Generalization in mathematics at all educational levels

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INTRODUCTION

From all processes involved in mathematics, generalization is considered one of the most important ones. For some researchers, generalization *is* what mathematics is about. Thus, whether it is viewed as part of a higher level process, like abstraction or as the core process involved in a particular mathematics field, like algebra, there seems to be an agreement on its significant role in advanced mathematical thinking. This is also acknowledged by most significant curriculum documents, which make an explicit reference on processes related to generalization.

The need for focusing on generalization might be also justified by the development of mathematics as a scientific discipline; this means that arithmetic and computational skills are not enough for the students to 'grasp' the deeper underlying structure of mathematics. The teachers should be well informed on that and should be prepared to create opportunities for their students to detect patterns, identify similarities and link analogous facts. But generalization does not appear just by performing the previous activities; to use John Mason's terms, a shift of attention should take place or, in other words, a shift in the way one sees things.

Contrary to what most people might think, generalization can be even observed in young children; such observations are signified by terms such as 'early algebra', which have recently appeared in the relevant literature.

This volume presents various approaches on how generalization is or should be treated in the mathematics classroom. The five parts offer only one way of differentiating between the views presented. Among them the reader may find chapters focused on the theoretical foundations of generalization, but also chapters focused mostly on the implementation of approaches based on generalization, e.g. by pattern recognition. There is a part dedicated to early generalization, in line with the current trends in research that we have mentioned, and another part focused on teachers' skills in generalizing.

According to John Mason *generalization is the life-blood, the heart of mathematics*; being aware of that fact and being able to accordingly adapt the classroom practices is a highly important aim of mathematics education. We hope that the present volume can offer to mathematics educators and researchers a means to a deeper understanding of the many possibilities existing within the approaches that highlight the role of generalization at all educational levels.

Rzeszow, June 2012

The Editors

Generalization from theoretical points of view

Part 1

"TO GENERALISE, OR NOT TO GENERALISE, THAT IS THE QUESTION"
(WITH APOLOGIES TO HAMLET AND WILLIAM SHAKESPEARE)

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From a very early age an ability to generalise makes our lives easier in many respects. Indeed, developing an awareness of pattern is an important step in becoming a proficient mathematician. Over the years as a researcher and teacher educator, however, I have observed many cases of 2 – 60 year-olds generalising when it is inappropriate to do so. Here I explore some of these and the possible reasons behind them. I then discuss some recent research on less successful teachers and how we might enhance their professional practice in order to capitalise on the rewards of generalising effectively.

INTRODUCTION

We all generalise in our everyday lives. We probably do it considerably more often than we realise. It usually saves time and it tends to make life easier but it is not always the best approach. I begin by considering examples of how and why we generalise. My particular focus is mathematics education in the earliest years of formal schooling. I will start with the children – the principal players – followed by their teachers. I will then discuss some fundamental groundwork I think is required – but frequently overlooked – before some teachers are able to move forward and begin contemplating enhancing their professional practice in the early years' mathematics classroom. Finally I will present some suggestions as to how teachers might use generalisation to their advantage and the wider implications arising from the question, to generalise, or not to generalise.

As a psychologist who happens to be interested in young children and their education I hope that my perspective will offer you new insights into some of the many challenges, and possible ways forward, in the pursuit of mathematics education of the highest quality.

THE YOUNG CHILD'S PERSPECTIVE

Research suggests that the average six-year-old has a vocabulary of 8,000 to 14,000 words (Woolfolk, Hughes and Walkup, 2008) and, indeed, Berger (2003) estimated that in the early school years children learn up to 20 words a day. Whether you favour behavioural (e.g. Skinner, 1957), nativist (e.g. Chomsky, 1957), social-interactionist (e.g. Bruner, 1983 and Piaget, 1969) or other language development theories, observation suggests that children generalise as they learn to talk and, although they make mistakes such as ‘I go, I goed’ rather than ‘I go, I went,’ generalisation tends to prove effective and efficient.

On arrival in the early years’ classroom life becomes potentially more confusing as the young child encounters further inconsistencies. For example in English some of the words they learn in the home, take on completely different meanings when children begin school such as ‘check’, ‘take away’ and ‘difference’ (Cockburn, 1999).

In some languages we further complicate matters by being inconsistent in our counting. Thus, for example, in English we say *twenty-seven* which we write in the same order as we say it, ‘27’, but *seventeen* which, adopting the same principle, one would expect to be ‘71’. The French describe ‘50’ and ‘60’ as *cinquante* and *soixante* - which suggests a pattern in their counting system - but ‘80’ is *quatre-vingts* which can be translated as four twenties.

Our various languages are not the only potential source of difficulty for children’s later generalising. How often, for example, have you heard someone say to a child, ‘I will be with you in a second’ and yet it is 60, or even 120 seconds, before they turn their attention to the child?

Very early on in my career I came to the conclusion that young children tend to have a very powerful inclination to try and make sense of their experiences and hence to generalise from them. For example, one of the kindergarten teachers I was researching found it hard to believe that her class of five-year-olds were unsure about why we use numbers so she initiated the following conversation:

Teacher: Why do we do these numbers?

Michelle: So that we can spell things.

Teacher: Spell things with numbers?

Antoinette: So we can count properly.

Teacher: What kind of things do we need to count?

Antoinette: You need to count the numbers.

Simon: We need to draw the numbers.

Teacher: Why do we need to draw the numbers?

Lisa: So we can copy them.

Teacher: But why do we need the numbers at all?

Lisa: So we can colour them in.

(Adapted from Desforges and Cockburn, 1987, p. 100)

One does not need to look very far – a quick trip to a classroom or a paper on mathematical misconceptions – to find children doing seemingly surprising things in mathematics. Although some of them may be amusing, adopting a child’s perspective, there are two important factors we need to consider. The first is that mathematics is potentially very confusing. The second is that, almost invariably, what the child has done makes sense to them and it is important that we recognise and accept that. One of the consequences of these combined factors being as Jordan, Kaplan, Oláh and Locuniak (2006) point out, 'Some children gradually learn to avoid all things involving math' (p. 153). Indeed, Margaret Brown and her colleagues (2008) went even further entitling a paper, *'I would rather die': reasons given by 16-year-olds for not continuing their study of mathematics*.

To summarise, generalising can serve young children extremely well but it can give rise to confusion. As educators our challenge is to capitalise on their propensity to generalise but to reduce the extent to which they do so inappropriately. As discussed below this does not necessarily mean avoidance but rather encouragement to question and challenge.

THE EARLY YEARS' MATHEMATICS TEACHERS' PERSPECTIVE

From the outset I want to emphasise that my aim is to *understand* behaviour rather than to pass judgement on dedicated practitioners as they endeavour to give of their best day in day out. Indeed I am a firm believer of William James' (1899) view that,

> The worst thing that can happen to a good teacher is to get a bad conscience about her profession…our teachers are overworked already...A bad conscience increases the weight of every other burden…(pp. 13-14)

Throughout my career I have been extremely impressed by the dedication and quality of the early years' teachers I have encountered. One of the factors which have impressed me most has been their almost universal desire to ensure that each and every one of the children in their care realised their full potential. Indeed early on in my life as a doctoral student I noted that the seven teachers I worked with put,

> ...considerable thought into their work, were anxious to promote their pupils' mathematical progress and had considerable insight into how their classrooms operated. (Cockburn, 1986, p. 253)

The strategies successful teachers need to adopt has been well recognised for over 100 years with William James (1899) advocating that they should,

- ✓ Capture the child's interest
- ✓ Build on what they know
- ✓ Teach and assess for understanding
- ✓ Provide plenty of oral and practical experience
- ✓ Adopt a varied approach
- ✓ Foster children's confidence in their mathematical abilities

The last of these seemed to be particularly prominent when I began my career as a researcher with Mrs T explaining,

> If they (i.e. children) get hung up about anything when they are five years old, what will they be like later? (Cockburn, 1986, p. 215)

The maintenance of pupil confidence continued to be a priority among early years' mathematics teachers in U.K. when we worked with them in 2005 (Cockburn and Iannone) and I have no reason to believe that their views have changed markedly since then.

In mathematics classrooms around the world I am confident that you can see many outstanding examples of teachers making effective use of generalisation. The repetition which is an integral part of the number system opening up a wide range of possibilities. Such activities can be very satisfying and confidence boosting as young children often display a great sense of achievement on realising the pattern 0, 1, 2...10, 11, 12...20, 21, 22.

Here, however, I want to focus on three examples where some teachers generalise without appreciating that it may be inappropriate.

The first arises when early years' practitioners do not have a thorough understanding of some fundamental mathematical concepts. In the past I do not think that this was perceived as an issue as, until recently in England for example, working with young children was perceived as a low status occupation requiring few formal qualifications. A common misconception among some early years' teachers is that zero (0) means *nothing* rather than *the absence of something* (Cockburn and Parslow-Williams, 2008). This can result in much confusion when their pupils endeavour to unravel the patterns of place value.

The second example of a generalisation seems to occur when teachers wish to simplify something for their pupils. This may be for a variety of reasons including pressure to get through a syllabus and children finding it difficult to grasp a new concept. A classic case in the early years' classroom is when subtraction is only taught as 'taking away' and pupils are encouraged to generalise by, 'always taking away the smaller number from the bigger one' (Cockburn, 1999).

The final example does not relate to mathematics specifically but rather to some teachers' tendency to generalise children's ability by putting them into groups for teaching purposes. This, in turn, can result in some unfortunate self-fulfilling prophecies (Rosenthal and Jacobson, 1968) and, in the later years of schooling, *RHINOS* or, in other words, children who are 'Really Here in Name Only' (e.g. Nardi and Steward, 2003).

PRELIMINARY GROUNDWORK

Before we can consider how we might encourage appropriate generalisations in early years' mathematics classrooms, I would argue that there may be a considerable amount of fundamental groundwork to be done.

In this section I will refer to a variety of sources. Initially, however, unless otherwise stated, I will reflect exclusively on data collected as part of a study funded by the Nuffield Foundation[1] for I think it provides some valuable insights as to where we should start. By way of background: this was a small scale – as yet unpublished - study designed to develop of understanding of less successful teachers. It involved 12 semi-structured interviews (Robson, 1993) with experienced head teachers and yielded a wealth of material which extended far beyond my original remit.

When discussing how to develop their colleagues' professional practice the head teachers explained that there were several issues which had to be attended to before any progress was likely. These are overlapping and interconnecting but, in essence, they involve,

Focusing on attitudes

Bob[2] explained of early years' teachers in general, '...a lot of them come in with their own baggage, don't they?' (592-593). This was echoed by Hannah who, on describing the attitudes of her staff to mathematics said, 'I still think as a culture we don't do maths terribly well. So easily people say "I'm not very good' (672-673).

It was clear that in some schools changing teachers' attitudes was not enough for, as Jean explained, 'If we don't work with the parents, there is no way we can get those children because it just isn't important' (475-476).

Building trust

Some of the head teachers I spoke to were aware that not all of their colleagues trusted them making it difficult to move forward. Clare, for example, recounted that,

> ... the trouble was that very often it (the teacher's planning) looked very good on paper, but actually it didn't translate like that into the classroom. And when I had supply teachers going in to cover for her, they said: 'I can't do all this' and then they felt like failures. And she would be telling her colleagues that, actually, she was doing it all. So they all thought, gosh, that she's this wonder woman. When, in fact – in reality – she wasn't. (Clare 268-273, brackets added)

[1] The author gratefully acknowledges financial support from the Nuffield Foundation (project # 39039)

[2] All of the participants' names have been changed.

Developing teachers' confidence

As with children (James, 1899) it is also important to foster some teachers' confidence in themselves. David reflected on the progress he was making with 'a worrier',

> ...we just want her to be a little bit more….braver. She will probably feel uncomfortable doing these things but…a lot of the new things that she feels uncomfortable about are the new initiatives that we've been driving through. Because they are new initiatives to everybody, she knows that everybody, you know, she knows that everybody else has similar anxieties so it has helped her. (David, 182-187)

Focusing attention

Clare succinctly explained that sometimes she has encountered teachers who, '...are extremely industrious but they are focusing on the wrong things' (88-89).

Recognising a need

The head teachers explained that there were a range of reasons why some people do not appreciate that there is a need to change their professional practice. The two, which are of particular relevance here, are sometimes related.

The first is that, on the face of it, a teacher may appear to be doing a good job with a beautifully organised classroom, contented pupils and complimentary parents. Closer examination, however, can reveal that the children may be significantly underperforming as Debra explained when she took over the headship of a middle class school, 'The kids were getting the equivalent of national expectations but they were bright kids who should have been far, far above that' (222).

A second obstacle to teachers appreciating that there may be a need to amend their practice is that, because their pupils are performing well in mathematics, they are unaware that they may be creating problems for the future. This became particularly apparent during a European project funded by the British Academy[3]. When the equals sign (=) was discussed it was clear that some of the early years' teachers thought of it in terms of an operator rather than a symbol of equivalence and that this significantly restricted the way they used it in their classrooms (Parslow-Williams and Cockburn, 2008). This observation prompted Marchini and colleagues to examine undergraduates' understanding of equality and discovered that, in some cases, it was significantly lacking (Marchini et al, 2009).

Uncovering any other underlying problems

In addition to the above, the head teachers explained that some extra support might be required from time to time as in the following cases:

[3] The author gratefully acknowledges financial support the British Academy (LRG-42447)

> ... there were occasions where I think she was physically and emotionally and mentally a bit tired and sort of, you know – not cruising to retirement, because she was too conscientious for that – but she had lost that kind of real spark (Maggie, 396-398)

> ... teachers go through all sorts of difficult things in their lives and that can affect how you perform at school. So, if suddenly a teacher has been… well, suddenly they have a family to care for and therefore their priorities can change. (Clare, 44-46)

In some cases, however, head teachers taking up a new post encountered colleagues who appeared to be doing an adequate job but, 'They are sitting very comfortably …in too much of a comfort zone.' (David, 100)

MOVING FORWARD

Once the groundwork is underway in early years' settings focusing on mathematics – let alone something as specific as generalisation – is not necessarily as straight-forward as one might imagine. Indeed the head teachers indicated that there were several further factors to take into consideration before they could make substantial progress. The following were discussed in the context of committed and experienced teachers although, you will note, that there are several similarities the examples I have already presented for their less successful colleagues.

Approaching professional development in a non threatening manner

In my experience I have often found that the very best early year's teachers often lack confidence in their abilities. Indeed Hannah remarked, 'The more self-critical people are generally the better I find them as teachers' (24-25). Accordingly she tends to work to people's strengths,

> There's all that sense of 'we're not very good…' the two teachers who are really good in school they are both passionate about literature so what we are trying to say is 'well, what is it you do, in teaching literacy that we can transfer to teaching mathematics? (672-676)

Ellen, recognising that teachers appreciate the opportunity to buy new equipment, invited them to bring their catalogues to the staffroom as this proved to be,

> ...a good way in because it meant that my Deputy could see...what they were planning to do and say 'If you are doing money then perhaps we could get this, you know, this equipment' and 'Had you thought of doing' 'Oh, we could do a shop' or 'We could do this, that and the other' and feed in ideas and appropriate equipment for the children to play with. So, that was a wonderful way in for Maths. (585, 589-594)

Building on teachers' interests

In discussion with the head teachers they were very honest about how they had acquired many of the techniques they used through trial and error. Thus, for example, Jean recalled how she had observed that the same topic could be presented to her staff in a number of different ways with varying degrees of success. By way of illustration she said,

> I think the key thing about them is that on pedagogy they are very, very strong and if we sit down and...look at it… with a pedagogical focus…they can go 'yes, actually, you're right'... Whereas if I started from saying like 'I'm not happy with your planning – do it like this' they would probably say 'no, I don't think I want to do it like that.' But if you can say, 'Look, this is the outcome. This is how ….how is that child learning within this?' they will go 'yeah, yes, I can see what you are saying. (184-190)

Maggie simply recounted, 'I think, as a whole, people took it on board, you know, very willingly because they could see the sense in it.' (633-634)

Working across age phases

Debra found working as a whole school team proved effective,

> It's actually getting them to really know their children and create the culture of team effort within the school and not, for example, to say 'Well, actually, the year 6 results belong to year 6'. Year 6 results belong to the whole school. And, all the time, looking for trends in things so, for example, if it's Maths, and you say 'well, you know, we dipped this year in our Year 6 SATS (national tests). Let's analyse all the SATS papers and see where they went wrong.' And 'OK, it's subtraction.' Right, the whole school, then, is going to have a push on subtraction. And let's have some staff training on that. Let's gets our targets in sight of what we are going to do with our kids as far as subtraction is concerned. And let's look at the difficulties, let's model to each other, let the whole school talk about how we are going to teach subtraction in different ways. Get different teachers to lead staff developments and then evaluate what the kids have done better. (149-165, brackets added)

Poised for action?

In some schools the above are likely to be much easier to achieve if you have an enthusiastic and able nucleus of staff such as,

> I have a superb Maths subject leader who gees (i.e. encourages and inspires) us all up and makes sure that we do Maths a great service (Janice, 347-348, brackets added)

It would be naive to suggest, however, that all schools are ready to move forward even if their head teacher is outstanding. Bob described how,

> ...with both schools where I've been a Head …there's been…you know a bell curve, you know, you've got some at each end and the majority are in the middle

> and it's being able to move the majority in the middle in the direction in which you want to go is…is the difficult part. (125-128)

Later he elaborated,

> I think one of the keys to it is actually getting the balance in the staff between…shifting the balance, shifting the core dynamic within the staff room away from the negative, you know 'we've done that before' and 'that hasn't worked' ... and you start to appoint staff. (Bob, 282-286)

THE ROLE OF INITIAL TEACHER EDUCATION

Over the years I hope I have been increasingly successful in preparing newly qualified teachers for Bob and his colleagues to appoint. In essence I have found much of my role has been similar to that described by the head teachers above. At the University of East Anglia we work hard to develop our students' mathematical understanding and confidence. Much of this is done through modelling and encouraging a range of techniques. Thus, for example, we introduce Haylock's model (Haylock and Cockburn, 1989) and invite prospective teachers to examine their understanding of a concept in terms of real objects, pictures, mathematical symbols and mathematical language. We work on developing students' mathematical knowledge, urging them to seek generalisations and, on finding them, to hunt for counter examples. Recently the work of Milan Hejný and his associates has proved particularly useful (see, for example, Hejný and Slezáková, 2007; Hejný, 2008; Littler and Jirotková, 2008) in demonstrating how learners can build up their conceptual understanding. This process may be summarised thus:

Individual experiences → generalisation → generic model → abstraction → abstract knowledge

CONCLUDING REMARKS

Almost without exception young children have a great capacity for learning and they generally embark on their earliest years' of schooling with energy and enthusiasm: the potential is all there and our task is to capitalise on it.

Bob, one of the head teachers I interviewed, said an expert teacher is someone who, '...is open to new ideas and fresh challenges' (40). Fortunately there are many such individuals in the profession. We know a considerable amount about them as they tend to be the teachers who volunteer to take part in research studies. In this paper I have concentrated rather more on their less confident and mediocre colleagues for we know far less about them and yet I believe that many of them have the potential to be considerably more effective mathematics educators than they currently believe. As a research community I would suggest that we still have much to learn about such individuals and how best to enhance their practice. We cannot ignore them for, as I have illustrated above, their capacity to generalise inappropriately has the potential to create considerable damage in the mathematics classroom and beyond.

So, to return to my original question: to generalise or not to generalise? My answer is ‘yes’ but only if you know what you are doing!

