledge must be established. In this paper we present a problem we categorise as a mere procedural one and it's beneficial for the students from the following aspects: students develop skills to reason mathematically, they are encouraged to discover new connections among different ideas in mathematics, they use and gain new knowledge of representing mathematical ideas in different ways, and most of all identify mathematical patterns and experience the challenge of generalisation which is one of the important goals of mathematical problem solving. Dorfler (1991) defines the generalization process as a social-cognitive process which leads to something more general and whose product consequently refers to an actual or potential manifold (collection, set, variety) in a certain way. According to Radford (2008) a crucial step towards generalization is being able to discriminate between the same and the different. To generalise a problem situation is to identify the operators and sequence of operations that are common among specific cases and extend them to the general case (Swafford, Langrall, 2000). A generalisation of a problem situation may be presented verbally or symbolically. Narrative descriptions of the general case are verbal representations of the generalisation, whereas representations using variables are symbolic representations (Swafford, Langrall, 2000).

It should be distinguished between two aspects of generalization: seeing the general in the particular or seeing the particular in the general (Kruetski, 1976). These two aspects allow the classification of students' generalizations. From the mathematical point of view the saying 'seeing the general in the particular' can be understood as inductive reasoning, which is a very prominent manner of scientific thinking, providing for mathematically valid truths on the basis of concrete cases. Many researchers have developed different stages of inductive reasoning (Reid, 2002; Cañadas & Castro, 2007; Polya, 1967). In this paper we are going relate our research findings to Cañadas and Castro (2007) who considered seven stages of the inductive reasoning process: observation of particular cases, organization of particular cases, search and prediction of patterns, conjecture formulation, conjecture validation, conjecture generalization, general conjectures justification.

Another important stage in the process of generalisation so called creative moment or abductive generalisation was proposed by Peirce (1958 in Rivera et al 2007) and it is widely used in the recent research.

Radford (2008) uses a term abductive reasoning. He defines the step of noticing a commonality and generalizing it to the rest of the terms of the sequence as an abductive reasoning. Inductive reasoning is in relation to abductive phase defined as a phase of testing and confirming the viability of an abduced form (Rivera et al, 2007). Radford (2008) distinguishes between algebraic pattern generalizations and arithmetic generalizations. 'Algebraic generalization refers to capability of grasping commonality noticed on some particulars, extending this commonality to all subsequent terms and being able to use commonality to provide a direct expression of any term of the sequence (deduction of schema or rule)' (p.84). If the step of forming a meaningful algebraic rule in generalisation is missing then we talk about arithmetic generalization. There is another type of situation to deal with when the abductions don't result from inferring a commonality among the particulars, but are a mere guesses. In that case abductions lead to guessing the expression for general case. Even if a general rule is formed it is not based on algebraic thinking but on guessing. Radford (2001) calls this type of generalization naive induction. Becker and Rivera (2006) also report about students' difficulties in producing a meaningful rule. They usually employ trial – error and finite differences as strategies for developing recurrence relations with hardly any sense of what the coefficient in the linear pattern represent. In the just explained terms this is not an algebraic but a naive induction. Similar results were found also by our recent research in this field (Manfreda Kolar, Mastnak and Hodnik Čadež 2012).

It is of course very important how students deal with the problems: do they work individually, in groups, are they guided by the teacher or not. Some research has been done in this manner, e. g. Rott (2013) who found out that there is a strong correlation between (missing) process regulation and success (or failure) in the problem-solving attempts (the pupils worked on the problems without interruptions or hints from the observers). We are in this research among others interested what is the role of the teacher if he is guiding the student through problem solving: what kind of hints are appropriate in the problem solving process and how do the problems solvers communicate about mathematical ideas.

EMPIRICAL PART

Problem Definition and Methodology

The mathematics curriculum for primary schools in Slovenia includes a number of goals related to problem solving. For example, a specific goal is articulated in the section about arithmetic operations stating that pupils should be able to use arithmetic operations in problem solving. In the section of the curriculum entitled didactical recommendations, it is explained that problems must be understood as tasks where the solver does not know the strategy in advance, but has to develop a strategy in order to solve the problem. For the purpose of this paper we used the problem 'Lines and Parts in a Geometric Shape' that would give deeper insight into the pupils' problems-solving abilities. Our principle goal was to examine how pupils of different ages attempt to solve a problem that requires the development of a strategy for generalization – how far do they go in the process, what kind of thinking do they develop, how do they communicate, and how much help do they need in order to progress when they are stuck but still motivated to solve the problem. It is important to note that the pupils were not solving the problem on their own, but were observed and guided individually by primary teacher students whose goal was not to be overly suggestive, but rather to provide appropriate hints, challenges, and comments.

The empirical study was conducted using the descriptive non-experimental method of pedagogical research.

Research Questions

The aim of the study was to answer the following research questions:

1. What is the teacher's role in the process of guiding pupils through problem solving?
2. How do pupils of different ages vary in their problem-solving strategies?
3. How much do the pupils delve into problem solving on their own, and what stage of inductive reasoning do they achieve?

Sample Description

The study was conducted at the Faculty of Education, University of Ljubljana, Slovenia in 2015. It encompassed 12 research reports, written by students training to become primary school teachers (2nd cycle degree). Each primary teacher student worked on the problem ‘Lines and Parts in a Geometric Shape’ individually with two to three pupils at different grade levels. Each student chose their own sample of pupils according to the following guidelines: one pupil from lower primary school (grades 4-6), one from upper primary school (grades 7-9), and one from secondary school. The total number of pupils in the sample was 35 (14 from grades 4-6, 11 from grades 7-9, and 10 from secondary school).

Data Processing Procedure

The pupils were given a mathematical problem and the primary teacher students observed how they solved it and what stage of generalization they achieved. The students then wrote a protocol for guiding pupils’ problem solving. They also wrote reports that included the pupils’ work, dialogues with the pupils, and hints given to the pupils in order to support their problem solving.

The following is the problem ‘Lines and Parts in a Geometric Shape’ that was given to the pupils:

> You have a rectangle. If you draw one straight line across the rectangle, you will have two parts. If you draw two straight lines across the rectangle, you will have three of four parts.
>
> Your task is to get as many parts as you can when crossing the rectangle with three, four, five straight lines. What about if you have 15 lines or n lines?

Results and Interpretation

Table 1 shows how successful the pupils were for each of the different cases of the problem: 3, 4, 5, 15 and n lines.

	3 lines	**4 lines**	**5 lines**	**15 lines**	**n lines**
correct	34	33	34	27	7
incorrect	0	1	0	0	5
no solution	1	1	1	8	23

Table 1: Success in solving the problem for different number of lines

The category “no solution” includes, among others, 5 pupils who wrote down the rule for calculating the number of parts for 15 lines, but did not present the solution.

Table 2 shows the rate of problem-solving success by grade level:

	3 lines	**4 lines**	**5 lines**	**15 lines**	**n lines**
Grades 4-6	100%	93%	100%	86%	0%
Grades 7-9	100%	100%	100%	73%	9%
Secondary school	90%	90%	90%	70%	60%

Table 2: Problem-solving success by pupils' grade level.

The results represent the rate of success of all pupils in each group, including those who did not solve the problem. There is one exception: a secondary school pupil who did not solve the first three cases, only working on the most advanced level of the problem.

In terms of success, there is almost no difference among grade levels for the first three cases of the problem (3, 4, and 5 lines). In the case of 15 lines, there is a significant difference, but it was skewed by the fact that there were many secondary school pupils who did not work on this level of the problem. The most significant difference in success rates by grade level is seen in the case with n lines. None of the pupils in grade 4 to 6 began to generalize this case. This was in part due to the decision of the primary teacher students who guided them to not present the case with n lines, because they predicted that pupils of that age and with limited knowledge of algebra would not be able to solve it.

Because the pupils did not work on the problems alone, but were guided by the primary teacher students, it is also important to analyze the students' protocols for guiding the pupils and to know how many pupils needed hints and what kind of hints were the most helpful. Table 3 shows how many students needed to be guided for the cases with 3 to 5 lines.

<table>
<tr><td colspan="2">3 lines</td><td colspan="2">4 lines</td><td colspan="2">5 lines</td></tr>
<tr><td rowspan="3">71%</td><td>86% grade 4-6</td><td rowspan="3">63%</td><td>86% grade 4-6</td><td rowspan="3">63 %</td><td>86% grade 4-6</td></tr>
<tr><td>82% grade 7-9</td><td>45% grade 7-9</td><td>55% grade 4-6</td></tr>
<tr><td>40% sec.</td><td>50% sec.</td><td>40% sec.</td></tr>
</table>

Table 3: Number of hints given for 3 to 5 line problems.

In general, more than 50% of all pupils needed guidance for all three cases. It is also clear that the younger problem-solvers needed more hints, the middle group needing the most hints for 3 lines but significantly less for 4 and 5 lines. We assume that the hints given for the first case helped the grade 7-9 pupils to solve subsequent cases. Deeper analysis of the problem solving indicates that pupils on this grade level were able to transfer strategies obtained for the first case to more advanced cases. For example, when solving the problem for 4 and 5 lines, they continued using the previous picture, not starting from the beginning as many of the youngest pupils did. It is interesting to note that the oldest group also needed hints, though significantly fewer than the younger groups.

We were also interested in discovering what types of hints the primary teacher students used. Drawing on their protocols, we classified hints into nine main types of hints as presented below:

Type of hint	3 lines	4 lines	5 lines
Lines don't have to be vertical or horizontal.	8	1	
Lines don't have to intersect.	9	6	3
Count again.	1	2	3
There are more parts.	11	11	7
Help yourself by using the previous case.	2	6	1
Draw more precisely.		1	
Draw a bigger picture.		3	4
Make a table/organize data.			6
Each line should intersect all the previous lines			4

Table 4: Types of hints

We categorized the hints into two main groups: procedural hints (marked green) and conceptual hints (marked yellow). In general, the primary teacher students gave more conceptual hints (40) than procedural (20) hints, but the type of hint also depended on the number of lines. The most useful hint (there are more parts) is a general hint as it informs the problem-solver that the correct solution has not yet been achieved, but provides no information about the type of mistake that has been made or directions for solving the problem.

We can see that the number of conceptual hints decreases as the number of lines increases (3 lines: 19; 4 lines: 13, 5 lines: 8), whereas the number of procedural hints increases (3 lines: 1; 4 lines: 6, 5 lines: 13). When pupils began to solve the problems, they were mostly preoccupied with the position of the lines – they mainly used vertical, parallel, and horizontal lines – and the intersection of the lines – they insisted on drawing lines that intersected at one point (see Figure 1). The pupils seemed to need conceptual hints to move from their established perceptions of lines, the result of prototypes received in the teaching process, to less predictable positions.

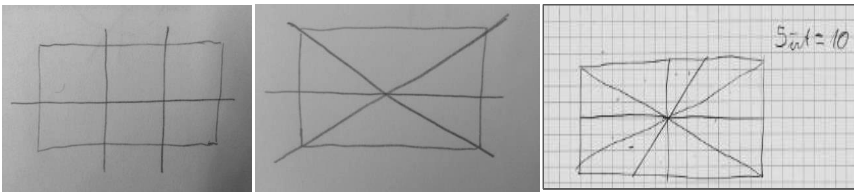


Figure 1. Typical mistakes with 3, 4 or 5 lines (horisontal lines, vertical lines and intersection of lines)

Once the pupils experienced this shift, they encountered another problem that was of a more procedural nature. Specifically, they had difficulties drawing more lines in addition to all the lines from the previous cases. These difficulties led the pupil, or the student assisting the pupil, to find other ways to present the solution, for example, by organizing data in a table (see Figure 2).

To summarize our findings in relation to our first research question (What is the teacher's role in the process of guiding pupils through problem solving?) we can conclude that the majority pupils were able to make progress in problem solving to higher stage only if they were given hints by the teachers (see Table 3). Without hints the pupils might stop solving the problem due to the misunderstanding of the problem's goal (for example focusing on parallel or intersecting lines).

If we consider the age of pupils when discussing their problem solving strategies (our second research question) we can conclude that all groups of pupils had similar conceptual problems with 3 lines and procedural problems with 5 lines. The stage of algebraic generalization was only achieved by the oldest pupils (secondary school pupils).

