Communication in the Mathematics Classroom

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MATTHEW'S STORY ABOUT FALLING TRIANGLE

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The domain we are dealing with is the understanding of geometric transformations. Our assumption was to start from student imagination related to the physical movement of objects, and on the basis of their own solutions lead the mathematization of motion. Studies have been carried out on a small group of selected students from classes V and VI (12 - 13 years old). We have planned to link teaching with what the student is doing spontaneously. We used reference to the actual motion of the rigid figure in the physical space to enter the student's understanding of mathematical isometric transformations. However, the student does not perform here a series of well-planned exercise, leading to a fast and reliable describe of the transformation. Analysis of this meeting highlights several important facts:1.in the domain of basic geometric intuition, the main tool for transmission of information is the image and gesture, 2. observations argue that the appropriate use of language can be a tool for supporting the transition to a higher level of understanding of the geometric concepts.

INTRODUCTION

Talk about geometric concepts in a class of is a difficult issue. This is particularly evident at the non-formal level, not only because of the narrow scope of student's scientific vocabulary. Theories about the formation of geometrical concepts emphasizes that the origins of geometrical knowledge proceed without words, and the flow of information often takes place through other channels: visual or through gestures (van Hiele 1986, Hejný 2001). These theories also emphasizes that a tool which assist in the transition to higher levels of understanding is the spoken language. This is why the use of language during geometry lessons is treated as didactical challenge.

The second important issue for us is associated with the formation of dynamic images for geometrical relationships. Dynamic cognitive processing of images is a key factor in solving geometrical problems (Szemińska 1991). However, the first geometrical knowledge is static – consists on shapes identification or perceiving position object-to-object (Vopěnka 1989, Swoboda 2006). To give a dynamism of images is another educational challenge.

Is it possible to take these two challenges at the same time? Attempt to deal with such challenges is presented in this paper.

The domain we are dealing with is the understanding of geometric transformations. Textbooks' approaches to transformations are necessarily static: textbooks' authors can use only drawings or definitions. Dynamic understanding of the transformation is often supported by computer programs (eg. Cabri or GeoGebra, various applets with possible observation of the effects of mirror-reflection or changes objects' position by rotation). Environment is designed so that the student uses the transformation already defined, does not create an image of figures (although it is possible) and the goal of learning is to observe the effects of the transformation and to distinct its property.

Our assumption was to start from student imagination related to the physical movement of objects, and on the basis of their own solutions lead the mathematization of motion.

MANIPULATIONS AS A BASIS FOR LEARNING GEOMETRY

There are many publications dedicated to child's actions for creation of geometric concepts. Some of them are related to the use of various objects to manipulate (manipulatives, or virtual manipulatives). However, in general, activities were organized so that the child rather focused its attention on the final effect of manipulation and not on the performed movements to achieve expected effect.

There is still not so many researches about how students represent the movement, although these are beginning to appear (Kotsopoulos et al, 2014 Kuřina, Tichá, Hospesová, 2008). Children's drawings depict different ways to express motion - may be arrows denoting the direction of the road (moving on the map), or attitudes of people and animals in motion.

Among the studies on the understanding of geometry there are also those that are focused on the analysis of the movement as such - they are either researches of the perception of movement by the student, or of the importance of children's movements during problem solving (Swoboda 2013, de Freitas, McCarthy 2013). This may indicate a grow of interest in this research area.

In general, research results show that the children's assessment of static symmetrical relations is largely correct (Bulf 2011, Xistouri 2007). Among isometric relationships spontaneously performed by 4-10 years old children axial-symmetry or parallel transformation are the most common (Jagoda, Swoboda 2010). Static reception of object to object relationship does not guarantee that these images are synchronized with dynamic interpretation. Although it is believed that each of the geometric transformations is, for children, a prototype of the movements and transformations of physical objects with the reality that surrounds us (Siwek 1998), but unfortunately, this belief is not supported by research. On the contrary, it appears that awareness of the use of movement can come to students with difficulty.

METHODOLOGY

Studies have been carried out on a small group of selected students from classes V and VI (12-13 years old). They are those students who have no problems with mathematics and are open in their contacts to others. The whole research consisted of a series of meetings aimed at bringing the students to describe the mathematical feature of these movements in the physical space, which can be the prototype of isometric transformations of the plane.

During each session only two students and two researchers were present. We treat them as a pilot study prior to the main stage of research.

According to the curriculum, teaching about geometric transformations starts from axial symmetry treated as a mirror reflection. In the subsequent years of schooling students learn about axisymmetric figures, then the concept of congruent figures is introduced. Therefore, for students participating in the study, concept symmetry was associated with the mirror reflection, because in this area they have the biggest experience. They were not familiar with any other geometrical transformations. If they had some intuitions in this field, they were built out of math lessons.

In the first part of the study students have worked in pairs. This was intentional, to give them the opportunity to talk with each other, to share ideas. In addition, the teacher and the observer could ask students questions, thus provoking them to verbalize their activities or to explain the meaning of certain actions. The conversation had to be the main source of information about how the task is realized.

Students received a small book (size 10cm x 10cm) with 30 clean sheets and a template of a triangle. During the work they had an access to a basic geometrical instruments (ruler, set square, compass, protractor).

Teacher has presented a problem to the students in the following way:

> Teacher: You are well familiar with cartoons, I guess. Creators of such films make a series of pictures, which, when quickly viewed, look like a movie. Look at the drawing of a butterfly in this book (shows that a butterfly drawn in the corner 'waving' wings). This butterfly's wings are waving. I have a booklet, which also will be well-leafed (she shows it) and I have prepared a template of triangle.
>
> We would like you to draw a triangle in this book and made the animations, that it changes its position.

The aim of the research was to find answers for following questions:

- Does the statement 'change of position' will be understand by students as a change in regular, continuous movement?

- Among these movements, can physical prototypes of geometric transformations such as rotation, parallel translation, possibly combination of these transformations be distinguished?
- If such prototypes occur, how far will their implementation adhere to the mathematical representatives of such concepts.
- What will depict this movement - words (what kind), gestures (what kind), drawings (what kind)?
- Which of these representatives will fall within the definition of a geometric transformation.

THE ANALYSIS OF MATTHEW'S WORK

In this study we will be focused on the selected parts of 13-year old student's work. Matthew was a dominant personality in the analyzed group of students. Only in certain parts of this session the collaboration with a colleague consisted of a common discussion. Apart from this, Matthew used friend's comments. Sometimes it resulted in paying attention to important parts of his own work, sometimes gave the opportunity to verbalize own views, to seek arguments.

Posing the problem before the students have provoked them to a discussion. From the beginning the students directed their action in continuous and regular movements. The first issue which they wanted to clarify was associated with the type of movement. These movements, however, they presented only by gestures:

Fig. 1a

Fig. 1b

> Matthew: the location, in the sense that ... (*here the student shows one movement on the bench – he puts two pointing fingers of both hands together and then one arm moves parallel*, Fig.1a), ... and maybe such that it turns? (*one finger resting on the bench, and the other is doing the rotation*, Fig 1b.)

Matthew, thus, has distinguished two types of movements, which are intuitive equivalent to parallel shift and rotation. He marked the trajectories of those movements: the first was a movement on the straight line, the second – on the circle. The trajectory was, thus, the first element differentiating rigid motions in the plane.

Later, the student had a desire to present a parallel shift. The second boy - Michael - accepted the idea of his colleague and let him realize it, despite the fact that he had a different idea for the task. Eventually boys created separate and different animations.

When Matthew decided on how his triangle will change its position, he started to draw. Course of his work indicates that he mixed random elements with planned ones. The boy was constantly accompanied by afterthoughts, he has checked his work, corrected, commented.


Fig. 2a


Fig. 2b

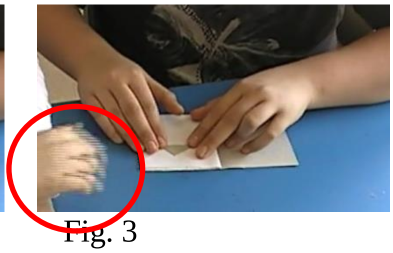
Fig. 3

The first element, which he decided to establish was a trajectory.

Matthew: But first, I will make a pencil line on which it would move (he moves his finger down from the top card, then shows the line by laying hands (Fig.2a, 2b)). The line, to know as it will be moved (he arranged a vertex of the triangle on the imaginary line and moves the triangle parallel down the line).

Matthew indicates two elements that are important for him. The first one is the track, which is represented by a straight line. He also justified why this line is needed - it is designed to maintain a constant direction. It means that he is thinking about the vector of displacement, which is constantly in the same direction, although, of course, he is not able to express it in this way.

However, although only by gestures, he express some of the properties of this shift - the movement maintains parallelism of sections in subsequent stages of the movement. He is fully aware of this fact, and such a goal he wants to achieve. This is confirmed by a conversation with a friend who suggested some modifications:

Michael: And it can make a little turn (doing movements in wrist over a fixed triangle, Fig.3).

Matthew: If so, it may be the animation as if falling (by both hands he alternately shows the movement as of a falling leaf) on so that it was, but rather it will go down like this (shows a parallel shift).

During further work, this topic came back again. After a few filled 'slides' Michael again suggested to return to a falling leaf motion (Fig. 4). Matthew, however, was not to be persuaded to change plans. He convinced his colleague and his verbal arguments were illustrated by movements of his fingers (Fig.4, 5).

Matthew: If we failed, it would be so (he shows an alternating movement with his two fingers), and I mean it would be all the time as such (holds two fingers stiff and makes the rigid motion from top to bottom). (....) All the time falling down.

Fig.4

Fig. 5

Matthew and Michael talk about different movements. Matthew wants to pursue a rigid motion, in which all points of figure move at the same pace. Physically, we can say that all the points of the figure in a specific unit of time are characterized by the same vector of displacement. The whole discussion is effectively supported with gestures. The expression of these movements by words is probably too difficult. It is clear that the vocabulary of boys does not refer to geometrical names. The word ‘parallel’ does not appear spontaneously even though it functions as an indicator of the accuracy of the solution. Matthew uses very inaccurate terms: it would be so ... and my point is that it was so .... - They have sense only in connection with the performance gestures. These gestures are a code, carrying basic information.

After discussing the task Matthew proceeds to draw his animation. He starts on the first page at the top - he draws the first triangle. Instead of drawing a line, he marks one point on the top edge of the book, going through tops of all following pages. Perhaps in this way he wanted to avoid drawing additional lines on cards.

Matthew: We will measure how many cm. from the wall, from the edge of the paper (he measures) so, as 5 cm, then we will look, we will ride here (he shows).

The statement confirms that the boy wants to maintain a constant distance of the highlighted points of the figure from the edges of cards, which determine the path of movement (trajectory) as precisely as possible. Implementation of this idea is as following: Matthew puts the figure slightly below the selected point and draws a triangle heavily. He focuses on two elements: the vertex of the triangle should be placed on the track marked by the highlighted point and subsequent drawings of triangles remained parallel position of the sides.

That parallel sides proved to be easy when the boy noticed that the drawings of the previous sheets reflected their mark on the following pages. This had a place already while drawing the second ‘slide’. He commented on this fact:

Matthew: super, how cool, do not even need ... (puts triangle slightly lower than marked contour, maintaining the parallel position of the sides of a triangle).

The statement is incomplete, the boy merely remarked that reflected contours of the prior position are treated as an unexpected hint for finding the current position of the triangle. From that moment, his attention was focused not so much on the movement of the triangle along a single line, but on maintaining a visual parallel position of the sides of consecutive triangles. In that case, the whole movement is broken into a series of relationships: geometrical figure - its image. Analysis of his work in this regard shows that these relations a constant vector of displacement remained.

TEACHER'S INTERVENTIONS

The questions asked by the teacher aimed at discovering and naming some of the features of movement realized by Matthew. The boy's reaction on teacher's questions shows that often he was not aware of some of their actions, nor their validity. Matthew often has repeated: I did not think ... and I have no idea. However, after a momentary thought, he could give the answer indicating the realization of these compounds and assessing their suitability. Conducted conversation with the teacher was a tool to assist in abstracting the relevant mathematical relationships, which are the basis for further defining the concept. It is also an opportunity to connect them with his previous mathematical knowledge.

Example 1:

Teacher: And these contours are something you need?

Matthew: Contours? No**, I even didn't know** that they will bounce, only this line ... but it's better for us because we can see that (he stops drawing, moves finger on the desktop down Fig. 6) that is pushed away from the contours, which we drew a second ago.

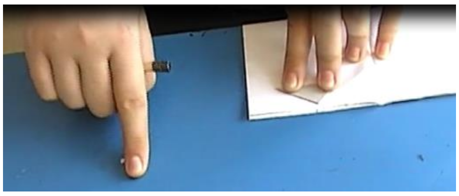

Fig. 6

Again, Matthew is in trouble with the description, he prefers to support his speech by the gesture. He states that he didn't plan such an effect, what he has planned was a trajectory for the vertex (only this line ..). However, he is satisfied with it – in this way it is easier to create the figure in relation to its previous image. In this relation, it is important to preserve the parallelism of corresponding sections. The word 'push away' is used here in a special sense, as 'to move with a constant vector'.

Example 2:

Teacher: So, here you think that the path, the track is important. What else is important?

Matthew: Oh yes, that as a triangle is so (shows one position) is that the triangle does not rotate so (he shows - slightly rotated triangle using one of vertex as a centre) that the animation will show the right fall.

Teacher: Can use the word 'parallel'?

Matthew: Oh, (surprised) ... but ... can be (smiles).

Matthew knew from school notion of parallel line, parallel segments. The use of the word in the new situation clearly surprised him. You might be wondering why. Probably the parallelism was associated only with the static relationship between two objects (straight segments) or he used it to describe the properties of figures (e.g. parallel sides of the quadrangles) and not the behaviour of the figure in motion, even in his animation he has kept this condition. Clearly he had to rebuild the relevance of parallelism, assimilate it to the new situation.

Example 3:

Teacher: Will it fall quickly or slowly?

Matthew**: I have no idea.** It will probably fall slowly ......

Teacher: How do you know it?

Matthew: I ... am moving very slowly .. because **every card** that I move a half ... or less than about half cm ... **4, 5 windows per second** ... it's ... I do not know how to count.

Teacher: And you think it will be as a regular move, or maybe will it have some steps - fall faster, slower ...

Matthew: Maybe it will, but looking at the fact that I'm **still doing almost the same**, in the same line at the same distance from the previous one, it probably will decline at a similar rate.

The intention of the teacher is clear - she wants to talk about the length of the vector of translation. She does not expect the only right answer, does not highlight the important things. She gives him time to think, to search for the right words. The student refers to his own actions and on the basis of this he is trying to give an answer. We can see that this is a process of reflection. Matthew supports his answer by the idea of an animated film in which the tape moves at a constant speed (per second, 4.5 windows). Therefore, the rate of the object's position change can be obtained only by changing the length of the displacement vector. For this vector Matthew wants to maintain a constant length. Thus, the translation vector with all its attributes: the trajectory, the direction and the length, unambiguously defines the changes of position that Matthew wants to realize in his animation.

STUDENT'S REFLECTION

While observing the work of the students we had the opportunity to observe how they reflect on their own actions. Often they were looking for ways to improve expected effect, sometimes – they have made new discoveries.

Example 4:

After a few drawn 'slides' the students decided to test the effect of their actions. Boys are trying to flick through a notebook but it does not go fluently. Finally, Matthew takes a book and starts flipping through the end. With astonishment, he states that the movement is in the opposite direction than he has planned and executed by drawing (Fig. 7).

Fig. 7

Matthew: As such, it is ... but then it will be followed up, because here there is this ... because yes, it should be from the bottom.

When planning his animation Matthew started from the first sheet of the notebook. Thumbing from the end he saw something different than planned. He found the explanation for this phenomenon. He commented that if someone was watching his book from the end, for an effect of a falling triangle it is necessary to start drawing from the final position, this means from the bottom of the card (because yes, it should be from the bottom). These are insights related to the properties of the shift. Matthew noted that the inverse transformation to the performed shift is also a shift, but with the opposite direction.

Example 4:

Matthew: (browsing a book to trigger the animation) Oh, my triangle probably fell a little to one side - dropped in this way. Well, a little removed, ajajaj .. It did another way, it went more to the side (he shows the edge of the triangle close to the greater angle, the right side of the triangle), and moves more to this side (shows two different edges of page in a booklet). Yet these lines would have helped, this grid would help, however, that would not descend from this line.

This is another example showing that the student is trying to adjust their own actions to imagine of rigid movement - parallel shift - taking place in accordance to the accepted intuitive vector displacement. For him the planned vector had to be parallel to the edges of the paper. Here, the real shift took place in a slightly different direction than we had planned. He estimates where a mistake was made (it went more to the side and moves more to this side) His animation doesn't present the vertical motion, but the combination of two movements: vertical downward and horizontal, in the direction of one of the edges of the cards. His further statement (and yet these lines would have helped me ..) is an informal term for the trajectory of one of these vectors.

MATHEMATICAL ANALYSIS OF MATTHEW'S WORK

Matthew is an example of a student who is able to retain features of the parallel transformation. His action is described by paradigms which are different than those found in the formal definitions. However, the mathematical analysis of these designations indicates that in their bases there are the same concepts that appear in the definition, or which result from the definition. For Matthew important were:

A. Features of the vector

1 The line of the vector, identified with the trajectory. The movement of highlighted points of the figure took place on the track. Matthew has marked the trajectory by hand movement or by its appropriate location. Speaking about the trajectory he has used the term ‘line’ in the sense of ‘straight line’. The trajectory of the movement is treated as a rail line. He could recognize the change of directions, especially those unscheduled.

2 Direction of the vector was presented by gestures or indicated by the initial and final position of the triangle. In addition, Matthew has described the direction in words using informal terms, treating the edges of the cards as highlighted strategic objects: from top to bottom. He was also able to see how to trigger the movement into opposed direction.

3 The length of the translation vector, associated with the ‘monotony of movement’. Matthew tried constantly keep the same distance between the position of the vertexes of the triangle on the following slides. In his view, this would give the effect of falling at the same speed. If another position of point A is denoted A ', A", ..., than |AA'| = |A'A"|.

B. Characteristics of the transformation

1 Maintenance of parallel segments from one slide in relation to the corresponding segments on the next slide. This feature was so obvious that it has become a major attribute of the whole movement. It obstructs using the other, initial characteristics. Mathematically, the relation of parallelism took place: AB | | A'B '

2 Maintaining the lengths of the segments. This fact has been imposed by the use of the template. Therefore, it has been undertaken intuitively. Matthew has not manifested any interest in enlarging or size-reduction (nor other deformations of shape), which is consistent with intuition of motion of a rigid figure. Mathematically it corresponds to the condition of the isometric transformation.

In Matthew’s work so many very important elements occurred. They have a direct relationship with the approach to teach an isometric transformation. Thus, building the concepts on that representation can make the image of this mathematical concepts not only mathematically correct, but dynamic, too.

Observations of other student’s work while performing this task confirm that the parallel shift is for them intuitively determined by the vector. The trajectory of the figure indicates the line of the vector, direction is exhibited by determining the initial and the final position of the figure. The hardest thing is to find proper words (specify verbalize, express by gesture?) the preserved length of the vector. Nevertheless, these three components exists while transforming figures.

But how far Matthew himself is aware of the validity of their actions? How far he is able to connect it into IMS[1] he possessed? It is obvious that this requires a large help from the teacher.

SUMMARY

Observations led in a larger group of students has shown that children's intuitions are different. It is a good phenomenon because this natural differentiation can help highlight those elements that are essential for geometric transformations. On the other hand, the results have shown that the answer to the research question is not clear-cut and their application in mathematics teaching – is not immediate.

In Matthew's work we could see that

• The statement 'changing position' was understood by him as a regular change, triggered by the continuous and steady motion on a straight line. It was a movement that may be taken as a prototype of a parallel transformation.

• Implementation of this movement is connected with all properties of mathematical concepts.

• The description of motion was made mostly by gestures. In this way, the trajectory of movement and the simulated position of the object in successive phases of the movement was presented. Possibility to create drawings that Matthew performed alone was very important. In those pictures he contained all the relevant intuitions. On the contrary - words that have used for describing the movement were far away from geometrical terms. Even those that were known from prior schooling, did not apply to the new situation. It turned out that the mathematical vocabulary is hardly engaged in the description of intuitive operations.

Analysis of this meeting highlights several important facts. They are described in the literature, and here were confirmed:

1 In the domain of basic geometric intuition, the main tool for transmission of information is the image and gesture.

2 Observations argue that the appropriate use of language can be a tool for supporting the transition to a higher level of understanding of the geometric concepts. In the described situation, Matthew has worked on an intuitive level. Appropriate naming of his performance, awarding items important for mathematics, should help distinguish them from all actions, give them the appropriate status. Here is the great role of the teacher, who often should prompt appropriate formulation that student can adapt to new situation.

It is not enough to provoke the student's own actions which we plan to use in schooling. A further, very thoughtful action of the teacher is needed. The role of

[1] Inner Mathematical Structure, in the sense describing by M. Hejný

the teacher is to find these components in student activities, which are used to define the concepts. Description built on earlier student's activities should highlight the mathematical properties of the concept and eliminate from its image everything which does not fall within the concept.

Hence, a conversation with a student can't proceed in a detailed, planned way. The teacher must assess student's actions from the perspective of the created concept, pays attention to these important elements of the student's work that will be useful for the intended purpose. Then induce reflection on these elements, which should be completed - if not on definition - then at least at the awareness of their use.

Individual classes (such as those described) must therefore find its continuation.

FINAL REMARKS

In the current approach to teaching of mathematics that refers to specific student activities, educational planning proceeded in a clearly defined framework. Teacher should conduct a first analysis of the mathematical concepts towards extracting those activities that were the basis for mathematizing. Then, he should plan student's activity in such a way that undertaken actions could create the basis for mathematical concepts. This tradition functioned in Poland as the concept of "Action Based Teaching of Mathematics[2]", developed by Z.Krygowska. In our paper the problem is treated in a little different manner, because we plan to link teaching with what the student is doing spontaneously. From the position of teacher, we used reference to the actual motion of the rigid figure in the physical space to enter the student's understanding of mathematical isometric transformations. However, the student does not perform here a series of well-planned exercises, leading to a fast and reliable describe of the transformation. Although he has used his experience which a teacher rated as useful for building a mathematical concept, he has acted in a way that made sense for him. Observation of such actions is telling us to what extent the student's experience can be referenced. Only the recognition of these activities may be the basis for the next stage of mathematizing which will be attended by both - the student and the teacher.

[2] In Polish „Czynnościowe Nauczanie Matematyki", translation of the title proposed by Stefan Turnau.

LANGUAGE AND MATHEMATICS – PROBLEMS OF COMMUNICATION

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The term CLIL (Content and Language Integrated Learning) was coined by David Marsh, University of Jyväskylä, Finland (1994): "CLIL refers to situations where subjects, or parts of subjects, are taught through a foreign language with dual-focused aims, namely the learning of content and the simultaneous learning of a foreign language." In this work I refer to the problem of communication on Math classes emerging from the use of a mother tongue (L1), a foreign language (L2) and the language of mathematics, as well as the specific features of the two languages (L1 and L2). I am also giving an example of communication in the course of students' unassisted task solving in French (L2), by analyzing the difficulties and their possible reasons.

INTRODUCTION

Double language (bilingual) classes with French were first established in Polish high schools in the school year 1991/92 (Gajo, 2005). The students of bilingual schools take their Polish "matura" Math examination as well as a Math exam in French. This is why bilingual teaching in Poland follows the curriculum requirements and assumes the acquisition of the subject terminology in both languages. One of the general curriculum requirements at all stages is the use and generating of information. A student should be able to interpret a mathematical text (and after having solved the task, should be able to interpret the obtained result), create their own texts of mathematical character and use mathematical metalanguage to describe their reasoning process as well as the obtained results.

Communication related to the language of math

The language of math can be compared to any other foreign language, with its lexis and grammar (Usiskin 1996). A very interesting view is presented by Weinzweig (1982) who called mathematics an "extension of language". According to him language of mathematics involves visual and sensory thinking which is not part of ordinary language. Another common opinion is that it is a universal language from a notional point of view, after all it describes the world we live in. However, the way the objects are described, can be different in different languages. The role of the mathematical language in the math learning/teaching process is the subject of didactical studies (i.e. Ellerton & Clarkson, 1996, Pimm, 1987).

Undoubtedly the language of math has its own characteristics, but similar to other languages it serves communication. In the course of teaching we observe continuous communication between a teacher and a student, which takes more or less active forms and is oriented to a didactical purpose. The communication is performed by the natural language as well as the one of mathematics. As Stefan Turnau (1990) writes:

> "The language serves the communication of information, most of all to another person, and in the course of mathematical activities, also to the person themselves, in order to make use of or to transform it by the receiver. The symbolic language of mathematics has also another, extra function: it serves to transform the information it includes."

Differences between the Polish and French language of mathematics

In bilingual teaching communication is additionally performed in a foreign language. In case of Polish and French mathematical languages there are certain differences in the symbols, mathematical terminology and in the applied algorithms. Even the name of the subject is different, since in Polish it is singular while in French it is plural. It is well known that the mathematical language is closely related to the colloquial one, it contains homonyms, which can cause some problems in understanding certain ideas (i.e. sens, face, milieu, centre…).

Teaching math in French I have observed the students' remarks: „*We do not understand this in Polish, not even mention in French*". This is why the mother tongue should be used at the notion introductory stage for better understanding, at the same time accompanied with learning the foreign terminology referring to that notion, also to facilitate the smoothest communication in class. These are also the students' suggestions participating in the bilingual math learning process.

Assessment test as one of communication forms at math classes

I would like to concentrate on one of the forms of communication occurring during math classes, that is on the assessment test, with the application of H. D. Lasswell's model of communication (Goban-Klas, 2009). It consists of five succeeding elements as the answers to the following questions:

Who?

Says what?

In which channel?

To whom?

With what effect?

The teacher is the Sender, who formulates the message (he/she prepares the tasks) in order to present it to the student in a written form (encoding the message). The student reads the meaning of the message through its decoding. The student tries to understand the task and to interpret it as well as to accomplish the task as it is instructed in the message. Then the receiver gives the sender information about the way of understanding the meaning of the message and performing the task. The feedback is in a written form.

That seemingly very poor form of communication (since it is lacking a natural interaction between the sender and the receiver in the course of solving the task) is an indispensible element in the didactical process, because it serves to control the students' behaviour in the area of mathematical activity. The feedback information concerning the mistakes made by the student stimulates the teacher to ponder on the course of the teaching process. Analyzing the solutions, the teacher learns about the extent to which particular topics have been acquired by the students, the teacher also deliberates on the causes of errors and the difficulties that have occurred. It allows to properly plan the lessons to come in order to fully perform the intended goals. With the use of evaluation, commentary and discussion, the teacher communicates the student very important information about their progress and gives the student further directions for their activities. That form of communication may favour motivation, since through it we make the students aware of our expectations, how they can improve their effectiveness, or what kind of errors they make.

The student's communication (interaction) with themselves in the course of unassisted problem solving, that is in the course of mathematical activity, is a very important stage of an assessment test. Its result - the output message - is the record of the task solution. G. Polya (1945) differentiates four stages of problem solving:

- understanding the task;
- making a plan for the task solution;
- performing the plan;
- „looking back" – checking the solution.

The understanding of the task is indispensable for the task solving. That activity very often is difficult for the students, both those who study math in their mother tongue, and those who do it in a foreign language. This is connected to a big extent with the abstract character of the language of mathematics and its specific features.

Research characteristics

The goal of the research is an attempt to answer the following question:

What is the communication process connected with unassisted task solving in a foreign language in the course of an assessment test characterized with?

The research has been performed on a group of 17 students of grade 3 at the Bilingual High School in Poznań (including 12 students of advanced math curriculum), who I taught this year (from January to April) a 13-hour (once a week) preparatory math course in French because of the absence of the teacher, who had taught them math in French. The research tool was assessment test tasks. I have applied the following methods in my research: the analysis of the students' tests and observation of their work in class. The aim of the classes was to prepare them to a bilingual *matura* examination with a particular attention to the subject terminology. Those students represent a good command of French (B2+ or some of them even C1 levels). During the course the students mainly solved the sample *matura* exam sheets and they performed lexical exercises. The sheet for the math *matura* exam for bilingual classes consists of tasks (the number of which is different depending on a year), which can be given maximum 30 points. The content of a sheet covers the math material for a basic level. It mainly checks the level of math French vocabulary acquisition. In 2013 the sheet contained 17 multiple choice and open (short answer) questions. The time to complete the exam was 80 minutes. Most third grade students solved the sample sheets within 45 minutes. To sum up the classes there was a test consisting of 15 tasks originating from *matura* exam sheets, (most of them had sub-questions), which could score maximum 45 points. So they had to solve more tasks than at the *matura* exam, but in a shorter time. In spite of that most students completed the task before the time limit.

Research results and conclusions

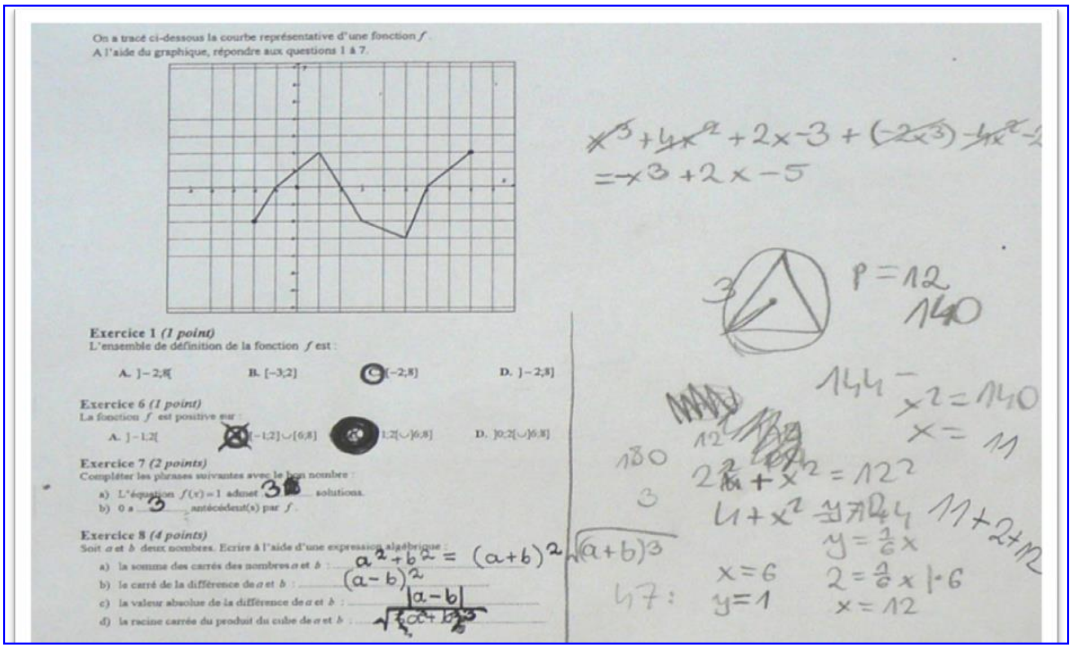
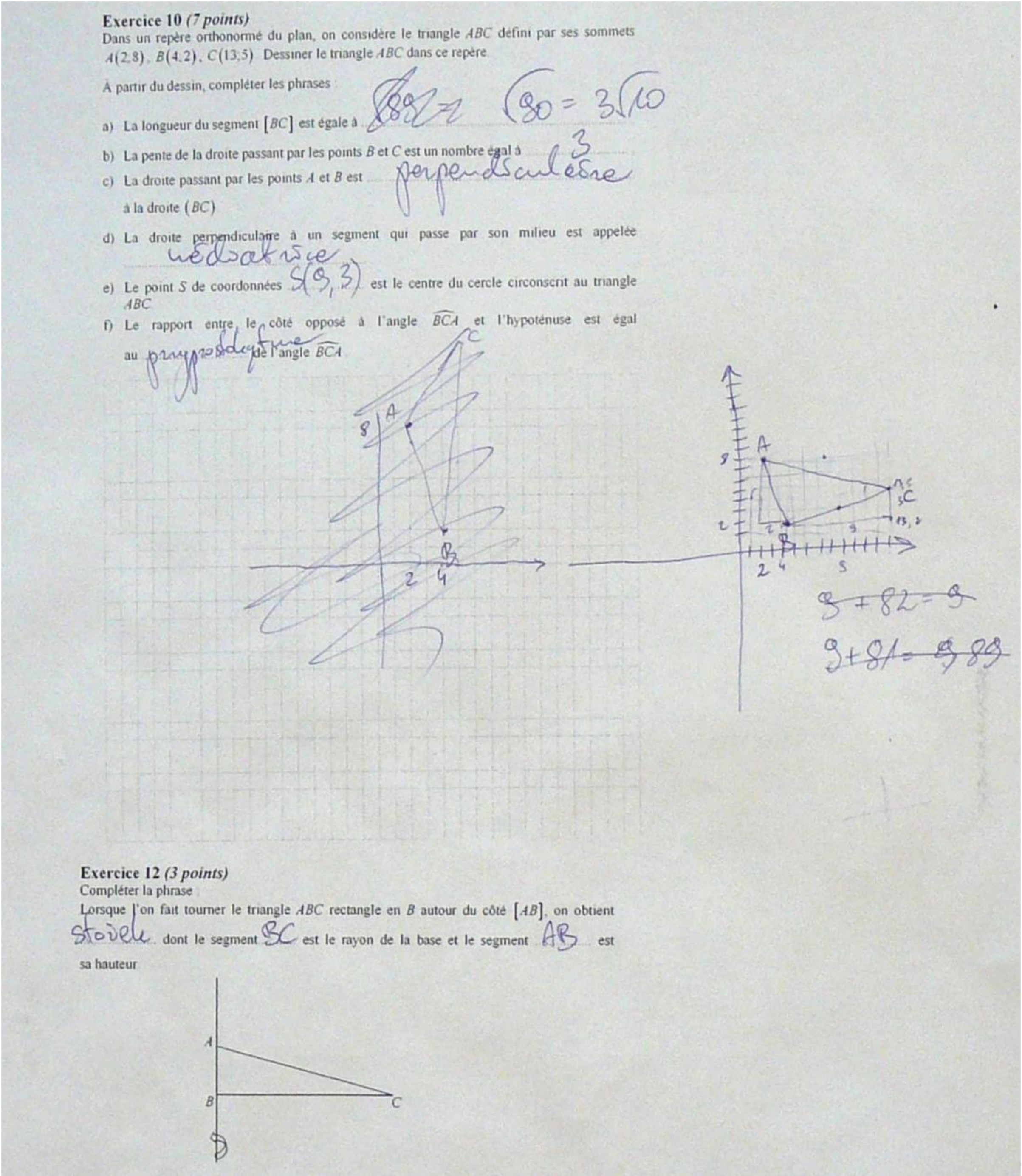


Figure 1- A student's 1 work, basic level (47%, M 3, Fr 4)

Here are examples of two students' work pieces from a basic level (Figure 1, Figure 2, Figure 3) and one from an advanced level (Figure 4)[3].


Exercice 10 *(7 points)*
Dans un repère orthonormé du plan, on considère le triangle ABC défini par ses sommets $A(2;8)$, $B(4;2)$, $C(13;5)$. Dessiner le triangle ABC dans ce repère.

À partir du dessin, compléter les phrases :

a) La longueur du segment $[BC]$ est égale à $\sqrt{90} = 3\sqrt{10}$

b) La pente de la droite passant par les points B et C est un nombre égal à 3

c) La droite passant par les points A et B est perpendiculère à la droite (BC)

d) La droite perpendiculaire à un segment qui passe par son milieu est appelée médiatrice

e) Le point S de coordonnées $S(9,3)$ est le centre du cercle circonscrit au triangle ABC

f) Le rapport entre le côté opposé à l'angle $\widehat{BCA}$ et l'hypoténuse est égal au [illegible] de l'angle $\widehat{BCA}$

Exercice 12 *(3 points)*
Compléter la phrase :
Lorsque l'on fait tourner le triangle ABC rectangle en B autour du côté $[AB]$, on obtient stoèle dont le segment BC est le rayon de la base et le segment AB est sa hauteur

Figure 2- A student's 2 work, basic level (76%, M 3, Fr 3)

The student commented on the last statement in the following way:

3 In the brackets there are percentage results of the students' exams, parts of which are presented, as well as their final grades in Math and French. The scale of grades is from 1 to 6 (6-excellent, 5- very good, 4- good, 3- satisfactory, 2- acceptable, 1- unacceptable).

„In Task 16 I was not able to come up with the right answer, although I translated the last sentence and, in general, it is also true."

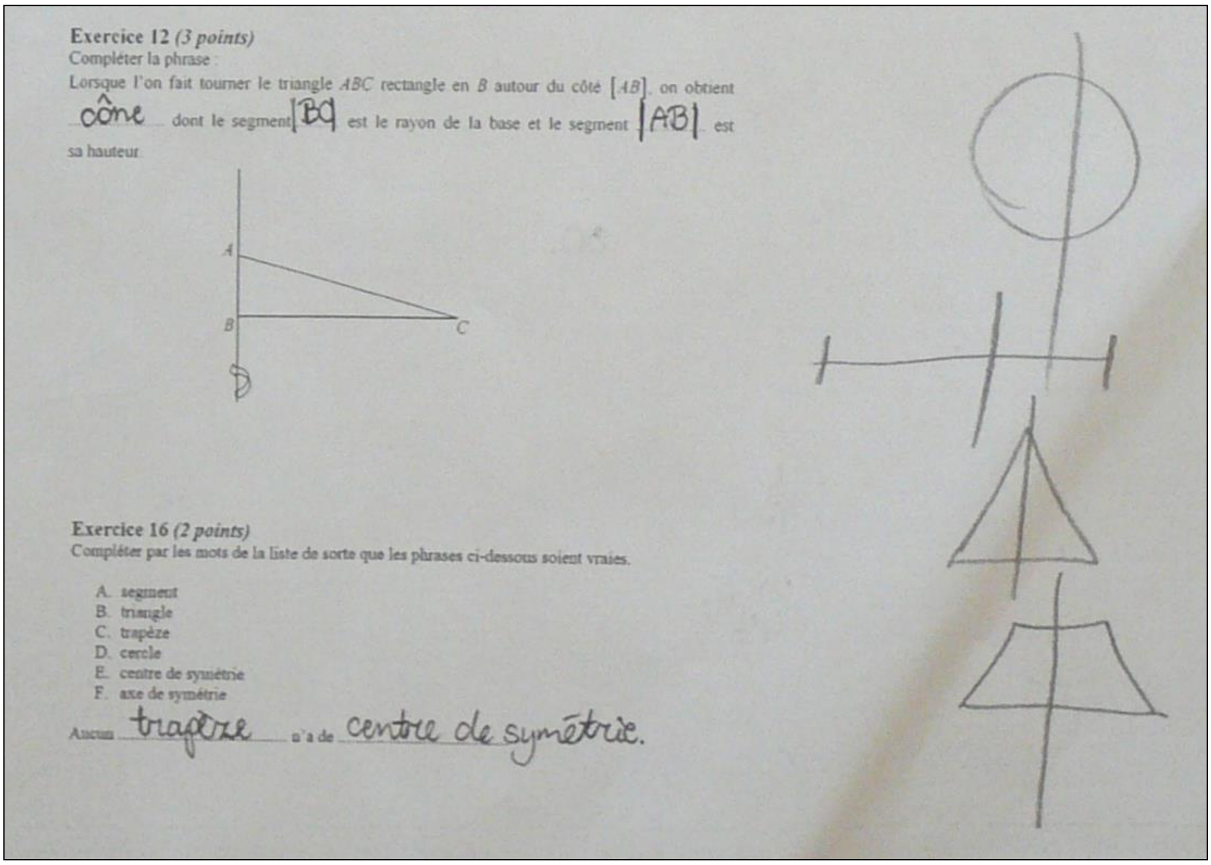


Figure 3 - A student's 1 work, basic level (47%, M 3, Fr 4)

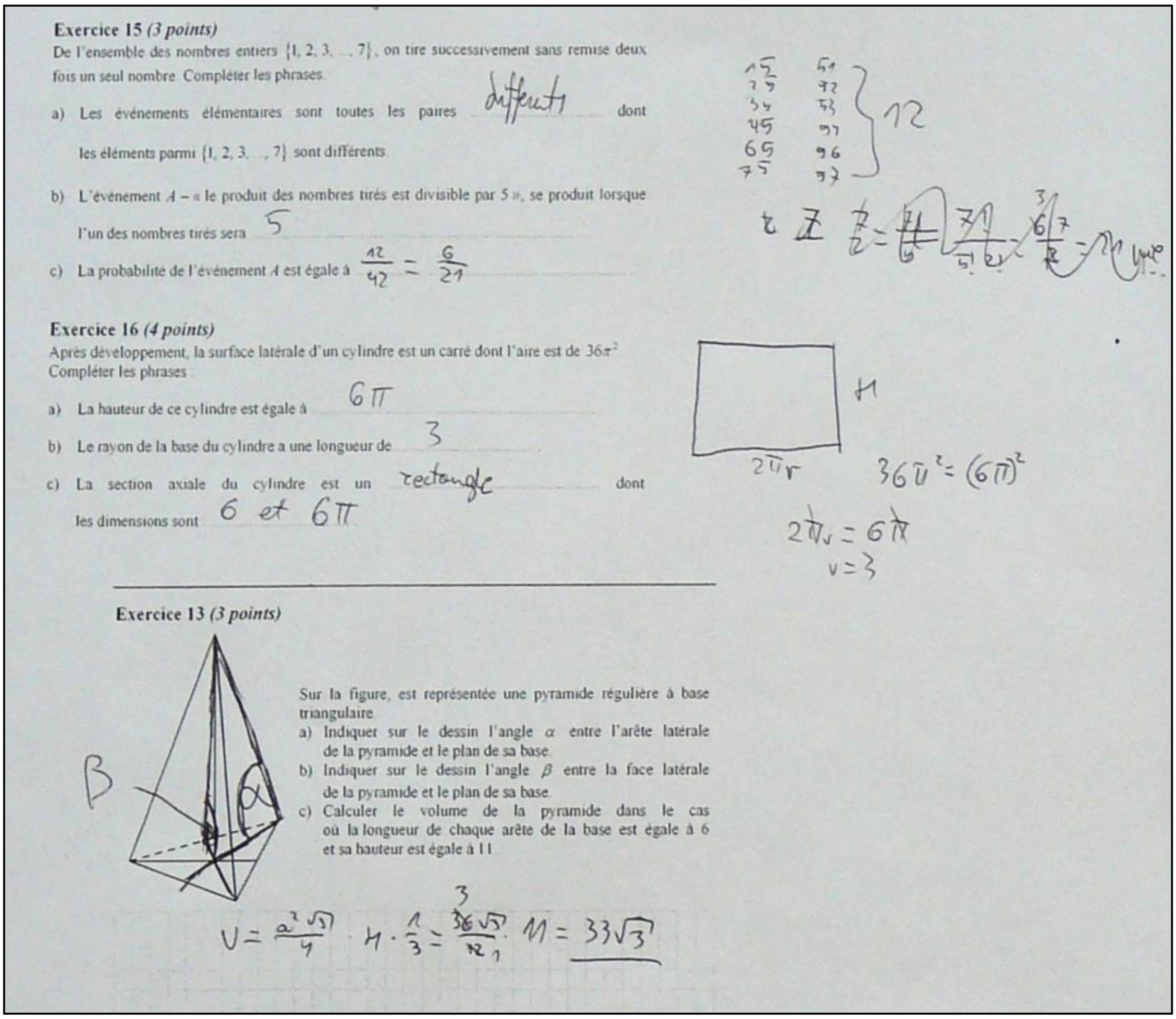
Task – Exercice 13 (3 points) for the calculus of probability, apart from sub-point b), caused a lot of trouble to the students. In sub-point a) only one of the students gave the correct answer referring to the ordered pairs. In the discussion afterwards the students pointed to an ambiguous form of that task. Moreover, that kind of tasks had not occurred in Polish math handbooks, although students learn about the idea of an ordered pair. In sub-point c) only 24% students correctly calculated the probability of event A. Figure 4 is an example of a correct solution to sub-point c). It can be assumed that there were the following stages of solving this problem (Figure 4):

- using combination formula instead of permutation formula (later he calculated the number of outcomes in the sample space, but we don't know in which way);
- verification and abandoning that strategy of solving (claiming eventually that "no" (in Polish "nie");
- writing out number pairs and counting them which are the number of outcomes in event A (the student understands more or less consciously that

those are ordered pairs, although he did not give that answer explicitly in sub-point a);

- giving the answer (he did not notice that the fraction could have been reduced).

According to most students the *matura* examination tasks are not difficult because of the mathematical content. Here are some of their opinions:



Exercice 15 *(3 points)*

De l'ensemble des nombres entiers $\{1, 2, 3, \ldots, 7\}$, on tire successivement sans remise deux fois un seul nombre. Compléter les phrases.

a) Les événements élémentaires sont toutes les paires différents dont les éléments parmi $\{1, 2, 3, \ldots, 7\}$ sont différents.

b) L'événement A – « le produit des nombres tirés est divisible par 5 », se produit lorsque l'un des nombres tirés sera 5

c) La probabilité de l'événement A est égale à $\frac{12}{42} = \frac{6}{21}$

Exercice 16 *(4 points)*

Après développement, la surface latérale d'un cylindre est un carré dont l'aire est de $36\pi^2$. Compléter les phrases :

a) La hauteur de ce cylindre est égale à 6π

b) Le rayon de la base du cylindre a une longueur de 3

c) La section axiale du cylindre est un rectangle dont les dimensions sont 6 et 6π

$36\pi^2 = (6\pi)^2$

$2\pi r = 6\pi$

$r = 3$

Exercice 13 *(3 points)*

Sur la figure, est représentée une pyramide régulière à base triangulaire.

a) Indiquer sur le dessin l'angle α entre l'arête latérale de la pyramide et le plan de sa base.

b) Indiquer sur le dessin l'angle β entre la face latérale de la pyramide et le plan de sa base.

c) Calculer le volume de la pyramide dans le cas où la longueur de chaque arête de la base est égale à 6 et sa hauteur est égale à 11

$V = \frac{a^2\sqrt{3}}{4} \cdot H \cdot \frac{1}{3} = \frac{36\sqrt{3}}{12} \cdot 11 = 33\sqrt{3}$

Figure 4- A student's 3 work, advanced math level (82%, M 4, Fr 4)

Student A (extended level since grade 2, 84% of points, M 3, Fr 5):

„The task for the matura examination are extremely easy, in my opinion even for the basic level. The only difficulty is that those are not standard tasks, they require thorough thinking and deliberation, even a deeper insight into the problem not only on mathematical grounds - simply speaking, that refers to the merits of the problems. They do not resemble the tasks from Polish handbooks, they look rather like simplified versions of the French handbook tasks - I had an opportunity to

> come across them when studying at a school in France for half a year - those were exactly the same type of tasks."

The opinion above includes information about the course of communication during task solving, which in fact is not disturbed by the misunderstanding of the mathematical content rather than by the way the tasks are formulated, which is a far cry of the traditional tasks included in Polish handbooks. It is a remark about the existence of a problem on the cognitive and interpretative levels.

Only one of the students signalled a problem on the mathematical level:

Student B (extended level, 83% points, M 3, Fr 5):

> „I had problems with the elements that I did not understand on Polish language math lessons. However, I noticed that more other students also mistook "ensemble des images"[4] for "ensemble des definitions"[5]. Nevertheless, general knowledge of French helped recollect math vocabulary and understand it. Sometimes the tasks required more concentration because they were tricky, i.e. when you had to grasp when to put down both numbers as squared, and when to square only one of them."

The task the student refers to are the first two tasks (Figure 1). Tasks of that type also occur in the Polish *matura* exam sheets. First task in the test was solved in 88 %, while second task in 94 %. Only one student gave the range of values instead of the domain (misunderstanding of the terms), while another student had problems with the symbols (the way of marking intervals differs from the Polish one). Both problems were observed during regular classes solving tasks. On the other hand, Task – Exercice 8 (4 points) was from the junior high school level and it referred to algebraic expressions. That type of tasks do not occur in the Polish matura examination. The problem with giving the correct answer (particularly in the last sub-point 47% of students solved that sub-point correctly) was caused by insufficient knowledge of lexical items as well as misunderstanding grammatical structures, which imposed a certain solution (that is if only one number, or both of them should be raised to a cube). That problem was clear in Student's 1 work. Particularly in the last sub-point there are modifications of the solution, which, however, did not lead the student to giving the proper answer.

Recapitulation

The teacher received in a written form a feedback to their message, which had not been correctly understood by all the students, and it did not cause the expected effect. The communication was disturbed on different levels of working on the task - message, starting with its reception, to its transformation and encoding in order to transmit the sender through:

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4 set of values

5 domain of definition

misunderstanding the content in a foreign language, which disabled further mathematical activity (most frequently it revealed in the lack of any answer);

improper interpretation of the task (the student solved their task or abandoned the solution);

inattentive reading of the text (in spite of correct calculations, the give answer was incorrect);

giving a wrong answer because of not knowing proper symbols (disability to encode according to the accepted system).

The former observation of students' work allows to elicit the following statements:

The students also had difficulty in reading the symbols referring to marking intervals (symbols different from the Polish ones, which did not cause misunderstanding of the task, however, occasionally there were only problems with a correct written record of the solution).

For some of the students basic mathematical vocabulary was not clear enough, like: product, quotient, radical, absolute value, degree of polynomial, coefficient, circumscribed circle, recording of algebraic expressions (sometimes, when they came across unknown vocabulary, they asked about it, not trying to guess/deduce the meaning of a mathematical term by themselves).

The analysis of surveys and the observation of the students in class, first of all revealed difficulties connected with the ignorance of mathematical terminology and problems with interpretation of the tasks in French, which both seem to cause communication disturbances during unassisted problem solving.

The conducted research, although only to a limited extent, encourage to further studies on the methods of working with students, which might improve the process of communication on math lessons performed bilingually.

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COMMUNICATION BETWEEN THE AUTHOR OF A MATHEMATICAL PROBLEM AND A STUDENT

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The contemporary didactics puts the emphasis on the development of effective communication skills. The information transfer can happen in different forms and ways, e.g. verbally, in writing, or by the usage of gestures. It is not only important to know how to transfer the information but also how to receive it, interpret and comprehend it. A perfect skill development opportunity for receiving the information transferred in a written form is given by mathematical problems, predominantly the word problems. Having that in mind it is good to look at a mathematical problem as a specific linguistic message. In my paper I shall present some causes of misunderstanding that happens between the author of a mathematical problem and a student.

INTRODUCTION

The learning of mathematics takes place mainly by solving specially selected problems. They offer learning of new terms and their characteristics, help acquiring the knowledge and improving the specific and complex skills, they are also favourable for developing the critical thinking as well as intellectual independence. By solving the problems, students get to know how to act in subjectively new situations and how to overcome the obstacles of emotional and intellectual nature.

The content of each mathematical problem is a mixture of natural and mathematical language: symbolic, graphic or verbal; bearing a characteristic structure. The specific characteristics of such content are the carrier of various elements of didactical instruction. The ignorance or lack of understanding of specific words or symbols, included in the content, the principles of mathematical notation or different comprehension of a problem meaning can be, and often is, the reason for misunderstanding and as a consequence student's failure in solving a problem.

Every mathematical problem can be perceived as a specific linguistic message[6] (see: Czajkowska, 2003). The author edits a problem in a language that will

[6] A linguistic message is a product created in a given language to establish the communication between the sender and the recipient. Every linguistic message (see: Jakobson, 1989) concentrates on: sender (emotive function) – it specifies the attitude a sender has towards the content he transfers; recipient (appeal function) – provokes the recipient's cognitive and emotional reactions; the form of message (poetic function) – describes the esthetic aspect of the message, the attractiveness in its visual form, external shape; the linguistic system (metalinguistic function) – determines the sense of signs present in the message which can be not understood by

cause desired behaviour and action of the recipient. An effective communication between the author of a problem and a student happens when the student can correctly decode the problem content and the iconic material behind it as well as interpret the information included. However, in practice, many students possess a poor decoding technique, and therefore they can incorrectly "read" the content which the author wanted to express. This creates misunderstanding on the author-student communication route, which results in an incorrect solving of a problem or solving a different problem than author desired.

In the process of problem solving a student refers to his entire in- and out-side school knowledge as well as to his experience coming from all the past mathematical problems that he was solving before. Sometimes, the situational context, being the authors' intention to facilitate the understanding of a problem situation, or to indicate the usefulness of a particular mathematical tool in solving a life problem, turns in reality to disturb and obstruct the comprehension of a mathematical problem. Some aspects of a situation completely not important for the author may get a prevailing importance in the eyes of student. It happens also that a student sets additional conditions, by doing so he changes the content of a problem completely. Other time, a student transforms a mathematical problem into a real life situation and by making use of life facts he solves it in a practical manner, without the usage of mathematics (see: Czajkowska, 2003).

RESEARCH METHODOLOGY

The herewith described, research problem refers to misunderstanding which results from the lack of a proper communication between the author of a mathematical problem and the recipient, a student. The source subject to this analysis is secondary, meaning not prepared for this particular research. One cannot plan such an experiment as this would mean we plan ahead the miscommunication between the sender and the recipient of a problem (see: Slezakova & Swoboda 2008).

The subjects to this analysis were therefore, the collected solutions of mathematical problems produced by an individual student work, in which the lack of proper communication between the sender and the recipient was observed. However, the situations in which the students were solving mathematical problems were different. Some work was done during a school lesson or during individual classes. Other comes from school tests, quizzes and external exams. The rest was carried out during students' participation in mathematical contests or educational research.

the recipient; the reality (reference function) – presents the described branch of knowledge in a thorough way; contact (phatic function) – confirms, sustains or breaks the the flow of information.

THE ANALYSIS OF STUDENTS' PROBLEM SOLVING

One can distinguish few types of cognitive obstacles present in the communication between the author of a task and a student; identical to those that teacher and student use while communicating during the lesson (see: Slezakova – Kratochvilova & Swoboda 2006, Slezakova & Swoboda 2008). This will be characterized and illustrated hereafter by means of examples taken from students' work.

Different interpretation of a situation described in a problem content

In many mathematical word problems, one can find a plot that to some extent is not fully specified. In practice, this means that in order to solve a problem a student needs to, more or less consciously, set up certain conditions. The authors usually try to construct their problems in such a way that the plot specification is consistent with their intention. It happens, however, that the student, referring to his outside school experience sets up conditions which are not even mentioned in the text itself or which were not expected by the author. The content of a problem is for the student not only all that he concludes from the layout of the information about the objects, events, processes and the correlation between them, but also what he associates with the described situation. In such instances the communication between the sender and the recipient is disturbed. In the end, the student solves a different task than the one its author meant for.

Example 1

The problem and its two solutions come from a test of a 6-grade student, test taken in 2011[7].

Problem

Magda has 56 PLN savings, Barbara 20 PLN. The girls decided to continue saving the money. Magda will lay by 9 PLN a month. How much Barbara shall lay by every month to get the same amount like Magda after 8 months of saving? Write down all your calculation.

Solution 1

$$56 + (8 \cdot 9) = 128$$

$$108$$

$$20 + (10 + 10 + 10 + 15 + 15 + 20 + 20 + 8) = 128$$

Answer: Barbara in the first month 10 PLN, in the second 10 PLN, in the third 10 PLN, in the fourth 15 PLN, in the fifth 15 PLN, in the sixth 20 PLN, in the seventh 20 PLN, in the eighth 8 PLN.

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[7] The solution has been taken from an unpublished Bachelor's dissertation of Martyna Sapińska (2012) Strategie rozwiązywania zadań otwartych z matematyki na sprawdzianie szóstoklasisty 2011 (The strategies of solving a mathematical problem during a 6-grader test in 2011) which was mentored by the author of this article. By courtesy of the Regional Board of Examiners in Łódź it was possible to perform the quality analysis of the three open problems taken from 118 works of students writing the 6-grader test in 2011.

Solution 2

Few calculations indicate that the analysis of the situation described in the text of the problem was being analyzed by the student in different ways; finally the student provides the answer: 13 PLN *each month, and in one month* 4 PLN *more*.

It is worth noting that the situation described in the mathematical problem was differently understood by the author of this task and differently by the student. The author connects the sentence of "how much money shall she lay by every month" to the previously mentioned message about Magda's manner of saving money and for him this indicates that Magda saves the very same amount every month. However, this was not so clear for the students solving the problem. The students solving this problem assumed that the amount Barbara is supposed to save does not need to be equal every month.

By thinking in this way, students have changed the mathematical sense of the problem. They analyzed the distribution of the number 108 as the sum of 8 parts, rather than as a division into 8 equal parts. To their way of thinking, the solution of the problem was not clear.

Example 2

The below problem solutions come from the observation of individual student work conducted during the personal research. Let us have a look at the solutions of the below problem, presented by 4-grade students: Wojtek and Marcin.

Problem

There is sand in two vessels. In the first vessel there is 11.2 kg, in the second there is 6.8 kg. 3.5 kg of sand was taken away from the first vessel. In which vessel there is more sand and how much more?

Wojtek's solution:

11.2 kg – 3.5 kg = 7.7 kg – this is the amount of sand that will be in the vessel no 1 after the sand being taken away

6.8 kg + 3.5 kg = 10.3 kg – this is the amount of sand that will be in the vessel no 2 after the sand being added

10.3 kg –7.7 kg = 2.6 kg

Answer: In the second vessel, there is now 2.6 kg of sand more than in the first one.

Marcin's solution:

Vessel no 1: 11.2 kg – 3.5 kg = 7.7 kg

Vessel no 2: 6.8 kg

7.7 kg – 6.8 kg = 0.9 kg

Answer: In the first vessel, there is now 0.9 kg of sand more than in the second one.

The difference between the solutions results from a different comprehension of the problem sense, from the layout of the objects and the relation between them. The word “take away” means “to deduct, to separate from something”, and this is how it is understood in the two examples. However, Wojtek understood that the sand taken away “needs to be added somewhere, something needs to be done with it”. By treating this situation in a practical way as a specific closed set (created by the vessel and the sand), he “took away” the sand into the second vessel. Marcin, on the other hand, was interested only in the information that connected the initial weight of sand in the second vessel with the final weight of the sand in the vessel no 1; and this is the only relation that the problem content described.

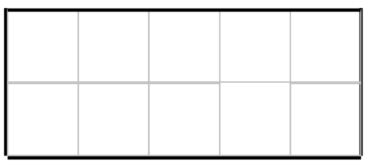
While in realistic mathematical problems the specification of the plot is accepted, in “pure” mathematical ones this is not the practice. Meanwhile, the analysis of students’ work shows that students tend to set up additional conditions also in the purely mathematical problems, just like in the following example.

Example 3

The problem was taken from the research Szkoła samodzielnego myślenia (The School of Independent Thinking)[8].

<u>Problem</u>

The rectangle presented below consists of 10 small squares. The length of the side of the small square is 1 cm. Paint two squares over so that the figure you will get, after deducting them, has the same circumference as the rectangle.



The 1-grade student of a secondary school (gymnasium) states that this is not possible, and provides this clarification: *rectangle can be a rectangle only when we paint its side over, but the circumference will be smaller then*. This means that the student set up an additional condition that the figure he will get after deducting two squares needs to be still a rectangle.

[8] The research Szkoła samodzielnego myślenia (The School of Independent Thinking) was conducted by the Educational Research Institute (IBE) in Warsaw as a part of the systemic project Researching the quality and effectiveness of education and institutionalisation of research infrastructure, realized by the European Social Fund under the Human Capital Operational Programme, Priority III: High Quality of the Educational system, sub-measure 3.1.1 Creating Conditions and Tools for Education System Monitoring. The goal of the research was to diagnose the skill levels for mathematical modeling, creation of the strategy for task solving, reasoning and argumentation.

Various meaning assigned to terms present in the problem content

In the process of solving a mathematical problem, different students give different meaning to the very same word, predominantly this happens when the words come from a colloquial language and can be here differently understood. This results in students' seeing some connections between the unknown value and the data which as a consequence makes them build different mathematical models.

Example 4

The below example is taken from a mathematical test, 5-grade of primary school.

<u>Problem</u>

One "Fortuna" (Eng: Fortune) lottery ticket costs 5 PLN. Pawel had 7 PLN, while Czarek 3 PLN. The boys clubbed all their money together and bought two lottery lots. It turned out that they won 200 PLN. They divided fairly the money they won between one another. How much did each get?

<u>Solution 1</u>

$200 : 2 = 100$

Answer: Each got 100 PLN.

<u>Solution 2</u>

Pawel: $\frac{7}{10} \cdot 200 = 140$

Czarek: $\frac{3}{10} \cdot 200 = 60$

Answer: Pawel got 140 PLN, while Czarek 60 PLN.

<u>Solution 3</u>

$3 + 7 = 10$

$200 - 10 = 190$

$190 : 2 = 95$

Answer: Each boy got 95 PLN.

Different solutions of the problem result from different understanding of the word "fairly". For the student who presented the first solution this meant "equally". The author of the second solution understood the word "fairly" as "in proportion to his contribution". While the third, 5-grade, student suggested to "divide the amount equally" only after their contribution was given back to each of the boys.

Attention to different excerpt of information

Every mathematical problem includes the information which is more or less crucial throughout the different phases of problem solving. Moreover, in the content of some problems (so called problem with excess of data) there appears information that is not significant from the perspective of the question asked. It introduces so called “information noise” and disturbs the identification of the situation. Such problems require, from a student, a thorough text analysis which would help him recognize the redundant portion of information as per the unknown value he is looking for (see: Bugajska-Jaszczołt & Czajkowska, 2012). However, sometimes, the redundant information becomes dominating and gains a special importance.

Example 5

During the mathematical lesson conducted in 4- grade of primary school, the students were independently working on a problem they received from a teacher during that class. One of them was the following.

<u>Problem</u>

Father made a costume of an alien for his son, Piotr, for a school carnival ball. Piotr weights 54 kg. The prepared costume weights 2.5 kg, shoes 0.75 kg, and the headgear 0.45 kg. How much does Piotr weight in the costume of an alien? Write down all your calculations.

One of the students underlined the excerpt of the text “Piotr weights 54 kg” and he wrote down:

54 –Piotr’s weight

$2.5 + 0.75 + 0.45 = 3.7$ – The costume of an alien (not Piotr)

Answer: Piotr dressed in his costume weights 54 kg.

The student put his attention to “Piotr’s weight”; however, the information about the character’s cloths was not so vital for him. In the real world, by providing the weight of a man one would not take into consideration the weight of clothes he is wearing. It is possible that the 4-grader had met before a problem that included the answer to the question inside the problem itself, and the problem aim was just to “read” it. The student did not understand the author’s intention and the aim of the problem which got suddenly changed. Therefore despite all the calculation he made the answer he provided was only Piotr’s weight. However, what was important for the teacher was the fact that “Piotr is in the costume of an alien”.