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GENERALIZATIONS IN EVERYDAY THOUGHT PROCESSES AND IN MATHEMATICAL CONTEXTS

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Generalizations are the engine which forms concepts in all domains and claims about almost any subject. It seems that it is possible to claim that generalizations are kind of a cognitive drive (if we use Freudian terminology) or cognitive need (if we prefer the terminology of Maslow). If we like to use evolutionary psychology it will be easy to point at the evolutionary advantage of generalizations. Namely, when we were still hunters in the wilderness, generalizations helped us to survive. The talk will point at the thought processes which lead to generalizations. All that is true about non-technical situations. Things are different in mathematical thinking. Here the ultimate goal is that the student will acquire the desirable mathematical behavior. Namely, in mathematical contexts we are supposed to train our mind to form concepts by relying on formal definitions and to establish claims by relying on proofs. This contradicts the spontaneous nature of thinking. Thus, some mathematics educators, in order to facilitate the learning of mathematics, offer to the students strategies which are supposed to imitate the assumed spontaneous way of forming generalizations. They do it by presenting to the students examples which will lead them to the correct generalizations. The talk will focus on the role of examples in everyday thought processes and in mathematical contexts.

1. INTRODUCTION

What I am presenting here is not a research report. It is an essay. In essays it is allowed to reflect, to speculate and, hopefully, to stimulate. Also, I consider it a mathematics education essay, the way I see mathematics education. Namely, as a discipline which is designed to help mathematics teachers and mathematics teacher trainers to understand aspects of learning and teaching mathematics. It is not designed, according to the way I conceive it, to come up with innovations in cognitive psychology, brain research, philosophy of mathematics or sociology. On the other hand, since I came to it from mathematical research, my way of looking at things is influenced by my mathematical background. Namely, in the back of my mind always sits a deductive theory in which there are concepts (or notions if you wish) and claims about these concepts and some inference rules by means of which we derive new claims (theorems, if you wish), from claims already assumed to be true or to be proven. I am absolutely aware that in a domain like mathematics education, as well as in other domains in behavioral sciences, there are different ways to establish various claims. However, pointing at the differences is quite often very helpful.

2. GENERALIZATIONS

I am not going to define here the notion of generalization. I consider it as a primary notion in the sense of primary notions in a deductive system (this is a reference to deductive systems which I mentioned earlier).

Before going on I would like to clarify the distinction between a notion and a concept the way I use these words. A notion is a lingual entity. It can be a word or a combination of words (written or pronounced); it can also be a symbol. A concept is the meaning associated in our mind with a notion. It is an idea in our mind. Thus, a notion is a concept name. There might be concepts without names and for sure there are meaningless notions, but discussing them requires subtleties which are absolutely irrelevant to this context. In many discussions people do not bother to distinguish between notions and concepts and thus the word "notion" becomes ambiguous. The ambiguity is easily resolved by the context.

So, back to generalizations: If we reflect about people's thought processes we realize that there is a tendency there to generalize. Here is a small sample of accidental generalizations: 1. Little children are cute. 2. Women are gentle. 3. Men prefer to watch football on TV than to have a conversation with their wives. 4. Trains in my country are always late (when I say my country it is not necessarily my country, it can be anybody's country). I am not claiming that these generalizations are true. I present them in order to support my claim about the tendency of human beings to generalize. Instead of the word "tendency" I would like to use a technical notion borrowed from the domain of psychology. The notion I have chosen is borrowed from the Freudian psychology. Freud spoke about drives (Trieb in German). He spoke about the sex drive and about the death drive. However, the notion of drive can be related to many more actions in our behavior. We can speak about the drive to protect, a drive to discover, aggression drive, competition drive, sadistic or masochistic drives etc.. A drive which is relevant to the theme of our conference, is the drive to generalize or, in short, *the generalization drive*[1]. One major outcome of generalizations is concepts. Another major outcome of generalizations is universal claims of the kind I mentioned earlier.

Before dealing with these two outcomes in details I would like to say few words about the origin of drives. The common approach to drives in evolutionary psychology is not to ask about the origin of our different drives or about our certain abilities. It is assumed that they were formed accidently during the course of evolution. However, these drives gave the creatures that had them an evolutionary advantage. For instance, the ability and the drive to generalize helped us to survive million years ago, when we were still primitive beings wandering in the wilderness. While wandering in the wilderness, when noticing a certain creature coming up against us we were supposed to decide very fast whether this creature is an enemy or a friend. In case it was an enemy we were supposed to decide immediately whether to *fight or to flight.* Our ability and drive to generalize form in our mind the concept of enemy. The ability to identify a specific enemy and to determine what should be the appropriate action at a given situation was critical for our survival. A more detailed discussion of this issue can be found in Goleman (1995, Chapter 2).

3. GENERALIZATIONS AND CONCEPT FORMATION

I would like to discuss now in a more detailed way the generalizations which are related to concept formation. Let us consider little children learning to speak. How do we teach them, for instance, the concept of chair? The common practice is to point at various chairs in various contexts and to say: ‘chair’. Amazingly enough, after some repetitions, the children understand that the word ‘chair’ is supposed to be related to chairs, which occur to them in their daily experience, and when being asked ‘what is this?’ they understand that they are expected to say: ‘chair’. Later on, they will imitate the entire ritual on their own initiative. They will point at chairs and say: ‘chair’. I would like to make a theoretical claim here by saying that, *seemingly*, they have constructed in their mind the class of all possible chairs. Namely, a concept is formed in their mind, and whenever a concrete object is presented to them, they will be able to decide whether it is a chair or not. Of course, some mistakes can occur in that concept formation process. It is because in this process two cognitive mechanisms are involved. The first mechanism is the one that identifies similarities. The mind distinguishes that one particular chair presented to the child is similar to some particular chairs presented to her or him in the past. The second mechanism is the one which distinguishes differences. The mind distinguishes that a certain object is not similar to the chairs which were presented to the child in the past and, therefore, the child is not supposed to say ‘chair’ when an object that is not a chair is presented to him or her by the adult. Mistakes about the acquired concept might occur because of two reasons. An object, which is not a chair (say a small table), appears to the child (or even to an adult) like a chair. In this case, the object will be considered as an element of the class of all chairs while, in fact, it is not an element of this class. The second reason for mistakes is that an object that is really a chair will not be identified as a chair because of its weird shape. Thus an object which was supposed to be an element of the class is excluded from it. More examples of this type are the following: sometimes, children consider dogs as cats and vice versa. These are intelligent mistakes because there are some similarities between dogs and cats. They are both animals; sometimes they even have similar size (in the case of small dogs) and so on.

The above process which leads, in our mind, to the construction of the set of *all* possible objects to which the concept name *can be applied* is a kind of generalization. Thus, generalizations are involved in the formation of any given concept. Therefore *concepts can be considered as generalizations.*

The actions by means of which we try to teach children concepts of chair are called *ostensive definitions*. Of course, only narrow class of concepts can be acquired by means of ostensive definitions. Other concepts are acquired by means of explanations which can be considered at this stage as definitions. Among these concepts I can point, for instance, at a forest, a school, work, hunger and so on. When I say definitions at this stage I do not mean definitions which are similar, or even seemingly similar to rigorous mathematical definitions. The only restriction on these definitions is that familiar concepts will be used in order to explain a non-familiar concept. Otherwise, the explanation is useless. (This restriction, by the way, holds also for mathematical definitions, where new concepts are defined by means of previously defined concepts or by primary concepts). In definitions which we use in non-technical context in order to teach concepts we can use examples. For instance, in order to define furniture we can say: A chair is furniture, a bed is furniture, tables, desks, and couches are furniture.

The description I have just given deals with the primary stage of concept formation. However, concept formation in ordinary language is by far more complicated and very often, contrary to the mathematical language, ends up in a vague notion. Take, for instance again, the notion of furniture. The child, when facing an object which was not previously introduced to him or to her as furniture, should decide whether this object is furniture or not. He or she may face difficulties doing it. Also adults might have similar difficulties. I myself have difficulties with the notion of recyclable items. Usually, they are defined by general notions like glass, plastic, aluminium or paper. In some countries you can see pictures of recyclable items which are placed on recycling containers. Well, are milk cartons recyclable? Are thin plastic bags recyclable? Are cottage cheese cups recyclable? I keep asking these questions the recycling department in my town and I do not get clear answers.

This is only one example out of many which demonstrates the complexity of concept formation in the child's mind as well as in the adult's mind. I have not mentioned yet the concept formations of abstract nouns, adjectives, verbs and adverbs. Nevertheless, despite that complexity, the majority of the children acquire language at an impressive level by the age of six (an elementary level is acquired already at the age of three). The cognitive processes associated with the child's acquisition of language are discussed in details in cognitive psychology, linguistics and philosophy of language. One illuminating source which is relevant to this issue is Quine's (1964) "Word and object." However, a detailed discussion of these processes is not within the scope of this lecture.

In addition to the language acquisition the child acquires also broad knowledge about the world. He or she knows that when it rains it is cloudy, they know that dogs bark and so on and so forth. In short, they know infinitely many other facts about their environment. And again, it is obtained in a miraculous way, smoothly without any apparent difficulties. Things, however, become awkward when it gets to mathematics.

4. THINKING IN MATHEMATICAL CONTEXTS, SYSTEM I AND SYSTEM II.

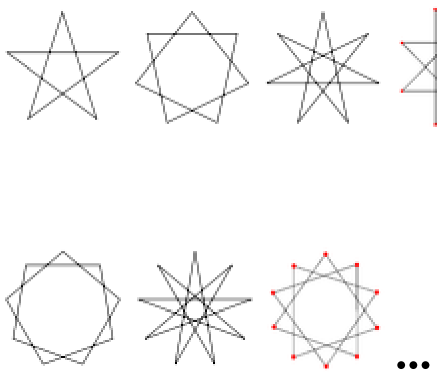
One possible reason for things becoming awkward in mathematics is that, in many cases, mathematical thinking is essentially different from the natural intuitive mode of thinking according to which the child's intellectual development takes place. The major problem is that mathematical thinking is shaped by rigorous rules and in order to think mathematically children, as well as adults, should be aware of these rules while thinking in mathematical contexts. This requires awareness. It requires the ability to reflect and to be analytical. In short, it requires *thought control.* Thought control has a negative connotation because of George Orwell's 1984, especially in countries which were under a communist regime. However, in the context of mathematical thinking and also in the broader context of rational thinking it should have a positive connotation.

Psychologists, now a day, speak about two cognitive systems which they call System I and system II. It sounds as if there are different parts in our brain which produce different kinds of thinking. However, this interpretation is wrong. The correct way to look at system I and system II is to consider them as *thinking modes.* This is summarized very clearly in Stanovitch (1999, p.145). System I is characterized there by the following adjectives*: associative, tacit, implicit, inflexible, relatively fast, holistic and automatic.* System II is characterized by: *analytical, explicit, rational, controlled and relatively slow.* Thus, notions that were used by mathematics educators can be related now to system I or system II and therefore this terminology is richer than the previously suggested notions. Fischbein (1987) spoke about *intuition* and this can be considered as system I. Skemp (1979) spoke about two systems which he called *delta-one and delta-two* which can be considered as *intuitive and reflective,* or using the new terminology, *system I and system II,* respectively. I myself (Vinner, 1997) have used the notions *pseudo-analytical and pseudo-conceptua*l which can be considered as system I.

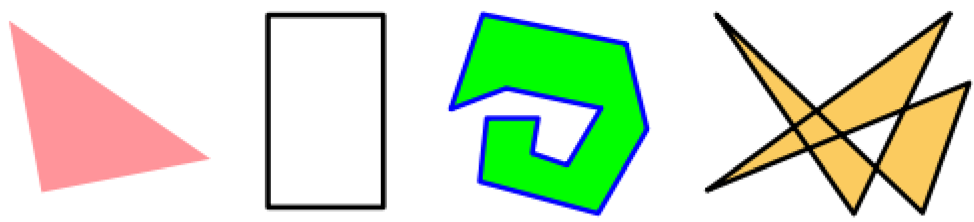
In mathematical contexts the required thinking mode is that of system II. This requirement presents some serious difficulties to many people (children and adults) since, most of the time, thought processes are carried out within system I. Also, in many people, because of various reasons, system II has not been developed to the extent which is required for mathematical thinking in particular and for rational thinking in general.

5. CONCEPTS AND GENERALIZATIONS; TWO ADDITIONAL EXAMPLES AND SOME PROBLEMS

Consider the formation of notions in different languages. Some actions occurred in a given culture. People, let us say, danced. Various dances were formed. At a certain stage, the people who were involved identified certain similarities between some dances. Identifying similarities is the first stage of a generalization. A concept has been formed. Then, somehow, a name was given to this concept - the concept name; the notion. Think for instance of the valse (waltz, in German). It is quite reasonable to assume that people used this notion to describe the occurrences of this dance before a formal definition was given to it. At a certain stage, when the question "what a valse is?" was introduced, an explanation (or a definition if you wish) should have been given. If you look for such an explanation now a day in various dictionaries you may find something which is similar to the following: *a ballroom dance in triple time with a strong accent on the first beat. Any of a variety of social dances performed by couples in a ballroom.* The Webster's Ninth New Collegiate Dictionary (1986) claims that official use of the verb "waltz" in German started in 1712, but only in 1781 the official use of "waltz" as a noun was started. This, by the way, supports Quine's claim (1964) that nouns were developed from verbs by a process which is called *reification*. In English, the noun "valse" appeared in 1796. If you listen to a valse by Johann Strauss you immediately get the impression that it is *a ballroom dance in triple time with a strong accent on the first beat.* On the other hand, if you listen to a valse by Frederic Chopin you can hardly say that it is a ballroom dance. It is not at all similar to the valse by Johann Strauss and many other well known valses which are quite similar to the valses by Strauss. So, why did Chopin choose the notion of valse as a title for his compositions? Well, I do not want to get into musicological discussions here, but by doing this Chopin extended the notion of valse to a bigger set of musical compositions. The authors of the above Merriam Webster dictionary were aware of that and they noted an additional meaning to this notion: *a concert waltz.* This example beautifully illustrates a development of a concept from a narrow set of examples to a broader set. We can find similar processes of concept development also in mathematics. So, the next example will be a mathematical example. Consider the concept of *polygon.* Again, it is reasonable to assume that already in ancient days people (not necessarily mathematicians) were aware of certain polygons. They dealt with all kind of triangles, with various sorts of quadrangles, with regular and irregular pentagons, hexagons etc. Then, similarities between these geometrical shapes were noticed and thus the first stage of generalization took place. A concept was formed. In order to discuss it a name was required. The word *polygon* was suggested by Greek mathematicians sometime in the fifth century B.C. When being asked what a polygon was, the answer could be: *A closed plane figure bounded by straight lines* (the above Webster's dictionary). It is reasonable to assume (although I cannot point at any historical document which can support it) that at the very beginning of the polygon concept people thought mainly about convex polygons. However, the above definition should accept also concave polygons as members of the polygon club. Later on, a refinement of the above polygon definition was formed. It started with the notion of a connected sequence of *line segments. A polygon is a connected sequence of line segments such that the second endpoint of the last segment is identical with the first endpoint of the first segment.* Note that this definition presents higher cognitive demands on the learner than the first one. Also, this definition should accept as members of the polygon club polygons which intersect themselves. When this was realized some mathematicians decided to be concerned only with polygons which do not self-intersect and thus the notion of *simple polygons* was formed. On the other hand, other mathematicians decided to study polygons that do self-intersect and thus the domain of star-polygons was introduced. According to the Wikipedia, the mathematician who started to study the star-polygons in depth was the English scholar Thomas Bradwardine (about 1290–1349). The Wikipedia also claims that only the regular star-polygon have been studied in any depth and it adds that star polygons in general appear *not to have been formally defined.* So, here is a mathematical concept that does not have a definition. It can be illustrated by the following picture:



And here is a picture of some other polygons which was downloaded from a Wikipedia page:



6. GENERALIZATIONS RELATED TO BELIEFS ABOUT CONCEPTS

The moment a concept is formed also some beliefs are formed about it. These beliefs can be formulated as universal statements. For instance, consider again a child who acquired the concept of dog. He or she knows that dog barks. Hence, there is an implicit claim here about dogs which is: All dogs bark. Some children experience a fearful event with a dog. This may lead them to the implicit belief that all dogs are dangerous. As a consequence of this belief they try to stay away from any dog they see. In an early work of mine (Vinner, 1983) I suggested to call the set all the concept examples in a certain person's mind together with all the beliefs about them the concept image of that person. Usually, the beliefs are generalizations formed by the generalization drive. Therefore, in most cases, there are products of the above system I. They are formed very fast, sometimes, relying only on a sample of a single element. If system II were involved the path from a statement about a single element to a universal statement should pass through the following statements: There is at least one element about which the predicate P is true. There are some elements about which P is true. There are quite many elements about which P is true. There are many elements about which P is true. P is true for almost every element. P is true for all elements under consideration. Thus, system II is supposed to stop at several stations before reaching, if at all, the final conclusion: P is true for all elements under consideration. However, the spontaneous tendency of our mind is to move fast and to reach a final conclusion in relatively short time. Therefore, quite often, we observe generalizations based only on a single example.

Before mentioning some wrong generalizations about mathematical concepts I would like to illustrate this point by mentioning generalizations made in everyday contexts. "Mathematicians are arrogant", some people claim. Well, there is at least one mathematician who is arrogant (I myself met one). Are they quite many arrogant mathematicians? Are there many? Are all mathematicians arrogant? A careful analysis by system II won't allow us to reach such a conclusion. However, there are people who believe that mathematicians are arrogant. Among them you will find victims of school mathematics. Their hatred to mathematics is a strong motivation for them to adopt negative views about mathematicians.

Another example: "Men are male Chauvinists", claim some feminists. It is true that quite many men are male Chauvinists, but is it true that all men are male Chauvinists? There are some feminists who believe in it. Among them you may find women who had a terrible experience with one man and as a result they developed hatred to all men. The view that all men are male Chauvinists is supported by their hatred to all men. Hence, we see from the last two examples that, in some cases, also emotions are involved in shaping concept images.

The last example is really a male Chauvinist generalization. The reason I present it here is that a wonderful music is associated with it, the famous aria from Verdi's Rigoletto. I should emphasize that, to the best of my knowledge, neither the libretto author, Francesco Maria Piave, nor the author of the play on which the libretto is based, Victor Hugo, can be considered as male Chauvinists. On the contrary and my claim is supported by the fact that the man who sings this aria, the Duke, is presented in the opera as a morally corrupted disgusting person. Here it is:

Woman is flighty

Like a feather in the wind,

She changes her voice — and her mind.

Always sweet, Pretty face,

In tears or in laughter, — she is always lying.

Always miserable

Is he who trusts her,

He who confides in her — his unwary heart!

Yet one never feels

Fully happy

Who on that bosom — does not drink love!

My claim about system I generalizations in everyday thought processes holds also for generalizations in mathematical contexts. Therefore, in mathematical contexts, quite often, we find concept images which are not coherent with the concept definitions. Among them one can mention the following: *multiplication increases; the altitude in a triangle falls always inside the triangle and it cannot be a side of the triangle; the elements of an infinite sequence which has a limit can never reach the limit; a function should be given by an algebraic formula.*

If developing system II in our students would be one of the goals of mathematics education then discussions about the above misconceptions should be part of the mathematics classes. Reflections about contradictions between concept images and concept definitions should be integrated in our lesson planning. Unfortunately, since almost the only goal of mathematics education now a day is to prepare our students for the crucial exams, system II will remain quite neglected.