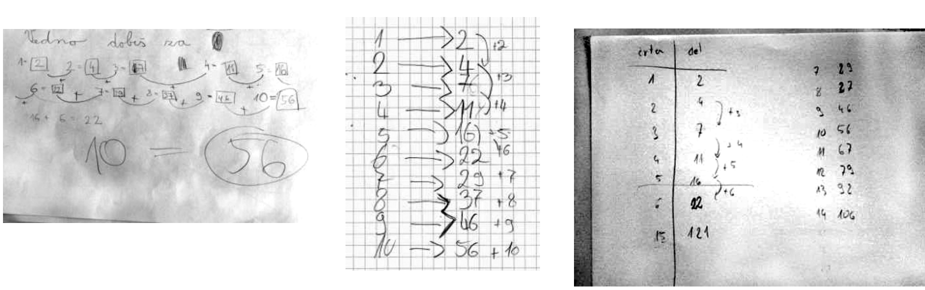


Figure 2. Examples of organising data

Table 5 lists methods for presenting the solution to the 5-line problem:

representation	**5 lines**
picture	19
table/organizing data	7
calculation	7
other	2

Table 5: Representations of solutions for the 5-line problem

The 5-line case represents the point when a geometrical problem begins to change into an arithmetical problem. When solving the 15-line case, almost all students (34 of 35) recognized how the number of lines increased relative to all previous cases.

At this point, we will summarize the results according to the stages of inductive reasoning by Cañadas and Castro (2007) which correspond to our third research question which deals with the pupils' ability to delve into problem solving and the stage of inductive reasoning they achieve. Pupils' strategies for solving the problems with up to 5 lines correspond to the first stage of inductive reasoning according to Cañadas (observation of particular cases). The 5-line case pushed pupils to advance to the second stage of inductive reasoning: organization of particular cases. The 15-line case already represents the next stage of inductive reasoning: the search for and prediction of patterns, and the making of conjectures.

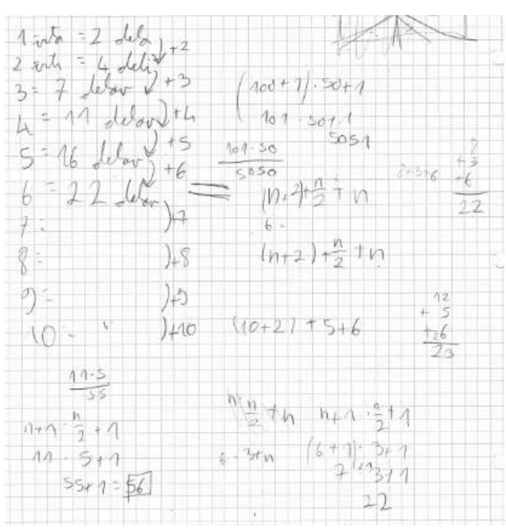
The following are oral explanations or conjectures generated by 34 pupils for the 15-line case. According to Radford (2008), these conjectures are examples of abductive reasoning.

- The number of parts from the previous problem + the number of lines from the next problem.
- Differences increase by one.
- The number that is added to the previous problem increases by one.
- Each new line must intersect all previous ones: the difference between the neighbouring pictures equals the number of lines that we drew in the last example.
- For 6 lines, we have to add 6 to the previous number of parts, for 7 lines we have to add 7, etc.
- For 15 lines, we should add 15.
- We arrive at the next number by adding the number of lines.

We came to the following conclusions after further analysis of pupils' generalizations. Pupils who produced oral explanations of the observed rule are at the stage of narrative generalization. Most of them also solved the problem symbolically. This means that they also arrived at the stage of either arithmetic or algebraic generalization. If a pupil solved the 15-line problem correctly but was not able to find a rule for the n-th case, we considered the stage of generalization to be arithmetic one. Only pupils who managed to articulate a general rule were considered to achieve the last stage: algebraic generalization. One of the pupils who produced a general rule for the n-line problem by guessing is considered to have relied on naive induction as defined by Radford (2008) (Figure 3). Table 6 shows the distribution of the level of reasoning achieved by the students.

Abductive reasoning	Narrative generalization	Arithmetic generalization	Algebraic generalization	Naive induction
34	23	27	6	1

Table 6: Distribution of students according to achieved stage of generalization by Radford (2008)



Primary teacher students' explanation:

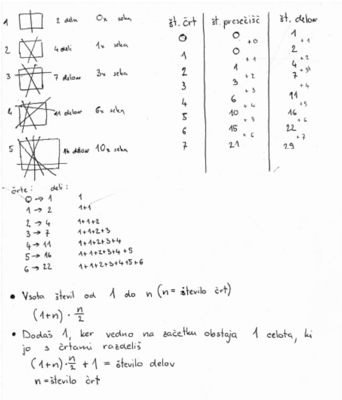
> She tries to compose an expression with variable n, which would give the result 22 for n= 6. She tries with different examples: (n+2) + n/2 + n (she gets 23, but she makes a mistake while calculating); n·n/2 + n (which equals 24). With further trials and changing the expression she manages to get the expression n+1 ·n/2 + 1 (she forgets the parenthesis), which gives the correct result. She checks the formula on another example (for n = 10).

Figure 3. An example of naive induction

Examples of algebraic generalization are presented in Figure 4. Only secondary school students were able to produce this kind of generalization and we can observe differences among them in their individual problem-solving records. Some of them already knew the rule for the sum of consecutive natural numbers, some of them developed the rule from the arithmetic mean, and some of them produced the rule in recursive form.

As mentioned above, those pupils who produced an oral explanation of the rule for the 15-line problem achieved the stage of narrative generalization. However, we observed some indicators of narrative generalization before. During the process of solving particular problems, there were some comments made about the process of making a picture:

- A new line must intersect as many previous parts as possible, and it mustn't go through intersections.
- When a new line intersects a certain part, we get two parts.
- Each new line must intersect all the previous lines; there is no point in representing the intersection of three lines.
- A new line must not go through intersections of previous lines.



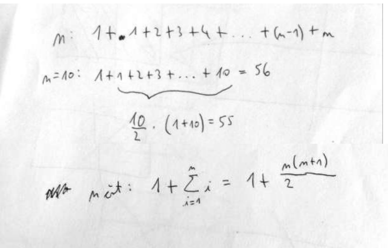


Figure 4. Examples of algebraic generalisation

These comments or guidelines provide the essence for proving the general rule. If we want to prove the rule, a mathematical induction is needed and the reasoning for it is based on the fact that the new line has to intersect all the previous ones: we assume that the n-th line contributes n new plane parts. So, what happens in the (n+1) step? The (n+1)-th line intersects previous n lines at different points, which means that it will be divided by those n points into (n+1) line-sections. Each of these line-sections represents a part of a border of a newly formed part of a plane, which means that the (n+1)-th line contributes (n+1) new parts to the plane.

DISCUSSION

We found that pupils were able to solve the presented problems in many different ways, which had not been expected. In our opinion, all the strategies were interesting, innovative, and emerged from the pupils' prerequisite knowledge. This is especially true of the forms of generalization they were able to make. Younger pupils needed more hints. It became clear that we had to deal with different types of generalizations or reasoning, and we discovered the same categories of generalizations proposed by Radford (2008).

Throughout the project, we were aware that guiding pupils through the problem-solving process could be problematic, but we were also aware that, with thoughtful guidance, we can help pupils build their mathematical knowledge, self-esteem, and autonomy. According to Lev Vygotsky, this kind of collaboration is called 'the zone of proximal development' (Vygotski, 1986).

As the project continued and we confronted examples of pupils' problem-solving, it also became evident that the greatest challenge to pupils was the development of generalizations where skills with numbers was also required. In such cases, only the older pupils were able to produce a general rule. As long as the problem allowed for a geometrical approach, differences among the age groups in terms of success were not that noticeable. The differences became more noteworthy for more complex examples where a shift to an arithmetical approach was required.

As can be seen from the protocols, the primary teacher students posed very similar hints or questions to the pupils. Only two refused any guidance, wanting to solve the problems by themselves. The students didn't want to help with the solution itself, only to guide the pupils in pursuing their own problem-solving approach. Some situations did occur where pupils became stuck, seemingly unable to make any further progress. In such situations, it is up to the teacher or researcher to decide how to proceed. Because the primary teacher students realized that the pupils were motivated, they decided to use their didactical and mathematical knowledge to help the pupils make a further step. We believe that this is an extremely important skill for professionals who guide problem solving: not to be overly suggestive, but to remain helpful. Observing pupils in their attempts to solve a problem on their own enables teachers to gain deeper insight into the way the pupils reasons, and to diagnose possible misconceptions and procedural difficulties with which pupils struggle. When the primary teacher students presented their work with the pupils, they were enthusiastic about the project and about the new knowledge they gained about the pupils' thought processes. They were surprised by what the pupils managed to accomplish, especially the younger ones. We believe that this is an important, though still modest shift towards moving teaching practices in the direction of problem solving as a tool for learning mathematics. We are aware that the research conditions under which the primary teacher students worked were ideal (i.e. working individually with the pupils), but nevertheless we do hope that it motivated them to start thinking about problem solving in the mathematics classroom in a different way – one that is more favorable to the learning process. For those who find the actual problem presented in this paper interesting, there are at least two extensions to it: 1) dividing the whole plane, and 2) coloring the regions. We hope this presentation will be motivating enough for our readers (especially the teachers among them) to begin solving these problems with a group of pupils in the classroom.

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TO WHAT EXTENT CAN A COMPUTER REPLACE GEOMETRICAL SOLID MODEL MANIPULATION? (ON CERTAIN ASPECTS OF TEACHING SPATIAL GEOMETRY IN MIDDLE SCHOOL)

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Solving spatial geometry problems is difficult for many students. It requires proficiency in geometric imagination and "selective vision" at an appropriate level. It is known that these skills can be developed by performing the right tasks. These include e.g. playing around by making buildings out of blocks, drawing geometrical solids on paper, cutting and gluing models of geometrical solids. Some adults believe that interaction with real-life objects can be replaced by observing phenomena on a monitor. They think that teaching with the use of a computer can, on the one hand, be more accessible and friendly to the modern student, and on the other hand, be less burdensome and time consuming. It is their opinion that working with computers stimulates the students to experiment, ask questions, and search for answers. In this article, I present partial results of my own research, the aim of which was to ascertain to what extent manipulating geometrical solids on a computer screen can help students aged 12-17 solve problems which require spatial imagination.

INTRODUCTION

Despite the fact that humans live in a 3D environment, issues regarding spatial geometry are disliked by students and reluctantly taught by teachers (Bakó, 2003). Many pupils and students encounter significant difficulties in solving such problems. An example can be an experiment conducted and described by de Lange (1986). He asked technology university students to assess what the lowest possible number of cubes can a geometrical solid be built from, for which the side and front view were known. The students had issues with solving the problem. They did end up providing a number of cubes from which such a geometrical solid can be built, but it was not the lowest possible.

The problem of "selective vision" in the geometry teaching and learning processes was the focus of multiple researchers (Ben-Chaim, Lappan & Houang, 1989; Bishop, 1980; Clements & Sarama, 2011; Duval, 2006; Fujita, Jones & Yamamoto, 2004; Jones, 1998; Krygowska, 1977; Kurina, 2003; de Lagne, 1986). Despite this, some mysteries of the "vision" phenomenon are yet to be discovered and explained (Panek & Pardała, 1999). The issue of how to shape, develop, and diagnose the spatial imagination of students is still being discussed. Ben-Chaim, Lappan & Houang (1989) (in: Panek & Pardała, 1999) state that it can be developed by performing appropriate tasks. This includes playful tasks of building geometrical solids out of cubes, drawing constructed plane geometrical solids and acquiring information from such drawings. Unfortunately, students very rarely get to do such activities during maths lessons. Sometimes performing actions on real items is substituted with observing phenomena on a computer screens. Some teachers and researchers consider that developing spatial imagination can solely involve performing specific experiments by using a computer. Although it is worth noting that adults already have certain enactive experience, which might be lacking in teenagers who have been making use of IT-related equipment most of their lives.

In literature, one can find many articles which describe how computers can help students in understanding situations and developing problem-solving methods (Hillel, Kieran & Gurtner, 1989; Kąkol, 2006; Kąkol & Pająk, 2009; Kąkol & Ratusiński, 2004; Kutzler, 2000; Parcia, 2004), as well as counteracting the student's mathematical helplessness (Czajkowska, 2009; Kąkol & Pająk, 2009). Although it is worth noting that if a computer is used as a didactic aid, the student is usually instructed accordingly by the teacher. The teacher asks the student to perform specific activities and draws the student's attention to elements essential to solve the problem at hand. A different situation, however, is when a problem which requires computer use is given to the student, and the student cannot opt out of using the computer to solve the problem. In such a case, the student is usually not being instructed by the teacher and can use the computer freely.