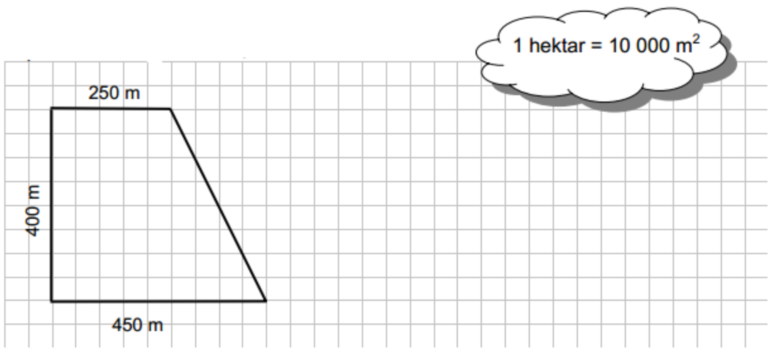
A picture is sometimes an integral part of a mathematical problem. It might play the role of: 1) a record of information, 2) means enabling the distinction of important and insignificant content, 3) supportive means, indicating what to look at and suggesting what next steps of problem solving to take, 4) the object which needs to be analyzed, 5) illustration. Sometimes, it can hold multiple functions simultaneously (see: Czajkowska, 2005). It happens, however; that the drawings instead of becoming the communication tool, they might become an obstacle on the communication route between the author and the student. Let us look at the below example.

Example 6

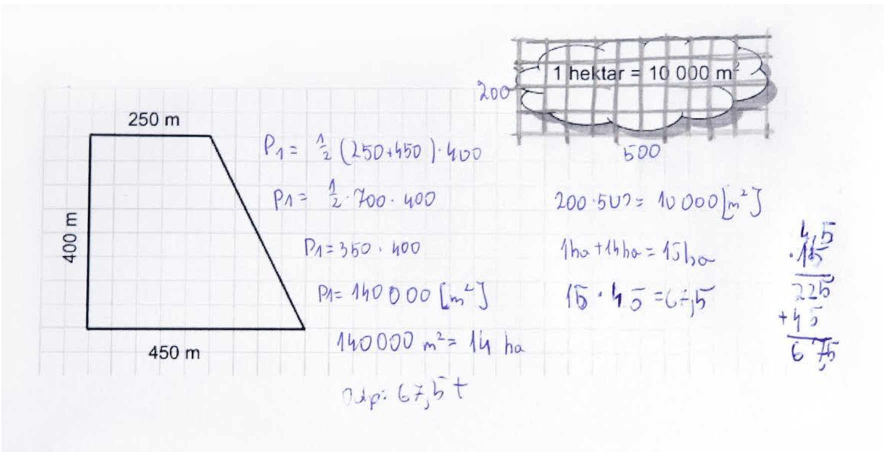
The below solution was presented by a 6-grader during a test.

Problem[9]

The size and shape of the land are provided in the below picture. A farmer sowed wheat on his land. From each hectare he gathered 4.5 ton of wheat. How much tons of wheat did he gather from this land? Write down all your calculations.



Solution



The cloud presented in the picture was of no importance for the author of the problem, it actually did not belong to the picture which was subject to analysis. However, the student noticed the two elements and treated them both as the

9 *The problem has been taken from a 6-grader test in 2004. The teacher used it again during a test 2010.*

objects the problem describes. What is more, he even thought that the area of a cloud shape element is given and he tried to check whether the both drawings are made in the same scale (trails of calculation of the "cloud" area are evident).

Other students' failures resulting from a drawing obstacle are described in the works of Czajkowska and Bugajska – Jaszczołt (see: Czajkowska, 2004, Bugajska-Jaszczołt & Czajkowska, 2013).

Different understanding of problem goals

While formulating a problem in a specific manner the author wants to cause a desired behaviour of a student and drive towards the changes in his thinking and acting. One of such goals can be the efficiency exercise (e.g. arithmetical), skills development of: the usage of an known scheme or algorithm, noticing the correctness or analogy, coding or decoding the information in a specific language, the usage of mathematical terms and symbols, mathematical modeling, carrying out mathematical reasoning. A student can however, "read" the goal of the problem in a totally different way. If he "reads" it incorrectly, then the task can lose its educational character, or even cause damage to his positive attitude towards mathematics and discourage him from further contact with that topic. For example it is observed that specific numbers, context in which the problem is set or the mathematical branch which the problem refers to, are of a basic importance to many students while solving the problem; the lesser importance is carried by the informational structure, reasoning which one needs to carry out, or the similarity towards other problems solved before.

Example 7

Problem

In the shop there are eggs sold in big packs of 10 eggs, and in small packs of 6 eggs. The big pack costs 6.40 PLN, while the small one costs 3.90 PLN. Which is more profitable to be bought, a big or the small one, and and why?

<u>Solution:</u>

> *Answer: None of them or the smaller one, because in my house one eats few eggs only and they would get spoiled.*

The intention of the author of that problem (in this case of the teacher) was that a student can specify the unit cost of eggs and can choose the most profitable price. The student, however, approached the problem by using a realistic standpoint; taking into consideration the family's eating preferences. In this way, he changed the mathematical problem into a real life problem.

Conclusions

Much as the cognitive obstacles can and should be recognized by the teacher during the lesson, e.g. by asking additional questions to the student group; in the case of individual task solving, they are not diagnosed on time.

The communication between the author of the problem and a student is actually a peculiar one. It is only one-sided: from an author to a student. This situation does not even change when the problem is presented in a digital form, more and more often popular in mathematical education. A student solving a problem using his/her PC usually receives only return information that indicates whether the solution is correct or wrong, without getting the details about the cause of mistake.

The awareness of the hereby described misunderstanding is very crucial in the view of the control of students' achievements. Sometimes, the incorrect solution comes not as a result of lack of knowledge or lack of student's mathematical skills, but as an improper understanding of situation described in the problem or as the act of assigning its own meaning to certain words or expressions, used in the content. However, it is also important to notice that the strategy of problem solving or the reasoning conducted by a student is adequate to the situation that he can understand.

USING OF CLIL METHOD IN MATHEMATICS EDUCATION IN THE FIRST GRADE OF SLOVAK PRIMARY SCHOOL IN HUNGARY

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Content and Language Integrated Learning (CLIL) can support the using of national language in different subjects. In this paper we present this method in the case of school mathematics. There is no analysing of the possibilities of this method in Slovak national schools in Hungary up to this time. The first insert must be in the preparing of future Slovak minority´s teachers. We show some practical examples of this method.

INTRODUCTION

The motivation for the implementation of CLIL - an integrated teaching of minority languages and non-language subjects are European Commission's recommendations, particularly the Office of Commissioner for Education and Culture. European Commissioner Ján Figel 2006 states:

„Multilingualism is at the very heart of European identity, since languages are the fundamental aspect of the cultural identity of every European. For this reason, multilingualism is referred to specifically – for the first time – in the brief of a Commissioner. I am honoured to be that Commissioner.“ (see CLIL (2006))

CLIL (Content and Language Integrated Learning) is an educational method for teaching non-language subjects through a minority language. It's an innovative approach that changes ways in which students are introduced to the curriculum, and that accelerates the acquisition of basic communication skills in a minority language.

WHY CLIL?

This teaching method has several advantages within minority schools. At this point we want to mention few of them.

- When teaching with CLIL method, the aim is on the particular activity and not the minority language itself.

- This approach provides the opportunity to learn to think in that language and not only learn the language as such. CLIL allows students to practice the minority language in learning another subject.
- CLIL is the opportunity for graduates to develop their skills using foreign or minority languages and therefore to increase their personal potential for an advantageous position in the labour market.
- The curriculum can be explained first in Hungarian and later extended in the Slovak language, or vice versa.
- The activities in both languages should be complementary.

From advantages that CLIL brings, we can mention following ones:

- overall improvement of student communication skills in a minority language,
- deepen awareness of the minority language, official language and other languages
- Increased student motivation through real educational situations in the teaching of minority languages
- increase the fluency of expression, a wider range of vocabulary
- active involvement in lessons,
- a positive attitude towards the minority language,
- development of own national and cultural awareness,
- preparation for practical life and work in a multicultural society.
- CLIL provides opportunities that allow students to use a minority language naturally, in such a way that they gradually forget about the use of minority languages and focus only on content.
- In the CLIL method the minority language is associated with other objects. In the classroom there are two main goals: one is the subject, topic, and another one is the language.
- This is the reason why CLIL is sometimes called a dual-focused teaching.
- CLIL can really do a lot, increases the willingness, wanting and ability to learn both - language minority and non-language subject.

CLIL METHOD IN MATHEMATICS

We organized in the year 2013 educational circle of Mathematics in Slovak language and we used CLIL method. The pupils were from fourth grader of Slovak minority primary school. Topic for the activities was roman digits. The pupils already know basic roman digits. We present now two examples how we teach pupils use roman digits in practical way and some operations with them (see Gunčaga-Lestyan(2014)).

The pupils received record of four-digital numbers through roman digits. They should transform this record to the record with Arabic numbers. We used following records:

MDXLII=1542 MDXXII=1522

MDCCLIX=1759 MCXCIV=1194

MCDXXVI=1426 MCCCXLIII=1343

MCMLXXXIX=1989 MDCXLVIII=1648

Later they solved opposite task:

2567=MMDLXVII 1999=MCMXCIX

963=CMLXIII 1046=MXLVI

1459=MCDLIX 2681=MMDCLXXXI

1843=MDCCCXLIII 797=DCCXCVII

Another motivational tool for pupils is a cross. We choose following:

1. 30+28=
2. 4 roky je koľko mesiacov? (How many months are 4 years)
3. 1000-374=
4. 999
5. Aký rok píšeme teraz? (Which year is now?)

The question was what I do with eyes? The solution in light was “VIDIM” that means “I see”.

1.	L	**V**	I	I	I		
2.	L	**I**	I				
3.		**D**	C	X	X	V	I
4.		**I**	M				
5.	M	**M**	X	I	I	I	

Table 1.

Pupils try to speak only in Slovak, but they had problems with mathematical expressions. The unknown words they try to find with the help of Slovak-Hungarian vocabulary.

TANGRAM AND CLIL

Another possibility for using CLIL was tangram figures. We used the figures prepared from tangram parts. First we analyze the tangram parts from mathematical point of view and find their names in Slovak and Hungarian language and later we try to formulate sentences in both languages.

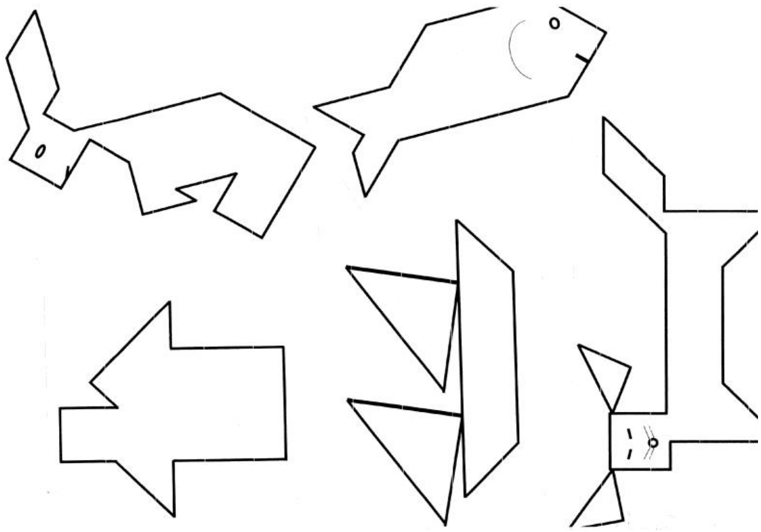

Figure 1

Štvorec	Négyzet	Square
Trojuholník	Háromszög	Triangle
Rovnobežník	Paralelogramma	A rectangle
Rovnoramenný trojuholník	Egyenlő szárú háromszög	Isosceles triangle
Pravouhlý trojuholník	Derékszögű háromszög	Right triangle
ObdĺŽnik	Téglalap	Rectangle
Dom	Ház	House
Komín	Kémény	Chimney
Pes	Kutya	Dog

Table 2 Figures and tangram parts

Zajac beží pred domom.	A nyúl fut a ház előtt.	Rabbit is running in front of the house.
Pes naháňa zajaca.	A kutya kergeti a nyuszit.	A dog is chasing a rabbit.
Ryby sú v akváriu.	A halak akváriumban vannak.	Fishe are in the aquarium.

Table 3 Sentences

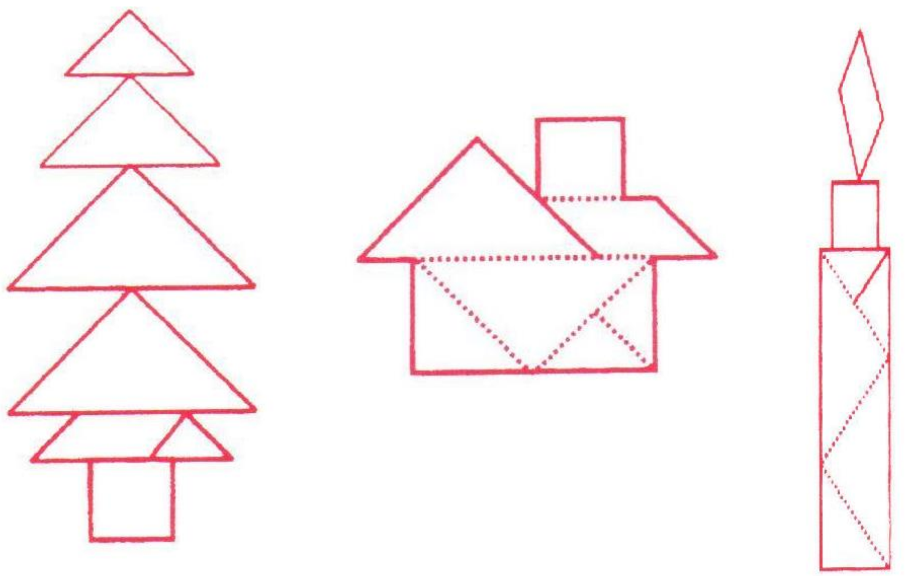

Figure 2

Dom	Ház	House
Komín	Kémény	Chimney
Strecha	Tető	Roof
Smrek	Fenyő	Spruce
Strom	Fa	Tree
Sviečka	Gyertya	Candle

Table 4 Tangram figures

Prasiatko vyšlo z domu a vyliezlo na vrchol stromu. Vlky ho nechytili.	A kis malac kiment a házból és felment a fenyő tetején. Farkasok nem csapták öt be.	The pig came out of the house and climbed to the top of the tree. Wolves did not catch him.
V dome svieti sviečka.	A házban ég a gyertya.	The candle shines in the house.

Table 5 Sentences

These activities support not only the building of mathematical notions by the students, but they have possibility to express their knowledge in Slovak and Hungarian language and develop their communicative abilities.

CONCLUSIONS

According Beardsmore (2008) bilingual children have a greater faculty for creative thinking at their disposal. They perform significantly better in tasks which require not the finding of the single correct answer to a question, but where they are asked to imagine a number of possible correct answers, for example, giving the maximum number of interesting and unusual uses for a cup. Nowadays it is important the interdisciplinary approach (see Tkačik (2007), Kopáčová (2012)). It is possible to introduce the Content and Language Integrated Learning (CLIL) in the minority school by the teaching of nonlinguistic subject. One good possibility is mathematics, because it has universal language (see Domínguez (2011)). In the future we would like to use also historical textbooks from Austrian-Ugrian monarchy because it was state with many nations and the textbooks showed how to teach more languages (see Páleš (1820), Gejdoš(2009)).

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ISSUES ON THE COMMUNICATION IN THE MATHEMATICS CLASSROOM – THE CASE OF A PRIMARY SCHOOL TEACHER

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This study analyses a primary school teachers' ideas and practice focused on mathematics communication. It addresses 2 questions: a) Which ideas do teacher have about communication in mathematics classroom? 2) How do teacher promote mathematics communication in the classroom? A semi-structured interview was conducted to reach teacher's ideas of mathematics communication in the classroom; classroom practice observation was carried out to identify teachers' strengths and difficulties when teaching mathematics. The results show that the tasks and the way the teachers organize the classes determines the development of verbal language. Discussion, the negotiation and the argumentation were encouraged, although without success in many cases.

SOME ISSUES ON MATHEMATICS COMMUNICATION

Learn to communicate and communicate to learn are related to two basic ways of looking at the learning: learning as acquisition of knowledge and development of skills or learning as a process in which the student has an active role in the construction of their own knowledge (Martinho, 2007). It is difficult to speak about mathematics communication without referring the teachers' role in its promotion. On the other hand, associated with this it often appears the term discourse. The discourse is seen by NCTM (1994) as "the way of representing, thinking, speaking, agree and disagree" (p. 36) and encompasses both how the ideas are exchanged and what ideas are portrayed. Thus, the teacher's role is considered essential for the discourse. The NCTM (1994) highlights three aspects that characterize this role. The first aspect relates to the students' reasoning. The teachers must stimulate the students' thinking through the tasks and the issues that arise when solving the tasks. Must listen with attention and interest students' ideas, ask them, orally and in written, for justifications and explanations for the solutions found. In this sense, it becomes relevant to ask "why?" regularly and consistently following the comments of the students, as well as other requests for justification mode. A second aspect that characterizes the teacher's role in guiding the speech is to be active, but not in the direction of the master. Teachers should lead the discourse with caution, prompting students' thinking, selecting the most relevant questions, providing information and deciding when to put guiding questions and when to promote students discussions.

A third aspect is to promote the control and organization of student participation. The teacher decides who is going to answer his/her question, how to implement group working and which communication means students should use to present their thinking. According to the NCTM (2001), speech materializes through the tasks. The selection of valuable tasks and challenge students to take positions and defend them, trying to convince the colleagues, is a very important aspect of math class.

Another important feature about teacher's role refers to the learning environment created. The control and authority exercised by the teacher will interfere with the learning environment created, preventing the development of student autonomy. Goffree and Bishop (1986) point out that the control exercised by the teacher is crucial in the negotiation of mathematical meanings and can be an impediment or facilitator. In this sense, Chazan and Ball (1995) argue that teachers should give more authority and autonomy to students.

Communicative practices also involve stimulating students' interest, thus contributing to enhance the interactions established. Communication as sharing is established through interactions between teacher and students and among students themselves. In this dynamic, students engage more with mathematical work than with mathematical content presented by the teacher, and that Bishop and Goffree (1986) relate to the concept of activity. The "activity" of students through the interactions established is one aspect that relates to this sense of mathematical communication.

Negotiation is another term related to communication, from the interactionist perspective. According to this perspective, mathematics is a social practice and knowledge is constructed in discourse through the production and negotiation of meanings (Menezes, 2004; Sierpinska, 1998). The speech, according Sierpinska (1998), is the language in action. As the author refers, attention began to focus around communication processes between students and communication with students and the question of sharing meanings that emerge through communication in the classroom (Sierpinska 1998).

Communication in the classroom

Language is a means by which to communicate. The communication is the main function of language (Menezes, 2000). Communication can be seen as a goal but also as a learning method (DGIDC, 2007). Ponte and Serrazina (2000) distinguish three categories for communication in the mathematics classroom, which are: presentation, questioning and discussion. The first is of expository nature; exposure of a story, an idea or an experience for the participants, teacher or student. The questioning is characterized by the use of successive questions by the teacher to the students in order to identify problems, motivate them to participate, stimulate reasoning, etc.. Regarding to the questioning in the mathematics classroom, Matos and Serrazina (1996) and Ponte and Serrazina (2000) consider three basic types of questions: focusing questions; questions of confirmation; and questions of inquiry. The focusing questions aim to "focus" the student's attention on what is essential, helping him to follow a certain reasoning. The confirmation questions put the focus on learning achieved. The aim is to explore the knowledge and possible feedbacks. For Ponte and Serrazina (2000) these questions when answered correctly by students increase their confidence and improve their internalization of ideas. The questions of inquiry allow the teacher to determine the students' way of thinking, how to solve a certain task and their opinion regarding a strategy or solution. Often these are a type of open questions that encourage divergent thinking and reflection.

This work enhances the language as a set of tools and ways of communicating between teacher and students, as it is through the various languages and representations that meanings are shared and developed (Bishop & Goffree, 1986; Pimm, 1991; Pirie, 1998) having a direct relationship with the tasks. The communication in the classroom as a goal and as a method assumes a wide nature. One values the development of students’ communication skills, with moments that help them express their thoughts and ideas clearly. In this scenario, interactions assume a relevant role. The interactions between students or between students and the teacher can take a form of discussion, in which usually the teacher assumes the role of moderator. A discussion between students can reveal their thinking (Ponte & Serrazina, 2000). Listening to the students interactions, the teacher not only appropriates their thinking as it gives them the opportunity to clarify their ideas. The discussion may provide an opportunity for students to learn to reason mathematically (Voigt, 1995). Thus, the teacher's role is highlighted in order to listen to students and stimulate their reasoning through questioning and selection of appropriate tasks. In addition, the teacher's role is also taken into account in creating an atmosphere that helps students to hear each other, creating room for discussion and reflection on the processes and products.

The discussion and negotiation of meanings either in small group or with the whole class is emphasized. The NCTM (1994) points out that the way teachers conduct the discourse in the classroom depends on their mathematical knowledge, but also on the curriculum and on students. The way the discourse is conducted, in math class, determines the knowledge and ways of thinking that are valued, as well as the processes of construction and validation of meanings.

Thus, it becomes relevant to know how mathematics communication practices are developed by primary school teachers. One tries to know more about the way these teachers value and promote the mathematics communication in their practices. One tries to address two questions: (a) what ideas and conceptions have primary school teachers about communication in mathematics? (b) What difficulties are identified when promoting mathematical communication in their practices?

METHODS

This study is a case study of a Primary School teacher. In agreement with Bogdan and Bicklen (1994), the case studies are justified for a specific knowledge of a given situation or phenomenon. The case study methodology, in general, requires the prolonged presence of the researcher in social contexts under study through direct contact with people and situations (Yin, 2009). The participant was the Primary School teacher Beth, who teaches 1st grade and has 9 years of teaching experience at this school level. All the names used here (teacher and students) are fictitious.

The study was organized around an individual interview and observation of teacher practices when teaching mathematics. A semi-structured interview focused on Beth ideas, beliefs and attitudes about mathematical communication was carried out. It addressed, among others, questions such as: Do you value the development of communication skills? As usually work the math, in large groups, small groups or individually? Do you think that students' interactions can improve learning? What kind of tasks do you usually explore in your classes? How do you select them? How relevant is for students to communicate their learning experiences? How do you stimulate this?

The observation of teacher practices comprised three lessons, in which the researcher was not a participant observer. The idea was to, eventually, find discrepancies between teachers practice (through classroom observation) and teacher's ideas. The observation of lessons focused on the analysis and interpretation of the teaching practices attending to the selected tasks, teacher's attitudes, mathematical knowledge and communication.

Data collection was carried out using an audio record for the interview; in the observed lessons, video and audio records were used as well as field notes taken by one of the authors of this paper.

RESULTS

The results are presented here in two moments: the first concerns the interviews comprising teachers' ideas and believes about mathematics teaching and learning and mathematics communication; the second focuses on the observed teachers' practices.

The interview

Beth seems to face teaching and learning of mathematics in a constructivist perspective. Knowledge is not transmitted but constructed, and student plays an important role in their own learning. Considers that it is important that students work in groups and justifies the less frequent use of this methodology due to having students from two grades in the same classroom, referring the difficulty she feels to meet the needs of all at a time. It states that her students have worked in pairs in solving various problems, but emphasizes that to have all these students in the same room "is a certain challenge because the difficulty of working with the two grades and meet successfully the needs of these students, sometimes prevents a little more to promote work group as I would like". The promotion of the development of reasoning ability, problem-solving and communication are understood by her as interconnected. Beth argues that:

Interviewer: How do you promote de development of problems solving? And reasoning? And communication?

Beth: Sometimes a single exercise reflects the mathematical communication and also the representation of reasoning because the child can reason and solve the problem orally, and often have a bit of difficulty translating it to an algorithm that was formerly most valued. So often, the mathematical communication arises from a picture or a scheme, and is up to them the translation of how they reasoned. But I greatly value sharing that justification of a resolution.

Interviewer: And how do you promote the sharing of these justifications?

Beth: Sometimes with a question for the large group, and there are children who can have their participation in a more accurate way than others, and I try to improve their answers, especially with the help of the justifications of those who have the right answer.

Beth considers that it is important for students to communicate their learning experiences, and communication in large group is the most emphasized in her speech and seen as important in learning. The exchange of ideas and discussion in the whole group is seen as an asset, especially for students with lower performance.

Beth: Normally, I opt for the correction in large group, self-correction in large group. Each child checks and sees where it went wrong and why is wrong, and with such communication and sharing among all, other children become able to reach a better understanding. Justifying their choices, students help those who have not understood so well.

The observed lessons

Beth used very often communication as presentation, often associated with a story emerged as a way of introducing the task. The communication as questioning was the most used by the teacher in all observed class, trying to promote students acquisition of mathematical concepts and mathematical vocabulary. For example, Beth tries to help students to identify some properties of geometric figures through questioning. The transcription below gives an example of this, when discussing if a rectangle was still a rectangle when in a different position.

Beth: Look, ... and what if I put it this way instead? [The smaller side as a base]. This is no longer a rectangle?! Do not tell me that!

Martha: No.

Beth: No... no, Marta?

Lucas: It becomes a square.

Beth: It is a square? Everyone agrees?!

Lucas: Yes [Only Lucas responds].

Beth: Is this a square? So this is like this as well? [The teacher compares this rectangle with square pointing to the figures].

Mary: No.

Beth: Why not Mary?

Mary: Because is upside down.

Beth: What? Is this still a rectangle upside down?

Lucas: No.

Beth: Who thinks that it is still a rectangle?... [Everyone agrees, including Lucas himself].

Beth: So this is also a rectangle? This and this are they rectangles?

Students: [in chorus] Yes!!!

Beth: Yeah, because just put them in a different position.

Teacher’s inquiring attitude in order to detect the students' conceptions about the rectangle was also common. The questions of inquiry "It is a square? Everyone agrees?!" or “Why not Mary? " are some examples. In another moment, it was also observed confirming questions such as "A square! What does a square have? The sides are ..." or "Who knows what figure is this?". The teacher also often uses focusing questions attempting to get the student's attention on important aspects to solve the task.

The group discussion as a means of sharing ideas, processes and results were poor in all observed lesson. However, in some of the class moments there were opportunities for it. The reflection on the work developed took place in Beth classes, but did not include discussion of mathematical ideas. Frequently, it was lost the opportunity for students to be encouraged to explain and justify their ideas, and increase their knowledge, as they were not given the opportunity to have a reflection and synthesis around possible solutions and solving processes. Concerning to the communication means, oral verbal language, visual and active representations were the most used in Beth observed classes.

Beth values whole class communication promoting verbal interactions student-class group. In her lessons, whole group communication can assume the form of discussion in which students share their conclusions and this has an important role in the task assessment. Interactions in small groups (student-student, student-class group) are less valued by Beth. The results are valued at the expense of resolution processes. The teacher promotes the iconic language beyond the symbolic, but the promotion of mathematical written language was not included in her speech.

When analyzing Beth observed practices, one tried to characterize the attitude of the Beth when promoting mathematical communication in the classroom. Beth uses verbal interactions. She promotes oral verbal language in teacher-student interactions in different phases of the lesson. In two of the three observed lessons, students were challenged to share their resolutions with the whole group, but they were not stimulated to reflect on their options. Being able to discuss resolutions/products without being encouraged to reflect on them, students were prevented from the systematization needed in learning. Oral communication, while the initiative of students, was attended by the teacher only when revealing discouragement or doubt. Communication between students was almost non-existent in the three observed classes. The teacher did not promote much this type of interaction.

In the observed classes it was visible the control set by Beth and the authority way she handled the error. At crucial moments, she presented no negotiation of meanings; Sometimes the teacher's corrections were not accompanied by explanations to students. The teacher also failed not questioning students about the reason of their findings. In some tasks presented to the students, the conditions for carrying out some tasks were not consistent with the guidelines due to a poor problem presentation.

It appears that there is a certain gap when comparing the results of the observed classes with the teacher’s ideas and conceptions about student-student interactions, solving processes and task assessment and teachers practices. The interview gave us the idea that Beth believes in a more constructive practices involving students more actively in the building process of their knowledge. The reality showed that Beth promotes individual work in the classroom, even when solving problems. Students are challenged only to compare solutions but no resolution processes.

An explanation for this discrepancy lies in the attitude of the teacher, who needs to take the class under control in order to be aware of everything that is said. This control is also present when she evaluates students work, being the authority. This discrepancy between Beth ideas and her practice seemed to be reduced when she refers to problem solutions, when analyzing students’ resolutions. Beth keeps saying to them "Ask the opinion of colleagues before saying whether it is right or wrong (...)". Beth reveals a certain lack of awareness of the importance of sharing opinions and discussions in the students learning process. This conception is present by focusing mostly in class group interactions when a task is completed. However, despite Beth considers the interactions between students facilitators of learning, she did not promote them in the observed classroom practices.

The classroom is very controlled by the teacher; the rules are strict and well defined and there is no room for transgressions. There is a classroom culture that privileges individual work but focused on "learning by doing". The student has an active role in learning process being able to use and explore manipulatives and is encouraged to develop discoveries. Thus, Beth can be characterized as a teacher who values dialogue between teacher-student that promotes the discovery choosing not only exercises but also problem solving tasks in her classes; but she can also be characterized by the development of an inhibitor control of the discovery at the level of student-student interactions and assessment. The written communication in mathematics was not valuable for this teacher as students were asked very seldom to explain their ideas or present a justification in a written mode.

Discussion and Conclusions

Beth easily understand that communication is important for students' learning, and that group interactions are important because students learn from each other, arguing that they promote interactions among students very often. However, despite this idea, it is difficult to find moments in which teacher was able to implement this in her classroom practices. This fact allow us to believe that special attention should be given to discrepancies between teachers believes and their practices. The participant of our study recognized the relevance of communication but never mention written communication in the interview or comments on lessons episodes; oral and written communication could be used as a way to understand students reasoning and improve it, but was she aware of it? The selection of tasks and materials, and the structure and organization of the classes created opportunities for the various forms of language and representation to take place. However, written and symbolic languages were the exception, having had virtually no expression.

Beth recognized interactions among students as a valuable issue, but her students were not allowed to solve problems in pairs or group. The NCTM (2007) refers that to encourage students to take positions and defend them, trying to convince colleagues is a very important aspect of math class.Was she really aware of students learning opportunities when implementing the problem solving tasks in her class? Teacher considers fundamental the promotion of argumentation and discussion with the whole class about strategies and/or resolution processes and products. Observed practices showed that Beth stimulates discussion and confrontation of opinions with the whole class, but she judges what is right and what is wrong, giving no room for students to assume that role. This suggests, in an implicit way, that she assumes permanently the role of authority in terms of knowledge. In this respect, Beth's ideas and practices are inconsistent, as she considers discussion and arguments were slightly promoted in the observed classes. In agreement with Goffree and Bishop (1986), the control exercised by the teacher is crucial in the negotiation of mathematical meanings and can be an impediment or facilitator. Also Chazan and Ball (1995) pointed out that teachers should give more authority and autonomy to students in order to promote reasoning and communication in mathematics classes. To be able to develop student-student interactions is an excellent opportunity to promote mathematics communication. It is of utmost importance for students to have an opportunity to work in an environment that enhance their ability to listen each other, creating discussion and reflection moments on the resolution processes and their solutions.

This study gives evidence that the main difficulties identified when promoting mathematical communication related, mainly, with didactic issues, but also the mathematical and curriculum knowledge. In educational terms, it is noteworthy the little incentive to clash results, to discuss strategies and arguments. The problem solving selected tasks created moments to promote communication and learning, but their implementation produced poor reflection and systematization moments. According to Bishop and Goffree (1986), DGIDC (2007) and Bridge and colleagues (1997), the reflection and systematization process about the developed work are essential to learning. The students had an important role in *doing*, but not to *arguing* and *discussion*. In some observed episodes, a weak mathematical knowledge was also observed and also a lack of rigor in mathematical language when exploring properties of the figures in the plan. Caution should be taken when assessing teachers' ideas about communication, as their concepts about communication might be very poor indeed. More important than listen their opinions would be the observation of their practices. In-service teacher training focused on mathematics communication would be relevant to help primary teachers to reduce these difficulties.

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CHILD-CHILD COMMUNICATION FOR REINFORCEMENT OF AMALGAMATION TRANSFER

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The goal of the research is to show the potential of the environment of Cube Constructions in children's' pre-mathematical activities (5 to 6 year old children). The presented research focuses on the use of child-child communication when reinforcing amalgamation transfer (the ability to perceive a processually described situation conceptually or conceptually described situation processually) in the environment of Cube Constructions. We show the importance of this communication for development of transition between conceptual and processual concept of a cube construction (the construction as a product and a process) as well as the level of ability of children of the selected age group to perceive cube constructions dually.

PROBLEM FORMULATION AND AIMS OF RESEARCH

Unsatisfactory results of Czech pupils in mathematics can be seen in a number of materials (e.g. PISA, TIMSS) and have been repeatedly subject to discussion. The number of teachers and educators who see Hejný's work as a possible way out of this impasse has been growing significantly. His belief that the traditional reproductive teaching style must be replaced by constructivist approaches in which it is the pupil who controls the activity is not new. The same idea had been discussed by a number of renowned educators before, e.g. by H. Freudenthal (1973) and E. Fishbein (1999). What makes M. Hejný's work unique is his simultaneous development of theory and its application and his development of scheme-oriented teaching based on theory of generic models (Hejný, 2007; 2012).

Theoreticians explain the concept of scheme by the fact that all information stored in human mind is not isolated but structured into meaningful units – schemes (Gerrig, 1991). Therefore also mathematical facts are not stored as separate concepts, statements, formulas, algorithms and proofs but as integrated in mathematics schemes. The method of scheme-oriented education is also based on the concept of learning environments introduced by E. Wittmann (1984), who defines an environment as a set of interrelated concepts, relations, processes and situations that allow posing problems guiding the pupils to discovery of deep mathematical ideas. The concept of a deep idea is elaborated in detail by Z. Semadeni (2002).

If Czech primary school teachers want to teach mathematics using scheme-oriented teaching, they can use the set of textbooks (Hejný, 2007–2011), officially approved by the Ministry of Education of the Czech Republic in 2008/09. The attention and effort of mathematics educators who work on the method has now turned to secondary and pre-school levels. At secondary level, the existing environments (about 30 environments) are further elaborated and deepened. At this stage problems for a new set of textbooks are piloted. At pre-school level, mathematics educators ask which of the environments are usable a and how.

Over the past two years experiments have been conducted primarily with environments of semantic nature, i.e. those that stem from a child's everyday experience. In the area of arithmetic, they are the environments Autobus (Jirotková, Slezáková, 2013), Stepping - Stairs and Farther Woodland, in the area of geometry the environments Sticks, Parquetry and Cube Constructions. The first of the authors cooperates with a group of 7 nursery school teachers from different areas of the Czech Republic. They have been meeting on a regular basis every month since November 2012. They also communicate in electronic form. In November 2012, the team selected environments for work with five year old children in a nursery school. All the teachers got acquainted with the chosen environments, both with respect to mathematics and didactics. Their task was to develop the environment for their children didactically and to carry out a teaching experiment. The experiment was to be recorded on video and in writing and the notes from observations were to be archived.

The goal of the reported research here was to test the potential of the environment Cube Constructions for development of amalgamation transfer in pre-school children. Our initial focus was on children aged 5-6 and the intention was to lower the age limit if the potential of the environment for 5-6 year old children proved high. Our partial goals were formulated in the following questions: 1) Is it possible to pose problems in the environment Cube Constructions that pre-school children are able to solve? 2) Can solving of these problems be used for reinforcement of amalgamation transfer, and if so, with what results? 3) What problems work best for the reinforcement of amalgamation transfer in pre-school children?

Amalgamation transfer is described by M. Hejný (Hejný, 2012) as an alternative to the proceptual transfer in those schemes. Our approach to procept is based on procept theory of E. Gray and D. Tall (PROcept, conCEPT). We understand proceptual transfer as the ability to perceive a processually described situation conceptually or conceptually described situation processually (Gray, Tall, 1994).

METHODOLOGY

The research was conducted in January 2014 with children aged 5-6 (the last year before they enter compulsory education) who had been attending nursery school at least since September 2013. The children were from different nursery schools from different regions, predominantly from North Moravia. Neither the length of their nursery school attendance nor their age were taken into account.

The teacher worked individually with randomly selected pairs of children that are referred to as child1 and child2. Child1 had six cubes of the same size (2 red, 2 blue, 1 green and 1 yellow), 4 crayons (red, blue, green and yellow) and a record sheet. The teacher had the same six cubes as the child and a record sheet.

We selected a cube construction built with six cubes of four different colours. One of the four cubes on the first floor is invisible in the front view, one cube on the second floor stands on a cube visible in the front view, the other stands on the cube invisible in the front view, see fig. 1, which shows the plan of the construction.

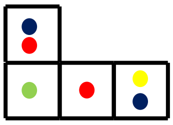

Fig. 1: Assigned construction – plan

The teacher placed the finished construction in front of child1 and asked him/her to build the same construction. Child1 then worked on the construction and the teacher gave verbal help if needed. This is the situation of amalgamation transfer. The teacher’s construction is the input concept, the child’s work on construction following the teacher’s model the process and the construction by the child the output concept (identical with the input concept). The construction was intentionally selected in such a way to force the children solve 4 key problems of increasing difficulty: (1) the front row of cubes in the first floor, (2) the yellow cube on the second floor of the front row, (3) the red cube “hidden” behind the front row of cubes, (4) the blue cube on the second floor placed on the “hidden” red cube.

After this the teacher asked child1 to record his/her construction on a sheet of paper (the child’s record sheet) in such a way that his/her friend (child 2) can build the same construction following this plan. Child2 was sitting in such a way that he/she could neither see the built constructions, nor the process of their construction. Child2 gets child1’s record, six cubes from the original teacher’s construction (the teacher does not need it any longer) and the task to build a construction whose plan he/she got from child1. Child1 does not watch his/her work and the teacher provides no help. When child2 finishes work, the teacher makes the plan of child2's construction into his/her record sheet. At the same time he/she puts this new construction in front of child1 and asks whether the construction is the same (child1 now has both constructions in front of him/her). His/her answer (yes – no) and the explanation of why they are not the same (in case they are different) is recorded into the teacher's record sheet. There are two additional amalgamation transfers: from the concept "child1's construction" over the process of "recording the construction" by child 1 (drawing a picture) to the concept of "picture", and from the concept of "recording of construction" over the process of "work on construction" by child2 to the concept of child2's "construction". Child1 subsequently compares two concepts.

If child2 failed to build the right construction in the first attempt, the set of tasks is repeated. Child1 is asked to record his/her construction into the record sheet again, child2 again tries to build a construction following the drawing, the teacher again records this new construction into his/her record sheet and places the construction in front of child1 asking whether the constructions are the same. Child1's answer (yes – no) and the explanation of why they are not the same in case they are different is noted into the record sheet.

If child2 failed to build the right construction even in the second attempt, the teacher asks child1 to have a look at the construction and to help child2 verbally to correct it. The teacher notes how this verbal help is formulated into his/her record sheet.

THE PRESENTATION OF THE RESULTS AND THEIR ANALYSIS

Record sheets of 83 pairs of children were analysed in this research. We focused on (1) how child1 records his/her construction graphically, (2) if and how he/she changes his/her recording of this construction if child2 fails to build the right construction using it, (3) what concepts children use in their verbal communication.

First we analysed recording of the front row of cubes on the first floor to find out how children draw cubes in plane. There were two different ways, either the cubes were represented by coloured squares (49 children) or by attempted drawings-of colored squares, or the cubes were represented by some kind of outline of squares in respective colours (34 children). 65 children drew squares or their outlines with no space between them, fig. 2, 18 children used spaces between the squares or their outlines, fig. 3. These data come from the first attempt. A change in how cubes are represented between the first and second attempt could be observed only in six cases which will be discussed in detail later. The frequency of different cases is given in tab. 1.

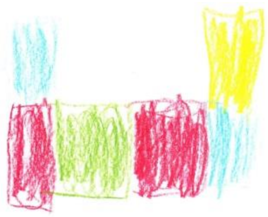

Fig. 2: Squares without spaces

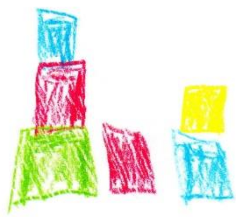

Fig. 3: Squares with spaces

	without spaces	with spaces	in total
square	43	6	49
square outline	22	12	34
in total	65	18	83

Table 1: Recording of the front row of cubes on the first floor

In the next step we focused on how children recorded the yellow cube on the second floor. In the first attempt, 76 children used the front view, fig. 2 and 3, 6 children the top view, fig. 4, and 1 child used translation of squares, fig. 5. 3 ways of recording the "overlay" of squares or their outlines in the top view could be identified: drawing one colour over another, fig. 4, placing one square inside the area indicated by outlines of another square, drawing the top part of a square or oblong using one colour and the bottom part using another colour, fig. 11 – the second recording. 8 children did not need to work in the second attempt (child2 built the correct construction in the first attempt). Out of the remaining 75 children, 65 used front view, 6 children top view and 4 children translation. The frequency of different cases is given in tab. 2. We can see that even in the second attempt children do not feel the need to change the way of recording the yellow cube from the front view to the top view even though they use the top view for recording other cubes in the construction in this attempt. The reason may be that child2 is able to place the yellow cube correctly even when following this "combined" way of recoding (front view and top view).

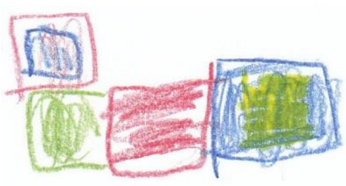

Fig. 4: Yellow cube in top view

Fig. 5: Yellow cube in translation

	front view	top view	translation
1^{st} attempt	76	6	1
2^{nd} attempt	65	6	4

Table 2: Recording of the yellow cube

In the next step we focused on recording of the red cube which is hidden in the front view. We expected this task to be most difficult for children but hoped it would guide them to discovery of a new method of recording. In the first attempt this red cube was recorded in top view by 35 children (method A), fig. 3 – 5, this despite the fact that in the same plan the yellow cube on the second floor was recorded in the same plan in front view, fig. 3. 25 children drew this cube as a red square which has one common side with the green square but in a different position than would correspond to top view (method B), fig. 2, 10 children as a red oblong which has one common side with the green square (method C), fig. 6, 5 children indicate free parallel projection (method D), fig. 7, 4 children use their own unique way of recording, (method F), fig. 11. 4 children left the cube out of their recording (it remained hidden). 8 children did not work in the second attempt. Out of the remaining 75, 33 children used method A, 28 method B, 4 method C, 4 method D, 3 children changed their strategy to the front view in which they drew the red square over the green square (method E), 1 child used method F and 2 children did not record the cube into their plan (the cube remained hidden in the record). The frequency of individual cases is given in table 3. This confirms our hypothesis on the need of a new way of recording of the “hidden” cube. Children depart from the front view dominant in recording of the yellow cube, table 2 – front view and look for new ways and methods. We know that their activity in the second attempt was connected to cooperation with child2. However, we did not consider this in the here reported experiment.

Fig. 6: Red cube as oblong

Fig. 7: Red cube in projection

method	A	B	C	D	E	F	Not recorded
1st attempt	35	25	10	5	0	4	4
2nd attempt	33	28	4	4	3	1	2

Table 3: Recording of red hidden cube

In case of recording of the blue cube on the second floor, the same methods were used as for recording the yellow cube on the second floor.

Then we analysed if and how children change their recoding of the construction in the second attempt in relation to mistakes in child2’s construction according to the first recording. We classified them as follows: change for better (A) – i.e.

the recording in the second attempt is closer to reality, fig. 8 (a – first attempt, b – second attempt), change for worse (B) – i.e. the recording in the second attempt is farther from reality, (C) – recordings are different but their quality is comparable (C), recordings are identical (D), there is no second attempt (D). The frequency of individual cases is shown in table 4. In total 38 children improved their recording (A), 8 children made it worse (B) – 4 out of them used a sophisticated recording in the first attempt which child2 failed to interpret and thus selected a method of recording more easily "comprehensible" for child2. The method of recording changed without loss of quality in case of 14 children (C), there was no change in recording in 15 children (D), 8 children did not work in the second attempt. The following changes were classified as improvement: omission of spaces between squares, unification of the method of recording of the cubes on the second floor, change in position of the square making it closer to reality (front view, top view), recording of the red cube hidden in the front view, use of symbols (outlines of square instead of full square, arrows, …), use of two colours in one square. It could be often observed that child2 worked with child1's recording as if it were in top view – i.e. they placed all the cubes on the first floor. Consequently children1 in their justification why the constructions were not identical described the need to place some cubes on the second floor (using different formulations) and tried to deal with this problem in their second recording (in different ways). The response (wrong construction) of the constructors forced them to look for improved ways of recording, which was our goal. In 4 cases children used arrows in their second attempt, fig. 9 (a – first attempt, b – second attempt). We regard this method as a highly abstract emphasis of the process of construction. We must bear in mind that the quality of the second recording depends on child2's construction. An exceptional case of change for worse is recording in fig. 10 (a – first attempt, b – second attempt), however, as it is the consequence of child2's activity, it is not taken into account in this research. The radical increase in number of improved recordings (38 children) confirms how high the potential of the environment Cube Constructions is in the studied age category.

A	B	C	D	does not work
38	8	14	15	8

Table 4: Changes between the first and second attempt

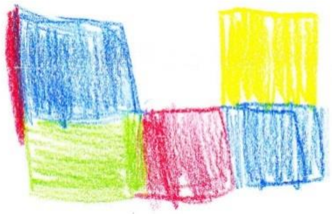

Fig. 8a: Changed recording, 1st attempt

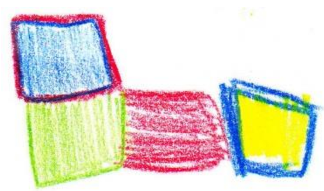

Fig. 8b: Changed recording, 2nd attempt

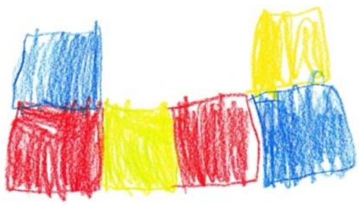

Fig. 9a: Rec. without arrows, 1st attempt

Fig. 9b: Rec. with arrows, 2nd attempt

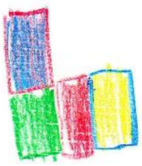

Fig. 10a: Recording, 1st attempt

Fig. 10b: Recording, 2nd attempt

Finally we analysed which concepts were used by the children in their verbal communications. The most frequent were **(1) adverbs, pronouns and prepositions of place, direction and order**: at the back, backwards, behind, behind the other; at the top, to the top, on, just on, above, from above; at the bottom, downwards, under, beneath; to the front, in front of; next to; on the right, on the left, by the window, by the door, towards our teacher; to; on the borderline, at the end; instead of; first, last; **(2) adjectives – attributes:** covered, hidden; flat, tall; horizontal, lying; colours – red, blue, green, yellow; **(3) verbs:** have a look, move; turn; swap, exchange, replace; put, stand; take away; be; **nouns:** cube; corner; row; chimney, like chimney; tower. Formulations of most children gradually improved (from brief explanations why the two constructions are not identical to more detailed instructions on how to build it correctly – also these instruction are refined if facing failure of the constructor, children use more concepts and more accurate instructions). The following are some selected interesting communications:

Marek: “If you look from the top, you see it as it’s there. Some cubes are on top of another …”

Elen: (draws the front view, child2 interprets and builds as if it were top view) “the cubes don’t look like my constructed cubes, they look like my picture.”

Aneta: (draws the front view, child2 interprets and builds as if it were top view) “This is lying and this is standing, that’s why it’s different.”

Nikol: (1st attempt) “The constructions are not identical because I didn’t draw it correctly for Kuba.”, fig. 8a.

(2nd attempt) “The constructions aren’t the same because Kuba didn’t understand, now I drew it correctly.”, fig. 8b.

Cyril: “If the paper was like a box or cube, I could draw it.”

Finally we present three recordings that were very different from all other. The first two recordings, fig. 11, represent a unique method of depicting the cubes on the second floor (the first recording was used by two children, the other by one child). The third recording, fig. 12, is work of a three year old child (not included in the experiment) and casts light on the issue of introduction of this problem at different ages, or on selection of alternative tasks if the child is not yet ready to master the methods of recording.

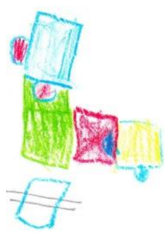

Fig. 11: Unique recordings

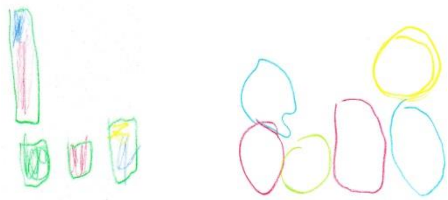

Fig. 12: Recording of a 3 year old child

CONCLUSIONS

The results of the experiment prove that the environment Cube Constructions is rich basis for posing a wide range of problems suitable for children of the studied age category. All children discovered some method of projection of the cube into a plane (although on different levels). In most cases, this method was then refined in communication with child2: with respect to mistakes child2 had made when constructing following child1’s recording, child1 corrected the original recording into a more “easily comprehensible plan”. Some of the children came very close to the ideal recording of the plan for construction when they produced the correct top view of the construction and depicted cubes on the second floor by drawing them on the cube from the first floor or by recolouring.

Child1 was able to interpret his/her less comprehensible / incomprehensible recording from the first attempt. They were able to reconstruct their construction following their own record.

In the follow-up to this experiment we plan to verify if and in what activities the children will be able to reach the ideal construction plan, fig. 1, or its analogy. We propose the use of construction with three floors when it will not do to work with two colours of a square. If this leads to a shift and unification in recordings, teachers could start assigning simple tasks like: draw a plan of a construction, build a construction according to its plan, move one cube and draw the plan … In the follow-up experiment we could also study the results of solution of problems on different levels of difficulty. Moreover, we could start taking into account the relationship between child1 and child2, which was disregarded in this experiment.

Intensive amalgamation transfer could be discerned in most children who were solving the task. It clearly shows in the high success rate (the construction was built correctly by child2 in the end) and in records from the experiment: child1 built the construction they saw correctly in most cases (concept – process – concept), recorded the construction (concept – process – concept), child2 built the construction correctly (concept – process – concept). Some children even used symbols – arrows (emphasis on process).

The results of the experiment show that the selected task was adequately difficult for the studied age group. All children tried to solve the task and only a small proportion of them could not cope with cubes on the second floor or with the hidden cube. In these cases the solutions are different but this divergence forms a solid basis for further research on optimal recording of the construction. In this further research we plan to work in groups which will guarantee transfer of thoughts and their joint development.

It would be very interesting to explore the relationship of selected method of recording of the construction with future school success. Our assumption is that these are closely related (the higher the level of work with schemes, the higher performance in school mathematics, namely geometry). In that case construction recording could become a possible a diagnostic method.

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Discourse on visualisation

Part 2

AN INVESTIGATION OF THE VISUAL-MENTAL CAPABILITY OF PRE- AND IN -SERVICE MATHEMATICS TEACHERS: A TALE OF TWO CONES

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This study investigated the visual-mental capability of pre-service and in-service mathematics teachers as well as university graduates making a career change to mathematics teachers with regard to manipulations of geometric shapes (from 2- dimensional to 3-dimensional). Moreover, it investigated whether there are differences between the visual-mental capabilities of these participant groups. The study included an assignment, which requested mental imaging. The participants were asked to explain their solutions and to describe their phases and their solution process. Findings illustrate that a very low percentage of participants manifested a visual-mental capability in the task. The study indicates that learners' high visual view should be developed in order to enhance their visual-mental capability.

INTRODUCTION AND THEORETICAL BACKGROUND

Mathematics educators concur about the importance of developing spatial orientation and visual-mental capability for the enhancement of mathematical thinking in general and visual thinking in particular (Patkin & Dayan, 2013; Halpern, 2005; Sorby, Wysocki & Baartmans, 2002).

Solid geometry is one of the topics of geometry studied from kindergarten, throughout the years in elementary school, through high school and up to mathematics teacher education in college in Israel. Quite a few studies explored children's thinking related to spatial orientation in general and perception of concepts associated with spatial geometric shapes in particular (Shaw 1990; Yackel & Wheatley, 1991; Hannibal, 1999). Spatial comprehension comprises two components: visual-mental capability and spatial orientation. Spatial orientation is the position of an object in space with regards to a frame of reference that is chosen. Visual-mental capability is the ability to imagine 2-dimensional shapes and solids as well as their movement or change in their attributes (Del Grande, 1990).

Developing visual-mental capability is based on learners' experiences and activities with 3D shapes, using illustration aids and actual experience of building 3D shapes (Patkin & Sarfaty, 2012). The National Council of Teachers of Mathematics (NCTM, 1998, 2000) defined several standards, principles and tools for teaching mathematics at different ages. The document explicitly states that the development of visualisation is one of the tools for solving mathematical problems:

Students should gain experience in using a variety of visual and coordinate representations to analyse problems and study mathematics… beginning in the early years of schooling, students should develop visualisation skills through hands-on experiences with a variety of geometric objects… students need to learn physically and mentally to change the position, orientation and size of objects…. One aspect of spatial visualisation involves moving between 2- and 3-dimensional shapes and their representations…(NCTM, 2000, pp. 42–43).

Walker, Winner, Hetland, Simmons & Goldsmith (2011) argue that, although the importance of visual-mental capability is recognised in the learning of mathematics, solid geometry is usually studied with a strong emphasis on the formal aspect and symbolic representation rather than on visual-mental capability.They noted that teaching which integrates formal aspect and visual aspect facilitates complete conceptualisation of geometric ideas and the ability to cope with geometric assignments. This attitude is supported by the work of the mathematician William Thurston, winner of the Fields Medal in 1982. He became famous after illustrating the power of visual representations in the conceptualisation of abstract mathematical ideas. He claimed that using visual tools for representing abstract mathematical ideas contributes more than formal proofs. Consequently, he developed ways for teaching geometry by means of visual arguments (Horgan, 1993).

According to Presmeg (1986) and Bishop, (1989), the ability to create mental pictures may improve the understanding of mathematics concepts and problem solving. In her study, Cohen (2007) discussed the analysis of geometrical thinking processes of pre-service teachers and in-service teachers while performing tasks in the 3-D space. Her main goal was to identify supporting and hindering factors which affect the cognitive processes and to examine the effect of combining visual imaging and analytical difficulties and misconceptions. Cohen's findings supported the premise that strengthening the use of analytical–visual integration enables improvement of the ability to imagine pictures.

Strengthening the analytical-visual connection facilitates a more comprehensive understanding of concepts and space relations. Walker et al. (2011) investigated whether students of art have high visual-mental capability, because art is an area of knowledge requiring high visual-mental capability. Their study consisted of two student groups: art students and psychology students. The participants were asked to respond to items designed to check visual-mental capability. Answering these items required 2- and 3-dimensional visual-mental capability. Findings of this study showed that the art students demonstrated better capabilities in solving the geometric assignments. Moreover, it illustrated that there was a considerable difference in their visual-mental capability versus those of the psychology students. All these findings corroborate the researchers' claim that developing visual-mental capability in art improves geometric thinking.

Hence, in-service and pre-service mathematics teachers should be aware that using a variety of assignments engaging in visual-mental capability as well as shifting from 2- to 3-dimension.

One of the ways to improve this capability is to advocate the use of a reflective written record (one of the communication manifestations) in order to promote mathematical thinking and the learners' ways of coping with mathematical material. Furthermore, writing serves as teachers' diagnostic tool so that they can monitor the development of knowledge and mathematical thinking, as illustrated by the learners' written explanations and arguments (Patkin, 2011). The process standard *Reasoning and Proof* calls for students to be making conjectures and generalizations and evaluating these conjectures and generalizations. Students cannot do this without effective communication. Communication also brings forth the teaching principle, which challenges teachers to know what it is the students already know and need to learn, and then to help challenge and support them in these endeavors (Sample, 2009).

Our study focuses on the visual-mental capability of pre- and in-service mathematics teachers. Therefore, the research questions are:

(i) What is the visual-mental capability of the pre- and in-service mathematics teachers regarding manipulations of geometric shapes (from 2-dimensional to 3-dimensional)?

(ii) Are there differences between the visual-mental capabilities of these participant groups?

This paper focuses on a assignment that dealt with an unfamiliar rotated solid (solid consisting of two cones with a common basis). The assignment does not depend on formal knowledge of geometry, e.g. definitions, theorems and proofs but rather focuses on geometric thinking. It requires reliance on vision and ability to imagine solids.

METHODOLOGY

Research Population

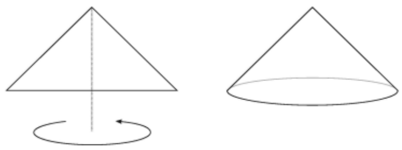
The research population consisted of 87 participants learning in an academic college of education. The participants were divided into three different sub-groups. The first sub-group comprised 46 pre-service teachers in their first year of studying to become elementary school mathematics teachers (1^{st} year). These participants have never taken any academic geometry course on the topic of solids (3D). The second sub-group consisted of 18 university graduates making a career change to mathematics teachers (career change) .These university graduates are in their first year (out of two years) of studies for obtaining a teaching certificate. The third sub-group included 23 in-service mathematics teachers studying towards an M. Ed. in mathematics education for elementary school (M.Ed.). These teachers have B.Ed. in mathematics and teach geometry including the topic of solids (3D) at their school.

The Assignment

The participants were asked to solve the following assignment: Imagine you are holding in your hand a paper square, holding it at two opposite vertices. Now rotate the square so that the diagonal between those vertices constitutes the rotation axis. Which image of a shape is created?

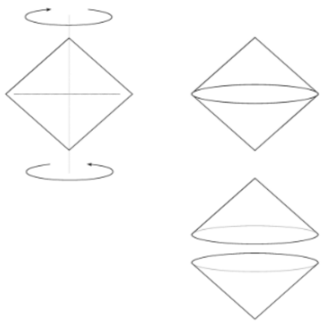
As already mentioned, this assignment does not depend on formal knowledge of geometry, but requires reliance on vision and ability to imagine solids. In order to investigate the in-service and pre-service teachers' ability to solve this assignment by mental imaging, without manipulation of external representations, the participants were asked to respond to the assignment in as detailed a manner as possible, using words but without using a drawing. In case they used a drawing, they were asked to explain why they did it, indicating at what stage of the answering process they decided to use a drawing (prior to the solution, during the solution or as a backup of the solution they suggested).

The assignment requires rotating a square page whose diagonal serves as the rotation axis, creating a 3-dimensional solid comprised of two cones with a common basis. It is important to point out that rotating an isosceles right-angled triangle around the altitude to the basis of the triangle creates an image of a cone (3-dimensional figure).



Picture 1a. Rotation of an isosceles right-angled triangle

The diagonal of the square divides it into two congruent isosceles triangles. Hence, rotating a square page around its diagonal creates a solid consisting of two cones with a common basis.



Picture 1b. Rotating a square page around its diagonal

FINDINGS

Table No. 1 presents the distribution (%) of correct and incorrect answers given by the participants to the assignment. Moreover, the table indicates the participants who did not respond to this assignment. The table refers to each of the three sub-groups participating in the study: pre-service elementary school mathematics teachers in their first year of studies; university graduates making a career change to mathematics teachers; and in-service elementary school teachers studying towards an M.Ed. in mathematics education.

Sub- Group	N	Correct	Incorrect	Did not respond
1^{st} year	46	0	41 (89%)	5 (11%)
Career change	18	7 (39%)	6 (33%)	5 (28%)
M.Ed.	23	4 (17%)	14 (61%)	5 (22%)
Total	**87**	**11 (13%)**	**61 (70%)**	**15 (17%)**

Table No. 1: Distribution (%) of answers in the cones assignment

Table No. 1 illustrates that only 13% of the research participants gave a correct answer and identified that the resulting solid was two cones with a common basis. These were the university graduates making a career change to mathematics teachers at elementary and junior high schools and in-service teachers studying for M.Ed. in elementary school mathematics education. 89% of the pre-service teachers in their first year of studying to become elementary school mathematics teachers did not respond correctly to this assignment. The other 11% failed to respond to this assignment. Among those learning towards an M.Ed. and the career-changing, the percentage of non-respondents was twice higher and even more than that (22% and 28% respectively).

As for the participants who answered this assignment correctly, the findings show that only five (out of the 11 who responded correctly) offered arguments. All the arguments presented were correct and related to essential attributes of the created solid. The participants pointed out that one diagonal of the square is the diameter of the obtained cones whereas the second diagonal constitutes the altitudes of these cones. The sides of the square are the creating line segments of the cones. Below are the explanations given by the participants.

“The diameter is the diagonal of the square. The creating line segment is the side of the square” [In-service teacher R];

“The radius of their bases is half the diagonal of the square and their altitude is half of the diagonal of the square and you connect them at their bases” [In-service teacher M];

“The ends of the square (the vertices which we are not holding) create a circular basis (collection of dots equally positioned from the centre of the square which becomes the centre of the circle” [In-service teacher A];

“The side of the square will become the creating line segment of the cone on each side” [Career change C];

“The vertex of the cones will be the opposite vertices of the cube which are held in both hands. The rotation around the diagonal axis of the square turns the diagonal of the cube in the connection of the two cone altitudes” [Career change F].

It is worth noting that some of the participants, including those who did not give any mathematical argument, described their decision-making process regarding the (correct) answer.

The participants indicated that for the purpose of making a decision or checking their answer, they used a tangible item available to them such as a notebook, a notebook page or a handkerchief. They did not rely only on the mental picture of the solids.

“I used the examination page in order to see what we get. It took me time to understand the problem. Only after rotating the page several time I saw these were two cones connected at the basis” [In-service teacher H];

“It is interesting to point out! Rotation of a square is done in the head, but as backup I rotated the exam notebook” [Career change O];

“I used a drawing and, mainly, a handkerchief in order to comprehend the meaning of rotating around one axis while the other axis (the second diagonal) is turning around” [In-service-teacher S].

Table No. 2 presents the variety of incorrect answers given by the participants. The table relates to the distribution of the incorrect answers about 3-dimensional solids and to the distribution of incorrect answers about 2-dimensional shapes.

Table No. 2 shows that 39 incorrect answers related to solids and 22 answers related to incorrect shapes. Among the 1st year participants, 13 types of incorrect answers were found. The most prevalent incorrect answer was the cylinder. This answer might have been given based on the knowledge that rotating a rectangular paper page around one of its sides results in a cylinder. This answer did not relate to the fact that the rotation axis in the assignment was not one of the square sides but rather its diagonal. Additional five types of incorrect solids (10 answers) and seven types of incorrect shapes (17 answers) were presented by the participants of this sub-group. Among the university graduates making a career change to mathematics teachers, three types of incorrect answers were given. Half of the university graduates (three out of six) indicated that the created solid would be a ball. Another type of incorrect solids (one answer) and one type of incorrect shape (two answers) were also presented by the participants of this sub-group. The in-service teachers studying for an M.Ed. presented a variety of eight incorrect types of answers: eleven participants of this sub-group related in their incorrect answer to different solids and three others related to two types of 2-dimensional shapes.

Figure	The answer	1st year N=41	Career change N=6	M.Ed. N=14	Total N=61
3- D	Cone	3	---	1	4
	Cylinder	12	---	1	13
	Ball	---	3	3	6
	Pyramid	---	---	2	2
	Cube	2	1	1	4
	Square prism - box	2	---	---	2
	2 square pyramids with a common basis – octagon	4	---	3	7
	A very thin body	1	---	2	3
2-D	Square	4	---	2	6
	Circle	2	---	---	2
	Triangle	3	2	1	6
	Rhombus	4	---	---	4
	Kite	2	---	---	2
	Parallelogram	1	---	---	1
	Other	1	---	---	1

Table No. 2: Distribution of incorrect answers in the cones assignment

DISCUSSION AND CONCLUSIONS

This study focuses on mathematics education and, therefore, engages in the knowledge of pre-service and in-service teachers. This population has been extensively researched in studies of mathematics education with the purpose of exploring the knowledge and capabilities of mathematics teachers. The research aim was to explore the visual-mental capability of pre-service and in-service mathematics teachers as well as of university graduates making a career change to mathematics teachers regarding manipulations of geometric shape.