7. GENERALIZATIONS AND EDUCATIONAL VALUES

Since I have recommended in various occasions in the past that educational values should be integrated in mathematics classes as a *by the way* habit I would like to demonstrate it also in the context of this presentation, the context of generalizations.

Dealing with educational values starts very often with the *Golden Rule*. The golden rule has many versions in different religions and cultures. For the sake of this discussion I have chosen one of the Jewish versions related to Hillel, an ancient Jewish scholar (first century, B.C.).

It says: *What you hate – do not do to your friend*. One can argue about it by saying that the rule should be: *Do not do to your friend what he hates*. A possible answer to this claim can be: How can we know what our friend hates? Hillel's suggestion is to generalize from what you hate to other people. Thus, here is a generalization based on a single element sample about the entire population of human beings. Surely, such a generalization must be wrong. However, in this context it is recommended because it tells you how to behave. Without it you will never know how to start.

8. DIFFICULTIES IN OVERCOMING WRONG GENERALIZATIONS

I mentioned in section 5 that if developing system II in our students were one of the goals of mathematics education then discussions about misconceptions should be part of the mathematics classes. A necessary condition for doing that is the student's capability of reflective thinking. According to Piaget and Inhelder (1958) this capability is acquired at the age of formal operations, namely, at the age of adolescence. *The adolescent's theory construction* (it is said there, p. 342) *shows that he* (the adolescent) *has become capable of reflective thinking*. This implies that reflective discussions with our students about their misconceptions are pointless before they reach the junior high level. Even if we do not accept all the theoretical claims of Piaget, reflective discussions are quite problematic at any age. They require from the teacher special skills of discussion management. They also require the students' cooperation. Usually, discussion management is not part of teacher training at any stage and usually students are not used to listen to each other and to reflect. Also, very often reflective thinking leads to cognitive conflicts. Piaget believed that cognitive conflicts will end up with appropriate accommodation. However, experience shows that this is not always the case. Thus, my recommendation to develop system II in mathematics classes is more of a vision than a practical advice. Nevertheless, I would like to elaborate a little about the desirable mathematical thinking[2] and the challenges it presents to children and mathematics educators. For children who only start studying mathematics at the elementary school with thinking habits that they acquired in their early age, desirable mathematical thinking is not a simple challenge. For instance, in kindergarten, they learned about some geometrical shapes as squares, rectangles, triangles and more. They understood that rectangles and squares have different shapes (in rectangles the adjacent sides are not congruent). All of a sudden, their third grade teacher tells them that a square is also a rectangle. When it happens, it is a kind of a cognitive conflict and it requires a conceptual change. Unfortunately, quite often, the desired conceptual changes do not occur. The task of the third grade teacher, whose mathematical background, sometimes, is not satisfactory, is to explain to the children why they should, from this point on, consider squares as rectangles. Later on, or at the same stage, they are required to consider rectangles as parallelograms, while their concept images tell them that parallelograms do not have right angles. In situations like this children may start developing ambivalent attitudes toward mathematics. I am not going to point at more situations in which ambivalent attitudes toward mathematics can develop. Also, I am not suggesting here cures to the problem. My only recommendation to handle conflicts between concept images and concept definitions is to borrow some advice from the relatively new social science discipline - conflict management. The advice is that while interacting with people with whom you have a conflict, try to focus, if possible, on issues about which it is relatively easy to achieve an agreement, and try to avoid, as long as possible, dealing with issues that are extremely hard to solve. I believe that at the school stages of learning mathematics, especially at the elementary level, it is quite possible to apply this advice.

Let us deal now with some conflict situations at the junior high level. At this age, in many countries, the students study some chapters in Euclidean geometry. In this context, definitions are indispensible since very often new notions are introduced to the students, such as median, altitude, perpendicular bisector and more. Also, some familiar notions, such as angles or parallel lines, for which the students have concept images, require certain clarifications. With new notions, there is no potential conflict between concept images and concept definitions. However, taking into account the fact that the students' mind (as well as our mind) tends to rely on concept images and not on concept definitions in thought processes, we should do our best to form the correct concept images in the students' mind. For instance, if we teach the concept of a median we should present it to the students in all kinds of triangle positions and not mainly in triangles in which one side is horizontal and the median is drawn to the horizontal side. If we use the practice of drawing the median only to the horizontal side of the triangle, we may find out that after a while, when the concept definition is forgotten or has become inactive, the students will find difficulties in identifying or drawing medians in triangles that do not have a horizontal side.

Anyhow, geometry at the junior high level is probably the best context to teach the role of definitions in a deductive structure. Here, students are expected to understand that the meaning of a concept is determined by its formal definition and it does not matter what their previous views about the concept were. They are expected to play *the game of mathematics* as deductive structure according to its rules. They are expected to follow the rules of the game. Some students may like it, others may dislike it. Here, individual differences play a critical role. It is similar to the fact that some people like the basketball game and others prefer football. We should respect individual differences and it is a pedagogical mistake to force changes in taste and inclinations. The differences have psychological reasons; some of them are structural, while some of them are acquired. As long as we are not concerned with moral issues, there is no justification for imposing on our students games they do not like to play. We should be especially sensitive since mathematics, to a certain extent, is an obligatory discipline for all school students. Sometimes, for the sake of ‘‘mathematical integrity’’, the curriculum includes topics for which the students do not have mathematical maturity or solid mathematical background. For instance, the case of irrational numbers Some curriculums insist on introducing this concept to the students at the end of the elementary level or in the beginning of the junior high level. Usually the following definition is suggested: an irrational number is a number that cannot be expressed as a ratio between two integers. A lot of mathematical ideas are required to understand this concept. The curriculum does not have the time to elaborate on it. The practice is to mention some irrational numbers, and the simplest practice at the junior high level is to mention π. Thus, π becomes part of the concept image of irrational numbers. On the other hand, at an earlier stage, in some countries, the students are told that: $\pi \sim 22/7$. The symbol "~" means approximately equal. Since, system I tends to ignore seemingly small differences the "approximately equal" becomes "equal" and "$\pi \sim 22/7$" becomes "$\pi = 22/7$." At this stage, the equality $\pi = 22/7$ becomes a part of the concept image of π. At a later stage, surprisingly enough, some students and some elementary teachers when asked to give an example of an irrational number, point at 22/7. The explanation is quite clear. Consulting definitions is a system II project. The definition of irrational numbers was, probably, too difficult to understand. Hence, it was forgotten or ignored and when being asked about irrational numbers, the students' concept image became active and an example, which obviously contradicts the concept of an irrational number, was given.

With regard to algebra and calculus at the senior high level, my advice is to maintain an informal way of teaching. This was the way that mathematics was taught at the elementary level also at the junior high level. Changing this, all of a sudden, causes a discontinuity in the learning process. Generally speaking, discontinuities are not desirable since, as I claimed above, they require a conceptual change. Such changes may, unnecessarily, cause more students to become victims of mathematical difficulties. A partial list of central concepts in algebra and calculus at the senior high level may include function, limit, derivative, continuity and more. These concepts can be introduced by means of examples, which can be followed by general explanations. Indeed, this approach may face some difficulties at certain intersections. If you do not introduce the Bourbaki definition of a function to the students, then they might not be able to deal with all kinds of weird functions presented to them in the curriculum. However, there is no need, in my opinion, to present to them all these weird functions. These weird functions will be presented to some students with special talent for mathematics at the university, in case they decide to be mathematics majors. If you do not present to the students, the ε, n definition for a limit of a sequence, they might have some difficulties with the question whether a constant sequence $a_n = c$ has a limit. Nevertheless, there are ways to smooth out this difficulty without presenting to the students the ε, n definition of the limit of a sequence. For example, one can simply say that mathematicians decided that the limit of the sequence $a_n = c$ is c and the reason for that decision is usually presented in more advanced mathematics courses. Similar advice can be given about the definition of a limit of a function. It is true that if the ε, δ definition were introduced, it will be easier to explain various cases of limits of functions. However, as in the case of the sequence, there are ways to smooth out the difficulties that can arise. Moreover, it is much easier to cope with these difficulties than with the conceptual difficulties caused by the need to understand the ε, δ definition of a limit of a function. Last, but not least, the continuity of a function can be characterized by its graph (a function is continuous if its graph can be drawn without lifting the pen from the paper). Although this is not an accurate definition (and there are continuous functions the graphs of which cannot be drawn at all), it is better to leave all the weird functions to the mathematics majors at the university level. There, they are supposed to be exposed to the ultimate rigor of mathematics. This kind of rigor is not suitable for high school students, even to those who study mathematics at the highest high school level. We should remember that only few of them will choose to be mathematics majors at the university level. Very often when rigorous proofs are discussed in mathematical education forums, it is recommended that they are not suitable for the majority of high school mathematics students. I would like to suggest that rigorous definitions are also not recommended for the decisive majority of high school students.

9. GENERALIZATION SKILLS AS TOPIC IN MATHEMATICS CLASSES

In the beginning of my presentation I spoke about the drive to generalize and I claimed that in spontaneous thinking the generalizations are formed by system I. Contemporary mathematics education undertook the task of teaching mathematics students generalization skills. Now, there is a huge difference between everyday situations which spontaneously lead to generalizations and artificial situations used as an *invitation to generalize*. Such situations are supposed to activate system II. Technological developments have given us the options of doing it elegantly. Thus, it has become a practice in the learning of mathematics to use computers as a means to trigger students to form generalizations (see for instance Schwartz et al., 1993; Perkins et al., 1995). I do not know to what extent this practice is common in my country or in other countries. Namely, I do not know the percentage of students who are exposed to this kind of activity. Also, I am not familiar with the particular micro-worlds provided by the many softwares used in different places. Therefore, I would like to make only a short comment about a potential misconception that might be caused by the use of these technologies. Sometimes, the procedure that is used by the software is like the following: the students are asked to examine some examples of a well-known mathematical theorem (about which they never heard in the past). After that, they are asked to make a generalization. A better notion for this context is "conjecture." The conjecture should turn out to be a mathematical theorem. The next stage is to ask the students to prove the theorem. However, since for many students the proof is only a ritual that occurs in the framework of mathematics (Vinner, 2007), but quite dispensable when we are out of this framework, then the conclusion about establishing generalizations might be the following: it is quite sufficient to examine some particular examples. If these examples lead us to a certain generalization then this generalization is necessarily true. I myself notice this line of thought in my mathematics education courses for elementary mathematics teachers in a master program. For instance, I asked my students about the number of all sets which are subsets of a set that has n elements. We counted them together for n=1,2,3,4,5. The class came to the conclusion that for any n, this number is 2^n. Then I asked my students whether they had any idea how we can prove it. I noticed a surprise expression on their faces. The eldest student, a 59 year old man who switched to mathematics education from an insurance company said: *Aren't the examples that we considered enough to establish the generalization. Is it possible that this generalization is not true? So, between the two of us, isn't the proof an unnecessary formality?* I was grateful to this student about his comment. His age and his past as an insurance agent gave him a lot of self confidence to express these thoughts. I distinguished some other students who nodded their heads in order to indicate that they agree with his view.

In order to avoid such misconceptions, it is quite desirable to present to the students ‘micro-worlds’ in which a set of particular examples supports a certain generalization, however, the generalization is false. For instance, in the context of quadratic equations, one can lead the students to think that the solutions of a quadratic equation of the form $x^2+bx+c=0$, where b and c are integers, should be divisors of c. There are infinitely many examples which support this conjecture. However, it is trivial to point at counterexamples. Thus, if we let students form a generalization and then let them realize that the generalization they formed is false, then they might understand why it is necessary to establish the validity of a generalization in the context of mathematics, as well as in other contexts.

10. A CONCLUDING REMARK

Since I do not want to end my presentation on generalizations in a pessimistic mood I decided to relate in my concluding remark to an old male Chauvinistic generalization from the days of Mozart and Da Ponte. It is the main theme of their opera Cosi Fan Tutte (Thus do they all). As a matter of fact, at least the two women in the opera, Fiordiligi and Dorabella, are counter examples to Don Alfonso's claim: Thus do they all. The two men in the opera, Ferrando and Guglielmo, were convinced by Don Alfonso to examine their belief that their brides are counter examples to above male Chauvinistic claim. This caused them to be involved in extremely unpleasant situations. Fortunately, there is a happy end to the enormous complications and it is summarized by the following lyrics:

Happy is the man who look

At everything on the right side

And through trials and tribulations

<u>*Makes reason his guide*</u>

What always makes another weep

Will be for him a cause of mirth

And amid the tempests of this world

He will find sweet peace.

Just notice the lines: "Happy is the man who … through trials and Tribulations, makes reason his guide". Isn't this a message sent to us by Mozart and Da Ponte, from the end of the eighteen century to use system II in the twenty first century, and by using it to achieve sweet peace?

Endnotes

1. The American psychologist Abraham Maslow (1908-1970) uses the term "need" in his motivation theory. Because of his disagreement with Freud's theory he suggested an alternative notion – "need." However, if we try to bridge between Freud's theory and Maslow's theory (Maslow, probably, won't approve this) I believe that a need and a drive are somehow equivalent. Here is a quotation from Maslow which is relevant to our discussion about generalizations: ‘Curiosity, cognitive impulses, the needs to know and to understand, the desires to organize, to analyze, to look for relations and meanings as an essential part of the human nature.’ (Maslow, 1970). The notions *impulses, needs, and desires* are clearly related to the notion of *drive.*
2. Desirable mathematical thinking includes, among other things, training our mind as well as our student's mind to form concepts by relying on formal definitions and to establish claims by relying on proofs.

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GENERALIZATION IN THE PROCESS OF DEFINING A CONCEPT AND EXPLORING IT BY STUDENTS

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Generalization is one of the most important processes that occurs in the construction of mathematical concepts, discovering theorems, and solving math problems. This process can be analyzed from two different viewpoints:

1. *the cognitive theory,*
2. *the mathematical activity of individuals.*

Both these aspects will be taken into account. In the first part of the paper I will present in outline two theoretical issues, namely Dörfler's theory of generalization (Dörfler, 1991) and Krygowska's types of generalizations of theorems (Krygowska, 1979).

The second part will include an analysis of examples of generalization activity disclosed in my research on solving math problems by students at different levels of mathematical knowledge and experience (Ciosek, 2005, 2010).

GENERALIZING FROM EPISTEMOLOGICAL PERSPECTIVE

1. Dörfler's theory of generalization

In the work entitled *Forms and means of generalizations* (1991) W. Dörfler gives the following explanation of generalizing:

> I understand generalizing as a social-cognitive process which leads to something general (or more general) and whose product consequently refers to an actual or potential manifold (collection, set, variety) in a certain way. (Dörfler, 1991, p. 63)

The author differentiates two forms of generalization: empirical and theoretical.

The basic process in **empirical generalization** is to find a common quality or property among several or many objects or situations and to notice and record these qualities as being common and general to these objects or situations. The common quality is found by comparing the objects and situations, with regard to their outward appearance, isolated mentally, and detached from the objects and situations.

In contradistinction to this form Dörfler introduces another one - called **theoretical generalization –** and describes it with the help of a theoretical model for processes of abstraction and generalization which can often lead to the genuinely mathematical concepts (propositions, proofs, etc.). Here is this model (Figure 1).

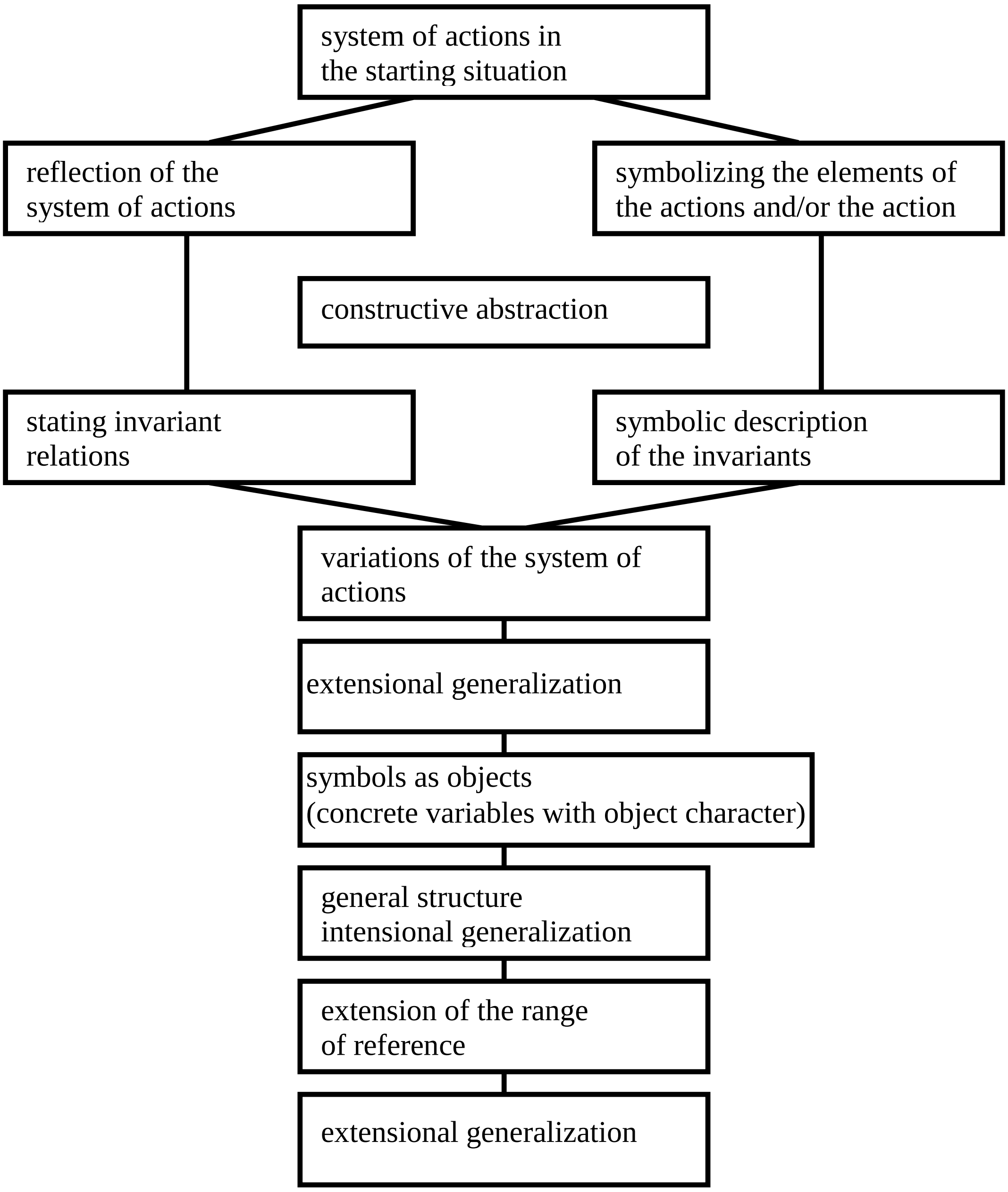


Figure 1. Dörfler's model of theoretical generalization.

Let us concentrate now on the upper part of this model – labelled **constructive abstraction**, as an opening stage for the process of generalizing. Dörfler characterizes it as follows.

- The starting point is an **action** or system of actions (material, imagined or symbolic). Elements of this action are certain objects (material or ideal).
- Course of this action direct one's attention to some relation between the elements of the actions. This relation proves to be steady when the actions are repeated. They are called **invariant** (or schema) of the action.
- Stating invariants need a symbolic description; one has to introduce symbols for the elements of the actions, and then describe invariants stated by means of these symbols. This stating of invariants and their

> symbolic description have the character of a process of abstraction. It is constructive abstraction because what is abstracted is constituted by the action.

I will use this model to analyze students' generalization activity further.