Inspired by reported problems, I have decided to check to what extent manipulating geometrical solids on a computer screen can help modern students aged 12-17, who grew up in a multimedia environment, in solving problems regarding geometrical solids created with cubes. I was seeking an answer to the question whether middle school students, when not being instructed by the teacher, can make use of the interactive nature of the tasks in order to solve them. I wanted to observe to what extent making use of the computer stimulates the students to conduct their own investigations, make discoveries, perform experiments, formulate hypotheses, and verify them. I was also interested in how much can a computer replace performing actions on real-life items.

Brief description of the study

In the first half of 2015, I conducted a study entitled Interactive problems in teaching mathematics as part of the *Use of IT in working with a mathematically gifted student project* of the Jan Kochanowski University in Kielce[1]. The study included 1055 students (53 class units, 16 middle schools). The criteria for a school to take part in the study were the permission given by the principal and

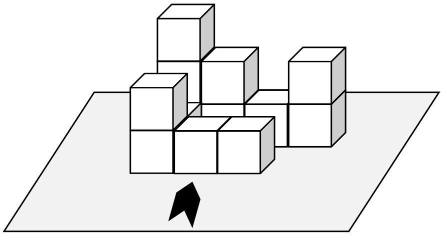
[1] The author of this paper was the initiator and the leader of this project. Special thanks from the author go to Paweł Fomalski for creating the interactive form of tasks (based on paper version tasks prepared by the author as well as based on scenarios of interactive tasks).

the mathematics teachers, as well as the school's possession of equipment and software which fulfils all requirements. In all classes subjected to the study, the students were divided into two groups in such a way so that both groups have students with varying mathematics abilities (as ascertained by the students' maths grades and the opinions of teachers). Students in one group solved problems digitally, while the other one used pen and paper. The digital tasks were being solved by 588 students (264 girls, 324 boys), while the paper tasks were being solved by 467 students (234 girls, 233 boys). The study took place during maths lessons, with the teacher attending. The students were, however, not instructed in any way by the teacher.

The test tools were constructed in such a way so that both versions of a given task (paper and digital) were identical in regards to their mathematical structure. They did differ in form, and, to some extent, in syntax as well as in the actions that the student had to or could have performed while solving the problems. In the digital version, the student could switch between tasks and modify previously provided answers.

One of the tasks of the test tool was related to geometrical solids. It included a header with a description and a set of four questions. This task in its digital form was properly interactive. A properly interactive task is what I call a task in which the student can or has to perform experiments by using the computer, as well as form hypotheses and verify them.

The problem is presented below.

> A geometrical solid, as shown in the figure, was created by using identical cubic blocks. The arrow at the base represents the front of the geometrical solid.
>
> 
>
> Question 1. How many cubic blocks does this geometrical solid consist of?
>
> Question 2. What is the smallest number of cubic blocks that has to be added to the geometrical solid for it to be a cube?
>
> Question 3. The figures present the front, side, and top view of a different geometrical solid created from such identical cubic blocks.

Front view Side view Top view

How many blocks could this geometrical solid consist of?

Question 4. The figures present the front and side view of yet another geometrical solid created from such blocks.

Front view Side view

What is the smallest number of blocks this geometrical solid could consist of?

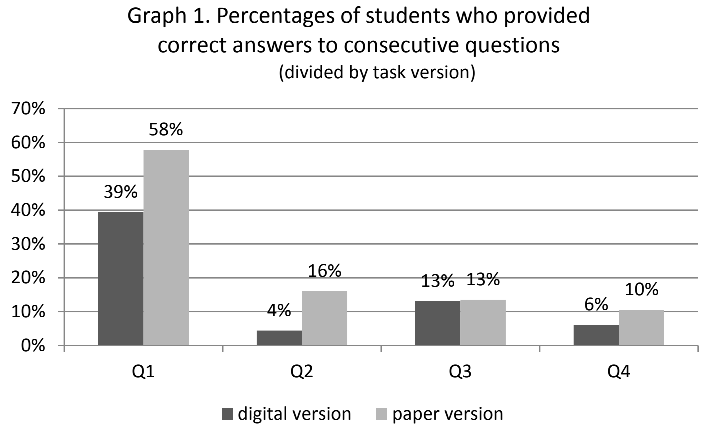
Two of the questions in the problem were identical in the digital and paper versions. The other two questions in the digital version included the problems of creating a geometrical solid.

In the paper version of the problem, the students were able to create additional drawings as well as perform calculations directly on the test sheet. But in order to answer the presented questions, it was necessary for the student to manipulate the geometrical solid by using their imagination. In the digital version, the student was able to rotate the geometrical solid, add blocks, and remove the ones they added (the blocks were of a different colour than the ones the geometrical solid was originally built of). This means that making proper use of the interactivity of the task could have been beneficial in obtaining the solution to the problem. But the students had to discover how to make use of the possibilities resulting from working on a computer on their own.

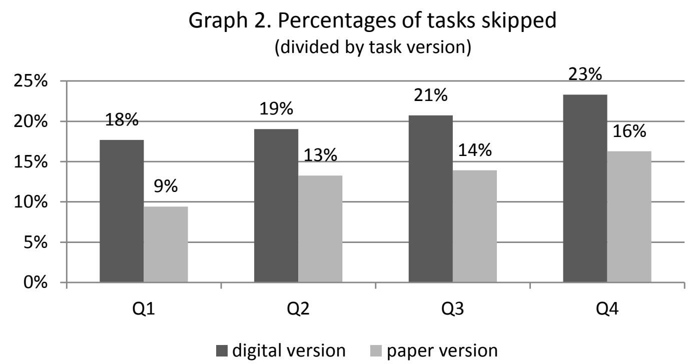
The main research method used by the author was the analysis of the students' products. What mattered was not only the provided answer, but also the sequence of actions performed by the students while solving the problem as well as making use of the newly-acquired knowledge of the situation to verify the correctness of the answers provided earlier. I have also conducted a statistical analysis of the results.

Results of the study

Graph 1 presents the percentages of students who provided the correct answers to consecutive questions (not taking into account the correct construction of a geometrical solid) in both versions of the test. As seen below, for three of the questions, the higher percentage belongs to the students who solved the problem on paper, and for the remaining question, the percentages are approximately equal.



It is surprising to note that the provision of the digital version of the tasks as well as their interactivity did not encourage the students to work on them. The number of consecutive skipped tasks in the digital version is considerably larger than its paper equivalents (see Graph 2). This phenomenon can be partially explained by the fact that some of the students turned off their computers even before familiarising themselves with the tasks. Although many of the students who took on the digital version of the test did have contact with the task, they chose to click the "next" button without making any effort to solve the problem.



The digital version of the task not only did not help some of the students solve the problem, but even hindered it. Instead of conducting experiments with the aim of solving problems, they used the interactivity of the task for fun. In certain cases, they did not even pay attention to the questions posed in the task and performed actions irrelevant to solving the problem. In the first problem alone, multiple students created geometrical solids based on their independent ideas or checked the possibilities of the software (see Figure 1). Some of the students solely played around, while others, aside from having fun, tried to find the answers.

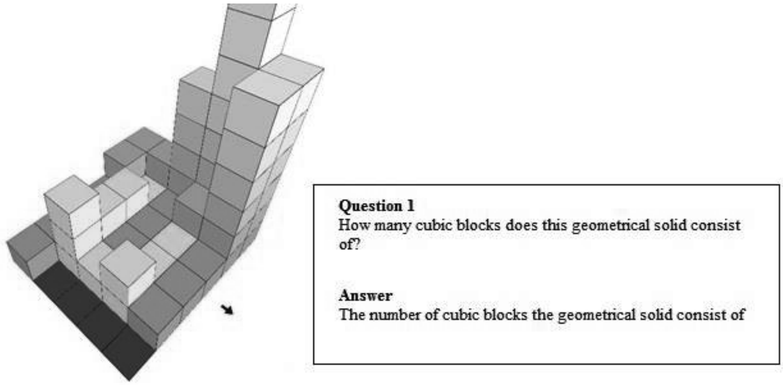


Figure 1.

Among the researched students, some decided to enter random content or to click random buttons. Similar behaviour (e.g. drawings unrelated to the task, guessing the answers) was also observed among the students who solved the paper version of the tasks, but such occurrences were considerably less frequent.

When analysing the work of the students who solved the digital version of the task, one can observe that they often did not pay attention to the fact that the figures provided in question 4 present different sides of the same geometrical solid. First, they constructed a geometrical solid, the front view of which complied with the first figure, often positioning the blocks in one row (Figures 2a and 2c). Next, they rotated the base and only paid attention to the blocks they were currently adding. Among the blocks placed earlier, they only noticed those that "fit" the second figure (Figures 2b and 2c). They did not pay attention to the fact that the blocks placed before rotating the base, which are not obscured by other blocks, are visible from another side. The students performed similarly in question 3.

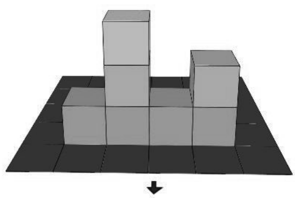


Figure 2a

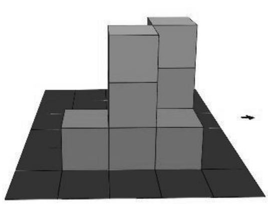
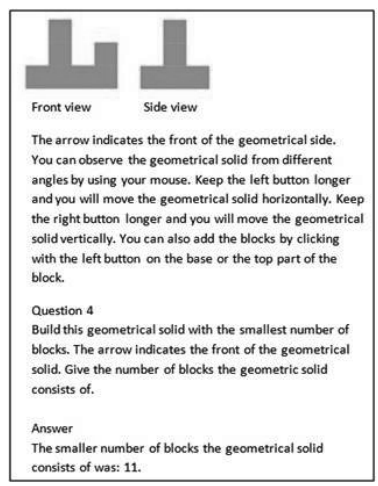


Figure 2b



Front view Side view

The arrow indicates the front of the geometrical side. You can observe the geometrical solid from different angles by using your mouse. Keep the left button longer and you will move the geometrical solid horizontally. Keep the right button longer and you will move the geometrical solid vertically. You can also add the blocks by clicking with the left button on the base or the top part of the block.

Question 4
Build this geometrical solid with the smallest number of blocks. The arrow indicates the front of the geometrical solid. Give the number of blocks the geometric solid consists of.

Answer
The smaller number of blocks the geometrical solid consists of was: 11.

Figure 2c

Discussion of results

The conducted research shows that the digital tasks were harder for the students to solve than the paper tasks. This was caused not by the fact that computers do not help in mathematical education, including the development of spatial imagination, but due to the fact that it is easy to stumble upon educational traps which I have outlined in (Czajkowska 2016) by presenting the students with interactive tasks. In this work, I will only describe some of them.

The observation of phenomena on a computer screen cannot replace the manipulation of real objects. A moving picture and the possibility of modifying it by using a mouse and keyboard is not enough to develop spatial imagination. According to Bruner's (1978) theory, learning mathematics requires performing and internalising specific, imaginary, and abstract activities. In working with a computer, mostly sight is involved. Which means that computers help in developing iconic and symbolic representations. Moving a cursor is primitive, and the learning process does not involve performing specific actions.

The research shows that multiple middle school students had an insufficiently developed ability of "seeing" a situation from their own perspective as well as someone else's. The students had a tendency of centration. When constructing a geometrical solid, they only paid attention to one of the sides. They perceived it selectively, only noticing the blocks they deemed important at a given moment, in accordance with their vision of the situation. They paid attention to those elements that "fit", and disregarded the rest. Some students had an insufficiently developed decentration ability. They were unable to perceive a situation from another point of view that looks at the geometrical solid at a different angle. Lack of such skills can be one of the reasons why the students were unable to visualise the situation described in the task by using a computer (Duval, 2006). The abilities of "seeing" a situation from their own and someone else's perspective are crucial for the development of spatial imagination. They should be nurtured as early as kindergarten and developed in primary school education (Gruszczyk-Kolczyńska, 2015). Unfortunately, the core curriculum and grade 1-3 primary school curricula have not contained any material related to geometrical solids for years. It is worth noting that this also occurs in other countries (Clements & Sarama, 2011). Thus, the poorly developed spatial imagination of middle school students could be caused by incorrect primary school education.