Only a few studies investigated the visual-mental capability of pre-service and in-service teachers at different stages of their education regarding manipulations of geometric shapes and transition from 2-dimensions to 3-dimensions. Studies focusing on the relation between visual capabilities (e.g. of art students) and formal aspect of geometry illustrate that visual art can serve as a starting point or even as a jumping board for learning geometry and enhancing geometric comprehension. Visual capability can be a useful tool for promoting geometric comprehension, mainly when focusing on dynamic geometric thinking instead of memorisation and application of static rules and geometric relations (Seago, Driscoll & Jacobs, 2010; Walker et al., 2011).

The study conducted by Walker et al. (2011) investigated students of psychology and art in order to check whether they have different visual-mental capabilities of solving assignments dealing with manipulation of shifting from 2-dimensions to 3-dimensions. This study, however, aimed to examine the visual-mental capabilities of pre-service and in-service mathematics teachers with regard to these unusual assignments which are not studied directly in the curriculum. All the participants in this study have high school education, including matriculation in mathematics. They were directly or indirectly exposed to solid geometry during their previous studies (the differences reside in the number of mathematics learning units they took at high school).

Findings show that the participants demonstrated low visual-mental capability in a less familiar assignment which is not so prevalent in their professional world: identifying the solid created by rotating a square around its diagonal (13% answered correctly and 70% answered incorrectly). This finding is typical of all the participants of this study regardless of the sub-group to which they belonged. The participants presented a variety of incorrect answers (70% and 17% did not respond). Some of the mistakes were characterised as a misconception in the transition from a 2- to 3-dimensional shape. Many of these participants related to a cylinder or cone on the one hand or to solids whose basis is not round, such as a rectangular prism or a cube on the other hand. Another typical misconception was that the rotation had no effect on the shape dimensions, namely the participants perceived that the created shape was 2-dimensional following the rotation.

Yackel and Cobb (1996) note that in classrooms, where students are challenged to think and reason about mathematics, communication is an essential feature as students express the results of their thinking orally and in writing. They emphasize that explanations should include mathematical arguments and rationales, not just procedural descriptions or summaries. The participants in this study were asked to communicate in writing. They were also required to add to their answer regarding the type of the obtained body some written explanations and a description of the decision-making process associated with the answer, which led to identification of these misconceptions. It is similarly to the study conducted by Patkin (2011).

In light of the difficulties encountered by the pre-service teachers, in-service teachers and the career-changing university graduates in this solid geometry assignment, we believe that using operational ideas such as the assignments included in this study can develop learners' spatial orientation as well as help those engaged in the teaching of geometry to enhance the learners' geometric thinking.

Findings of this study illustrate the importance and contribution of visual-mental capability to geometric thinking, even in studies of another area of knowledge, art for example. Hence, it is recommended integrating in geometry teaching activities which nurture and develop a visual-mental capability at all stages of learning and in different pathways, from kindergarten to academic education. Learners' visual-mental capability might be developed not only within the framework of geometry teaching but in other areas as well and thus their spatial orientation and geometric thinking would be enhanced.

COMMUNICATION OF STUDENTS WITH EDUCATIONAL MULTIMEDIA PROGRAMS TO LEARN MATHEMATICS[1]

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The learning process is a constant exchange of information between the teacher and a student. If the teacher be substituted by a computer – a programmed machine – it not only needs to act as sender, but also to be sensitive to the feedback from the user. According to the research, it is the sensitivity for user's feedback that proves educational efficiency of modern computer programs.

Communication is the exchange of information between a sender and a receiver. The transmitted material is a certain information unit – the message. In order for the message to be received correctly its sender needs to prepare the information in an appropriate way (so that its content is understandable) and decide on an adequate way of transmitting it – choose the information channel.

In education, the communication between the teacher and the student is the most important process determining learning. If the message is unclear (eg. too complicated), the receiver is not able to render it, understand it and as a result remember it. Messages can also be sent with an inaccessible channel, eg. in a different language or using symbols which are not yet known to the child. The way of sending the message plays a vital role in the learning process. If the teacher-sender transmits the pieces of information too quickly one after another, the receiver may be unable to receive them all, understand them and master them. Distracting stimuli, accompanying the message and disrupting its reception, are factors disturbing the whole communication process.

What plays an important role in the communication (and learning) process is the right, reception-focused approach of the receiver – attention. If the receivers-receptors (eyesight, hearing, etc.) are not focused on receiving and processing information, the efforts of the sender trying to convey the message are in vain.

The goal of the communication process is exchange of information. In order for this process to take place both the sender and the receiver need to show signs of understanding and assuring, which can be described as *I understand, please continue*. If the receiver does not send such information to the sender, the latter

[1] The text comprises abstracts from the unpublished doctoral thesis written under the direction of Prof. Edyta Gruszczyk-Kolczyńska, PhD, at the Faculty of Pedagogical Sciences of the Maria Grzegorzewska Academy of Special Education in Warsaw.

does not know, if he/she was understood correctly. Therefore, the role of the receiver is sending some feedback.

Teachers, whose main role is sharing information, transmitting it to others in an understandable way, need to master the art of sending messages and pay attention to feedback information. Moreover, they need to focus on logical ordering of the message contents and adjusting them to the possibilities of the receiver.

Mathematics teachers have a specifically difficult task, where the language of the message they use is encoded in the special language of mathematic symbols. Introducing the system of symbols to the students requires from the teacher to properly structure the mathematic content. An additional hindrance is the fact that mathematical competences are measured by solving difficult tasks, requiring the knowledge of complex symbols. As the system of symbols is, for obvious reasons, not accessible to children, the teacher needs to introduce it gradually. The first stage is creating the ability to solve tasks relating to concrete objects that the students can manipulate with. The teacher writes the operations in a symbolic way. Learning the secrets of math the students are introduced into the world of encoding first on the level on concrete objects and operations connected with them (adding/removing), then on the level of signs substituting the concrete operations – symbols (addition/subtraction).

Considering the communication aspect, the role of a math teacher is not only giving the students more and more difficult tasks to solve. They also need to pay attention to understanding the content of the task and to control how the students solve it. It is of special importance when there are weak students in the class, who cannot use the information from the task in order to find the solution by themselves.

In a class the teachers support their students with various forms of help. Offering individual help the teacher:

- points out the information skipped by the student, which is *the missing link* to solve the task;
- tells the missing information that the student does not know or forgot;
- solves a part or the whole task, showing the student the correct algorithm and allowing the student to use it in another task.

Due to the fact that in a class at school the teacher needs to deal with many students at the same time, he/she encourages the students to try to find the solution by themselves. However, the students can reach the solution using different strategies. If it is an arithmetic task, the student may:

- establish the solution mentally;
- use an abacus or concrete representations (eg. finger count);

- hide the way of solving the task and copy the answer from a colleague;
- guess at random and write an approximate number without calculating.

The strategies used by students depend to a great extent on their previous experience and mathematical skills. However, regardless of the skills, the teacher needs to pay attention to independent task solving because it is the only way of acquiring task-solving skills (see J. Bruner, 1978, p. 676-677). In order to encourage the students to choose the right strategies the teacher applies a system of punishment and reward in form of school grades. However, students are also subject to social pressure (from peers, parents and the teacher).

The aim of the research

School education has been supported by multimedia resources for a long time, most of these resources are computer-related. Modern multimedia programs are an alternative to the typically frontal lessons, where the teacher controls the learning process of all the students in the class. Multimedia programs, acting as a teacher, need to influence the receivers in a programmed way. On one hand the messages (instructions, tasks) need to be understandable and adjusted to the students' skills and on the other hand the punishment and reward need to be adjusted to the motivation level. It seems that fulfilling these criteria should guarantee high efficiency of multimedia programs. Unfortunately there are still different opinions among specialists concerning their efficiency (see Raszka R., 2008; Kengfeng K., 2008). The most popular condition of educational efficiency of a program is comparing the students' results before they started to use the program with the results after a series of computer sessions. In such a comparison the way, in which the students use the computer program – the process of communication with the program, is not taken into account. Further it will be proven a mistake. Including the way, in which students use the multimedia programs, allows us to check if the means of solving tasks on computer screens are the same as those encouraged by the teacher (independent ones). Based on the above it was tried to establish what is the communication between a student and a computer program for learning mathematics.

Methodology

The research was based on the best-known and most popular program for learning mathematics in Poland: „Klik uczy liczyć" (see Kłosińska T. and Włoch S., 2002). The program is aimed at children aged five to nine. All of the Polish research (Kaczmarek Ż., 2003; Watoła A., 2006; Raszka R., 2008) showed that the program is very efficient for gaining mathematical knowledge and skills.

The research was conducted on 25 first grade students. None of the researched students had known the computer program „Klik uczy liczyć" before and all of them were familiar with the computer to the extent allowing to use the program.

The research was based on the technique of one group of the experiment. In the beginning of the research a test of mathematical knowledge and skills was conducted and then the students were split into two groups – experimental (12 students) and control (13 students). The experimental group was to use the educational multimedia program during ten sessions. None of the sessions were time-limited and the ways of using the program were also unlimited. During the sessions the students were watched by a researcher and recorded with a program working in the background of the educational program. It recorded not only the picture of the child but also screen prints, the instruction sound and the keys clicked by the student. Whereas the students from the experimental group used the educational program, the control group only took part in educational classes in the classroom (they did not use the program). After five months (the period of research), when the series of computer sessions were completed, all of the students were tested once more regarding their mathematical knowledge and skills.

That research was a quality research as the main focus was on analysing the behaviour of every one of the twelve students from the experimental group. Repeated analyses of the film material allowed to describe how the students use the educational program. How they deal in difficult situations, how they approach the more and more difficult tasks, in what situations they resign from executing the speaker's instructions and what they pay special attention to when using the program. All of the aspects of the communication-process-related behaviour shall be shown on the example of Dominik. The boy was chosen due to the fact that among the twelve students he was the only one assessed by the teacher as the weakest in the class, and those – according to the common opinion – should solve much more tasks. He was repeating the grade, according to the teacher, mostly due to frequent absence from classes. The teacher described the eight-year-old as willing to learn but unable to gather enough experience in solving tasks due to difficult family situation.

The description of the way of using the computer program by Dominik will analyse: time of using the computer, the number of tasks solved and their difficulty level and the way of solving the tasks (arithmetic and content-based tasks) in the „Klik uczy liczyć” program. This way the communication of a student with a very low level of mathematical skills with a computer program will be described. Finally the results of the boy's first and second mathematical knowledge and skills test will be compared to establish the educational efficiency of the computer program.

The presentation of the results and their analysis

Dominik took part in nine sessions with the program „Klik uczy liczyć”, lasting 319 minutes altogether. During those 5 hours (and 19 minutes) he solved 297 tasks. Most of them (118) were level two (addition 1 to 10 tasks). He also solved many tasks (140 altogether) on higher levels (4-6)[2]. The difficulty level of the tasks was frequently modified by the boy. Dominik initially chose the most difficult tasks – probably to check what they look like. Solving them he overused the strategy of trial and error method resulting in lowering the difficulty level by the program. He also often moved the difficulty slider by himself, so that he would solve easier tasks, resigning from intellectual efforts. The observed tendency is shown by data in Table 1. Based on the data the trend to pick easier tasks is very clear. Based on a thorough analysis of the boy's behaviour, his way of using the computer program can be described as *focused on the program*.

Session (min.)	**1** (45)	**2** (27)	**3** (52)	**4** (44)	**5** (38)	**6** (26)	**7** (29)	**8** (36)	**9** (22)
Difficulty levels	Number of tasks solved during every session ordered by difficulty levels								
17									
…									
6	7								
5	3	4	8						
4	28		39	3					
3			20	22		5			
2		19	20		36			40	3
1							17		22

Table 1. The number of meetings, their duration, number and difficulty level of the tasks solved by Dominik in the „Klik uczy liczyć" program. Own elaboration.

Dominik's behaviour showed that he adjusts the task solving strategy to his subjective estimation of the task's difficulty. If the task seemed easy for him, he tried to solve it himself establishing the result mentally or finger-counting. When the task was too difficult or not understandable (eg. due to a complicated instruction, too complex background picture for the task or including the multiplication sign – unknown for him), the boy gave up his own strategy for the strategy suggested by the program authors (counting objects in a program window). If the task did not involve objects to be counted Dominik solved it with trial and error method or resigned from solving the task and exited it (escape strategy). Situations when Dominik changed the strategy depended on the difficulty level of the tasks shown on the computer screen.

Analysing the way of solving tasks by Dominik in the „Klik uczy liczyć" program it needs to be noticed that the methods of establishing the result used by the boy were not consistent with the assumptions of the authors of the program.

[2] Level three (subtraction 1 to 10) – 48 tasks, four (addition and subtraction within 20) – 70, five (addition and subtraction to full tens) – 15 and 7 from level six comprising addition and subtraction double digit numbers within 100).

He often used techniques helping him avoid intellectual efforts. He did it in spite of the help options provided by the program, such as *lifeline*. In the software instruction the option was described as a way of solving the task by the program. After choosing it the program wrote part of the answer (eg. the units digit). Dominik never used that option, even though its icon was visible on the computer screen all the time.

When comparing the strategies used by the student to the type of the task a clear difference can be noticed if we group the tasks depending on if they gave the possibility to count concrete objects or not. The result of the comparison shows that Dominik preferred solving tasks by counting the objects in easy tasks (up to level four according to the scale defined by the author of the program), even though he was able to solve them mentally. In the tasks without objects to be counted (from level five up) the student preferred establishing the result by trial and error method. This phenomenon is shown clearly by data in Table 2.

		Easy tasks	Difficult tasks
The authors' strategies	Counting objects	**206**	1
	Mental calculations	16	
The students' strategies	Finger counting	9	2
	Trial and error method	21	**20**
	Escape	using it did not provide a solution	

Table 2. The number of using five strategies for solving easy (with objects to be counted, levels one to four) and difficult tasks (without objects to be counted, levels five to 17)[3]. Own elaboration.

I shall quote the transcription of meeting 3, where Dominik frequently used the strategy of counting objects resigning from finger counting:

> *(…) stopped the instruction with a mouse click and started solving the first task. He solved the operation 4+5 by finger counting (there were also objects to be counted on the screen). All of the following operations (3+7, 6+3, 3+3, 3+4, 2+2, 1+8) he solved by counting the objects (…). He chose the next book. He solved the operations 5-2, 4+5 by finger counting. He solved the following ones (3+4, 4+2, 4+4, 1+5, 4+6) by counting objects. (…). During counting he moved his head closer to the screen and moved the cursor as a pointer on the screen.*

The objects to be counted on the screen were small and the program window only took a half of the whole screen, what additionally made noticing the smaller objects more difficult.

[3] As it was not possible to establish explicitly what strategy was used by the student (eg. mental calculations or quiet counting objects on the screen), therefore the numbers in the table are not consistent with those in Table 1 and 2.

Dominik was very absorbed by the program. He was excited about everything it offered: rewards, animations, pictures, music, activities and the tasks themselves, but the last ones were treated as a means of getting a reward. He tried to solve the tasks the best that he could, however, his low level of mathematical skills made him use strategies accessible for him, such as trial and error, in order to get the reward. One has the impression that with every subsequent session with the program Dominik paid less attention to the tasks and focused on finding pleasures such as: films, animations, pictures, hidden tasks in the program, etc. He also tended to look for easier tasks. It was clearly visible during the last sessions. Following there is an example of his behaviour observed during the sixth session:

> *(...) chose the task „family picture”. However, after starting it he resigned. He did similarly with the task „Uncovering” (...), and also later when he picked the „park” environment from the main menu and then the tasks: „traveller in the park”, „memory” and „uncovering” were all exited immediately after being started. The first task completed by him was „trash” – grouping of trash to the appropriate containers. Dominik's behaviour – going through the program and solving chosen tasks is a sign of looking for tasks fulfilling precise criteria. Tasks omitted by him show that he tried to avoid tasks based on calculating or doing memory activities (such as „memory” task).*

It is clear that during the following meetings not only did the student's motivation to solve difficult tasks drop, but so did also his motivation to use the program at all. The effect was that Dominik asked (in session nine) to stop the experiment[4]. He expressed that wish even though he had not opened all the tasks in the program yet, did not collect all the stickers in the album and did not see all the movies – rewards in the program. The wish to stop the research earlier applied to all the remaining students from the experimental group. It shows that the students were bored by the program before uncovering all its secrets.

The final, most important aspect of assessing the efficiency of using an educational program by students is establishing what change occurred in the students as regards their mathematical knowledge and skills. In order to establish it the researcher compared the pre-test and post-test results in the experimental group and the control group. Dominik's results are shown in Table 3.

The data from Table 3 show that Dominik was able to solve tasks on the high – fourth level of difficulty (it comprised operations such as adding and subtracting 1 to 10)[5]. In spite of the ability to solve more difficult tasks he solved most of the tasks on easier levels – first three difficulty levels (205 out of 297 tasks solved).

[4] To remind: the students were allowed to use the program for ten sessions

5 The students practiced such tasks in the class during the research.

Difficulty levels of the tasks	1	2	3	4	5	6	…	17
Pre-test results				X				
Number of tasks solved	39	118	48	70	15	7		
Post-test results				X				

Table 3. Number of tasks solved by Dominik from the experimental group according to 17 difficulty levels of the tasks, including the results of the first and the second test of mathematical skills and knowledge. Own elaboration.

The results obtained from the second test of mathematical knowledge and skills show that the boy did not improve his ability of solving tasks. But the data gathered from the analysis of the film material show otherwise. On the computer screen Dominik solved 22 more difficult tasks (levels five and six). This shows that the student did not do them by himself. Solving these tasks Dominik used the trial and error method. Having this information one cannot speak strictly about independent task solving. But it needs to be remembered that this way of solving the task is treated as correct by the computer. A computer program, unlike the teacher, can be cheated on by the student as it is unable to recognise when the student establishes the result by a random answer and when they in fact do the operation. The authors of the „Klik uczy liczyć" program tried to find a way out by creating a program blockade. It was impossible to write an answer when the student gave a wrong answer three times one after another in short periods of times. It was unblocked automatically after a few seconds. It was not described in any part of the program or on the leaflet. Therefore, the students were surprised when they came across it and saw no solution on the screen in spite of having written all the digits. Dominik, who used the trial and error method, came across the blockade early. But he quickly avoided it by slowing down the pace of writing the answers.

The student described in this article experimented with the program, avoided difficult tasks and looked for attractive parts and reward with minimal intellectual efforts. The program was not able to use the attractiveness of the rewards to encourage the student to make more efforts. Due to many negative experiences connected with solving tasks Dominik gave up the effort of solving difficult tasks where there was the danger of failure. In this aspect the program should adjust the help options better in order to minimize the fear connected with failure. But in this case the only form of help (apart from counting the objects in the easier tasks) was the lifeline. But it was not discovered by the user as this option was not introduced in a clear way. Based on the behaviour analysis of the students taking part in the project it needs to be stressed that the sensitivity to feedback from the users is the deciding factor in shaping the mathematical knowledge and skills by an educational program.

As shown in Table 3, even though Dominik solved additionally 297 tasks in a multimedia program, he did not prove a change showing gaining new knowledge and mathematical skills. Such a change would indicate the educational efficiency of the program. Lack of change suggests that the multimedia program was not effective in case of Dominik. It needs to be stressed that among 12 students from the experimental group only three indicated the probability of influence of the multimedia program on the learning process (see Jelinek J., 2014).

Conclusions

As shown in the introduction, solving tasks is the basic way of learning mathematics. Task situations are difficult and require intellectual effort from the students. In order to avoid difficult situations the student may use different strategies. As shown by the research, a *weak* student, who needs a large number of experiences in task solving, prefers the strategies of avoiding intellectual efforts. Based on the research it was concluded that the more a child knows about mathematics, the more he/she uses the program according to the assumptions of the authors of the program. In other cases the child uses strategies not leading to gain new information or mathematical skills. These results are also confirmed by Kengfeng Ke (2008), who analysed the use of computer programs and games for learning mathematics. He concluded that the students are not able to use information gathered during the usage of these programs in tests outside of the computer screen (eg. paper and pencil exercises).

As shown by B. Skinner[6] – researcher of teaching machines of the 1930s

> between a program and a student there is a constant exchange (…), the machine insists that the student fully understands the task, or rather every unit of it – smaller or larger, and only then it allows them go to the next point of the program (…). The machine shows only what the student is already prepared for and it makes them only make the step with the best chances of success at that time. The machine helps the student find the correct answer and it does it to some extent with the ordered structure of the program (…). The machine rewards the student for every correct answer, just as a private teacher, and it uses instant feedback for that.

The communication between the program and a student is aimed at gaining skills by the students. However, unless the student is focused on receiving the messages from the program-teacher and willing to make efforts in solving tasks or is subject to external control of an adult, using a multimedia educational program will be inefficient.

It needs to be remembered that the communication of the teacher with the students in the classroom (at the board or at the desk) has a personal character.

[6] B. Skinner, The Technology of Teaching, New York, Appleton-Century-Crofts, p. 37-39

The effort made in front of the computer screen has a different dimension. The child does not have to anticipate unpleasant social consequences caused by using wrong strategies, even though they know that they are using an educational computer program. A *weak* student, whose experience is full of failures, treats the multimedia program as a source of fun. Therefore, the student requires external control because the educational program itself is too attractive for him/her and they are not able to focus on the educational content of the messages.

PRODUCING AUDIO-PODCASTS TO MATHEMATICS

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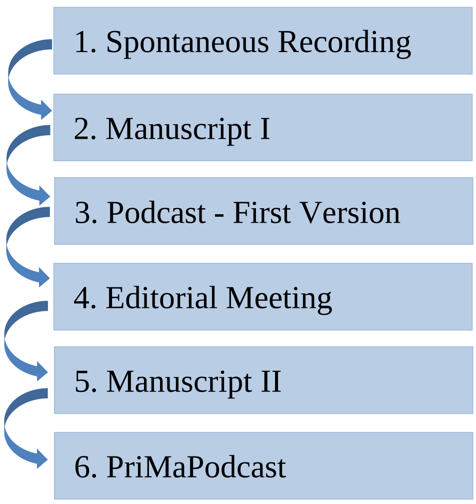
The use of digital media in the mathematics classroom affords different means of communication. With the recording of mathematical audio-podcasts, oral communication and representation are focussed. This paper deals with the use of mathematical audio-podcasts in primary education. First, the steps of production and the aim of mathematical audio-podcasts are presented. This leads to the depiction of our research interest in the fields of mathematical language. The theoretical explanations shall be followed by an empirical example.

AUDIO-PODCASTS TO MATHEMATICS

Podcasts have been a popular medium for several years now. Since many subjects are suitable for being dealt with, they can be well used for mathematical and educational purposes, too. Audio-Podcasts, which are produced by primary school children, are called 'PriMaPodcasts' (s. Schreiber 2013). According to the latest conception, the development of PriMaPodcasts consists of six steps in which oral and written communication and representation are interwoven permanently.



Steps of production

1. **Spontaneous Recording:** The pupils split up in groups of three and spontaneously answer a question of mathematical content, e.g. *'What is a triangle?'* or *'What is so special about a cube in comparison to other 3D shapes?'* Thus, their pre-knowledge is activated. Their response is recorded by a voice recorder.
2. **Manuscript I**: As second step, the pupils plan an audio-podcast, which is to communicate their mathematical topic to an audience. For this, they create a manuscript. The children are free to make it more or less detailed, to decide about structure and form. At this point, they can gather more information about their topic. Therefore, appropriate printed material or the Internet can be used.

3. **Podcast - First Version**: Based on their manuscript, the children record their first podcast version. Beforehand, they get time to practice their presentation.
4. **Editorial Meeting**: This first podcast version is presented to another group and the teacher. The children get feedback and reflect upon their recording in terms of certain aspects like content, style, language and performance.
5. **Manuscript II**: In the following, the children revise their first manuscript. Therefore a new manuscript may be created.
6. **PriMaPodcast**: On the basis of this second manuscript, the children record their final podcast version, which is called PriMaPodcast.

The different podcast versions and documents are published in a blog on the Internet. The advantage of the publication in a blog is the categorising of several podcasts in main-theme and sub-theme categories, which facilitates the handling for the audience. We produce PriMaPodcasts in different languages: in German, English, Spanish and Turkish. PriMaPodcasts in English are available in this blog: http://blog.studiumdigitale.uni-frankfurt.de/primapodcast-en.

Aims

Different aims concerning the recording of mathematical audio-podcasts are important:

Learning: Mathematics learning is supported by the requirement of not using any written or graphical representation, but only oral means of communication. By talking aloud and for others, learners express their mathematical ways of thinking. Thus, they can clarify und organize their thoughts themselves and reflect upon certain topics (cf. Pimm 1987). Apart from talking and explaining, oral communication is about listening, questioning, defining, justifying and defending, too. These activities may further pupil's mathematical understanding (cf. Ontario Ministry of Education 2006, 66).

Research: Since mathematics is rather a written and graphical-based discipline, it is very promising to see what happens when oral communication has priority. The research interest we want to pursue in this context is: What mathematical language is available for young learners in order to convey meaning?

CHARACTERISTICS OF MATHEMATICAL LANGUAGE

Technical language

Mathematical language is often identified by its specific terminology and vocabulary. Wittenberg as an example titled his doctoral thesis "from a thinking in terms" (Wittenberg 1957; translation R. Klose). Indeed, mathematics and its technical language (Maier & Schweiger 1999; Pimm 1987) take a prominent position, especially among other school subjects. Whereas everyday language is rather subjective and context-bounded, technical language consists of generalizations and definitions, which are intersubjectively comprehensible (cf. Heitzmann 2010). It is crucial that mathematical language goes beyond the learning of new terms and vocabulary.

Mathematics register

Halliday (1978) refers to a further set of meaning and therefore uses the notion of the ‘mathematical register’. The features of the ‘mathematics register’ used at school are here outlined according to Schleppegrell (2007, 141f). First, there are multiple semiotic systems such as mathematics symbols, oral and written language and visual representations. These systems combined are necessary for the learners in order to construct meaning. Moreover, there are the linguistic features of mathematics language, which include technical vocabulary as well as further grammatical patterns. ‘Technical vocabulary’ not only consists of mathematics words (e.g. *sum*), but also of words that are not solely mathematical. The latter refers to words of multiple meanings depending on the context (e.g. *product*). Since mathematics also uses many words that are already familiar to children in their everyday language, ‘technical vocabulary’ needs to be used in meaningful grammatical patterns of language in order to be correctly learnt. It typically comes along with ‘dense noun phrases’ *(e.g. the length of…)* as well as ‘being and having verbs’, which construct different kinds of relational processes (attributive and identifying ones). Whereas attributive clauses refer to objects and events and are non-reversible (e.g. *A cube is a solid shape*), identifying clauses define technical terms and are reversible (e.g. *A prime number is a number which can only be divided by one and itself).* Additionally, Schleppegrell lists ‘conjunctions with precise, technical meaning’ (e.g. *if, when, therefore*) and points on differences of their use in mathematical language and ordinary everyday language. Finally, she names ‘implicit logical relationships’ in mathematical reasoning, which are not further explained, but rather assumed to be already known.

Academic language

Apart from ‘technical language’, the notion of ‘academic language’ becomes increasingly important for the teaching and learning of mathematics (Gogolin & Lange 2010, Meyer & Prediger 2013). As interdisciplinary language it is also characterized by precise and elaborated vocabulary and complex sentence structure, but in a more functional manner: It is this ‘academic language’, which affords the participation in schooling activities (Cummins 1981, Schleppegrell 2012). There are multiple Discourse practices in a mathematics classroom since learners and teachers bring, while interacting, various perspectives to a situation. Academic mathematical Discourse practices aim for learners to become mathematically literate and therefore refer to practices as generalizing, abstracting or making claims (Moschkovich 2007). Different language activities such as discussing and hypothesising are in accordance with the acquirement of a mathematical language, which is “more broadly conceptualised” (Morgan 2005, 103). As presented above, the production of PriMaPodcasts affords different language activities, which is exemplified in the next chapter. Moreover, the focus is on the children’s use of mathematical language while interacting and discussing their mathematical issue.

EMPIRICAL EXAMPLE

The example, which is presented here, deals with the question of ‘What is symmetry?’. It was produced in a bilingual Maths class in Frankfort/Germany. Three boys of grade 4 (aged 9-10) first made a spontaneous recording, of which the transcript is depicted and commented on in the following. Second, the manuscript and transcript due to their first podcast version will be illustrated. Finally, we want to present the boys’ second manuscript and the transcript of their final version. All citations of the transcripts are marked in squared brackets <like this>.

Spontaneous recording:

sp	min.sec		
01 02	00.02	p1	Explain what is symmetry … ehm symmetry is w h e n . things are . mh l i k e . mh the same looks like the same on both sides
03 04	00.21	p2	Symmetry is ehm l i k e w h e n . it´s eh s o m e when it´s both the same *(he mumbles)* be that´s what . Kilian said it´s . the same
05	00.37	p3	*(he speaks low)* I don´t know what symmetry is\ *(he breathes out)*
06 07 08	00.40	p2	Yes a symmetry is would be like ehm . a h o u s e and the neighbour has also a house that looks . the same . just the same .. b u t ehm not everything can be symmetry ..
09 10	01.01	p1	When you draw a picture and . and you wa and you m have a .. mirror it looks symmetric
11 12 13	01.13	p2	Yes like e h m .. m h h how I can say that . symmetry is symmetry *(S3 laughs*) . like e h like this . like eh a blue order from the same market ehm . yes#
14	01.33	p3	#the same colour

Table 1: transcription of the spontaneous recording

Pupil 1 reads out the task and continues giving an answer. By using an identifying clause combined with the conjunction ‘when’, he tries to define the technical term ‘symmetry’. He talks about things that are the same and emphasizes this fact in the following <sp02>. Then he reformulates this statement, by saying that something is same on both sides. It is difficult to analyse which conception of symmetry he has got in mind because his explanations remain unspecific: technical terms are not used. Moreover, he seems to be uncertain about the topic, which is signalized by his way of speaking with some fillers (mh, l i k e) and pauses <sp01/02>. Pupil 2 alludes to two things which ‘both’ are the same. In doing so, he adopts the grammatical patterns of pupil 1 and also refers to his answering. In between he mumbles something, which lets his ideas remain unspecific, too <sp03/04>. Pupil 3 answers with a complete sentence and confirms that he does not know what symmetry is <sp05>. Then pupil 2 makes another attempt by giving an everyday example. The unknown ‘things’ are replaced by two houses, which are neighboured. He underlines again that both houses look the same. Therefore, he starts his statement with an attributive clause. Then he makes a restriction, claiming that ‘not everything can be symmetry’ <sp07/08>. As second language learner he uses the noun ‘symmetry’ instead of ‘symmetric’ in this context <sp08>. Pupil 1 seems to think of an approach to line symmetry: the mirroring. By using the conjunction ‘when’ instead of ‘if’, he tries to describe this procedure. However, he only names two parts of construction, ‘drawing a picture’ and ‘having a mirror’. By having both components, he suggests it would look symmetric. Even though one will understand what he means, this description is again incomplete <sp09/10>. Pupil 2 continues with a statement, which makes pupil 3 laugh. While thinking about the way of explaining, he simply states: ‘symmetry is symmetry’ <sp11>. This attributive clause defines symmetry by itself, which expresses his incapability to find a better explanation. Then he refers to a blue anything in particular from the same market <sp11/12>. This aspect might indicate a translation movement. Pupil 3 seems to know what his classmate means and puts emphasis on the same colour <sp14>. At this point the recording ends. The boys seem to reach an agreement in the end despite of their incomplete statements. Even though they use some grammatical patterns of the mathematics register, it lacks more technical vocabulary and precise explanations.

Manuscript 1:

As described above, the pupils do research on their topic and create a manuscript. This serves as basis for their first podcast version.

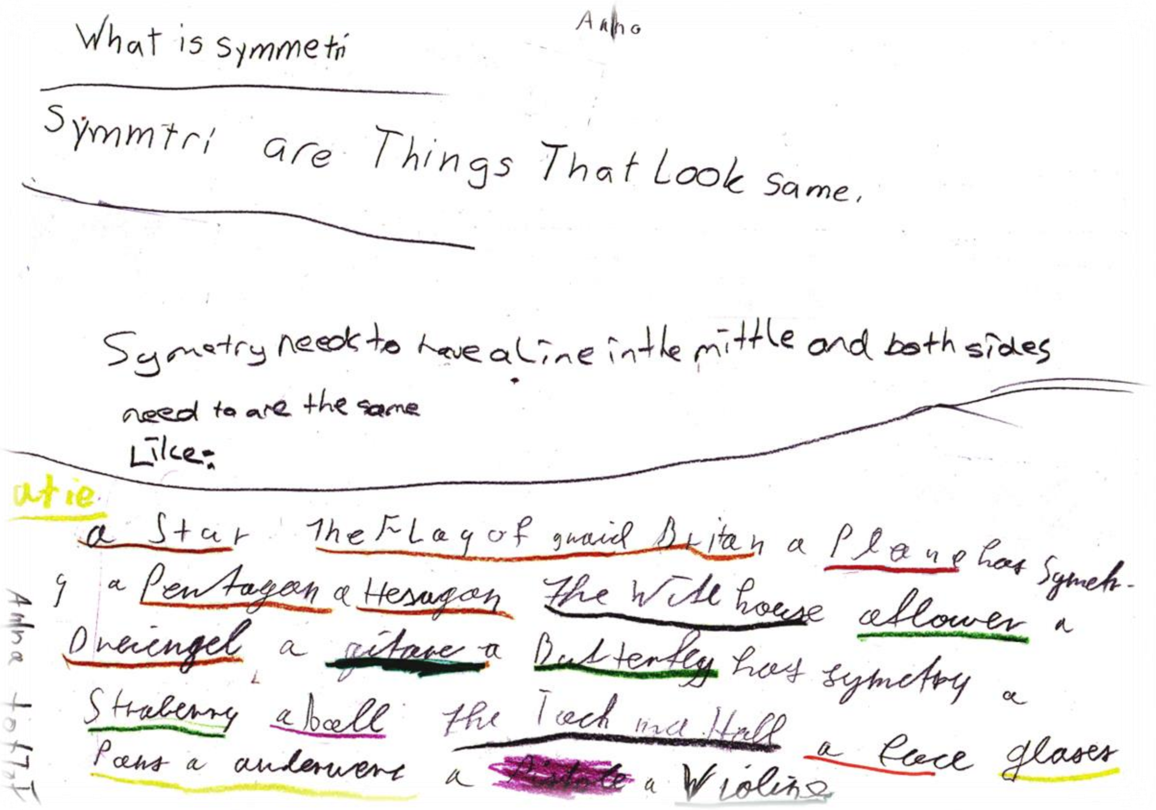


Figure 1: Manuscript 1

Podcast – First Version:

fv	min.sec		
01	00.01	p1	Symmetry are things l that look same
02 03	00.04	p3	Symmetry means you have a line in the middle and both sides need to are the same like /
04 05 06 07 08	00.11	p2	A star has a symmetry . the flag of Great Britain has a symmetry a plane has symmetry - a pentagon a hexagon the White House has symmetry a flower a triangle a guitar a butterfly . ehm . symmetry has also a strawberry a ball . ehm . then underwears has symmetry and p a n t s and the f a c e #
09	00.42	p3	#And Scarfs
10	00.43	p2	Yes . scarfs and glasses has symmetry .. *(he speaks low)* That's it.

Table 2: transcription of the first version of the Podcast

In this version pupil 1 starts off with a general statement towards symmetry by using an attributive clause. This statement is comparable to the first utterance of the spontaneous recording. The emphasis is again on the sameness of things <fv01>. Pupil 3 seizes the idea of having 'a line in the middle' which generates two equal sides. This idea, described by a common expression, seems to relate to the aspect of line symmetry <fv02/03>. In order to express meaning, an identifying clause is used. However, it is not explicitly conveyed which aspect of symmetry and which objects are meant. Only in the following statements, pupil 2 presents many examples concerning line symmetry <fv04-08> by using common words as well as technical terms, e.g. 'hexagon' and 'triangle' <fv05/06). Apparently, he forgets to mention one example, which is added by pupil 3 <fv09>. Pupil 2 repeats the suggested word and names another example <fv10>. In contrast to the spontaneous recording, the statements are now better conceived and more structured. Only the beginning and the end remain rather informally.

Manuscript 2:

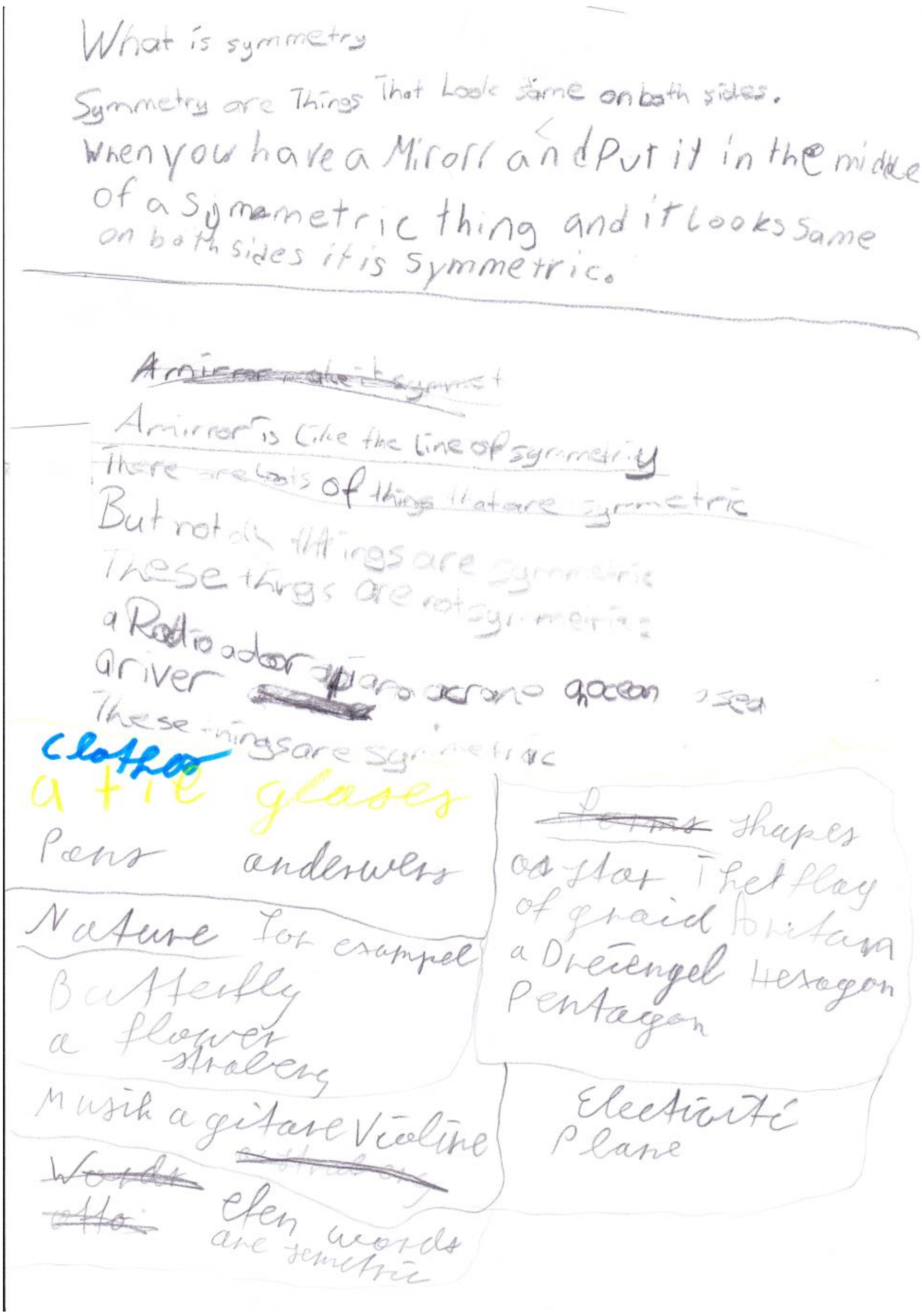


Figure 2: Manuscript 2

In the editorial meeting the peers especially praise the chosen examples. The boys get some hints in terms of content, style and language and thus, they can revise their manuscript. Their second manuscript can be regarded in figure 2.

PriMaPodcast:

pr	min.sec		
01	00.01	p3	What is symmetry
02 03 04	00.03	p1	Symmetry are things that look same on both sides . when you have a mirror and put it in the middle . of a symmetric thing and it looks same . on both sides it is symmetric ... A mirror is like a line o of symmetry
05 06 07	00.24	p3	These . there are lots of things . that are symmetric\ but not all things are symmetric\ these things are not symmetric\ a radio a door a piano a crane an ocean a sea a river\ these things are symmetric
08 09 10 11	00.45	p2	Clothes a tie glasses pants and underwears shapes a star the flag of Great Britain a triangle a hexagon a pentagon . nature for example butterflies flowers and strawberries\ . music a guitar/ a violin\. electricity a plane even words can be symmetric otto

Table 3: transcription of the PriMaPodcast

Pupil 3 reads out the question <pr01>. Pupil 1 gives a definition by using an identifying clause, similar to the beginnings of the other recordings: he equals symmetry with ‘things that look same on both sides’ <pr02>. Then he returns to the idea of mirroring which was originally uttered in the spontaneous recording. The instruction on how to mirror is more detailed than before. Moreover, the technical term ‘line of symmetry’ is used for the first time. In his explanations pupil 1 relates to general ‘symmetric things’ <pr02-04>. So in terms of language, the German sounded expression ‘it has symmetry’ is changed into ‘it is symmetric’ <pr04>. Before these things are further explained by giving lots of examples, pupil 3 emphasizes that ‘not all things are symmetric’ (pr05/06). Having taken up the idea of the spontaneous recording, now he names seven things, which are not symmetric. He leads over to pupil 2 who reads out the examples of symmetry <pr08-11>. Some of them, which are presented in the first podcast version, are not mentioned here any more (e.g. the ‘White House’ or ‘a scarf’). Another difference is that the examples are classified into categories like ‘clothes’, ‘shapes’ or ‘music’. Thus, they are more structured. A new example of symmetry is highlighted in the end: words like ‘OTTO’ (pr11) can be symmetric, too.

CONCLUSIONS

As presented in the empirical example, the production of PriMaPodcasts opens up various possibilities of oral communication in mathematics learning. Language activities and mathematical Discourse practices such as explaining, defining, generalizing and justifying enable learners to express and discuss their mathematical thinking. The reflection upon certain content deepens the pupils' mathematical understanding. Through the different steps of production, it becomes possible to specify the use of mathematical language, too. This interactive procedure is in accordance with the requirement of a mathematical conceptualization, which goes beyond the strict learning of technical terms and certain vocabulary. Thus, learners are supported to become mathematically literate.

In the context of a research project, we investigate the mathematical language skills of bilingually taught pupils in Germany (Klose 2013, Klose & Schreiber 2013). Therefore the children produce PriMaPodcasts in English and in German language, by which a comparison can be drawn. Although bilingual forms of teaching and learning are meanwhile common practice in Germany, they are more realized in secondary school forms than in primary education (Elsner & Keßler 2013). The usage of a second language in German mathematics lessons remains rather unconsidered in all fields of education (Rolka 2004, Viebrock 2013). However, already young learners are capable of communicating mathematical content in a second language, as our empirical example demonstrates.

IS IT A CUBE NET?-3RD GRADE STUDENTS TALKING ABOUT GEOMETRY

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Geometry plays an important role in the process of developing and creating students' mathematical thinking. But in school curriculum there is not enough space for geometric issues, especially in lower primary school. The main problem is: how to introduce young students into the world of geometry? In this paper some episodes from math lesson in 3rd grade of primary school are described, which focused on three-dimensional geometry. The aim is to demonstrate how students deal with cubes and their nets and which language they use when they talk about spatial geometry

INTRODUCTION AND THEORETICAL BACKGROUND

In the latest trends in the early education a large emphasis is placed on developing skills needed for a child to explore and understand the world, to cope with different situations of everyday life. The skills that are particularly useful in various situations include analysing, critical thinking, putting and verifying hypotheses. The tasks of the school according to the new curriculum include the opinion that a child could acquire the knowledge and skills needed for understanding the world and equip the child needs in math skills in real-life and school situations and for solving the problems.

The most important skills acquired by the student in the course of general education in elementary school should be, inter alia, mathematical thinking, comprehension as the ability to use the basic tools of mathematics in his or her daily live and carrying out the elementary mathematical understandings.

Several studies highlight the need to develop mathematical thinking and skills such as creativity and criticality of thinking.

On the higher stages of education mathematics is presented as a structured and ordered formal knowledge, with specified chapters of mathematics. This gives you the ability to work on both advanced learning and the development of mathematical knowledge, as well as o the development of a formal (symbolic) mathematical language. At lower levels of education (especially in the lower grades of primary school) students do not submit a finished, formal knowledge from various fields of mathematics, but it introduces them to the world of arithmetic and geometry (Hejny, Jirotkova, 2006) on the basis of their own activity - often taking place in the physical world. However, while the arithmetic world is ultimately structured, governed by clear rules and individual markings and symbols used in the world are read equally by all, it is slightly different in the geometry. As Hejny and Jirotkova wrote (2006):

> The world of geometry is a community of individuals or small families and there is a large diversity in the linkages between them. From the didactic point of view, arithmetic is suitable for developing abilities systematically, and geometry is more suitable for abilities such as experimenting, discovering, concept creation, hypothesizing and creating mini-structures. (p.394)

By analyzing the historical development of geometry we can notice that it was accompanied by a human in his activities since the dawn of time (much like arithmetic). Initially, the geometry was not a theoretical science, but it appeared from the need and desire of man to land the space around them, solve many practical problems - from construction by travelling to the ornamentation (Hejny, 1990). However, this geometry was the first "scientific field within mathematics" which was created by human. Its significance for the study of the ancient world was great. It had an important role in mathematics. This historical trait also points the way for didactics approaches to teaching school geometry: geometrical knowledge arises by the action. It is important here to gain experience, and practical problem solving.

The importance of geometry in the education of children and young people was the theme of many researchers' considerations. There is a belief that geometry can support the overall development of the child's competence in mathematics. About the importance of geometry in teaching children and adolescents writes E. Swoboda:

> „… quasi-geometrical activities can develop widely understood children's mathematical competence. On the one hand, since geometrical approach to mathematics is closer to children than arithmetical one, geometry can open doors to a world of mathematics. Geometrical cognition starts from a reflection upon the perceived phenomena and in this way correlates with the basic ways of learning among children. On the other hand, it gives a chance to develop such ways of thinking that are typical for mathematical thinking. Skills like generalization, abstraction, perceiving relations, understanding rules are the base for this aim. Early geometry is in-between physical and abstract worlds. By this, it enables to mathematize this world." (Swoboda, 2009, p.29)

Although the geometry has a great potential to develop mathematical thinking of students in the school teaching it is not treated with due care. It consists of a number of factors. As write Karwowska-Paszkiewicz, Łyko, Mamczur and Swoboda (2001):

> Teaching geometry is conducted in a very limited number of lessons and so children do not realize why they are learning it. It is just part of mathematics without any references to reality. Children were taught abstract geometry items and they did not

> have the opportunity to learn the properties of the figures through manipulating and even if they had it, it was only apparent. Such a style of teaching geometry does not give any chance for problem solving teaching. It also does not help a child to make links between the problem, the procedure of solving it and the solution. (p. 86)

To take advantage of the full capabilities of the geometry in the education of children and youth, you need to change the approach to its teaching. Geometry was born out of the action and of human needs for development and structuring of space around them. Therefore, an important element in the teaching of geometry should be acting. As we can see in Swoboda (2001):

> Action play an important role in the formation of geometrical concept because there is always correlation between concept and the activity addressed to the concept. The object from the real world is perceived as the gestalt. The way of gathering information is perception, but after that the action with the object leads to the verbal description in their properties. (p.151)

So an important didactical issue is organized activity in mathematics lessons in such a way as to encourage students to actually participate in the lesson and to give them a chance to creative thinking and discovering mathematics.

Language and communication

The experience plays an important role in the learning process. Specifically writes about this M. Hejny in his theory of the General Model (Hejny 2001). Children creating their own knowledge primarily based on the experience they have. For a description of these experiments they use language - as it is close to them. The closest experience for a child is the language of gestures. It is the first language, how a child learns. This language is very helpful during learning of mathematics. How Cook and Goldin-Meadow (2006) say: “children who produce gestures modeled by the teacher during a lesson are more likely to profit from the lesson than children who do not produce the gestures”. Only at a later stage of learning a child meets the formal language of mathematics. It is important that the experience and action in the acquisition of mathematical knowledge appears first, followed by the language, and only at the end there is a formal knowledge. As B. Burton (2009) says:

> The evidence from language points to the conclusion that mathematics arise after, not before, human activity. The development of mathematical language is consistent with the idea that mathematical concepts, objects and relationship arise through language, and within particular socio-cultural environments, in response to human thinking about quantity, relationship and space.(p.88)

Many researches highlight the importance of gestures in learning of children. We can find references in the literature (Stevanoni, Salmon, 2005; Broaders, Cook, Mitchell, Goldin-Meadow, 2007)

Language has its external form: this is both a visual (writing and gesture) or sound (spoken word). Meanwhile, the thinking is invisible and inaudible.

Therefore, it is important how we pass our thoughts. The transition from thinking (internal speech) to external speech (Wygotski) causes some dissonance:

- between what implicit and what is clearly expressed
- between what implicit and what is clearly expressed.

To overcome these problems, the various form of expression are used during communication. The interaction between students and teacher taking place in the classroom are enriched with different forms of communication: verbal and nonverbal.

> In social situation encompassing mathematical learning, a variety of linguistic forms will be used within a broad communicative environment. Aspect of language used will be specifically associated with conventional mathematical ideas, but much will be less precise, supporting other facets of the exchange. In most learning situations we are concerned with activity taking place over periods of time comprising personal reflection making sense of engagement in this activity. The learning process entails more than a local concern of getting to grips with clearly defined "mathematical" concepts. Indeed there are many forms of mathematical discourses each flavored by their particular social usage. (Brown 1997, p.27)

Several contemporary approaches emphasize, for various theoretical reasons, the embodied dimension of thinking (see, e.g. Arzarello, 2006; Lakoff, Núñez, 2000; Nemirovsky, Ferrara, 2008) and the role of artifacts (Bartolini Bussi, Mariotti, 2008).

Gestures can be important components of semiotic means of objectifications, whether used when communicating directly with others, or to highlight aspects of artefacts and symbolic representations of mathematical concepts.

METODOLOGY

Data for this paper were collected during classes with the third grade students from primary school. It was a series of meetings, which main purpose was to develop the students' interest and talents of mathematics. Classes took place once a week and last one school hour (45 minutes). Twenty students from the third grade of primary school (9-10 years old) took part in these meetings. The theme of these meetings was fun with spatial geometry. By playing the students learned about spatial figures and they built their nets. The aim of these meetings was the development of spatial imagination and, in particular, to bring students to the concept of a cube and its net. Also worked on the development of students' mathematical language, and, in especially, to bring students the concepts such as cube, cube net, walls (opposite, adjacent).

During each classes, students have access to a cubic wooden blocks with a side length of 3 cm, cardboard in the shape of a square of side 3 cm, scissors, duct tape, crayons, markers. Classes were recorded with a video camera. After each meetings the protocol was drawn up.

Classes were divided into thematic blocks. Initially, students were gluing squares to cubic blocks forming colored houses. After that they cut the made models and analyzed the relative position of individual elements. In these classes much emphasis was put on analyzing the relationship and interactions between elements. Heavily stressed was the correct justification for own opinion, which led to work on the development of mathematical language.

Another series of activities related to the implementation of the model prepared according the description.

At the beginning of the classes the teacher together with the students read the tasks, and then they began to analyze its contents. They considered the number of walls, windows, doors. They turned their attention to the elements located inside the House of Winnie the Pooh and their location. Then students prepared components of house and glued its model. After completing the task the students explored each other, whether received houses are in accordance with the description. During these activities the emphasis was put on the ability to read with understanding, follow instructions, ability to analyze the data. The description of these classes can be found in Pytlak (2014a, b, 2011).

In a further step, students worked without cubic models, but only with their flat nets. They had only cube nets with missing one wall. And on the one square there was drawn a door. The task was to draw other elements of houses: windows, floor, celling. Students also had to answer the question, which the element (i.e. which of the walls) is missing.

When this task was completed, students received a "clean" cube net. This time they created the patterns on the net. There was only one rule: squares should be coloured in such a way that after making a cube the opposite walls will have the same design. Students had to work on a net which was lying flat on the desk. Only after the painted all net they could to cut it and see if they were able to do the task correctly.

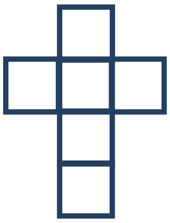
The last meeting of the whole cycle related to the cube nets. During this class students tried to find all possible (and different) cube nets. They had squares and used them to arrange the cube net. Students worked together, with whole class. Each of them could proclaim its proposal and presented it on the blackboard. The whole class together resolved whether the presented schema could be the new "ribbed band clothes for the Mr Cube”. Students during these classes not only discovered different cube net, but also learned how to argue, making hypotheses and verifying them.

The convention used during these activities was taken from Hejny, Jirotkova (2001). The grid has been presented to the students as "clothes for Mr. Cube".

DESCRIPTION OF STYDENTS' WORK

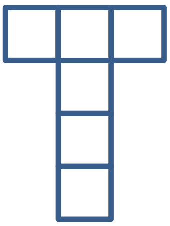
At the beginning of the lesson the teacher clipped to the blackboard classic (typical) cube net. The task of the students was to select a colored marker, which edges meet after folding the cube. Students in turn approached the blackboard and marked the appropriate edges. Everyone had to justify their choice. During this task students assisted themselves with gestures, first showing respective edges and next painted them. Here it was very important to refer to the previous classes, when students created from preparing squares "the dress for Mr Cube". The net was identifying with the ribbed band clothes, and overlap the edges was a place of stitching forms. This was a preparation for the next task, which they received. This time students have to prepare another form of dress for Mr. Cube.

The students have prepared six equal squares. They pin them to the blackboard by magnets, thus creating a cube net. As the first drew up the following net:



Picture 1

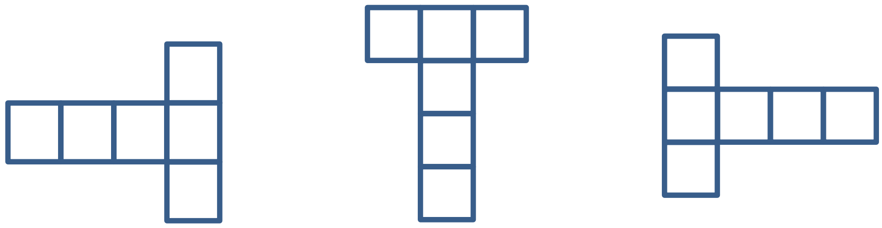
But, as we all have found that such an arrangement has already been, only the "inverted". Finally, one of the students changed the schema on the following:



Picture 2

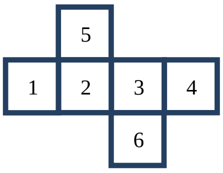
Then students justified whether it will be possible to stitch „dress for Mr Cube" from this kind of dress form. Students would come up to the blackboard and pointed out which edges should be stitch together to receive a cube.

Students rightly noted that it is important the position of elements relative to each other, and not the same arrangement of net. They eliminated alone such nets, which could be received by rotation of the initial net around their axis, regarding them as the nets of the same kind.



Picture 3. Nets treated by students as the same

Each time when the students proposed a new arrangement, the teacher asked the whole class, whether it is correct. This was also after positioning the following net:



Picture 4

Teacher: Is this arrangement correct?

Students: Yes!

Weronika: [very definitely] No, it may not be!

Teacher: And why? Weronika, come to blackboard and explain.

Weronika: Because when we take these walls, so here will be missing.

Students: No!!!

Teacher: Who are explaining to Veronica? Please, Maciek.

Maciek: When we fold these two [he is pointing on the 3 and 4 squares], then this one will be here [he is pointing on the 3 and 5 squares], and this one will be here [he is pointing on the 4 and 5 squares]. And then those two will be here [he is curling up the 5 and 6 squares]

Teacher: And now everything is ok?

Students: Yes!!!

Picture 5. Maciek explains to Veronica, how to make a cube

While justifying that the arrangement is or not the net of cube, students appealed to the language of gestures. By analyzing only verbal statements, we are not able to say what they are talking about. Their verbal language, which they had used, was quite poor. It lacked a formal mathematical language. Although terms such as: walls, adjacent walls opposite walls were used during the earlier classes, for students they were still the new ones. Much more convenient it was to them to use the language of gestures. And this language was understood by all students. It does not require any additional comment and additional explanations. The usual "this and that" supported by appropriate gesture indicating that the elements in question is enough for students to determine whether this is the correct reasoning. In the explanations presented by the students gestures played a key role. Equally significant was the movement and action. Describing how to flat net fold the spatial figure, students not only pointed out which the edges will connect with each other, but also pointed out how this connection should be made. To this purpose they made a movement of bringing hands together (that was supposed to symbolize the joining of walls) and applying one to the other (what was supposed to symbolize the overlapping-double faces).

As a result of common work students created sixth different nets of the cube.

Picture 6. The nets of the cube created by students during the 7th classes

Summary

For 8-9 years old students the classes were the new form. For the first time they have seen the net of the cube. They created their own nets basing only on their own intuition. It turned out that they are able to cope with the task demanding the usage of three-dimensional imagination in very good way. They could, in fast way, notice the correlation between the sides and the edges in the cube and use the perceived earlier correlations in their work. They did not approach the full net of a cube, however, there appeared two ideas how it should look like. The first one concerned the structure of the connected elements in one raw, what can be identified as the ring road of the cube. The second approach was connected with the awareness of cutting the cube along the perpendicular edges till the basis and putting it flat.

During the classes the students turned out to be open to new challenges. They approached creatively to the presented to the problem. Care about the details of the created houses can testify about huge interest in the presented subject.

At the beginning of classes students did not refer to the reciprocal arrangement of the walls and while describing the cube they used the colloquial language (at the top, at the bottom, on the side, corner of the walls). During the last classes in the description of the reciprocal arrangement they used the terms in more precise way (opposite, common edge, vertex). It seems that the series of classes can be good introduction to the work concerning the development the three-dimensional imagination and connected with it students' mathematical language.

For students taking part in this classes it was the first meeting with the cube and its net (mathematically). They did not know the respective mathematical concepts to help in describing a cube, its properties, and the cube net. They very active participated in the classes. During the argumentation and justifying their solution they appealed to the language of gesture. Using gestures, they talked about cube and its net. Gestures helped them to justify whether the net is correct or not.

M. Hejný (2004) in his theory of the development of student’s mathematical knowledge writes that it is very important in the learning process to gain experience, which is the basis for the creation of formal mathematical knowledge. For third grade students presented a series of activities has been very good opportunity to gain just such experiences.

FIRST GRADERS' GEOMETRICAL SPEECH
THE SLANTING GIRAFFE'S NECK

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This paper analyses pupils' locutions for detecting in their speech the implicit recognition of geometrical relations between segments ('parallelism' and 'perpendicularity'). By means of a social-constructive approach, we leave the pupils free to decide how to describe artefacts through their spontaneous language.

INTRODUCTION

Our aim is to detect whether the younger pupils' spontaneous language and class communication may reveal their possible understanding of 'parallelism' and 'perpendicularity'. These relations could be considered a sort of embodied knowledge of everyone; but our experiment shows that first graders unaware of these geometrical aspects are present. The choice of the topic is justified through the following main arguments.

- We did not find in the literature inquiries focused on the early pre-concepts of these geometrical relations.
- In 1st grade the geometrical context is suitable since it relies on visual mediators, objects and drawings, which help to think. In this grade the concepts of 'parallel or perpendicular straight lines' are approachable, at least from visual and graphical points of view. This assumption is justified through the history of humankind since, as Keller (2004, p. 137) shows, in Palaeolithic times we may find drawings in which such mutual relations between straight lines are present (Lascaux caves).
- In 1st grade the training with parallel and perpendicular lines could be helped through the learning of the capital letters writing.
- The geometrical standard approach in Italian school is mainly grounded on classification of polygons and their names. This way of teaching may be ascribed to the philosophy in fashion at the time of Euclid, with its pre-eminence to unary predicates.
- The relations 'parallelism' and 'perpendicularity' are at more abstract stage than polygons, since they are expressible using binary predicates which translate the mutual positions of two objects.

- The role of these two relations is fundamental in order to identify the main figures through their principal features.
- These relations are uncommon in 1st grade of Italian Primary School, therefore these topics allow to verify the role of spontaneous language without the influence of school language.

THEORETICAL FRAMEWORK

With the aim of detecting pupils' thinking about their geometrical understanding, we investigate their own language. Our starting point is the following quotation (emphasis is in the original):

> […] *it is necessary to postpone the systematic use* […] *of specific words*, i.e. the typical concepts of geometry (circle, square, sphere …) […]. Children must also *learn to speak about spatial events* with their own words (i.e. to describe a path, a figure, a movement) (Speranza, Vighi & Mazzoni Delfrate, 1988, p. 14)

The previous quotation is according to model of Gawned (1990), that shows four stages of the construction of a mathematical specific language: "real world language", "the language of the classroom", "the specific domain of the language of mathematics", "construction of meaning in mathematics".

Our investigation of children's real world language makes use of some artefacts and the proposal of activities and discussions about them. Following Nonaka & Tageuchi (1995, quoted in Lester & William, 2002, pp. 494-495) the dialogue is fundamental for bringing a tacit knowledge to an explicit one. In this way a non-standard description emerging from the class discussion would be the background as a first step to arrive to the standard mathematical definitions. The speech helps the building of concepts; subsequently the used word is a symbol for the concept itself (Vygotsky, 1987).

The point of view of Sfard (2001) integrates the previous considerations:

> The conceptualization of thinking as communication is an almost inescapable implication of the thesis on the inherently priority of social origins of all human activities. Anyone who believes, as Vygotsky did, in the developmental priority of communicational public speech (e.g. Vygotsky, 1987) must also admit that whether phylogeny or ontogenesis is considered, thinking arises as a modified private version of interpersonal communication (Sfard, 2001, p. 26).

Our experiment considers also the theory of semiotic representation registers (Duval, 1993), which shortly may be described with the following quotation:

> We may distinguish three main group of semiotic representation: material representation (in paper, cardboard, wood, plaster, etc.), a drawing (made either with pencils on a sheet of paper, or on a computer screen, with use of a geometrical software, etc.), and a discursive representation (a description with words using a mixture of natural and formal languages). Each register bears its own internal functioning, with rules more or less explicit. Moreover, students have to move from one register to

> another, sometimes implicitly, sometimes back and forth (Dorier, Gutiérrez & Strässer, 2004).

Our artefacts promote the transition from visual to verbal semiotic registers. Moreover, Duval (2005) deals with the different ways of look at figures: perception is the first, it is routed on the "merely qualitative properties" (Poincaré, 1963, pp.134-135) in which the metric is not used. Furthermore "[…] it is the task that determines the relation with figures. The way of seeing a figure depends on the activity in which it is involved." (Duval, 2006).