2. Types of generalization of statements by Krygowska

Generalization through induction

A formula $f(n)$ for natural n is to be found. One first finds $f(1)$, $f(2)$, $f(3)$ and notices that the results can be obtained when applying a general rule. This rule is a conjecture only. Though being naive, it's often an important step toward the solution.

Generalization through generalizing the reasoning

One notices that the reasoning carried out in a single case will remain correct in a different setting or minor modifications will be needed only to get a more general result. This often happens as the result of "variation of constants" or spontaneously resulting in the analysis of the proof.

Generalization through unifying specific cases

A bunch of statements, each referring to one case of a setting, proves able to be replaced by one general statement, the original ones being its special cases. E.g., Pythagoras theorem, formulas for acute-angled, obtuse-angled, and "flat" triangle can all be generalized to the so called cosine formula.

Generalization through perceiving recurrence

As in the case of generalization through induction, a formula $f(n)$ for natural n is to be found. But in this case, $f(2)$ is obtained using $f(1)$, $f(3)$ using $f(2)$, and a regular way is noticed to pass to the next n: the recurrence rule. Applying it backwards one obtains the sought formula.

Illustrations of the first three types will be shown further while presenting examples of students' activity. Now I explain the last type of generalization with the help of the following problem:

PROBLEM 1

How many common points at most can have n lines in the plane?

(A common point here is meant as an intersection point of two different lines.)

We start from a concretization. Two different lines can have one common point at most (Figure 2).

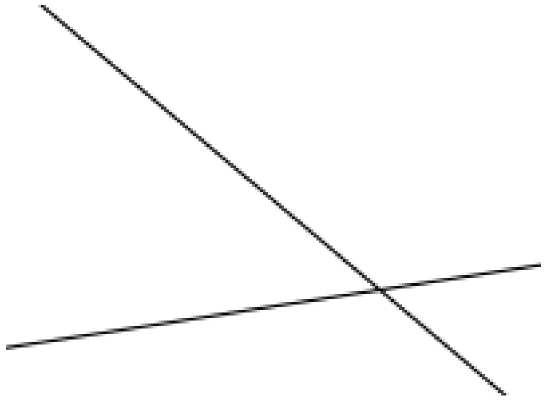


Figure 2

Three different lines have no more than 3 common points (Figure 3).

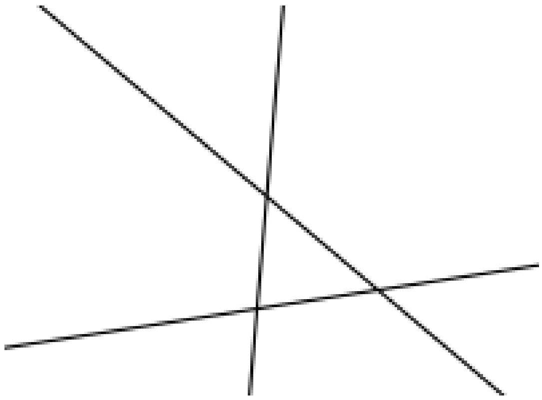


Figure 3

We can ask how to draw the fourth line so as the number of common points be maximum. Of course, it should cross each of the previous lines but not pass through any of their intersections. This is possible as shown in Figure 4.

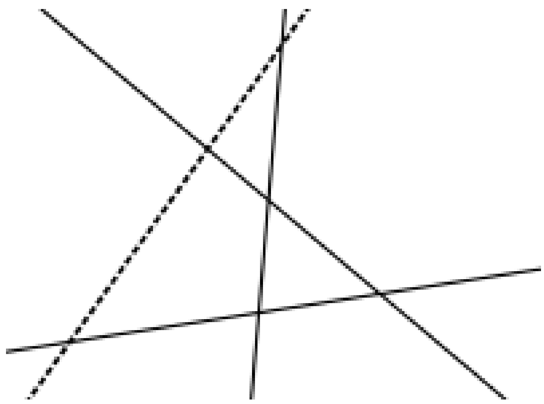


Figure 4

Next we must draw the fifth line so as to cross the four and so on. We become aware of the recurrence: to know the number of common points of some number of lines we need to know that of the less by one number of lines.

If L(n) is the required number of common points of *n* different lines, then

$$L(2) = 1,$$

$$L(3) = L(2) + 2,$$

$$L(4) = L(3) + 3,$$

....

$$L(n) = L(n - 1) + (n - 1).$$

Of course, finding the compact formula for L(n) is also possible:

$$L(n) = 1 + 2 + 3 + ... + (n - 1) = n(n-1) / 2.$$

EXAMPLES OF STUDENTS' GENERALIZATION ACTIVITY

During the last two decades, research on the activity of generalization focused on the phenomenon of noticing by the learner regularities in special-type contexts (e.g. Garcia-Cruz, Martinon, 1997; Iwasaki, Yamaguchi, 2008; Legutko, 2010; Pytlak, 2006, 2007; Stacey, 1989; Zaręba, 2004, 2006).

The student was shown, for example, a series of pictures drawn according to a certain rule. The student's task was to discover that rule. As a help, some tasks were given:

- draw one or a few subsequent pictures conforming to the given series
- find a number characteristic for the pictures (e.g. find the number of some elements)
- represent algebraically the number characteristic for picture number *n*.

A representative of this kind of problems is the following:

PROBLEM 2 (a modification of PISA problem, 2003)

A farmer plants apple trees in the square garden. In order to protect them against the wind he plants coniferous all trees around the orchard. Here is a scheme that illustrates the situation. It presents the pattern of apple trees and coniferous trees when there is *n* rows of apple trees. (Figure 5)

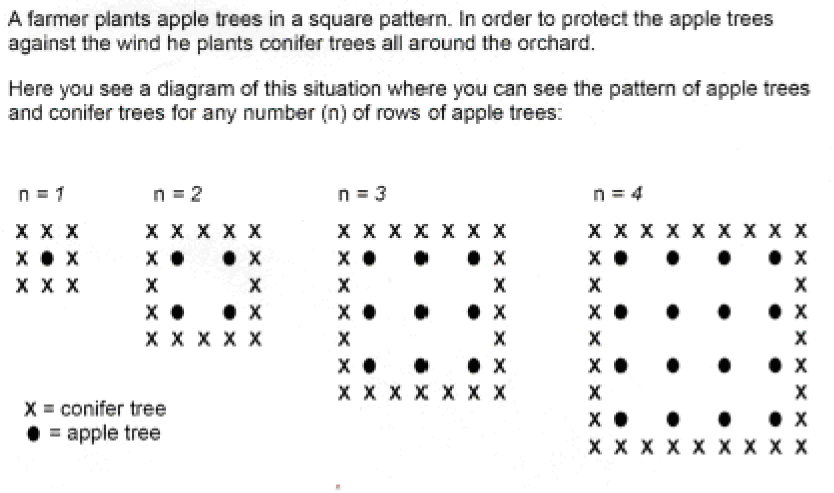


Figure 5. Problem 2

Task 1.

Fill out the table.

N	Nr of apple trees	Nr of coniferous trees
1		
2		
3		
4		
5		
10		
25		

Task 2.

Try to write the numbers of apple trees and coniferous trees for an orchard in which there are *n* rows of apple trees.

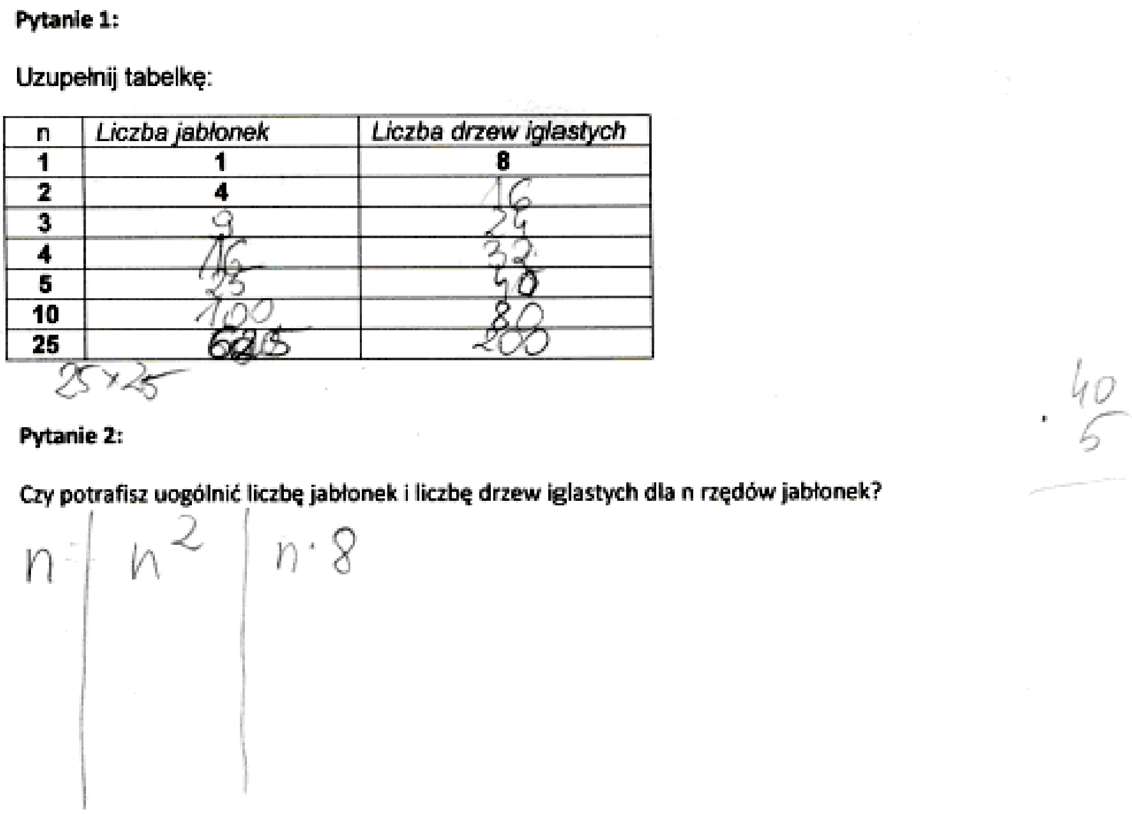
I'll discuss now two approaches to these tasks: by one lower secondary and one upper secondary students[4]. The analysis of paths leading to the solution indicates clearly the differences in the thought processes toward a generalization required in Task 2.

Example 1 – Reasoning by Adriana (9th grade)

Adriana fills the first four rows of the table for the apple trees. She counts the circles in each picture and puts down the number. Then she similarly fills the second column for *n* 1 to 4.

[4] The observations on solving the task have been made by a student preparing a Master's thesis under my supervision: P. Matras, *Generalizing theorems of elementary geometry by secondary school students*, IM, Pedagogical University of Cracow, 2012.

Next, she concentrates on filling the remaining boxes in the apple trees column. She says: "For n=5 there will be 10 times 10 or 100, and for n=25 there will be 25 times 25" (she puts down 625).


Pytanie 1:

Uzupełnij tabelkę:

n	*Liczba jabłonek*	*Liczba drzew iglastych*
1	1	8
2	4	16
3	9	24
4	16	32
5	25	40
10	100	80
25	625	200

25×25

· 40
5

Pytanie 2:

Czy potrafisz uogólnić liczbę jabłonek i liczbę drzew iglastych dla n rzędów jabłonek?

n | n^2 | $n \cdot 8$

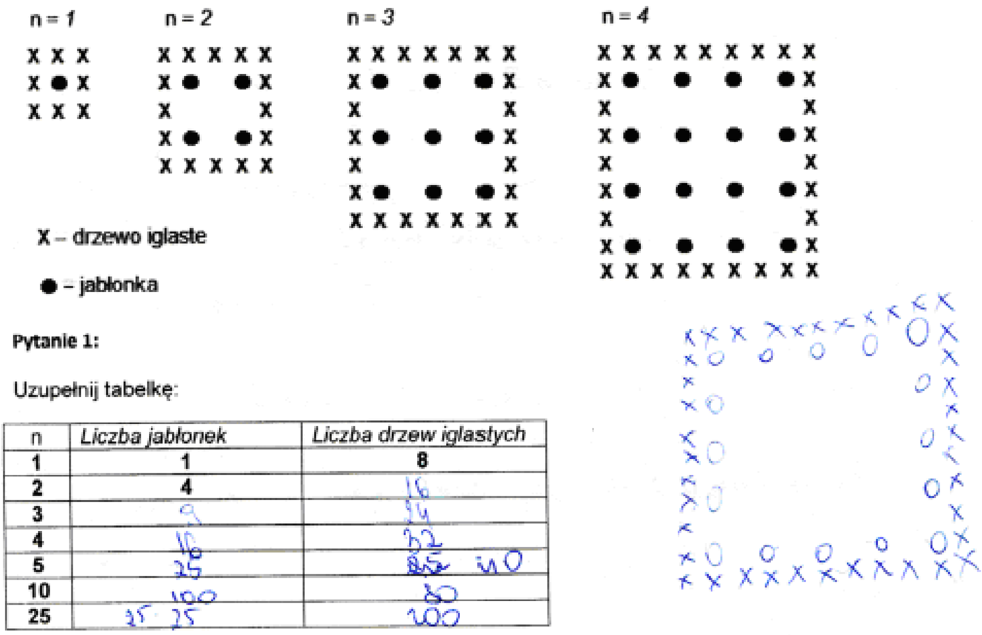
Figure 6. Adriana's work

She announces to start dealing with the coniferous trees. She points the row for *n*=5 in the right hand side column and, after a while of watching consecutive numbers in that column, she says: "then here 40". Asked why she explains: "because 8+8=16, 16+8=24, 24+8 makes 32, and 32+8 is 40." She continues: "And for n=10 we should put down 80 as 80 is 40 times 2; for n=25 it will be 40 times 5 or 200."

After this Adriana passes to Task 2. She is thinking for a while watching the pictures, then she writes: in the left column $n \cdot n$ as the number of apple trees and $n \cdot 8$ as the number of coniferous trees.

Example 2 – Reasoning of Asia (12th grade)

Asia fills the first four rows like Adriana, i.e. as the result of counting crosses and circles on the pictures. She puts down in each row both the number of apple trees and the number of coniferous trees. For *n*=5 she writes 25 as the number of apple trees explaining that it's 5 times 5. She fills the right column up to the end.



n = 1 n = 2 n = 3 n = 4

X – drzewo iglaste

● – jabłonka

Pytanie 1:

Uzupełnij tabelkę:

n	Liczba jabłonek	Liczba drzew iglastych
1	1	8
2	4	16
3	9	24
4	16	32
5	25	~~85~~ 40
10	100	80
25	25 · 25	100

Figure 7. Asia's work (part 1)

The student announces the intention of drawing the garden nr. 5 to correctly reckon the number of coniferous trees. She begins with putting the circles as shown below, then looking at the given pictures she completes her scheme with crosses. Pointing at the left hand side of the scheme Asia says:

> There will be 5 coniferous trees adjacent to apple trees (she applies 5 crosses) and 6 more between the apple trees, and 2 at the corners. There will be 11 altogether or 2 · 5 + 1. It will be same here (she points the right hand side vertical row on the scheme). On one of the remaining sides (pointing a horizontal row) there will be less by 2, so 9. The number of all coniferous trees for n = 5 will be 40.

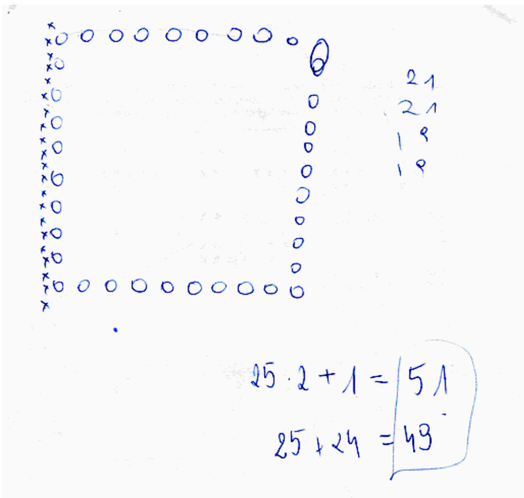


Figure 8. Asia's work (part 2)

Asia draws a fragment of the garden's scheme for n=10 writing next to it numbers 21, 21, 19, 19, then putting down the number of coniferous trees in the 10th row. For n=25 she so calculates the number of coniferous trees:

$$25 \cdot 2 + 1 = 51$$
$$25 + 24 \quad = 49$$

then adds up the results and multiplies the outcome by 2. In the table she writes 200.

As the answer to Task 2 Asia writes

$n \cdot n$ – the number of apple trees

$n \cdot (2 + 1) \cdot 2 + (n + n - 1) \cdot 2$ – the number of coniferous trees (Figure 9).

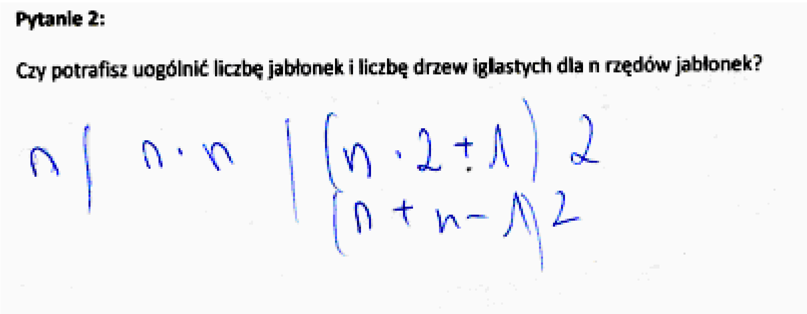


Figure 9. Asia's work (part 3)

Comparison of performances by Adriana and Asia

Generalization by the lower secondary student Adriana was of the **induction type.** The student was concerned about relations between the numbers that quantitatively characterized considered objects, separately for each one (numbers of apple trees in relation to *n*, and numbers of coniferous trees in relation to *n*).

The upper secondary student Asia was interested in mutual relationships between objects of the two kinds. We can say that she discovered the "structure of the orchard", the arrangement of one species of the trees with respect to the other one. One example made her aware of the structure of the orchard. She reproduced what she noticed in this example in one more picture, for another *n*. I think that her reasoning illustrates the type of generalization called by Krygowska generalization **through varying a constant**.

In mathematical point of view the problem consists in finding formulas for two functions $f(n)$, $g(n)$ when their values for four consecutive natural numbers are given. The solution requires some generalization acts. To compare the thinking processes of both students Dörfler's model can be applied. To do so, the examined subjects' actions should be identified as well as invariants they found. Finding the formula for f was easy for both students, but more difficult for g. **With respect to the function *g*** generalization made by both students was essentially different.

Here are the actions taken by the students and the invariants they had noticed.

Adriana's actions

A1. Counting both kinds of elements in figures 1 to 4 (writing results in the tables)

A2. Finding the relationship "+8" among numbers in subsequent rows and its application to find $g(5)$.

A3. Finding the relation between the object nr. n ("orchard" in the picture) and the number of its elements ("coniferous trees" - crosses), that is the formula $g(n)$ in terms of n.

Scheme of the invariant resulting from Adriana's action: $g(n) = n\cdot 8$.

Asia's actions

A1. - same as Adriana's.

A2. Sketching the considered object (fragment of the orchard for $n = 5$).

A3. Finding a way of mutual disposition of two kinds of elements for the object nb 5 (for n=5).

A4. Application of the discovered disposition to objects nb 5, 10, and 25, that is calculating *g(5), g(10),* and *g(25).*

A5. *Imagining* object nr. n.

A6. Finding the relation between the object's number the number n of its elements, that is the formula $g(n)$ in terms of n.