Despite the fact that the study took place on a mathematics lesson, the form of the tasks and the method of solving them differed from the everyday school methods (of typical mathematics and IT lessons). 60% of subjects declared that computers are not used during mathematics lessons, 14% stated that this occurs less than once a month, and for the remaining students, it happens at least a few times per month. It can therefore be assumed that working with computers was uncommon, with collective teaching being the focus. In such a way of teaching, the student's actions consist of looking at pictures on a screen and listening to the teacher's or another student's explanations. Therefore, the subjects either did not have the possibility of solving interactive tasks on their own during maths lessons, or it was a very rare occurrence. When faced with a new, unknown situation, they sought to change it so that it seemed more familiar and safe. Some tried to solve the problems as they would on paper, without making use of the computer's capabilities, others tried guessing, and the rest of the students decided to play around with the mouse and keyboard, mindlessly clicking buttons on the screen. The interactive task seemed like an excuse for playing around and having fun for them, instead of being a way of learning mathematics and nurturing their development.

CONCLUSIONS

To sum up, in the modern world, in which computers are prevalent, the inclusion of digital tasks in mathematics education is unavoidable. Teaching, including mathematics teaching, should include all aspects of the times the students are living in. Although it is worth noting that computers can facilitate the work of those students who have a sufficiently developed geometric imagination and are able to solve spatial geometry problems without using a computer. It is not enough to "see"; it is also necessary to know how to interpret what is being seen. In order to be able to interpret what is being seen, a sufficient amount of enactive experience is required. An essential prerequisite is the manipulation of specific objects; making use of a moving picture on a monitor is insufficient. This is why it is very important for children in kindergarten and primary school to play with blocks under adult supervision, to create geometrical solids in accordance with the given instructions, and to inspect the geometrical solids from different angles. Therefore, a change in primary school curricula and the inclusion of geometrical solids in the core curriculum for grades 1-3.

A methodology of introducing students to making use of multimedia is required in teaching mathematics. As rightly noted by Bakó (2003), a convergence of traditional (using geometrical solid models) and modern (using computers) teaching methods is necessary. The current form and content of mathematics and IT lessons in Poland are far from it.

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Establishing deep mathematical knowledge

Part 6

SPACIAL MODELLING AS MOTIVATION FOR STUDYING MATHEMATICS

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Motivation for learning is one of the most important elements of education. At present, both researchers and teachers constantly aspire to further develop this matter. It can be observed that, for the last two decades, mathematical education is being applied in different contexts of human knowledge, and not only in science. Also, school textbooks contain so-called real-life tasks, in which students can notice that the theoretical background which they are learning is really important for solving real-world problems. The aim of this paper is to present an analysis of students' solutions to a task concerning architectural problems, which really engaged and the motivated students, as well as the manner in which the students made use of IT in the proposed solutions.

INTRODUCTION AND THEORETICAL BACKGROUND

"Is mathematics really boring?" ask the authors at the beginning of their article (Sendova et. al, 2005). In ordinary teaching of mathematics, the subject is described as being full of rigid, closed-off rules intended to be memorised. But with a bit of will and engagement of both the teacher and the student, mathematics can be interesting, e.g. by correlating it with other, non-mathematical topics.

The most visual part of mathematics is most likely geometry, which "plays a special role in helping students represent and make sense of the world" (Sendova et. al, 2005). Further in Sendova's paper, she states that "geometric models provide a perspective from which students can analyse and solve problems".

As studies have shown, students sometimes have problems with some of the geometric shapes. As shown in (Pinker, 1997), the axis can be a powerful organiser of some of the shapes and forms.

As mentioned in (Sendova et. al, 2005), one of the tools which are very helpful in constructing shapes for use in special movement is the computer environment. Undoubtedly, using such an environment may be more exciting if it is used for solving problems concerning modelling. The students can get a deeper insight into the structure of the models, especially if they are in 3D.

The importance of the increasing matter of teaching and learning the relation between the real world and mathematics is considered by the author of (Blum, 2002), which was presented in PISA (Program for International Student

Assessment) and during some congresses dedicated to mathematical education – especially applications and modelling in mathematical education (ICMI). As Blum writes:

> Today mathematical models and modelling have invaded a great variety of disciplines (...) This has been substantially supported and accelerated y the availability of powerful *electronic tools* such as calculators and computers (...).

In this paper, following (Blum, 2002), we accept the following definition of *real-world* and *mathematical modelling.*

> Real-world – everything that has to do with nature, society an culture including everyday life as well as school and university subjects or scientific and scholarly discipline different from mathematics.
>
> Mathematics modelling – the process leading from a problem situation to a mathematical model.

In this sense, a model is a product of modelling.

The authors of (Erbas, et al. 2014) emphasize that mathematical modelling has been increasingly used in mathematical education from elementary to higher education. It is the authors' opinion that it is a way if improving the students' ability to solve real-life problems.

In mathematics textbooks used in different stages of education, one can find tasks concerning applying particular parts of mathematics in real-life situations, although most of them are a rather artificial "mathematisation" of such matters, as the problems presented in the tasks are often non-realistic and without a proper representation in real life. For instance, a task in which the average cost of goods is described by a function f(x), where x is the amount of units of the goods, can be, in the case of incorrectly choosing f(x), a negative or irrational number when an attempt is made at trying to find the maximal cost. The students feel confused at such a result, aside from the fact that they may not understand what average cost is and how the function f(x) was obtained. A similar situation occurs when, in a task concerning the calculation of the number of bacteria, it is presented as a very complicated increasing exponential function, in which the result is an irrational number. Such tasks usually cause misunderstandings and confusion, and a student cannot really see the significance of mathematics by enduring such tasks. They can also discourage students not only from solving them, but from learning mathematics altogether, and finishing the mathematics course with a positive grade is only one of the causes for learning.

Often such tasks contain are explained in a complicated way, and either are intricate or contain words or phrases which are unknown to the students.

That's why it is really important to use those tasks for which the interpretation is realistic and easy to verify and understand. It is worth emphasizing that textbooks contain a very small amount of such tasks.

The most interesting aspect in mathematical modelling is that there is no strict procedure for reaching a solution by using the provided information, although in NCTM (1989) (National Council of Teachers of Mathematics) (see: Erbas et al. 2014), one can find that modelling is a process that includes five interrelated steps: 1. Identify and simplify the real world problem situation, 2. Build a mathematical model, 3. Transform and solve the model, 4. Interpret the model, 5. Validate and use the model.

The research questions are:

1. Do students follow the steps of mathematical modelling proposed by NCTM? Are there are any new elements?
2. Is using IT (graphic display calculators (GDC) or computer software) useful for mathematical modelling? What are the advantages of using IT in these tasks?
3. Are tasks concerning mathematical modelling educational for the students?

DATA COMPLETION AND ANALYSIS

In the research, the following methods were used: observation, interview, and qualitative analysis of documents (students' solutions).

Five 17-18-year old second-year high school students took part in the research. Before the research, the students were taught about particular aspects of mathematics, such as algebra, elementary geometry, and calculus. For these aspects of mathematics, the students were taught how to use GDC (for example in calculating integrals and solving polynomial equations of a degree of at least two). Over the course of their entire education, they were only familiar with short real-world related tasks which usually contained one short problem to solve. Some of the tasks were poorly formulated (as mentioned earlier). The students who took part in the study were not taught how to use computer solutions for modelling or even creating graphs and graphical pictures which were used to illustrate the ideas of the tasks.

The task considered in the research (Appendix 1) was sourced from www.ibo.org (a website dedicated to teaching in the International Baccalaureate Diploma Programme) and it was a part of an examination (so-called internal assessment).

The students were given 10 working days (i.e. excluding weekends) to prepare the final version of their solution for this task and deliver it to the teacher (the author of this paper). The role of the teacher was very limited and restricted to a general discussion regarding the task without giving any particular or partial solutions. The teacher was forbidden to give any further instruction concerning using IT. The students were expected to work individually. Although they discussed the problem posed in the task in the classroom, they were made aware by the teacher not to provide each other any suggestions.

After the ten working day period, the students delivered the final versions of their solutions, which were then subject to analysis. Excerpts of the students' solutions and the conclusion are presented below.

All of the students who were chosen to take part in the research were very interested in solving this task, especially once they knew that IT was required to solve it. All of the students finished the task, solving all of its problems.

The aim of the task was to model a building as shown on the included pictures, and to produce a report for the contractor with all the necessary specification requested in the task.

Firstly, the students recognized the shape of the roof of the building as a parabola and represented the front of the construction in systems of axes, where the y-axis was the axis of symmetry of this parabola. This helps them find the formula of this parabola in the form of

$$y = \frac{-h}{1296} \times x^2 + h$$

where h is the height of the roof.

Then, the students found the cuboid located under the roof and its maximum possible volume. As the length of the building was set (at 150m) and

$$V_1 = 2xy \times 150$$

the students tried to find such dimensions of the rectangle so that it could be inscribed in the parabola and x-axis, i.e. the rectangle for which the area

$$A_1 = 2x\left(\frac{-h}{1296}x^2 + h\right)$$

was the biggest. Almost all students used the appropriate theorem of calculus regarding the extremes of functions for this task (where x is a variable (half the length of the cuboid) and h is a parameter (height of the cuboid)). This way, they obtained the solution for $x = 12\sqrt{3}$ and $h \in [36; 54]$. In their attempts, they used GDC to obtain the results shown in Table 1.

No.	Height	x	Y	Z	Volume of cuboid
1	36	20,78461	24,00	150	149649,2
2	38	20,78461	25,33	150	157963,0
3	40	20,78461	26,67	150	166276,9

4	42	20,78461	28,00	150	174590,7
5	44	20,78461	29,33	150	182904,6
6	46	20,78461	30,67	150	191218,4
7	48	20,78461	32,00	150	199532,3
8	50	20,78461	33,33	150	207846,1
9	52	20,78461	34,67	150	216159,9
10	54	20,78461	36,00	150	224473,8

Table 1: Dependence of the cuboid volume inscribed in the building to the height and the value of xyz

The next part of the task concerned finding the ratio of the volume of wasted space (between the roof and the outside of the cuboid). In order to find it, the students used a definite integral in the form of

$$A_2 = 2\int_0^{36} (-\frac{h}{1296} \times x^2 + h)\ dx$$

for the area of the front of the cuboid, or

$$V_2 = z \times 2\int_0^{36} (-\frac{h}{1296} \times x^2 + h)\, dx$$

for the whole volume. As a result, they obtained the following (Table 2).

No.	Volume of cuboid	Volume of structure	Wasted Space	Ratio
1	149649,2	259200	109550,8	0,732051
2	157963,0	273600	115637,0	0,732051
3	166276,9	288000	121723,1	0,732051
4	174590,7	302400	127809,3	0,732051
5	182904,6	316800	133895,4	0,732051
6	191218,4	331200	139981,6	0,732051
7	199532,3	345600	146067,7	0,732051
8	207846,1	360000	152153,9	0,732051
9	216159,9	374400	158240,1	0,732051
10	224473,8	388800	164326,2	0,732051

Table 2: Calculations of the ratio of wasted space to the volume of the cuboid

The next part of the task was to determine the total maximum office floor area in the block for different values of roof height. One of the proposed results is given below (Table 3).

No.	Height	y	No. of floors	Maximum office floor area
1	36	24,00	9	56118
2	38	25,33	10	62354
3	40	26,67	10	62354
4	42	28,00	11	68589
5	44	29,33	11	68589
6	46	30,67	12	74825
7	48	32,00	12	74825
8	50	33,33	13	81060
9	52	34,67	13	81060
10	54	36,00	14	87295

Table 3: Cuboid office floor area change depending on the height and the number of floors

The second section of the task concerned a new situation, in which the base of the building was turned in such a way that the façade was placed on the longer side of the base. Hence, the students produced this new formula for the parabola which forms the front of the roof

$$y = \frac{-h}{5625} \times x^2 + h$$

where h means the height of the roof. Proceeding as before, they recognized that the ratio of wasted space is still the same and is approximately 73.2%. But when they calculated the maximum office area, they obtained the following result (Table 4).

No.	Height	y	No. of floors	Maximum office floor area
1	75	50,00	20	124708
2	79	52,67	21	130943
3	83	55,33	22	137178
4	87	58,00	23	143414
5	91	60,67	24	149649
6	95	63,33	25	155885
7	99	66,00	26	162120
8	103	68,67	27	168355
9	107	71,33	28	174591
10	112,5	75,00	30	187061

Table 4: Cuboid office floor area change depending on the height and the number of stories.

DISCUSSION

The examples of the students' calculations and the obtained results shown in the previous section have uniquely indicated that the students' experience of solving real-life tasks in their previous stages of education are rooted too strongly rooted in their assumptions related to solving such tasks. Undeniably, in this situation they did not care about interpretation of their results. For example, when finding the maximum volume of the cuboid in the first part of the task, they had indicated that the best suitable cuboid in this situation has the dimensions

$$\boldsymbol{x = 12\sqrt{3}\,m\,,\ y = 36\,m,\ z = 150\,m.}$$

As far as the author is informed, the dimensions used in architecture should be natural numbers. Also, the building lacked stability. Moreover, such numbers would generate other irrational dimensions which were shown in Tables 1-4.