In particular, visual perception may hinder the ways of seeing figures. Gestalt theory deals with laws of visual data organisation that leads us to see certain figures rather than others in a picture. With reference to the reading of images, Mazza (2001, p. 58) writes about behaviour of children 4-8 years old:

> Even in presence of abstract art works pupils show a "referential need", i.e. to identify a likeness, to find out which object "hides itself" hidden through the apparent oddity.

Following these hints, we chose to take advantage of the "need to find similarities" to lead pupils speak in their "real world language" about mutual position of segments.

Moreover the literature offers us another way of analysing the child's approach to geometrical understanding e.g. Pyshkalo, 1965 (quoted in Varga, 1976, p. 31) writes about five different levels of appropriation of geometrical concepts. His ideas may be compared and integrated with some other studies about visual perception made from pedagogics and psychologists. Visual level is also assumed as a first step in the Van Hiele's theory.

The ways of considering the space intervene also in the geometrical understanding through an appropriation more and more elaborated of the concept of space. Piaget & Garcia (1983) give us the possibility to distinguish different kinds of space: in particular, they study the psychogenesis of "intra-inter-trans-figural stages of learning", referring to different ages of pupils. They assume "intra-figural stage" as the first stage of geometrical learning, typical of 8-9 years old pupils. The next one is the "inter-figural stage" (11-12 years old pupils). In both these stages the authors assume that no understanding of the invariant properties for isometry is shown. We may adopt, for short, the same qualifying adjectives that Piaget & Garcia adopt for the stages of learning as attribute of the pupils' intuition of space.

Our experience (Marchini & Vighi, 2011) in other research advised us that preconceptions of isometries and their geometrical role may be present earlier than Piaget & Garcia (1983) affirms. Sbaragli (1999) confirms our statement showing that the three stages are interwoven and not sequential.

The research questions

First question: may we extend ideas of Speranza *et al*. to the descriptions of binary relations of parallelism and of perpendicularity?

Second question: are parallelism and perpendicularity perceived from a qualitative point [7] of view through very young pupils or not?

Third question: which concept of space is revealed through children locutions?

METHODOLOGY

The experiment involved three 1st grade classes, in which we presented our artefacts made through poor stuff as grey cardboards, straws, buttons, glue, adhesive tape. We prepared 8 squares cardboard with 23 cm side and 8 round cardboards with 30 cm diameter. In each of them we glued two straws, one red and one green, in different positions, as representation of caterpillars. We used pleasant colours for caterpillars and a 'neutral' colour, light grey, for the background. On the squared cardboards, we glued a small button as head of the caterpillars, at one end of each straw, for enhancing the 'naturalistic' environment [8]. We are aware that in this way we may induce the idea of 'oriented segment' but we used also straws without 'heads' on the round cardboards. We chose straws as representation of animals instead of drawing segments; as matter of fact, the presence of animals, introduced through a narrative, may draw the pupils' attention to the 'particular' of the geometric relations existing between the two straws. The tale context could direct pupils' language towards an imaginative language. In this way we try to avoid that first graders may consider each cardboard and the geometrical configuration on it as a whole (Pyshkalo, 1965). We prepared the cardboards for the study of geometrical relations between two 'straight lines'; nevertheless it is possible that child focuses her/his attention on the whole cardboard space, instead of the space 'between' the two straws.

From theoretical point of view, with these tools and narrative we aim to detect possible anticipations of the pupils' conception of interfigural space, with the overcoming of the intrafigural space.

We are aware that straws could be unsuitable for representing segments and straight lines, since straw are 3D objects with a thick. In a similar research Vighi (2008) shows that some pupils reproduce caterpillars as simple segments, overcoming the problem of thickness. In this way drawing helps the

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[7] There are at least two ways for introducing 'parallelism' and 'perpendicularity'. From Posidonius (c.a. 1st century BC) on, customarily parallelism is identified through speaking of a constant distance between two straight lines. Perpendicularity is associated to 90° degrees angles, both by the use of measures and therefore through quantitative aspects. In our opinion only a qualitative approach is suitable in 1st grade.

[8] The choice of this kind of animal had shown itself suitable in (Vighi, 2008). The two shapes of cardboards are functional to assess the role of didactic variable 'form of the background'. For the didactic variable 'oriented segments' in the round cardboard we skipped the round buttons.

'transformation' of a straw in a segment or in a straight line in pupils' understanding. With the chosen tools it is impossible to show a complete superposition of straws and it is difficult to present crossing straws, but this technical problem has a positive side since it allows sticking to Euclid's definition:

> I.10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands. (Euclid, 2007, p. 6)

Many Italian textbooks present perpendicularity using the right angles which are created through particular couple of segments. In our cardboards, right angles are present, but we never speak about them or we ask to observe them. Moreover, as regard as 'parallelism' Euclid writes:

> I.23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions). (Euclid, 2007, p. 7)

The previous definitions I.10 and I.23 emphasize the qualitative aspects.

Both 'parallelism' and 'perpendicularity' need at least two objects and their suitable reciprocal positions (Table 1); cardboards 2, 3, and 6 are about parallelism (or 'superposition'), cardboards 5, 7, and 8 are about perpendicularity (with segments in different orientations as regards as the cardboard sides), cardboards 1 and 4 present straws in neither parallel nor perpendicular situations. Our cardboards represent in a concrete way some possible mutual positions of segments. These choices are justified in order to promote the thinking through analogies and differences. We recorded the sessions through a video-camera.

THE EXPERIMENT

Our experiment involves class 1A (23 pupils), and 1B (25 pupils) of the Vicofertile school in which we used squared cardboards (Table 1). In the third class, i.e. 1A of Fognano school (23 pupils) we presented the straws without buttons in the same position on round cardboards [9]. We present here the first steps (work in progress) of this mid-term study, performed in the school years 2013/14 and which will continue in school year 2014/15.

We propose to observe straws representing caterpillars placed on cardboards in appropriate mutual positions, and to give descriptions of showed configurations

9 We acknowledge the teachers G. Barantani (Vicofertile) and B. Riccardi (Fognano) for kindly giving us the permission of experimenting in their classes, and for their collaboration and presence during the experiment sessions. We gave their, in advance, an outline of the aims and of the methodology of our research, together with the script of the class intervention.

through these visual mediators. This requires the mastery of the visual (image reading) and verbal (wording) semiotic registers.

In particular the researcher named green caterpillar as Pelù [*P.*]; children named the red caterpillar Mangiamela [*M.*]. In the first two interventions pupils start with the observation of our artefacts, and their verbal description (verbal semiotic register). Pupils allocated many and different names for the same geometrical relation. During the second slot learners' suggestions are recalled and the names of the relation represented on cardboards are put in the vote in order to build the "language of classroom" of Gawned (1990).

Sometimes in order to favour the verbalization we proposed two different cards for comparison. We tested also whether the orientation of the cardboard influences pupils verbal description. Children's descriptions of the cards are recorded through videotape and analysed.

RESULTS AND THEIR DISCUSSION

This experimentation gave us many occasions of reflection about the pupils' pre-conceptions and the standard classroom activity. We present in Table 1 a possible categorization of the children's locutions, in order to detect in them some geometrical aspects. In two columns of Table 1 we quote pupils' locutions which are pertinent with intra- or interfigural space conceptions, borrowing from the nomenclature of Piaget & Garcia (1983). I.e. the use of adjectives as 'horizontal', 'vertical' and 'slanting' makes reference to the whole cardboard and its sides, therefore we consider these locutions as the presence of an intrafigural interpretation of the space. On the contrary we assume that the interfigural space is revealed through a child's statement regarding distance or the evocation of the space between the straws and also, generally, if the subjects of statements are both the caterpillars [10]. We find unforeseen presences of the transfigural space, which are revealed through the invariance of a configuration under symmetry or rotation. We report the locutions concerning this conception in a specific column.

The analysis of the children's 'definitions' in Table 1 may provide clarity to our purposes. The underlined locutions have been chosen through class open vote. Remark that the majority of polls choices reveal an intrafigural stage. A consequence of the intrafigural space consideration is that the cardboards inner space is anisotropic, i.e. there are two privileged directions, 'horizontal' and 'vertical'[11]. On the contrary, in the interfigural space consideration these privileges disappear (isotropic space).

[10] Our interpretation of the intrafigural – interfigural dichotomy of Piaget & Garcia (1983) may be disputable; we are aware that other researchers propose different readings for the same concepts, e.g. Charambolos (1991), Aglì, D'Amore, Martini & Sandri (1997), Sbaragli (1999). We made a choice and using it, we produce our categorization.

11Many locutions reveal that the medium on which straws are glued is not 'neutral' as states Sbaragli (1999).

Cardboard		Intrafigural space	Interfigural space	Transfigural space	Independent space
1		Being horizontal; Crocodile's mouth; Triangle (?); Mountain; Pink is horizontal	Here closer, there far off; Their faces are far; They collide with each other; They meet; Triangle (?)	If you rotate it, it looks like an 'A'	*M*. goes in another place; They collide with each other
2		A pencil with rubber; Are horizontal; Double-barrelled pistol; In a line; Letter 'I'; They are going to the same side; They are walking; Thread	Falling down the tree; Looking like as one caterpillar; *M*. is going towards *P*.; *M*. labours more since it stays behind; Sticky treads	A rotated 'I'	Dive; Going down; Going underground; *M*. is going to another friend
3		A trunk of a tree; As stripes of bees; Crocodile's close mouth; Horizontal stairs; Mast; Striped; They are horizontal straight and striped; They are horizontal; Two lollipops	A railway track; Crocodile's close mouth; Railway; They do not meet since they are going straight; Train	A fallen trunk; Rung ladder	They are going straight; They do not meet since they are going straight
4		Beak; Elephant's proboscis; Mouse's muzzle; Mouse's whiskers; One straight and the other slanting	Joining together they are a boat; The faces are closer; They are more detached	Joining together they are a boat	
5		As a 'T'; *M*. helps *P*. for bringing an apple on the tree; *P*. climbs up *M*.		Turning on the other part it looks like a 'T'	Each one goes home; One goes down, another one goes left
6		Slanting giraffe's neck; Slanting railway track; Slanting vertical line; Curve; Escalator in a supermarket; Slope; Stairs	Competing in a race; Racing; Slanting railway track	Crocked vertical; Rows; Slanting railway track	Competing in a race
7		Slanting; Letter 'Y'; *M*. is falling down; *P*. is standing up	A kind of 'L'; Turning in that way it looks like a 'L'	Slanting; P. is lying down and M. is on it; Turning in that way it looks like a 'L'	
8		Letter 'L'; Two attached lines; Car seat	Half rhombus; Two colliding cars; M. sniffs the feet of P.	In this way it looks like a hut a few turned	

Table 1: Analysis of pupils' locutions describing cardboards 1 - 8

We observed that in presence of a rotation the pupils locutions change: e.g. description of 'parallels straws' as "railway tracks" (card 3) and as "slanting giraffe's neck" (card 6).

Therefore, we conjecture that in general the invariance through rotations is not grasped through children and this testifies the lack of appropriation of the transfigural stage. Nevertheless (Bulf, Marchini & Vighi, 2013 and 2014) prove that, with suitable artefacts and tasks, pupils of the same age reveal the presence of pre-concepts regarding the isometries.

In the last table's column we quote also locutions denoting the preconception independent space which is revealed through the statements expressing the possible continuation of 'trajectories' out of the cardboard, therefore it is a way of thinking which requires a form of abstraction from the straw to the straight line. The dichotomy independent space – non-independent space is studied through Speranza (1994). In our experience, these two interpretations of space may be identified in younger pupils (Marchini, 2004).

CONCLUSION

The experience confirms that younger pupils use their out of school knowledge for describing geometrical configurations with their own language, "the real world language" of Gawned (1990). In the physical world, horizontal and vertical positions are dominant; therefore they may condition the language. E.g. for cardboard 6, a child suggests "slanting vertical line"; s/he introduces a new locution to make up for the lack of a suitable word.

Some pupils interpret the straws configuration as representations of 2D or 3D objects (e.g. trunk or neck) and use metaphors:

> Since, however, language is a medium through which metaphor comes into being and influences our thinking. I view the discourse on metaphor as inseparable from the discourse on language (Sfard, 1997, p. 343).

For instance in the case of cardboards 3 and 6 the polls gave, respectively, "railway track" and "slanting giraffe's neck". We interpret the use of different metaphors, as the fact that pupils are unaware of the likeness (i.e. parallelism) of the two cards. Moreover the (rare) presences of "slanting railway track" as description of cardboard 6 could be interpreted as the recognition of parallelism and the attainment of the transfigural stage.

About perpendicularity, children describe it taking advantage of the learning of capital letters writing. They overcome the fact that our cardboards 5, 7 and 8 do not present letters in the 'canonical position' (cf. "Turning on the other part it looks like a 'T'"). We consider the learning of writing an useful support of the pre-conception of perpendicularity. But in the case of letter 'Y', cardboard 7, the usual shape of this letters gets the better of perpendicularity (and also of the lengths of straws).

We may affirm that our tools offer the possibility of rich linguistic and geometrical experiences about parallelism and perpendicularity, supporting our first and second research questions. About the third question, Table 1 ratifies that examples of locutions that show warning signs of different conceptions of space are present.

Our research shows the importance to pay attention to children language and it suggests that a meaningful experience may gradually enrich the mathematical vocabulary:

> The most important single factor influencing learning is what the learner already knows. Ascertain this, and teach him accordingly. (Ausubel, 1968, p. vi)

Therefore a didactical consequence is that teacher must should be attentive to the pupils way of speaking, taking it into account. In classroom discussion teacher might exploit child's tacit knowledge in order to improve "the language of classroom" (Gawned, 1990). The teacher's goal could be to favour the acquisition of transfigural stage, since the mathematical nomenclature is meaningful only with the intuition of transfigural space. But it is not enough. Indeed, as a corollary of this study, we remark that the use of traditional names 'parallels' or 'perpendiculars' requires with the transfigural stage of knowledge coupled with an isotropic space preconcept.

The pupils' polls show the permanence in an intrafigural stage, therefore in an anisotropic space conception. This is typical of the everyday experience of the physical space, in which perpendicularity and parallelism, appear mainly in presence of horizontal and vertical directions. Therefore, teacher should take into account this children's attitude and s/he is allowed to use the mathematical standard words only whether s/he is aware that pupils achieved this more complex and ripe stage (transfigural and isotropic space).

About the third research question, Table 1 ratifies that examples of locutions that show warning signs of different conceptions of space are present.

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Discourse in various mathematical contexts

Part 3

THEMES OF PUPILS' CHOICES FOR THEMATIC APPROACH IN MATHEMATICS IN PRIMARY SCHOOL

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The article describes and analyses theoretical materials, pupils' choice about the aspects of the thematic choice in the acquisition of mathematics content using the thematic approach in primary school. Teaching mathematics thematically emphasises the use of applications of mathematics around a central theme whereas teaching in topics predominantly emphasises mathematical content. In the thematic approach mathematics content involves objects, information, topics and themes. The topicality should be linked with happenings in their personal lives as well as the latest developments in community life, socio-economic processes or a scientific context as well.

INTRODUCTION

Mathematics is a necessity for the majority of humans' actions as well as activities. In each culture mathematics content is used on a daily basis, for instance, in measurements, construction, cooking meals; also, mathematics is used in various fields of commerce, i.e., food trade and last but not least, in solutions of global financial issues. Thus, it is possible to introduce such activities and objects in mathematics content that help learners appreciate the significance of mathematics teaching/learning performing diverse lifetime actions. The report published by the European Commission „Mathematics Education in Europe: Common Challenges and National Policies" states that teachers do not provide pupils with sufficiently clear explanations how mathematics should be linked with everyday life and lifetime actions (European Commission, 2011). In the process of teaching/learning mathematics we should change the attitude and opinion that mathematics is complicated and not related to real life. One of the possibilities how to implement it, is to organize teaching/learning mathematics around „a big idea" and cross-curricular themes which will help to show the connection with everyday life and other subjects (Van den Heuvel-Panhuizen, 2001).

It explains why it is essential to create such mathematics content during which learners would comprehend, understand and link the acquired mathematics skills as well as knowledge with the existing situations in reality; apart from that, the content of mathematics should ensure proper learners' development, satisfy learners' needs as well as interests within the study process; also, holistic approach in learners’ development should be carried out. One of the existing opportunities in order to do it is the thematic approach (Helmane, 2011).

AIM OF RESEARCH

To analyse the pupils’ chosen themes in the acquisition of mathematics content within thematic approach and determine the compliance of pupils’ chosen themes with thematic aspects within thematic approach in primary school.

ESSENCE OF THEMATIC APPROACH

The thematic approach involves the integration of various content fields exploring an exciting idea which is closely linked with the content of different subject areas. This approach arranges the study content in such a way that learners comprehend the link among different subject areas as well as interconnection with real life (Volsa, Konflina, 1998). Mathematics content in the framework of the thematic approach is associated with the development of skills in practical activities the so called “hands on” as well as the intercorrelation of the acquired knowledge based on the theme or a concept; also, skills that can be applied in lifetime actions as well as the development of a personal sound attitude, values and goals.

Teaching mathematics thematically envisages linking mathematics with the central theme which emphasizes and comprises mathematical content. For instance, if the central theme is „Sports”, then the thematic units could be organized in various content areas: percentage, measurements, statistics or algebra, thus strengthening mathematics content (Handal, Bobis, Grimison, 2001). Therefore, the themes should be included in mathematics curriculum and content in such a way that skills and knowledge would be taught and mastered in accordance with the central theme, thus attaching significance and direction to the educational process (Freeman, Sokoloff, 1995). The thematic learning links mathematics with real, existing life situations (Handal, 2000) and the knowledge is acquired in a meaningful and practically oriented context (Putnam, Borko, 2000).

Taking into account that a pupil at the early school age has a difficulty concentrating if the content is not interesting, if it is abstract and does not stimulate thinking (Cooper, 1998), also, when a pupil does not see the sense in doing what he has to do, then the anticipated does not give satisfaction. In such a state the pupil feels disappointed, stressed, internally alarmed which is frequently accompanied by isolation, protest and unwillingness to learn. It is especially dangerous in those educational stages when learning as a meaningful process has just started (Abramova, 2003; Sousa, 2001). Then, within the framework of thematic approach it is possible to show the practical importance of mathematics content by explaining to pupils where mathematical skills and knowledge are encountered in life and how significant it is to obtain and apply correctly each new mathematical skill and knowledge. When facing such practical study content linked with real lifetime activities, pupils develop a positive attitude towards mathematics and the content to be mastered as well as interest and motivation to acquire mathematical skills and knowledge.

ASPECTS IN THE SELECTION OF THEMA IN THE ACQUISITION OF MATHEMATICS CONTENT

In the thematic approach to mathematics content it is essential to use such topicality that a learner may encounter in his/her real lifetime activities; moreover, the topicality should be linked with happenings in their personal lives as well as the latest developments in community life, socio-economic processes or a scientific context as well: learners' personal experience in accordance with learners' daily activities; social processes and learners' roles in them, behavioural norms, ways of socially important activities, generally accepted symbols as well as the economic aspect; calendar year with seasonal changes, anniversaries and traditional holidays, specific features of the period as well as objects; science including the technological process, a theoretical explanation or a precisely defined maths problem as well as widening the outlook; themes in accordance with the National Standards of Basic Education with the content integration of other subject areas, i.e., cross- curricula acquisition of the mathematics content (Helmane, 2011).

Taking into account the specific character of mathematics content in primary school and the aspects of thematic choice when mastering the mathematics content, the thematic approach should be implemented in levels (see Table 1).

Level	Component	Essence of level
Level 1	Objects	any object, article, phenomenon, living being according to the calendar time from pupils' private and social life, socio-economic processes or with a scientific context
Level 2	Information and event	any real event, its procedure, real life phenomenon about private and social life events, socio-economic or scientific processes, calendar passage of time
Level 3	Topic	a short definition of content viewing, discussing and investigating private and social life events, socio-economic processes or also with a scientific context, calendar passage of time
Level 4	Theme	general narration as the whole of phenomena, ideas, vital issues according to the calendar time about pupils' private and social life events, socio-economic or scientific processes

Table 1: Levels of thematic approach in matematics (Helmane, 2012).

The levels of thematic approach in mathematics are implemented successively providing a gradual transition from objects to information and then to a topic and theme. The thematic approach levels are closely connected with the content saturation of the chosen thematical aspect. By choosing a higher level of thematic approach, the content included in the thematic aspect enhances and enriches. Thus, the number of tasks, exercises and activities related to the chosen thematic aspects, increases in the acquisition of mathematics content. Also, the time allocated for doing the tasks increases where the majority of tasks, exercises and activities in a certain period are connected with the chosen thematic aspect.

It is characteristic that when selecting a thematic aspect and the first level of implementation of the thematic approach, the acquisition of mathematics skills and knowledge is related to the objects appropriate to the thematic aspect, not envisaging a wider substantive discussion about them. However, when selecting a thematic aspect and, for instance, the third level of implementation of the thematic approach, the acquisition of mathematics skills and knowledge should be implemented within the framework of the topic, envisaging a short, substantive statement about the events of pupils' private and social life, socio-economic or scientific processes as well as the opportunity to use objects and information in accordance with the chosen thematic aspect.

RESEARCH OF PUPILS' CHOSEN THEMES IN MATHEMATICS IN PRIMARY SCHOOL

The research of pupils' chosen themes in mathematics in primary school was carried out in Form 4 (aged 9-10) of 3 different schools in Riga: one secondary school, one private basic school, one basic school branch of secondary school. The research was done over the period of autumn semester in academic year 2013/2014 where the pupils' selected themes for the acquisition of mathematics content within the thematic appoach in primary school were explored. During the research the data were obtained by applying such empiric research methods as questionnaires and a semi-structured interview. The aim of questionnaires was to specify the pupils' chosen themes for mastering mathematics content in primary school. The aim of a semi-structured interview was to define more precisely the pupils' choice of themes for mastering mathematics content in primary school.

The pupils of Form 4 were asked to fill in a questionnaire about the themes to be used in the acquisition of mathematics content in primary school. The pupils independently marked the themes topical to themselves about which they would be willing to do mathematical calculations in the lesson. When marking the themes, the pupils took into account that each offered theme must be evaluated as interesting, topical for himself/herself and applicable in mathematics, as partly interesting, partly topical, partly applicable in mathematics or not interesting, not topical, not applicable in mathematics in primary school.

Proceeding with questionnaires the pupils individually marked at least one theme which could be interesting, topical for him/her, and applicable in mathematics in primary school. The questionnaire data were analysed according to the previously set thematic aspects as research criteria: learners' personal experience, situations; socio-economic processes; calendar year; science and technological process; themes in accordance with the National Standards of Basic Education (see Table 2).

Criterion	**Indicator**	**Index**
Learners' personal experience, situations	School, hobby groups	1
	Pupils' daily activities	2
Socio- economic processes	Shops, buying, selling	3
	Customer,passenger, cook	4
	Socially significant activities	5
	Generally accepted symbols: digits,currency signs, symbols of measurements etc.	6
Calendar year	Annual and traditional festivals	7
	Seasons, change of seasons, seasonality	8
Science and technological process	Technological process, situation	9
	Precisely defined mathematical problem	10
	Outstanding inventors	11
Themes in accordance with the National Standards of Basic Education	Related to natural sciences	12
	Related to home economics and technologies	13
	Related to other subjects	14
	Animal world in different continents	15

Table 2: Criterion and indicators of the research.

The obtained data in questionnaires were made more precise in a semi-structured interview where the pupils supplemented their answers to the proposed questions about the themes, their topicality in mathematics in primary school in the context of thematic approach. A peculiarity of this interview was that the questions were not previously defined precisely, also, their succession was not strictly specified, however, during the interview it was explored which themes seem binding, topical and which themes do not arouse interest. The pupils also specified the chosen themes based on the criteria set for the research. The length of each semi-structured interview was approximately 20 minutes and they were transcribed and coded.

RESULTS

The data obtained from the questionnaires by 77 primary school pupils of Form 4 prove that the pupils in most cases evaluate the offered themes as interesting, topical for themselves and applicable in mathematics in primary school. It could testify about the pupils' needs to realize, notice and link the skills, knowledge obtained in mathematics with the situation existing in real life. It is characteristic that among the pupils' chosen themes the dominating is such a thematic aspect as themes in accordance with the National Standards of Basic Education, socio-economic processes. Interesting, topical for pupils and applicable in mathematics are the themes in the following thematic aspects: socio-economic processes and science and technological process. However, the pupils of Form 4 evaluate the themes related to calendar year as not interesting, not topical and not applicable in mathematics (see Figure 1).

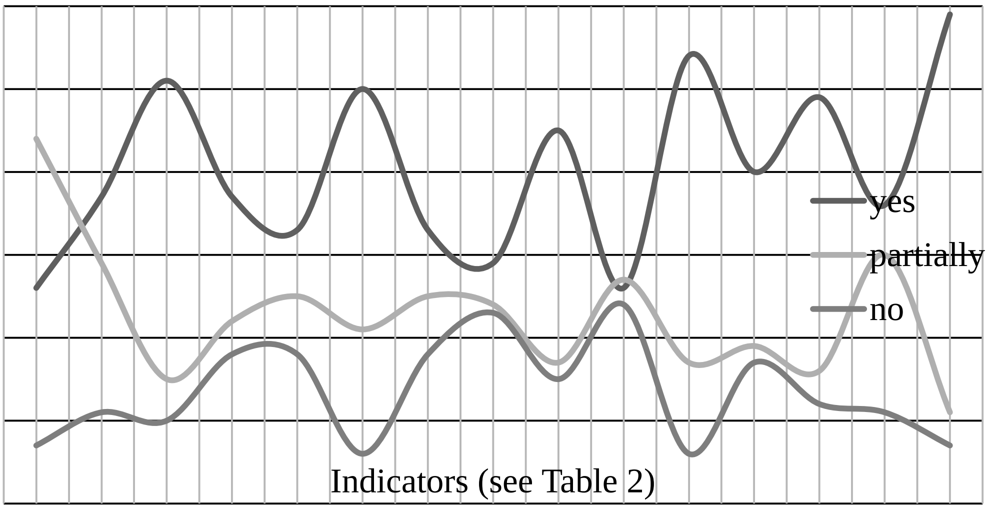

Figure 1: Pupils' chosen themes for mastering mathematics content.

The collected pupils' answers in the thematic aspect *Socio-economic processes* comprise common trends, interest about various socio-economic processes in society, for instance, buying-selling processes, the actions to be performed in it, transport, its movement, distances between cities, the population in different cities, popular and in labour market required professions. The pupils have a desire to explore extensively their town, area, country.

In a semi-structured interview most of the pupils acknowledged that in mathematics lessons they would prefer to do such tasks of socio-economic aspect which would be related to the theme about shops, buying and selling processes. The following pupils'comments testify to that:

Interviewer: What tasks would you like about shopping?

Anna: About shops. I like shopping. There are so many things to buy. I would like to calculate so that I could calculate better in the shop.

Edvard: I like tasks where I can choose and buy. My mum also buys a lot.

It is typical that the pupils' answers in the thematic aspect *Learners' personal experience, situations* show a pupil in various situations performing varied lifetime activities, taking part in educational process. However, there is a desire to replace this thematic task by children's pastimes, out-of-school activities. For instance, in a semi-structured interview a pupil indicates topical alternatives to himself/herself for the themes about school and hobby groups thus, enhancing the thematic aspect *Learner's personal experience, situations:*

Interviewer: You marked (in a questionnaire) that only occasionally you want tasks which would be about school and different hobby groups.

Peter: Yes, I already know what is going on there. Why should I calculate about it ?

Interviewer: How do you know that?

Peter: I myself go to football, sing in a choir, what else is needed?

Interviewer: What is your suggestion about the theme?

Peter: It might be about...films, film stars?

Interviewer: Yes.

Peter: Also about famous sportsmen.

Conversely, the themes of the thematic aspect *Calendar year* about seasons, change of seasons, seasonality are the ones which pupils evaluate as the least interesting, not topical, not applicable themes in mathematics in primary school. The critical evaluation of the theme is explained by unconformity with the specific age as well as the excessive uniformity, seasonality in each academic year. However, the pupils who characterize the theme about seasons, change of seasons and seasonality as topical, interesting and applicable in maths tasks, emphasize the thematic link with really existing, typical phenomena, objects in a particular period of time.

Interviewer: Don't you really want tasks about seasons?

Ivo: It is usually done in kindergarten, but I am already in Form 4.

Interviewer: What should be calculated?

Ivo: I want to do the tasks containing complicated things, for instance, about constellations.

Interviewer: Why does it seem interesting to you to do tasks about seasons?

Eva: I like to calculate about the same what is going on outside: in autumn about rain, in winter about snow.

The pupils evaluate the themes of the thematic approach aspect *Themes in accordance with another subject* as topical, interesting and applicable in maths tasks. The majority of pupils emphasize that they would rather do tasks related to home economics and related to nature, animals, plants which are acquired in natural sciences. For instance,

Interviewer: What should be the tasks about?

Kate: Very interesting and not too hard text tasks

Interviewer: What would be an interesting text task?

Kate: I would like about painting, fruit, flowers.

It is typical that pupils evaluate a more precisely defined theme, for instance, *Animal world in different continents* more positively than the theme with general guidelines on connection with a particular school subject – natural sciences. In a semi-structured interview pupils willingly comment and substantiate their choice explaining that in natural sciences there are both exciting and also boring and dull themes.

The analysis of pupils' opinions obtained in the research revealed a close connection between the thematic approach and the acquisition of mathematics content in primary school. Pupils closely link the acquisition of mathematics content with the thematic approach offering themes which could be used in processes of both the acquisition of new skills and knowledge and the development of the mastered mathematics content.

Interviewer: About what else would you like the tasks in mathematics?

Kolja: Simplify the expresions about dancers.

Interviewer: Simplify the expresions and dancers?

Kolja: Yes... Somehow boring, only numbers and numbers. I like dancing. Then it would be more interesting to calculate.

Thus, it is possible to raise the probability that in the acquisition of mathematics content it is advisable to use themes in all levels of implementing the thematic approach: the levels of objects, information and event, topic and theme. Therefore, as a result of the research, it is possible to specify the significance of thematic approach in the acquisition of mathematics skills in primary school. Within the framework of thematic approach it is possible to show the practical importance of mathematics content, explaining to the pupils where the mathematics skill and knowledge are encountered in life, and how significant it is to master and apply correctly each new mathematics skill, knowledge. Facing such practical content based on real life, pupils develop interest about mathematics and the content to be acquired. Complying with this prerequisite the content of mathematics comprises a less obvious contradiction between the reality of life and the content to be acquired in mathematics, moreover, the mathematical skills and knowledge obtain a greater confidence level, the vision of connection with life. The reflection of real life provides the opportunity to supplement the content of exercises and tasks, problems and reasonableness of the acquisition of measurements.

CONCLUSION

Teaching mathematics thematically envisages linking mathematics with the central theme which emphasizes and comprises mathematical content. Mathematics content in the framework of the thematic approach is associated with the development of skills in practical activities the so called "hands on" as well as the intercorrelation of the acquired knowledge based on the theme or a concept; also, skills that can be applied in lifetime actions as well as the development of a personal sound attitude, values and goals.

Within the framework of thematic approach it is possible to show the practical importance of mathematics content by explaining to pupils where mathematical skills and knowledge are encountered in life and how significant it is to obtain and apply correctly each new mathematical skill and knowledge.

In the thematic approach to mathematics content it is essential to use such topicality that a learner may encounter in his/her real lifetime activities; moreover, the topicality should be linked with happenings in their personal lives as well as the latest developments in community life, socio-economic processes or a scientific context. Taking into account the specific character of mathematics content in primary school and the aspects of thematic choice when mastering the mathematics content, the thematic approach should be implemented in levels.

The pupils in most cases evaluate the offered themes as interesting, topical for themselves and applicable in mathematics in primary school. It could testify about the pupils' needs to realize, notice and link the skills, knowledge obtained in mathematics with the situation existing in real life. It is characteristic that among the pupils' chosen themes the dominating is such a thematic aspect as themes in accordance with the National Standards of Basic Education, socio-economic processes. Interesting, topical for pupils and applicable in mathematics are the themes in the following thematic aspects: socio-economic processes and science and technological process.

THE USE OF GRAPHIC DISPLAY CALCULATOR AND COMPUTER SOFTWARE IN MATHEMATICAL MODELING

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When technologies are used in teaching and learning mathematics it is important to assess how activities will support the process of mathematical understanding and skills. In the literature one can find a lot of information about using computer software and graphic display calculator (GDC) in teaching and learning mathematics. The aim of this paper is to show how students overcome difficulties with solving a task concerning statistical data by using IT.

INTRODUCTION

The use of graphic display calculator was introduced in 90s of the last century and is still widespread. In Diploma International Baccalaureate exams in mathematics it was introduced since 2004 as a mandatory device. (Some information about International Baccalaureate Diploma one can find in (Jureczko, 2012a, 2012b)). Till now there are other examinations which the use of GDC is compulsory. In the literature there are a lot of reviews of attempts of classification of ways in which GDCs can be used to maximize learning and achievement. We can find also propositions of teaching practices during lessons, see (Graham at.al, 2008), (Lee at.al, 2010). Some of the modes of GDC are used more often. Students and their teachers usually use GDC as a tool for numeric and graphic strategies even such which can be made using paper and pencil. One can find in (Jureczko, 2014) a feedback of questionnaire in using GDC during mathematics learning. More and more teachers recognize that computer software and GDC are powerful tools for the mathematics learning. But most of papers authors describe only observations and work of students during the lesson in limited time and with little help of the teachers. It seems to be important to observe and analyze how students use GDC and computer software in solving challenged tasks at home without time limitation. An International Baccalaureate (IB) Program is good opportunity to do such analysis.

The research analyzed in this paper was a task from portfolio for 2011 and 2012 (type II - modeling) solved by students during 10-day period (further information about portfolio one can find in (Jureczko 2012a, 2012b)). According to procedure of International Baccalaureate Organization a teacher could help students in very limited way (without giving even a partially solutions). The role of the teacher was to say general information about task assessment criteria and make students meet deadlines. Students solved this task as their homework and after 10 days gave the teacher finished work. It is impossible to follow which buttons in GDC were pressed during the work of student, but we can observe in which part and why students used GDC.

The task considered in this paper is related to statistics and analysis of statistical dates. In literature there are papers concerning the same part of mathematics (Gordon at.al 2009), (Foster 2007). In paper (Obara, 2009) we can even read how students collect and represented their data to statistical analysis. In (Doerr at.al, 2000) authors divided type in using GDC computational tool, transformational tool, data collection and analysis tool, visualizing tool, checking tool. This paper shows how use of technology assisted students in solving task in (compare (Foster , 2007)):

- reading data (to looking at difference between data in each part of the task)
- estimating lines of best fit (not having knowledge about regression)
- changing models for fitting to farther data
- recognizing trend to data
- finding limitations.

DATA COLLECTING AND ANALYSIS

Data collecting

The task was originally taken from portfolio Mathematics Standard Level (SL) for use in 2011 and 2012 published by International Baccalaureate Organization. In research 8 female students of International Baccalaureate class took part. During attending this class students did not learn using computer software for solving tasks of statistics. Before the experiment students were learnt descriptive statistics (especially with using GDC: box plot, cumulative frequency, median, quartiles, histogram). Students were not inform about regression at all (which occurred important in this task) because this topic does not exist in syllabus of mathematics SL.

During one lesson I gave general information about task, but did not solve any part of it (because of procedure published by International Baccalaureate Organization). Students solved this task during 10-day-period without time limitations (as their homework) and after this time gave the teacher final version of their work. Each student was equipped in GDC with which they worked one year before the research. At home students had their own computers with access to the Internet. Teacher knew limitations of using GDC and computer software for this task but did not inform students about it. The original task titled "Population trends in China" is in Appendix 1.

Analysis

Following the instruction given in the task all students plotted data properly and all of them noticed that this trends grew in those years. Almost all of students used a functions of spreadsheet but some of used Autograph or MatLab or something like that. Then students tried to find the line of best fit. Four of them found that the line should be exponential function, but the other decided that the best line is a polynomial (only one girl decided that it should be polynomial of degree 9, the others seeked polynomials of degree 6). There are example of student's works.

Original equation for the exponential growth function is:

f(x)= ae^{x}

However, here it can be altered into:

$\boldsymbol{P(t) = ae^{d}}$,where *P(t)* indicates the number of population in millions in each year (based on the table 2.), *a* is parameter and *d* represents difference between initial year (1950) and any subsequent year.

Table 2. Differences between years.

Year (t)	1950	1955	1960	1965	1970	1975	1980	1985
Difference between initial year and subsequent year (d)	0	5	10	15	20	25	30	35

Example 1.

P(1950)=$a \times e^{d_1}$,where $d_1 = 1950 - 1950 = 0$

554.8 =$a \times e^0 = a \times 1 = a$,thus *a*= 554.8

Example 2.

P(1955)=$a \times e^{d_2}$,where $d_2 = 1955 - 1950 = 5$

609 = $a \times e^5$ = 554.8×148.4131591

$609 = 82339.62$,therefore the outcomes greatly differ.

Because of this, the equation need to be amended and thus:

P(t)=$a \times e^{d\delta}$,where δ is a parameter.

P(t)= $554.8e^{d\delta}$

$e^{d\delta} = \frac{P(t)}{554.8}$

$$d\delta = \ln\left(\frac{P(t)}{554.8}\right) \quad , \quad \boldsymbol{\delta = \frac{\ln\left(\frac{P(t)}{554.8}\right)}{d}}$$

All values of δ are counted with the use of GDC.

To obtain one constant value of δ, the best solution is to conclude the mean value.

Hence, $\delta = \frac{0.019+0.017+0.018+0.020+0.021+0.020+0.019+0.018+0.018}{9} = 0.019$

This gives the final formula:

$$\boldsymbol{P(t) = 554.8e^{0.019d}}$$

It enables to draw a following graph with the use of the program *Autograph.*

Graph 2. Comparison of the former function and exponential growth function.

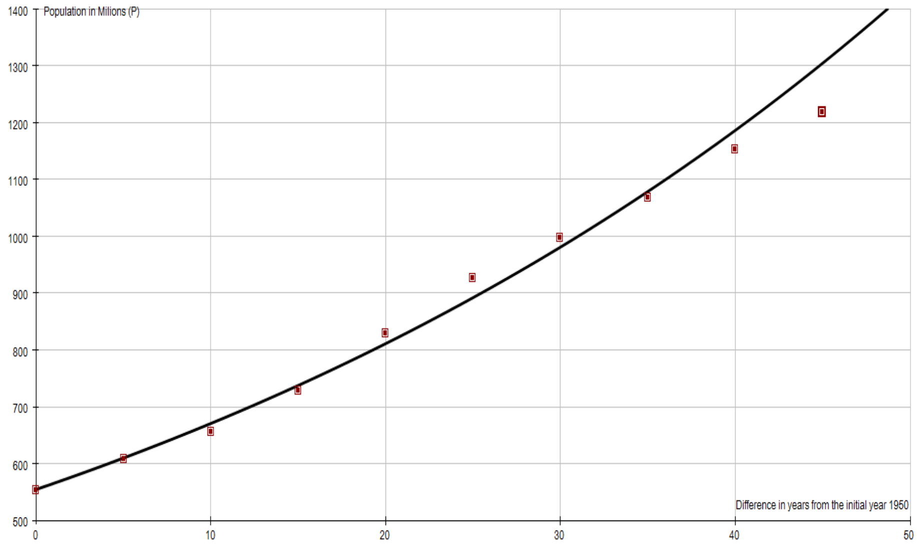


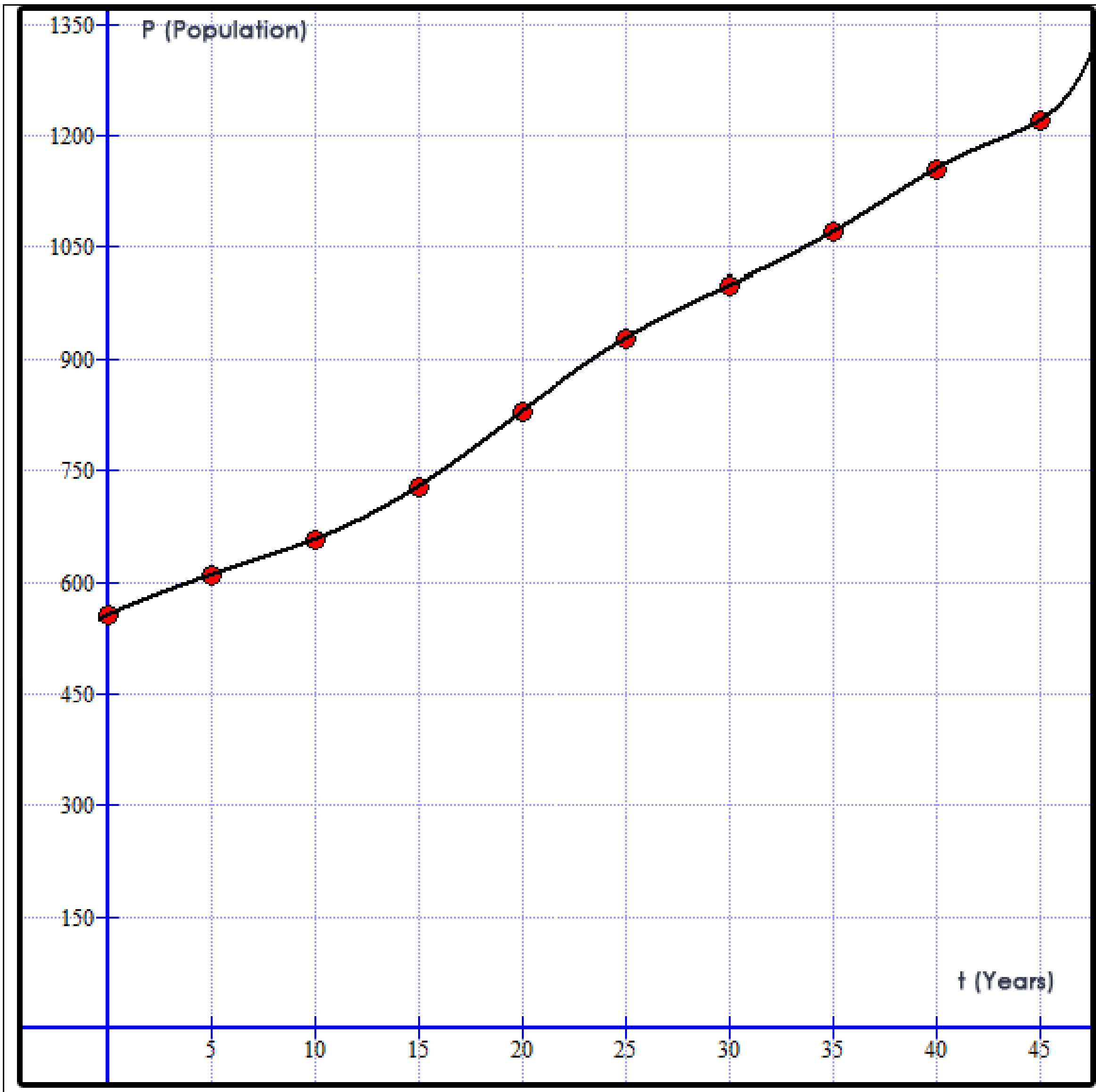
Table 1: Original student's work

Basing on all the analysis of the graph above, we can now develop one model function that will fit the data points in the graph. The polynomial function of ninth degree will be the most accurate with the original data. The equation for such function is:

$y = ax^9 + bx^8 + cx^7 + dx^6 + ex^5 + fx^4 + gx^3 + hx^2 + ix + j$

Where in this investigation, y is P, population, and x is t, years.

The coefficients of x are going to be calculated with the use of technology, alongside with plotting the graph.



The equation of the model above is:

$$y = 2.5227E - 10x^9 - 4.4501E - 08x^8 + 3.0871E - 06x^7 - 0.0001x^6 + 0.0017x^5 - 0.0135x^4 + 0.0558x^3 - 0.4082x^2 + 12.3508x + 554.8$$

The equation was determined by the software, Graph 4.3. The value of coefficients is given to four significant figures. The value of j is the value of P in t = 0.

This graph also has a correlation of the original data and model of $R^2 = 1$. Therefore, it is a perfect fit. This proves that the analysis of data graph was correct.

Table 2: Original student's work

Another one even explained how to do using GDC

Looking on the graph and changes between next points it can be predicted that the function that the best fits the data will be the polynomial function. Thanks to many parameters that can change the behavior of the curve (in points of inflection) it can fit changing the data the best.

USE OF THE GDS

1. In STAT select F1-GRAPH and then F1-GRAPH1.

2. Select F1-CALC and then F2-x, F1-ax+b

3. Parameters and correlation coefficients for given function will appear on the screen.

4. Repeat the same action for every other function

5. Present obtained data in a table and choose the one function with the biggest correlation coefficient.

The r2 coefficient of given function are:

Function	The correlation coefficient r^2
y=ax+b	0.9947
$y=ax^2+bx+c$	0.9955
$y=ax^3+bx^2+cx+d$	0.9985
$y=ax^4+bx^3+cx^2+dx+e$	0.9986
y=a+blnx	0.9945
$y=ax+be^{bx}$	0.9903

As predicted the highest value of r2 has the polynomial function $y=ax^4+bx^3+cx^2+dx+e$. It means this should be the function that the best fits the given data.

Question 3.0

"Analytically develop **one** model function that fits the data points on your graph."

Answer 3.0

To find an exact parameters of function the we will use once more the GDC.

USE OF THE GDS:

1.In MAIN MENU select STAT.

2Select F1-GRAPH and then F1-GRAPH1.

3.Select F1-CALC, F6 to move list further and then F2-x^4.

4. The calculator will show the exact parameters of the function on the screen.

The parameters given by the calculator are:

$a=-2,589*10^{-6}$

b=0.01496

c=-25.29400

d=10091.98657

e=6456286.18429

Therefore the function $y=ax^4+bx^3+cx^2+dx+e$ will look as following:

$y=-2,589*10^{-6}x^4+0.01496x^3+-25.29400x^2+10091.98657x+6456286.18429$

Question 4.0

"On a new set of axes, plot your model and the original data. Comment on how well your model fits the original data. Revise your model if necessary."

Answer 4.0

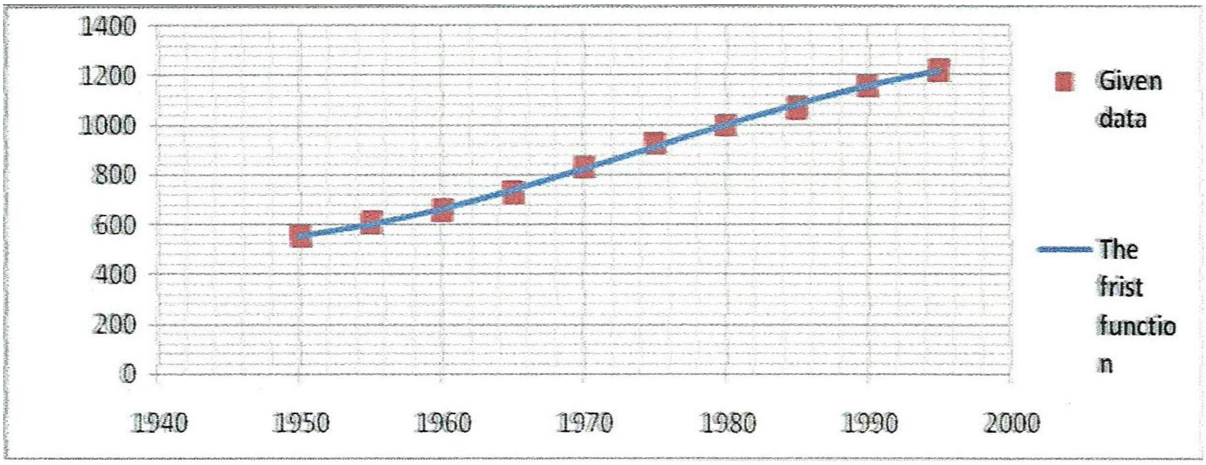
To properly compare new model and the set of data they should be place on the same set of axes.

USE OF THE GDC:

1. In STAT select F1-GRAPH, F1-GRAPH1,

2. Select F1-CALC and then F2-x^4.

3. Select F5 to copy equation and F6 to draw a graph.



4. In the MAIN MENU go into GRAPH.

5. Select F6 to draw copied graph.

6. Select F1 and write a value of x you are interested in to obtain the y.

To determine if the given function actually fits the graph we can use relative and absolute error. They will show us the difference between given data and value obtained from graph and the pertencage change in the values.

Relative error : $z=[x_d-x_n]$

In which:

x_d - given data

x_n- theoretical value

[ ] – absolute value

Absolute error: Relative error*100%/x_d

If the absolute is smaller than 3% we can say that the graph model the given data well.

Years	Given data	Theoretical values from first function graph	Relative error	Absolute terror [%]

1950	554,8	556	1,2	0,21629416
1955	609	601,6	7,4	1,215106732
1960	657,5	664,4	6,9	1,049429658
1965	729,2	740	10,8	1,481075151
1970	830,7	824,1	6,6	0,794510654
1975	927,8	912,3	15,5	1,670618668
1980	998,9	1000	1,1	0,110121133
1985	1070	1082,9	12,9	1,205607477
1990	1155,3	1156,4	1,1	0,095213364
1995	1220,5	1215,9	4,6	0,376894715

Table 3: Original student's work

It is interesting that students tried to find the line of best fit using technology only. Almost all of them used GDC. They tried to fit linear and logarithmic to but they quickly gave up such solutions. None of them tried to find the solutions algebraically. Observing coefficients it would be rather difficult. Students probably tried to find a pattern of linear function but they gave up this idea.

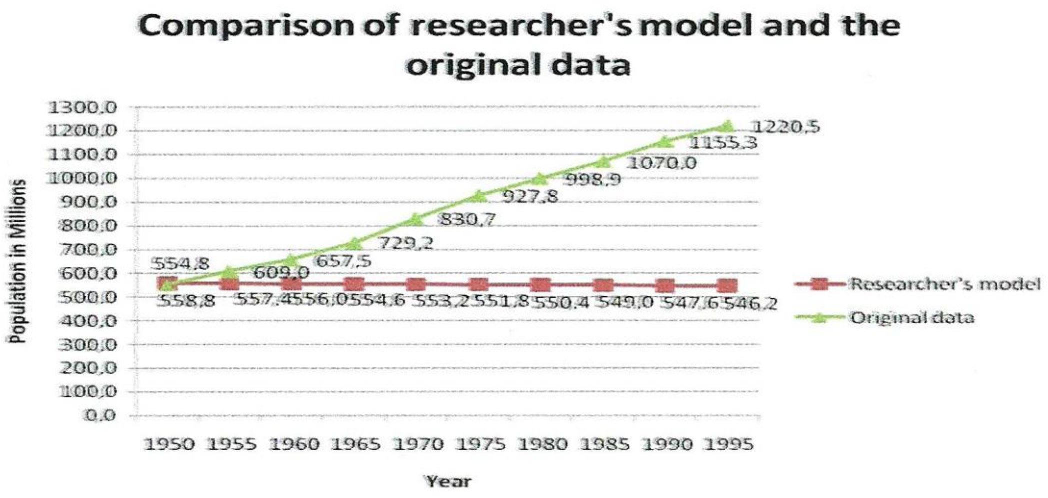
The model P(t) proposed in the task was recognized by all of students as a logistic model. They found the adequate function in their GDC. But, it is very important that each student gave this function which other parameters K, L and M, but if they put a graph of the same screen they obtained almost ideal line of best fit. Below there are propositions of such functions (or parametres) of all students taking part in the research.

$f(x) = \dfrac{750}{1+0.85e^{-0.23x}}$	$P(t)=\dfrac{3224.22413}{1+4.76773238\times e^{-0.02432536t}}$
K = 3227.9068 L= 4.7747 and M = 0.0243	$P\ (t) = \dfrac{3224.22}{1+4.77e^{-0.02t}}$
$y = \dfrac{470.258722}{1+4.6022 * 10\text{^}19 - 0.02226\text{x}}$	$P(t) = \dfrac{3224.22}{1+4.77e^{-0.02t}}$
$\boldsymbol{P(t) = \dfrac{2045}{1+2.77e^{-0.032t}}}$	$P(t)=\dfrac{4228.11312}{1+4.6022*10^{19}e^{-0.02225828}}$

Table 4: Pieces of original student's work

Four of the functions mentioned above (in grey boxes) are very similar, but two of them were interpreted as follows

$$P\ (t) = \frac{3224.22}{1+4.77e^{-0.02t}}$$



$$P(t) = \frac{3224.22}{1 + 4.77e^{-0.02t}}$$

Table 5: Original students' work

And were recognized as unfitted lines. It is strange situation, because for other students the line was of best fit.

The last questions of this task was to observe how the populations was grown in the future.

Four to eight students claimed that the best one occurred the logistic model (but with changed parametres). They tried to fit their own models, but exponential and polynomials modedls tended to infinity, hence they gave them up.

$\frac{750}{1 + 0.85e^{-0.23x}}$	$P(t) = \frac{1846.69011}{1 + 2.31173963 \times e^{-0.03216208d}}$
K = 1869.1956 L= 2.3564 and M = 0.0319	$\boldsymbol{P(t) = \frac{2045}{1 + 2.77e^{-0.032t}}}$

Table 6: Original students' work

One student tried to analyze more than two models. Her solutions she wrote in a table (below) and she chose the polynomial.

Function	Formula	Coefficient of determination R^2
Linear	y = 43,593x + 1058,9	$R^2 = 0{,}8292$
Exponential	$y = 1061{,}9e^{0{,}0366x}$	$R^2 = 0{,}8027$
Logarithm	y = 149,14ln(x) + 1051,7	$R^2 = 0{,}9753$
Polynomial	$y = 0{,}0875x^6 - 1{,}9583x^5 + 16{,}45x^4 - 60{,}583x^3 + 63{,}663x^2 + 183{,}14x + 829{,}3$	$R^2 = 1$
Power	$y = 1053{,}8x^{0{,}1265}$	$R^2 = 0{,}9633$

Table 7: Original student's work

One student claimed that the polynomial which she found in the first pair of the task it was the best one. Students who found the logistic model as the best one for these data they noticed that this model became stabilized. Students who chose polynomials did not think about future data because each mentioned polynomial tended to infinity. Students in this task still used GDC as a main device.

CONCLUSIONS

At least two types of technology were utilized for this task. The main role in solving played GDC which were used as a data collection and analysis tool and visualizing tool (to find the line of best fit) and as a checking tool (some students passed photo of the screen of GDC in their papers). Although GDC was enough device to solve problem given in the task students were made use some other software usually to show the graphs. (A screen of GDC is to small and imprecise). It is important to repeat that students were not taught about regression but they could find it as a useful function in their GDC. But students could not use the device fully, because they did not think about predictions for the future (how chosen by them line will behavior in the future). Hence technology is not able to substitute human thinking.

One it is important to admit that author of (Foster, 2006) was right that

> GDC offer sequential viewing of linked representations, while spreadsheets allow simultaneous viewing of linked representations.

It is important reason for which students have to use computer software instead of GDC.

The task was interesting for students as real-life although it was obligatory they solved it willingly. They did not any instructions from the teacher but their work occurred with precise analysis (they gave step-by-step-solutions). As researched students said the task was inspiration for examining other data sets especially in natural sciences.

Appendix: original Chinese task

POPULATION TRENDS IN CHINA **SL TYPE II**

Aim: In this task, you will investigate different functions that best model the population of China from 1950 to 1995.

The following table[1] shows the population of China from 1950 to 1995.

Year	1950	1955	1960	1965	1970	1975	1980	1985	1990	1995
Population in Millions	554.8	609.0	657.5	729.2	830.7	927.8	998.9	1070.0	1155.3	1220.5

Define all relevant variables and parameters clearly. Use technology to plot the data points from the above table on a graph.

Comment on any apparent trends shown in the graph. What types of functions could model the behaviour of the graph? Explain your choices.

Analytically develop **one** model function that fits the data points on your graph.

On a new set of axes, plot your model and the original data. Comment on how well your model fits the original data. Revise your model if necessary.

A researcher suggests that the population, P at time t can be modelled by

$$P(t)=\frac{K}{1+Le^{-Mt}}, \text{ where } K, L \text{ and } M \text{ are parameters.}$$

Use technology to estimate and interpret K, L and M. Construct the researcher's model using your estimates.

On a new set of axes, plot the researcher's model and the original data. Comment on how well this model fits the original data.

Discuss the implications of each of these models in terms of population growth for China in the future.

Here are additional data on population trends in China from the 2008 World Economic Outlook, published by the International Monetary Fund (IMF).

Year	1983	1992	1997	2000	2003	2005	2008
Population in Millions	1030.1	1171.7	1236.3	1267.4	1292.3	1307.6	1327.7

Comment on how well each of the models above fit the IMF data for the years 1983–2008.

Modify the model that best fits the IMF data, so that it applies to all the given data from 1950 to 2008. Comment on how well your modified model fits all the data.

[1] Data from the "land use change and agriculture program", published by the *International Institute for Applied Systems Analysis*

AN EXAMPLE OF AN UNUSUAL POLYHEDRON

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Line segments contained in the boundary of faces of a polyhedron and not containing, except their ends, other vertices are called "edges" of this polyhedron (seeWaliszewski, 1987). In school mathematical education student has to deal with some examples of polyhedra, such as prisms and pyramids (see e.g. Kłaczkow, Kurczab, Świda 2005). However, the concept of a polyhedron and the related definition of its edges, vertices and faces are not fully understood by students (see Lakatos 2005). From my own observations I noticed that students very often identify the notion of a polyhedron as the notion of a simple prism or, what is more worrying, with the notion of a normal prism. This paper contains an interesting example of a polyhedron and a related research concerning problems mentioned above.

THE ORIGIN OF THE RESEARCH PROBLEM

I was a co-organizer of the mathematical workshop for children devoted to polyhedra, which took place during the "Małopolska Noc Naukowców" in 2013. Participants of this workshop, after a short introductory lecture, were given the task to build, using blocks (Reko or Geomag), an Euler's polyhedron chosen by themselves. As a next step they had to count the number of vertices, edges and faces of the created polyhedron, and check that the Euler's formula held in that case. One of the workshop participants built an interesting polyhedron, for which (according to his opinion) Euler's formula did not hold. The proposed construction turned out to be so interesting that prompted me to start a research based on this particular case (see Grunbaum (1994) for a discussion regarding related topics concerning "unusual" polyhedra).

THE EXPERIMENT

The aim and the organization

The main aim was to observe, with the help of the polyhedron mentioned above, how students perceived the notions of edges and vertices of polyhedra.

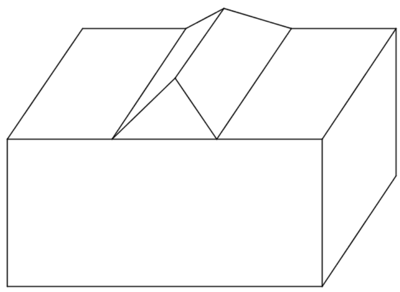
For this purpose, for each participant of the experiment I prepared:

- a model of the polyhedron made from paper,
- a projection of the polyhedron in the form of a computer visualization on a sheet of paper,

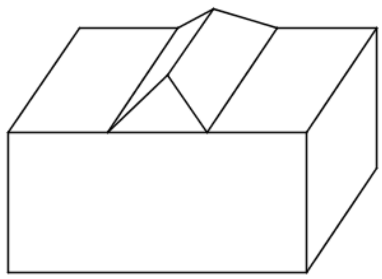
- a research task divided into two parts (containing questions about the number of vertices, edges and faces of the considered polyhedron, and also some questions concerning Euler's formula, including its usefulness in that situation).

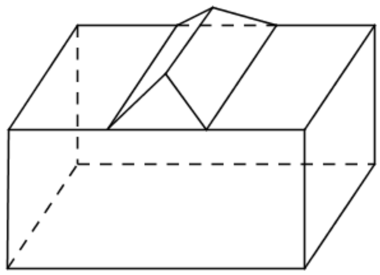
The experiment was carried out on a group of 100 students from one of the top high schools in małopolska. Each person participating in that experiment had to solve the task using only a given model of the polyhedron and its projection in the form of a computer visualization on a sheet of paper.

Materials used in the experiment



Picture 1: A model of the polyhedron





Picture 2: A projection of the polyhedron in the form of a computer visualization

*Magdalena Kubat, **Zadanie z pewnym wielościanem***

Data urodzenia: (dzień, miesiąc, rok) ☐☐ ☐☐ ☐☐☐☐

Aktualny etap edukacji:

☐ gimnazjum ☐ liceum ☐ studia I stopnia ☐ studia II stopnia
☐ studia III stopnia

Klasa/rok studiów: .....
Klasa: ☐ matematyczna ☐ niematematyczna ☐ nie dotyczy

Oznaczenia:
K – liczba krawędzi
W – liczba wierzchołków
S – liczba ścian

1. Ile krawędzi ma prezentowana bryła?

   K = ...

2. Ile wierzchołków ma prezentowana bryła?

   W = ...

3. Ile ścian ma prezentowana bryła?

   S = ...

4. Jaki związek zachodzi pomiędzy liczbą krawędzi, wierzchołków i ścian w <u>dowolnym</u> wielościanie zwykłym (nazywanym inaczej wielościanem Eulera)? (podaj wzór)

   ....

   Jeżeli nie pamiętasz wzoru Eulera: W-K+S=2.

5. Czy Twoje obliczenia zgadzają się z powyższym wzorem?

   ☐ tak ☐ nie

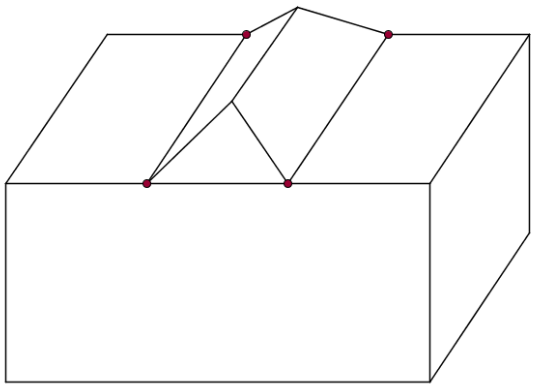
6. Jeżeli na pytanie nr 5 opowedź brzmiała „nie", to co to oznacza?

   a) wzór nie zachodzi dla prezentowanej bryły ☐ tak ☐ nie
   b) błędne obliczenie liczby krawędzi, wierzchołków lub ścian ☐ tak ☐ nie
   c) coś innego (podaj co): ....

Picture 3: A research task

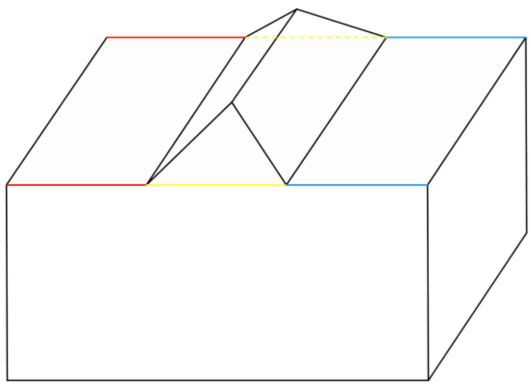
THE NUMBER OF EDGES, VERTICES AND FACES OF THE PRESENTED POLYHEDRON

For simplicity, the number of vertices, edges, and faces of the polyhedron is denoted by V, E and F, respectively. The polyhedron presented in this article has exactly 4 concave-convex vertices.