Scheme of the invariant resulting from Asia's action:
$g(n)= (n\cdot 2 + 1) \cdot 2 + (n + n - 1) \cdot 2.$

The question could be asked if Adriana really found the invariant of actions in the set of considered objects or rather a "candidate" for such an invariant. Asia doubtlessly found such an invariant. So we can say that - differently than Ariadna - Asia accomplished a generalization, which in Dörfler's model is called **intentional generalization** as she discovered and described a general structure of considered objects. In my opinion none of the two students made the extensional generalization because variables are referred by them to one kind of objects only.

I think that the analyzed model of generalization presents first of all the scheme of thinking processes that may lead to the formation of a mathematical concept. The process of such generalization is long lasting. All elements indicated by the model should occur in it. In the process of generalization that a researcher (or teacher) initiates with a problem for "finding a regularity" thinking often consists in reflective **abstraction** only.

Interesting observations concerning elementary school students' attitude with respect to Problem 2 are reported in (Pytlak, 2006, 2010).

Example 3 – Reasoning of Michał

Michał - students of the 4th elementary grade were assigned to find the sum of all integers form 1 to 100. The teacher suggested the possibility of using the following table (Table 1):

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
41	**42**	**43**	**44**	**45**	**46**	**47**	**48**	**49**	**50**
51	**52**	**53**	**54**	**55**	**56**	**57**	**58**	**59**	**60**
61	**62**	**63**	**64**	**65**	**66**	**67**	**68**	**69**	**70**
71	**72**	**73**	**74**	**75**	**76**	**77**	**78**	**79**	**80**
81	**82**	**83**	**84**	**85**	**86**	**87**	**88**	**89**	**90**
91	**92**	**93**	**94**	**95**	**96**	**97**	**98**	**99**	**100**

Table 1. Table of numbers 1 - 100

A student used that table in the following way. He decided to add numbers in each row and add up the results. So he added the first row in memory getting 55. Then he calculated the sum in the second row using the column algorithm, getting 155. Both results he put down in the table. After some reflection he wrote, without any calculations, numbers 255, 355,... up to 995 as the remaining row sums. Asked why he knew that the other sums would be so he explained:

> Every number in the second row is by 10 greater than the one in the first row above it as the units digit is the same and the tens digit in the second one is greater by one. So the sum of the numbers in the second row is by 10 times 10 or 100 greater than 55. The same will be with rows 2 and 3. When we move down one row we add one hundred...

It's worth noticing that in the student's reasoning a generalisation occurred, which was not of the induction (empiric) character. Referring to Dörfler's model we can say that the student (12 years old), the first two operations having been done, recognised the invariant of going one row down: increase of the sum by 100. He saw that it results from the structure of the table. Though he did not formalise it, it was – in my view – an important element of the process of **reflective abstraction.**

Example 4 – Reasoning of Dominika

Dominika (aged 15) - 2nd class of the lower secondary solving Problem 3

PROBLEM 3

Examine the truth of the sentence: If a natural number n can be presented as the sum of the squares of two natural numbers then the number $2n$ can also be presented as the sum of two natural numbers.

Having read the problem, Dominika wrote:

$$n = x^2 + y^2$$

$$2n = z^2 + r^2 \quad ?$$

Then she made a few sums of the squares of two numbers (Figure 10).

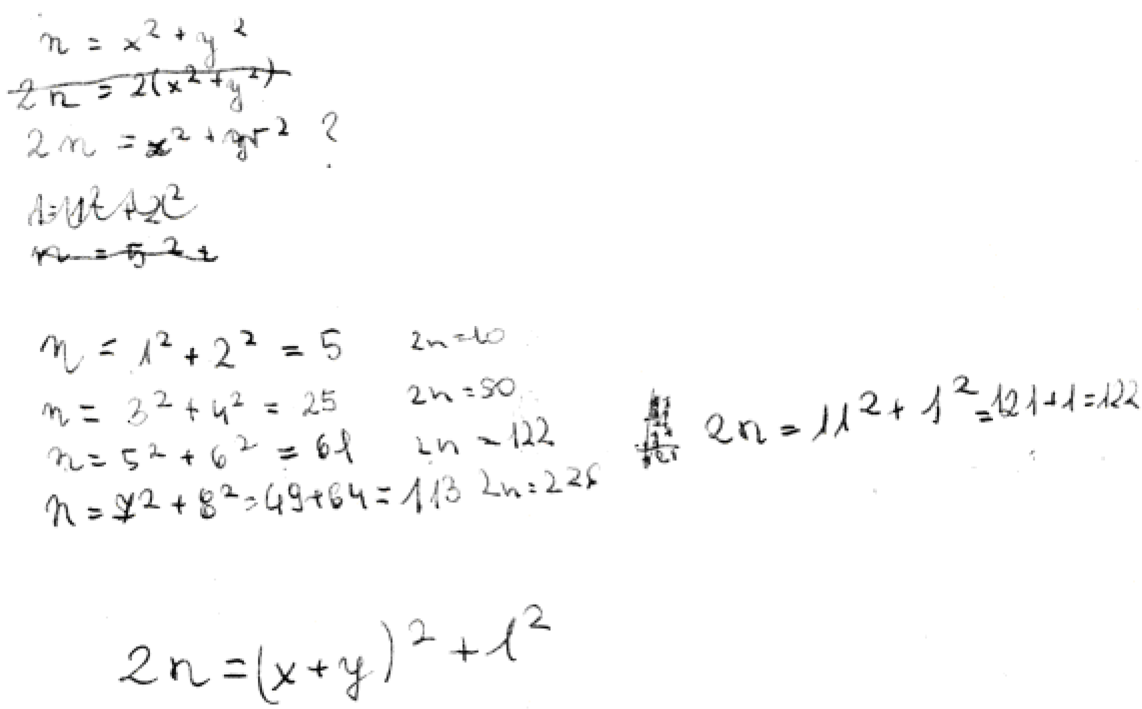


Figure 10. Dominika's work (part 1)

Next to each of the calculated sums she put its double:

$n = 1^2+2^2=5$	$2n=10$
$n=3^2+4^2=25$	$2n=50$
$n=5^2+6^2=61$	$2n=122$
$n = 7^2+8^2=113$	$2n=226$

She tried to present each of the doubles as the sum of two squares. As to the numbers 10 and 50, she said they are the sums of two squares, because 10 is $3^2 + 1^2$ while 50 is $7^2 + 1^2$. She said that 122 is not likely to be so presented, because she failed to do it in her memory calculations. In a while she changed her mind and wrote $122=11^2+1^2=121+1$. Then she said: „For a number n which is the sum of two consecutive numbers, number $2n$ will also be the sum of squares. It will be so:

$2n = (x+y)^2 + 1^2$.

After this the student checked on an example if the devised way of presenting number $2n$ as the sum of two squares works in examples where n is the sum of two non-consecutive numbers. She considered the case of $n = 5^2 + 7^2$. She calculated: $(5+7)^2 + 1^2$ getting 145, and not – as she supposed – 148. She decided that number $2(5^2 + 7^2)$ cannot be presented as $(5^2 + 7^2) + 1^2$ (Figure 11).

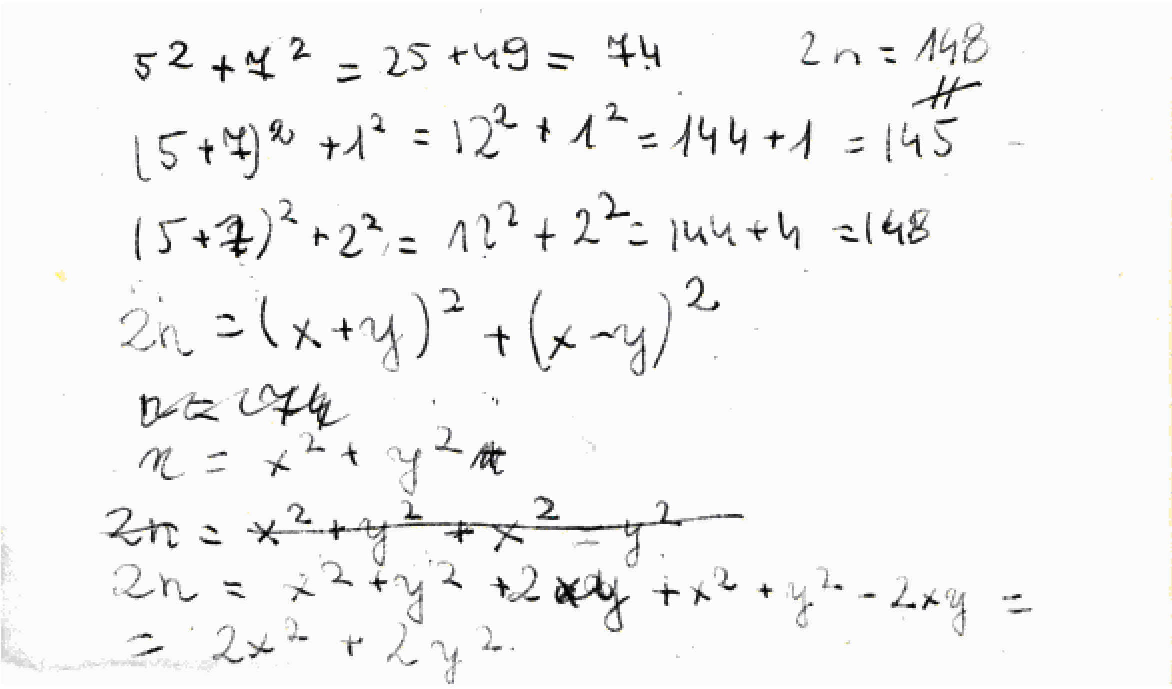


Figure 11. Dominika's work (part 2)

After a while she added: "But in this case number $2n$ can be presented as $(5^2 + 7^2) + 2^2$." A bit later she noted:

"I know already how it is going to be. If $n = x^2 + y^2$ then $2n=(x+y)^2 + (x-y)^2$".

Dominika decided that having uttered the last sentence she finished her work on the problem. Only after the observer's remark: "Explain please why you think that number $2n$ that you have put down equals $2(x^2 + y^2)$" caused her to transform the sum of squares of (x + y) and (x – y) to the form $2(x^2 + y^2)$.

Analysis of Dominika's work

In the Dominica's work 3 acts of generalization can be discerned.

The first begins at the moment when Dominika has checked that the sentence being examined is true for four natural numbers chosen for n, each being the sum of squares of two consecutive numbers. She is looking at these examples seeking their common property. She wants to find a relationship between x, y and z, r. She puts down her observation as:

$$2n = (x + y)^2 + 1.$$

She does not treat this representation of 2n as a hypothesis; seemingly she has no doubts that it will hold for every n being the sum of two consecutive natural numbers. This kind of generalization is called by Krygowska inductive generalization.

The second act begins with Dominika's questioning herself if the way of selecting numbers z and r invented in the previous case cannot be applied to a number n, for whom x and y are no longer consecutive numbers. This behaviour shows that the girl is treating the relationship found as a hypothesis, now concerning an arbitrary number n. She verifies the hypothesis with examples and rejects it. She notices another possibility of representing the double of n:

If $n = 5^2 + 7^2$ then $2n = (5 + 7)^2 + 2^2$.

She realizes that 2 is the difference of 7 and 5. Probably here the next (induction) generalization act is taking place and – worth noticing – based on one example only. This time it is related to numbers n such that their difference equals 2; Dominika does not express it in words nor in writing, she is just thinking.

The third and last generalization act results from juxtaposing the form of the number 2n with the subsequently considered cases: $y = x + 1$, $y = x + 2$. Again, based on the two examples the student formulates and puts down in the general form the way of representing numbers z, r using x, y. Again she makes a generalization in the induction way.

It is worth noticing that Dominika was applying the strategy of considering special cases in a – say – model way. The special cases considered were not taken at random but from among a special type. It was a *systematic choice of examples* (according to Mason at al., 2005), and this probably helped her to guess the relationship which proved to be conclusive for the solution. Yet, she did not spontaneously undertake any algebraic verification of the conjecture to make sure it always works.

Example 5 – Reasoning of Beata

Beata – 3rd year of mathematics for teacher students. Solving Dominika's problem 3.

The student also starts by considering examples (Figure 12):

n =13 = 4 + 9 = 22 +32 ; 2 · 13 = 26 = 12 + 52

25 = 16 + 9 = 42 + 52 ; 50 = 12 + 72.

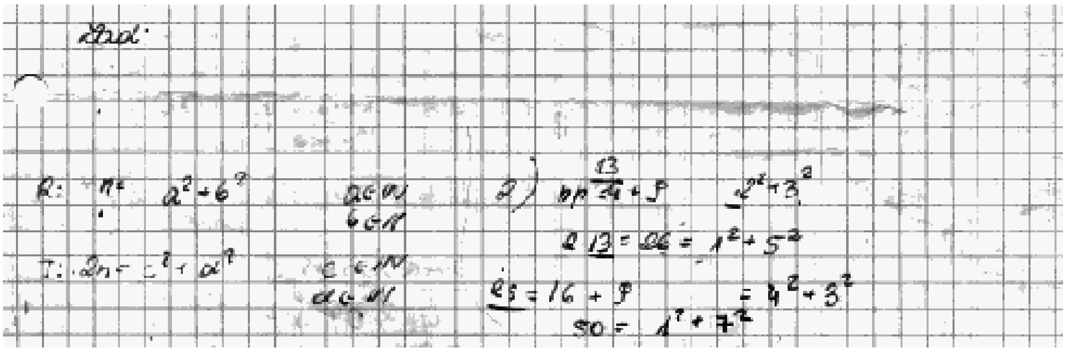


Figure 12. Beata’s work (part 1)

Then she says: “Let’s look at the examples." After a while she adds:

> “Let $n = a^2 + b^2$. Is it so that $2n = 1^2 + (a + b)^2$?”

Now she transforms the latter equation to an equivalent form: $(a - b)^2 = 1$ (Figure 13).

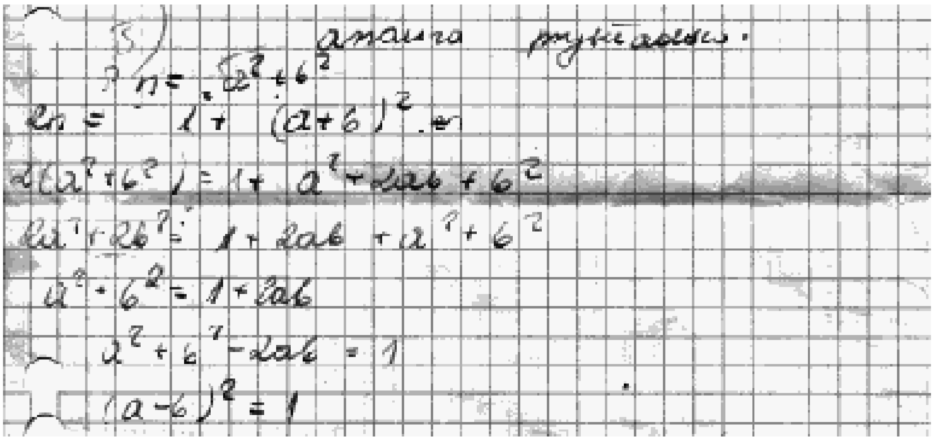


Figure 13. Beata’s work (part 2)

The last result can so be conceived:

> We already know that if n is the sum of squares of two consecutive natural numbers a, b (a greater than b) then 2n can also be written as the sum of two consecutive natural numbers, namely the square of 1 and the square of the difference of numbers a and b.

She carries on:

> What would happen if the difference of a and b were different than 1?
>
> Let us assume that $n = a^2 + b^2$. May be $2n = (a - b)^2 + (a + b)^2$? (Figure 14).

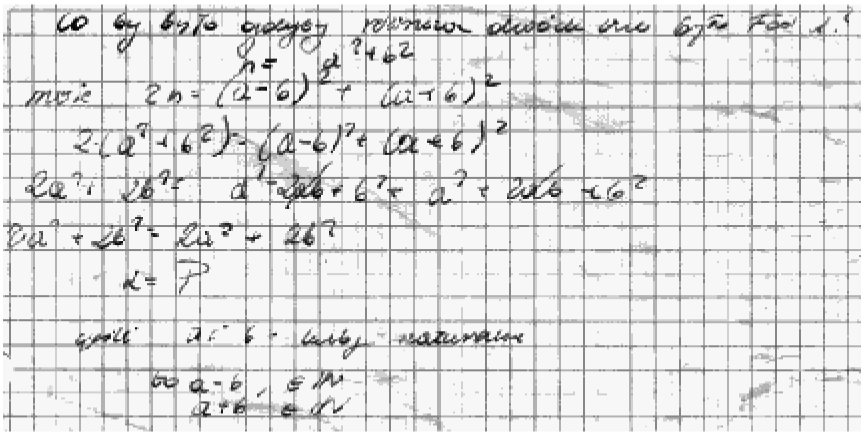


Figure 14. Beata’s work (part 3)

Beata transforms the last equation using the assumption on *n* and known formulas, arriving at the identity

$$2a^2 + 2b^2 = 2a^2 + 2b^2.$$

To finish up she concludes:

> The sentence occurring in the problem is true because if a number is the sum of the squares of two natural numbers we can sort out two numbers whose sum of squares equals the doubled initial number.

After a short while Beata reflects:

> Of course, it must be verified that those two numbers are natural. … But they are so as for natural a, b, a+b and a-b are natural.

(In the last expression she commits an error; she may have thought that *a* is greater than *b* which needn’t be so.)

Analysis of Beata’s work

In Beata's reasoning two generalization acts take place. The first stage of her work on the problem is analogical with Dominika's one. Beata analysed examples of the same type. But, otherwise than Dominika, the condition for 2*n* resulting from her empiric generalization was treated by her as a hypothesis to be verified with algebraic calculus.

The next generalization act which led to a new hypothesis concerning the decomposition of 2*n*, now without an additional assumption concerning *n*, she most probably formulated as a result of reasoning and also verified with algebra. This independent undertaking of verification of hypotheses and the way it was carried out differentiates her reasoning from that of the lower secondary student.

If the two reasonings are referred to the Dörfler's model, we can say that the two persons, who undertook similar actions at the initial stage of work, verbalized and formulated in the algebraic language the same invariants (Domika's was rather a "candidate" for an invariant). If in both cases for the invariant the status of theorem for the special class of *n* such that ... with *b*=*a*+1 was acknowledged we could say that both students accomplished the extensional generalization as they referred the thesis of the formerly found theorem - with a certain modification - to all cases of considered objects. So they accomplished an extension of the reference range.

CONCLUSIONS

In all the examples analyzed here generalization acts were taking place during the solution of the problem, within the short time of observation by the researcher. Indeed, it proved that with each of the problems taken into account identifying some regularity was associated.

The differences among the considered problems consist in the following. In some of them examples of objects to be analyzed were imposed from above (Problem 2), while in the remaining ones the solver was to select them according to the problem situation. This choice may essentially influence the generalization process.

There are other situations associated with generalization. I mean processes that go on out of the observer's (researcher's, teacher's) reach where generalization results from many experiences of the learner, during a long learning period, and are false. We can say that those are *hidden generalizations*, happening *in the background*. They can come out to light unexpectedly or remain undisclosed forever. Of this character are false convictions such as "multiplication increases, division decreases", "raising to power increases, taking the root decreases" etc. Such false convictions result from unjustified extension of the reference range of mathematical operations and their results, from the domain of natural numbers to a wider domain of integer, rational or real numbers.