One can interpret it as a kind of trap in using IT, as performing calculations on irrational numbers using the pen-and-paper method is uncomfortable.

Aside from this aspect, IT turned out to be very helpful in solving such tasks, because one can find a number of examples and examine different versions of the same model, as well as transform the model to new conditions in a short amount of time. Using the pen-and-paper method would demand considerable mental effort and would make it hard to actually finish the task. But it is worth emphasizing that without the mathematical background which the students taking part in the research had (especially calculus and geometry), they could not have solved this task. This, it can be concluded that IT is a tool which makes the solving quicker, but not easier.

CONCLUSION

Undoubtedly, such tasks were very interesting and educating for the students who solved the task willingly and, as the teacher observed, "for ten days, they thought and breathed only for this task." As the task was divided into particular parts, the students proceeded according the steps proposed by the NCTM. When they had built the mathematical model, they tried to solve it using the pen-and-paper method first, and then, after checking the model, they used IT. Using IT undoubtedly accelerated some parts of their solutions, especially when the students had to calculate rather complicated integrals and polynomial inequalities of the third degree. Undeniably, computer software helped the students in preparing the final version of their solution. The GDC was used in these tasks as a calculation and transformation tool (for modifying the nature of the task), but computer software was used as a tool for visualisation (especially graphs). As mentioned in the first section, the use of the system of axes helped the students in organizing the problem and calculating the required properties of the given building. The task was also educational for the students in the following way. The most important thing is caring about the solution being proper, which means that in regard to the length or height of the rooms in the building, one must create a model without any irrational numbers. The students did not seem to care about this due to their previous experiences (mentioned at the beginning of this paper). This, it is very important to pose a good question and care about the interpretation conforming to realistic situations.

The final conclusion we can give is in the following points. Real-life tasks can motivate students to learn mathematics because:

a) Authentic real-life contexts are more interesting for the students, there is no exclusion from the real world;

b) In the modelling process, the students try to use mathematics for the description of a real-life situation;

c) The problem posed in the tasks can have different interpretations;

d) There is no single algorithm/strategy for solving such a task, that's why more than one solution (model) is possible;

e) The students can see that mathematics is really important in various parts of life.

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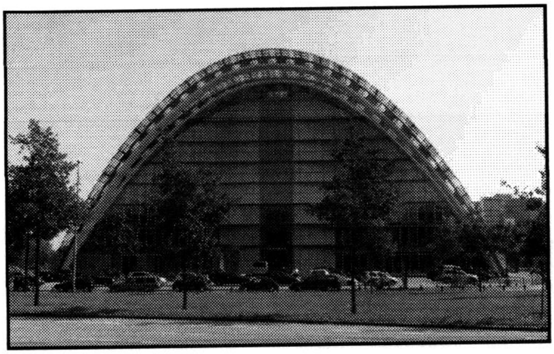
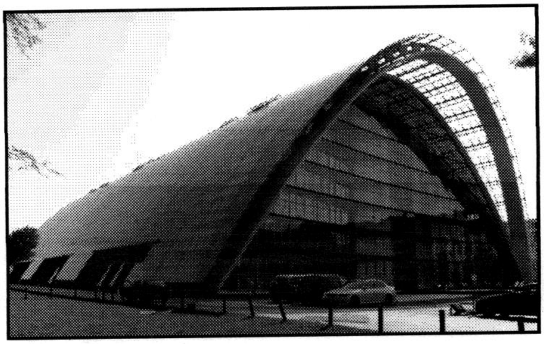
Appendix: The authentic text of the task used during research

MODELLING A FUNCTIONAL BUILDING **HL TYPE II**

To the student: The work that you produce to address the questions in this task should be a report that can stand on its own. It is best to avoid copying the questions in the task to adopt a "question and answer" format.

As an architect you have been contracted to design a building with a roof structure similar to the one shown below.





In this task you are to model such a building and produce a report to the contractor with all necessary specifications.

(This task continues on the following page)

Appendix 1 –cont.

You are to design an office block inside a structure with the following specifications:

The building has a rectangular base 150 m long and 72 m wide. The maximum height of the structure should not exceed 75 % of its width for stability or be less than half the width for aesthetic purposes. The minimum height of a room in a public building is 2.5 m.

- Create a model for the curved roof structure when the height is 36 m.
- Find the dimensions of the cuboid with maximum volume which would fit inside this roof structure.
- Use technology to investigate how changes to the height of the structure affect the dimensions of the largest possible cuboid.
- For each height, calculate the ratio of the volume of the wasted space to the volume of the office block.
- Determine the total maximum office floor area in the block for different values of height within the given specifications.
- Given that the base remains the same (72 m × 150 m), investigate what would happen if the facade is placed on the longer side of the base.
- You now decide to maximise office space even further by not having the block in the shape of a single cuboid.
- Review your model and calculate the increase in floor area and the new volume ratio of wasted space to office block.

DEEPENING MATHEMATICAL KNOWLEDGE BY USING A "BLACK BOX"

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For students who lack the motivation to learn mathematics and who are looking for answers to the question 'For what do I learn this?', activities related to inquiry based mathematics education can be organized with a "black box" functioning at work. The motivation may be a need to modify the operation of such a device, where the mathematical knowledge is useful.

INTRODUCTION

According to many mathematics educators (Hull, 1999; Murray, 2004; Harris, 1991; Wood, 2010) learning mathematics in context favors the deeper meaning of the procedures and concepts bonded with the context. One can extend this statement by saying that adding context to learning mathematics creates greater motivation to learn mathematics as such. Mathematics "in context" becomes a tool to solve problems. In particular, such an approach seems to be a very good idea for teaching mathematics in vocational and technical schools.

The ability to reach students from such schools with mathematics and its effective implementation is a problem raised quite often (Wedege, 1999; Lindenskov & Valero, 2002; Wedege, 2002). In many countries it has become apparent that the teaching of mathematics in vocational schools cannot rely on a narrow content only (Groenestijn, 2002; Lave, 1993). In the technical professions mathematical skills are becoming increasingly necessary, regardless of the degree of automatization of many tasks performed in the workplace. In the Polish schools' tradition it is still difficult to convince teachers that the question often asked by students: "why am I learning this?" should be taken seriously, and the reference to the "rough and cold beauty of mathematics as a scientific discipline" is not a good argument.

THEORETICAL ASSUMPTIONS OF "MATHEMATICS FOR WORK"

The aim of Wake's (2014) article: '*Making sense of and with mathematics: the interface between academic mathematics and mathematics in practice'* was to find proposals for the math curriculum for working. The author relied on the findings of researchers talking about the fact that '*there is a fundamental difference between the nature of the role of mathematics in the different settings of school and workplace That has a major impact on mathematics as practised in each*.' (Wake, 2014) In the aforementioned article, the author refers to a study, during which scientists explored workplace practices that involved some aspects of mathematical activity. The objective of these studies was to ‘focus closely on the nature of the mathematical understanding at issue’ (Wake, 2014).

It is worth to mention that some of the findings from various studies are referring to the relationship between mathematics at school and mathematics in the workplace (Coben & Weeks 2014; Hahn, 2014; Hoyles, Noss, Kent & Bakker, 2010; Hoyles, Noss & Pozzi, 2001; Roth, 2014; Triantafillou & Potari, 2010).

The main problem, outlined in mathematics for workers (current or future) is the observation that the students consider theory as one thing, practice as another. Some students can solve problems in the school-based material, but they are not able to use the theory and methods in practice. Their skills are not functional. Other students experience just the opposite. (Wedege, 2007, p.18). Linking these two aspects is not easy. There is no natural transfer of school mathematics to situations of the workplace. This is due to deep differences between the deductive, logical structure of mathematics as a scientific discipline, and those aspects of reality in which the elements of mathematical knowledge are useful. In the domain of vocational education and workplace training, the nature of mathematics is abstract; generally applicable and confronted with practical constraints and concerns in concrete situations. Mathematics in the workplace is often referred as "horizontal", as opposed to the "vertical" abstract mathematics, which means that it applies a lot of loose links between sometimes very distant fragments of knowledge. Noss and Hoyles (1996) introduced the term situated abstraction to highlight the necessity of bridging these two aspects. How this should be done, however, remains an open question that has only received partial answers from small-scale intervention studies in vocational education (LaCroix, 2014; Wake, 2014).

Therefore, searching for links between such distant worlds is a natural challenge for educators. Research in this area can help us understand how to bridge the gap between abstract and general mathematics typically taught at schools, on the one hand, and situated workplace mathematics as typically found in Workplaces on the other. This seems necessary for all students who want to see the point of learning mathematics.

To zoom the math problems to students directed at professional preparation, repeatedly their actual work environment is analyzed, to explore situations in which mathematical concepts are rooted. So this is the reverse to the traditional approach, in which the starting point is the structure and scope of the concepts of ‘vertical’ mathematics. In the ‘horizontal’ approach the starting point is the work context. Based on the phenomena occurring there one is looking for opportunities for their mathematisation. Another idea is to use such real tools of work, for which you need to apply some mathematical knowledge. It is expected that the objects that will be used will be those which are in actual use in practice (keeping statistics, computers, etc.). It should be emphasized that users of mathematics (hidden in instruments, readers, switches) do not see it, *mathematics at work is often black-boxed* (Williams & Wake, 2007) or *invisible* (Bakker, Hoyles, Kent & Noss, 2006). Learning can be mediated in such cases by specially created boundary objects—artefacts that are sufficiently simply to maintain meaning across communities, but also flexible enough to be used effectively in each (Bake, 2012).

INQUIRY BASED MATHEMATICS EDUCATION AND ITS RELATION TO VOCATIONAL TRAINING

One of the consequences of adopting a constructivist approach to teaching mathematics is the concept of inquiry based mathematics education. In general, it assumes that students come to see mathematics as more than simply a sequence of skills but as a tool with which to understand the world, to analyze complex issues in society and to communicate mathematical ideas in order to contribute to and participate in a democratic society (Dorier & Maass, 2014). Usually, students have a lot of freedom in choosing how to resolve the problems raised, while the role of the teacher is reduced to monitoring their work.

For students who lack the motivation to learn mathematics and who are looking for answers to the question ‘For what do I learn this?’, activities related to inquiry based mathematics education can be organized with a "black box" functioning at work. The motivation may be a need to modify the operation of such a device, where the mathematical knowledge is useful.

Although it is not assumed that inquiry based mathematics education is directed to gifted students, the truth is that this type of work requires some change in the working style of both the teacher and the students. Their engagement in math classes is different than in typical classroom activities. First of all, the student must be active and must want to take on solving the problem posed in front of him: to put hypotheses, to verify them. The student must be ready for attempts and for the lack of clear tools to use. There is a risk that for the student who is not prepared for this type of activities and in addition, a student who does not have the motivation to deal with mathematics, this approach may be ineffective (Kirschner, Sweller & Clark 2009). And there are many such students in vocational schools. Therefore, in the classroom, which implies self-seeking, the role of the teacher is special. On the one hand, s/he must encourage students to deal with the problem, and on the other, s/he must find a way to lead to the achievement of the results.

HOW DOES THE ALTIMETER WORK

The altimeter is used for measuring angles (inclination) and in particular can be used for measuring the height of trees; for this purpose, it produces special clinometers (altimeters) calibrated already in meters. The measurement is made by looking through a viewfinder. It should be remembered that targeting to the top or to the base is carried out with one eye, the other eye must observe the scale placed inside the altimeter. The measurement can be carried out within the distance 15 or 20 meters.

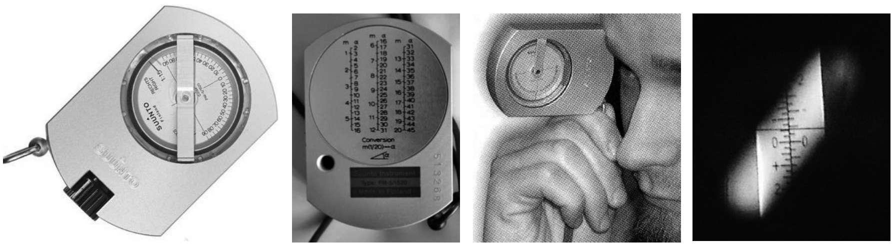


Figure 1. Altimeter SUUNTO, a-obverse; b-reverse; c-a way of holding during measurement, d-scale, view through the viewfinder.