Picture 4: Four concave-convex vertices of the polyhedron

According to the definition of an edge of a polyhedron (an edge connects two vertices – no more!) (cf. with definitions given in (Coxeter, 1987, Cromwell 1997, Grunbaum 2003, and Cohn-Vossen, Hilbert 1956, Janowski 1978, Krygowska, Treliński 1972, Waliszewski 1997) there are two lines and each line contains exactly 3 edges of the polyhedron from Picture 5.



Picture 5: Colinear edges of the polyhedron

Summarizing, it is easy to see that V=19, E=23 and F=11.

THE RESULTS OF THE EXPERIMENT

Before starting the research, I thought that the students would have difficulties mainly with noticing three edges of the polyhedron located on one line. Therefore, the most expected answer to the question about the number of edges was 19. Another problematic situation foreseen by me was the question of recognition by the students of the convex-concave vertices in the polyhedron. I suspected that omission of these vertices would lead students to the wrong number of vertices in the polyhedron equal to 10 instead of 14. Moreover, I assumed that the polyhedron would prove to be so unusual that at least some students would have doubts whether the Euler's formula held in that case.

However, the results turned out to be much more surprising. The below table presents the results without the division of students according to age and profiles of school classes.

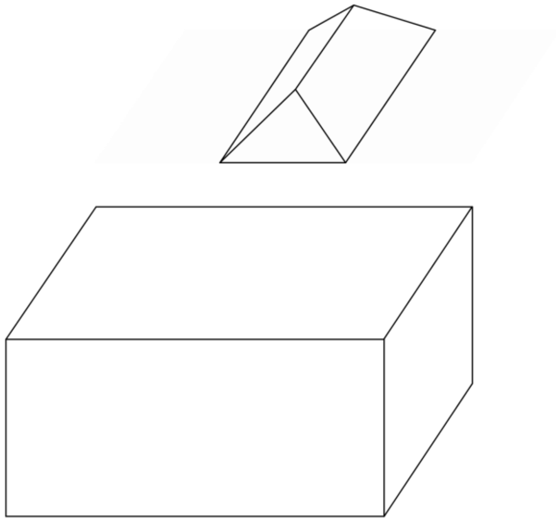
V	**E**	**F**	Number of results
10	19	11	15
14	19	11	14
10	17	11	14
14	17	11	10
10	17	10	6
10	19	10	4
14	23	11	3
14	21	11	3
14	19	10	3
10	13	11	3
14	17	10	2
14	16	11	2
14	16	10	2
14	15	11	2
10	13	10	2
10	23	9	1
13	22	11	1
10	21	11	1
10	20	10	1
14	18	12	1

V	**E**	**F**	Number of results
14	18	10	1
6	17	11	1
13	17	16	1
10	16	11	1
14	15	10	1
9	13	11	1
10	12	11	1
8	12	6	1
10	11	11	1
20	10	12	1

Tabele 1: The results of the experiment

During the research project I had the opportunity to talk with the participants. More than once, students expressed their surprise that there existed the formula which expressed the number of edges, vertices and the faces of the polyhedron. Moreover, which also was confirmed in written sheets of the research task, the high school students felt presented solid as “strange” and were not certain about counting anything on it (counting both of vertices, edges and faces of the polyhedron, and thinking about the applicability of Euler's formula for the solid). Nevertheless, the biggest surprise for me turned out to be such a large range of received results. I was expecting results such as: $E = 19$, $V = 10$, $F = 11$ or $E = 19$, $V = 14$, $F = 11$, however received a large variety of outcomes, including some very low numerical results, which were particularly surprising to me.

I would also mention about the repeated results of the number of faces equal to 10. I asked one of the students how she had been counted, and with counting again beside me, she said that the polyhedron was formed by gluing two solids. Therefore, the upper base of a cuboid was, according to the student, one face (not two) of the newly created solid. Curious of such student considerations, I asked the same thing the next research participants, according to whom the number of faces in the presented polyhedron was also equal to 10 and gave a similar explanation.



Picture 6: The polyhedron as the solid that was formed by gluing two solids

CONCLUSIONS

The research provided some informations on how students cope in a situation not previously encountered at school. By observing the process of the research and talking with the participants I came to the conclusion that the high school students were not certain about the correctness of their calculations due to the uniqueness of the task. The research problem turned out to be so interesting and extraordinary for the students, that it led to a lot of interesting discussions about both the result of the calculations of the number of edges, vertices and faces of presented polyhedron, as well as the definitions of those notions in a polyhedron. The research participants who observed some of the problematic situations that were described in section "*The Number of Edges, Vertices and Faces of the Presented Polyhedron*" of this article were very curious about the subject of defining an edge and vertex of a polyhedron (cf. discussion of such phenomena stated in Lakatos (2005). I believe that the research problem submitted in this paper can be an excellent introduction to entering the definition of a polyhedron in schools, and concepts associated with it.

PROBLEM POSING BY PRESERVICE TEACHERS

Konstantinos Tatsis, Bozena Maj-Tatsis

University of Ioannina, University of Rzeszow

Problem posing is a mathematical activity closely related to problem solving but also to creativity. In the present study we engaged preservice mathematics teachers in a situation that involved posing, solving and evaluating problems that were related to two given contexts. The results show differences in the difficulty of the problems as it was perceived by us and the students; moreover, the importance of the wording of problems has emerged because of the different interpretations of the posed problems.

INTRODUCTION

The importance of engaging students of all ages in problem solving situations is manifested in numerous contemporary curricula and research reports. Since Polya's (1957) *How to solve it?* there have been considerable changes in the methods for implementing problem solving in the mathematics classroom. However, Polya's contributions are still considered in many contemporary approaches. A characteristic example comes from Polya's heuristic that suggests to the solver to initially solve a problem which is related or analogous to the proposed one. In other words, the solver is asked to *pose* a different problem. Thus, problem posing, which lies at the core of this paper does not constitute a novelty in mathematics education; however, its systematic implementation and analysis have started rather recently. The most frequent method of implementing problem posing is as a companion to problem solving (Kilpatrick, 1987). In all cases, researchers tend to agree not only on the relationship between solving and posing problems, but also on the creative nature of problem posing activities (Silver, 1997).

A considerable amount of studies has been devoted to problem posing by students (Kontorovich, Koichu, Leikin & Berman, 2012; Silver & Cai, 1996) or preservice teachers (Crespo & Sinclair, 2008; Koichu & Kontorovich, 2013). Among these, we find some which offer a systematic approach for implementing problem posing and for analysing its outcomes (i.e. the created problems). Following Kontorovich et al. (2012) we may claim that:

> ... learning effects of problem posing activities (e.g., improving motivation or creativity development) are, as a rule, impressive even when the problems produced by students and teachers are not of good quality. Second, problem-solving skills are necessary but not sufficient in problem posing. Third, problem posing as a complex learning activity is still not sufficiently understood... (p. 150)

The present study aims to contribute to the existing body of knowledge on problem posing of students-preservice teachers, by offering a somehow different approach for the implementation of the process itself, in a way that allows for the evaluation of students' work, together with their self-evaluation in an interactive way. From a research perspective, we aimed to answer the following questions:

a) What type of problems did the students create when they were given a semi-structured situation?

b) How did the students evaluate their colleagues' problems, based on predetermined criteria?

c) What was the degree of similarity between the solutions provided by the posers and their peers and how was it explained by the posers?

d) What type of context did the students favour for their problem posing and why?

Before we proceed with the description of our research, we will present our theoretical background, which consists of particular approaches on methods to implement and analyse problem posing.

THEORETICAL FRAMEWORK

As mentioned before, problem posing is (or can be) related to problem solving in many ways. We have already mentioned one of Polya's heuristics, according to which the problem solver poses and then solves a problem which is similar to the initial problem he is facing. According to whether problem posing takes place before, during, or after problem solving, it can be categorised as following:

- Prior to problem solving when problems were being generated from a particular stimulus (such as a story, a picture, a diagram, a representation, etc.)
- During problem solving when an individual intentionally changes the problem's goals and conditions (such as in the cases of using the strategy of "making it simpler")
- After solving a problem when experiences from the problem-solving context are modified or applied to new situations. (Silver, 1994, as cited in Bonotto, 2013, p. 39)

Generally, problem posing can be defined as the creation of a new problem or as the reformulation of a given problem (Silver, 1994), where the initial conditions may vary. Particularly, Stoyanova & Ellerton (1996) refer to three problem posing situations:

a) free, where there are no restrictions for the students; they can pose any problem they wish;
b) semi-structured, where the students are provided with an initial situation and they are asked to create a problem based on it;
c) structured, where the students are provided with an initial problem and they are asked to reformulate it.

There are many aspects of problem posing that make it a very useful tool in mathematics education and also in (mathematics) teacher education. The basic aspect is the close connection of problem posing with creativity (Silver, 1997). Additionally, by problem posing, students are put at the centre of the learning process, which is usually not the case in traditional teaching/lecturing or even in problem solving, where the problem is posed by somebody else, e.g. the teacher or the author of the textbook (Kilpatrick, 1987). Finally, it provides the teacher or the researcher with a concrete view of the poser's mathematical knowledge base (Kontorovich et al., 2012).

Moreover, the request from the students to pose problems within realistic contexts (English, 1997) provides them a great opportunity for critical thinking, since they have to interpret the given data, discriminate between significant and insignificant information and "investigate if the numerical data involved are numerically and/or contextually coherent" (Bonotto, 2013, p. 40).

The process of problem posing itself has been investigated in various studies (e.g. Bonotto, 2013; Koichu & Kontorovich, 2013), while some of them extend their analysis to possible relationships between the posed problems and the posers' mathematical knowledge (Leung & Silver, 1997), and more particularly problem solving (English, 1998) or creativity (Silver, 1997). Among these, of particular interest to us was Kontorovich et al.'s (2012) study, where they focused – among other facets – on the notion of *aptness* for analysing problem posing:

> Considerations of aptness are the poser's comprehensions of explicit and implicit requirements of a problem-posing task within a particular context; they also reflect his or her assumptions about the relative importance of these requirements. (Kontorovich et al., 2012, p. 151)

Aptness can be differentiated among aptness to the poser herself or himself, aptness to the potential evaluators, aptness to the potential solvers and aptness to the group members in cases of problem posing in groups. Kontorovich et al. (2012) claim that "Considerations of aptness can be elicited based on one's utterances concerning quality of problem-posing ideas and the resulting problems" (p. 154).

The studies aforementioned, together with a previous study conducted with Greek preservice teachers (Tatsis, 2014) have offered us a valuable source for the design of our own research. Our methodological considerations are presented in the following section.

CONTEXT OF THE STUDY AND METHODOLOGY

Design of the study

We have designed our study according to our research questions, at the same time considering the relevant literature and the particular characteristics of our setting. Concerning the type of problems that were posed, we were actually interested in what Kontorovich et al. (2012) refer to as the *knowledge base* of the posers, which

> … can be explored by examining mathematical validity of the posed problem, that is, the unambiguity of the problem formulation and the solvability of the problem. The examination can be done from both the posers' perspective(s) and researchers' perspective(s). (p. 152)

We decided to examine the solvability of the problem by asking the posers and a different pair of "solvers-evaluators" to solve the problem. By giving the problem to a different pair of students we also aimed to identify its possible ambiguities. As will be discussed later, we, as researchers, categorised the problems by also considering the previous parts of the process.

Concerning the evaluation of the posed problems, it is related to the notion of aptness that we mentioned before; particularly, we aimed to identify possible *success stories*, which are cases "in which a posed problem is perceived by somebody (e.g., the problem posers themselves or their peers or problem solvers) as interesting, meaningful or surprising" (Koichu & Kontorovich, 2013, p. 72). Our decision was to design a peer-to-peer evaluation within the session, which included the solution and evaluation of the posed problems according to a predetermined set of criteria: originality, difficulty and realistic character. Originality refers to the degree of similarity of the problems with those usually contained in textbooks; difficulty refers to the mathematical actions that are expected by the solver to successfully solve the problem; the realistic character refers to the degree of resemblance of the posed problem to an everyday situation. There were two purposes for providing these criteria: firstly, we aimed to assist the students on their evaluation, by narrowing the possible factors to be considered (in the same line we provided a numerical scale from 1 to 5, followed by space for comments); secondly, we were interested to observe how the students would perceive these notions.

Another novelty of our approach was the possibility that was given to the posers to study and reflect on their own problems and their peers' solutions. This is also related with the identification of success stories by the posers themselves, but with a focus on the solver.

Following Leung & Silver (1997) who, among other, investigated the relationship between task format (particularly, whether it contained numerical information) and success in problem solving we decided to offer two different types of initial contexts (see Image 1 in the next section), one containing numerical information and the second not. Additionally, we posed a question to the students, on which of the two contexts was their favourite and why.

Participants and setting of the study

The participants of our study were twenty-two undergraduate students (18 female and four male) of the Department of Mathematics and Natural Sciences in University of Rzeszow, in Poland. The students formed eleven working pairs, with the colleague that was sitting next to them. All of them were at the third year of their studies in mathematics with a specialization in teaching. These students had attended courses in didactics of mathematics for the two previous semesters. That group of students can be characterized as a mixed-ability group, since it contained students with high as well as average or even low marks in their studies. The authors of the paper were both present in the session (which took place during a normal class period), and the second author was known to the students from previous courses. All material was presented to the students in Polish, with the exception of some initial theoretical comments which were presented in English and translated into Polish. Part of that presentation contained a description and examples of the revised Bloom's taxonomy (Anderson & Krathwohl, 2001), with a focus on the higher-order thinking activities: analysing, evaluating and creating. That taxonomy was presented as an aid for posing challenging problems and for evaluating the problems posed by their colleagues. The session lasted two hours, and after the presentation of some theoretical points, a warm-up activity took place, taken from Silver & Cai (1996):

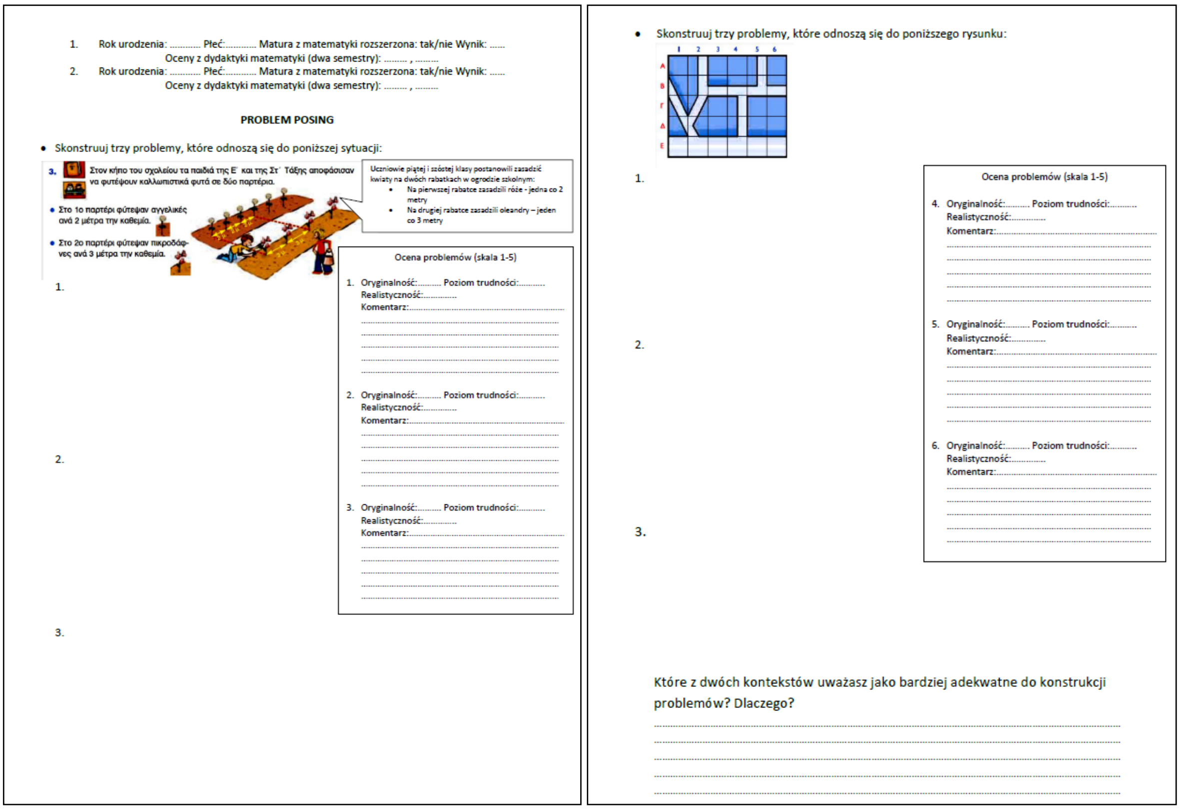
> Write three different questions that can be answered from the information below.
>
> Jerome, Elliot, and Arturo took turns driving home from a trip. Arturo drove 80 miles more than Elliot. Elliot drove twice as many miles as Jerome. Jerome drove 50 miles. (p. 525)

The students were then given the worksheet shown in Image 1. At the top of the first page (shown on the left) the students were requested to fill in some information concerning their age, sex and their marks in mathematics at the GCE exam as well as their marks in their university course of didactics of mathematics. Then we read the following:

> Construct three problems that are related to the image below.
>
> In the school garden the students of the 5th and 6th class have decided to plant flowers in two flower beds.
>
> In the 1st flower bed they planted roses, one every 2 metres.

In the 2nd flower bed they planted oleanders, one every 3 metres.

The same instruction is written in the second page, which contains only an image. Both pages contain a frame, in which the students are asked to put a grade from 1 to 5 for the problems' originality, level (of difficulty) and realistic (character). At the bottom of the second page there is the question: Which of the two contexts do you consider more adequate for constructing problems? Why?



1. Rok urodzenia: .... Płeć:........... Matura z matematyki rozszerzona: tak/nie Wynik: ......
   Oceny z dydaktyki matematyki (dwa semestry): .... , ...
2. Rok urodzenia: .... Płeć:........... Matura z matematyki rozszerzona: tak/nie Wynik: ......
   Oceny z dydaktyki matematyki (dwa semestry): .... , ...

PROBLEM POSING

- Skonstruuj trzy problemy, które odnoszą się do poniższej sytuacji:

3. Στον κήπο του σχολείου τα παιδιά της Ε΄ και της Στ΄ Τάξης αποφάσισαν να φυτέψουν καλλωπιστικά φυτά σε δύο παρτέρια.
- Στο 1ο παρτέρι φύτεψαν αγγελικές ανά 2 μέτρα την καθεμία.
- Στο 2ο παρτέρι φύτεψαν πικροδάφνες ανά 3 μέτρα την καθεμία.

Uczniowie piątej i szóstej klasy postanowili zasadzić kwiaty na dwóch rabatkach w ogrodzie szkolnym:
- Na pierwszej rabatce zasadzili róże - jedna co 2 metry
- Na drugiej rabatce zasadzili oleandry – jeden co 3 metry

1.

2.

3.

Ocena problemów (skala 1-5)

1. Oryginalność:......... Poziom trudności:..........
   Realistyczność:.............
   Komentarz:..........
2. Oryginalność:......... Poziom trudności:..........
   Realistyczność:.............
   Komentarz:..........
3. Oryginalność:......... Poziom trudności:..........
   Realistyczność:.............
   Komentarz:..........

- Skonstruuj trzy problemy, które odnoszą się do poniższego rysunku:

1.

2.

3.

Ocena problemów (skala 1-5)

4. Oryginalność:......... Poziom trudności:..........
   Realistyczność:.............
   Komentarz:..........
5. Oryginalność:......... Poziom trudności:..........
   Realistyczność:.............
   Komentarz:..........
6. Oryginalność:......... Poziom trudności:..........
   Realistyczność:.............
   Komentarz:..........

Które z dwóch kontekstów uważasz jako bardziej adekwatne do konstrukcji problemów? Dlaczego?

Image 1. Worksheet given to the students (front and back page)

Thus, the problem posing situation can be characterised as semi-structured (Stoyanova & Ellerton, 1996), since the students were provided with two contexts. The students were asked to work in pairs in order to:

a) Create three problems for each context, which should be challenging for the solver; we clarified this on the basis of higher-order activities that should be expected by the solver.
b) Solve their problems in a separate piece of paper.

After this phase was completed, we collected the worksheets (without the solutions) and handed them randomly to other pairs, the "evaluators". Although the number of students was small, it was not easy for them to know the poser of the problems they had to solve. After the problems were solved for a second time, we handed the worksheets with the second solutions to the initial posers, and asked them to answer the questions: "Is your solution of your problem the same with the solution of the other pair? If not why?".

Analytical scheme

Due to the variety of data that we collected and the diversity of our research questions, our analysis was characterised by a variety of approaches. The 64 problems that were collected were firstly evaluated by the students and then categorised by us. The students' evaluation of their colleagues consisted mostly of numerical data (since most students had merely put a grade without adding any comments), thus we created some tables which provided us with a first view on how students evaluated their colleagues' problems. At the same time, we performed a qualitative analysis of these evaluations: we looked into each problem (together with its two solutions), solved it and tried to look for consistencies across the students' grades. For example, we examined the problems that had the best grades in realistic character and tried to see whether these problems could indeed be characterised as realistic. For the purpose of the present paper we examined the problems concerning only the perceived level of difficulty and the realistic character. In the next stage, our categories emerged from the problems themselves and in order to construct them we have basically considered the type of operations included in the provided solutions, but also in our own solutions of the problems.

The solvability of the posed problems was not examined only in the sense of a problem being solvable or not (Silver & Cai, 1996), but also on the degree of agreement between the poser's and a colleagues' solutions. Moreover, the posers were asked to comment on the cases of different solutions.

Finally, concerning the role of the context that served as the basis for the problem, we collected the students' answers on the question "Which of the two contexts do you prefer and why?".

RESULTS

All participants showed big willingness to participate in the activity, although, for some of them the time provided was not enough (that is the reason why we have collected 64 instead of 66 problems). From the total of 64 problems which were posed, 55 were solved by the posers and 56 by the evaluators. For Context 1 (flower beds) the most of the problems posed contained questions on quantities, e.g. "How many flowers...?" , "How long is the flower bed...?", "How much money...?", "What is the area of...?", etc. For Context 2 the most striking feature of the problems posed was the clear distinction between "pure" mathematical problems and "realistic" problems; in the first category the problems referred merely to geometrical figures, while in the second category there was a clear attempt on behalf of the posers to include some contextual information that is supposedly related to reality.

Students' evaluation of their colleagues' problems

Concerning the students' evaluation of their colleagues' problems, Table 1 summarises the numerical data that was collected; O stands for originality, L for level of difficulty and R for realistic character. We have to note that the scale provided was from 1 (poor) to 5 (excellent) for every factor.

	Context 1: Flower bed									**Context 2: The map**									
	Problem 1			Problem 2			Problem 3			Problem 1			Problem 2			Problem 3			Average
	O	L	R	O	L	R	O	L	R	O	L	R	O	L	R	O	L	R	
1	3	3	4	4	4	4	-	-	-	1	3	1	2	3	1	-	-	-	2,75
2	1	1	1	2	2	3	4	4	5	5	4	5	5	3	3	5	5	3	3,39
3	3	1	5	3	1	5	5	4	5	5	5	2	3	3	3	2	3	1	3,28
4	2	3	1	5	2	4	5	4	5	3	2	3	1	1	2	3	2	3	2,83
5	2	2	5	4	3	5	3	4	5	2	2	4	2	1	3	3	4	2	3,11
6	2	1	5	2	2	3	4	4	3	1	5	2	1	5	2	5	2	2	2,83
7	5	4	5	4	4	5	4	3	5	3	2	5	5	5	5	4	3	5	4,22
8	2	2	5	3	4	5	1	2	5	2	2	5	2	2	5	1	1	5	3,00
9	5	5	5	5	5	5	5	5	5	3	1	4	3	1	4	3	4	4	4,00
10	3.5	3.5	4.5	4	4	5	4.5	5	3.5	4	3	5	3	3	3	4	3	4	3,86
11	4	3	4	4	4	4	5	5	5	4	4	4	4	3	5	5	5	5	4,28
avg.	Context 1: Flower bed									Context 2: The map									
	Originality			Level			Realistic			Originality			Level			Realistic			
	3.52			3.23			4.34			3.09			2.97			3.44			

Table 1. Students' evaluation of their colleagues' problems.

In the above table we may firstly observe that the students generally evaluated their colleagues' problems rather positively, since the total average of the 64 problems is 3.43. At the same time, we may see that only three pairs have a score of 4.00 or over. Among these pairs, there are cases for which some of their problems have received the best grade from their evaluators. Here are two examples of problems that received the best grade in all three factors (originality, level of difficulty and realistic character). The coding represents the number of group (the letter G followed by a number from 1 to 11), the number of context (1: flower beds, 2 map) followed by the number of problem (1 to 3).

G9 – 1.1: Students are planting flowers alternately (one row of roses and one row of oleanders) in a flower bed with dimensions 14 m × 24 m. The distance between the edges of the flower bed and the first row is equal to 2 m, the distance between the rows is also 2 m. Calculate how many roses and oleanders they need to fill the whole flower bed.

G11 – 2.3: This is a map in a scale 1:2000. It was decided to build a park on the area C5:D6. What are the maximum dimensions it can have if its distance from the path should be 5 m?

The first problem was solved by both pairs by few simple calculations, although the two solutions were different; in both cases though, the solutions were not correct. Image 2 below shows the solution of the posers to their problem:

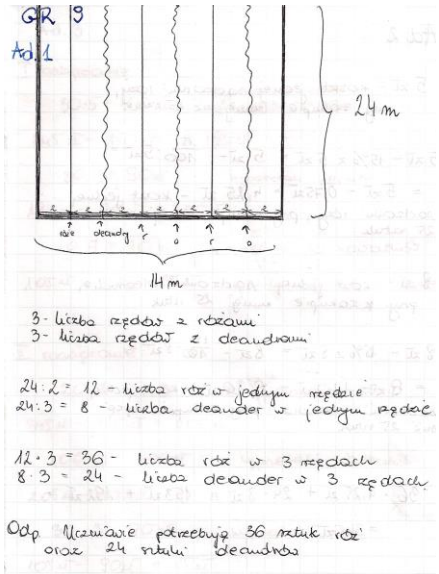


Image 2. Posers' solution of G9 – 1.1

We may note that the students by performing e.g. the division 24: 2 they ignored two facts: first, the fact that there should be a distance of 2 m between the row and the edge of the flower bed and second, the fact that if planting is done in a different way than the one shown, the type of flowers in the first row affects the total number of flowers needed. These considerations render that problem a rather challenging one, although the students perceived it in a different way.

For the second problem there were two unsuccessful attempts for a solution by the evaluators (the posers did not solve it), but the fact that they did not succeed in solving it did not stop them from grading it with three 5's and writing as a comment that it is "a very creative task with a high level of difficulty; the length of the square's side was not given". We regard this as a not so challenging task, which requires by the solver only to make calculations by using the given scale. Moreover, we assume that the use of scale might be the reason for its high score.

The realistic character of the problems was the next factor that we analysed; what follows is a problem that received the best grade in that aspect:

> G7 – 1.1: Kasia plants roses and Ula plants oleanders. We know that you can plant one rose and one oleander in 45 seconds. In what distance from each other the girls will be after one and half hour of work, if they plant in separate flower beds?

The posers solved the problem by using proportions and the evaluators by some simple calculations. What both pairs failed to consider was the time needed for the girls of the story to move from one flower to the other and the possible delay because of fatigue. One might claim that both girls could take a break simultaneously, but we believe that the posers of the problem had not considered such an option. Additionally, the problem received 5 points for its originality, although it does resemble some "traditional" textbook problems which lack of realistic considerations.

Our analysis of the students' problems

In order to investigate the type of problems posed by the students we analysed them according to their mathematical content, and eventually according to the mathematical activities that were involved in the solutions of the posers, the evaluators and our own. In the cases where there were discrepancies between our solutions and the students' ones, we chose to use our solutions for the categorisation. This process has led us to the categories shown in Table 2. The levels shown represent the type and the difficulty of mathematical activities involved.

Level	Context 1	Problems	Context 2	Problems	Sums
1	simple calculations, a number of simple calculations	16	counting , simple calculations based on the image, use of coordinates	4	20
2	calculations of the areas of figures, simple equations, proportions, percentages	8	manipulating areas of figures, calculations of the areas and the perimeters of figures, percentages, possible routes	14	22
3	Pythagorean Theorem, calculations involving multiples and common multiples	1	Pythagorean Theorem, calculations involving scale, estimations	5	6
4	text interpretation, decision making	3	text interpretation, regularities	5	8
5	open problems, considering cases, justifying	4	open problems, considering cases, justifying	4	8

Table 2. Problem categorisation.

Table 2 shows that half of the problems related to Context 1 were categorised in the first level of difficulty, while for Context 2 there are only four problems. Our calculations led to an average of 2.09 for Context 1 (compared to 3.23 in students' evaluation) and 2.72 for Context 2 (compared to 2.97 in students' evaluation), thus generally the problems posed were not perceived as challenging by us as they were by the students-evaluators.

Concerning the levels that were established, the number of operations involved in a solution was not considered by us as a factor that increases the level of a problem (Silver & Cai, 1996). What is important to note is that there were many cases when our perceived levels did not match students' evaluation concerning the level of difficulty:

> G11 – 1.3: The class collected 100 PLN for buying flowers to be planted in the flower beds. One rose costs 5 PLN and one oleander 5 times less. The students want to plant two flower beds with a length of 40 m. Will the money collected be enough for filling both flower beds?
>
> G6 – 2.3: Is it possible to create a quadrilateral by connecting all elements of the image? [*the students refer to the blue figures*]

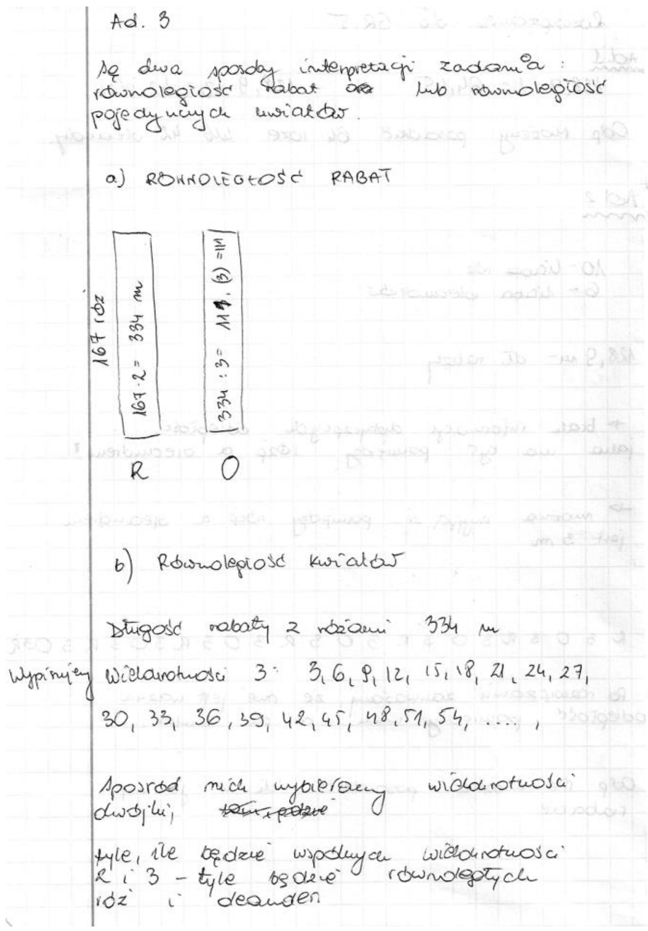
The first problem received 5 points for its level of difficulty by its evaluators, while according to us it belongs to the first level: it can be solved by few simple calculations, thus it cannot qualify as a challenging problem.

The second problem received 2 points for its level of difficulty by its evaluators, while according to us it belongs to the fifth level of difficulty. Its posers did not solve it, while its evaluators wrote the answer: "Yes, you can create a quadrilateral by connecting all elements of the image". We assume that the students answered by mentally making translations of the blue figures, without considering some significant elements of these figures, e.g. their angles. We see this task as involving an investigation of the relationships between the given figures, considering possible cases and mostly, justification of one's opinion.

Finally, there were cases in which even the posers did not grasp the potential of their own problems, at least as it was demonstrated in their solutions. A characteristic one was the following:

> G5 – 1.3: We have 2 parallel flower beds (see the image). We plant the roses on the first one and the oleanders on the second one. If we plant 167 roses then how many oleanders will be parallel to the roses?

The posers solved their problem by performing a division, while the evaluators considered two cases: the first was named "Parallel flower beds" and the second "Parallel flowers". While the first was solved by two operations, the second contains the concepts of multiples and common multiples, as shown in Image 3:


Ad. 3

Są dwa sposoby interpretacji zadania: równoległość rabat lub równoległość pojedynczych kwiatów.

a) RÓWNOLEGŁOŚĆ RABAT

167 róż

167 · 2 = 334 m

334 : 3 = 111,(3) = 111

R O

b) Równoległość kwiatów

Długość rabaty z różami 334 m

Wypiszmy wielokrotności 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, ...,

Spośród nich wybieramy wielokrotności dwójki, ~~[illegible]~~

tyle, ile będzie wspólnych wielokrotności 2 i 3 – tyle będzie równoległych róż i [illegible]

Image 3. Evaluators’ solution of G5 – 1.3

At the bottom of the page we read that “the parallel flowers will be as many as the common multiples of 2 and 3”.

Similarity of provided solutions

In a total of 47 problems that were solved by both the posers and the evaluators, 30 of them had the same solutions and 17 of them had different solutions. The different solutions appeared for two main reasons: mistakes in calculations and different interpretations of the problem, which were mostly due to lack of precision in the formulation of the problem by the posers. However, there were cases, like G5 – 1.3 mentioned before, where the evaluators saw the problem in a deeper manner than its posers intended. A characteristic example of misinterpretation due to lack of precision is the following:

> G3 – 1.1: The students plant flowers on two flower beds. They plant violets every 2 m on the flower bed of dimensions 4 x 6 m and roses every 3 m on the flower bed of dimensions 9 x 5 m. What is the maximum number of flowers they can plant?

The evaluators solved the problem in a simple way by performing few calculations on their drawings. However, the posers in their solution have shown that, e.g. the expression “they plant violets every 2 m” should be interpreted as follows: the distance between each violet and all neighbouring flowers should be

2 m. This interpretation renders the problem a quite challenging one, since the solver is expected to devise a method to solve it (e.g. by drawing circles, which was the posers' suggestion).

Students' preference of context

Seven pairs of students expressed their clear preference for Context 1 (flower beds), three pairs favoured Context 2, and one pair (G2) wrote the following answer:

> We think that context 2 is more adequate for constructing problems. Context 1 contains a small amount of information and because of that you can add some more, which makes the problem more attractive and helps creating an interesting problem. But context 2 doesn't impose anything and you can maximally use creativity in constructing problems. In spite of that we like context 1 more.

Here are some more examples of the justifications provided by the students for their preference for Context 1:

- Related to the real life, you can think up more things.
- It is more adequate for constructing problems because it is directly connected to real life.
- It is more adequate for constructing problems because the answers for the questions require higher mathematical knowledge.
- It gives more possibilities to create problems.

Below are some examples of the justifications provided by the students for their preference for Context 2:

- It doesn't impose anything and you can be more creative in constructing the problems.
- It is more adequate for constructing problems, better activates imagination.
- The situation required bigger creativity, you have to add some information by yourself.
- It was easier to situate it in real life.

Generally, although the majority of students favoured Context 1, the more challenging problems were created for Context 2 (according to our categorisation of the level of difficulty of the problems). Probably, the students felt 'safer' with the "realistic" context but at the same time it somehow limited their creativity, since many problems (especially at the first level of difficulty) were similar to each other.

CONCLUSIONS

Our study has provided us with a rich set of data; but mainly what we have witnessed was students' willingness to participate in an activity which was different that what they usually experienced in their university studies. The main characteristics of that activity were the focus on their own productions (posed problems), the peer evaluation and the possibility for self-reflection on their problems.

Indeed we have observed a great variety in the problems produced, especially concerning their perceived level of difficulty and realistic character. In many cases, students' views of what constitutes a challenging and a realistic problem were different than our views – and in some cases than their colleagues' views. We believe that this is not a surprising finding and it provides us, as teacher educators, with some valuable insights into the students' understandings.

The importance of verbal expression was another issue that came to the fore; most differences in students' solutions of the same problem could be explained usually by the lack of important information or by improper wording. This fact was noticed by some students when they received their colleagues' solutions and they realised that there was information missing from their problem.

The initial context that was used as a basis for the creation of problems seemed to affect the process of creating problems, but not in a straightforward way; what seems to emerge from our data is that the less "structured" situation (in the sense of containing less information or, in other words, less conditions) has proven more fruitful for our students' problem posing.

Summing up, the preservice teachers that participated in the present study had been given an opportunity to be engaged in an authentic mathematical activity and our results have shown that such activities should be provided for all teachers of mathematics.

"NARRATIONE DE RECHERCHE" METHOD IN MATHEMATICAL EDUCATION AT THE SECONDARY SCHOOL LEVEL IN POLAND

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The article describes a research on the implementation of the French method “Narratione de Recherche” in mathematical education on the secondary school level in Poland. In this paper we examine if the secondary school students in Poland are able to use the “Narratione de Recherche” method at all. If yes, we pose another research questions: Is the usage of the method helpful for them and what are its limitations? We formulate our conclusions regarding the selection of tasks appropriate for this method and point out its advantages and disadvantages. We conclude on the basis of the results that the method has proved to be very helpful, especially if the problems given were difficult enough for students. In particular, the method can be widely implemented for underperforming students in mathematics. It makes such students feel contented with the work independently done by them. The method turned to be a strong motivating factor for them and it aroused their creativity.

MOTIVATION TO UNDERTAKE THE RESEARCH AND ITS GENERAL OBJECTIVES

Developing the students’ mathematical activity and finding ways to encourage them to keep undertaking an intellectual effort is still a current challenge for didactics of mathematics. The guiding principle of education is the conscious and active participation of a student in this process. What is more, that rule is treated as an indispensable condition for the implementation of other principles.

Students who achieve worse results in learning mathematics are exposed to failure not only because of their lack of knowledge and mathematical skills. The reason of such a situation is rooted also in their emotions. Students who usually do not succeed in learning mathematics are ashamed of their doubts not being able to distinguish if they are elementary or rooted in the task complexity. They are also afraid of making more mistakes and receiving more criticism by the teacher, and they also want to avoid compromising themselves in the eyes of their peers. It is worth of our care to help the students to overcome such obstacles. For this purpose different methods of communication are established to ensure the comfort of students. Apart from the various traditional methods, there are those which are using new technologies. One of them is e. g. a personal response system (PRS, a voting system), which is designed to provide an anonymous signaling difficulty in understanding the content (e. g. Błasiak, Godlewska, Rosiek, Wcisło, 2012) or other tools to measure psychophysiological data of responders like the eye-tracker (e. g. Błasiak, Godlewska, Rosiek, Wcisło, 2013).

We approach the problem from another point of view. The aim of our research is the verification of the effectiveness of *Narratione de Recherche* method, which has been established on the basis of mathematics education in France. We want to explore if the method may concur to develop students' active and creative attitude towards solving mathematical problems and to shape a positive attitude towards mathematics as a school subject. The first research on *Narratione de Recherche* method in Poland in the context of mathematics teaching was undertook by Czernek – Dąbrowska (2009). The description of the method can be found for example in the article written by Bonafé et al. (2002)[1] and Lubczański (1991).

Our research is an attempt of implementation of that method in the conditions of Polish mathematics education at secondary school. We have posed the following general research questions: Are the Polish secondary school students able to use the method of *Narration de Recherche* at all? If so, is this method useful for them? What are the limitations of this method?

GENESIS OF THE *NARRATIONE DE RECHERCHE* METHOD (NDR)

In the late eighties of the last century the Research Group of Geometry Teaching in *Institut de Recherche sur l'Enseignement des Mathématiques* (IREM) in Montpellier, under the leadership of Gerard Audibert conducted research on new ways of mathematics teaching. In July 1989, Arlette Chevalier on behalf of the Group during the CIEAEM 41 meeting in Brussels presented a new model for working with mathematical problems in classrooms, which has been tentatively named ***Narratione de Recherche*** (we use the abbreviation **NDR**). The results of French experiment were very encouraging for the Group. Pupils taking part in the experiment showed active attitude toward solving mathematical problems and they started achieving higher scores than the others.

The method has been adopted enthusiastically in France and began to be regularly used by a number of teachers, mainly at secondary schools and on university level, but also sometimes at the higher grades of primary school. The method also became the topic of numerous publications and conferences on mathematics education.

[1] We would like to thank a lot to Professor Maciej Klakla for the inspiration to undertake the research problem and for the translation of the article Bonafé (2002) from French to Polish.

THE ASSUMPTIONS AND OBJECTIVES OF *NARRATIONE DE RECHERCHE* METHOD (NDR)

NDR is a method of student's individual work on solving a mathematical problem. It consists of the creation by a student written description, a detailed report of his own thoughts, insights, conclusions and any activities that have been taken by him or her while solving a problem. This method is a kind of record of the mathematical activity undertaken by a solver.

The main objective of the NDR method is to give students a tool for communication, a certain area of freedom, where they can comment on given topics in a natural way, unlimited by mathematical language and restrictive symbols.

In order to use the method students should be able to:

- ask themselves the right questions and conduct an internal monologue to express in words the content, that would normally remain hidden regarded as negligible, and which in fact may be crucial to discover the solution of a problem;
- take a risk and initiate various mathematical activities, even being not sure enough about their correctness;
- be guided by imagination and intuition, to put themselves in the role of explorers;
- admit to their dilemmas and doubts, knowing that wandering is a natural step in finding problem solutions and learning process.

THE ROLE OF THE TEACHER IN *NARRATION DE RECHERCHE* (NDR) METHOD

Teachers who use the NDR method to teach mathematics should encourage students to face the problems without fear nor discourage stemming from the possibility of failure. In this method the role of a teacher is as a guide for students in the process of learning mathematics. It is manifested through convincing students to adopt an open attitude towards problems, to set themselves a variety of questions related to the task and to approach a mathematical problem in various ways, that may - but do not have to - lead to its solution. Teacher's role focuses on providing necessary clarifications and tips to continue work and overcome obstacles.

ADVANTAGES OF *NARRATION DE RECHERCHE* (NDR) METHOD ACCORDING TO ITS AUTHORS

The authors of the paper (Bonafé et al., 2002) formulated three main reasons why NDR method is worth of introducing at mathematics classes.

Stimulation of genuine mathematical activity of students

Students themselves should look for ways to solve a problem, verify them, be able to create their own mathematics and have the opportunity to experience a "glare"; this is what makes this method stimulating the mathematical activity. A student leading an inner monologue is not limited to known algorithms, because his/her own questions and doubts unexpectedly guide him/her on the way for new solutions, new methods of dealing with problematic situations. The student ceases to be impersonator, repeater of well-known schemes or procedures but begins to define the way to get a solution. Besides bringing the narration with himself or herself cannot remain passive, is not able to "switch off", and therefore is forced to demonstrate an interest in solving problem. This method requires student’s full involvement, concentration and exploration of his/her own thoughts deeply enough, to capture fleeting thought processes.

An additional factor stimulating the activity of the students in NDR method is the use of a natural language, which helps to search for the solution of a problem. Expressing of certain hypotheses loudly and intelligibly makes it easier and understandable for students. In addition, the use of everyday language helps them to bring together facts from the world of mathematics, and therefore helps to create a clear image of the issue.

It is very important for the students’ process of discovering to not limit them to mathematical formalism and symbolism and strict rules of deduction but to allow them to express themselves spontaneously. The fact that students are not distracted and have no pressure of correctly formulating their thoughts in mathematical language definitely helps them to catch and write down more proposals and observations. With this freedom of expression, the process of exploration and discovering solutions becomes spontaneous and encourages taking further effort on searching the answer to a problem.

According to Bonafé et al. (2002) the NDR method helps:

- students to understand and experience by first hand what is a genuine mathematical activity,
- to appreciate the phase of heuristic search for the solution to a problem and the ability of detecting new facts and finding relationships between them, and
- to shape a creative attitude towards mathematical problems like in scientists – by experimenting, formulating hypotheses and verifying or proving them.

Individualization of work with a student

NDR method is inherently individualized, directed to the unique thought processes of each student. Thus it is a method highly mobilizing to take his intellectual effort in the process of problem solving. A student expresses his/her thoughts and judgments by recording an inner monologue not as a reviewer and rapporteur solving the problem, but as a searcher, creator and explorer. This helps students to believe in their own abilities and encourages them to undertake further mathematical challenges.

Provoking students reflective attitude towards problems

NDR is a method that requires students to think intensively, to draw conclusions continuously, to be reflective all the time and to control their reasoning. It requires a two-pronged process of thought.

> Chronological recording, as opposed to synthetic research reports, obliges the student, step by step, **to think about their own thinking** and **to become an observer of himself or herself**.
>
> (Bonafé et al., 2002)

On the one hand it is a very personal work model, in which a student is working for himself, without any interference from others. On the other hand, it requires the student to think about his/her approach to the problem, which means that he/she has to be able to express the own findings to other people, to argue his/her views, which demands continual monitoring and analyzing the work.

AIMS OF THE RESEARCH

The aim of our research is to answer the question: How NDR method could be implemented in Polish mathematical education? We want to check how it can be adopted by students, how they will apply it and what will be the results delivered of using this method by Polish youth, which is taught from early childhood education in a completely different way than the French youth.

Moreover, on the basis of the research we are going to answer the following specific questions:

1. Is this method suitable and achievable for every student? Can this method be used by students at different levels of mathematical knowledge and skills?
2. Can we use it to work with any kind of a mathematical task? Are there any restrictions on using this method? What are its advantages and disadvantages?
3. Is the method effective? Is NDR method going to be the authentic facilitation for students, a help in solving mathematical problems?
4. What are the results of the implementation of that method in Poland? Does NDR method have the potential to be used permanently in Polish schools?

DESCRIPTION OF THE RESEARCH GROUP

The research group consisted of six students in the second grade of secondary school (17-18 years old) attending the same class at a school in southern Poland. The students were selected by the mathematics teacher. They had different levels of knowledge and skills. The selected group consisted of:

- two students manifesting troubles in mathematics learning, having problems in mastering of the knowledge and skills described by the minimum requirements of curriculum at the second grade;

- two students of rather ordinary skills in mathematics, having no special involvement during mathematical classes or activities, but not having major difficulties in understanding mathematics;

- two students with mathematical aptitude, easily assimilating new content and willingly deepening their knowledge by solving extracurricular problems.

Such differentiation was important in order to examine how the method was adopted and what results it brought among students representing different levels of knowledge and mathematical skills.

An additional lesson was run before the experiment in order to remind the study participants the most important facts on stereometry. It was done to eliminate, as far as possible, the discomfort of students.

RESEARCH DESIGN

Research was run in two complementary stages:

1. PRELIMINARY STAGE, in which pupils became familiar with the NDR method both in practical and theoretical way,

2. MAIN STAGE, in which students solved individually problems using the newly learned method.

This design of the research is illustrated in the following figure (see fig. 1).

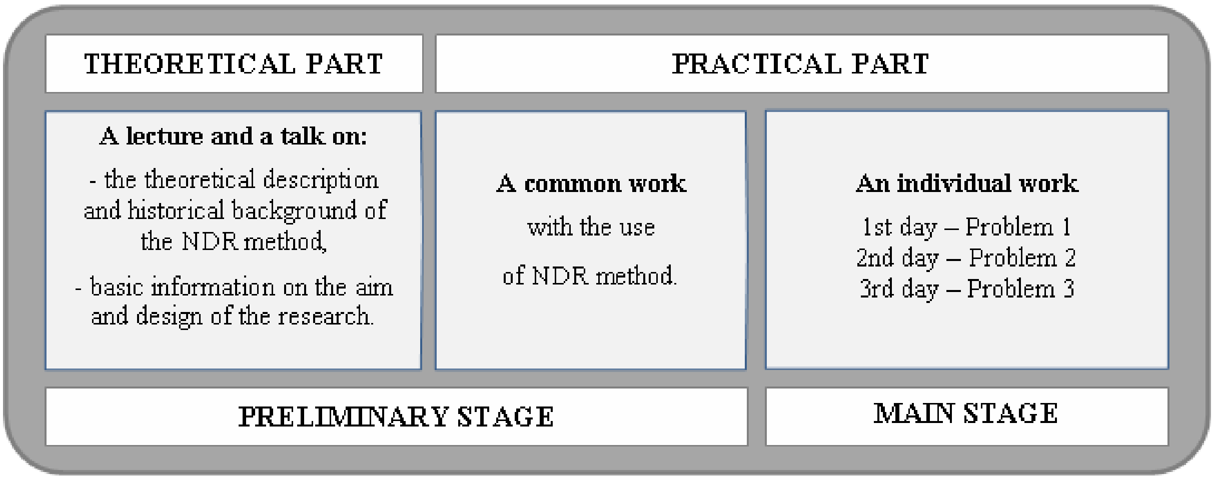

Figure 1. The research design

Preliminary stage consisted of the two following parts:

- THEORETICAL, containing brief historical summary of NDR method and presenting its basic assumptions. This step was aimed to present the method and how it should be used by students. The second aim of this part was making clear to the students what is the purpose of the study, what range of curriculum requirements will be covered, in what timeframe it will be carried out and how the results will be used;

- PRACTICAL, in which a task was solved in common, by the participants, with the use of NDR method. The idea was to check if the theoretical part was understood by learners and to help them develop a model of work by using the newly learned method, to inspire and prepare them for the main part of the research. The students for the first time tried to write together a report of their common mental activity.

REPORT AND THE ANALYSIS OF THE INITIAL STAGE OF RESEARCH

Having familiarized students with the stages of the experiment, a brief historical background and the most important assumptions of NDR method were described. Students learned therefore what the method is and how to use it. An important part of this step was to clarify to the students what is actually expected of them. A particular accent was placed on honesty and courage to describe their thoughts.

Talking to the youth the experimenter tried to convince them to express their own thoughts and insights bravely, even if it meant admitting to erring or going to dead ends. The experimenter tried to convince them to capture their thoughts in a fullest way, even if they appeared just for a moment and were incorrect from mathematical point of view. It was repeated as a mantra that every idea, every product of mind is valuable, regardless of the correctness, and every mistake stands as an important research material, so under no circumstances should be eliminated from their description.

There is no denying that the hardest part of this stage consisted in convincing the group to stop thinking about errors as a failure. On the faces of all the respondents was visible some kind of confusion; it seemed that even thinking about such a description aroused their anxiety. This one aspect of the method seemed very unnatural, since it was even contrary to the instinct of “self-preservation” at school.

Two days later the practical part of the initial stage of research was carried out. Prior to solving the problem we agreed that the handwritten notes of the common activity will be placed on the blackboard, so that everyone could at any time return to the previous notes. Once again, the experimenter asked the participants to loudly speak out their thoughts and doubts.

The following Initial Problem was given to the whole group of participants:

Initial Problem

Four steel balls with a radius of 3 cm are melted down and shaped into a cylinder with a base radius of 2 cm. Calculate the height of that solid.

The first struggle of students with the method was difficult. Below we present a complete record of their formulations, as they were written on the blackboard, which were an attempt of their own mathematical activity description. It is rather a record of brainstorm, but it gave students the taste of what would be the main part of the research, where they would have work independently.

- The first thing I see is that the three balls will have the same mass as the cylinder, because it was created by melting.

- Here we have volume rather, and mass is probably not the same as volume - although in our task it is probably the same.

- We have solids so surely we have to count area or volume.

- OK, so let’s settle what we know: we have three balls whose volume is the same as the volume of the cylinder. Let's calculate how much it is, then we should be able to calculate its height... (after a long moment staring at the problem formulation) if there is not missing some data.

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi 3^3$$

$$V = \frac{4}{3}\cdot 27\pi$$

$$V = 36\pi$$

- I don’t like this π!

- It is rather not going to be a problem. Then it should be reduced and if not we will approximate it by 3,14…
- Let's go on… we have got already the volume of the cylinder – and now just an equation and we have got the solution…
- Here we have "V" of one ball – this we have to multiply by 3

$V_{cylinder} = 3 \cdot V$

$V_{cylinder} = 3 \cdot 36\pi$

$V_{cylinder} = 108\pi$

- Ok! And now the height!
- Wait for a moment. We have an error here… We had four balls and not three… the radius was 3

(an experimenter asked to not delete the improper calculations)

$V_{cylinder} = 4 \cdot 36\pi$

$V_{cylinder} = 144\pi$

- We can calculate the volume by using the ordinary formula: $V_{cylinder} = \pi r^2 \cdot h$, and because this is the same figure so we can compare it : $\pi r^2 \cdot h = 144\pi$

-We know the " r" and the π will be reduced.

$r = 2$

$\pi r^2 \cdot h = 144\pi$

$\pi \cdot 4 \cdot h = 144\pi \quad / : \pi$

$4 \cdot h = 144 \quad / : 4$

$h = 36$

- And the answer: The height of the cylinder is equal to 36 cm.

The task was fairly easy for the students, so they formulated fake comments. In case of doubt they preferred to gently pass by the issue to wait for its clarification. It was seen that students were afraid to deduce specific requests. When they made a mistake resulting from a careless reading of the tasks formulation it could be noted that they glanced uncertainly at the experimenter. At that moment the researcher reiterated the request to not delete any incorrect writing.

They explained that it is such "a crevice on the glass" for them - nothing significant, but very irritating and spoiling - in their opinion - the effect of the resolved problem. Here we can observe the power of the habits and the impact of education based on reproducing mechanical schemes, schemes finely structured and faultless, duplicated according to formulas, lacking individuality and creative activity, but safe and reliable.

This stage turned out to be difficult. Students learned that NDR method, which at the previous meeting was perceived by them as a straightforward one, only seemed to be simple. For the first time they had to be faced not only with the solution of the task, but also with the chronological organizing their thoughts and their writing. They noted that its application requires a lot of involvement and self-control.

The first impressions of the respondents just after the first use of the NDR method, are well described by the words of one student:

> ...At a certain point I started to be afraid that thinking on thinking will make the solution of the problem impossible. What is more, we worked together in the group! I cannot see using it alone - I can feel that it will be very poorly ....

RESEARCH TOOLS OF THE MAIN STAGE OF THE RESEARCH

The research tools used in the research were selected to not to force respondents to demonstrate complex knowledge, formulas and complicated mathematical relationships. Their goal was to encourage them to mathematical activity, to allow them focusing on the thought process. The necessity of remembering formulas and complex facts could distract respondents from the main tenets of the method, which is analyzing and recording the thought process. To prevent students from this they received extra help – a list of stereometric formulas.

Below we present the formulation of the research tools used within the main stage of the research.

Problem 1

On the adjacent plots Mr. Nowak and Mr. Kowalski have been building houses of different shapes of roofs (see figure 2). Which of the men will have to purchase a greater number of tiles to cover the roof of his newly built house?

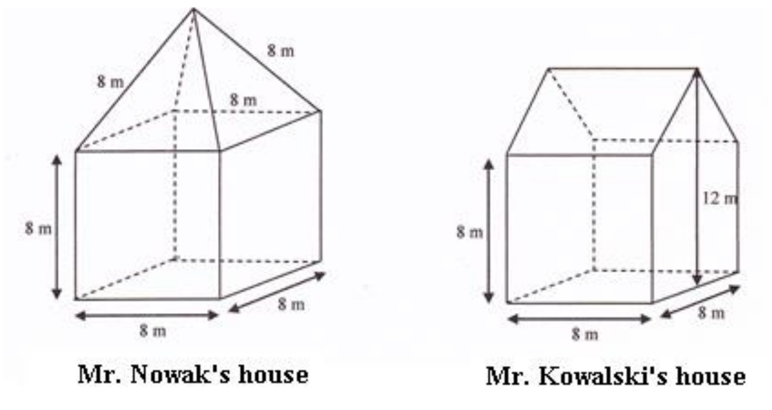

Figure 2. Shapes of houses built by Mr. Nowak and Mr. Kowalski (Problem 1)

Problem 2

a) How many oranges with a diameter of 6 cm need to be squeezed out to completely fill 6 glasses in the shape of a cylinder with the juice? We know the

internal dimensions of glasses: 8 cm of height and 5 cm of bottom diameter. We assume that all oranges are ideally spherical, and their juice content is 60%.

b) How many wine glasses we can completely fill with the same amount of juice, if the bowl has the shape of a cone having a base diameter of 5 cm and the height three times greater than the height of the cylinder?

Problem 3

From 1331 wooden cube blocks with the edge of 2 cm was composed a large cube. The obtained solid was painted in blue, and then decomposed into single blocks again. Calculate how many blocks were obtained, whose walls were not painted at all.

The main stage of the research was held at the week following the initial part. It was carried out at regular intervals, with a one-day break, namely on Monday, Wednesday and Friday. Each day the students worked on one problem.

THE GENERAL RESULTS OF THE MAIN STAGE OF THE RESEARCH AND ITS ANALYSIS

To be able to assess the correctness of the provided by students writings we singled out for each problem the necessary steps to obtain a solution. We decided for the correct answer of each step one point, and zero for wrong answer or the lack of it. If a student at some stage made a mistake, but successive operations performed correctly he or she was given 1 point from the further stages. Following this criterion students' writings were assessed from the mathematical correctness point of view.

To better demonstrate the difficulty of the problems for students, we calculated the easiness coefficient of each stage of solution for every task and for every student. A size of the coefficients allows drawing conclusions about the difficulties caused for examined students at the various stages. The easiness coefficient of a task represents the number of points obtained by the respondents divided by the number of points possible to obtain. We define the level of tasks' difficulty using classification according to the Table 1.

The easiness coefficient of a task	**0,00 – 0,19**	**0,20 – 0,49**	**0,50 – 0,69**	**0,70 – 0,89**	**0,90 – 1,00**
	Very difficult	Difficult	Mildly difficult	Easy	Very easy

Table 1: The easiness coefficient of a task.

The efficiency of Problem 1 was 64%, which means that the task was for participants mildly difficult. Problem 2 proved to be for them a very simple task with the easiness coefficient of 0,98, while Problem 3 efficiency was 10%, which makes this in the category of very difficult tasks.

CONCLUSIONS

NDR method versus problems selection

NDR method was not liked by all participants. The tasks proved to be too simple for some students, too obvious. The application of the method is possible and reasonable when the tasks to be solved are not trivial. Too simple problems do not enforce a creative attitude; they only require imitation and involve the usage of the already known procedures. The preparation of the report on the process of reasoning in this case is practically not feasible. Browsing the students' writings we can conclude that they made many mistakes due to a misinterpretation of the formulation of problems or drawings with the additional data. Therefore, we believe that the level of prepared tasks was not too low.

NDR method versus mathematical skills

A very interesting relationship can be observed while analyzing the writings. Namely the reports of the students who usually receive good and average results in mathematics were less interesting in the context of the examined method than the reports delivered by students of weaker skills. The talented students had difficulty and resistance to the idea of the method. Admitting a mistake or a spontaneous relation blocked them. After talking to the students we found out that they felt fear. They were much worried of losing the image of a good student who always achieves high results or, in other words, they felt that their positive face was threatened (e.g. Tatsis, 2013). They did not understand that searching for the solution of a problem is the most natural process of learning mathematics, and there is no reason to be ashamed of being lost at this stage.

These students tried to put comments in the report containing unfortunately only explanations "for us", only in order to satisfy our desire to know their way of thinking. Thus their thought process has remained hidden for us. These students' work was focused on the final score and not on revealing the activity process. In their opinion they did not want to waste time and energy for unnecessary activities, which did not give them measurable benefits i.e. points. They did not take into account the fact that such a trial could provide them a hint and a long-term profit.

The NDR method was much more effective for students who achieve the weaker results in mathematics. They were much more open. We had an impression that they did not care so much to keep some semblance. Without resistance they confessed difficulties. Their statements were spontaneous. Their thoughts were formulated straight-forwardly and not in beautifully arranged sentences, sometimes they wrote a bit chaotically.

At the end of the research all students were asked to express their opinions on the NDR method. Their views were rather favourable, but among students achieving good marks in mathematics a kind of reserve could be felt. Below we cite examples of such statements:

> I personally like this method but only when solving difficult problems. If I had an extensively described command and a problem that requires thought and precise calculations I would use this method. But it will be only for my comfort and to facilitate resolving the task.

> I think that this method takes a lot of time, but by the fact that you have to write everything in details it is much easier for me to check the results or find a mistake. The method is not complicated, and if only I had more time to solve a problem I would use it.

Opinions of students were divided to the question of whether they are going to apply the method while solving problems in the future. The more talented students argued that will try, only when the problem is really difficult. Their declarations were not entirely convincing - it seemed that they were rather polite and not willing to use that method again.

Instead the students who usually manifest troubles in mathematics learning were very enthusiastic about the NDR method. During the talks they said for example: *I felt very well not giving back an empty worksheet*. They felt satisfaction with the effort done, and the written down report *was the evidence that the thought process really took place in their minds*. This was a motivating factor for them. As they claimed: *since this way is a dead-end I will look for another one*. By saving their thoughts they sometimes unexpectedly dropped in ever newer ideas.

They claimed that by working in a traditional way they would not have even started solving such a problem. But by using this method, without feeling the pressure of obtaining the solution causes that they not only started solving the problem but also started, surprisingly for them, noticing things that could be helpful. This experience has allowed them to draw the conclusion like that the following:

> ... I will come back to this method, because as I somehow hit on an idea maybe it allows me to get some points. And if not then at least I will gain satisfaction from the fact that I tried.

These students enthusiastically assured that will use this method as much as possible in the future. The confirmation of what may be the following statements:

> I noticed that by analyzing and writing the contents of the task, step by step, I began understand something of it. For sure I'll use this method!

> In my opinion, this is a very facilitating method. By using it you can quickly understand the issue. When you itemize everything by yourself you can get a better view of what to do. The only downside is that you need to write a lot and during exams there is always not enough time. But surely I will yet go back to it.

Advantages of NDR method

On the basis of our research it can be concluded that the NDR method:

- conduces to developing an open attitude of the learner, open to discovery, to search for solutions,
- induces and develops a creative attitude of students towards mathematical problems,
- incentives to take efforts, attempts to solve, eliminates to some extent the discouragement before solving a difficult problem, which can be at the first glance not enough understandable and whose solution requires an atypical approach,
- organizes thoughts, allows to return to the previous considerations, thus avoiding ineffective duplication of proceeding and wandering over and over the same paths.

Disadvantages of NDR method

The research has shown that NDR method, as each teaching method, has disadvantages, which include:

- very extensive descriptions,
- a huge time-consuming,
- the need for increased concentration, necessity to be able to note at least selected thoughts, though capturing the majority of them is, *de facto*, impossible,
- *metacognitive shift*, i. e. the need to continually redirect attention from the mathematical problem to your own thinking process and to control writing.

SUMMARY

The research has shown that the NDR method can be and should be used in Polish schools, as a complementary teaching and learning method. It allows developing students' creative attitude towards mathematical problems, raises their motivation for learning and animates imagination. The appropriate selection of active teaching methods, such as the NDR, provides a valuable aid for mathematics teachers. The NDR method is particularly helpful in the work with students achieving weaker educational outcomes and for everybody but working only with suitably challenging and problematic tasks.