False convictions have also become the object of research in Mathematics Education. Interesting reports can be found in (Howe, 1999; Pawlik, 2003, 2004; Tirosh, Graeber, 1989; Żeromska, 2010).

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GENERALIZATION PROCESSES IN THE TEACHING/LEARNING OF ALGEBRA: STUDENTS BEHAVIOURS AND TEACHER ROLE

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We give an overview of the literature on generalization with particular reference to the studies about the students' ways of thinking in the development of generalization in algebra. We discuss the teacher's role in guiding students to face algebraic generalizations and we report on our methods and tools to improve teachers'competence in teaching this kind of tasks in a socio-constructive perspective.

> To learn mathematics involves learning to think mathematically… The essence of thinking mathematically is recognition, appreciation, expression, and manipulation of generality. …
>
> The future of Arithmetics and Algebra teaching lies in teacher awareness of the fundamental mathematical thinking processes, most particularly, generalization. (J. Mason, 1996a)

1. THEORETICAL ASPECTS ON GENERALIZATION

A metacognitive teaching practice is necessary to give mathematics strength and meaning as a subject. In this type of teaching practice, the main tasks of the teacher are to lead the students to reflect upon how meaningful the procedures they choose are in front of the various situations, to make verbally explicit the strategies they implement, to compare them, to distinguish what is common and essential from what is not, to check the effectiveness of the representations they use. The aim of all this is to help the students focus on the unifying elements that emerge from the activity, getting to incorporate a variety of cases or situations in one single vision, to consider the strength of representations and to become aware of the process - object dynamics (Sfard, 1991) which governs the reification of mathematical objects.

Basic elements of this type of teaching are generalization processes. By 'generalization process' we mean, briefly, a sequence of acts of thinking which lead a subject to *recognize*, by analyzing individual cases, the occurrence of common peculiar elements; to *shift attention* from individual cases to the totality of possible cases and *extend* to that totality the common features previously identified.

Detecting patterns, identifying similarities, linking analogous facts are all at the base of generalization processes; the key element in these processes is not the detection of similarities between cases, but rather the *shift of attention* from individual cases to all the possible ones, as well as the extension and adaptation of the model to any of them.

Generalization processes are natural: they emerge from our way of looking at things, of capturing them and of elaborating the products of our observations and experiences. They pervade human activities, although they are peculiar of the mathematical activity. Enriques (1942), writing as Giannini, discussing on the role of the error in the development of knowledge, writes:

> The path of the human mind is essentially inductive: that is to say, it goes from the real to the abstract. The understanding of the general should be conquered as a higher degree of something already known and easier, that is to say as a ‘generalization’. On the other hand, the example has a clarifying property and, so, it is a strong instrument in scientific research and, at the same time, an invaluable tool for verifying and correcting theories. …The heuristic value of examples is even more evident, because everyone knows that the comparison between two different cases in which something in common appears is able to suggest to our mind the most beautiful generalizations and to show to us the best positions of problems …

It is also possible to generalize from the examination of one single case, when, regardless of its peculiar features, one sees it as representative of a whole area. The case is ‘exemplary’, i.e. it exemplifies the totality of cases. As in Hilbert’s renowned aphorism

> The art of doing mathematics is finding that special case that contains all the germs of generality.

Mason (1996a, 1996b) claims that ‘*generalization is the heartbeat of mathematics’* and that in the teaching of mathematics the students have to be brought to gain a double awareness: of ‘seeing the particular in the general’ and of ‘seeing the general through the particular’. As to the latter, he states the importance of the experience of ‘*examplehood’*, which brings the students to become aware of how a multitude of details can be subsumed under one generality. He writes (1996a, p. 21):

> One of the fundamental forms or experiences of a shift in the locus, focus, or structure of attention is the sense of ‘*examplehood*’: suddenly seeing something as ‘merely’ an example of some greater generality. To experience examplehood, in which what was previously disparate are now seen as examples of something more general, has an effect like cristallization or condensation (Freudenthal 1978[1],

1 Freudenthal, H. (1978) Weeding and Sowing: Preface to a Science of Mathematics Education, Reidel, Dordrecht.

p. 272): it is releases energy and reduces the amount of attention required to deal with similar situations.

Mason underlines that the students' recognition of a thing *as* an example requires that they grasp the sense of what the example expresses, the enhancement of the features which makes it 'exemplar' and the shading of the features which make it particular. Moreover he says that if the teacher is, at present, unaware of what makes exemplary the example, (s)he may not provide students with adequate support to appreciate the examplehood being offered.

Without disclaiming the efficacy of generalization as a didactical instrument, with reference to the inference of mathematical facts from the observation of few examples, Radford (1996a p. 107-109) poses the problem of the logical validity of the assumptions that come from that generalization[2]. He deplores the abuse of generalization in teaching, since the students may get the idea that the fact that a regularity occurs in few cases is enough to claim that it is valid as a 'general rule'. It is therefore necessary to spend time working towards the recognition of the limitations of generalization, to distinguish between inductive and deductive processes and to become aware that the validity of an inductively inferred sentence can only be established through a proof.

However, it should be noticed that generalization processes in mathematics not only concern particular mathematical contents; they also involve meta-aspects, linked with the organization and structuring of the gradually acquired knowledge.

Harel & Tall (1991) reflect upon the modalities in which students, progressing in their studies, link together pieces of knowledge and enlarge their horizons. They detect how these moments of reorganization depend on the features of the students' mental constructions and on the type of understanding (relational or instrumental) which underlies their knowledge. They distinguish between three types of generalization: 1) *expansive generalization* in which one extends his or her scheme without reconstructing it; 2) *reconstructive generalization* when a subject reconstructs an existing schema in order to widen its applicability range; 3) *disjunctive generalization* when, on moving from a familiar context to a new one, the subject constructs a new, disjoint, schema to deal with the new context and adds it to the array of schemas available.

They underline that expansive generalization is more frequent and easy to apply than reconstructive generalization, that the latter is delicate and subjective but also more effective, that disjunctive generalization is cognitively poor and turns

2 Radford introduces the issue by making reference to a renowned scene of 'La cantatrice chauve' (usually translated as The Bald Soprano) by Ionesco: at the Smiths' they ring at the door, Mrs Smith opens but she doesn't find anyone; the same happens at the second and third doorbell, at the fourth one she blurts with her husband, making a nonsensical inference, generalization of the previous cases "Do not send me to open the door! You have seen that it is useless! Experience has shown us that when we hear the doorbell, it implies that no one is here".

out to be a real ‘recipe for failure’ for weak students: they are not able to see linking schemes and are helplessly submerged by the amount of notions.

Dörfler (1989, 1991) is interested in the modalities of construction of knowledge in the students, and he theorizes on the processes of generalization. He sees the generalization as a combination of cognitive processes at a double level: the subjective-psychological one, related to the individual-reflective dimension and the objective-epistemological one, related to the social dimension (sharing, communication and use of language). He considers knowledge as the result of the structuring and the organization of one’s own experience and he views it as stemming from *appropriate actions* on certain objects through *reflection* upon both actions and transformations produced in the objects. In order to consolidate knowledge, he considers crucial the representation of a process ‘*by the use of perceiveable objects, like written signs, of the characteristic and stages, steps and outcomes of the actions’.* In this way a *protocol of actions* is generated which allows for a cognitive reconstruction and conceptualization of the process itself.

On these premises he develops a “*model of the processes of abstraction and generalization*” (Dörfler 1991). This model has its roots in Piaget’s construct of ‘reflective abstraction’, a process where the actions are seen as genetic source of the (mathematical) concepts, but Dörfler enlarges the meaning of ‘action’ including also the symbolic actions. Two phases can be distinguished in the model: the first one, which leads to the emerging of invariants as well as the birth of representations for them; the second one, more meaningful from the mathematical point of view, where the focus is on the representations: through a reflection on them, the way of viewing them evolves and this leads to the reification of new mathematical objects.

More in details, the starting point of this model is an action or a system of actions (which are material, imagined or symbolic) upon certain (material or ideal) objects. In these actions one’s attention is directed to some relations and connections between elements of actions. In many cases the actions combine the original elements in a certain and invariant way; when, repeating the actions (as often as one likes), the relations prove to be steady, these combinations and basic transformations emerge as “*invariants of actions*”, defining the “*schema*” (of actions). Dörfler underlines: “*the emergence of the invariants* needs *a certain symbolic description*”. This is a key point for the model. Symbols are used for the elements of actions or for quantities relevant for them, and for transformations or combinations on the objects induced by the actions. This representation of the invariants may include variable elements related to objects on which actions are carried out. The symbols (of verbal, iconic, geometric or algebraic nature) initially play a purely descriptive role: they represent either actions or transformations. This first phase can be summarized as one moment of *constructive abstraction*, where the original elements are substituted by *prototypes*, which better highlight properties or relationships we want to focus on (they gain meaning and ‘existence’ via the actions). The second phase develops through other two important moments:

One moment of *extensional generalizations*, when the use of prototypes leads to determine the domain of variability of the patterns, which enhances the interchangeability of the objects with respect to the actions upon them. At this point the symbols lose their initial meaning of generic representatives and they acquire that of *variables with properties of substitution*.

One moment of *intensional generalization*, when by reflecting upon the symbolic representations of the invariants, the used symbols lose their meaning of representatives (of variable elements of the actions), and they become elements of the action themselves and ‘carriers’ of the invariants: at this point symbols are detached from their range of reference and acquire a new meaning, intrinsically connected to the invariants, of *variables with the feature of objects*: so, a new mathematical object is born.

Dörfler claims that once a generality of this type is constructed, it becomes the basis for further generalization. He stresses that his model is a ‘theoretical generalization’ model, juxtaposed to ‘empirical generalization’ (EG), that is the Aristotelian basic process of finding a common quality or property among several objects or situations from sense perceptions. He states that EG does not contribute to the construction of the meaning of the concepts because it is mainly a recognition process, he criticises the use of EG in mathematics teaching and the fact that usually the ability to recognize the generality is postulated.[3]

Dörfler offers also an interesting sequence of examples of his model from both elementary and advanced mathematics. In these examples, however, the focus is uniquely on the mathematical contents, without specific reference to either the students or the teacher.

On the contrary, Dörfler explicitly does not take into account the problem of what the appropriate starting situations for the students may be, and he devolves their choice to the teacher, since, he says, “*it is only she who knows the students and their interests*”.

Later, Hejny (2003) proposes a model of construction and structuring of knowledge organized in six stages (see the table below) where generalization is viewed as a basic element, but still at a lower level than abstraction and functional to this. Hejny, referring to what Sierpinska[4] thought about the development of mathematical understanding, considers her vision as reductive,

3 As to this Dörfler considers the derivative concept and the ‘examples’, such as velocity, gradient, density, usually used to show the derivative as their common structure but- he stresses- this structure is not developed by the students themselves.

4 Sierpinska, A. 1994, Understanding in mathematics, London & NewYork: The Falmer press

and he claims to agree with her, only if abstraction[5] is juxtaposed to generalization. An original element in Hejny's model is the fact that the student's motivation is seen as the first step of the process.

Comparing Hejny's model to Dörfler's one a first important difference can be noticed: Dörfler does not make a distinction between generalization and abstraction, he rather describes processes of generalization with moments of abstraction; on the contrary Hejny states that generalization is prior to abstraction.

The stages of development and structuring of knowledge in Hejny's model

1. *Motivation*. By motivation we mean a tension, which appears in a student's mind as a consequence of the contradiction between *I do not know* and *I would like to know*. This tension steers the student's interest towards a particular mathematical problem, situation, idea, concept, fact, scheme,...
2. *Stage of isolated (mental) models*. The acquisition of an initial set of experiences. At first, these experiences are stored as isolated events, or images. Later on, it might be expected that some linkage between them occurs.
3. *Stage of generalisation.* The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme that generalizes all these models is discovered. The process of generalisation does not change the level of the abstraction of thinking.
4. *Stage of universal (mental) model(s).* A general overview of the already existing isolated models develops. It gives the first insight into the community of models. At the same time, it is a tool for dealing with new, more demanding isolated models. If stage 2 is the collecting of new experiences, stages 3 and 4 mean organising this set into a structure. The role of such a generalising scheme is frequently played by one of the isolated models.
5. *Stage of abstraction.* The construction of a new, deeper and more abstract concept, process or scheme which brings a new insight into the piece of knowledge.
6. *Stage of abstract knowledge.* The new piece of knowledge is housed in the already existing cognitive network, thus giving rise to new connections. Sometimes it ends up in the reorganisation of either the mathematical structure or a part of it.

A second difference concerns the role of representations. In Hejny's model the representation issue does not even appear, while for Dörfler it is essential, since the role played by symbols in the representation of invariants and the progressive change of meanings associated with them allows for the reification of mathematical objects. Another element of difference in the work of the two authors concerns the nature of the examples given for their model. While Dörfler presents examples focused on the mathematical content, with no reference to the subjects involved in the process, Hejny analytically shows the

5 Hejny (2003) writes: "In her analysis of the act of understanding, Sierpinska considers four basic mental operations: identification, discrimination, generalisation and synthesis. 'All four operations are important in any process of understanding. But in understanding mathematics, generalisation has a particular role to play. Isn't mathematics, above all, an art of generalisation? L'art de donner le même nom à des choses différentes, as Poincare used to say?' Sierpinska (1994, p. 59). We agree with this statement provided that 'donner' covers both our terms generalisation and abstraction".

ongoing process of construction of knowledge through excerpts from the students' activities and dialogues which testify the moments when generalizations and abstractions are generated. In this sense, drawing on Sfard's (2005) classification on the time periods that mark the evolution of mathematics education research, Dörfler's study can be placed in the 'content's era' whereas Hejny's research is fully placed in the student's era.

Regarding students, a broad and interesting piece of research is due to Ellis (2007), a teacher-researcher. The research object is the identification of students' key behaviours in the generation of generalizations. Ellis starts from the analysis of studies in mathematics education dealing with students' processes of generalization and she identifies three categories of actions that are typical of generalization: (a) the development of a rule that serves as a statement about relations or properties;

THINKING ACTIONS IN THE PRODUCTION OF GENERALIZATIONS (Ellis 2007)

ACTIONS OF GENERALIZATION

I *RELATING*

- *relating situations:* the formation of an association between two or more problems or situations. a) *connecting back* (the formation of a connection between a current situation and a previously-encountered situation); b) *creating new* (the invention of a new situation viewed as similar to an existing situation);
- *relating objects*: the formation of an association of similarity between two or more present objects. a) *property* (the association of objects by focusing on a similar property they share); b) *form* (the association of objects by focusing on their similar form)

II *SEARCHING*

- *searching for one same relationship*: the performance of a repeatead action in order to detect a stable relationship between two or more objects
- *searching for one same procedure:* the repeatead performance of a procedure in order to test whether it remains valid for all cases
- *searching for one same pattern:* the repeatead action to check whether a detected pattern remains stable across the cases
- *searching for the same solution or result:* the performance of a repeatead action in order to determine if the outcome of the action is identical every time

III *EXTENDING*

- *Expand the range of applicability*: the application of a phenomenon to a larger range of cases than that from which it originated
- *Removing details*: ther removal of some contextual details in order to develop a global case
- *Operating*: the act of operating upon an object in order to generate new cases
- *Continuing*: the act of repeating an existing pattern in order to generate new cases

FORMULATION OF GENERALIZATION

IV. *IDENTIFICATION OR STATEMENT*

- *continuing phenomenon*: the identification of a dynamic property extending beyond a specific instance;
- *sameness*: a). *common property*: the identification of a property that is common to objects or situations; b) *objects or representations*: the identification of objects as similar or identical; c) *situations*: the identification of situations as similar or identical);
- *general principle*: a statement of a general phenomenon. a) *rule*: the description of a general formula or fact; b) *pattern*: the description of a general pattern; c) *strategy or procedure*: the description of a method that can be extended beyond a specific case; d) *global rule*: the statement of the meaning of an object or idea).

V. *DEFINITION:* the definition of a class of objects all satisfying a given relationship, pattern, or other phenomenon.

VI. *INFLUENCE:* a) *prior idea or strategy*: the implementation of a previous generalization); c) *modified idea or strategy* (the adaptation of a existing generalization to be applied to a new problem or situation).

(b) the extension or expansion of one's range of reasoning beyond the case or cases considered, and (c) the identification of commonalities across cases.

The scholar regrets that these studies essentially address the students' difficulties regarding the production of a law which is predetermined by the researchers, and that, consequently, the latter neglect to consider possible generalizations that are partial or not fitting with what is expected from the students[6]. She puts herself in a wider perspective and, in her observation of the students, she considers processes of generalizations as well as processes of transfer through which the students autonomously transfer and adapt their knowledge to new contexts, acting under different conditions.

Ellis investigates how students extend their reasoning, examines the sense given by students to their general claims, and explores which types of common characters the students might perceive throughout the cases. The activities proposed to the students are various and very diversified and allow for the analysis of processes and outcomes. The wide range of the collected data (students' protocols, interviews, video transcripts of the class processes) allows her to develop a taxonomy on two macro levels: that of the *generalizing actions* and that of the *reflection generalizations* (see the previous table).

Several other studies concern the processes of generalization in algebra which we refer to in the next section.

> What matters is how our eyes combine the images that have chosen to assent to be captured, how we are able to associate them playing back and forth, how we follow intuitions, alternative paths, possibilities […] (Davide Enia, *Palermo-India*, 2010).

2. GENERALIZATION AND THE TEACHING OF ALGEBRA

Processes of generalization are dominant in a teaching of algebra which gives room to generational and meta-activities in the sense of Kieran (1996). At the international level few studies address processes of generalization at an advanced level, on non standard problem solving activities (Papadopulos & Iatridou, 2010, Zazkis & Liljedal 2002). The majority of the studies concern processes of generalization in generational activities and are intertwined with the introduction of letters to encode the observed regularities in general terms. Kaput (1995) writes:

6 Ellis writes: "studies examining students'generalizations often report students'difficulties in recognizing, using and creating general statement. Because work on generalization predetermines what types of knowledge counts as general, it may fail to capture instances in which students may perceive a common element across cases, extend an idea to incorporate a larger range of phenomena, or produce a general description of a phenomenon, regardeless of its correctness. … Focusing on correct mathematical strategies, mental acts that cut across strategies may be overlooked and generalizing processes that result in incomplete or incorrect generalizations may be omitted."

> both the means and the goal of generalising is to establish some formal symbolic objects that are intended to represent what is generalized and render the generalization subject to further reasoning.
>
> […] acts of generalization and gradual formalization of the constructed generality must precede work formalism – otherwise the formalism have not source in student experience.

Kaput is recognized as one of the fathers of early algebra, a disciplinary area which is now well established, which proposes the early use of letters intertwined with a relational teaching of elementary number theory as well as a valorization of algebraic language as an instrument to represent relations and properties, to carry out reasoning patterns and produce justifications. His studies gave birth to interesting experiments in the US which invested both the curriculum, by making students get closer to the generalization of facts, procedures and reasoning patterns, and teacher training (Kaput & Blanton 2001, Blanton & Kaput 2001, Carpenter & Levi 2001, Carpenter et al. 2003, Carraher et al. 2000, 2001, Schliemann et al. 2001). Influences of these studies can be found in the NCTM's proposals for the curriculum, where there is a strong emphasis on students' learning to make generalizations about patterns. Regarding this topic, the anticipatory studies carried out by Stacey (1989), Lee (1996), Orton & Orton (PME 1994, 1996) and the books by Mason et al. (1985) and by Orton (1999) must be mentioned.