For making the measurement it is necessary to appoint a distance of 15m or 20m from the tree (in such a way as to be easily visible). Looking by the viewfinder by one eye, but with both eyes open (Figure 1c), aim altimeter up so that the horizontal line scale covered the top of the tree; then read the height (in meters at once), for a distance of 20 m on the left scale, for a distance of 15 m on the right. If we stand on flat ground, it is needed to add height at which altimeter is kept - an average of 1.60 m.

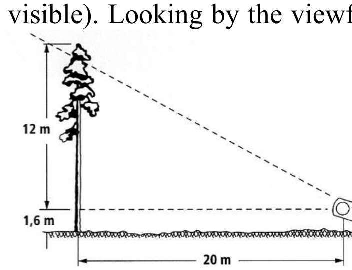


Figure 2. Relationships between amounts when reading the height of the tree

In the Polish school, the problem of deter flagship issue for the use of Thales Theorem, talking about the relative lengths of the respective sections designated on straight lines cut by parallel lines.[1]

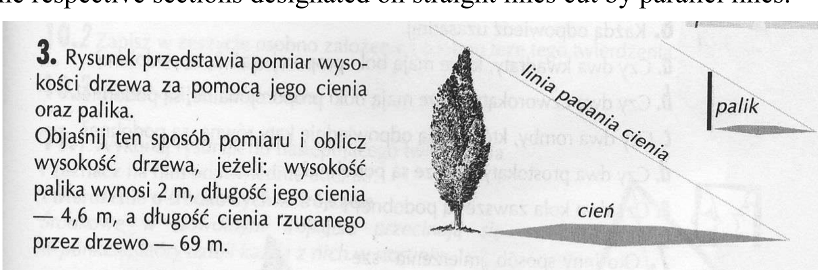

3. Rysunek przedstawia pomiar wysokości drzewa za pomocą jego cienia oraz palika.
Objaśnij ten sposób pomiaru i oblicz wysokość drzewa, jeżeli: wysokość palika wynosi 2 m, długość jego cienia — 4,6 m, a długość cienia rzucanego przez drzewo — 69 m.

Figure 3. The task of measuring the height of the tree, placed in the chapter 'application of Thales theorem' from one of the Polish textbooks for middle school students (Zawadowski, 2006)

[1] Formulation for junior high school students: when two non-parallel lines cut several parallel lines, then ratio of respective sections designated by these lines on non-parallel lines are the same.

We assume that students - even if they know this theorem - see such an application too ‘far-fetched’. This is consistent with findings that ‘*mathematical activity in workplaces, where and when it occurs, looks very different from that in educational settings’*. (Wake, 2014). It is unlikely that a modern woodsman has hammered pegs and measures the shadow. Regardless of the fact that for the measurement of distance in the real area the ratio properties are actually used, such a problem should be introduced differently.

MATHEMATICAL BACKGROUND OF THE ALTIMETER’S FUNCTION

Already by looking at the device, you can see the use of mathematical concepts. At the front of the altimeter there is a shield on which are shown respectively located angles. This suggests the use of the apparatus for measuring angles. The altimeter is equipped with some kind of a protractor. The movement of the device allows us to determine at what angle it is tilting. When the altimeter is in the horizontal position, the red indicator shows a value of 0, which means that the angle of inclination is zero degrees (Figure 4).

Figure 4. Altimeter in a horizontal position

Figure 5. Altimeter inclined at the angle 40^0

By changing the level of the altimeter, we do not change the orientation of the circle with angles - it will remain in the same position but the indicator will show on what angle the altimeter was leaning (Figure 5). The wheel does not rotate with the device, because it is balanced (heavier at zero, indicating level).

The maximum angle that can be read is 90 degrees - which from a mathematical point of view is sufficient, because wanting to measure the angle / height of the object standing on any distance, the angle of measuring is always less than 90° (the sum of acute angles in right-angled triangle is 90^0).

At this stage of the analysis of the device you can already investigate the following issues:

- How does the reading of the angles work, how does it differ from reading using a protractor?

- Why is the reading of the angle limited to 90^0?

Another issue that can be discussed when analyzing the construction of the device is the design of the scale and the principle of measuring the height of the tree.

Figure 6. Instruction

Figure 7. The scale

The scale which is located on the reverse side shows how the angle is converted to meters. A thorough analysis of the recording enables us to find a hint: under the table the instructions are shown (Figure 6). The correct interpretation leads to the conclusion that it is related to the measuring done from a distance of 20 m. Mathematisation of the measurement suggests that the distance from the object acts as a one hypotenuse, and the height of object as the other one. The reading on the scale shows the relationship between the height of the tree and the viewing angle. This instruction can be confronted with the scale placed on the reverse (Figure 7).

And so, at an angle of 450 you can see the value in meters: 20 meters. This is an information relating to the relations in a right-angled isosceles triangle. These compounds can be further tested. For example, for a 350 angle the value of 14 m appears. In a natural way the attention is directed to the relationship between hypotenuses in right-angled triangle and a measure of the angle - this relationship is expressed by the tangent function. The ratio of 14/20 = 7/10 = 0.7 is therefore the tangent of the angle, which is found in the trigonometric tables. Indeed, the tangent of the angle 350 is 0,7002 (Figure 8).

α	sinα	cosα	tgα	ctgα
0°	0	1	0	-
1°	0.0175	0.9998	0.0175	57.29
2°	0.0349	0.9994	0.0349	28.6363
...	...	...	...	...
34°	0.5592	0.829	0.6745	1.4826
35°	0.5736	0.8192	0.7002	1.4281
36°	0.5878	0.809	0.7265	1.3764

Figure 8. Trigonometric tables for the tangent function

By looking through the viewfinder, however, we see a double scale. On the left we have defined the height of the measurement carried out from a distance of 20 meters, while on the right, standing at a distance of 15 meters - according to information on the altimeter obverse (LEFT 1:20 RIGHT 1:15)

Figure 9

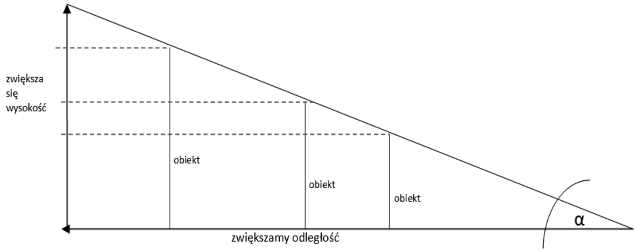


Figure 10

Altitudes read from the left scale are provided more densely than on the right. The difference is due to the fact that with the same viewing angle a greater distance will affect a greater height (Figure 10). This fact is related to the mathematical understanding of the similarity of the triangles. In the figure beside it (Figure 9) you can see that the place on the scale on the left side corresponding to the size 2 is balanced by scaling on the right side at 1.5. This is obvious, since 2/20 - 1.5/15. You can also relate them to a particular interpretation of the Thales Theorem in which the measure of corresponding segments are proportional to each other.

The concept of proportion is one of the most important in mathematics, used in many mathematical concepts and theorems. Among them we find the concept of rational numbers (and as a consequence, irrational numbers), percentages, scale, similar figures, trigonometric functions, proportionality (as a logical consequence the inverse proportionality). Students are not much aware of these links, especially when each of these concepts is developed separately, and when they have no need or ability to look for links between them.

OBSERVATIONS AND ANALYSIS OF THE MATHEMATICS LESSON BY USING THE ALTIMETER

Sebastian, a mathematics student at Rzeszów University, during his work with students from a technical and economic high school, used a device - altimeter - known to foresters, who use it to determine the height of trees. His inspiration came from an article by Wake (2014). The activity took place in May 2016. For school students there was an additional lesson (after their compulsory school activities), aimed on improving their mathematical skills. Therefore one of the aim of activities was a repetition of already-known topics. Classes were not carefully planned, their other aim was to stimulate student's interest in mathematics by showing its usefulness.

Here, we will analyze how he run these classes, during which the altimeter was a "black box" –this means was used as an instrument, which principles of operation are based on uncovered by students mathematical concepts.

The situation described here is not intended to be a proposal for general use; it is just an attempt to show that problems for mathematics lessons can be emerge from different contexts. This is an attempt to move away from the common lessons, an opportunity for the pupils to seek independently for the relationship between the surrounding world and mathematical knowledge acquired during school.

The lesson should start with the question: is there a way to measure the height of certain objects, for which the measurement with the ordinary way is quite burdensome because of the excessive size or the distorted view (e.g. high buildings or trees). It can be also suggested that, in practice, some simple tools are often used.

The actual realistic context can be a stimulant to an authentic exploration. An employee, who uses the device, can treat it as a "black box" without the need to understand why and how some parts act. However, during math classes students can try to analyze the work of such a device and explore its mathematical foundation.

The main question is: how does this reader work?

For students who are accustomed to "button" automation, such a question can be senseless, but it can provide a better stimulus than learning the theory. The instrument is authentic, appropriate, and on the other hand, is so simple that its mathematical background can be described by the students.

Below are excerpts of the transcripts from such activities, together with the commentary.

Episode 1

The teacher sketches on the board a tree and in a distance a person who has to measure the height.

Ex: With the help of the tape measure and the altimeter (which we do not know how it works), is it possible to measure the height of the tree?

S: e.g. to measure the shadow, or any of Pythagoras can insert there .....

T: In which manner?

S1: you must have at minimum 2 distances.

S2: the delta (laughs)

S3: when he will know how far away he is, it will be enough that he will know under what angle he looks at it.

T: where do I have to draw this angle?

S3: from the ground to the top of the tree.

T: As we already have an angle, what it

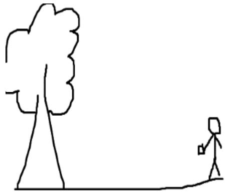


Figure 11. The sketch done by the teacher as an introduction to discussion

gives him?

S3: well, and then a sine or something ...

S5: counts of tangent!

S4: so, he is able to calculate ...

T: how?

S3: because there is 90 and there (... mumble ....)

S4: and generally it ... I do not know. It has 180 degrees, or 360, well, and it adds to the another, well, and there is and everything comes out nicely, beautifully….

Ex: Let's say that the measure of angle, e.g. what?

S1: 30 degrees.

Ex: Well, we have an angle of 30 degrees, how to calculate the height of the tree?

S2: So, from the tangent. Well, because there will determine ... hmm ... It is the ratio of the hypotenuse opposite to our acute angle to the second hypotenuse.

Ex: I'll write it for all to see: tan 30˚ = . Height of the tree we do not know, a distance we can measure, what about this tangent?

S3: we read it from the tables and solve the equation.

Analysis: The preliminary mathematisation of the problem went smoothly. The initial associations are those typically made at school (e.g. to measure a shadow, or any of Pythagoras can insert there....), referring to book-pencil activities. Then, after the first misguided proposals, critically evaluated by the students themselves (with Pythagoras, but you must have two lengths), they noted quite easily that it is worthwhile to determine the size of the viewing angle. The mere use of the theory is incoherent, the students had difficulties both with the correct formulation of the relationships between the values of the angles in a triangle, and with the expression of these relationships that define the tangent function. But they are aware that such knowledge exists (it is 180 degrees, there is no 360 and it adds to the well, and there is, and all goes nicely, beautifully). With the help of the teacher, they slowly recreate the part of the school knowledge. Assuming the specific size of the angle (30^0) they clearly relate their knowledge to the real situation, which requires the use of certain theoretical relationships. They noted that helpful is to know the angle defined by a horizontal line and the line between the top and measuring point. Then, they could use the tangent function, the value of which can be read from the corresponding tables.

Episode 2: The students watch the altimeter

Ex: What is there, what the indicator shows, what happens when you tilt?

S4: So, it shows zero degrees .

Ex: So, what can be measured by this ?

S1: degrees.

Ex: How do I know the degrees?

S5: Yes, 90 is located where is the right angle from zero.

S2: You need to use a tripod to this while making the measurement?

Ex: No, just to hold in the hands, but we'll come to it soon to (back to the drawing). How do we measure the angle of the device?

S3: from the ground to keep the right angle.

Ex: Okay, but can we measure from the height of the eyes, because from the ground is a little uncomfortable.

S3: you only need to maintain the level.

Ex: Yes. So what about the rest of the tree?

S4: He might just know how much is the height from the eyes to the ground.

Ex: Good. Any other ideas?

S1: He can measure by the tape and add, or measure the angle down, then add.