K-CHILDREN PLAYING TANGRAM IN TWO ENVIRONMENTS: A TOOL FOR LEARNING SHAPES

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The issue of early childhood mathematics has recently come to the fore, together with the importance of teaching geometrical concepts and reasoning from a young age. This paper explores how an intervention using a tangram-based puzzle game in technological and tangible environments affects children's concepts of triangles. The results suggest that in general tangrams can be used as a tool for learning geometry. Children can form different geometrical shapes using tangram pieces. Moreover, playing in collaboration with a knowledgeable adult may promote the level of children's geometrical thinking (Van Hiele levels).

INTRODUCTION

Over the last decade interest in mathematics education for preschool children has grown. This is evidenced by governmental efforts to introduce a curriculum for young children (e.g., EYFS, 2008). In addition, the mathematics education research community has become involved in research efforts geared at this specific age group (e.g., the POME[2] conference and the 2013 special issue edited by Krummheuer). In the US, for example, the National Association for the Education of Young Children (NAEYC) and the National Council of Teachers of Mathematics (NCTM) jointly published a position statement advocating "high quality, challenging, and accessible mathematics education for 3- to 6-year-old children" (NAEYC & NCTM, 2002, p.1). This statement affirms that high-quality, challenging, and accessible mathematics education for preschool children is a vital foundation for future school success (NAEYC & NCTM, 2002). The curriculum focal points for preschool mathematics in the US specifically mention that children should be able to identify and describe a variety of shapes presented in a variety of ways and to use geometrical concepts (NCTM, 2000). Since then, many national, state, and local organizations in the US have embraced this new vision. The Practice Guidance for the Early Years Foundation Stage framework in England suggests ways of fostering mathematical knowledge among children aged 0–5 years. This framework aims at providing useful advice and detailed information to support

[2] http://cermat.org/poem2012/

children's learning and development and examines how to implement the learning and development requirements in more detail (EYFS, 2008).

In the mandatory mathematics curriculum for pre-school children issued by the Israeli Ministry of Education, geometry is one of three major knowledge domains for kindergarten children (Israel National Mathematics Preschool Curriculum, INMPC, 2008). This preschool mathematics curriculum explicitly refers to the ability to identify shapes (for example, triangles). The curriculum suggests that in order to develop children's mathematical understanding and thinking skills, one should ask the children to explain their actions. These explanations enable children to justify their actions and at the same time allow the teacher to better understand what the children meant.

Thus, in this study we examine how kindergarten children learn about triangles in two environments by means of a tangram-based puzzle game. We focus on preschool children between the ages of 3 and 5. The main question we ask is: To what extent can a tangram-based puzzle game promote geometrical thinking among k-children?

THEORETICAL BACKGROUND

In this section we outline the theoretical framework underlying this study. First, we describe the Israeli mathematics curriculum for pre-school children, which is built upon Vygotsky's theory. Next, we refer to the Van Hiele (1986) model of children's geometrical thinking. Finally, we survey some studies carried out with this age group.

Learning according to Vygotsky's theory

In Israel, the Ministry of Education issued a mandatory mathematics curriculum for pre-school kindergartens (2008). This curriculum is explicitly based on Vygotsky's theory (1978). A major theme in Vygotsky's theoretical framework is that social interaction plays a fundamental role in the development of cognition. Vygotsky (1978) states: "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological) and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relationships between individuals" (p. 57). According to Vygotsky's theory, the social environment is important and the role of the adult is to help children tackle challenges that are just a bit beyond what they could accomplish alone. That is, while working with an adult, children are working in what Vygotsky termed *the zone of proximal development*. Hence children are given an opportunity to perform at levels they cannot achieve on their own.

Levels of Geometrical Thinking

Understanding how children's geometrical thinking develops is essential for the effective teaching of geometry, which is a major branch of mathematics. One of the most important studies of children's geometrical thinking was conducted by Van Hiele (1986). Van Hiele (1986) proposed a five-level model describing levels of geometrical thinking in geometry. These hierarchical levels are the product of experience and instruction: visualization, analysis, informal deduction, formal deduction, and rigor. On the first (visual) level, learners visually recognize shapes and figures by their global appearance. For example, learners recognize triangles by their shape, but they do not explicitly identify the properties of these figures. At the second (analysis) level, learners start analyzing the properties of figures and learn the appropriate technical terminology for describing them, but they do not interrelate figures or properties of figures. (For example, a triangle is a figure with three sides, but the child may not relate this to the property of having three vertices.) At the third (informal deduction) level, learners can identify relationships between classes of figures and discover properties of classes of figures by simple logical deduction. (For example, a square is considered a rectangle because it has all the properties of a rectangle.) The other two levels are beyond the scope of this study.

Clements, Sarama and DiBiase (2004) proposed a different hierarchical model for geometrical thinking with the following levels: pre-recognition, visual and analysis. It should be noted that the first and second levels in the model proposed by Clements et al. are included in the first level defined by Van Hiele, which is a visualization level.

Research on k-children's concept of triangles

Young children's conceptions of shapes have become the focus of attention in several countries. Maier and Benz (2012), for example, compared the abilities of 6-year-old English and German children to name shapes correctly. They found that at the beginning of first grade 60% of the English children were able to use the accepted geometric terms for two-dimensional shapes and explain their reasoning formally. Their German peers did not use these names and explained their reasoning informally. These differences were explained by the children's experience. Sinclair and Moss (2012) studied a class of 22 children ranging in age from 4 to 5. A dynamic geometry environment that included a teacher's computer and a data projector was used to discuss the following question: what is a triangle? The researchers found that the environment's dragging mode enabled the children to advance their thinking regarding triangles, as can be deduced from analyzing the transcript of the whole-class discussion. These are only two examples of recent research efforts in the domain of shape identification and geometrical reasoning.

METHOD

Participants and procedure

The research participants comprised 23 kindergarten children aged three to five. All were native Arabic speakers living in the center of the country. The study took place at an early childhood academic community center.[3] The parents and the children gave their consent to participate in the study.

The research design included two intervention groups and two control groups. All the participants underwent pre- and post-intervention interviews, each 15 minutes in duration. In addition, children from the intervention groups underwent three semi-structured interviews of 15 minutes each. One intervention group and one control group were interviewed in a computerized environment, while the other two groups were interviewed in a tangible environment (see Table 1). All the interviews with all the children were videotaped and transcribed.

Group	Computerized Intervention	Computerized Control	Tangible Intervention	Tangible Control
Pre-interview (15 min.)	+	+	+	+
Semi-structured interview (15 min.)	Tangram game	--	Tangram game	--
Post-interview (15 min.)	+	+	+	+

Table 1: Research procedure

Tools

The pre- and post-intervention interviews included the same three tasks. The first task involved sorting shapes into examples and non-examples of triangles. The researcher asked each child to put the triangles on one side and the other shapes on the other side. This task included 16 shapes (Figure 1), chosen to provide a range of intuitive and non-intuitive examples and non-examples of triangles (Levenson, Tsamir, & Tirosh, 2008). After the child indicated a particular shape as an example or non-example of a triangle, the interviewer asked: How do you know that this shape is a triangle (or not a triangle)?

[3] Website Center: http://bidayat.qsm.ac.il/English/Default.aspx

In the second task the child was asked to complete a house-shaped puzzle using given shapes. The researcher asked the child to try and cover the house-shaped puzzle in different ways by using different shapes. This task had three different solutions. The third task involved completing a parallelogram-shaped puzzle using given shapes. As in the second task, the researcher asked the child to try to cover the parallelogram-shaped puzzle in different ways by using different shapes. This task had four different solutions.

The semi-structured interviews took place during the tangram-based puzzle game.

A tangram is a well-known Chinese puzzle that consists of seven geometric shapes, called *tans*: a square, a parallelogram, and five right-angle triangles – two big ones, a medium-sized one, and two small ones. The *tans* can be fit together in various ways to form polygons or a variety of figures, such as birds and animals. The children were given a series of figures and the *tans* and were asked to use the *tans* to assemble puzzles in computerized or tangible environments. While the child played, the interviewer mediated the child's actions by discussing the names of the figures and their properties.

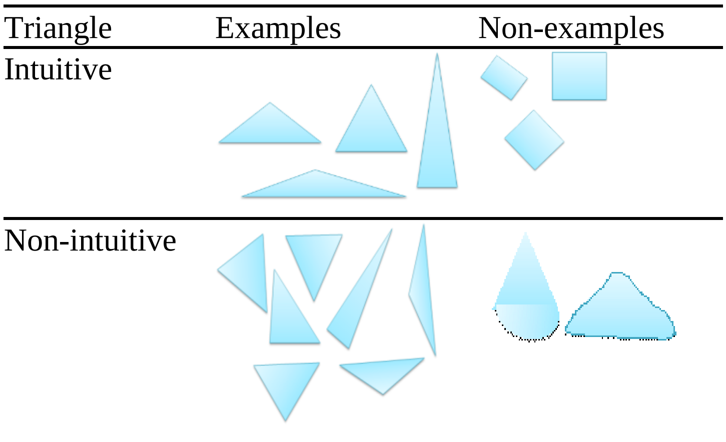

Figure 1: Shapes presented for the first task

Data analysis

For the first task, shape identification during the pre- and post- intervention interviews was coded as either correct or incorrect. In addition, the children's answers regarding their reasoning were analyzed to identify which of the Van Hiele levels this reasoning reflected. For example, utterances such as "because it looks like a triangle" were categorized as first level, while utterances such as "it has three sides" were categorized as second level. For the second and third interview tasks, coding involved counting the number of solutions for each puzzle.

FINDINGS

In this section we describe the results of the pre- and post-interviews. We begin by presenting the overall results for all of the groups. After that we describe certain results for one child from the computerized intervention environment as an illustrative case study.

Results of pre- and post- intervention interviews

The analysis reveals two major findings regarding the differences in children's answers between the pre- and post-intervention interviews. First, among the children in the intervention groups a change occurred in their identification of the triangles and in their reasoning about them (based on Van Hiele levels), as compared to their peers. Second, the results were similar for all the interventions groups. Both intervention groups achieved progress, but it is not possible to determine the significance of this progress due to the small sample.

Table 2 depicts the children's identification of examples and non-examples of triangles in the first pre-intervention task. Almost all the children correctly identified the triangles. For all the groups, identifying intuitive figures was indeed more successful than identifying non-intuitive figures. Yet when the researcher asked the children: "How do you know that this shape is a triangle?" a typical answer was: "I see it." Such an answer was identified as being on the first Van Hiele level - the visualization level. On the second task, only seven children were not able to provide any solution to the house-shaped puzzle, while the others provided at least one solution. The third task was more complex, as 13 children were not able to find a solution for it.

		Tangible Control N=7	Tangible Intervention N=4	Computerized Control N=4	Computerized Intervention N=8
Triangles	Intuitive	M=100	M=100	M=100	M=75
	Non-intuitive	M=84.3	M=94.6	M=87.5	M=73.4
Non-triangles	Intuitive	M=100	M=100	M=75	M=79.3
	Non-intuitive	M=75	M=71.4	M=62.5	M=56.5

Table 2: Means for correct identification of shapes (first task in pre- interview)

During the post-intervention interview, children from the two intervention groups changed their reasoning regarding shape identification (the first task). This time, a typical reason given for identifying a triangle was that it has three vertices and three straight lines, or that it has three vertices and three sides. This reasoning was identified as the second Van Hiele level – the analysis level. Children from the two other groups did not exhibit such a change in reasoning. In addition, we observed a change in correct identification of shapes compared with the pre-intervention interview results (see Table 3).

Tangible	Tangible	Computerized	Computerized

	Control N=7*	Intervention N=4	Control N=4	Intervention N=8
General score pre-intervention	89	93.7	84.3	72.2
General score post-intervention	95.3	97.3	78.1	90.62

Table 3: Correct identification of triangles on the first task, pre- and post- interviews.

* One child asked to stop playing and we of course respected his request.

During the post-intervention interview, most of the children in the intervention groups completed the puzzles in the second and third tasks in three different ways, forming three solutions (see Tables 4-5). A similar finding was not observed among the children in the control groups.

Number of solutions	Tangible				Computerized			
	Control N=4		Intervention N=7		Control N=4		Intervention N=8	
	%	No.	%	No.	%	No.	%	No.
Zero	50	2	0	0	25	1	0	0
One	0	0	0	0	0	0	0	0
Two	25	1	0	0	50	2	0	0
Three	25	1	100	7	25	1	100	8

Table 4: Correct house-shaped puzzle solutions for post- interview

Number of solutions	Tangible				Computerized			
	Control N=4		Intervention N=7		Control N=4		Intervention N=8	
	%	No.	%	No.	%	No.	%	No.
Zero	100	4	0	0	75	3	0	0
One	0	0	14.2	1	0	0	12.5	1
Two	0	0	0	0	25	1	0	0
Three	0	0	85.7	6	0	0	87.5	7

Table 5: Correct parallelogram-shaped puzzle solutions for post- interview

Clearly, three short interventions of 15 minutes each had an impact on the children's solutions and on their reasoning. To understand this change in the level of children's thinking according to the framework proposed by Van Hiele, we now qualitatively examine the recorded dialogue between the researcher and one of the children.

The case of Eli:

Eli is a 4-year-old boy. We chose to focus on his discussion with the researcher because his performance differed considerably between the pre- and post-intervention interviews. Also, he was one of the more fluent children, a somewhat unique characteristic in this age group. Episode 1 is a typical interaction during the pre-intervention interview.

Interviewer:	What is special about all the shapes that you put on the left?
Eli:	They are all triangles.
Interviewer:	Why are they triangles?
Eli:	(Does not answer.)

Episode 1: Eli and the researcher in the pre-interview session

After the pre-intervention interview, Eli and the researcher met for three separate semi-structured interviews in which the researcher mediated the learning experience while Eli solved a tangram-based puzzle game in a computerized environment. The researcher explained the properties of a triangle to Eli, as can be seen in Episode 2.

Interviewer:	How do you know that this shape is a triangle?
Eli:	According to its color. It is colored red.
Interviewer:	Are triangles only red shapes?
Eli:	No, this is a triangle, and this is a triangle (pointing to others triangles in the puzzle).
Interviewer:	Right. But I want to tell you a secret about triangles. We call it a triangle because it has three sides. Triangle means three. Three vertices. This is the first, and this is the second, and this is the third. One, two, and three (points to each of the vertex points). Please point to the vertices of the triangle.
Eli:	This, and this, and this (points with his hand to the vertices of the red triangle).
Interviewer:	In addition, a triangle has three sides. This is the first side, and this is the second side, and this is the third side. One, two, three. Please point to the three sides of the triangle.
Eli:	This is one, and this is one, and this is one (pointing with his hand to the sides of the triangle sides).
Interviewer:	Now let's look at another triangle. Let’s take the green triangle. Please point to the vertices and the sides.
Eli:	One, two, and three (pointing to the vertices).
Interviewer:	And where are the sides?
Eli:	This and this and this.
Interviewer:	And how many sides does a triangle have altogether?
Eli:	One, two, and three.

Interviewer: Correct. Three vertices and three sides. And so this shape is called a triangle because it has three sides and three vertices.

Episode 2: Eli and the researcher in the first semi-structured interview

The following short discussion from the third semi-structured interview demonstrates the change in Eli's geometrical thinking. The next puzzle (Figure 2) is composed of four small triangles, two squares, and a large triangle. Eli assembled two small triangles and two squares in the right places. The following discussion (Episode 3) then took place between Eli and the researcher.

Eli continues to assemble the puzzle, and solves it by finding the vertices and sometimes by finding the sides of the shapes. At times he turns the shapes around or straightens them in order to fit into the puzzle, until he finishes assembling the puzzle.

Interviewer: How did you locate the place of the triangle in the puzzle?

Eli: According to its sides.

Interviewer: And what else does a triangle have?

Eli: Three vertices.

Episode 3: Eli and the researcher discussing Fig. 2

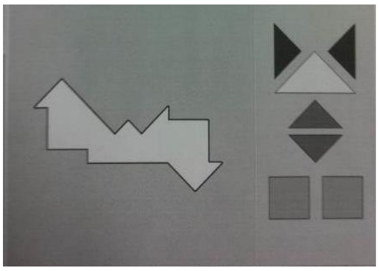

Figure 2: Puzzle No. 14

DISCUSSION

The aim of this study was to determine how an intervention affects the way in which kindergarten children learn about triangles in technological and tangible environments through the use of a tangram-based puzzle game. In general, we found that the way kindergarten children learn about triangles in the two environments was similar. Nevertheless, a marked difference was noted among the children in the intervention group between their pre-intervention interview and post- intervention interview reasoning in identifying triangles.

When it came to identifying examples and non-examples of triangles during the pre-intervention interviews, the identification of children from all the groups was relatively good. Yet their explanations of the identifications reflected Van Hiele's first thinking level – the visualization level. This identification was followed by three mediated learning interviews (Feuerstein, Klein &

Tannenbaum, 1999) that included playing with *tans* to assemble puzzles, along with discussion between the researcher and the child using mathematical language. In the post-intervention interviews, all the children from the two intervention groups (computerized and tangible) were able to justify their identification of examples and non-examples of triangles. The justifications reflected a shift in the children's geometrical thinking to the second Van Hiele level (1986). Such a change was not observed among the children in the control groups.

We believe that the change seen among the children in the intervention groups can be explained by their semi-structured interviews with the researcher. During the interviews, the interviewer exposed the child to the correct mathematical language describing the properties of a triangle. She also asked the children to explain their actions. In so doing, she gave the children an opportunity to explicitly express their reasoning. This type of explanation allowed the researcher to evaluate the children's level of thinking about triangles. She encouraged the children to think about the properties of the shapes. By doing so, she collaborated with the children in their zone of proximal development (Vygotsky, 1978).

The K-teacher is the adult responsible for children's development in general and for their mathematical thinking in particular. By providing similar 45 minute experiences and mediation to k-children, she can promote their level of geometrical thinking. Also, by pointing to a variety of examples and non-examples of triangles (Levenson, Tsamir, & Tirosh, 2008), the k-teacher can broaden children's concepts of triangles.

The researchers recommend conducting a larger-scale research study using a tangram-based puzzle game to investigate the effect of different environments on the development of kindergarten children's thinking. It is also recommended that kindergarten teachers receive some training in this domain.

TYPICAL MISTAKES MADE IN A CLASSROOM WHILE TEACHING PROBABILITY AND STATISTICS

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The research contains a description and systematization of the most typical mistakes in representing facts and methods of the probability theory and statistics with examples of such mistakes taken from teachers' publications, lesson materials for Moscow schools, and experience of vocational training for math teachers in the MCCME and Moscow Institute of Open Education from 2005. In addition we provide examples from similar courses in USA and Japan based on conferences and research made by APEC in the area of mathematical education and training of teachers of mathematics in 2011–2013.

INTRODUCTION

In Russia the probability theory and statistics are gradually included into school curriculums since 2004 with accordance to the educational standard. Since 2012 problems on the probability theory and statistics are included into the unified state exam. Chapters on the probability theory appear as attachments to mathematical textbooks since 2002. The first regular textbook for 7–9 grades was published in 2004, and for high schools in 2014. Since 2005 regular retraining of math teachers on the subject "The probability theory and statistics" takes place. During last years a vast experience of training is accumulated, and beside evident progress one also can see tendencies of mistakes in perception of probability concepts by math teachers, who then transmit wrong understanding to students, slowing growth of the probabilistic culture of the population.

The aim of this research is to determine and systemize typical mistakes that happen in the process of teaching the probability theory and statistics in school. The research is based on studying publications of math teachers, learning materials used in classrooms in schools, and experience obtained during vocational training and retraining of teachers in the Moscow Centre for Continuous Mathematical Education and the Moscow Institute for Open Education since 2005. The paper contains descriptions of the most typical mistakes with examples. Besides we provide examples from similar practices in USA taken from APEC materials and surveys in the area of mathematical education and preparation of teachers of mathematics in 2011–2013.

Our research results in outlining and systematizing the most common misconceptions.

- Not understanding the nature of impossible and certain events as subsets of a random experiment; mixing of events in common and formal sense.
- Overestimating the role of combinatorial calculus.
- Poor understanding of probabilistic background for construction of forecasts and forecast intervals. Mechanistic approach to usage of characteristics of random phenomena.
- Lack of understanding of significance of the probability theory in applications to quantitative design tasks in mass consumption services, security, economics and social events.
- Non critical use of algorithms that does not consider specific properties of data, goal of research and character of obtained results.

The second goal of our research is to develop approaches for teacher preparation in the area of the probability theory and statistics that take into account the found misconceptions. This will help to overcome difficulties in the interpretation of probabilistic methods in the daily work of a teacher.

THE PLACE OF PROBABILITY AND STATISTICS IN THE SYSTEM OF MATHEMATICAL KNOWLEGE OF SCHOOL STUDENTS

We often hear from mathematicians that the probability theory and statistics are too difficult, and therefore should not be included in school curriculums. This opinion grows out of complications that follow studying the probability theory in universities, which traditionally is deductive and based on combinatorics and wide knowledge of calculus. A mechanistic approach to the probability theory is typical for many generations of our tertiary education. In fact, combinatorics and calculus are not directly related to base ideas of statistics and the probability. They are just a way to enumerate elements of vast probabilistic spaces and prove theorems. The probability theory is secondary. Experience and intuition are primary, and no theory is useful without them. One should meaningfully consider chances of events, especially unlikely ones that play significant role in daily life.

The problem is that events are less obvious than figures or numbers, while concepts of chance and volatility are not as intuitive as length, area or volume. An event and its chances make special types of imaginary objects and their formalization into mathematical notions is much more complex than formalization of a picture (transition to geometry) or quantity (to arithmetic or algebra).

The second problem is that the majority of children until a certain age are alien to the concept of volatility and instability of events. At what age a child is ready to perceive changeable models and which models should they be, is still to discover. In the Soviet period the science of the laws of cognitive activity was destroyed [P1991]. Its place was taken by the paradigm of “a clear sheet of paper”. But if in the early childhood the rejection of variability possibly serves as a defensive mechanism which simplifies adaptation to social and natural conditions minimizing the number of necessary rules of behavior, then later an absence of general ideas of randomness and volatility start to hinder in life. Formal study of statistics in adulthood does not improve this situation (Kakihana,Watanabe, 2013).

Because of all that, the conceptual distortions of perception of stochastic events are common for grown-ups as well as for school children. Trying to conserve a rigorous algorithmic approach and simplicity of teaching, teachers create mistakes and whole incorrect concepts while teaching probability and statistics.

Since 2003 the international assessment research PISA includes questions that check knowledge of probability and statistics. According to the data published by the Centre of Assessment of Quality of Teaching (www.centeroko.ru), Russian school students showed low results on this section (Kovaleva, 2007).

In Makarov, A.A., Tyurin, Y.N., Vysotsky, I.R., & Yaschenko, I.V. (2009), Vysotsky, I.R., & Yaschenko, I.V. (2014) authors note that school education in Russia traditionally is aimed at developing ideas about rigid connections between events of the surrounding world. As it goes, the school mathematics remains a conservative subject. On the contrary, statistics and probability can provide clear applications of mathematics to changing events of a day. The goal of scientists, trainers and teachers is not to abandon statistics and probability, but to free them from redundant technical layers and create a coherent school subject. The first thing to take care of is to make basic concepts of probability and statistics clear and familiar to math teachers who have difficulties in transition from teaching abstract facts of math to application of mathematical concepts and laws to solving practical problems.

Complications that arise in teaching statistics and probability theory are common not only for Russia. In many countries (USA, Great Britain, Switzerland, South Korea, Japan etc.) where statistics and the probability theory stay in school curriculums for long time, misunderstanding of methods and misinterpretation of facts often happen (Isoda M., Gonzales O., 2012).

All said above motivated us to research the accumulated experience with the aim to determine the most common misconceptions and mistakes in teaching of probability and statistics. Below we analyze them and suggest applicable methodological solutions of appearing problems.

ANALYSIS OF TYPICAL MISTAKES AND MISCONCEPTIONS

1. Impossible, certain and random events.

Usually a random event is treated as an event that may or may not happen as an outcome of some experiment. It creates a false opposition between random events and impossible and certain events. Below see a typical example from a textbook.

Determine which events are certain, which are impossible and which are random:

A. During next year there will be snow in Moscow.

B. A football game between two teams will end as a dead heat.

C. You will receive a talking crocodile as a birthday present.

D. There will be a math midterm tomorrow.

E. On February 30th it will be raining.

First, here is a terminological mistake: random events are considered as opposite to impossible or certain. This approach will lead us into a serious trouble when a union or an intersection of two random events may be a not random event.

Example. *When tossing a coin, events* $A = \{Head\}$ *and* $B = \{Tail\}$ *certainly are random, but their union is certain, while their intersection is impossible.*

Second, statements *A–D* are not related to any random experiment, but are "common events". Such approach is destructive for further formalization of random events. Therefore the second mistake is mixing notions of "events" in usual sense and of random events from a random experiment.

In a random experiment a set of elementary events must be determined. All subsets of this set are random events. Among them there are *certain* and *impossible* events. The certain event consists of all elementary events of the experiment, therefore it will happen for sure and has probability 1. But it does not cease being random because of this. The impossible event does not contain elementary events at all. It is an empty set. It will not happen for sure and has probability 0. But it still is random. There also exist events with undetermined probability. One can call them uncertain, or leave them without a special name.

We can somehow reformulate events *A* and *B* from the list above, and we rather not speak about statements *C*, *D*, and *E*, because *C* and *D* require determining too many conditions of the experiment. The probability of the event *E* cannot be found, because *E* corresponds to an impossible experiment.

2. What if a coin falls at its edge?

Answering an old student joke "And what if a coin falls on its edge?" teachers often enter long and fruitless discussions. Here a misunderstanding appears again. In fact, the coin does not fall on its edge. The reason is not that "we do not consider" such possibility, there is nothing we can consider or not consider: the outcome "Edge" is not contained in the experiment. What to say to a student? In the experiment "A toss of a coin" the situation "Edge" is not an event at all. By definition, the experiment "A toss of a coin" contains exactly two elementary outcomes: "Head" and "Tail", and there are no other elementary outcomes. Uniting two possible outcomes, it is not possible to obtain the event "Edge". Therefore the coin will not fall on its edge, will not stay hanging in the air and will not roll into a floor crack.

In real life, when we toss a coin, we model a random experiment using available tools. A real coin can roll under the skirting, and in this case we must admit that our model of a random experiment failed and toss the coin one more time.

3. Why do people need the probability theory?

Especially, why do students need it? The usual answer to this question is: "probabilistic and statistical methods nowadays deeply permeated into applications: they are used in physics, engineering, economics, biology and medicine. Their role especially increased with development of computers". This is a very good observation.

How to tell the truth in a not very complicated way? Recall the problem about gloves.

A drawer contains 20 left gloves and 20 right gloves. How many gloves one needs to take out of the drawer not looking into it in order to be certainly able to make a pair of gloves?

Deterministic mathematics claims that to take out 21 gloves is certainly enough. This is right, but...what will the probability theory say? After taking out one glove, the probability of taking out four more gloves for the same hand is

$$\frac{19}{39}\cdot\frac{18}{38}\cdot\frac{17}{37}\cdot\frac{16}{36}\approx 0.047 .$$

So the probability to obtain a pair when we randomly take out five gloves is 0.95. It is almost certain. If you are ready to believe in it, please take out five gloves instead of 21. And if you are not ready, take out… six. The theory of probability practically guarantees the success. This case is funny and more a game than a problem, but here we provide serious tasks that might be solved:

- *Determine the minimal amount of medical supplies for a city such that will be certainly enough in the case of a natural disaster typical for the given region.*
- *How many portions of chicken or fish should be prepared for a flight meal that will almost certainly guarantee absence of displeased passengers?*
- *How many employees should be in a bank office in order to guarantee that queues of customers will happen very rarely?*

- *How much cash should be loaded into an ATM machine that it almost certainly will be enough for a workday?*
- *Determine a minimal insurance payment that an insurance company needs in order to almost certainly get a calculated profit.*

Instead of determined but meaningless answers the probability theory often gives reasonable and practically certain answers.

4. Overestimating the combinatorics.

One common wrong tendency is that many teachers overestimate the role of combinatorics when teaching the probability theory. Often teachers formally state facts and formulas from combinatorics and then suggest problems that contain the word "probability" as examples to apply the formulas.

Example. *Pinocchio has 4 silver coins and 2 gold coins in his right pocket. Not looking he moves three of these coins to his left pocket. Find the probability of the fact that both gold coins are now in the same pocket.*

This problem appeared in different formulations in many textbooks and exams and provoked a lot of discussions. As we found, many specialists consider it as a combinatorial problem with the following solution.

The total number of combinations of three out of six coins is $\binom{6}{3}$. *One can choose two gold coins out of two and one silver out of four and place them into the left pocket by* $\binom{2}{2}\cdot\binom{4}{1}\cdot$ *different ways. The number of ways to place chosen coins into the right pocket is the same. So one gets:*

$$p=\frac{2\cdot\binom{2}{2}\cdot\binom{4}{1}\cdot}{\binom{6}{3}\cdot}=\frac{2\cdot 1\cdot 4}{20}=\frac{2}{5}.$$

This is a typical manifestation of the common problem: in universities the probability theory is often taught as an application of combinatorics and this approach is being projected on schools. Let us give an appropriate solution.

Assign to the gold coins numbers 1 and 2. Suppose that the coin #1 comes to some pocket. Two randomly chosen coins out of five remaining might get to that pocket. Hence the probability for placing the coin #2 to the same pocket equals $\frac{2}{5}$.

It is important to note that the absence of combinatorics does not narrow the area of problems. The combinatorics should play an auxiliary role in the probability theory; it is needed when vast probability spaces come into play.

5. Forecasting.

Here is another textbook problem.

Two companies assemble computers. The first company uses cheap details whose probability to break is 2.4%. The second company uses details of good quality. Their probability of malfunction is 0.6%. The first company had sold 6000 computers, second 22500 computers. Which company will receive more complaints about quality of their production?

***Solution**. Let x_1 be the number of malfunctioning computers made by 1st company, x_2 by 2nd company.*

Then $\frac{x_1}{6000} = 0.024;\ \frac{x_2}{22500} = 0.006$. *So* $x_1 = 144;\ x_2 = 135,\quad x_1 > x_2$.

One can see some carelessness in posing the problem. But even disregarding the absence of rigorousness in the condition, and leaving only the essence of the problem, one must admit that this problem and its solution are not stated correctly. The correct answer to this problem in the form in which it was posed can be only one: it is unknown which company will receive more complaints.

Where is the mistake? Authors of the problem use mechanistic approach and consider mean values of quantities as if they were certain values. 144 and 135 are not x_1 and x_2, but their mathematical expectations, i.e. theoretical mean values. It is more probable that the 1st company will receive more complaints. Our intuition suggests that and it can be proven. But intuition also says that averages 144 and 135 are close enough, while distributions of both quantities are large, therefore in reality 2nd company can obtain more complaints. Calculations show that the probability of event $x_1 > x_2$ is 0.696, of $x_1 < x_2$ is 0.283 and there also exists the event $x_1 = x_2$ with probability 0.021. One can discard the event $x_1 = x_2$ as unlikely, but the event $x_1 < x_2$ (2nd company gets more complaints) is quite possible. One cannot state with assurance that the first company will receive more complaints.

6. A volume of a random sample.

From a paper written by a math teacher and included into a textbook published by the institute of development of education of Yakutia we took a lesson about sample surveys. Here we provide its fragment omitting unimportant moments.

There is a difficulty assessing the level of knowledge for 9th grade students in some region of Russia, say, in the Sakha republic (Yakutia). It is obvious that for valid results of such assessment the participation of 9th grade students from the whole region is necessary. Assume that we gave a test that consists of six problems and calculated how many students solved correctly some number of problems. ... In order to obtain completely reliable information on the subject it is enough to make a sample survey and check the knowledge level of a relatively small part of students. In such situations usually 5 – 10 % of all target population is measured. Let the number of 9th grade students in some small rural area be 716. Then choose randomly from them 50 students (6,9 %).

Where is the mistake here? In real life in order to achieve a valid result while studying a vast target group, specialists take a sample of some determined volume that does not depend on the number of members in the target group. It is not correct to measure the volume of a sample as a portion of such group. If there are 716 students in the region, then one needs a sample with volume much bigger than 50 people. But if we consider, say, 70000 students, then it is enough to survey 1600 of them (for the given accessible error and reliability), not bothering about what part of the target group it makes.

AND WHAT IS GOING ON IN OTHER COUNTRIES?

Above we already said that in many countries the probability theory and statistics were included into school curriculums for a long time. But misunderstanding of methods and misinterpretation of facts of probability also often happen. Usually, and one can see it in textbooks, the study is skewed toward the application techniques of probabilistic methods without understanding the essence of a problem. Let us give two examples from a large set.

7. **What is more probable?**

A working group of an APEC project created a test to check readiness of students of last years of study to be math teachers. In the section “The probability theory and statistics” this problem was offered.

A basketball player claims that he hits the basket in average 70 times out of 100. His trainer wants to check it and finds out that in 100 throws the player hits the basket only 55 times. What is more probable: that the player overestimates his achievements and 55 hits is his usual result, or that he says the truth but has no luck today?

Solution provided by authors. *Assume that the probability to hit the basket is 0.7. Then the standard deviation of the number of hits for 100 throws is* $\sqrt{100 \cdot 0.7 \cdot 0.3} = 4.58$.

Consider the most probable event “55 hits or less happened while the average number of hits is 70”. By the Central Limit Theorem, this probability is approximately equal to the probability of the fact that the random variable ξ*, which has standard normal distribution, does not exceed* $\frac{55-70}{4.58} = -3.27$.

Now assume that the real average is 55, i.e. the probability of a hit in reality is close to 55. Then the standard deviation is $\sqrt{100 \cdot 0.55 \cdot 0.45} = 4.97$. *The probability of the event “the average is 55 and 70 hits or more happened in 100 throws” is close to the probability of a random variable* ξ *to be not smaller than* $\frac{70-55}{4.97} = 3.02$.

Obviously, $P(\xi \le -3.27) < P(\xi \ge 3.02)$. *Therefore the event "the number of hits is not smaller than 70 if the average is 55" is more probable than "the number of hits is not more than 55 if the average is 70". It looks like the player lies.*

A ready recipe has been applied to a more or less natural problem. It seems that everything is fine. Is a mistake here? No. Here one can see a profound misunderstanding of obtained results. Why? Let us find these probabilities:

$$P(\xi \le -3.27) = 0.00054, \qquad P(\xi \ge 3.02) = 0.00126.$$

So the comparison does not make sense. It is the same as seriously consider which probability is bigger: to meet a wild penguin or a wandering elephant when walking on streets of London.

8. Amy prepares for an exam.

Here is one more problem from the same source.

Students in your class got a problem: Amy prepares for a math exam and passes a test each week. Her results (in percents of the maximal grade) are shown in the table.

Week	1	2	3	4	5	6
Grade, %	38	63	67	75	71	2

A. Construct the dispersion diagram using data from the table.

B. Determine the best model that describes how Amy's grades change in time.

One student suggested a linear model, another – quadratic model, third – logarithmic model. Which answer is correct? How to convince other students that their models do not work?

***Suggested solution**. A. Considering the distribution diagram, one can see that the logarithmic model fits because it follows the shape of the diagram. It seems that Amy's grades grow more and more slowly with time.*

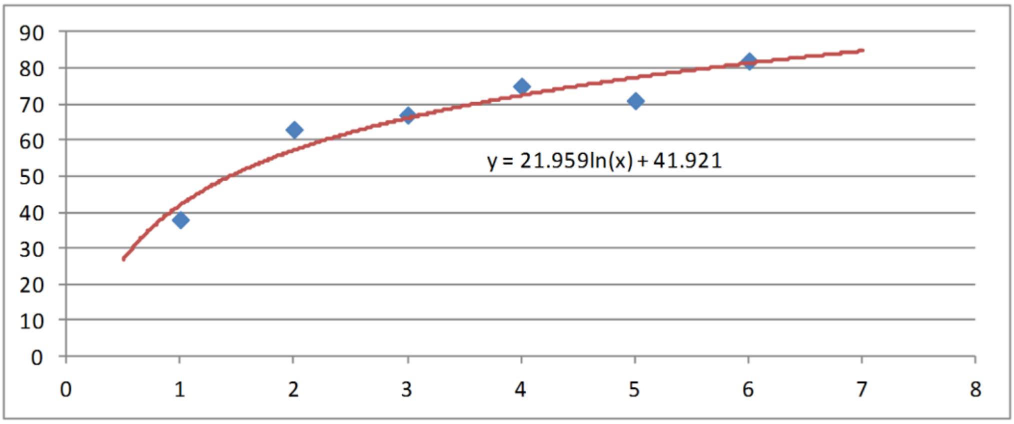

Figure 1

B.The linear model does not work here, because it predicts that soon Amy's grades will exceed 100%, which cannot be. The logarithmic model also shows that grades will exceed 100%, but much, much later; one can think that Amy will finish her

> *preparation before that. In the quadratic model the corresponding quadratic polynomial should have a negative leading coefficient, in which case grades will decrease from some time moment and reach negative values, which is unlikely.*

What is wrong? Authors of the problem did not say anything about the desirable properties of the model in question: amount of difference between theoretical and observed values, continuity, possibility to extrapolate etc. If none of properties is mentioned, then the best model is the exact model, for example, the table itself. Or one can construct Lagrange polynomial that will pass through all points of the diagram. There exist many good and different models.

CONCLUSIONS

Authors studied the most general mistakes that appear during teaching in schools, processing data obtained from Russian schools and from some foreign sources. We had found that typical mistakes come from the deficiency or absence of traditions of teaching stochastic in schools. As the experience shows, *direct adaptation of university courses on probability and statistics to school curriculums does not produce desired results but often leads to vast misconceptions*. Despite the increase in volume of good quality textbooks, educational and methodical literature we observe stable existing mistakes in teaching. Frequency of these mistakes increases with growth of the number of school teachers who specialize in the probability theory and statistics.

Once the most common mistakes based on misconceptions of random events, wrong approach to the interpretation of random characteristics, not critical use of algorithms are found and determined, *it would not be difficult to develop compensating techniques aimed on correcting said misconceptions*. Recently such work takes place as a part of teachers' vocational courses in many regions of Russian Federation. Authors came to the conclusion about necessity of vast increase of publications on the considered area of problems in methodological literature for teachers.

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. Thinking processes related to mathematical discourse

Part 4

LANGUAGE IN CHANGE: HOW WE CHANGED THE LANGUAGE OF MATHEMATICS AND HOW THE LANGUAGE OF MATHEMATICS CHANGED US

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Mathematics is usually understood as the language of science. It is seen as a tool by means of which disciplines like physics or economics reach their exactness. We are therefore often not paying attention to the fact that mathematics itself has its own linguistic dimension: the same mathematical content can be expressed in many different ways. These different ways of expression come clearly to the fore in the study of the historical development of a particular mathematical discipline. In texts that come from the past the same or very similar mathematical contents are usually formulated in strikingly different ways. We see that the language in which the texts are coined changed. But what exactly can change in the language of such an exact discipline as mathematics?

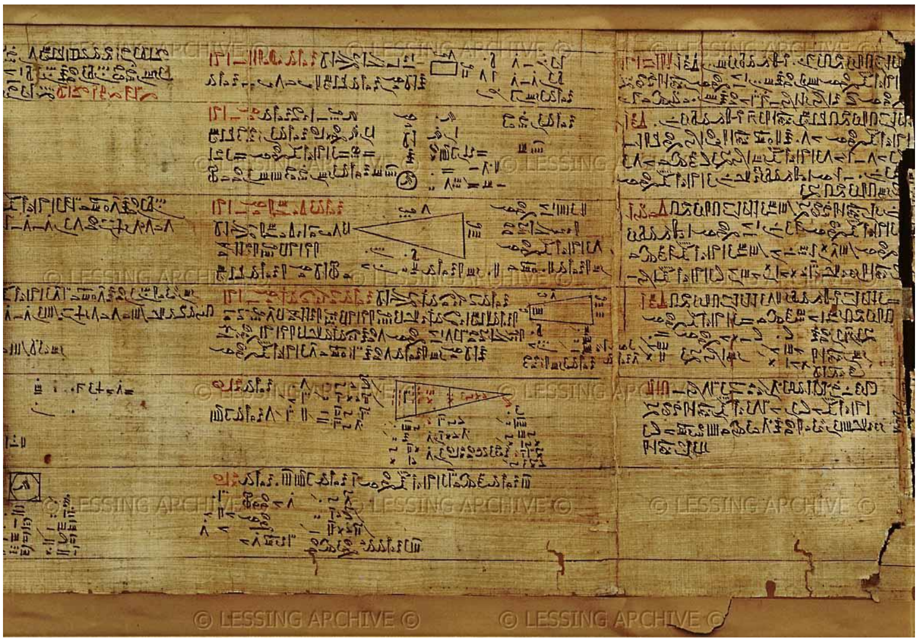
To see the linguistic changes more clearly it is useful to look at one of the oldest surviving mathematical texts—the *Rhind Papyrus*. If we compare this text with our contemporary mathematical practice, we will notice several deep differences. In order to capture them in a systematic way, we can introduce six important features of the language of mathematics—*the logical power*, *the expressive power, the methodological power*, *the integrative power, the explanatory,* and *the metaphorical power*. These features are not taken at random. On the contrary, they characterize one important pattern of change that occurred several times in the development of mathematics.

In the course of the history of mathematics we witness a gradual *increase of logical power* (we can prove much stronger theorems), *increase of expressive power* (we can study more complex phenomena), *increase of methodical power* (we develop stronger analytic methods), *increase of integrative power* (our theories display deeper unity), *increase of explanatory power* (we can explain our previous failures), and *increase of metaphorical power* (we can apply the language to unforeseen situations). In order to obtain a better understanding of these changes, I will illustrate the increase of the six features on the transition from the synthetic geometry of the ancient Greeks to the algebra of the Arabic and early modern period.

Nevertheless, as philosophers we cannot be satisfied by discovering of some regularities in the development of mathematics. Our task is to try to understand how these regularities are constituted. For each of the six features mentioned above we shall therefore try to identify the particular linguistic innovation by means of which it was formed. Thus we will try to find an explanation of how the language of mathematics achieved its *logical power*, *expressive power*, *methodical power*, etc. It will turn out that, due to the strict requirements on logical consistency and exactness, there are very limited possibilities how mathematicians can change the language of their discipline. An identification of them enables us to understand the sources of the huge impact which mathematics had on western culture in general, and on science in particular.

1 The Rhind papyrus

The *Rhind papyrus* is one of the oldest surviving mathematical texts.



The manuscript that is kept in the British Museum in London was written in the 18th century B.C. and it is a copy of an even older original. Let us take the following problem from it:

‘Find the volume of a cylindrical granary of diameter 10 and height 10.

Take away 1/9 of 10, namely 1 1/9;

the remainder is 8 2/3 1/6 1/18.

Multiply 8 2/3 1/6 1/18 times 8 2/3 1/6 1/18;

it makes 79 1/108 1/324 times 10;

it makes 790 1/18 1/27 1/54 1/81 cubed cubits.

Add 1/2 of it to it;

it makes 1185 1/6 1/54, its contents in khar.

1/20 of this is 59 1/4 1/108.

59 1/4 1/108 times 100 hekat of grain will go into it.'

(Fauvel and Gray 1987, p. 18)

This text is fascinating—it is so different from anything to what we are used in mathematics today. The text is so incredibly old, and still when we read it we clearly recognize it as a mathematical text. It poses a problem and solves it—just like any mathematical texts does. But at the same time, despite this fundamental similarity we have also a strange feeling. This feeling is an important clue because a feeling is a spontaneous reaction of the mind. As philosophers we have to try to articulate, analyze and understand our feelings. So let us try to describe as precisely as we can, in what respect does this text from ancient Egypt differ from our contemporary mathematical practice.

Several historians of mathematics tried to articulate this difference. For instance Jeremy Gray wrote:

> "However, the differences between Greek and Babylonian or Egyptian mathematics of around 300 BC are manifest. The Greeks were doing geometry, they were proving things, their methods were deductive and there are signs of a lively interest in questions of rigor and logical validity. The Babylonians, on the other hand, had procedures but **no proofs**." (Gray 1979, p. 3 – stress added)

"*For these reasons rhetorical algebra* [Babylonian mathematics] *is without proofs and can accommodate different and* ***incompatible*** *answers."* (p. 5) "*The disadvantages of rhetorical algebra are that it is difficult to think in it for an extended period, that it is* ***non-explanatory****, and that it even contains contradictory estimates of areas and volumes*." (p. 14) But where the historian stops, the philosopher begins. We have to reach a conceptual understanding of these differences, i.e. to create concepts that would enable us to understand their nature and origin. So let us start with a systematization of the differences. Gray mentioned three of them, but some important differences have to be added.

First of all the text **lacks any trace of a proof**. In contemporary mathematics we prove every theorem that we state. For us mathematics is to a great extent synonymous to proving. The author of the Rhind papyrus, on the other hand, does not make any attempt to prove the correctness of his solution. He simply states it. The second sense in which the text differs from our practice is that it **lacks generality**. In mathematics we try to present our results in maximal possible generality, to state them for the broadest possible range of values. The above text, on the contrary, presents the solution of the problem only for particular values of the parameters, without any attempt to generalize it.

The third important difference is the ad-hoc nature of the solution—the text **lacks any method**. In contemporary mathematics we are concerned with developing and standardizing methods, by means of which we can solve widest possible classes of problems. Here we cannot decipher any trace of this concern.

The solution of the problem is presented without any methodical concern. A fourth difference consists in how the papyrus is organized. The problems **lack any integration** into a system. In contemporary mathematics we classify the problems that we can solve into a system in order to see the effectiveness of the different methods of solution. In the papyrus, on the other hand, we see no such integration. The problems are listed, one after the other, without any attempt to classify them. The results of different problems are sometimes even incompatible.

The fifth difference can be seen in the total **lack of explanation**. In contemporary mathematics we are interested first of all in understanding why things are as they are. We are not satisfied with a solution until we understand why it works. In the papyrus, on the other hand, no attempt of any explanation is made. We are not told why the steps of the calculation are taken. They are simply made, without any explanatory comment. The sixth difference, to which I would like to draw attention, is the **lack of metaphoric meaning**. The language of modern mathematics, despite its exactness and logical strictness, uses many metaphors. The number zero is considered an even number; straight lines are considered as going to infinity; negative numbers are considered to have square roots. These and many other metaphoric uses give the language of mathematics its flexibility. The Egyptian text, on the other hand, does not make used of metaphors. It is straightforward; the words mean what they stand for.

These six points, or at least some of them, concern not only the Rhind papyrus, but also many other historical texts. Perhaps some other differences could be found but I will end my list here. The six features mentioned above were discovered in the course of a systematic analysis and comparisons of several texts from the history of mathematics. I have chosen the papyrus only because, due to its age, the six features are particularly conspicuous. Nevertheless, they characterize rather well ancient mathematical text. The old texts usually lack, or at least display to a much lesser degree, the six above features. Of course, it is not them, who lack these features; it is rather us, who gained them.

2 Potentialities of the language of mathematics

After systematization we can turn to the second step, which consists in trying to understand the source of this lack of *proof*, lack of *generality*, lack of *explanation*, etc. in the old texts. After a while we will probably realize that it is their language. The language in which the older texts are formulated simply does not make it possible to prove, to generalize, or to explain in the way as we are used to do today. So we can interpret the above-mentioned differences between our and the Egyptian mathematical practice, as differences in the ***potentialities of the language of mathematics***. By turning to language we can reach a conceptual understanding of the historical phenomena mentioned above.

One possible way to describe the development of mathematics is to describe the development of its language. We can characterize the language of mathematics of a particular period of time by means of the six features mentioned above. These features are not arbitrarily chosen; on the contrary, I believe, that they represent six basic properties of the language of mathematics. Taken together they unequivocally characterize the language of mathematics of a particular historical period. They are:

1. ***Logical power*** – how complex formulas can be proven in the language**,**

2. ***Expressive power*** – what new things can the language express, which were inexpressible in the previous stages,

3. ***Methodical power*** – which methods enables us the language to introduce there, where on the previous stages we saw only several unrelated tricks

4. ***Integrative power*** – what sort of unity and order the language enables us to see there, where we perceived just unrelated particular cases in the previous stages,

5. ***Explanatory power*** – how the language can explain the failures which occurred in the previous stages,

6. ***Metaphorical power*** – which shows how the language, by transgressing the rules of its own syntax, can create analogies for situations that defy normal expression

It is important to realize that these features are fully objective attributes of a particular language. They are not related to the psychology of mathematical discovery or to the sociology of the linguistic community. Independently of how was the language discovered and for what purposes it is being used, the question "which propositions can be proven", "which things can be expressed", "which problems can be solved", etc. is an objective property of the historically existing language itself. The language has its syntactic and semantic rules and these rules determine what we can prove, express, or solve.

The evolution of mathematics consists in the growth of the *logical* and *expressive power* of its language—the later stages of development of the language make it possible to prove stronger theorems and to describe wider range of phenomena. The *methodical* and the *integrative power* also gradually increase—the later stages of development of the language enable us to solve wider classes of problems by means of standard methods and offer a more unified view of its subject matter. The *explanatory* and the *metaphorical power* of the language also increase—the later stages of the development of mathematics offer a deeper understanding of its methods and provide us with powerful metaphors that shed light on new situations. I call the six features as "***potentialities of the language of mathematics***", because they express different things that the language enables us to do.

We can illustrate the growth of the *logical*, *expressive*, *methodical, integrative, explanatory,* and *metaphorical power* of the language of mathematics on the case algebra. It is worth mentioning, that two of these potentialities, the *methodical* and the *metaphorical power* are not discussed in the *Patterns of Change*. They present a further step in the development of this theory.

3 The example of algebra

In order to see more clearly that the six potentialities are indeed characteristics of the language of mathematics it is useful to examine in detail one particular example—the example of algebra. On the one hand, algebra is sufficiently complex to offer interesting illustrations of the six potentialities. On the other hand, at least substantial parts of algebra still belong to elementary mathematics, and so the illustrations are sufficiently simple to be accessible to the general public.

Algebra is a creation of the Arabs. The ancient Greeks did not develop the algebraic way of thinking. This does not mean that they could not solve mathematical problems which we today call algebraic. They translated them into a geometric setting and then solved them by means of a geometric construction. The ancient Greeks excluded calculative recipes from mathematics and reduced almost the whole of mathematics to geometry. So they lost contact with symbolic thinking and calculative manipulations. This reduction of algebraic problems to geometry has a fundamental disadvantage. The second power of the unknown is in geometry represented by a square and so quadratic equations have to be treated as relations among known and unknown areas. The third power of the unknown is represented by a cube and so cubic equations correspond to relations among volumes. But for higher powers of the unknown there is no geometric representation. So the language of synthetic geometry made it possible to grasp only a small fragment of the realm of algebraic problems, a fragment that was perhaps too narrow to stimulate the creation of an independent mathematical discipline.

The first who entered the land of algebra were the Arabs. There is no doubt that they learned from the Greeks what is a proof, what is a definition, what is an axiom. But the Arabic culture was very different from the Greek one. Its center was Islam, a religion that denied that transcendence could be grounded in the metaphor of sight. Therefore the close connection between knowledge and sight, which formed the Greek *epistéme*, was lost. The Greek word *theoria* is derived from *theoros*, which was a delegate of the polis who had to oversee a religious ceremony without taking part in it. A *theoria* was what the *theoros* saw. This means that for the Greeks a theory is what we see when we observe the course of events without participating in them. Thus the metaphor behind the Greek notion of theoretical knowledge was *view from a distance*. According to this metaphor, in order to get insight into a (mathematical) problem one has to separate oneself from everything that bounds one to the problem and could disturb the impartiality of his view. This shows that algebra with its manipulations was alien to Greek thought. Algebraic manipulations are based not on insight. Algebraic thought is always in flux; its transformations are incessantly proceeding. When Arabic algebra reached Europe, a dialogue started. It was a dialogue between the western spirit and a fundamentally different but equally deep spirit of algebra. This dialogue was led by the effort to visualize, to see; the effort to bring the tricks and manipulations before ones eyes and so to attain insight. As a tool of this visualization a new language has born: the symbolic language of algebra.

The language of algebra was developed by Italian, French and German mathematicians during the fifteenth and sixteenth centuries (see Boyer and Merzbach 1989, pp. 312–316). The main invention of these mathematicians was the idea of a symbol representing an unknown quantity, which they called *cosa*, from the Italian word meaning ‘thing’. They called algebra *regula della cosa*, i.e., the rule of the thing, and understood it as a symbolic language, in which they manipulated letters just as we manipulate things. For instance, if we add to a thing an equally great thing, we obtain two things, what they wrote as *2r* (to indicate the thing they usually used the first letter of the Latin word *res*). They gradually developed notation for the different powers of the unknown, notation for root taking, notation for the four arithmetical operations and equality and so created the symbolic language which we all learned at high school under the heading algebra. Let us now look closer on the potentialities of this language. To be able to recognize them more clearly I will systematically compare the language of algebra with the preceding language, which was the language of synthetic geometry.

3.1 Logical power

In comparison with the language of the problem from the *Rhind papyrus* that we presented in the first section, the language of algebra has a fundamental innovation—it contains a symbol for the unknown. In our times it is most often expressed by the letter x. The symbolic language of algebra surpasses in logical power the language of geometry. It surpasses it because the introduction of the symbol for the unknown brought with it an important aspect that is lacking in the language of geometry. Let us take for example the formula for the solution of a quadratic equation

$$\mathrm{x}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (1)$$

The parameters a, b, and c confer to this formula a generality analogous to that which we encounter in geometric proofs. But the formula represents not only the general solution of a quadratic equation, but it also represents the steps of the calculation of the value of x. In this way *the procedure of calculation gets explicitly expressed in language*.

The language of synthetic geometry represents the components of all particular steps of a geometrical construction. Each line or point used in the process of a construction remains a constituent of the resulting picture. Nevertheless, what gets lost is ***the order of the particular steps***. This is the reason why a geometrical construction is usually supplemented by a commentary written in ordinary language, which indicates the precise order of its steps. In algebra, on the other hand, we are able to *express the order of the steps of a calculation within the symbolic language itself*. Thus we need no further commentary on the above formula similar to the one we need on a geometrical construction. The formula represents the process of calculation. It tells us that first we have to take the square of *b*, subtract from it four times the product of *a* and *c*, then calculate the quadratic root of the result, etc. So the process of solution becomes expressed in the language. An algebraic formula indicates not only the particular quantities entering the calculation, but it also indicates the relative order of all steps.

Thanks to this feature of the language of algebra, ***modal predicates***, such as insolubility can be expressed within the language. The insolubility of the general equation of the fifth degree was proven at the beginning of the nineteenth century by Paolo Ruffini, Niels Henrik Abel and Evariste Galois. This proof was a major achievement in the history of mathematics because a totally new kind of theorem was proven. Later the proofs the insolubility of the problems of trisecting an angle, duplicating a cube, or constructing a regular heptagon followed using analogous technical means.

In order to understand the subtlety of these new theorems, we have to realize, that for each angle there is an angle that is just one third of it, or that each equation of the fifth degree has five roots. The insolubility does not mean the non-existence of the objects solving the particular problem. It means that these objects, although they do exist, cannot be obtained using some standard methods of construction. The language of algebra is the first language that was able to prove the insolubility of a particular problem. It was able to do this because it can express the order of the steps of a construction explicitly within the language. In this way the process of solution becomes represented by a linguistic expression. Then the question of solubility becomes a question about the existence of a particular linguistic expression. The insolubility proofs are thus simply arguments (based usually on properties of symmetry) that expressions of some kind cannot exist.

We can now understand why the language of synthetic geometry was unable to produce a proof of this kind. The process of a construction and thus also the set of all construable objects, cannot be explicitly expressed in the language. So it is fair to say that the proof of insolubility illustrates the logical power of the language of algebra. And we see also that this power is in deed a potentiality of the language. It was not due to some lack of ingenuity which prevented the geometers from proving theorems of this kind. It is a feature of the language of algebra that makes it possible to prove such theorems. The ingenuity of the mathematicians like Ruffini, Abel, or Galois consisted in realizing and using this feature, but the feature itself is an objective property of the language of algebra.

3.2 Expressive power

In geometry the unknown quantity is represented as a line segment of indefinite length, the second power of the unknown quantity is represented as a square constructed over this line segment, and the third power of the unknown is represented as a cube. Three-dimensional space does not let us go further in this construction to form the fourth or fifth power. The language of algebra, on the other hand, is able to transcend the boundaries of space and to form higher powers of the unknown. Thus nothing hindered algebraists from going beyond the third degree, beyond which Euclid was not allowed to go by the geometrical space. The algebraists called the second degree of the unknown *zensus* and denoted it *z*. That is why they wrote the fourth degree as *zz (zenso de zensu)*, the fifth as *rzz*, the sixth as *zzz*, and so on. In this way the language of algebra transcended the boundaries placed on the language of geometry by the three dimensions of space. The language of algebra offers us rules for manipulation with these expressions independently of any geometrical interpretation.

The turn from geometrical construction to symbolic manipulation made it possible to discover the method of solving cubic equations. This was the first achievement of western mathematics that surpassed the ancient heritage. It was published in 1545 by the Italian mathematician Girolamo Cardano in his *Ars Magna Sive de Regulis Algebraicis* (Cardano 1545).

HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aũt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarẽtur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

Cardano formulated the equation of the third degree as: '*De cubo & rebus aequalibus numero*' and gave its solution in the form of a rule:

> *'Cube one-third of the number of things; add to it the square of one-half of the number; and take the square root of the whole. You will duplicate this, and to one of the two you add one-half the number you have already squared and from the other you subtract one-half the same. You will then have binomium and its apotome. Then subtracting the cube root of the apotome from the cube root of the binomium, that which is left is the thing.'*

In order to *see* what Cardano was *doing*, we present the equation in modern form $x^3 + bx = c$ and we express its solution using modern symbolism:

$$x = \sqrt[3]{\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}} \, .$$

Cardano never wrote such a formula. In his times there were no formulas at all. Algebra was still *regula della cosa*, a system of verbal rules used to find the thing. Nevertheless, even in its verbal form the language had the basic ingredients that are needed for the solution of a cubic equation—names for the different powers of the unknown, names for the operation of root extraction, and the terms *binomium* and *apotome*. Cardano called $\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}$ a *binomiom* and $-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{b}{3}\right)^3}$ an *apotome*. (I took the liberty of transcribing his verbal rules into modern algebraic notation.)

The language of algebra makes it possible to solve problems which in the language of synthetic geometry it is difficult even to formulate. So we can take the solution of the cubic equation as an illustration of the expressive power of the language of algebra. And just like in the previous case we see that the increase in the expressive power is due to a technical feature of the language of algebra, its ability to form and freely manipulate powers and roots of quantities.

3.3 Methodical power

Euclidean geometry is a collection of disconnected construction tricks. Each problem is solved in a particular way, which has to be remembered. Algebra replaces these tricks by a universal method. This innovation was introduced by François Viète in his *In Artem Analyticam Isagoge* (Viète 1591).


FRANCISCI VIETÆ
OPERA
MATHEMATICA,
In unum Volumen congeſta,
ac recognita,
Operâ atque ſtudio
FRANCISCI à SCHOOTEN Leydenſis,
Matheſeos Profeſſoris.
LVGDVNI BATAVORVM,
Ex Officinâ Bonaventuræ & Abrahami Elzeviriorum.
cIↄ Iↄ c XLVI.

Before him, algebraists used different letters for different powers of the unknown (*r*, *z*, *c*, *zz*, *rzz*, …), and so they could write equations having only one unknown, whose different powers were indicated by all these letters. Viète's idea was to represent the different powers of the same quantity using the same letter and to indicate its power by an additional word. He used *A latus*, *A quadratum*, and *A cubus* for the first three powers of the unknown quantity *A*. Similarly he used *B longitudo*, *B planum* and *B solidum* for the powers of the parameter. In this way the letters expressed the identity of the quantity, while the words indicated the particular power.

Algebraists before Viète worked only with equations having numerical coefficients. This was a consequence of the use of different letters for the different powers of the unknown. Viète liberated algebra from the necessity to calculate with numerical coefficients only. His idea was to express the coefficients of an equation with letters as well. Thus he introduced the distinction between a parameter and an unknown. In order not to confuse parameters with unknowns, he used vowels (*A*, *E*, *I*, *O*, *U*) to express unknowns and consonants (*B*, *C*, *D*, ...) to express coefficients. So the equation which we would write as $ax^3 - by^2 = c$ he would write as

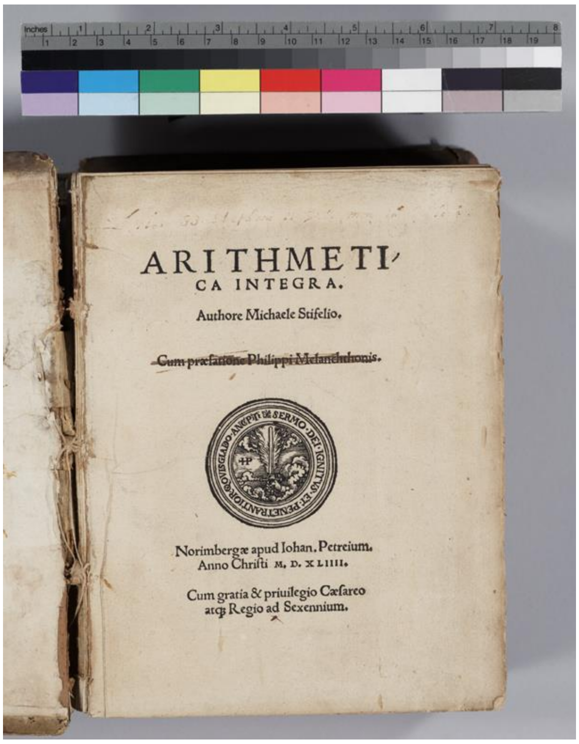
***D** latus in **E** solidum — **F** quadratum in **I** planum equatur **G** quadrato-quadratum.*

Viète's *analytic art*, as he called his method, was based on expressing the unknown quantities and the parameters of a problem by letters. In this way the relations among these quantities could be expressed in the form of an equation containing letters for unknown quantities as well as for parameters. Solving such an equation we obtain a general result, expressing the solution of all problems of the same form. In this way generality becomes a constituent of the language. The existence of universal methods for the solution of whole classes of problems is the fundamental advantage of the language of algebra.

The language of synthetic geometry does not know methods of any comparable universality. Geometry can express universal facts (facts which are true for a whole class of objects), but it operates with these facts using particular methods of construction. We cannot draw a general triangle. Any triangle that is actually drawn is always concrete. Algebra developed universal methods, which played a decisive role in the further development of mathematics. From algebra the analytic methods passed to *analytic geometry* (Descartes 1637), *mathematical analysis of infinitesimals* (i.e., calculus, Euler 1748), then to physics in the form of *analytic mechanics* (Lagrange 1788) and *analytic theory of heat* (Fourier 1822) till they reached logic in *mathematical analysis of logic* (Boole 1847). In the analytic method we can see an illustration of the methodical power of the language of algebra.