As a rule, international studies about the approach to algebra that involves the processes of generalization concern the study of: patterns, algebraically representable functional correspondences between pairs of variables, equations, structural aspects of arithmetic operations, simple numerical theorems (formulation of conjectures and their justification). However the study of patterns is the more practiced one, as it is also documented by the ZDM special issue "*From Patterns to generalization: development of algebraic thinking"* (2008).

Dörfler, in his comment to this issue, makes a few remarks we agree with (see Dörfler 2008). First of all, he claims that the knowledge and mastery of algebraic notations do not develop simply by generalizing patterns of various kinds. In particular, he observes that it is not enough for pupils to be able to translate expressions from the verbal to the algebraic register, if they are to grasp the meaning of formal expressions; he points out the importance of the "negotiation of the intended meaning of the algebraic terms, specially of their ascribed generality", because it is "the habit of usage of, of operating with, of talking about, etc, the marks/letters on paper" which makes the students aware of the meanings they bring. About the figural sequences he stresses the importance that the students become aware that a given visual cue can be seen in different ways and then look for its different views. Moreover, both to give room to the students' creativity and not to determine in them the stereotype of the existence of one ‘unique law’, given a series of figures, he suggests that it should be asked “*how can you continue*?” or “*what can you change and vary in the given figures*?”

Similarly, about the activities of modeling of functional relationships he states that “verbal or quasi-variable generalizations[7] will not easily permit one to even think of those properties of a functionl relationship. They describe the respective generality but they are not amenable to operate in it or with it”. He also stresses that what makes productive the use of letters that allow to transfer the reasoning on the facts at stake into the calculations, is the chance to operate with the letters according to the common rules of arithmetic (condensed in the notions of ring or field); yet if the students are not aware of the possibility of actions, such as “adding” or “multiplying”, on the letters, the sentence “*n stands for an arbitrary number*” remains void and difficult to be accepted.

Moreover, he claims that the papers presented in the ZDM issue do not clarify the relationship between this kind of activity and the mastery of algebraic calculations, which the students need to practice in order to become able to develop reasoning and produce proofs through algebraic language. Last but not least, he stresses that many papers are only focused on the difficulties met by the students, but that is reductive: the students’ behaviours and cognition can be influenced by the teacher’s methods and ways of posing problems. On these aspects we shall come back later.

As to the literature, due to space reasons, we only take into account some among the most wide-ranging and consolidated studies, precisely those by: Cooper & Warren, Rivera & Rossi Becker and above all by Radford. Before dealing with them, we would like to mention a particular study by Ferrari (2006) about the generalization and formalization of solution processes for numerical problems in a primary school; here children are guided to make a distinction between data and numerical value of the data and are faced with the task to express the procedure followed to solve the problem in general terms. In this process the letters are adopted by the pupils as short names for a voluntarily not defined quantity of data to emphasize the expression of arithmetic relations among them; each expression is made according to the operational acts needed to solve the problem, getting to represent the solution procedure in an algebraic expression. The results not only show the effectiveness of the approach: they also prove a strong involvement of the pupils which generates motivation to study the discipline.

2.1 Cooper and Warren studies

The studies by Cooper & Warren (2008, 2011) concern the devlopment of an Early Algebra Thinking Project (EATP) aimed at placing early algebra activities in the Queesland Years 1-10 syllabus. They consider three main topics:

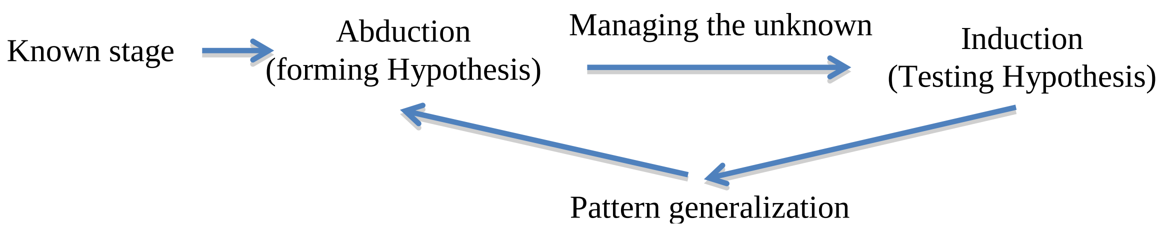
7 This construct is defined later.

a) patterns and functions; b) equivalence and equations; c) arithmetic generalization. The scholars, in the line of Radford, do not see algebra as the manipulation of letters but rather as a system charaterized by: indeterminacy of objects, analytic nature of thinking; symbolic ways of designating objects. Their obiective is the development of students' mental models based on relationships between real world instances, symbols, language, growth phenomena and graphs, particularly those that enable the modelling of real situations that contain unknowns and variables. In EATP they have studied the students' acts of generalization, in particular, pattern rules with growing patterns, change and inverse change rules with function machines and tables of values, balance principle in equivalence and equations, compensation principles in computations, abstract representation of change (e.g. tables, arrow diagrams, graphs) and relationship (equations), particularly looking at the relationships between representations and growth of algebraic thinking. These studies have reinforced their convinction that generalization is a major determiner of growth in algebraic thinking and preparation for later learning of studies. (Cooper & Warren, 2011). These authors, in analogy with the 'quasi variable' notion (Fuji & Stephens, 2001) - which espresses the students' recognition that a number sentence or group of number sentences can indicate an underlying mathematical relationship - introduce the *quasi-generalization* (QG) notion to indicate 'a step very near towards full generalization', i.e. the state where the students are able to express the generalisation in terms of specific numbers and can apply a generalisation to many numbers, and even to an example of 'any number', before they can provide a generalization in natural language and in algebraic notation. They have found that QG appears to be a needed precursor to the expression of the generalization in verbal or symbolic terms.

From the points of view of the classroom activities and of the students' side these studies are in tune with ours (see Cusi & Malara 2008, Cusi, Malara & Navarra 2011 and related references). But, as we shall show later, we take into account both the teachers' role in the class and, more in general, the issue of the development of their competence in leading the students to face algebraic generalization tasks.

2.2. Rivera and Rossi Becker's studies

The studies by Rivera (2010) and Rivera & Rossi Becker (2007, 2008, 2011) focus on the mental processes enacted by junior high school students to grasp and express linear (or quadratic) rules generated by the analysis of (figural stages of) non elementary patterns. The authors are interested in the students' construction and justification processes of their own generalizations. They focus their attention on the 'visual perception' as the result of sensory perception combined with cognitive perception, meaning, as far as the latter is concerned, the capacity of the individual to recognize a fact or a property as related to an object. They claim, like Radford, that the processes of exploration of a pattern are abductive-inductive, but differently from Radford[8], they incorporate in their model trial and error processes, accepting that cycles of abduction-induction may be repeated to refine the initial hypotheses, up to the definition of a rule which is suitable to generalization. The model produced for this process is the triangle indicated below.



In particular, Rivera (2010) investigates in an analytical way the evolution of students' cognitive visualization, at the basis of the produced algebraic modeling. Concerning this latter point he refers to: Giaquinto (2007)[9] who maintains that the detection of the structure of a pattern arises from the association due to the natural 'visual power' of each one and from the use of a 'visual or perceptible template' which directs the exploration aimed at the recognition of either constant or redundant parts of a pattern; Davis (1993)[10] who conceives the "eye" as a "legitimate organ of discovery and inference" and who considers the discovery not only as the result of a logical reasoning path but also of noticing; Arcavi (2003)[11] who sees a visual template as a strategy to allow the students to see the unseen of an abstract world, dominated by relationships and conceptual structures not always evident; Metzger (2006)[12] for the "law of good gestalt" or "gestalt effect" concerning one's ability to perceive, discern and organize a figure. The author uses the expressions "patterns high (or low) in gestalt goodness" to express their high or low effectiveness to highlight the structure of a sequence. He shows the existence and effectiveness of visual templates in dealing with patterns which have linear or simple quadratic structures but he states that further research is needed in order to ascertain the possibility of visual templates in all figural patterns which have a not linear structure[13].

In Rivera & Rossi Becker (2011) the authors classify the procedures used by the pupils to reach an algebraic model of the sequence in three categories: 1) *Constructive standard generalizations* (CSGs); 2) *Constructive non standard*

8 We present later the Radford model.
9 Giaquinto, M., 2003, Visual thinking in mathematics, Oxford University press.
10 Davis, R., 1003, Visual theorems, Educational Studies in Mathematics, 24, 333-344.
11 Arcavi, a. (2003), The role of visual representation in learning of mathematics, Educational Studies in Mathematics, 52, 215-241.
12 Metzger, W. (2006). Laws of seeing, Cambridge, MIT press.
13 Rivera realizes also a refinement of the previous model considering the starting triangle <abduction, induction, generalization> as a common base of two opposite tetrahedrons, where the top vertex represents 'the gestalt effect' and the bottom vertex 'the knowledge/action effect'. He considers a new research question, i.e. how this new model can be used in other algebra tasks involving generalization.

generalizations (CNGs); 3) *Deconstructive generalizations* (DGs). *The constructive generalizations* refer to those polynomial formulas that learners directly construct from the known stages of a figural pattern as a result of cognitively perceiving figures that structurally consist of non-overlapping constituent gestalt or parts. The *Deconstructive generalizations* refer to those polynomial formulas that learners construct from the known stages as a result of cognitively perceiving figures that structurally consist of overlapping parts (in some cases also embedding the pattern in a larger configuration that has a well known or easier structure).

The deconstructive ways of seeing a pattern imply that some elements (sides or vertices) of a figure can be counted two or more times and therefore the correspondent formulas involve a combined addition-subtraction process where overlapping elements have to be subtracted from the total. The terms "*standard*" and "*non standard*" refer to the algebraic expression of the rule: applying respectively if it is already simplified or not. From their studies CSGs appear to be dominant with respect to the DGs ones. The authors, even if they identify in the students' work the 'factual', 'contextual' and 'symbolic' Radford steps (see later), focus their analysis on the evolution of the students'work from figurally to numerically-driven (de)constructions. They document four types of justifications to support the formulas produced: extension generation; generic example use, formula projection, formula appareance match. They link the student's success with the classroom socio-cultural mediation which allows them to engage in multiplicative thinking and, in some cases, to simplify their justifications.

From these results the students' difficulty to produce CNGs is hardly understandable: since CNGs reflect faithfully the students'cognitive visions, in our opinion they should precede GSGs. Probably this behaviour shown by the students, depends on a clause of the didactical contract.

2.3. Radford studies

Radford develops a very refined set of studies (Radford 2003, 2006, 2008, 2009, 2010, 2011) where the ways in which 12-14 years old students immersed in a socio-constructive teaching, generalize linear patterns, are analyzed and theorized. We recall here some key points of Radford's theory.

The author claims that generalization implies two main processes which involve phenomenological and semiotic aspects: *grasping a generality,* a phenomenological act enacted through noticing how a local commonality holds across the given terms[14]; and *expressing a generality*, a semiotic act enacted through gestures, language and algebraic symbols.

14 Dorfler (2008) critically reflects on the conception of 'grasping a commonality' as based only on an empiricist understanding. He considers the notion of circle and he underlines that it does not fit in with this vision because "nothing observable have (exactly) the form of circle … in many situations the empirical

Grasping is seen as the enactment of an abduction from noticing some cases, i.e. the identification of a commonality meant as '*general prediction'* in the sense of Peirce. The abdution becomes a hypothesis through which, if positively verified, a new object emerges: a '*genus'*, i.e. a general concept arising by generalisation of the noticed commonality to all the terms of the sequence. An algebraic generalization occurs when the genus crystallizes itself into a *schema*, i.e. a rule providing one with an expression of whatever term of the sequence. This is Radford's model of this process (Radford, 2008, p. 85)

Particulars
$p_1, p_2, \ldots, p_k$ → Noticing commonality *C* → Making *C* a hypotesis → Producing the expression of p_n

abduction — Trasforming the abduction — Deducing p_n from *C*

Later he argues that "the identification of the genus cannot be considered the result of an algebraic process" (Radford, 2011). He claims that thinking development occurs both at the mental and the social plane, generated by material (gestures, language, and perception) and immaterial (imagery, inner speech…) components, which altogether constitute its '*semiotic texture'*. He considers that algebraic thinking is characterized by indeterminacy and analyticity which can be distinguished by the signs on which the student draw. As to the emergence of algebraic thinking he claims: a) that expressing generality algebrically does not imply necessarily the use of the letters (they can be used without any general meaning), instead of the *way of reasoning* which is made explicit in grasping and expressing vagueness in some way; b) the emergence of the algebraic thinking occurs when the students succeed to shift their attention from calculating a number of certain elements to the "*way of calculating*" such number.

Noticing students' behavior he distinguishes three levels of approach to generality. A first level, which he defines as 'naive induction', where there is no actual, aware generalization. It is characterized by pupils' trial and error processes, by the possible occasional discovery of generalities, by germs of abduction which are falsified in the checking stage. At this level, even though a rule may be expressed in the alphanumeric system the generalization is not algebraic. A second level that he calls *arithmetic generalization*, where generalization is seen locally, in a recursive way, and expressed in the different cases through the addition of a constant term. A third level, which he defines as *algebraic generalization*, is a very mazy and complex one, marked by gradually more and more advanced phases. Regarding this latter level the author talks about a whole working area called *zone of the emergence of algebraic generalization*, which develops through '*layers of generality'*. The first layer,

generalization or abstractions need a complementary support by epistemic processes like idealization and hypostatization".

defined ‘*factual*’, is the one where the generalization appears by means of *concrete actions* on the examined cases, but it is not coagulated in a statement. The second layer, defined *contextual*, is reached when indetermination enters the discourse, pupils talk about the 'number of a figure' but they make space-time remarks on it, in a general perspective and a rule is expressed in various ways drawing on words, gestures, rhythms and signs. The level of the algebraic generalization is reached when pupils detach themselves from the figural context and shift towards the relations between constant and variable elements (numbers and letters). Important elements which intervene in this last process are *iconicity*, i.e. a manner of noticing similar traits in previous procedures, the shifting from a particular unspecified number to the level of variables *summarizing* of all the local mathematical experiences, the *contraction* of expressions which testifies a deeper level of consciousness. This is a synthetic representation of the processes (Radford, 2006, p.15)

Radford’s model of the students’ strategies in dealing with pattern activities

Naïve Induction	**Generalization**			
Guessing	Arithmetic	Algebraic		
(Trial and Error)	(local recursion)	Factual	Contextual	Symbolic

In the most recent works by Radford (2010, 2011) the author adresses his attention to very young students (7-8 years old) and he studies in details the relationship teacher-pupils in classroom processes where the pupils are brought to detect and express generalizations in the exploration of figural sequences. In (Radford, 2010) the scholar claims that “learning can be theorized as those processes through which students gradually become acquainted with historically constituted cultural meanings and forms of reasoning and action”. In particular he focuses on the ‘*way of seeing’* and states that “*the mathematicians’ eyes have undergone a lengthy process of domestication*” in the course of which people come to see and recognize things according to “efficient” cultural means.

Radford considers “seeing” not a simply physiological act but as a fruit of the cultural milieu where one is imbedded; he stresses that “*generalization rests on synthesizing* resemblances *between* different *things and also* differences *between* resembling *things”,* and that this game of visions has to be conveniently educated by the teacher. He highlights the social character of the teaching-learning processes, the role assumed by the teachers in it and focuses on “the way in which teachers create the possibility for students to perceive things in certain ways and encounter a cultural mode of generalizing”; he claims that “perceiving sequences in certain efficient cultural ways entails a transformation of the eye into a sophisticated theoretician organ”.

In the analysis of classroom transcripts he highlights the teacher's behaviours (questions, guided reflections, gestures, tone of the voice, silences, looks) through which she succeeds to address her little students to become aware by themselves of the incorrectness of their visions and to autonomously correct them. As to this, he writes:

> …Poësis is a creative moment of disclosure – the event of the thing in consciousness … The poetic moment of disclosure of the general structure behind the sequence discussed in this paper was the result of a joint student-teacher interaction. This moment – the event of the thing in consciousness – was much more than a negotiation of meanings and an exchange. It was rather a Bakhtinian heteroglossic merging of voices, pointing gestures, perceptions, and perspectives … (Radford, 2010, p. 3)

From the examination of the studies we have considered, clear common elements appear about the articulation of the phases through which generalizations emerge, but there are also some elements of difference, for instance the different position of the trials in the models by Radford and Rivera about the students' behaviours in front of the exploration of figural sequences. Radford's studies stand out for the sharp intertwining between aspects of practice and theoretical aspects, and moreover for the consideration of the socio-cultural and epistemological dimension of both mathematics and its teaching. The experimental studies do not give esplicit indications about factors which contribute to the students'construction of the semantic basis for generalization. A study devoted to this aspect and carried out by my collaborators Cusi & Navarra, is presented in this conference.

In most of the studies we know, the teacher's role remains in the shadow. Warren (2006) states that more research needs to individuate teachers' actions and ways to pose questions which can facilitate the students' generalizations and Radford (2010, 2011) highlights the teacher's actions in guiding the students to 'see' analogies and differences among various stages of a pattern, but they do not mention that the majority of the teachers meet big difficulties to manage this type of teaching even when (s)he is convinced that it is appropriate to practice it. This problem has been an object of our studies and it is discussed in the next section.

3. OUR STUDIES ON THE SIDE OF THE TEACHERS' EDUCATION FOR A SOCIO-CONSTRUCTIVE APPROACH TO EARLY ALGEBRA

Since the nineties we have addressed questions of the teaching-learning of algebra and we have set up several experimentations of didactical innovation in collaborations with expert teachers -researchers. Our aims were to individuate the conditions of real applicability in the schools of didactical innovations in algebra, centered on algebraic problem solving, generalization, modeling and proof in the frame of a socio-constructive teaching. Our several studies have given birth to *ArAl Project: arithmetic pathways to favour pre-algebraic* *thinking*[15] (Malara & Navarra, 2003) which proposes a revision of the arithmetic teaching in a relational key and an approach to early algebra of a linguistic-constructive type. The project involves students and teachers from kindergarten to the first biennium of upper secondary school but it is mainly devoted to primary and lower secondary school in a perspective of continuity between the two school levels.

The ArAl project is based on the hypothesis that there is a strong analogy between modalities in which natural language and algebraic language are learned. As we know, a child learns natural language through a large variety of situations which he experiences with an experimental attitude, gradually mastering the meanings and supporting rules of the language, up to the school age, when (s)he will learn to read and reflect on grammatical and syntactic aspects of the language. Similarly, the mental models of algebraic thinking and language should already be constructed in an arithmetical environment, even from the very first years of primary school, bringing a child to face pre-algebraic experiences in the arithmetical realm (grasping regularities, generalizing and expressing relationships, giving and comparing representations, extending properties by analogy...). In this way (s)he can progressively develop algebraic thinking, in a strict intertwining with arithmetic, exerting a continual reflection on the meanings of the introduced symbols and of the implemented processes in classroom work.

As reported in Cusi & Al. (2011), our perspective of work in the classes is based on the following principles:

The anticipation of generational pre-algebraic activities at the beginning of primary school, and even before that, at kindergarten, to favour the genesis of the algebraic language, viewed as a generalizing language. From these activities the pupil is guided to reflect upon natural language; it is from the analogy between the modalities of development of the two languages that the theoretical construct of algebraic babbling comes out16.

The social construction of knowledge, i.e. the shared construction of new meanings, negotiated on the basis of the shared cultural instruments available at the moment to both pupils and teacher. Arithmetic and algebraic knowing are both central, but they need to emerge and strengthen themselves through the

15 ArAl is an acronym for “Arithmetic and Algebra”. The ArAl Project is led in collaboration with Giancarlo Navarra, a teacher-researcher who co-ordinates the organizational aspects of the Project and contributes to its scientific program.