Analysis: The students state by themselves that the device they firmly hold in hands is suitable for measuring angles. They can see that at a particular position the indicator shows zero, and at others it shows different numbers. The analysis of the changing positions confirms their belief that the measurement refers to the size of the angles (Yes, 90 is located where is right angle from zero). An element of curiosity appears; the device is new, unfamiliar, they do not really know how to use it (do you need to use a tripod?). Quite unexpectedly, they state the need to comply with the mathematics rules in using the device (you need to measure from the ground to keep the right angle). Then, in a natural way they create various proposals on how to avoid the necessity of measuring the angle from the ground.

Episode 3:

Ex: There's just a problem, because we will not be able to read the angle at the front of the altimeter, because we cannot perfectly set it on top of the tree. Try to look through this device and describe what you see there.

S1: digits. (laugh)

S2: degrees.

Ex: And beyond graduation and digits there is anything else you see?

S1: Well, there are some pluses and minuses.

S2: like a thermometer.

Ex: What do you think the pluses and minuses mean?

S1: Positive and negative.

Ex: But what positive and negative?

S1: Angles. When you measure up to have positive angles, and when you go to reverse negative.

Ex: Where are plus and minus located in the altimeter ?

S1: Well, as you move up than the scale goes down.

Ex: Why is that?

S1: Because there is a gyroscope. And it always works in the opposite direction.

Ex: But please explain this phenomenon by watching this device.

S1: Well, as I move it, the device is moved, and the wheel is in the same position. There must be something heavy.

Ex: then looking through the altimeter, why plus at the bottom?

S1. Well, because as this circle rotates, this rotates inside.

Analysis: The mere reading of the scale arouses surprise. The decoding is not so obvious. So far, they do not know what the scale shows - measuring angles or height. The students only see that there are some numbers on the scale. The double-sided indication of the scale is associated with the thermometer. By moving the altimeter, they are able to see the movement of the pointer on the scale. This convinced them that this is related with reading of the size of the angles. Probably they treat this measure as a measure of the rotation, assuming that 0 is located on the scale as a reference point. And by this, the measurement of angles is extended on the positive and negative part. They note, however, an important connection between the way how to move the device and direction of the movement of the pointer on the scale: this circle as rotates as this inside. They can therefore interpret the change of the reference system: by lifting up the device, which is indicated by the symbol + on the axis , we cause the looking level to be over the axle or the axle to be below the level of vision - the subjective perception axis marked level of 0 will go down. One student, commenting on this phenomenon, was not surprised by this fact at all (because there is a gyroscope, and it always acts in the opposite direction). In that case he used his ‘technical’, non-mathematical experience. The teacher did not comment at all and he let the students play the role of experts. He created the space for their own experimentation.

Episode 4:

Ex: What do you think are the numbers in the middle?

S1: maybe it is just the height of what we see here, for example?

S3: exactly, for example as we move away, say, one meter, then as you move, what is the height.. Ex: Good thinking, but from where to measure in order to read this height?

S2: from any given distance.

Ex: There is a small hint written on the front. Who can find it? (..) Is the LEFT and RIGHT and what is written there?

S5: 1 to 20 and 1 to 15

Ex: Uh-huh, and what that might mean, what is this anyway?

S1: Scale.

Ex: Yes. If you will read the right side of a certain value, it is the height of an object when I'm away from it for ..?

S5: 15 metres.

Analysis: The first interpretations of the scale are proper – the students assumed that the scale shows the height of the object being measured. It is still not clear how - this is a typical 'black box'. Students know that they need to move away from the measured object (as we move away, say, one meter...), but it seems that this distance is only a technical matter, to cover the measured object through the viewfinder. Then they presume that, however, this distance should be set (well, with some given distance). The teacher suggests paying attention to a hint placed on the cover. It becomes clear that in order to read data from the scale to the right, you must be at a distance of 15 meters. They assume that the scale reading on the left side is linked to a distance of 20 meters. Still they do not know why, and how these scales are created. You may also note that the scale on the left side is denser than on the right, despite the fact that it corresponds to the same angles. So we can conclude that at the same angle a greater distance corresponds to a higher tree, and a smaller to a lower. But what about the other distances? While working in the real environment it is not always possible to keep the desirable distance. An opportunity to analyze this problem is created by the students themselves, by considering the problem described below.

Episode 5:

Ex: Can we measure the height of a tree while standing closer?

S1: For example, when we are in 10 meters, is it enough to divide the result in half?

S2: no, no, definitely not ...

S1: total no chance.

Ex: Let's say that we are far from the tree ..?

S1: 6 meters.

S2: 6/20 of the distance will be.

Ex: And the altimeter will read the height from a distance, eg. 20m. Is it possible to somehow estimate …...the height of the tree? (Draw)

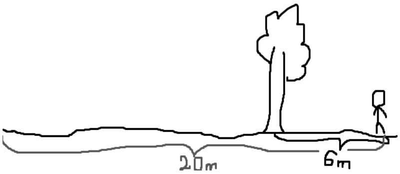


Figure 12. Illustration to the problem: Can we measure the height of a tree standing closer?

Ex: (...) No ideas?

S1: The angle will increase.

Ex: how to enlarge the angle?

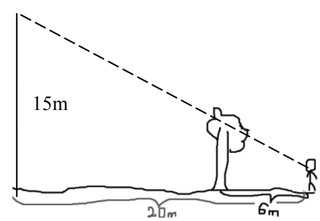
S1: it will increase by itself.

Analysis: The first attempts to solve the problem: how to use the altimeter, standing at a different distance than the one suggested by the instrument, present a common sense approach by not referring to the mathematical idea of preserving the aspect ratio. Although actually, while standing at a distance of 10 m (instead of 20) it would be enough to divide in half the value read on the scale, but the students themselves reject such a possibility. There are other attempts to manipulate the numbers, but also it did not rouse the enthusiasm of others, specifically students do not see the sense of such calculations. By forgetting about the device, they use the intuition of looking on the object (the top) with different distances - when by approaching to the object changes the view angle. But foresters, who use this instrument in the environment, are able to cope with it. The discovery of this method is another opportunity to go into the mathematical content, performed in school.

Episode 6:

Ex: Let's say that we measured the tree. We read ... what value?

S1: 15 meters.



Ex: Let it be 15. So it is a 15 meters high tree, if is measured from a distance of 20 meters. Do you recognize this drawing now?

.... (no reaction)

Ex: Let's look at what data we have. The device shows the 15 meters as the height of the tree, if we are 20 meters away. We know that we are in distance of 6 meters from it and we also know that it is the same angle, right? Can we in some way calculate the height?

S1: probably.

Ex: How?

S1: by dividing ...

S2: But we have 6. We do not have the angle, right? As if we try to count using Pythagorean theorem, this distance at the top, and then marked the shorter distance (from the person to the top of the tree) as x, then it could be by subtracting this x.

Ex: And how do we calculate the x?

S1: I do not know, some equation would do that.

Ex: can you come and try?

S1: No, it is not suitable, however ...

S2: It will be a congruent triangle.

S3: The similarities are needed.

S2: This is similarity, 15 by ..., it is as much as twenty-sixth.

S1: 6/20, it is like x to 15, and no game...

Ex: (...) Does anyone see maybe some relationship, through which we can facilitate this calculation, not saving every time ratio? What is enough to do, if we know how far we are and we have the measured value from the altimeter?

S5: you have to multiply the distance at which we are measured by height divided by twenty and will get x.

Analysis: In this section we can see an authentic exploration, culminating in the formulation of an important proposal. Finally, the students were clearly satisfied, which is apparent in their slang. The initial ideas suggest trying to fit different pieces of knowledge, fitting different tools - students did not have a clear plan of how to solve the problem. The overwhelming mathematical knowledge was fruitful, since it lead to associations with prior knowledge so that it becomes a tool for problem solving. With the adoption of some specific numerical size (15 m - height read on the altimeter on the left side, which is where the measurement is taken from a distance of 20 m, 6 m - the actual distance of the object being measured) they built a ratio, based on the properties of similar triangles. Then, this result can be generalized to form a useful pattern for measurements at any distance.

SUMMARY

Regardless of the number of independent trials, the role of the teacher was significant in each of the described scenes. Despite the fact that the teacher suggested the issue around which the discussion took place, it could happen that the self-extraction of mathematical facts could fail. An inquiry based education does not mean to leave the students by themselves; this new role of the teacher is significant. As highlighted in many studies "when students learn science in classrooms with pure-discovery methods and minimal feedback, they often become lost and frustrated, and Their confusion can lead to misconceptions' (Kirschner, 2006). The teacher does not only watch over the correct interpretation, but directs the investigation.

Not all the mathematical relationships associated with the construction and use clinometer Sebastian was able to touch. This simple device can be an opportunity to pose many other problems. One might ask, what objectives were reached within the framework of these activities through the "black box". In our opinion, these were:

- Showing the relationship between theory and practice on mathematics (searching for relationships between theoretical mathematics and mathematics in applications).
- Linking different pieces of mathematical knowledge into a coherent whole, including:
    a. various elements of the knowledge about angles (angle as a measure of the rotation, measuring angles, finding its measure in the different reference systems),
    b. trigonometric functions,

   c. similarity of triangles, Thales Theorem (additionally the use of Pythagorean Theorem),

- Changing attitudes towards mathematics as a school subject by showing that it can be a useful discipline at work.
- Building mathematical knowledge on students' own extracurricular experiences.

During these classes, the students raised many mathematical problems, prior to their learning in school but treated them as a very distant from each other. Some of the ideas were regarded as nothing telling them catchword ('Thales'). Now they took on other content, turned out to be useful. One may wonder whether such aims are important in the learning of mathematics, but in our opinion they are. In addition, from a research point of view, we were able to test a proposition of activities carried out in the spirit of "inquiry based mathematics education", associated with the problems of teaching mathematics in vocational and technical schools.

The students were asked to comment on these classes. One of the survey questions was*: should the lessons of mathematics in technical school contain more technical problems? Do you like these lessons? Explain*. Almost all the students responded positively. Here are some answers:

- In the technical schools we should solve technical problems because it helps us to understand what is really calculated and in the future it will be easier to solve a similar problem.

- I think that most of the material that we currently take is unnecessary to life or to work. Actually we are trained to be prepared to Matriculation exam.

- Yes, because we are able to remember more, to learn more.

- Yes, because it is not boring, practice is always better than theory.

It is hard to say that these statements express an enthusiasm for mathematics. But if students generally found that they could see the meaning for such actions, this could be the first step to the awakening of the motivation needed to learn the subject.

Acknowledgement

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FROM A PROBLEM TO AN OPEN PROBLEM

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In the present note we report on the research type activity conducted with a group of middle and high school pupils. Generalizing what students learn at school about the Euclidean plane geometry, we consider parallel problems in finite planes. The open problem mentioned in the title is stated here as the Conjecture. We hope it might be a convenient point for resuming the study both from the mathematical as well as from the educational points of view. The case study presented here argues for involving school students in open ended projects; in mathematics open problems are often more interesting and appealing than solutions. Also, this approach provides an opportunity to show to the children that mathematics is not given once and for all but it is created/discovered in a slowly developing research process.

INTRODUCTION

This paper is a case report on a part of a larger grant entitled “Configurations of lines and conics” run from November 2014 to December 2015 within the University of Young Researchers program funded by the Polish Ministry of Science and Higher Education. The grant involved 9 children in grades 7, 8 and 11 (i.e., the classes 1 and 2 in middle school and class 2 in high school in Poland, age 14, 15 and 17 respectively), three teachers, a number of graduate students from the Pedagogical University of Cracow and the authors of the present note as the scientific coordinators. The achievements of the project as well as the working methods are fully reported on in (Szemberg, 2015).

Two months before the project started (even before it was approved by the founding agency) the leaders of the project visited both participating schools, the Jan Matejko Middle School in Zabierzów and the Zofia Nałkowska High School in Cracow. During two meetings with groups of around 30 students held in each school we presented mathematical backgrounds on the subject of (Euclidean) line configurations without revealing the possibility of carrying out a research type project. Such visits of academic staff holding lectures or mini workshops in schools around Cracow are quite common. After the project was approved, the leaders visited the schools again. All students participating in the first meeting were given a chance to participate in the project. There were only 2 students in the High School interested in further meetings and 8 in the Middle School (one of them dropped off after two weeks). All students were male. They had all good and very good marks in mathematics but were not outstanding in their achievements. It was one of objectives of the program to test how far one can go with reasonably good students but with no prior contact to non school mathematics. As it turned out, one can go pretty far.