3.4 Integrative power

Cardano considered the equations $x^3 + bx = c$, $x^3 + c = bx$ and $x^3 = bx + c$ to be three different problems. In his *Ars Magna* he devoted to each of these types of equations a separate chapter in which he presented a method of solution fitting the specific type of equations, to which the chapter was devoted. He was forced to treat these equations separately because he allowed only positive values for coefficients and solutions. For equations of the third degree this represents only a small complication (instead of our unique type of cubic equations Cardano had three), but in the case of the equations of fourth degree we have seven different types of equations, and in the case of quintic equations fifteen. The three, seven, or fifteen types of equations are not independent. Cardano had shown that simple substitutions can transform an equation of one type into another. Therefore it is natural to try to reduce this complexity. It was Michael Stifel, who first saw how this might be accomplished. In his book *Arithmetica integra* (1544) he introduced rules for the arithmetic of negative numbers, which he interpreted as numbers smaller than zero, and he started to use negative numbers also as coefficients of equations. This enabled him to unite all fifteen kinds of quintic equations, which formerly had to be treated separately, into one general form: $x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0$.



Before Stifel attempted to solve an equation, he transferred all the expressions to one side, and in this way obtained the equation in a *polynomial form* $p(x) = 0$. Thus a polynomial as a mathematical object was created by Stifel, thanks to the idea of reducing all the different ***formulas*** to a single ***form***, by allowing the coefficients to be negative. The form common to all cubic equations was hidden for Cardano, because he bound the algebraic language too closely to reality. Only when Stifel stopped observing a distinction between positive and negative coefficients did this deeper unity become visible. Many important theorems, as for instance the fundamental theorem of algebra, stating that a polynomial of the n-th degree has exactly n solutions, it is possible to state only on this deeper level. Later several similar unifications occurred in algebra, for instance, when all coefficients of a system of linear equations were integrated into a matrix. Objects like polynomials or matrices are formal objects. They were created by mathematicians by integrating several independent expressions of the language into a form. By means of such (polynomial, linear, quadratic) forms the language of algebra is able to integrate different situations, which originally occurred in separate contexts, into a unique representation. By this means algebra achieves a deeper understanding of the unity of its subject matter. We can see this unity as an illustration of the integrative power of its language.

3.5 Explanatory power

The language of algebra makes it possible to understand why some geometrical problems, such as the trisection of an angle, the doubling of a cube and the construction of a regular heptagon, are insoluble with ruler and compass. All problems solvable by means of ruler and compass can be characterized as problems in which only line segments of lengths belonging to some finite succession of quadratic extensions of the field of rational numbers occur. This observation, even if without a rigorous proof, is present already in Descartes' *Geometry* (Descartes 1637),

Thus in order to understand the reason of the insolubility of the three mentioned problems it is sufficient to realize that their solution requires line segments whose length does not belong to any finite sequence of quadratic extensions of the field of rational numbers. This can be easily done (see Courant and Robbins 1941, pp. 134–139). Nevertheless, it is important to realize that the borderline that separates the problems that are constructible with ruler and compass from those, which are not, is drawn by means of polynomials. As polynomials are linguistic creations in the sense that they were introduced not as names of some independently existing entities but as formal expressions integrating particular algebraic terms, we see that the whole explanation of insolubility rests on distinctions originating in the language itself. We understand insolubility because the language of algebra makes it possible to draw distinctions, which we cannot draw except by means of this language. The language of geometry does not make it possible to understand why the three mentioned problems are insoluble. To try to trisect an angle is a rather natural task. From the algebraic point of view it is clear. The ruler-and-compass constructions take place in fields that are too simple. Thus it is fair to consider the explanation of insolubility of the three mentioned problems as an illustration of the explanatory power of the language of algebra.

3.6 Metaphorical power

Besides the equation of the form '*cubus and thing equal number*', the solution of which was discussed above, Cardano presented rules for solution of the other two forms of cubic equations. The rule for solution of the equation '*cubus equals thing and number*' ($x^3 = bx + c$) is similar to the first case:

$$x = \sqrt[3]{\frac{c}{2}+\sqrt{\left(\frac{c}{2}\right)^2-\left(\frac{b}{3}\right)^3}}+\sqrt[3]{-\frac{c}{2}+\sqrt{\left(\frac{c}{2}\right)^2-\left(\frac{b}{3}\right)^3}} .$$

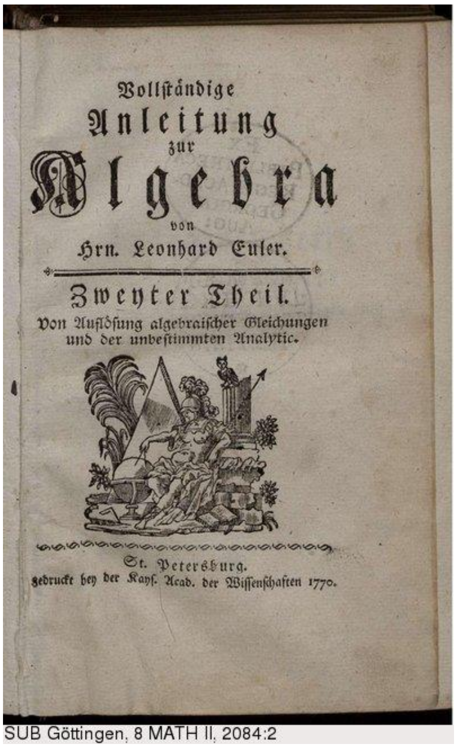
Nevertheless, when Cardano tried to apply this rule for the particular equation $x^3 = 7x + 6$, he obtained a result which we would express as:

$$x = \sqrt[3]{3+\sqrt{-\frac{100}{27}}}+\sqrt[3]{3-\sqrt{-\frac{100}{27}}} .$$

Below the sign for the square root a negative number appeared. The rule required him to find $\sqrt{-\frac{100}{27}}$, something he was not able to do. Cardano discovered something from which *complex numbers* evolved. The rules brought

Cardano into a situation that was beyond his comprehension, a situation where he had to do something impossible.

When we free ourselves from the understanding of the algebraic expressions as pictures of reality, and start to understand them as independent formal objects, it becomes possible to interpret the square roots of negative numbers metaphorically as a special kind of numbers. Even though we do not know what they represent, we know how to calculate with them. Maybe the most pregnant expression of this view can be found in Euler's book *Vollständige Anleitung zur Algebra* from 1770, where imaginary numbers are called *numeri impossibiles*, because they are not smaller than zero, not equal to zero, and not greater than zero.



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Euler writes:

> 'But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces –4; for this reason also nothing prevents us from making use of these imaginary numbers and employing them in calculations.' (Euler 1770, § 145).

Thus even though in reality there is no quantity whose square is negative, we have a clear understanding of the meaning of the symbol $\sqrt{-4}$. In a metaphorical sense we can consider it as a new kind of number. Thus the introduction of complex numbers can be seen as an illustration of the metaphorical power of the language of algebra.

3.7 Summary

So far we discussed six potentialities of the language of algebra. We have seen how the introduction of this language increased in a fundamental way our ability to prove theorems, to express complex situations, to solve problems, or to explain phenomena. Similar changes accompanied the introduction of the languages of analytic geometry, of the differential and integral calculus, of fractal geometry, of the predicate calculus, or of set theory. In each of these cases the way we prove, represent, or explain the phenomena of our world changed in a radical way. Thus we can say that the development of the language of mathematics changed in a profound way our thinking. We recognize in the world many very complex patterns, which without the language of mathematics we simply would not be able to perceive. We perceive deep unity where our predecessors saw only unrelated phenomena—as for instance the unity of electricity and light. We understand the world much better and can explain things, which for our ancestors appeared rather mysterious. The language of mathematics is a powerful intellectual tool and features like logical, expressive, methodical, integrative, explanatory, and metaphorical power characterize rather precisely the ways in which the language of mathematics enhances our intellectual capacities. It is the language of mathematics, which gives our thought its logical, expressive, etc. power.

After the introduction of our six potentialities of language we see much clearer ***how the language of mathematics changed us***. Now it is important to examine this link also in the opposite direction and try to explain the other line in the title of this talk—***how we changed the language of mathematics***. In the title the order of these links is natural. First we had to form the language of mathematics in order to allow the language of mathematics to change us. Nevertheless, as you have perhaps noticed, in the presentation I have reversed this natural order. We had first to identify the different potentialities of the language of mathematics to be able to turn to the question of how are they constituted. So let us pose this fundamental question: How acquired the language of mathematics its logical, expressive, methodical, integrative, explanatory, and metaphorical power?

4 How are the different potentialities constituted

The language of mathematics is not a mysterious power that forms our thinking in some hidden, incomprehensible way. The language of mathematics is a human construction and thus all the increase of logical, expressive, methodical, integrative, explanatory and metaphorical power is due to specific innovations made by concrete mathematicians. As so often in history, the authors of these innovations usually did not and even could not foresee the full impact of their work. When Newton and Leibniz introduced the foundations of the differential and integral calculus, they possibly could not foresee the tremendous impact that the calculus had on the whole development of theoretical physics. Therefore it is interesting to look back and to try to understand the way how the language of mathematics is constituted. We will go back to the language of algebra and for each of its six potentialities we will try to identify the linguistic innovation that constitute them.

4.1 Logical power

In the case of the logical power we are in a fortunate situation. This potentiality has been already analyzed by the great German logician Gottlob Frege in his paper *Funktion und Begriff* published in 1891. In this paper Frege characterized the evolution of the symbolic language of mathematics from elementary arithmetic through algebra and mathematical analysis to predicate calculus*:*

> 'If we look back from here over the development of arithmetic, we discern an advance from level to level. At first people did calculations with individual numbers, 1, 3, etc.
>
> $$2 + 3 = 5 \qquad\qquad 2.3 = 6$$
>
> are theorems of this sort. Then they went on to more general laws that hold good for all numbers. What corresponds to this in symbolism is the transition to the literal notation. A theorem of this sort is
>
> (a + b).c = a.c + b.c.
>
> At this stage they had got to the point of dealing with individual functions; but were not yet using the word, in its mathematical sense, and had not yet formed the conception of what it now stands for. The next higher level was the recognition of general laws about functions, accompanied by the coinage of the technical term 'function'. What corresponds to this in symbolism is the introduction of letters like f, F, to indicate functions indefinitely. A theorem of this sort is
>
> $$\frac{dF(x).\,f(x)}{dx} = F(x).\frac{df(x)}{dx} + f(x).\frac{dF(x)}{dx}.$$
>
> Now at this point people had particular second-level functions, but lacked the conception of what we have called second-level functions. By forming that, we make the next step forward.' (Frege 1891, p.30; English translation p. 40)

We see that Frege characterized each symbolic language by means of a specific sort of symbol that it introduced. In the case of algebra it is the use of letters that express generality and to the use of implicit individual functions. It is not difficult to realize that it is precisely these letters and functions, which make the language of algebra able to express the order of the steps of a procedure, and so they constitute the logical power of the language of algebra. In a sense our task is to complement each of the five remaining potentialities by an analysis analogous to Frege's analysis of the constitution of the logical power of the language, given hundred and twenty years ago.

4.2 Expressive power

The logical power of the language of algebra is constituted by the introduction of *letters* as symbols representing arbitrary numbers. The increase of its expressive power is due to the introduction of rules for a formal operation with these newly introduced letters. I have in mind the rules for the formation of the powers:

$$x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, \ldots.$$

That the three initial terms x, x^2, x^3, are well rooted in geometry: x is the length, x^2 the area, and x^3 the volume of a line segment, a square and a cube respectively. But in geometry there is a firm boundary—the three dimensions of space—that prevents us from going further to the higher powers. It was the audacity of the algebraists, which was present already in the work of al Chwarizmi, to ignore the limitations of space, to transcend its boundary and in a purely formal way go beyond the third power of the unknown.

We see that the boundary of the three-dimensional space was broken linguistically, by creating a formal operation that can be iterated indefinitely, beyond any boundary. This operation proceeds freely towards the horizon. It seems that in the case of the other languages we can identify a similar operation, which can be indefinitely iterated. In arithmetic this operation consisted in the addition of a unit and thus creating the number sequence 1, 1+1, 1+1+1, …, that transcends any conceivable quantity. In geometry this operation consisted in the indefinite production of the straight line, and it was guaranteed by Euclid's second postulate. Thus it seems likely that we would be able to produce for the expressive power of the language a story analogous Frege's story for the logical power. Just like in the case of algebra, also in the case other the operation transgresses some previously existing boundary and in that way it increases the expressive boundaries of the language.

4.3 Methodical power

While the logical power of the language of algebra is constituted by the introduction of a new sort of symbols and its expressive power by means of a free, unbounded iteration of a formal operation with the symbols, the methodical power of the language of algebra is based on a further innovation—on the introduction of a conventional discrimination between two groups of these symbols. It is a rather strange discrimination that exists only inside of the language itself, and does not correspond to any factual difference in the world represented by the language. It corresponds to an epistemological difference in our access to this reality. I have in mind Viète's distinction between the unknowns and the parameters. This distinction does not correspond to anything real; we cannot see on a quantity whether it is an unknown or a parameter. It is simply that an unknown is unknown to us while the values of the parameters are known. This subtle distinction, for which Viète introduced his convention to represent the unknowns by vowels and the parameters by consonants, forms the foundation of the analytic method. Due to this convention we can form a symbolic representation of almost any problem using equations and then we can try to solve the problem by means of formal manipulations.

Thus the methodical power of the language of algebra is constituted by the introduction of a conventional distinction among letters used in formal representation of quantities. This convention introduces into the language an internal boundary that separates what we know from what we don't. The solution of a problem consists the in a systematic shifting of this boundary as long as the unknown becomes expressed in terms of the parameters, just like in the formula (1) for the quadratic equation. Therefore it is fair to say that the distinction between unknowns and parameters constitutes the methodical power of the language of algebra.

It is not difficult to find analogous conventional distinctions also in the other languages of mathematics. For instance in the differential and integral calculus when we expand a function into its Taylor series, we usually write a formula like

$$f(x + h) = f(x) + h.f'(x) + \dots$$

Here both letters, x and h, represent variables and the difference between them is again mostly epistemological. The letter x represents the point in which ***we have decided*** to expand the function into the series, while h is the distance from that point. Thus in a sense both of them are parameters (in the sense of Viète) and it would be foolish to try to express x in terms of h (as we would proceed if x were the unknown and h a parameter). Here it is rather so, that x represents a point in which ***we already know*** the value of the function f (together with its derivatives), and h is a small parameter the powers of which ***we can easily handle***. Thus we decompose the value of the function in the point $(x + h)$ into components that are ***epistemologically accessible to us***. So also in the case of the methodical power we could write a story analogous to that of Frege.

4.4 Integrative power

The process of solution of a particular algebraic problem passes usually through several more or less equivalent representations of the same situation to reach the final one from which we can easily read off the solution. Let us just mention the different methods of elimination used in linear algebra. But besides this "methodological" use of the richness offered by the language, when ***we connect different representations of the same situation***, there is another way how this richness can be used. If we take Cardano's cubic equation: "*Cubus and things equal number*", i.e. $x^3 + bx = c$, the language of algebra offers us four different ways how to express it:

$$x^3 + bx = c \qquad x^3 = -bx + c \qquad x^3 - c = -bx \qquad x^3 + bx - c = 0$$

Another equation considered by Cardano as a completely different kind of equations, namely the "*Cubus equals things and number*", i.e. $x^3 = bx + c$ can be again represented in different ways

$$x^3 = bx + c \qquad x^3 - bx = c \qquad x^3 - c = bx \qquad x^3 - bx - c = 0$$

It was the idea of Michale Stifel to unite the last equations in each of the above rows, i.e. the equations $x^3 + bx - c = 0$ and $x^3 - bx - c = 0$ into one formal object, that we call a ***polynomial***. So a polynomial appears when ***we connect similar representations of different situations***. It was not possible to unite the original equations, which were studied by Cardano, because they had different number of terms on each side, and so were formally not sufficiently similar.

First it was necessary to use the richness of the language, offered by the introduction of the negative numbers to produce *formally similar* expressions representing the *different equations* (which was the core of Stifel's achievement), and then formally unite them into one expression. Thus polynomial forms are new objects created totally inside the language. They represent the type of a class of formal expressions. In reality we have always to do only with specific equations—the recognition that they represent a polynomial is a linguistic one. A polynomial integrates different equations not because they represent similar things, but because they represent different things similarly. And this similarity constitutes the integrative power of language. The language enables us to see formal connections between phenomena, between which on the semantic level there is no similarity at all.

Analogous integration of representations of semantically different situations just on the basis of the similarity of their formal representations occurred also in other languages. For instance in the differential and integral calculus the classification of partial differential equations into parabolic, elliptic, and hyperbolic unites also equations which describe physically rather different situations. Vibrations of an elastic medium and the spreading of light are from the physical point of view rather different, while both of them are described by equations that belong to the same class. Thus we could write a Fregean story also about the integrative power.

4.5 Explanatory power

The polynomial forms introduced in the previous step can be easily turned into predicates. By means of the polynomial $p(x)$ it is possible to create the predicate $\boldsymbol{P}(x)$ expressing the property that the quantity x is the root of the polynomial $p(x)$:

$$\boldsymbol{P}(x) \text{ is true} \qquad \text{if and only if} \qquad p(x) = 0.$$

In this way we project the distinctions that were introduced purely by formal means inside the language onto reality (as attributes of objects). We start to characterize reality by means of properties that are purely linguistic ones—different quantities are classified into classes on the basis of polynomials, which they satisfy. For a quantity, say a side of a triangle, the property "to be a root of a cubic equation" is a very strange property indeed. Here we characterize a geometrical object by means of an artificially created linguistic expression that it should satisfy. It sounds strange, because language seems to be a system of arbitrarily taken conventions, and so the properties characterizing objects should be independent of any language.

It is as if someone would try to classify cars according the special sorts of poems, by which they could be described. One sort of car would be considered as especially safe, because it could be described by a sonnet, while another sort would be considered as utterly risky, because its best description would be a haiku. But in geometry we are doing precisely this. A polynomial is rather similar to a sonnet or a haiku—it is a linguistic construction defined by some purely formal rules. And what is even more surprising is the fact, that if we project these purely linguistic predicates onto the world, they enable us to express some fundamental distinctions, which we were unable to notice without them. These newly introduced distinctions play crucial role in the explanation of some facts, as for instance the insolubility of the problem of trisection of an angle. Thus it is fair to say, that they constitute the explanatory power of the language. Now we see why this power is tied to language. It is the linguistic constructs and not our experience, which enable us to make these explanations. And again, just like in the previous cases, it is not difficult to find analogous predicates also in other branches of mathematics.

4.6 Metaphorical power

The metaphorical power of the language requires us to go even further. If we take the formal linguistic predicates that constitute the explanatory power of the language, usually we can find some objects that satisfy them. Nevertheless, there are some exceptional cases, like that of the polynomial $p(x) = x^2 + 1$, where no object satisfying the corresponding predicate can be found. Simply there are no numbers, the square of which would be equal minus one. And what happens is, that mathematicians extend the universe represented by the language by postulating new objects—in this case the complex numbers—so that also this predicate would be satisfied. Thus here we adapt reality to our language. It seems that the operation of turning a linguistic expression into an object lies at the basis of this adaptation. One could think that mathematicians lost contact with reality when they try to subdue reality to language. But a look into books on physics and chemistry shows, that the Schrödinger equation is coined in the language of complex numbers. Thus it seems that nature follows mathematicians in this step.

Thus even though the complex numbers were perhaps in the 17th century considered as some strange toys, in the ensuing time they were turned into an important tool. The complex numbers are a kind of numbers, but in order to see them as such, we have to take the notion of a number in a rather metaphorical way. And this ability to twist the rules constitutes the metaphorical power of the language of algebra. It seems that also in other areas of mathematics we could find several examples of such metaphoric usage—it is sufficient to consider the points at infinity in geometry, or Dirac's famous function in the calculus.

5 Conclusion

We can sum up the process of constitution of a new symbolic language in the following steps:

- first – a new kind of symbols is introduced;

- second – for these symbols an operation, which can be iterated indefinitely, is introduced;
- third – a conventional distinction representing an epistemological difference, is introduced;
- fourth – the different expressions thus created are united into linguistic forms;
- fifth – these forms are transformed into formal predicates; and
- sixth – reality is adapted to these new predicates by postulation of new objects.

It seems that these six steps were repeated several times in the history of mathematics and gradually they turned mathematics into a strong and efficient tool of human thought. The understanding of the process of construction of new symbolic languages can help us, just like it helped Frege, to construct new languages or at least to use the existing ones with more insight.

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ASSESSING PRESCHOOL CHILDREN'S UNDERSTANDING OF MATHEMATICAL EQUIVALENCE THROUGH PROBLEM SOLVING

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Mathematical equivalence is one of the most important concepts for developing young children's algebraic thinking. The paper presents a study conducted with kindergarten teachers and preschool children in which a weighting scale was used in order to solve a problem on mathematical equivalence. The problem was analysed according to the strategies children used, number of solutions they found and the role of gender. The research results show that most of the children solved the problem with insight by shifting the objects from one to another side until they achieved equivalence; some of them were able to find even more solutions without being encouraged from the teachers. We believe that a weighting scale is a very useful device for introducing rules for solving equations, namely: the role of a scale for transforming equations into equivalent equations and the role of a scale for understanding the meaning of the equal sign.

THEORETICAL BACKGROUND

Many studies in the factors influencing the understanding of the number concept in children were conducted in the past (Beckman, 1924; Desceudres, 1921; Gelman, 1972; Gelman and Tucker, 1975; Gelman and Gallistel, 1978; Gelman and Meck, 1983; Lawson, Baron in Siegel, 1974; Smither, Smiley in Rees, 1974; Manfreda, 2006). The results showed that preschool children can identify the number of items in a set when the size of the set is relatively small, but when it exceeds 5 the accuracy of numerical estimates markedly drops. Gast (1957) established that pre-schoolers identify numbers more accurately for homogeneous than for heterogeneous arrays, whereas Gelman's and Tucker's (1975) findings show that item heterogeneity does not have any effect on the number estimates of pre-schoolers. Gullen (1978) studied K-2 students' strategies while comparing the number of elements in two sets. His findings also suggest that students' performance may depend on the materials used in the task as well as on the children's age.

The fact that young children can accurately represent the number only when the numbers involved are small has led to the conclusions that the young child's concept of the number is intuitive (Piaget,1952) or perceptual (Gast, 1957;

Pufall, Shaw and Syrdal - Larsky, 1973), which means that young children obtain their representation of the number of items by a direct perceptual-apprehension mechanism (Jensen, Reese and Reese, 1950; Klahr in Wallace, 1973; Neisser, 1966; Schaeffer, Eggleston and Scott, 1974) and they have yet to develop the ability to reason about the number. Reasoning about numbers involves understanding of counting principles, which define the counting procedure: the one-to-one principle, the stable–order principle, the cardinal principle, the abstraction principle and the order-irrelevance principle. It is an open question whether young children realize that counting can be applied to minds, or is it a pure product of imagination, or even heterogeneous sets of objects are considered. The number becomes an abstract concept when children come to understand that the object of counting is irrelevant. The process in which children realize that three pencils and three marbles represent the same abstract concept of the number three is not trivial for children. They should understand that numbers are not bound by the physical elements being counted. The research of Tirosh et al. (2011) shows that even preschool teachers had problems when comparing sets with various types of unknown abstract elements. Some teachers were more concerned with the nature of the set elements than with the number of elements in each set. Linchevsky and Vinner (1988) also reported that elementary school teachers could not adopt the concept of a set as an arbitrary collection of elements. The mathematical definition of a concept provides for critical attributes that determine whether an instance is an example or non-example of some concept (Tirosh et al., 2011, p.120). The comparison of sets is based on the critical attribute of the number elements in the sets, and that number is an abstract concept, which is independent of the type of elements in each set.

In order to find out whether two sets are equivalent or not a method for comparing the number of elements in the sets is needed. There are different methods of how to do that. If the compared sets contain a small number of elements, then a child may visually recognise the set with more objects. The method of seeing may also be acceptable if the difference between the numbers of elements is big and one is only interested in establishing which set has more elements. If one cannot spot immediately which set is bigger or if one is not only interested in determining which set is bigger, but also in the number of the elements in a set, then one needs a more precise verification method thereto. Children usually use the method of counting objects, but the question posed also by Tirosh et al. (2011) is: 'Is counting always preferable?' If for example we would like to find out whether the number of boys is equal to the number of girls at a dance party another, more efficient method can be used: a one-to-one correspondence. If one set is a proper subset of another one, another preferable method of comparison is introduced: a subset method. The teacher as well as children should be aware that not every method is appropriate for every situation. So, when comparing the number of elements in different sets, the attributes of the elements are irrelevant to the result, but when deciding on the method to be used for the comparison, the attributes of the elements involved become important, because they determine the most efficient method for the comparison of the sets. In the situation when two homogeneous sets of objects with identical weight are to be compared, another method may be used, i.e. comparison by using a weighing scale. This device becomes very useful when addressing the problems of mathematical equivalence.

Mathematical equivalence is one of the most important concepts for developing young children's algebraic thinking (e.g. McNeil and Alibali, 2006). Children's difficulties with mathematical equivalence have been shown to be long term, persisting among some middle school, high school, and even college students (Knuth at al., 2006, McNeil and Alibali, 2006). Several researchers now argue that difficulties are due, at least in part, to children's early experience with mathematics (McNeil et al., 2011). This experience is narrow as claimed by McNeil, 2011) in a way that arithmetic problems are always presented with operations to the left of the equal sign, and the 'answer' to the right of it. As a result children extract operational patterns which are derived from experience with arithmetic operations and reflect operational rather than relational thinking (e.g. Jacobs et al., 2007, Cross et al., 2009). The authors also concluded that even differences in relatively specific, micro level factors, such as the amount of mathematically relevant speech used in preschool classrooms or the frequency of young children playing number board games can exert large effects on children's cognitive development (e. g. Ramani and Siegler, 2008). We believe that children develop operational understanding of the equal sign in everyday experience and that formal setting of learning mathematics (kindergarten and school) should provide for the experience to broaden their understanding of mathematical concepts and to help children overcome their misconceptions.

Our focus was to establish the manner in which preschool teachers and children address the problems of mathematical equivalence. When planning the research the following issues were addressed:

What kind of problem is appropriate to be presented to preschool children in order to understand the mathematical equivalence to some extent?

How to organise the activity in the kindergarten to get as good as possible an insight into children's thinking?

Considering these aspects the following decisions were made:

Ad 1 We chose to present a problem of mathematical equivalence with a weighing scale.

Ad 2 Children acquire concepts trough three types of learning experience: naturalistic experience, informal learning and adult-guided learning (Charlesworth, Leali, 2012). We opted for the adult-guided learning, at which experiences focus on specific concepts that an individual or group of children are ready to explore. We proposed to our group of kindergarten teachers the adult-guided learning when working with an individual child.

EMPIRICAL PART

Problem Definition and Methodology

In the empirical part of the study conducted with kindergarten teachers and preschool children the aim was to explore their competences of problem solving.

The empirical study was based on the descriptive, casual and non-experimental method of pedagogical research (Hartas, 2010; Sagadin, 1991). The teachers were posed the problem and asked to plan the problem solving activity with an individual child (each teacher worked with 4 children) considering the following issues: choosing the sample of children, preparation of the problem solving activity, etc. Children, on the other hand, were presented with the problem and asked to solve it. The teachers made notes of their steps of solving the problem and of the dialogue they had with each child. Children did not see each other's solutions of the problem. The kindergarten teachers had to write the reports on their problem solving activities with children, including all the mentioned issues.

Research Questions

The aim of the study was to gain a deeper insight into the way children are able to solve a particular problem and to answer the following research questions:

1. What strategies do preschool children use to solve a particular problem of mathematical equivalence?
2. How many solutions do they manage to find?
3. Do their problem solving competences in solving a particular problem differ according to gender?

Sample Description

The study was conducted at the Faculty of Education, University of Ljubljana, Slovenia in 2013 and consisted of 43 research reports, written by 43 preschool kindergarten teachers coming from different regions in Slovenia. Each of the kindergarten teachers choose 4 children for solving the problem. The number of children involved in the research was 172 (85 girls and 87 boys), aged from 5 to 6 years and coming from different social backgrounds, all of them were involved in the program of preschool education. The sample was occasional.

Data Processing Procedure

The teachers were posed a problem which provided for the use of different strategies and various ways of reasoning in order to reach the solution. The problem was, as follows: there are 4 marbles on one side of the weighing scale and 6 on the other side. The marbles are of the same size. There are also some other marbles which can be used, but do not suggest to children they should use them.

The child was asked the following question: What can you do to establish the balance on the weighing scale?

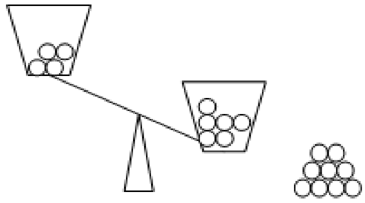


Figure 1: Problem situation.

We established the objectivity of our research by giving the kindergarten teachers equal, precise instructions about the activity they carried with the children. Contextual validity of the template for the teacher's research report and problems for the children was examined by two experts, one for didactics of mathematics and one for general didactics. Reliability of the research was provided with a prepared template for the teachers for writing their observations, which we categorised (categories are clearly different from each other) and analysed.

In the problem solving process the children were neither specifically reminded of multiple outcomes of the problem, nor did the teachers encourage them to produce more solutions. We wanted to establish whether the children would be able to come up with various possible solutions on their own. The children were solving the problem individually, the teachers were noting down the steps of children's problem solving strategies and the dialogue they had with them.

The data gathered from preschool teachers' research reports were statistically processed by employing the SPSS 20 programme for descriptive and inference statistics with the calculation of frequencies, percentage and the chi-square test for the examination of correlations between some of the variables.

In continuation the results are shown, which are analysed as per various observation aspects.

Results and interpretation

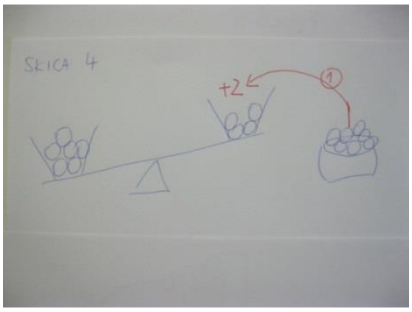
a) <u>Children's strategies</u>

The children were presented with the problem and asked to solve it. From the teachers' research reports it can be concluded that:

1. The children either solved the problem in one step (the insight strategy) or in more than one step;
2. The children used different strategies based on addition, subtraction or both operations.

Let us present each of these conclusions in details. The child's strategy was defined as the insight strategy if a child did something of the following: moving 1 marble from one side to another side of the weighing scale (Figure 2), adding 2 marbles (Figure 3) or taking away 2 marbles (Figure 4), and if he did it in one step. Children who needed more steps did something of the following: using a trial and error approach (Figure 5a), translating the problem to the situation with no marbles on one side of the weighing scale and then gradually adding the marbles (Figure 5b) or gradually taking away/adding one marble until reaching the balance (Figure 6).

Figure 2: A one-step strategy: 6-1=4+1.



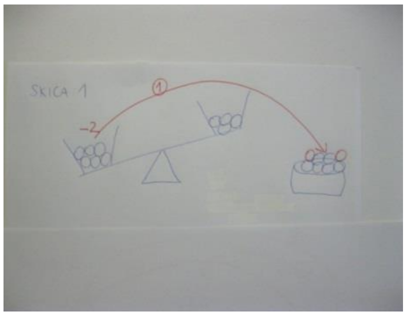
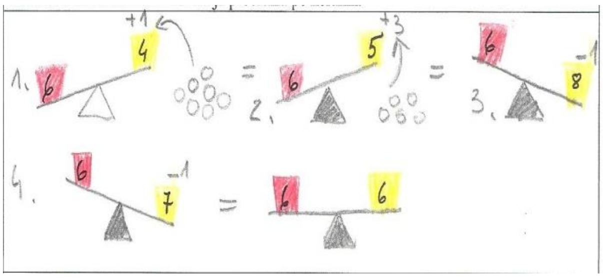
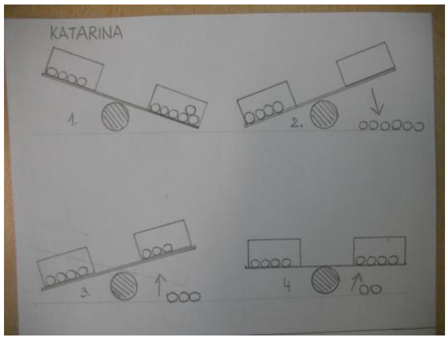


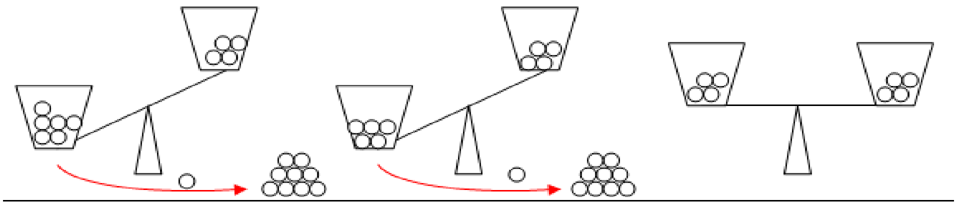
Figure 3: A one-step strategy: 6=4+2. **Figure 3: A one-step strategy: 6-2=4.**

a) b)





Picture 4: Strategies in more than one step: (a) 6>4+1; 6<5+3; 6<8-1; 6=7-1; (b) 4>6-6; 4>0+3; 4=3+1).



Picture 6: Strategy in more than one step: 6-1>4; 5-1=4.

Considering a one-step strategy or strategies in more steps it can be concluded that more children found the solution to the problem by insight (55%, see Table 1), so they made only one step. It is obvious that these children have a good sense of numbers and also of equality. We consider a weighing scale to be a good device for establishing the concept of equality, but there is a diference in its role for children with a well or less developed concept of the number: if a child takes away, for example, two marbles using only a one-step strategy, it is obvious he knew how many marbles he was supposed to take away to balance the weighing scale. On the other hand, a child who does not see the solution at first sight can make use of the weighing scale for immediate feedback on what he should do or how close he is to the solution.

Number of steps	Frequency [f]	Percentage[f%]
Solving in more than one step	72	42
Solution with insight	95	55
Other	4	2
No answer	1	1
Total	172	100

Table 1: Strategy and steps.

Considering the arithmetic operations (Table 2) the children used, it can be concluded that most of the children (44 %) used both arithmetic operations (a combination of addition and subtraction of marbles, which includes also the operation of moving a marble from one side to the other one), 33% of them used addition (adding marbles) and the minority of them (21%) used subtraction (taking marbles away).

Arithmetic operations	Frequency [f]	Percentage[f%]
Addition and subtraction	76	44
Addition	56	33
Subtraction	37	21
None of them	2	1
No answer	1	1
Total	172	100

Table 2: Strategy and arithmetic operations used.

We wanted to establish the proportion of children who used a particular strategy of one or more steps. We named the strategy when both operations were used and the problem solved by insight as 'insight, + and –', the strategy where only addition was used and the problem solved by insight as 'insight, +' , etc. The similar naming was used for the strategies in more than one step, e.g. the strategy where a child used both artithmetic operations was named 'more steps, + and –'. From Table 3 it is possible to conclude that most of the children solved the problem using both operations and by insight (24%). This means that a child perceived the situation on the weighing scale and moved one marble from the side with 6 marbles to the side with 4 marbles. This is a very surpising result, but only at first sight. The results of the study conducted by Tsamir et al. (2010) also show that shifting from one set to the other one is the most preferred method in order to achieve the equal amounts of objects. According to the authors this is not so surprising, because this method is most similar to the everyday context of fairness, familiar to kindergarten children. If one child has more of something than the other ones, then it makes sense for such a child to give some to a child who has less (p. 229). Thus shifting one cap from the set of six to the set of four may have linked children's everyday knowledge to the mathematical situation of the problem. We considered this strategy as the most advanced one from the mathematical point of view, because children were able to see the relation between the numbers 6 and 4: although the difference is 2, there is only one marble needed to be moved to achive equality.

17 % of the children used the insight strategy by adding two marbles to the side with 4 marbles on the weighing scale. They were able to establish the difference between the numbers 4 and 6 and took 2 marbles from the box beside the scale and put them to the side with 4 marbles. 14 % of children used subtraction and insight, so that they took 2 marbles from the side with 6 marbles on the scale.

Strategy	Frequency [f]	Percentage [f%]
Insight, + and -	42	24
Insight, +	29	17
Insight, -	24	14
Steps, + and -	34	20
Steps, +	27	16
Steps, -	11	6
Other	4	2
No answer	1	1
Total	172	100,0

Table 3: Strategy: steps and arithmetic operations.

b) Number of solutions

Regarding this criteria we can conclude that children either found only one solution or proposed more solutions.

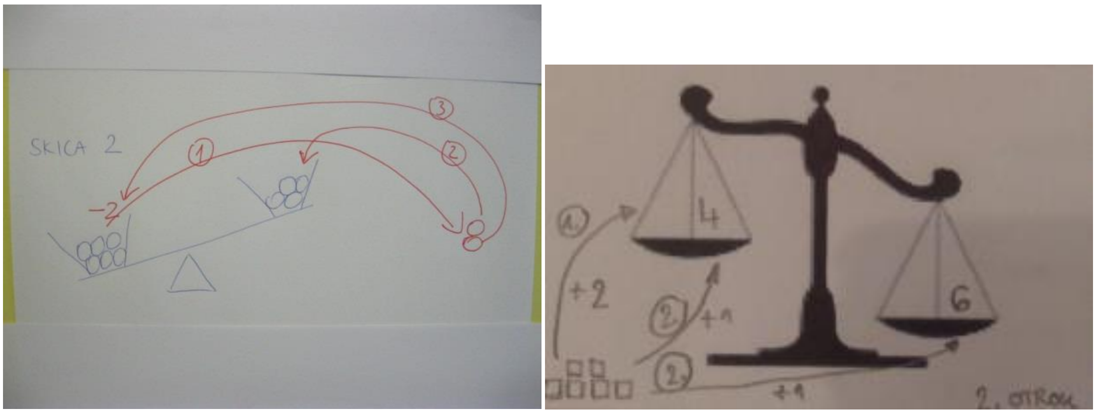


Figure 7: More than one solution: 6-2=4; 4+1=4+1 and 4+2=6; 6+1=6+1

From Table 4 it can be inferred that most children (82 %) were happy with only one solution of the problem and only 16 % of them indicated that there were many solutions. For example, a child first added two marbles to the side with 4 marbles, then established a balance, and proceeded with one marble to one side, one to the other one, two to one side, two to the other one, etc... until he used all the marbles from the table.

Number of solutions	Frequency [f]	Percentage[f%]
One solution	141	82
More solutions	27	16
No solution	3	2
No answer	1	1
Total	172	100

Table 4: Number of solutions.

The study of Tsamir et al. (2010) showed that kindergarten children are willing to search for more than one outcome and are flexible enough to employ more than one method (p.228). But the children in their study were prompted to search for additional outcomes. In our research children were not especially encouraged to do this, but there were still 16 % of them who searched for more solutions on their own. In our opinion children's own interest in searching for different solutions of the problem is a sign of developing mathematical thinking or their interest in mathematics.

c) <u>Gender and problem solving</u>

We were also interested if there were some differences in problem solving among boys and girls when considering the following relations:

- gender - strategy and steps
- gender – strategy and arithmetic operations used
- gender – number of solutions

From the Table 5 it can be concluded that there is no significant difference between the boys and girls with regard to the steps used in their strategies (χ^2 = 0,005; g = 1; P=0,946). Girls as well as boys more frequently used the insight strategy in comparison to the strategy with more than one step. There is as well no difference between boys and girls (χ^2 = 0,245; g =1; P=0,620) regarding the number of the produced solutions (Table 6). As already mentioned, the majority of the children found only one solution to the problem. According to the arithmetic operations used for solving the problem (considering the steps and insight) it could be concluded that both arithmetic operations (Table 7) were used by 51 % of girls and 40 % of boys, but the difference is not statistically significant (χ^2 = 2,400; g = 2; P=0,301). Subtraction was used by 26 % of boys and 18 % of girls, addition by girls (35 %), and also by almost the same number of boys (31 %).

		Strategy		Total
		Solving in more than one step	Solution with insight	
Gender	Boy	36	48	84
		43%	57%	100%
	Girl	36	47	83
		43%	57%	100%
Total		72	95	167
		43%	57%	100%

Table 5: Comparison of children's strategies in correlation to gender.

		Number of solutions		Total
		one	more	
Gender	boy	71	15	86
		82,6%	17,4%	100,0%
	girl	70	12	82
		85,4%	14,6%	100,0%
Total		141	27	168

Table 6: Comparison of children's' number of solutions in correlation to gender.

		Arithmetic operations			Total
		Addition and subtraction	Addition	Subtraction	
Gender	boy	34	30	22	86
		40%	35%	26%	100%
	girl	42	26	15	83
		51%	31%	18%	100%
Total		76	56	37	169
		45%	33%	22%	100%

Table 7: Comparison of arithmetic operations in correlation to the gender.

The results of our study, which show that there is no difference in gender, match with the results of the following researches (Clements and Sarama, 2008; Lachance and Mazzocco, 2006; Levine, Jordan and Huttenlocher, 1992; Sarama et al., 2008) and are in contradiction to the research of Jordan et al.(2006). Jordan et al. (2006) identified small, but statistically significant gender differences in favour of boys regarding the calculation with objects and numerical estimation. Coley's (2002) analysis indicated that girls are better at recognizing numbers and shapes and boys are somewhat better at numerical operations.

We can conclude that there is no clear evidence as to gender differences in mathematics competence of preschool children, although some differences were established, accounted for by socialization and motivation, e.g. boys are encouraged more than girls to address number tasks, even at an early age (Aunola et al., 2004; Eccles et al., 1990).

Discussion

Similar research with kindergarten children was conducted also by Tsamir et al. (2010), in which children had to compare the sets of 5 and of 3 caps and look for the ways to make the two sets equal. Our research differed from theirs in the following aspects:

- We used slightly larger numbers, one also larger than five, which are not so easily recognisable, so a child has to be able to reason about numbers in order to solve the problem,
- The role of the teacher: they did not encourage children to look for other possible outcomes,
- In Tsamir's study no additional objects were used, so when children were searching for other solutions the possibility of taking caps away was prioritized over the possibility of adding the caps.
- Different presentation of the problem: we introduced a weighing scale for comparing two sets of an identical object;
- Gender issue regarding mathematical problem solving competences of preschool children

Considering the results of our study in terms of the strategies used for solving the problem we can conclude that 55 % of all children solved the problem by insight and only 3 % of children did not come up with the solution. The majority of the children (44 %) used both arithmetic operations, 33 % used only addition, and the minority (21 %) used only subtraction, in general. If we consider the operation from the points of view of 'solution with insight' and 'solution in more than one step' we come to the similar conclusion: in both cases the use of both operations prevails, followed by addition and subtraction in the same order.

This may be consistent with the idea that in everyday life it is very natural to use both operations. As already mentioned, the shifting from one set to the other one corresponds to the use of both operations and this method is most similar to everyday context of fairness.. On the other hand, a strategy of taking away two marbles is linked to subtraction and adding two marbles is linked to addition. So, it can be assumed that this problem had the potential to link the children's knowledge both in the scope of mathematics and to their everyday experiences. We may discuss the results also in terms of the numerical counting schemes developed by Steffe (2002). He defined the initial number sequence, the tacitly nested number sequence, the explicitly nested number sequence and the generalized number sequence. If we apply these schemas to the relations between two numbers and to the strategies used by children to achieve equivalence, we may suggest that children who used the strategy named 'Insight, + and –' possessed the explicitly nested number sequence, meaning that they were able to see the inclusion relation between the two numbers (in our case 4 and 6), or, in other words, were able to perceive the set of 4 marbles as a subset of the set of 6 marbles. The strategies named 'Insight, +' and 'Insight. –' can be attributed to children who developed the tacitly nested number sequence, meaning that they were able to make decisions whether to subtract or add and they could justify their performance (Steffe, 1994). They perceived the set of 4 and the set of 6 marbles as disjunctive sets. Children who used other three strategies ('Steps, + and –', Steps, +' Steps, -') were at the stage of the initial number sequence with little or no awareness of why they added marbles or took them away and why that worked (Steffe, 1994). We understand that this categorisation is possible only for that particular problem and we cannot claim that children developed particular numerical counting schemes in general. Further research is needed to assess our proposal to relate Steffe's numerical counting schemas to children's understanding of the relations between numbers.

With extending our problem we would get better understanding of children’s strategies used for solving a problem on equivalence. Possible extensions of our problem which are worth investigating with young children are: bigger difference between the numbers of marbles, odd difference between the two numbers and bigger numbers. These extensions would give us an insight into children’s ability to transfer their strategies from one problem to another or to accommodate them if they are not appropriate for an extension of a problem.

Considering the number of solutions it is even more interesting that 16 % of the children produced more solutions, bearing in mind that most of the children mentioned before were not encouraged to come up with more than one solution. Our results, similarly to the results of the research conducted by Tsamir et al. (2010), suggest that problems characterised by multiple solution methods and multiple outcomes should be promoted already in kindergarten. Clements and Sarama (2007) also stated that young children are able of logical reasoning, which is needed for problem solving (reasoning forward, assessing the current situation and looking ahead towards some goals, as well as reasoning backward from the goal to set subgoals).

As already mentioned our respective research differed from the research of Tsamir also in the way of presenting a problem. In our case a weighing scale was used, offering a child immediate feedback about the solution, so fewer mistakes or wrong solutions were thus produced by children. A child can use a trial and error method and through guidance by the scale he gradually produces the correct solution. Similar is true for those children who take or add marbles using more than one step. We cannot say whether or not a child would stop with adding or taking away marbles at a certain point when not working with the scale. However, we claim that those children who solved the problem by insight or in one step would solve it even without using the scale. Another advantage of using the scale is also in searching for more solutions. If a child finds only one solution, the scale could lead him towards producing more of them: 'if I add equal marbles to each side of the weighing scale, there will be no change in the balance of the scale'. This clearly shows the role of the scale in solving the equations – when a child adds or removes marbles from the scale and keeps the balance, he is intuitively aware of transforming equations into equivalent equations. We therefore believe that we can use a weighing scale for introducing rules for solving equations, namely:

- the role of a scale in transforming equations into equivalent equations
- the role of a scale in understanding the meaning of the equal sign.

Such a problem with the weighing scale in primary school, when children are already able to calculate to 10 and know the symbols for addition, subtraction and the equal sign, could be used for the introduction of the concept of equation. From the results of this research it can be concluded that the idea of introducing the equal sign as a relation between numbers before moving on to a variety of arithmetic problems should be promoted (McNeil et al., 2011). Let us explain our proposal more in details. We usually present ideas of addition and substraction to children by adding and taking away objects. Such operations are dinamic in their nature because we change the initial stage of the number of objects. On the other side operations could be demonstrated also with a static situation where we compare two quantities in order to find out where is more or less. Introduction of simple arithmetics operations in a dinamic way is easier for children to understand because they see the effect of the operation. In this case the equal sign shows an effect of the operation and therefore has an operational function. Situations of comparison of two sets are less present at children's initial phase of arithmetic operations because they are less explicit. As a consequence, the relational function of the equal sign is less acceptable by the children. The use of a weighing scale in our opinion plays a role of a bridge between both functions of an equal sign. It enables children to look for different ways of balancing a scale, thereby changing static situaition (comparing two sets) into dynamic one (adding and taking away objects). They still compare two sets or in other words determine the relation between the quantities and at the same time they try to find a way of getting equality between the quantities by influencing the initial stage in a dinamic way. Equality therefore gets beside operational also relational meaning what is important for children for their further learning of mathematics where equality in general plays a very important role.

FUNCTIONAL OR PREDICATIVE? CHARACTERISING STUDENTS' THINKING DURING PROBLEM SOLVING

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The article presents a part of a research, whose goal was to study and describe the ways of applying the graphic calculator by 14-year-old students and GeoGebra computer software by students of Department of Mathematics and Computer Science in Adam Mickiewicz University during solving a particular kind of tasks. This article attempts to answer the question: Can the recognition of students' ways of thinking and the discovery of their dominant cognitive structure cause the improvement of communication during Math lessons? Can we, by analyzing the students' output, discover and characterise the way of thinking they have applied? In the article we use Schwank's theory of predicative and functional thinking.

INTRODUCTION

Good communication between a teacher and a student during Maths lessons is one of the most important factors influencing the quality of Maths education.

Effective communication in the area of both verbal and non-verbal behavior in educational situations is characterized by the contents of the school subject being presented by means of a clear language in a stimulating way. However, the teacher not only delivers the knowledge, but also formulates questions and tasks, as well as explains, explicates, answers the students' questions, analyses students' output, then comments on them, describes and evaluates (Sztejnberg, 2002). The teachers caring about good communication during their classes try to actively listen to each of their students.

Uncommonly, nevertheless it happens, few students in the same class will tell us a totally different story. These may be ideas told in another mathematical language, maybe other ideas of solving the same problem, may be different work methods, and, first of all, these may be different ways of reasoning.

I the book "Fundamentals of Communication in Education" (Sztejnberg, 2002) we can read: "In order to assure an effective performance of knowledge acquisition, students should be given such didactic tasks that are compatible with their sensory system. Such an approach will enable the students to learn with pleasure. They will discover the ins and outs of various school subjects with a great joy" (p.83).

Can the hypothesis referring to a student's dominant sensory system be applied to a student's dominant cognitive structure as well?

Can the recognition of students' ways of thinking and the discovery of their dominant cognitive structure cause the growth of students' motivation for studying and the improvement of communication during Math lessons? Can we, by analyzing the students' output, discover and characterise their way of thinking?

PREDICATIVE AND FUNCTIONAL THINKING

Inge Schwank (1995, 1999, 2001) is the author of the theory of functional and predicative thinking. Schwank's study shows that in an every human being we can observe a relatively stable tendency to represent the way of thinking characteristic of one of the two cognitive structures: either the predicative or the functional one. However, it does not mean that there is any hierarchical dependence between them, or that any of these types of thinking is more advantageous for mathematical thinking. What is more, the inclination for a specified way of thinking does not exclude reasoning in the other way. Besides, one can think functionally while arguing predicatively. The predicative thinking structure is more effective for solving more complex cognitive problems (Schwank 1995, 1999, 2001).

Predicative thinking is oriented to a structure analysis, looking for common features and similarities between objects, their systematization and creating structural connections. It is thinking in the categories of relations, which leads to generating the static mental representations of the considered ideas (Nowińska, 2008).

Functional thinking involves a tendency to searching the differences and changes interpreted as the results of the ongoing process. It leads to the systematization of the observed objects according to the functional criterion and to the creation of dynamic mental representations of the considered ideas. It is thinking in categories of functions and processes (Nowińska, 2008).

The figure below illustrates the differences between the ways of thinking described above.

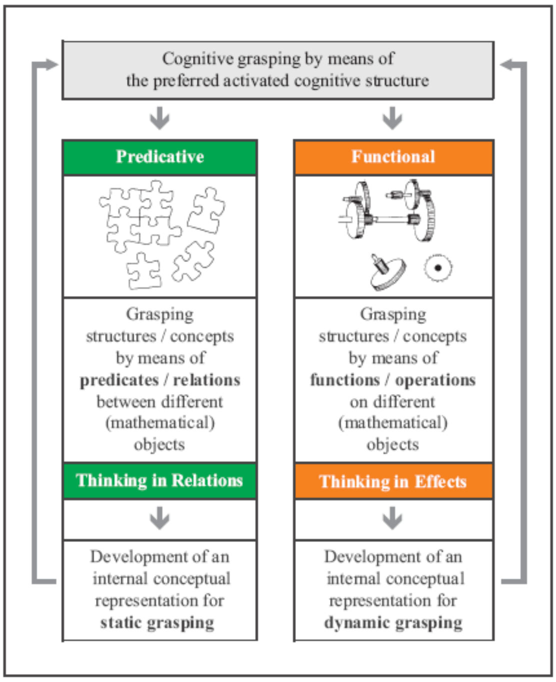

Figure 1 Predicative versus functional cognitive organisation (Schwank, 1995)

For diagnosing the way of thinking Schwank used the tasks from the Raven's Matrix Test1. In each of the test tasks the ninth, missing element needs to be included. The analysis of the way to find the missing element from the test allowed Schwank to attempt to define the tested person's way of thinking. The research material consisted of the record of eyeball movements, which was obtained with an advanced research method using the EYEtracker. For the task presented below, the analysis of the material obtained from the device has shown that people with the predicative thinking structure had focused on looking for similarities - they had always analysed the same top in each column and line of the arrangement of drawings, looking for similarities in their structure, while those with dominant functional thinking focused on the changes which had been the result of a certain process, happening in the structure of the drawings in the columns and lines of the arrangement, which has been shown in the picture below.

[1] **Raven's matrix test** (progressive matrix test) – used to measure the index of general intelligence. Thanks to that test the information about a particular person's value of so called g factor, a general intelligence factor, is obtained.Raven's matrix test in a standard version consists of 5 scales A, B, C, D, E, in each scale there are usually 12 tasks, which involve an examined person to grasp the relations among the elements of a matrix and point to the missing element from the matrix from the elements presented below the matrix. The level of difficulty is the smallest on the scale A and it increases in each following scale in such a way that the level of difficulty in the last scale is the biggest he test diagnoses so called non-verbal intelligence, independent of the examined person's experience, education, origin, etc. It checks the abilities of an individual to a logical induction, noticing the principles of continuity of patterns (scale A), observing analogies between pairs of figures (scale B), progressive changes of patterns (scale C), translocation of figures (scale D), taking figures apart (scale E).This tool is often used for the selection of employees, for instance in police forces, banks (in an advanced version). (http://pl.wikipedia.org/wiki/Test_matryc_Ravena)

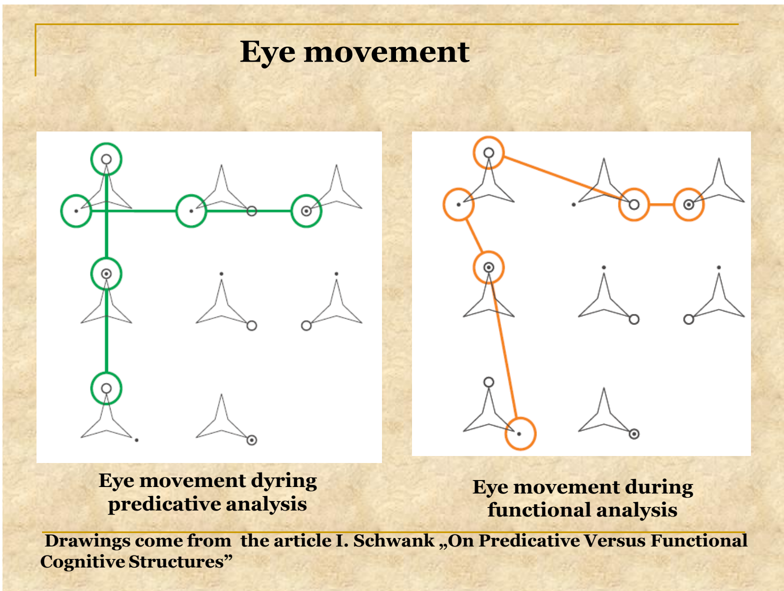

Figure 2 Eye movement

METHODOLOGY OF THE STUDY

The study of literature relating to predicative and functional structures of thinking provoked me to analysis accumulated research material, which were the effect of investigations the relating different ways of solving non-typical tasks by pupils and university students. I took the trial of the answer the question:

Is the analysis of the students' solution to the task sufficient to define their way of thinking, and discovery of their dominant cognitive structure - functional or predicative?

I analyzed, so far, five of the junior high school students' work and five University student's work. And therefore, the presented in this article data are the result of small analysis the investigative test yet.

The first grade junior high school students, whose work has been analyzed, took part in an examination referring to the methods of work on non-typical tasks with the use of a graphic calculator. The tasks were solved by four students, in their free time, that is after school classes, in the period covering their first grade. The extra classes took place once a week. During their work on the task the students used a graphic calculator, which was recording all the students' activities step by step. The research material consisted of a film of the students' work on the task with the use of a graphic calculator, the recording of a discussion with the students which took place after they had finished their work on the task, and also the students' work sheet.

The students of Math and Computer Science Department of Poznań Adam Mickiewicz University who are specializing in teaching, have participated for many years in a research referring to the ways of working on non-typical tasks with the use of new technologies as well as in a classical, "pencil and paper" way. The research group consists of about 10 students every year, who during the academic year attempt to solve a dozen or so tasks. The first attempt to solve a problem happens without a IT, while the second one is performed with a chosen tool, that is with either a graphic calculator or a free computer software called GeoGebra. The research material collected during the performance of a certain work stage without using the IT, that is only paper sheets, has been analyzed.

ANALYSIS OF STUDENTS' WORK

The analysis of the way in which the examined student had found the ninth element in Raven's test, either on the basis of a discussion with the examined student on the detailed strategy of work which they had applied or in the course of using advanced research tools such as EYEtracker, which enables to follow eyeball movements, very thoroughly allows us to define which of the two ways of thinking, the predicative or the functional one, is dominant for a particular person.

Is the analysis of the students' solution to the task sufficient to define their way of thinking?

In order to answer that question I have analyzed the work of first grade junior high school students as well as the Math department students specializing in teaching. In both groups I have analyzed the solution to just one particular problem, different for each group.

The remaining part of this article will present the essence of the tasks under the research, random solutions and the analysis of the research material, on the basis of which I have attempted to define the dominant thinking structure.

Analysis of the junior high school students' work

<u>Task</u>

For what values of a the graphs of the function $f(x) = ax$ are perpendicular to each other?

A description of two students' work has been presented below. This characterization consists of a scheme including information about the students' work stages, accompanied by a commentary on the student's activities. Additionally, there is a detailed description of the part of the problem, which makes it possible to attempt to define the dominating way of thinking when working on that task. The final part of the characterization is devoted to the thinking structure with an attempt to support the choice with proper arguments.

Student 1

The starting point to his reasoning was a graph of mutually perpendicular lines $y = x$ and $y =-x$. The student noticed that the lines form four angles of the same measure, which means right angles. That enabled him to obtain the graph of one of the straight lines as the result of the rotation of another one by an angle of $90°$ regarding the origin of coordinates.

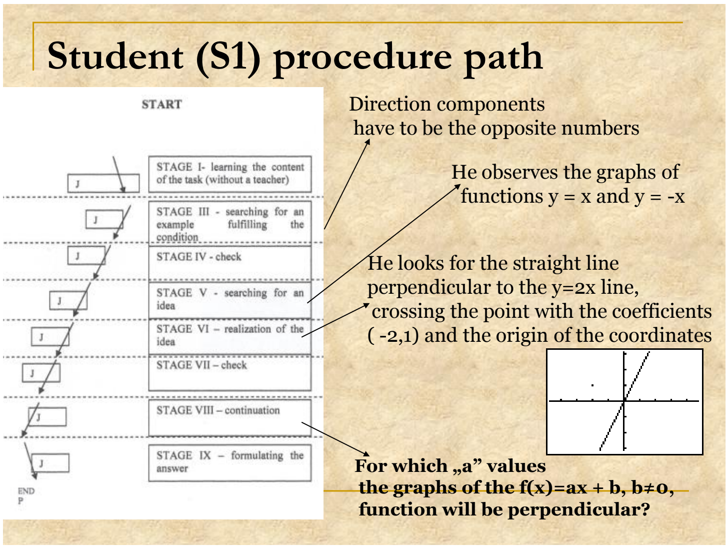

Figure 3

The method, consequently, allowed him to find the image of a point belonging to the straight line on the line perpendicular to the first one. The student completed his reasoning by comparing the coordinates of the chosen point and its image, he looked for changes which appeared as an effect of transformation.

This hypothesis of transformation in result which for given point of straight line the foundling his image on straight line to her the perpendicular he applies for concrete case. On straight line the y = 2x he chooses the point with the co-ordinates (1, 2), then he marks the point with the co-ordinates (-2, 1) (he tells that he follows on a previous case) and he looks for the formula of function, to its graph which second point would belong and this line would cross by beginning of origin of the coordinates. Student found the formula of the function after doing some calculation on the sheet of paper. The last stage of working it is finding the example of straight line crossing by this point.

Realization of several trials with put rule assures the student in conviction that the coefficient a of the second function has to be an opposite number to an inverse number of the coefficient of the first function. The student observing the graphs of the pair of functions y = x and y = -x, that is such, which fulfilled the conditions of task, he tried to find transformation in result which for given straight line foundling straight line perpendicular to its.

S1's work was characterized by their strong concentration on planning and performing the process, whose result would be several trials special cases of solution of task, and then their generalization. The student's thinking was dominated by looking for and analyzing the changes which had resulted from their applying of transformations within the frame of the thinking and acting process that had been planned by them.

In S1's work the functional thinking way is dominant.

Student 2

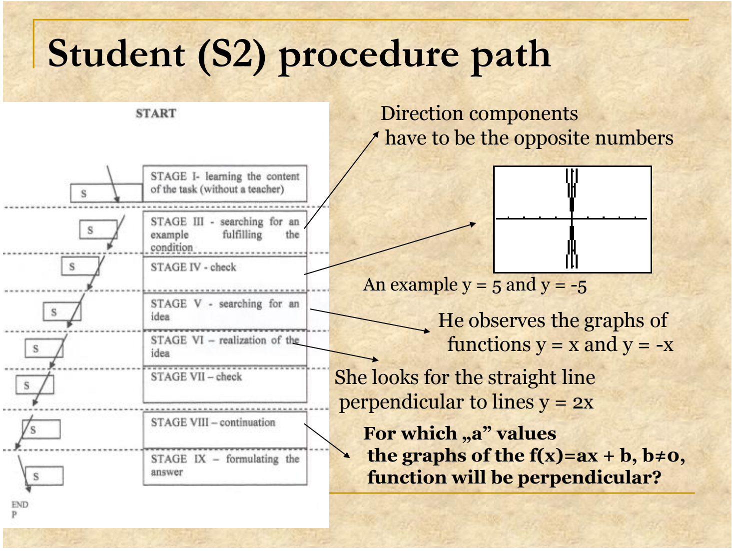

Figure 4

Student S2 draws in arrangement of co-ordinates the graphs of function, for these which look on perpendicular, it studies in what relationship are coefficients formulas of these functions. This hypothesis student checks on next examples, until to result. Working on that task student has made 22 attempts. The student in the course of his work looks for common features and similarities among the observed objects, that is the values of numbers and position of graphs in arrangement of co-ordinates, their way of thinking is performed in the relation categories.

S2's work on the task is characterized by their focusing on the discovery of the relation the looked for pair of numbers need to occur, and then generalization of features of these numbers.

S2's work is dominated by the predicative way of thinking.

Analysis of University students' work

The task content

Find all the integer pairs (x, y) satisfying the equation: $x^2 + x + 11 = y^2$

The description of the students' work has been presented below. This characterization consists of a detailed description of the part of the problem, which makes it possible to attempt to define the dominating way of thinking when working on that task. The final part of the characterization is devoted to the thinking structure with an attempt to support the choice with proper arguments.

University Student 1

Student 1 prior to the beginning work on the task had assumed that they have to transform the equation given in the content of the task in such a way to obtain such its form that would enable them to calculate all the integer pairs satisfying the equation. Each transformation is immediately interpreted with the reference to the instruction given in the task.

As a result of the first transformation there are many fractions in the equation, which disturbs the student's thinking process.

$\cancel{\left(x^2+2\cdot\frac{1}{2}\cdot x+\frac{1}{4}\right)-y^2}$ [zła myśl bo pojawiają się ułamki]

[bad thought because fractions appeared]

Figure 5

The second transformation leads the student to an equivocal distribution of the task part, so he quit the solving process again and start looking for a new idea.