THEORETICAL BACKROUND

The story told below is a testimony of two research mathematicians and hobby educators. The focus is therefore on mathematical problems and ideas rather than on analysing the students' participation in the project. Engaging students into solving problems is a well-known strategy advocated for by Alan Schoenfeld, see (Schoenfeld, 1985) and many others. Schoenfeld himself got inspired by George Pólya's book (1945), see also (Schoenfeld, 2013). Of course in order to be able to solve any mathematical problems, one needs first to pose them. Whereas it is very easy to come up with a very hard problem, it is very hard to come up with an interesting and challenging problem which might prove feasible, see (Silver, 1994). It is an even bigger rarity if it is students who propose such a problem. The story below revolves exactly around a rarity of this kind. For a theoretical point of view on the aspects of posing and solving problems with students of approximately the same age as involved in the project see (Arikan, Ünal, 2015) and (Silver, 1997) for general framework. The Conjecture posed in the last section is stated here rather as a sample of a good problem to engage other groups of students than merely out of professional interest in its solution.

METHODOLOGY

The activities of the project consisted of regular meetings of project coordinators with the teachers and children. Some meetings were led by graduate students, who were trained before, and supervised by senior staff. Starting from a lecture style introduction of a couple of minutes the meetings quickly turned to open discussions. Initial hesitations to participate in these discussions on the part of the children evaporated after the first few meetings. An important factor in gaining the trust and building a partner relationship, in our opinion, was the openness for students' questions even if they led to problems which were initially not planned to be discussed. A nice feature of working in the project was that there was only a very general research framework so we had complete freedom to follow those paths which at the moment seemed most exciting or which contributed to better understanding of the facts already handled. Whereas there was a scenario for each meeting, i.e. it was known how it was going to begin, we were not able to predict how the meeting was going to end. It was quite exciting for everyone involved, regardless of their mathematical experience, to see where the discussions would lead.

Another nice feature of the project was that funds were available to perform two day intensive workshops, where we were able to work on certain questions in a focused manner and with basically no time limit imposed upon us.

UNDERLYING MATHEMATICS

Mathematically the project was based on the well-known Sylvester-Gallai Theorem.

Theorem (Sylvester-Gallai)

Given a finite set of lines in the real projective plane either all lines belong to the same pencil (i.e., they all meet in the same point) or there is a point in the plane through which exactly two of the lines pass.

The assumption that the plane is real is important. The statement fails over the complex numbers and also in finite characteristic. One of the purposes of the project was to introduce finite projective planes and to study configurations of lines defined in these planes. (For the notion of a finite projective plane see the Definition below). Note, that if K is a field with $q=p^r$ elements, where p is a prime number, then all lines in the projective plane defined over K give a counterexample to the Sylvester-Gallai statement. Indeed, through any point in this plane there are exactly q^2+q+1 lines. However the following is an intriguing and to the best of our knowledge an open problem.

Problem 1

Find the *minimal* number of lines in the projective plane over K for which the S-G statement fails and determine what are the combinatorics of this configuration of lines. In particular: is this configuration uniquely defined up to projective change of coordinates?

This problem was not solved in the project. It provided however nice motivation to introduce the concept of finite geometries and to study projective planes over finite fields from both the geometrical and algebraic perspectives. Surprisingly, these concepts, after some training, were not only quickly grasped by the participating pupils but pretty soon they liked this kind of geometry even better than the usual, Euclidean, school geometry. The reasons they provided were:

- in the finite geometry we can draw everything (after a while they seemed to ignore the initial oddity of lines consisting of finitely many points only and another fact that although points belonging to a single line sometimes were collinear (see Figure 1) in the usual sense, sometimes their graph, formed what they called “broken lines” (see Figure 2)), and
- we can compute everything (again, after some initial awkwardness the pupils had no problems with computing modulo a prime number).

Surely enough, these bold claims were to be given a blow (as we describe in the section “Hitting the wall”).

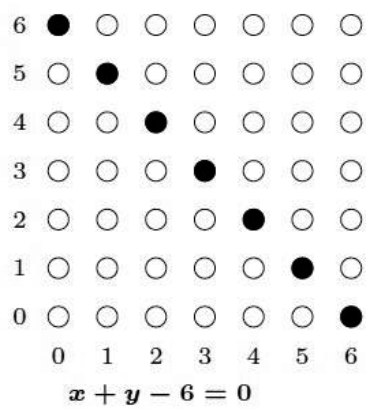

Figure 1. A line

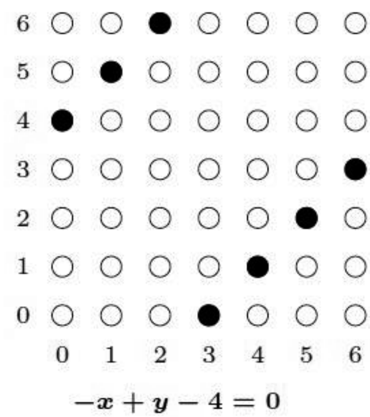

Figure 2. A “broken line”

BRINGING THE COLORS INTO THE GAME

During one the two days workshops students were discussing some problems they encountered recently (not necessarily connected to the project) and trying them as quizzes on us, their teachers. One of these exercises came from the Mathematics Olympiad for Medium Schools in Poland in 2013:

Problem 2

Paint a plane in such a way that every line in the plane has points of at most *two* different colors. What is the highest possible number of colors, which can be used to paint the plane? Justify the answer.

In the view of what was discussed earlier we should complete the problem by indicating that the plane here is the real Euclidean plane. It is implicitly understood in our setting as the only plane known to school pupils, which is quite regrettable, see (Brumbaugh et al. 2006, Section 12) and (Lenart 1993).

There is no unique solution to the problem. Maybe the simplest one is to paint one point in the plane with one color, all other points in a line containing this point with the second color and all remaining points of the plane with the third color. It is not hard to show that four colors cannot be used in order to fulfil the problem conditions. Making a rigorous argument requires however some effort and is proposed to the reader as an amusing challenge.

Problem 2 is a good problem. It allows various generalizations. One possibility is to consider the problem in three dimensional space. Then the solution presented above generalizes easily.

Another direction of thinking would be to increase the number of points allowed on every line, i.e.:

Problem 3

Paint a plane in such a way that every line in the plane has points of at most *three* different colors. What is the highest possible number of colors, which can be used to paint the plane? Justify the answer.

It took all of us a number of false guesses and ideas leading nowhere before the pupils realized that the answer is very surprising: the number of colors allowed now is *infinite*!

Solution to Problem 3

Let C be a smooth conic in the plane. And let every point in the conic be painted with a different color. Thus the number of colors used to paint the conic is infinite. All points in the plane not lying on C are painted with yet another color. It follows for example from a toy version of Bezout's Theorem, see (Gurjar, Pathak 2010), that every line in the plane then meets the conic in at most two points, so that it contains points of at most three colors.

As the project was concerned with finite geometries it is natural to consider Problems 2 and 3 in this setting. It turns out that the solution to Problem 2 does not depend on the underlying field. On the other hand the solution to Problem 3 certainly does, since there are only finitely many points to be painted.

HITTING THE WALL

The construction outlined in the Solution to Problem 3 works of course also in finite characteristic. Hence if K is the field with $q=p^r$ elements, then there are in the plane over K smooth conics containing q+1 points, so that the lower bound on the number of colors which can be used is q+2.

It seems much harder to come up with good upper bounds. Thus the investigations were restricted to the special case of q=5 (i.e., p=5 and r=1). Even in this case the students were able to bound the number of colors possible to "only" at most 12, see (Niewiara at al. 2015). They were quite disappointed especially that the quest for an improved bound took a couple of weeks. Finally they got an idea to solve the problem by "brute force", i.e., writing a computer program in order to check all possibilities.

This idea was not bad, because it is not easy to program a correct algorithm for this problem. They certainly expanded their computer skills while writing up executable code. On the other hand once the program was there, it became clear that on available machines it would take years before it runs through all possibilities.

This was the point where the pupils learned, to their astonishment, that there are a number of perfectly valid computer proofs in contemporary mathematics. The best known is probably the proof of the Four Color Theorem (4CT for short):

every map in the plane can be colored using at most four colors. Again a colorful problem!

Whereas the 4CT can be now proved on a home computer, there are other problems, which still require the most powerful machines or even are resistant to currently available computational tools. One of these problems addresses a very natural question which was asked at the beginning of the project we report on here.

Question

Are there finite planes for which the number of points is of the form q^2+q+1 for q *not* a power of a prime?

In order to understand the question correctly, we recall briefly the basic notion. It is convenient to do this in the projective setting.

Definition (A Finite Projective Plane)

A finite projective plane is a pair of sets: X (the set of points) and L the set of subsets of X (the set of lines) such that:

- for an arbitrary set of two distinct points in X, there exists a unique line in L, which contains both of them;
- for an arbitrary set of two distinct lines in L, there exists a unique point in X, which belongs to both lines;
- there exists a set of four points such that no three of them are contained in a line.

It is easy to check that a finite projective plane must contain a^2+a+1 for some integer a. This integer is called the *order* of the plane. If a=q is a power of a prime number p, then there exists a field K of characteristic p with q elements. The projective plane defined over K has q^2+q+1 points. Thus the first number interesting from the point of view of the Question is a=6. It was established classically by Euler that there is no plane of order 6. It took over 2 years on a CRAY machine to check that there is no plane of order 10, see (Lam 1991) for a very enjoyable account on this history. And it is still *not known* if there is a finite plane of order 12!

The students were very surprised when they learned that even though their results on Problem 3 for the plane of order 5 are incomplete, they are still worth publishing. In fact they have been not only published in the proceedings of the Euromath Conference held in Athens in 2015 but with this paper they won a number of prizes in nationwide competitions for writing and presenting a mathematical paper.

HEADING OUT

There are at least two points which are worth mentioning here in the way of conclusions. First, the students taking part in the project were all pupils from average schools and their former mathematical performance was also not that outstanding.

Second, thanks to the project they had an opportunity to learn that mathematics

- is not a dead subject, there is a lot of activity and a lot of research going on in mathematics;
- as taught at school is a manifestation of an accumulation of centuries of research coupled with didactical approaches which varied over the centuries;
- is full of problems which are not solved or solved only partially or whose solutions demand for simplifications and deeper insights.

We dare to say here that this kind of knowledge is never obtained in mathematical competitions, mostly because they contain hard but solvable problems. Additionally the solutions are typically short, once one learns the "trick" (or just the missing piece of knowledge required to solve the particular problem).

There are various ways we can meet needs of gifted students and there are more gifted students than usually assumed.

WHY SHOULD EDUCATORS CARE?

Before we conclude, we would like to make a few comments about the methodology. These comments might appear naïve and we are looking forward to discussing them with professional mathematics educators. The reflections come long after the project has been finished and are prompted by referee's comments. The approach taken on in our project was intuitive and therefore very simple. The main problem in teaching mathematics (and in fact any other subject) observed by us is that the students get easily bored. They fall into the bored mood if they cannot follow the discussion, they do not understand what is going on and why. In the worst case scenario, they are exposed to a frontal lecture with the lecturer focused on passing over certain topic or piece of theory rather than in students' comprehension and appreciation for the beauty of mathematics involved. In a somewhat better scenario, the students can follow the topic, are prompted pretty frequently to answer questions and notified if their answers were correct or not. However these answers do not influence the course of the discussion, the subject remains the same as determined before the class has even started. In our project we were able to wake the creativity and curiosity on the part of students. They were prompted not to answer our questions but rather to ask questions to their fellows or to their teachers. At the beginning it was encouraging and easier for them not to ask their own questions but rather to try to anticipate which questions the teacher might ask at the moment. Of course these guesses were their own, but psychologically it was easier for the participants to formulate them as if they would come from a teacher. Making a wrong guess seems to be easier to live with than making a wrong own statement. After a couple of meetings the participants observed that there were no "stupid" questions and that the teachers in the group frequently also had difficulties in answering all mathematical questions on the spot. They discovered that there is much more in mathematics than the teachers and professionals actually know or than is written in textbooks, no matter at which academic level. The moment they realized that not only they but also us actually discover new mathematics was the moment when there was no more them and us. There was a group of people joined and driven by common curiosity about some mathematical statements.

Were we lucky with the problems we came across and with the students? Could it work with any other similar group in similar circumstances? Educational studies look for common patterns and ways to reproduce certain effects. It might be however a delusion that all people exposed to the same education circumstances would react in the same way. The agenda we would prefer to state is that education (not the schooling) similarly to research often takes unpredictable turns. The process is far too complex to be completely controlled, see (Thomas, 2013). The experiments are hardly possible to repeat. It might be that the feeling of novelty, the curiosity, the uncertainty have to be genuine also on the side of the tutors. One cannot explore the same story again and again, even though one actually does and has to do so in the schooling. Working in off school groups offers extra freedom in choosing initial topics and in just letting the things roll.

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