$x+11=(y-x)(y+x)$

Liczby ~~x+11~~ oraz

[tak też nie bo nie będzie jednoznaczności rozkładu]

[this way also is not good, because of lack of unique factorization]

Figure 6

The third transformation, which again results in fractions in the equation, initially seems to be the wrong way once more, however, the student observes that as the result of subsequent two transformations and applying the binominal expansion formulas they would obtain the following equation form (2x+1-2y)(2x+1+2y)=-43, on the basis of which, immediately after solving four systems of equations they will be able to come up with the right solution to the equation. The student finds all the solutions.

$x^2 + x + 11 = y^2 \quad / \cdot 4$

$4x^2 + 4x + 44 = 4y^2$

$(4x^2 + 2 \cdot \frac{1}{2} \cdot 4x$

[tez ułamki]

[also fractions]

Figure 7

chociaż...

although…

$(x^2 + x + \frac{1}{4}) + \frac{43}{4} = y^2$

$(x + \frac{1}{2})^2 - y^2 = -\frac{43}{4} \quad / \cdot 4$

$(2x + 1)^2 - (2y)^2 = -43$

$(2x + 1 - 2y)(2x + 1 + 2y) = -43$

43 jest liczbą pierwszą, zatem jedyne możliwe rozkłady to: $-43 = -1 \cdot 43 = 1 \cdot (-43) = 43 \cdot (-1) = -43 \cdot 1$

43 it is a prime number, therefore the only possible distributions are:

Figure 8

The first stage of the student's work on the task, during which S1 looks for an idea to solve the problem, is characterized by their searching for such a strategy that would bring about a predicted result. A detailed analysis of the course of that stage allows to notice the signs of the student's dominant cognitive structure.

S1's work was characterized by their strong concentration on planning and performing the process, whose result would be such a form of the equation given in the content of the task that would enable the application of one of the familiar patterns for solving equations with two unknowns.

The student's thinking was dominated by looking for and analyzing the changes which had resulted from their applying of transformations within the frame of the thinking and acting process that had been planned by them.

In S1's work the functional thinking way is dominant.

University Student 2

Student S2 looking for an idea to solve the task for a given value of x, x being an integer, belonging to the interval [0,11], calculates the value of y.

Figure 9

The student observes that for the 12 cases the values of y^2 always have the unit digit equal 1, 3 or 7. They conclude that integer radicals ending with 1,3 or 7 are for instance 121, 361, 441… Following this track the student obtains two out of four solutions to this equation (-11,11) and (10, 11).

The student in the course of their work looks for similarities among the observed objects, that is the values of numbers, their way of thinking is performed in the relation categories.S2's work on the task is characterized by their focusing on the discovery of the relation the analyzed numbers need to occur, on the basis of which they formulate the solutions.

S2's work is dominated by the predicative way of thinking.

University Student 3

The student tries to transform the left and the right side of the equation to such a form that they become similar, or identical.

Figure 10

They observe that for x= -11, $x^2 = y^2$,and then y = -11 or y = 11. Consequently they obtain two solutions. Assuming now that y^2 = 121, the result is that the equation is valid for x = 10 or x=-11.

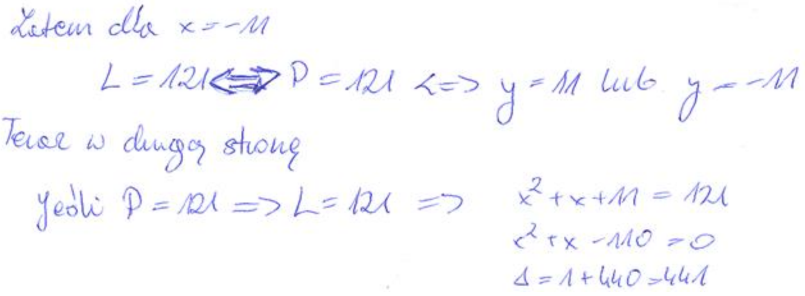

Figure 11

The student has obtained four pairs of solutions, all of which had been looked for. However, by using the empiric method of concluding, their reasoning should have resulted in the answer to the question whether those were all the possible solutions.

S3's work, like S1's, was characterized by a strong focus on planning and performing the process, whose result should be such a form of the equation given in the content of the task, which would enable the application of some heuristic task-problem solving strategy. The student's thinking was dominated by searching for and analyzing the changes which were the result of the transformations they had applied in the course of their thinking and acting process.

U1's work is dominated by the functional way of thinking.

ADDITIONAL RESEARCH MATERIAL

Unlike case of the junior high school students, where there was no premise referring to the dominating cognitive structure, the university students had solved few selected tasks from Raven's test before they participated in the research. Having finished those tasks they described on a sheet of paper the way in which they had been looking for the missing element, and then they also justified their strategy.

On the basis of each of the examined students' way of thinking when solving the tasks from Raven's test, the conclusion is that the dominant thinking structure of S1 and S2 was the predicative structure, while S3's was a functional one. The above analysis of the task solution brings about a conclusion that S1 and S3 were thinking functionally, whereas S2 was thinking predicatively. So in one case the diagnosis of the thinking structure based on Raven's above task. That confirms the fact, which had already been observed by Schwank, that the tendency to a specific way of thinking does not exclude thinking in other ways.

CONCLUSIONS

An attempt to define the way of thinking of the examined junior high school students and university students on the basis of their solutions has turned out to be not an easy task, particularly in case of the university students' work. The working process of the junior high school students has been thoroughly recorded and described due to the possibility of recording their work step by step; this enabled a thorough description of the problem solving process as well as the description of the students' way of thinking, and then identification the dominant cognitive structure. The documentation of the students' work on the task (without a calendar or a computer) was only a worksheet, which was much poorer that the documentation of the junior high school students. This impossibility of observing every step of the students' work narrows and restricts the way a teacher sees the thinking process engaged in the solution. Thus, a better documentation of the students' work on the task increases the chances for a proper definition of the cognitive structure. The application of IT together with the tools recording the work on a calculator or a computer allows to observe the process of solving and thinking in a better way, and therefore to better define the students' way of thinking.

In this article are results of analysis of two junior high school students' work and three students' work. In total, so far, I have analyzed ten solutions. It is the small test yet, but the attempt to answer the question whether on the basis of a task solution we can conclude about the students' dominant thinking structure, has completed successfully. However, the process itself was not equally difficult for each of the examined works. We have to remember, that "All of them are tought in the same way to such an extent as the children's nature is one and unified, and the methods is suitable to the development laws and the subject. However, even here there are differences and departures, which take into consideration the children's properties. One child has an easy access to ideas, while the other one the viewing" (Sztejnberg, 2002, p.87). I may add that one student prefers predicative thinking another functional thinking during solving specific mathematical problem. Not all can and should become the same and do the same.

Thinking is never achieved in the vacuum. It is influenced by both the cognitive and emotional atmosphere, independently from whether you are conscious of it or not. If you have to think mathematically in an effective way, you need to have enough confidence in order to try out own ideas without any fear. Teachers must necessarily understand how important it is for the students to have that self-confidence. They should make an effort to create such conditions in which a student can be successful (Mason, 2005).

The understanding of the existence of the aforementioned differences and departures in the students' ways of thinking and acting, the capability of diagnosing and defining them certainly may influence positively the quality of communication between a teacher and a student. It may cause a more thorough and deliberate selection of the material for the class work, which may enable an active functioning of students during lesson, therefore active communication.

CHILDREN'S EGOCENTRIC DIFFICULTIES WITH CONCRETE OPERATIONAL PROBLEMS IN LEFT-RIGHT SPATIAL SITUATIONS

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The study deals with children's egocentric difficulties arising in those spatial situations which require objectivization, decentration, and mental inversion of static relations and transformations. The tasks given to 78 children between 5 and 8 years of age concerned identifying left and right side (of the subject or of a third person), position of an object (to the right or to the left of a person), direction (moving, actual or potential, to the left or right of a place), left and right turns. The responses were analysed in terms: low, intermediate (Vygotskian zone of proximal development), high. Some children manifested a lack of an apparently obvious cognitive invariant – awareness that the left hand of a man must remain left despite a change of his position or a movement of the observer.

INTRODUCTION

There is a huge amount of literature on children's difficulties with simple problems that involve distinguishing the left and right sides. Much is known on children's ability to use the labels "left" and "right" correctly for naming their own body parts and parts of a person *who looks away*, i.e., *in the same direction as the subject*. Also much is known on the crucial question of how children transfer their own orientation to objects in the milieu and to the orientation of *a person facing them* (i.e. *directed towards them*). The latter situation is *operational* as it requires operational (more precisely: *concrete operational*) level of subject's thinking. However, relatively little information can be found on natural daily-life operational *complex* problems dealt with in this paper.

Those children who make incorrect judgements are believed to use a stationary egocentric reference frame, whereas more advanced subjects perform imagined mental rotations to align the self's frame of reference with the other's. Typically, studies in this area are based on laboratory tests according to some psychological paradigms and models of functioning of mind. Researchers make distinctions between egocentric and allocentric representations, hypothesize about processing spatial information, measure the response times, compare children speaking different mother tongs, and interpret such data in terms of the assumed framework. Further studies concerned children rotating movable objects and mental rotation of more advanced shapes. We do not discuss those questions here.

The purpose of the present paper is to deal with *the left-right problems from an educational point of view*. A wide range of tasks have been formulated to systematically show various operational, dynamic, left-right situations which potentially could be used in the school. Diagnostic interviewing in Piagetean clinical style made it possible *to identify children's difficulties* and *to formulate suggestions of how to organize their activities as to help them explore the problems*. This objective was regarded as more important than finding out, e.g., how the success rate depends on the age of the child or on other factors. Special attention was paid to cases of didactically interesting, unexpected behaviour of children.

THEORETICAL FRAMEWORK

The paper is based on Piagetean approach to spatial cognitive development as presented in Piaget and Inhelder (1948), Piaget, Inhelder and Szeminska (1948), Piaget and Inhelder (1989, Chapter 4), Wadsworth (1996) and on its neo-Piagetean reinterpretations (Donaldson, 1978).

Piaget (1924) examined 5- to 11-year-old children dealing with series of right-left tasks based on the same conception and varied only in the level of comprehension required for their solution. The results revealed characteristic developmental progress. This line of research was continued by Elkind (1961), DeVries (1971, pp. 1-14), Presson (1980), Cox and Willetts (1982), Newcombe (1989), Roberts and Aman (1993), Rigal (1994, 1996), Grush (2000).

Generally, most children 7 years of age correctly use the left and right terms for naming their own body parts and represent the location of objects in space relative to the three body axes of the self (front-back, up-down, left-right). They can also correctly name parts of the body of a person who looks away. However, even at 10 years of age many children have great difficulty in naming left and right of a person facing toward the subject.

Imagining a 180º rotation of perspective is still very difficult for a quarter of 11-year-olds. In a Piagetean task, three strung big beads A, B, C in three colours are introduced into a cardboard "tunnel" in a given order; then the "tunnel" is rotated by 180º and the child's task is to figure out which colour will appear first when the beads are pulled out at a given side. Only children who have reached a mature concrete operational level are able to understand such transformations (Szemińska 1981, 2.1.5; Wadsworth 1996, Chapt. 5).

Egocentrism

This is a crucial concept for the present study. However, its meaning varies. In his early works, Piaget was originally concerned with two aspects of egocentricity in children: language and morality, which are not considered here. We only deal with the spatial aspect of the problem: *the child's inability to imagine taking a different perspective in space.*

Piaget and Inhelder (1948) did the celebrated three mountains test. They put children (living in a mountainous region) in front of 3 mountains: small (green), middle (brown rocks) and large (white snow), built in a row on a table and showed them four pictures, described as taken by a doll photographer from the four sides of the table. Children were asked to show which "photo" was taken from a given side. Those before age seven (during the preoperational stage of development) systematically showed the picture of the view they themselves saw at the moment. Piaget regarded this as an evidence of the children's lack of ability to accept a viewpoint different from their own, labelled as egocentric; he thought that only when entering the concrete-operational stage of development children became capable of decentering and could appreciate viewpoints other than their own. Later the mountains test was criticized; other, differently devised studies showed that Piaget overestimated the egocentrism in children (Donaldson, 1978, Chapter 2).

Szemińska (1981, section 2.5.5) pointed out that younger children gave egocentric answers *even in situations when they themselves had seen otherwise*. When they were prompted to go to the place where the "photographer" was placed, they showed the right picture, but after coming back to their previous place they again showed the same as at the beginning.

This means that egocentrism is not simply the child's inability to see other people's viewpoints, as those children disregarded *their own* previous experience. Apparently they *did not understand the difference between what was objective and what they were seeing at the moment*. However, Szemińska not only described the consecutive levels of the development of understanding such questions; she also stressed that *after exploring various positions* of themselves and of the photographer at that table *the children made considerable progress*.

METHOD

Clinical interviews were designed by the author and administered by three graduate students (Sylwia Klepczyńska, Aneta Lewińska and Wioletta Pasik). At the beginning each child was asked to show left and right parts of his/her body; those who could not do it and those generally reluctant were not included in the further part of the study, *whose core concerned operational tasks*. The children were also asked to say whether they were right-handed or left-handed.

The tasks were given to a sample of 78 children aged 5 to 8. Seventy of them were interviewed individually; the remaining were pairs of children sitting one in front of the other. The interviewer showed them some tasks from the series listed below; *she was free to choose tasks and to modify or amend them*, depending on the subject's behaviour. Thus, the number of children who have performed a given task varied (from 8 to 28).

The children were prompted to explain their answers and to say why they thought so. The responses were analysed – for each task separately – in terms of three levels: *low* (such child could not correctly answer the operational questions; moreover, the fact that he/she just watched various pertinent configurations was not convincing), *intermediate* (which could also be interpreted as Vygotskian zone of proximal development), and *high* (the child either gave correct answers immediately or did it after making some motions which imitated turns of an imagined object).

Each child took part in one session which lasted from 15 to 70 minutes, depending on how quick he/she was and on the number of tasks chosen; when the child appeared tired, the talk was not continued.

DESCRIPTION OF THE TASKS AND AN OUTLINE OF RESULTS

The tasks prepared for the study are here divided into seven types. Some types may be further divided into subtypes or variants, depending on how the details are specified. The type describes *the mathematically essential features* of the task. After having chosen a suitable type one had to think of its *pertinent concretizations*, i.e., real-life stories with the mathematical structure embedded in them in a relevant way, natural for children. In this study the context of each story was devised so as to mildly clarify the intended mathematical meaning of the task and to avoid abstract words in the conversation with the child.

Type I. *The child, without moving, watches a person who either* (a) *looks away, or* (b) *faces the subject, or* (c) *turns round. The task is to show the left hand of this person (or to describe it if the person holds something or keeps a hand up).*

This is the well-known type mentioned above. Case (a) is not operational. Children interviewed in the study who could identify their own left hand simply transferred this orientation to the object; performing imagined mental rotation was not needed. Some children spontaneously peeped at their palms or put their hands on the chest to recall the position of the heart.

In case (b) most children aged 5 and 6 and a significant part of those aged 7 and 8 gave incorrect answers. *Specific success data depended*, in particular, *on the way the object of the task was presented*. It could be: a drawing of a single person shown to the subject; a drawing of several persons, part looking away and part facing the viewer; a movable cardboard figure standing on the table in front of the subject; the other child (in case of a pair); the interviewer herself.

The egocentric behaviour of several 6-year-olds was intriguing. E.g., girls X and Y sat facing each other. X wore a watch on her right wrist and said so. Then Y said: “No, it’s her left hand”, and they started to quarrel. Y was asked to go round and look from behind of X; then she admitted: “Yes, indeed, this is her right hand”. After coming back to her previous place, Y said that it was the left hand. The girls repeated the experience, but could not solve the enigma; the inteviewer left it so without explaining it.

An interviewer tried to help a boy by standing before him, with her back to him. She raised her right arm and asked: "Which arm is it?". After the answer: "right arm", she turned round keeping the arm up and repeated the question. The boy answered that it was now the left arm. Several children (including some 8-years-olds) behaved this way. This was also the case when a girl after the turn of the interviewer correctly said "right arm", but when they sat down facing each other and she was asked: "Show me my right hand", she pointed at the left one.

Several children immediately knew the correct answer. Several other children carried out the move in question or imagined it; this was helpful and after some trials they overcame difficulties.

Type II. *The child watches a picture in which certain objects P_1, P_2 ,... (imprints of hands and bare feet in the damp sand) are arranged in various directions; moreover, each object P_k is definitely left or right. The task is to label the left objects with an L. In a more difficult version, the subject and the picture are not supposed to move; or he/she may change freely his/her position so as to see some P_k in the easiest direction up-down; or he/she may rotate the picture.*

This type differs from the previous one in that the objects are things rather than persons; those things, however, are related to the human body and the attributes "left", "right" do apply to them.

Each child was first asked to put the left hand on the table (so that the backside of the palm was seen) and to look at it carefully. Then he/she was asked to scrutinize carefully the imprints of hands and feet in the picture, to identify the left ones, and to label each of them with an L (children who did not know letters marked dots). Some subjects spontaneously turned the picture (to see a hand or foot in the direction up-down) or turned their hand; this helped them to identify the left imprints. Some others found the task too difficult and labelled the imprints at random. In some cases hands were correctly labelled but feet turned out more difficult.

Another task of type II concerned pictures of top sides of mittens (a mitten is a glove with a separate part for the thumb and a single part for four fingers together). The child got (a) pairs of movable small slips of paper with pictures of mittens, each pair in a different colour, and (b) a sheet of paper with identical pictures but now unmovable and arranged in various directions. On the sheet, however, there was only one mitten from each pair (left or right). The task was to find the matching mittens among those in (a). In both tasks of type II and in each age group (5 years, 6 years, 7 years) some children succeeded, whereas some other could not manage in cases when a hand or foot was turned around.

Type III. *Two objects P and Q (e.g., a pencil and a rubber eraser) lie on a table. The first part of the task is to say on which side of P (to the left or right) is Q. Having answered this, the child goes to the opposite side the table, takes a look at the objects from the other side, and then is asked the same question.*

The objects P, Q have *no consciousness*; therefore the phrase "on which side of P" may only refer to the sides with respect to the subject S. After the child's move to the opposite side of the table, the *objective* positions of P and Q remained unchanged, but what S had seen to be to the left was now to the right, and vice versa.

Some children were surprised by this experience and could not understand how it could happen. Some other grasped the situation and reasonably articulated it.

Type IV. *The child S does not move and watches a person P standing or sitting by an object Q. Here P is a human being or, e.g., a fairy tale personality; Q may be a person or a thing. The point is this: P is now attributed with consciousness. The task is to find out what P would say: on which side of him/her (to the left or right) is Q.*

Thus, the question does not concern the side of P as seen by S, but what P sees himself/herself. The phrase "what P would say" is intended to help the subject understand that it is the viewpoint of P that matters here. If P faces away in the same direction as S, the left sides of P and of S agree, so the task is not operational. However the case when P faces S requires imagining a 180º rotation of P.

This abstract setting was concretized as follows. The child was shown a picture of a girl standing with a dog in a park. The girl was facing away, one saw her back. She held the dog in leash with her left hand; the dog was sitting on the left of her. When the subject had described the relative positions of the girl and her dog, he/she was shown a second picture. The surroundings (details of the park) were the same, but the girl had turned round and was now facing the viewer; the dog was still sitting in the same place – to the left of her as seen by the subject. The subject was asked to find out what the girl would now say: on which side of her (to the left or right) was the dog sitting now. In the second picture she held the dog in leash with her right hand; this was intended to help the subject to grasp that from the girl's point of view the dog was on the right of her. Difficulties of the children dealing with this task were akin to those described above. For some children the task became easier when the interviewer had suggested: "Imagine that you are this girl and you hold the dog in leash".

In another task of type IV the child got four pictures of four girls named Ala, Ela, Ola, Ula, sitting cross-legged, each dressed differently. Two girls were facing away and two were facing the viewer. By each girl there was a ball; two balls were to the left of the girl and two were to the right, in the four possible combinations. The subjects were to find out what each girl would say about the position of the ball. The interviewer's hint: "Look, I am now this girl" helped some of them to imagine the situation and to perform a mental rotation. One child spontaneously crossed her hands, as if she wanted to help herself with a gesture of interchanging the hands of the person in the picture.

Type V. *Persons P and Q are motionless; the observer S changes his/her position. The situation is somewhat similar to those in types III and IV, but now both P and Q are regarded as conscious.*

As a concretization of this general scheme, two successive pictures were shown to the child. The first of them (Figure 1a) depicts a family consisting of mother on the left, father on the right, and a boy between them. They were standing in a row, facing away. Mother and the boy held hands; the boy and father also held hands; mother held a rolling shopping bag with her left hand and father held a baby pushchair with his right hand. The subject was to specify who was to the left or right of whom and who held what with the left or right hand. Moreover, a girl was standing in front of the family and was taking a picture of them.

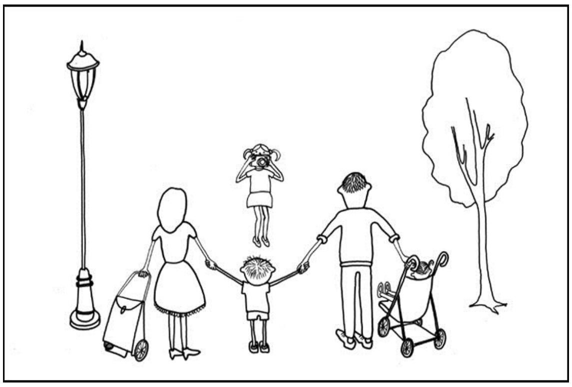

Figure 1a

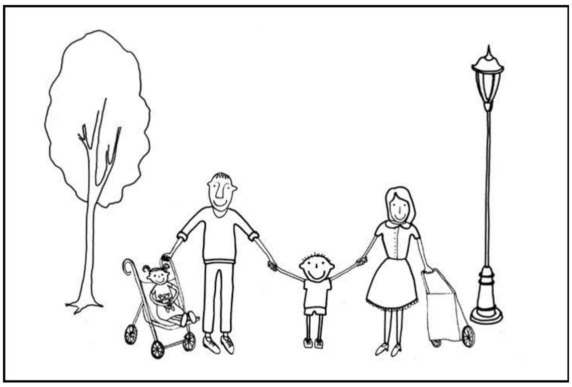

Figure 1b

When this phase of the task had been over, this presumed "photo taken by the girl" was shown to the subject (Figure 1b). The family was now seen from the other side, so they faced the viewer. The main difference between type IV (girl and dog) and type V lies in that in IV the surrounding park looks identically in both pictures (it is seen from the same place) and the girl moves (she turns round), whereas in V the family is (objectively) standing without move (posing to the photo, the father holds the baby pushchair with his right hand), but in the second picture the family, the tree and the lamp are seen from the other side.

Some children had answered the non-operational questions concerning the family seen from behind and then were also asked to say what the photographer saw: "Did she see the baby pushchair to the left of the father or to the right?"

The children's difficulties with the operational parts of the tasks V were akin to those experienced in dealing with the tasks IV and the success rate was similar.

Type VI. *According to traffic rules, cars should drive on the right side of the road, whereas the pedestrians (in absence of a sidewalk) should walk on the left side. Children watch pictures with left-right operational traffic situations, take part in activities simulating such situations, and are asked relevant questions.*

A mathematically essential feature of this type is that cars and people in question do not make left or right turns. They *go straight ahead*; possibly also may do U-turns to go in the opposite direction. Children were shown pictures of a road with cars and/or people moving *correctly* in one or two directions. When a car or a person was moving towards the subject, the situation required operational thinking. Some children said that the approaching car was driving on the left side or that the approaching person was walking on the right side. Some activities were designed to help children gather experience needed to grasp such situations. They were asked to walk on the right side of a lane and then to turn round and go back. Some of them were surprised: they had walked on the right side of this "highway" and after the turn on the spot they found themselves unexpectedly on the left side; they could not understand what had happened.

In another task a group of children walked in file along the left side of the school corridor (as if they were walking on a public road), by classroom doors, keeping their left arms up. When they reached the end of the corridor, they turned round and were supposed to go back and continue along the left side of the corridor, but this was now along the windows (Figure 2). Children were asked what they thought about those walking on the other side, whether the children on their way back were going on the proper (that is, left) side. The non-operational answers "They are going on the right side" were confronted with other evidence (which arm was up, at what side were the windows) to help the child. In a more difficult version of the task the subject was only shown a picture with children so going.

Figure 2

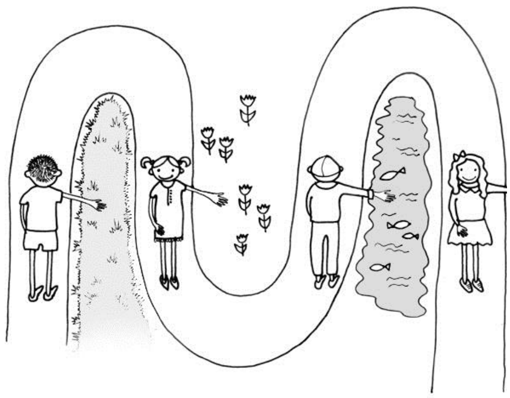

Figure 3

Type VII. *Left and right turns performed by the child or shown in a picture*.

Children went outdoors. The interviewer drew "mini-streets" on the ground with several turns, Y-shaped road forks and cardboard road signs "left turn", "right turn". Children got cardboard steering wheels and walked over there, *pretending to drive*, making suitable turns and spinning the wheels accordingly. After such experience each child was shown a drawing which depicted a girl and a boy holding such wheels, heading on a street in opposite directions, and both willing to turn at an intersection to the same side street. The subject was asked several questions: "On which side of the road does the girl/boy go?" and "What turn – left or right – is the girl/boy making?" The task was fairly difficult for subjects 8 years of age and much harder for younger ones – a cumulative effect of operational left-right troubles and lack of experience with road traffic. An exception were two 5-year-olds who performed flawlessly; it turned out that both had toy roads at home, with cars, traffic signs, lights, toy people, and had played with their parents.

In another task the interviewer *drew a winding road on the ground*. Children walked there, *pretending to ride a bicycle*. Before each turn they gave standard hand signals to indicate their intention to turn. After this practise each child got a picture of a winding lane in a park (Figure 3); two girls and two boys (dressed in different colours to facilitate the discussion) walked there along the lane, signalling the intended turns. The task was to say which girl or boy signalled left or right turn. The difficulty lay in that in two cases the hand directed towards the right edge *of the picture* signalled right turn whereas in two other cases – signalled left turn. Subjects who had reached an operational level of thinking explained that this depended on whether the girl or boy was facing away or towards the viewer; for other subjects this was incomprehensible. The winding road – *bending alternately* to the left and to the right – also caused difficulties.

A problem arouse with the linguistic diversity of the meaning of the word "direction". The girls and boys went into the same direction (i.e., they followed the lane and aimed at reaching its end) and at the same time they changed the direction at each bend. The wording of tasks and hints should take this into account.

CONCLUSIONS

Overall children's responses to the tasks of the study followed a similar pattern, with great individual differences. As anticipated, many children had serious difficulties with all operational tasks, particularly when the situation was presented in the form of a static picture. This applies to almost all 5-year-olds and to many 8-year-olds (specific data depended on the task and its secondary details). Generally, it was confirmed that success rate increases as children grow; however, in some tasks the results of 5-year-olds were better than those of 6-year-olds. Previous personal experience with left-right situations, performing movements and looking at objects from different points was crucial. Some children learnt about left and right turns when they were driven and heard adults' comments.

Several children manifested a *lack of an apparently obvious cognitive invariant – they were not aware that the left hand of a man must remain left despite a change of his position or a movement of the observer*.

One of the goals of the study was a description of the levels of the individual behaviour of children (for each task separately) in terms: *low* (explanations and hints do not help), *intermediate*, *high*. The intermediate level was particularly important, as it could be interpreted in Vygotskian terms of the zone of proximal development – as what the child can do with little assistance from an adult. The most efficient help was to generate a cognitive conflict and then advise the child to imagine suitable movements or actually make them and then to reflect on that.

Standard education pays little attention to left-right situations, reducing them to proper naming of people's body parts and understanding descriptions such as “at the upper left corner of the picture”. However, correct interpreting left and right situations is important in daily life, in traffic situations, while reading maps etc.

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UNDERSTANDING THE INVERSE RELATION BETWEEN QUANTITIES RELYING ON CHILDREN'S WRITTEN COMMUNICATION

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This study analyzes children's understanding of the inverse relation between quantities when division situations and the concept of fraction in quotient and part-whole interpretations are involved. A survey by questionnaire was conducted with 42 Portuguese fourth-graders. Children's written justifications and explanations were analyzed in order to get an insight on their reasoning. Results suggest that ordering and equivalence of fractions in quotient interpretation promote children's understanding of the inverse relation between size and number of parts more than when part-whole interpretation is involved or even partitive or quotative division situations. Children's written justifications support the idea that the correct answers were not obtained randomly.

THEORETICAL FRAMEWORK

This study investigates the understanding of the inverse relationship between size and number of parts in division situations and when fractions are presented to children in part-whole and quotient interpretations. To explore this understanding, children's performance is analyzed as well as their written justifications when solving the tasks.

Understanding the inverse relation between quantities

Considering the mathematical contexts approaching the inverse relationship between quantities, fractions and division situations are emphasized. Literature presents several studies focused on the students' understanding of the inverse relationship between quantities. Some are focused on the concept of division (see Correa, Nunes & Bryant, 1998; Mamede & Silva, 2012), others focused on the concept of fraction (see Behr, Wachsmuth, Post & Lesh, 1984; Kornilaki & Nunes, 2005; Mamede, Nunes & Bryant, 2005; Mamede & Cardoso, 2010).

In order to understand the aspects involved in division situations, it is important to distinguish the difference between partitive division and quotitive division. In partitive division, the quantity is divided between the number of recipients, and the part received by each recipient is the unknown part (e.g., John has 8 sweets to be shared between 4 children. How many sweets each child will receive?). In quotitive division, a quantity is divided and what each recipient will receive is already known; what is left to know is the number of recipients (e.g., Mary has 6 sweets and will give 2 sweets to each child. How many children will receive sweets?). Exploring the inverse relation between divisor and quotient in division situations, it makes sense to consider two kinds of tasks, when one of the dimensions is held constant (dividend or divisor) (see Correa, Nunes & Bryant, 1998). For the first situation, the divisor can be held constant and the dividend can be changed. In this case, children must understand that the bigger the whole, the bigger the parts, if the number of parts is held constant. For the second situation, the dividend is held constant and the divisor is changed. The divisor corresponds to the number of recipients or the size of the quote. In either case, the inverse relationship is applied - the bigger the number of parts, the smaller the size of the part, or vice-versa.

Kornilaki and Nunes (2005) argue that children understand more easily partitive division than quotitive division, because they use term-by-term correspondence as the procedure to solve this type of division, once it is more simple thinking about the inverse relationship than building each quote.

More recently, Mamede and Silva (2012) investigated children's understanding of partitive division with discrete quantities with 30 children aged 4 and 5. In individual interviews, children were asked to make judgments in tasks with inverse relationship between divisor and quotient when the dividend is the same. The tasks involved division of 12 and 24 discrete quantities by 2, 3 and 4 recipients. Results showed that children aged 4 and 5 have some idea about the division, are able to estimate the quotient when the divisor changes and the dividend is constant, and are able to justify their answers.

The inverse relation between quantities is essential to understand the concept of fraction. Research has been giving evidence that children struggle with the concept of rational number (see Behr, Wachsmuth, Post & Lesh, 1984; Mamede & Cardoso, 2010). Studies focused on different interpretations of rational number suggest that these interpretations affect differently children's understanding of fractions. Some authors argue that the quotient interpretation favors the understanding of the inverse relationship between numerator and denominator of the fraction (see Mamede, Nunes & Bryant, 2005). Nunes et al. (2004) suggest that this understanding is facilitated in quotient interpretation because numerator and denominator are variables of different natures. Nevertheless, traditionally fractions are introduced to children using the part-whole interpretation of fractions. In quotient interpretation, $\frac{a}{b}$ can represent the relation between the number of recipients and items to be shared (e.g., $\frac{2}{3}$ can represent 2 chocolates to be shared fairly by 3 children), but it also can represent the quantity of an item received by each recipient (e.g., $\frac{2}{3}$ corresponds to the quantity of chocolate received by each child). In part-whole interpretation, $\frac{a}{b}$ represents the relation between the number of equal parts into which the whole is divided and the number of these parts to be taken (e.g., $\frac{2}{3}$ of a chocolate bar means that this was divided into 3 equal parts and 2 of them were considered).

Mamede, Nunes and Bryant (2005) investigated whether the quotient and part-whole interpretation of fraction influence the children's performance in problem solving tasks. Eighty children participated in the study aged between 6 and 7 years-old, who haven't had formal instruction on fractions, but some of them were already familiar with the words "half" and "fourths" in social contexts. The authors analyzed how children understand fractions in part-whole and quotient interpretations, in tasks related to equivalence, ordering, and labelling. Results indicated that children performed better in quotient interpretation than in part-whole regarding ordering and equivalence of fractions; children performed similarly when solving labelling tasks presented in quotient and in part-whole interpretations. Children's success levels in ordering and equivalence of fractions in quotient interpretation suggests that they have some informal knowledge about the logic of fractions, developed in their daily life, without school instruction. These results emphasize the idea that different interpretations of fractions create distinct opportunities for children to understand the inverse relation between quantities.

As children possess an informal knowledge that allows them to understand the inverse relation between quantities in division situations and understand the logical invariants (ordering and equivalence) of fractions it particular situations, it is important to explore children's ideas about these issues and how these aspects are related. For that it becomes relevant to analyze children's performance but also their justifications, either oral or written, when solving problems.

About the written communication

When Children are encouraged to talk, write, read and listen they learn to communicate mathematically and they communicate to learn. Communication is therefore a curriculum goal and a way of learning (DGIDC, 2007; NCTM, 2000). As a curricular goal, children must be able to communicate their ideas and to interpret and understand others ideas, organizing and clarifying their mathematical thinking. Thus, they should be challenged to discuss ideas, processes and solutions. As a teaching methodology, mathematical communication should assume a prominent role in the teachers' classroom practice. Through oral discussion in class, children have the opportunity to compare their strategies to solve problems and identify the arguments produced by their colleagues. Through written texts and explanations, children have the opportunity to clarify and elaborate in greater depth their strategies and their arguments, developing their recognition of the importance of rigor in the use of mathematical language.

In mathematics classroom, communication is usually analyzed through the discourse of the intervenients (Ponte, Boavida, Graça & Abrantes, 1997), and the teacher's role is considered crucial in driving the discourse. For the NCTM (2000), to lead the discourse, among other aspects, the teacher should keep in mind, that students' reasoning should be provoked. Teachers should encourage students' thinking through the tasks they provide and the issues they raise in the classroom. The question "why?" as well as requests for explanations, should be presented regularly and consistently after students' presentation of solutions. To listen carefully to their ideas and to ask them to give justifications and explanations for their solutions using written communication are also ways of stimulating mathematics communication.

According to previous official curricular guidance (see DGIDC, 2007), the promotion of written communication, in particular concerning the explanations and reports related to the tasks they develop should be part of the classroom practice. Written communication allow children to reflect on the developed work as it demands a review of their procedures, think about how to organize their reasoning, and how to present it in a clear way (Fonseca, 2000). To be able to create and understand mathematics, it is important for children to have the opportunity to write in their own words and using their own symbols (Fonseca, 2000). Children should be encouraged to communicate by means of graphs, tables, diagrams and drawings, but also to present arguments and explain their mathematical ideas as children's justifications can give information about their reasoning. Thus, children's justifications can be a powerful tool when exploring children's mathematical ideas.

This study aims to analyze how the inverse relation between size and number of parts in division situations is related to the concept of fraction in quotient and part-whole interpretations. Two questions were addressed: (1) How do children understand the inverse relation between size and number of parts in partitive and quotitive division situations? (2) How do children understand the inverse relation when fractions are involved in part-whole and quotient interpretations?

METHODS

Participants

To assess the children's understanding of the inverse relation between quantities in division and fraction situations, a survey by questionnaire was carried out with 42 fourth-graders of 9- to 10-years-old (mean age 9 years, 6 months), from a public school in Braga, Portugal.

Tasks

The questionnaire included 22 tasks: 6 division problems (3 partitive division problems and 3 quotitive division problems); 16 problems with fractions (8 problems with part-whole interpretation (4 ordering; 4 equivalence); 8 problems with quotient interpretation (4 ordering; 4 equivalence)). All fractions involved in the tasks were less than 1 and were the same for the problems proposed with quotient and part-whole interpretations. Tables 1 and 2 show an example of a task presented for each type of division and fraction situation, respectively.

Division	Problem
Partitive	Mary and Louise have the same quantity of sweets. Mary will distribute her sweets by 3 children and Louise will distribute hers by 4 children. Will the children at Mary's group receive more sweets than, less sweets than, or the same quantity of sweets as the children at Louise's group? Explain your answer.
Quotitive	John and Paul bought the same quantity de marbles. John will put 3 marbles in each bag and Paul will put 6 marbles in each bag. Will John need more bags than, less bags than, or the same quantity of bags as Paul? Explain your answer.

Table 1: Examples of tasks presented in division situations.

Fraction	Equivalence	Ordering
Part-whole	Marco and Lara have each a pizza with the same size. Marco divided his pizza into 5 equal parts and ate one part. Lara divided her pizza into 10 equal parts and ate 2 parts. Did Marco eat more pizza than, less pizza than, or the same quantity of pizza as Lara? Explain why.	Ana and Rita have each a chocolate bar with the same size. Ana ate $\frac{1}{2}$ of her chocolate bar and Rita ate $\frac{1}{3}$ of her chocolate bar. Did Ana eat more chocolate than, less chocolate than, or the same quantity of chocolate as Rita? Explain why.
Quotient	Children share two same-sized cakes. Two girls share one cake fairly; three boys share the other cake fairly. Does each girl eat more cake than, less cake than, or the same quantity of cake as each boy? Explain why.	Two girls will share a chocolate bar and each one will eat $\frac{1}{2}$ of the chocolate. Three boys will share a chocolate bar and each one will eat $\frac{1}{3}$ of the chocolate. Does each girl eat more chocolate than, less chocolate than, or the same quantity of chocolate as each boy? Explain why.

Table 2: Examples of tasks presented with fractions.

Procedures

The questionnaire was solved individually and lasted for 40 minutes, being implemented in one session in the classroom and followed by the class teacher. The survey took place in January, and in the same day for all students. By this time of all academic year all the children already contacted either with division or fractions. According to the Portuguese curriculum, children are introduced to division and to fractions in the 3rd grade. Each child received a booklet with one problem per sheet to be solved. In each problem, multiple-choice questions were present, and the judgment for relative value of the quotients by using relations "more than/ less than/ same quantity as" was favored. Questions were presented to the class and read by the researcher using PowerPoint. Each child had to indicate the right answer on the booklet. The tasks used were adapted from the studies of Mamede, Nunes and Bryant (2005) and Spinillo and Lautert (2011).

RESULTS

Results of the children's performances when solving the proposed tasks were analyzed, by assigning 1 to each right answer and 0 to each wrong answer. Table 3 presents the proportion of means for the right answers and standard deviation according to the type of problem.

Quotient		Part-whole		Division	
Ordering	Equivalence	Ordering	Equivalence	Partitive	Quotitive
.75 (.29)	.58 (.29)	.49 (.37)	.35 (.33)	.38 (.36)	.48 (.37)

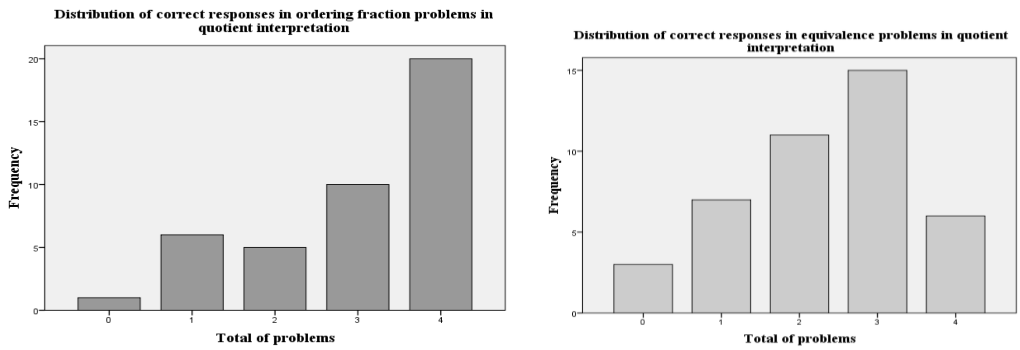
Table 3: Mean and (standard deviation) of the proportion of right answers.

Results suggest that, in quotient fractions problems children seem to understand better the inverse relation between quantities. Surprisingly, they also suggest that quotitive division seems to be easier than partitive division for children.

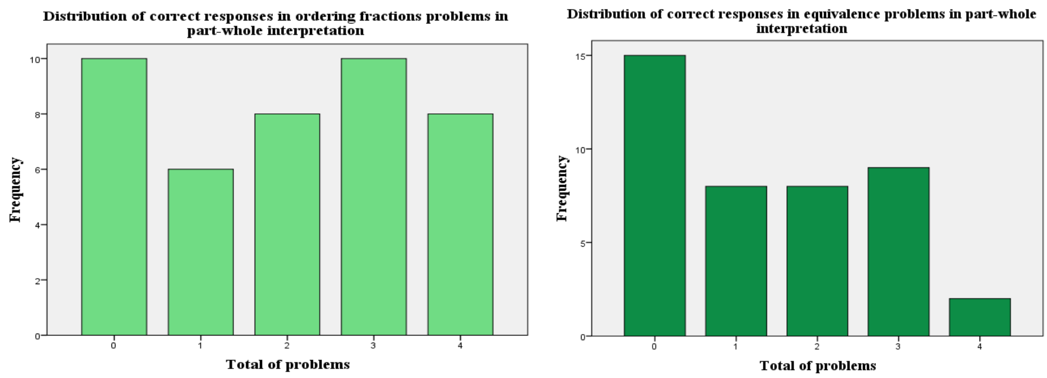
The analysis by type of problem proposed allows a better understanding of the children's performance on the tasks presented. Graphs 1A-B present, respectively, the distribution of the ordering and equivalence fractions problems correctly solved in quotient interpretation. Ordering problems seem to be more accessible to understand the inverse relation between numerator and denominator. About 47.6% of the children answered correctly to all ordering problems and 14.3% to all fraction equivalence problems presented to them in quotient interpretation of fractions; and 35.7% solved correctly 3 of the 4 problems of this type. A T-test indicates that, in quotient interpretation, children's performance solving ordering problems (Prop. Mean = .75; S.E. =.04) was significantly better than their performance solving equivalence problems (Prop. Mean = .58; S. E. = .04), ($t_{(41)}$= 3.15, $p<.05$).

The fraction problems presented in part-whole interpretation seem to be more difficult for children to understand the inverse relations between numerator and denominator. Graphs 2A-B presents the number of correct responses given by children when solving ordering and equivalence fractions problems presented in part-whole interpretation, respectively.

In ordering problems presented in part-whole interpretation, only 19% of the children answered correctly all problems and about 24% answered correctly to 3 of the 4 presented problems. In equivalence problems, about 5% of the children answered correctly to all problems and 21.4% answered correctly to 3 of the 4 problems.



Graph 1A-B: Distribution of correct responses in ordering and equivalence fraction problem in quotient interpretation.

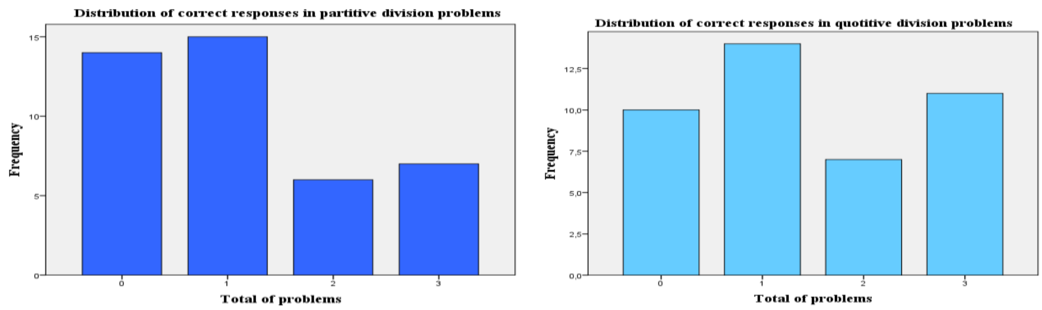


Graph 2A-B: Distribution of correct responses in ordering and equivalence fraction problem in part-whole interpretation.

A T-test indicates that, in part-whole interpretation, children's performance solving ordering problems (Prop. Mean = .49; S.E. =.06) was significantly better than their performance solving equivalence problems (Prop. Mean = .35; S. E. = .05), ($t_{(41)}$= 2.71, $p<.05$).

Concerning the division, children seem to struggle with partitive division situation problems on the inverse relation between quantities. Graphs 3A-B present, respectively, the number of correct answers in partitive and quotitive division problems. In partitive division problems, about 17% of children answered correctly to all problems; 14.3% answered 2 of the 3 problems correctly, and 35.7% answered correctly to only 1 problem. In quotitive division problems, 26.2% of the children answered correctly to all problems; 16.7% correctly answered to 2 of the 3 problems; and 33.3% gave a correct response to only 1 problem.

A T-test indicates that children's performance solving partitive division situations (Prop. Mean = .38; S.E. =.06) and quotitive division situations (Prop. Mean = .48; S. E. = .06) were not significantly different, ($t_{(41)}$= 1.87, n.s.).



Graph 3A-B: Distribution of correct responses in partitive and quotitive division problems.

These results indicate that the exploration of fractions in quotient and part-whole interpretation and the partitive and quotitive divisions contribute differently for the understanding of inverse relationship between quantities.

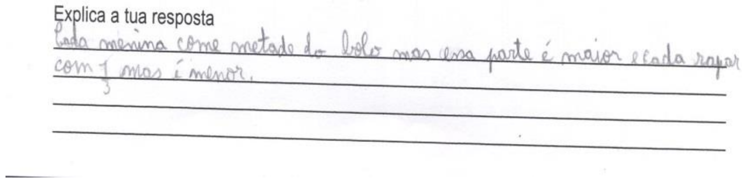
The written justifications of the children's answers were analyzed to reach a better insight of their reasoning and their ideas about the inverse relations between quantities. While systematizing the explanations given by the children, 4 categories of their justifications were distinguished: 1) *inverse relationship* – it attends to the inverse relation between the quantities involved in the problem, producing a valid justification (e.g.,"[…] because he divided his pizza into 2 equal parts and she divided hers into 4 equal parts and hers become smaller."); 2) *proportional reasoning* – it comprises a establishment of a proportional relation between the quantities of the problem, producing a valid argument (e.g., "They eat the same because there are 2 girls for 1 chocolate bar and the boys are the double of girls and they have the double of chocolate bars."); 3) *direct relationship* – it sets a direct relation between the quantities (e.g., "He eats more because he has more cake, thus he eats more cake."); and 4) *inconclusive/ invalid* – it comprises all inconclusive, inappropriate, or blank explanations.

	Quotient (%)		Part-whole (%)		Division (%)	
	Ordering	Equiv.	Ordering	Equiv.	Partitive	Quotitive
Inverse relationship	88.0	53.6	33.3	28.6	26.2	42.8
Proportional reasoning	2.4	19.0	4.8	11.9	0.0	4.8
Direct relationship	7.2	21.4	45.2	51.2	57.1	38.1
Inconclusive	2.4	6.0	16.7	8.3	16.7	14.3

Table 4: Percentage of types of justifications presented.

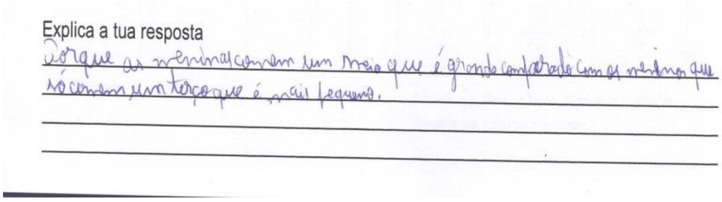
Table 4 summarizes the percentages of each type of argument used by the children according to the type of problem presented to them. It suggests that the success levels regarding the children's performances for the problems presented were not obtained randomly, since they seem to be followed by explanations supported by valid arguments.

Figures 1 and 2 illustrate, respectively, examples of valid justifications presented when solving ordering and equivalence of fractions problems in quotient interpretations.



"Each girl eats half of the cake but that part is bigger and each boy eats 1/3 but is smaller."

Figure 1: Justifications in an ordering problem in quotient interpretation.



"Because girls eat one-half which is big when compared with the boys who eat only one third which is smaller."

Figure 2: Justifications in an equivalence problem in quotient interpretation.

DISCUSSION AND CONCLUSIONS

This study suggests that children understand better the inverse relation between quantities in quotitive division situations than in partitive division. It is an interesting result, because it was not reported in previous studies (see Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005). Kornilaki and Nunes (2005) suggest that 5- to 7-years-old children have some ideas on the inverse relationship between divisor and quotient in partitive division tasks, when asked to judge the relative size of the shared sets. The results of the present study suggest that 8- to 10-years-old children also understand the inverse relationship between quantities, but quotitive division seems to be easier for them. This idea is supported by children's written justifications which indicate that a correct reasoning was established when solving the tasks. Surprisingly, children understand better the inverse relation between quantities when this relation is associated with fractions instead of partitive or quotitive division situations. To understand the inverse relation between quantities when fractions are involved in quotient interpretation is easier than when the part-whole interpretation is involved. Again, children's explanations support the idea of their understanding of this relation. Consistent with previous studies (see Mamede, Nunes & Bryant, 2005), quotient interpretation still reveals to be important for the children's understanding of inverse relation between quantities, regardless the children's age differences. More research is needed concerning these issues using not only children's written explanations but also oral communication, as children may provide different levels of performance when communicating.

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. Forum of Ideas

HOW HELP WEAK STUDENTS TO LEARN MATHEMATICS?

Krystyna Dałek,

Fundacja Rodziny Maciejko

Fundacja Rodziny Maciejko (FRM) started a project aimed at helping weak students and their teachers in learning and teaching mathematics. The projects called "Mathematics - do it yourself" is supported by modern technology. It is based on registering by means of interactive whiteboard short (5 – 10 minutes) scenes in form of dialogue between teacher and student. It is assigned mainly for individualized learning of lower secondary students (13 – 16 years old). The idea is to set mathematical problems in an informal, loose way. The aim of the project is to investigate to what extent such a form of assistance creates students activity and their understanding of mathematics.

MACIEJKO FAMILY FOUNDATION.

Maciejko Family Foundation (www.maciejko.org) was established in Warsaw in 2009. Its main aim is to help teachers of mathematics in their everyday work, to increase the effectiveness of learning mathematics in schools. The Foundation has undertaken various activities. It cooperates with schools and Universities (e.g. Warsaw Technical University and quite recently with Warsaw University), Association of Mathematics Teachers, promotes using modern technologies in teaching of mathematics. It supports various mathematics competitions as well.

WHY WEAK STUDENTS?

Observing for several years various activities in the domain of teaching mathematics in schools we realized that there exists an important offer for clever students, interested in mathematics. Many different competitions, special seminars and study visits have been organized. Each school is eager to have as many students as possible participating in such activities and being the winners or at least taking the highest places. On the other hand the mathematically weaker students are very often a burden, lowering the level of educational statistics, their teaching is difficult, time consuming, tiresome and quite often teachers do not have time for them.

There are various causes of students bad results in mathematics – sometimes they are individually justified and sometimes they are caused by family situation. Students regarded as weak ones, generally do not accept formal language. Using formulas and algorithms usually results in routine activities, but at the same time destroys mathematical understanding of problems (Benezet, 1935). Unfortunately, new technologies such as calculators, computers, interactive whiteboards entering to schools have not changed a lot. Thus, it is not the lack of technical school equipment that is responsible for the problem. (Jacobsen, 2012)

Additionally, what is underlined by many pedagogues and teacher trainers – weak students need individualized training. Teachers know very well how difficult it is to use individualized methods while teaching in a class of 30 students, that is to take care of students on different levels of interest, knowledge ant capability. It was demonstrated clearly by the experiment “Didactic Cards”, of which I was the leader in the years 1991-93 (Dałek, 1993, 1997). Besides many interesting results obtained, I would like to name one. This method of teaching needs special teaching material.

MATHEMATICS – DO IT YOURSELF

Basing on the obtained results and observations, Maciejko Family Foundation prepared a project aimed mainly on the weak students: “Mathematics - do it yourself”. The most important part of the project is to create many short (5 – minutes) films concerning school mathematics. Each film is prepared by means of the interactive whiteboard (Betcher, Lee 2009) supplied to us by the Department of Mathematics and Information Science of the Technical University of Warsaw. The recordings are in the form of dialogues based on the previously prepared scenario. Two persons discuss about various problems and subjects concerning school mathematics. At present, eight subjects on the level of lower secondary school are covered, that is: Fractions, Decimal numbers, Percentage, Rational numbers, Irrational numbers, Plane shapes I, Plane shapes II, Plane shapes III. Subjects concerning functions are in preparation. Each subject consists of a couple of parts. The recordings are successively displayed on You Tube and on the internet pages of the Foundation.

METHODS OF STUDENTS WORK

Each of the films can be used in different ways – it can be treated as a repetition or as a new subject. Also methods of use can be different. The teacher should discuss with students the methods of work, taking into account their needs and preferences. Here are several possibilities.

1. Students can watch them at home treating them as a homework, at the same time solving problems given by the teacher. The homework can be also prepared by the student, matching the current subject (e.g. the student can prepare for his fellow student a set of exercises). There must be imposed a certain self-discipline on students, but, on the other hand, it enables more concentration and multiple repeating of non-understanding fragments. Thus, elements of self-dependent thinking as well as the different point of view can be introduced (Hechinger, 1986). In this way, we have a variety of presentations.
2. Another way of using the films is to watch them in the classroom, during the classes. In this case the teacher should ask the students for comments, finding examples, asking questions and even creating new scenarios on the same subject. Good idea is also, for a chosen student, to explain in his own

words what new he learned watching the film. Using this method, the students ideas, their questions, reservations, misunderstandings can be expressed (Jacobsen, 2012).

Another possibility is to show a film to students with the sound switched off and then to ask them what was it about. What are their comments? These three ways are certainly not the only one and everything depends on the teacher's and student's invention.

One of the rules which we should follow is to avoid the formal discussion, but talk to each other in an informal way. We make mistakes, trying to correct them immediately! The sketches may be sometimes rough and the writing as well. We try hard to ask for questions in our recordings, questions possibly asked by students, especially the weak ones, e.g. "Why the proof is needed, it is evident" (especially in geometry). Various difficult for children problems are introduced and explained on examples, creating scenes an situations (e.g. percentages, VAT calculation, credits calculation, etc.). We are trying to use simple language, simple sketches and simple technology (Jacobsen, 2012)

Example.

1. Person 1: Now, in stores are all kinds of price reductions and sales. Yesterday I saw a dress with price reduced from 120 zł by 15%. What is the price now?
2. Person 2: It seems to be easy: 15% of 120 equals 120x0,15 = 18. Now it is possible to substract: 120 – 18 =102. New price is 102 zł.
3. P1: OK, your calculation is correct, but it is too long.
4. P2: Why?
5. P1: Just think. If the price of the dress is lowered by 15%, it means that the price of the dress will be 100%-15%=85% of the previous price. So it is easier to calculate at once 85% of the initial price.
6. P2: Exactly. 85x 120 = 102. One operation is enough to get the result.

We are only at the beginning of our experiment. Part of our recordings was tested with students from the school ZSO nr 6 in Radom and their teacher Ms. Grażyna Śleszyńska. The students that tested the recordings were good and active students. We obtained 20 students opinions in writing. The students found that the recordings were easy to use. Their opinion was that they may be used for repetition and they liked the recordings being non-formal. They appreciated the simplicity and clarity of explanations. In many opinions were used the words: simple, clear, intelligible. They liked also many examples taken from everyday life. Their remarks helped us to prepare new recordings.

On the other hand, the teacher remarked a very important matter. After watching the recordings, the students encouraged by the simplicity of presentation start talking about what they saw and heard. We observed this phenomenon also in some case studies concerning weak students (e.g. from hospital schools). These students start talking about mathematics. It means that there are realized three main postulates of L.P. Benezet: why learning mathematics, the students should "read, reason and recit" – famous 3R of Benezet (1935).

At the same time, it appeared how much the formal language of mathematics is the reason why the weak students reject the learning of mathematics. In some cases, regardless of our opinions that we went very far in the direction of non-formal presentation of mathematical ideas, the students asked form less formal language.

METHODOLOGY OF THE RESEARCH.

The further plans of the Foundation is to prepare such recordings for all subjects of the school mathematics for the lower and upper secondary school. At present, we are at the beginning of the road. The main research method is observation of behavior and results of students. Teachers choose a group of students who, according to their opinion, need some help (using sometimes tests and sometimes different diagnostic tools) and choose for them one or more methods using of prepared films. Teachers observe continuously students, prepare tests, compare results. Thus, they obtain initial and final data. All results are being sent to the Foundation. The plans of Foundations are personal contacts of cooperating teachers and schools as well as the students.

We expect to obtain the sources of the difficult mathematical problems resulting during students work (Jacobsen, 2012). The observation of several cases seems to be appropriate for our "Mathematics – do it yourself", but it is too early to prepare the full statistics. At present we cooperate closely with Public School nr 2 in Ustrzyki Dolne (the school is sport oriented) and with several hospital-schools in Warsaw. We observe, together with the teachers the effect of films on the progress of the students. In our opinion, giving teachers teaching material (films) to support their work with weak students should in some time result with the amelioration of the mathematical education effects.

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INFLUENCE OF CLIL METHOD TO MATH TEACHING

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A method called Content and Language Integrated Learning (CLIL) has appeared recently, targeted at different types of schools as well as at teaching of different subjects. The article deals with this method and outlines forthcoming research on lower grades of secondary schools, which will examine the influence of teaching using the CLIL method.

INTRODUCTION

One of the trends of the contemporary society is migration because of study and work opportunities. Therefore, one of the inter-subject links, that is the combination of specialised subjects and foreign language courses, is becoming increasingly more important. While subject-specific topics are seldom discussed in language courses at elementary and secondary schools as standard, the combination of a foreign language and a specialised subject is very beneficial. The method which deals with this integration is called CLIL.

THE CLIL METHOD

Content and Language Integrated Learning (CLIL) method is not new. The concept of CLIL was first officially used by David Marsh at a university in Finland in the year 1994. But it is relatively new in the Czech Republic. The official document which served to apply the CLIL method in the Czech environment was the National Programme of Teaching Foreign Languages in the Czech Republic for the Period Between 2005 and 2008. It was prepared as a reaction to the Action Plan for the period between 2004 and 2006 issued by the European Commission. In 2009, the Ministry of Education, Youth and Sports issued a document entitled Content and Language Integrated Learning in the Czech Republic. Since then, several projects have been ongoing in the Czech Republic, aiming at the introduction of CLIL into Czech schools and training teachers to use the method properly.

Advantages and Disadvantages

Like every method, CLIL has some advantages and disadvantages. The CLIL method may bring many advantages. A major advantage is the improvement of student communication skills in a foreign language. The method also helps students to expand their subject-specific vocabulary. Students are thus prepared for their future studies abroad, since they are more experienced in subject-specific communication in a foreign language and need not fear any communication blocks in a foreign country. The absence of such fear is very useful as regards studying and also possible job seeking abroad. Students are more likely to find jobs in the future, not only in the Czech republic but also abroad. The CLIL method does not necessarily have to be used for the entire duration of each class. It may be applied only to certain parts of the lesson, for example "mathematical warm-ups" or revisions at the end of the lesson. This brings an effective innovation and improves the attitude of students towards the given subject. Students may also be more motivated in terms of acquiring new knowledge (Hofmannová, Novotná 2003).

One of the disadvantages of the CLIL method lies in its demands. It is more time-consuming as regards the necessary preparation for classes and it is also more demanding in terms of teacher language skills. The preparation of materials is also difficult. However, their availability is increasing via various projects, portals and trainings.

Using of CLIL Method in Olomouc Region and South Moravian Region

A project which aims to determine the use of the CLIL method at primary schools and high schools in the Olomouc Region and the South Moravian Region is in progress and here is part of current results.

First part of my pre-research was for directors of schools. Questionnaires were sent via e-mail to the mentioned schools. The objective was to find out whether the schools use the CLIL method, and if they do, in which subjects. The return rate was between 16% and 18% in both regions. Out of the total number of replies regarding use of the CLIL method in the Olomouc Region, 10% were positive. These replies included the current use, use in previous years or preparation of the. In most cases, the foreign language intended for integration into other subjects was English. Only in one case it was French. Almost all subjects were represented that were taught or to be taught in the foreign language and no subject significantly prevailed over others. In the South Moravian Region, the situation was different. There were 13% of replies positive. The English language was dominant and a wide scope of subjects were represented. A major difference was in the dominance of some subjects. The most frequent were Mathematics, Arts, Music, Civics and Physical Education. Among other frequent subjects were Informatics, Social Sciences, Natural Science and History. These findings indicate that the CLIL method is not completely integrated. There is a frequent fear that such lessons are too difficult and the preparation for them too demanding.

Influence of the CLIL Method on Teaching Mathematics

In my pre-research, I focused on the effect of the CLIL method on teaching mathematics. With regard to the fact that it was a pre-research, the sample consisted of 79 respondents, the pupils of the secondary level at a South-Moravian elementary school where the CLIL method had already been used for some time. Out of the 79 respondents, 32 were boys and 47 were girls. The pupils were given a questionnaire containing 22 statements with respect to which the pupils were supposed to express agreement or disagreement using a four-point scale. In terms of the covered topics, the questions focused on the climate in classes, the popularity of the individual subjects and whether it is worth it to use the CLIL method in classes. It was obvious that the pupils had good mutual relationships and there was a friendly atmosphere; the pupils also denied any derision or ridicule in the event of failure of one of their classmates. This indicates a good climate for learning. The pupils also stated that the teachers were devoted to the subjects they taught and that the pupils were given tasks which they considered solvable. No classes thus indicated any problems, not even as far as mathematics teachers were concerned. The latest mid-year mathematics marks ranged from 1 to 4, which means that no pupil failed the subject. The general interest in mathematics was rather below average; however, when assessing mathematics taught by means of the CLIL method, there was a positive shift compared to lessons in which this method was not used. One of the frequent misgivings in connection with the integration of a foreign language into the teaching of non-language courses concerns the fear that pupils will not understand the task set in a foreign language. In the case of these particular respondents, however, this was not confirmed, and hardly any of the pupils expressed any fear of non-understanding. What was also above average was the assessment of the pupils’ activity; according to them, their activity in such lessons was increased. On the other hand, the lessons taught using the CLIL method were less focused on the actual subject in the pupils’ opinion. This may be attributed to the fact that if foreign language activities take the form of games, pupils learn from such activities but do not see them as traditional lessons. This discrepancy between activity and focus on the actual subject deserves further research. Nevertheless, the assessment of the benefit of the method for the pupils’ future lives was positive and slightly above average. In the use of a foreign language in mathematics, the pupils thus saw an advantage, whether in terms of their prospective studies abroad or their ability to discuss the issue of mathematics in the given foreign language.

CONCLUSION

Although the CLIL method is not entirely new in the Czech Republic, it is apparent that its introduction into actual teaching is not so frequent yet. All the same, despite its shortcomings and disadvantages, this method might make lessons more interesting and increase pupils’ ability to discuss subject-specific topics in a language other than their mother tongue.

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