16 We call algebraic babbling the experimental and continuously redefined mastering of a new language, in which the rules may find their place just as gradually, within a teaching situation which is tolerant of initial, syntactically “shaky” moments, and which stimulates a sensitive awareness of formal aspects of the mathematical language. We employ the “babbling” image because when a child learns a language, (s)he masters the meanings of words and their supporting rules little by little, developing her/his knowledge gradually by imitation and self-correction or with the adults’ support.

coordinated set of individual competencies, which are the main resource on which they are constructed.

The central role of natural language as the main didactical mediator for the slow construction of syntactic and semantic aspects of algebraic language. Verbalization, argumentation, discussion, exchange, favour both the understanding and the critical review of ideas. At the same time, through the enactment of processes of translation, natural language sets up the bases for both producing and interpreting representations written in algebraic language. From this centre, attention is then extended to the plurality of languages used by mathematics (iconic, graphical, arrow-like, set-theory language, and so on).

Identifying and making explicit algebraic thinking, often 'hidden' in concepts and representations in arithmetic. The genesis of the generalizing language can be located at this 'unveiling', when the pupil starts to describe a sentence like $4\times2+1=9$ no longer (not only) as the result of a procedural reading 'I multiply 4 times 2, add 1 and get 9', but rather as the result of a relational reading such as 'The sum between the product of 4 times 2 and 1 equals 9'; i.e. when he/she talks about mathematical language through natural language and does not focus on numbers, but rather on relations, that is on the structure of the sentence.

In an approach of this type the teacher has a key role. In fact (s)he needs to set up a teaching strategy that allows for the implementation of an authentic socially shared mathematical activity, where space is given to linguistic aspects, to the representation of information and processes, as well as to meta-cognitive aspects. The latter are important to monitor the appropriateness and suitability of representations, to recognize and identify equivalent ones and select the best ones. All this requires a deep restructuring of the teachers' conceptions about both the contents to be taught and the teaching methodology in the classroom: a real 'culture of change' is entailed.

For reshaping teachers' professionalism several scholars stress the importance of a critical reflection by teachers on their own activity in the classroom (Mason, 1998, 2002, 2008; Jaworski, 1998, 2003; Lerman, 2001; Shoenfeld, 1998). Mason, in particular, proposes the study of the discipline of noticing. He claims that the skill of consciously grasping things comes from constant practice, going beyond what happens in the classroom, and recommends the creation of suitable social practices in which teachers might talk-about and share their experience. Also Jaworski stresses the effectiveness of communities of inquiry, constituted by teachers and researchers, emphasizing how teachers' participation in these groups helps them develop their individual identity through reflective inquiry. Our teacher education model follows these conceptions and modalities. But it represents the outcome of research and training practices developed in Italian universities since the 1970's.

Instruments, methods and activities outlined and tuned in the ArAl project, work as a support for teachers to propose early algebra activities in the classroom, using a socio-constructive methodology, and, at the same time, as a training to become metacognitive teachers through a reflection upon their own action in the classroom. Follow-ups of the basic activities are twofold:

- on the pupils' side: the aim is to analyze the conditions under which pupils, since grades 4-6 manage, at a first level, to generalize, formulate properties and produce formal representations and, at a second level, to appropriate the meaning of algebraic expressions and become aware of their expressive strength;
- on the teachers' side: the aims are on two levels as well. One aim is to refine their ability to guide the class in the approach to early algebra following these ArAl modalities; a second aim is to foster their professional development through stimuli deriving from participation in at least two-year collaboration projects, characterized by the immersion in a community of enquiry on one's own practice, in a continuous interplay of reflection, exchange, sharing.

Our hypothesis for the promotion of the teachers' professional development is to bring them to be embedded in an 'environment' where they can acquire a new way to operate in and for the class, work actively and reshape their professionalism through frequent exchanges of studies, experiences and reflections. Our modalities of work in teachers' education are aimed at both bringing the teachers to analyze their didactical processes to assess their results and guiding them to reflect on these processes according to three different points of view: the development of the mathematical construction; the teacher's actions; the participation of each individual in the collective construction of the knowledge.

We believe that by observing and critically reflecting on socio-constructive teaching/learning processes, the teachers are led to become aware of the different roles they are supposed to play in the classroom, of the best ways to interpret them and can also get useful suggestions about how to behave in the classroom. Moreover, we believe it is crucial for teachers to be familiar with research results that can be useful for practice and to become aware of the importance of studying them for their own professional development.

The teachers who choose to participate in ArAl teaching experiments are mainly motivated by their 'first encounter' with the project through publications, congresses or events in the schools. Often these teachers have already studied the project and in particular its units17 and the glossary that can be found in the

17 The units can be viewed as models of sequences of didactical projects, open to the teacher's choices and focused on a specific strand of activities. They provide information on the mathematical meaning and the objectives of the single activities presented, report excerpts that exemplify class discussions, as well as

project's website18. When they actually face the teaching experiment, they nevertheless show uncertainty towards class discussions, felt as open and unpredictable situations, difficult to be managed.

Through our studies we became aware of the difficulties that the teachers meet both in planning and in guiding classroom mathematical discussions. Our studies highlighted how during a classroom discussion often the teachers assume not adequate behaviours or fall back to a trasmissive teaching model. Therefore often they do not share with the students the goals of a problem exploration, they do not give room to some potentially productive interventions, they tend to ratify immediately the validity of some meaningful contribution without giving the class any opportunity to validate them. An example of a discussion where the teacher has this kind of behaviour is reported in appendix with a comment.

As a support to teachers and an answer to their needs, a mentor-researcher is associated with each group of teachers involved in the same teaching experiment: teachers and mentor share some moments of work face to face together with a dialogical relationship via e-mail. There are also regular working sessions of small groups with their mentor and the researcher in charge of the group, but also collective sessions, involving all the researchers and teachers experimenters, all held in schools or at the university.

Believing that observation and critical-reflective study of classroom-based processes help teachers become aware of the processes involved in every discussion and of the variables that determine those processes, our objective is to lead the involved teachers: a) to become increasingly able to interpret the complexity of class processes through the analysis of the inner *micro-situations*, to reflect upon the effectiveness of their own role and become aware of the effects of their own *micro-decisions*; b) to be in a better and finer control of both behaviours and communication styles they use; c) to notice, during classroom activity, the impact of the critical-reflective study undertaken on pupils' behaviour and learning.

In order to achieve this objective, we involve teachers in a complex activity of critical analysis of the transcripts concerning class processes and of reflection upon them, aimed at highlighting the interrelations between knowledge constructed by the students and behaviour of the teacher in guiding the students in those constructions. The analysis is carried out by building up what we call '*Multi-commented transcripts* (MT), or 'the diaries'. They are realized after

comments on both pupils' behaviours (meaningful constructions, frequent attitudes, difficulties) and on teachers' behaviour (appropriate interventions, ways of introducing and managing issues, attitudes etc.).

18 The units are supported by the theoretical framework and, most of all, by the glossary, available online on the project's website <www.aralweb.unimore.it>, where teachers can find clarifications and further material on mathematical, linguistic, psychological, socio-pedagogical and methodological-didactical issues and also find prototype didactical sequences, aimed at giving them a stimulus for their-own elaboration of the highlighted teaching processes.

transcribing in a digitally formatted text the audio recordings[19] of lessons on topics that were previously agreed with the researchers. They are completed by the teachers-experimenters who send them, together with their own comments and reflections, to mentors-researchers, who make their own comments and send them back to the authors, to other teachers involved in similar activities, and sometimes to other researchers. Often the authors make further interventions in this cycle, making comments upon comments or inserting new ones. This methodology is characterized by a sort of web choral participation, due to the intensive exchanges via e-mail which contribute to the construction of the MTs, and to the fruitfulness of the reflections emerging from the different comments.

Here we only propose a short excerpt from an MT, trying to show how this instrument enables one to highlight the behaviours enacted by teachers, the difficulties they meet and the awareness they achieve after the work of analysis and reflection has been carried out on the basis of the received comments. We are well aware that this excerpt cannot fully express the richness and the variety of the questions which arise from the classroom transcripts, the type of interactions with the teachers that the comments allow and how these can help them to refine their actions in the class, so we refer to other examples which can be found in Malara (2008), Malara & Navarra (2011), Cusi et al. (2011), Cusi & Malara (in press). In order to preserve the discussion flow, analytical comments are reported in the same order in which they were made. Authors of comments are labelled as: T: teacher; M: mentor; R1-R2: team researchers.

A short example of MCT

The teacher proposes a topic concerning the exploration of a sequence, given the first three terms (it is the arithmetic progression with initial element 4 and step 7). The activity is aimed at determining a general representation of the sequence. In the following excerpt, the class (grade 6) had already identified the sequence's recursive generating law. The teacher writes the following table on the blackboard and opens up a discussion to introduce the class to the study of a representation for the general correspondence law (T represents the teacher; S, J and A represent the students involved in this part of the discussion).

19 We chose to analyze audio-recordings instead of videos of classroom processes because we believe (Malara & Zan, 2008) that, while watching the video may not enable teachers to completely capture the details of the verbal interaction, analyzing transcripts, instead, fosters the crystallization of interactive processes and highlights gaps, crucial decision making moments and also omissions, oversights, carelessness.

Sequence ranking number	Sequence number	Operations made to jump from the place number	'Mathematical recipe[20]' to construct the number
1	4	4	
2	11	4 +…	
3	18	4 + … + …	
4	25		
5	32		

1 T: How do we get to 11?

2 S: + 7.

3 T: We make 4 + 7. What about the third place, S? We make…?

4 S: 4 + 7 + 7.

5 J: Wouldn't it be better to make 4 × 2? **(1)**

6 T: What about the fourth place?

7 S: 4 + 7 + 7 + 7.

8 T: What about the fifth?

9 S: 4 + 7 + 7 + 7 + 7.

10 T:What if we had a sixth place?

11 S: 4 + 7 + 7 + 7 + 7 + 7.

12 T: Correct. So, now we find…

13 A: I didn't get it. What do I put in the first place?

14 T: Well, there is 4 in the first place.

15 A: I put 4 × 1. **(2)**

16 T: Well, but there is no '×' there. The first place is 4 **(3)**

Comments

(1) M. Why doesn't T comment upon J's intervention?

R2. I agree. Probably J grasps a regularity but doesn't express it correctly, instead of saying 4 + 7 × 2 he packs everything in 4 × 2. T should have clarified this.

(2) R1. Also this intervention might have been investigated. Why does A think about the product of 4 and 1?

R2. Again we are in front of a badly expressed intuition. The student probably wants to 'fill the gap' he sees in the representation of the first term as compared to the others. Here T misses the chance to change the representation of the first term, 4, into one that fits with the situation, for example writing 4 as 4+0 and getting back to the class posing the problem to find a representation for the first term, similar to the other ones.

20 The expression "mathematical recipe" is a metaphor used by the teacher to convey the idea that pupils should use a representation of the sequence's number in function of the place number.

(3) R2. This intervention by T suggests that she excludes the possibility of representing 4 in another way, thus showing little algebraic farsightedness. It would be extremely appropriate to encourage these intuitions, although imprecise, trying to redirect them.

T. All these remarks make me think I am really close-minded and I didn't realise it before. I don't know whether this is a matter of attention, of being used to seeing things in different ways, of fearing to get out of the scheme to be followed (or the one I thought I should follow).

Analysis of the excerpt

This excerpt documents a number of rigid behaviours by the teacher in her action. She does not manage to productively value the intuitions of some pupils, blocking their emerging mathematical explorations (lines 5, 13, 15) and to direct pupils towards a relational reading of the correspondence, which implies the use of the multiplicative representation (line 16). If we look at the comments she proposes, we notice that she only makes remarks about her action in the class after reading both mentor's and researchers' remarks. Her *a posteriori* comment shows awareness of her own rigidity and of her tacit fears to leave usual schemes to approach innovative activities (note 3-T).

Comments made in this excerpt reflect some of the categories we already highlighted (Malara, 2008) and that seem to be strictly interconnected here: (1) conceptions linked to cultural and/or general educational issues (note 3-R2); (2) methodological issues concerning mathematical aspects (notes 1-R2; 2-R2; 3-R2); (3) management of discussions in the classroom (notes 1-M; 2-R1). Further categories strongly emerged in MCTs- not documented here for space reasons- refer to the distance between theory and practice (difficulty in drawing on elements of the theoretical framework) and to a wide range of linguistic issues.

The example we presented shows the role of MCTs in the training program in which teachers are involved, reminding that this analytical work is carried out on the transcripts of all the episodes that constitute the teaching-experiment. It is through the comments that teachers: (1) actually realize how the development of pupils' mathematical constructions is strongly affected by the teacher's language, choices, attitudes and actions; (2) reflect upon their difficulties in managing a discussion and receive suggestions about how to face micro-situations of interaction; (3) express their own difficulties, doubts, awareness.

The collectively-written critical analysis is a particularly important methodological tool for the development of the teacher's awareness: divergent comments to a micro-situation lead to grasp a range of possible interpretations of both behaviours and interventions enacted; converging comments enable one to amplify the critical points of the management of the activity, on which it is necessary to (re)construct competences and refine one's sensitiveness.

We wish to underline the determinant conditions for the effectiveness of our MT approach. One first condition is the non-episodic nature of the situations for reflection and exchange: by progressively accumulating these moments of autonomous and interactive reflection, characteristic of our methodology, the teacher becomes more receptive and, in the long term, is led to develop new conceptions, attitudes and ways of acting. Another fundamental condition, crucial for the teacher's development process, is the enactment of a relationship between the members of a team, based on mutual trust, and the construction of a sense of belonging to a group that shares common values.

Moreover, the analysis of several MTs related to the implementation of a path designed with the teachers and aimed at the development of students' proving ability through algebraic language, allowed us to identify the specific characters which constitutes the profile of an 'effective teacher', who poses him/herself as a model of aware and effective attitudes and behaviours for students (Cusi & Malara, 2009). The defining elements of this model, are as follows: the teacher must (a) be able to assume the role of "investigating subject", stimulating an attitude of research on the problem being studied, and of an integral element of the class group in the research being activated; (b) be able to assume the role of operational/strategic leader, through an attitude towards sharing (as opposed to transmission) of knowledge, and as a thoughtfulness leader in identifying efficient operational/strategic models during class activities; (c) be aware of his or her responsibility in maintaining a harmonized balance between semantic and syntactic aspects during the collective production of thought through algebraic language; (d) seek to stimulate and provoke the building of key skills in the production of thought through algebraic language (be able to generalize, translate, interpret, anticipate, manipulate), acting as an "activator" of algebraic processes (generalization, traslation, manipulation, interpretation, anticipation); (e) also have the aim to stimulate and provoke meta-level attitudes, acting as an "activator" of thoughtful attitudes and "activator" of meta-cognitive acts, with particular reference to the control of the global sense of the processes.

The work developed with trainees teachers (Cusi & Malara, 2011), suggested us to conceive this construct as a possible theoretical lens for the analysis of classroom discussions to be used in specific workshops for/with in-service teachers. In the future we wish to verify the effectiveness of this construct also as a tool for the teachers' self analysis.

4. CONCLUDING REMARKS

In this paper we presented a brief overview of the literature and we sketched out some research results which offer meaningful indications about recent points of views on generalization processes. Then we focused our attention on some recent studies about generalization activities in early algebra teaching describing the position of some scholars.

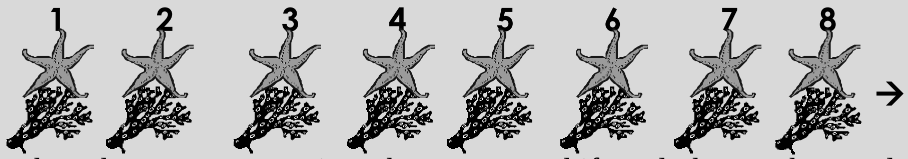
In this frame we have considered the issue of the role played by the teacher in leading the students to engage in this kind of activities and through some short excerpts of classroom work we have shown the sharp relationship between the teacher's actions and the students' behaviours. We have also sketched a profile of a teacher who acts as an effective guide for the students to promote the development of a meaningful and aware approach to algebraic thinking.

To conclude we stress the importance of the teacher's awareness at different levels to gain consciousness and control about the effective ways of posing him(her)self in the class and, above all, we underline the need of a refined teacher's education on this delicate aspect of teaching which requires a deep study of classroom episodes and above all a systematic careful self-analysis of the teacher's own practice.

APPENDIX

A problem situation presented in primary school (grade IV)
In the great reef life is very intense. You can possibly meet several types of animals: sponges, jellyfishes, octopuses, multicolour fishes. In the far eastern part of the reef a very numerous family of sea stars lives, each of them attached to a coral:

Alessia Angela Elena
Loretta Patrizia …



When the new moon arises the sea stars shift and change the coral following a very old rule. Try to discover the rule looking at how the sea stars in the first positions move: Alessia goes to n° 3; Loretta goes to n° 5; Angela goes to n° 7; Patrizia goes to n° 9 Elena goes to n° 11
1) On the n. 78 coral the little star Valeria lives: which will be the number of the coral on which it will move? 2) Which will be the number of the coral where the sea star living at the 459th place will move?
Justify your answers.

The discussion (the teacher's interventions are in italic)

At the beginning some pupils give numerical answers without any justifications or by chance.

1. Teacher *I asked you to justify your answers.*

Alex the stars move: from 1 to 3, from 3 to 5, then 'plus 2', from 5 to 7 'plus 2'…

Alessia I have added the number that says how much all the stars move: 2, 3, 4, 5… because from 1 to 3, it is +2; from 2 to 5, it is +3, then it moves from 3 to 7, it is +4; and then from 4 to 9, it is +5, from 5 to 11: +6 and adding 2+3+4+5 we obtain 15

Beatrice I have done in this way. (She goes to the blackboard and clearly describes her reasoning representing all the various cases with arrows) The star Alessia has to move from place 1 to place 3 and then it is +2; Loretta has to move from n. 2 to n. 5, it makes +3; Angelica moves from n. 3 to n. 7, it makes +4; Patrizia moves from n. 4 to n. 9, it makes +5; Elena moves from n. 5 to n. 11, it makes +6. Then, in my opinion, [the answer for the coral n. 78] is 78+79, that is 157, because I have added to the number of the place of the star in the initial position, the number which follows it.

★n.1 $\xrightarrow{+2}$ 3

★n.2 $\xrightarrow{+3}$ 5

★n.3 $\xrightarrow{+4}$ 7

★n.4 $\xrightarrow{+5}$ 9

★n.5 $\xrightarrow{+6}$ 11

2. Teacher *Really good! What do you think about this? One of you said that Valeria arrives at n.80, another one said at n. 93, another one at 84, another one at 157.*

Nicola I have not understood well Beatrice's reasoning.

3. Teacher *Beatrice, you have to help Nicola (and addresses the class), whether you do not understand, you ask.*

Beatrice Yes. The star Alessia stayed at n.1 and she moved to n.3 … (Beatrice starts from the first sea star and she retraces the arrow oriented from 1 to 3, she continues analogously with the other stars, indicating them while she is speaking).

4. Teacher What has Beatrice done with respect to the classmates who have spoken before her?

some pupils: She has represented … . Others: She has outlined a scheme…

The teacher suggests Beatrice to write in red the value of the arrow operators. While Beatrice colours she explains.

Beatrice … then for getting to 5 the star 2 makes +3; then from 3 for getting to 7, I have added 4; from 4 for getting to 9 I have added 5, from 5 for getting to 11 I have added 6

Nicola She has to put 6 because it is 5+1; she has to put the [number of] the star's address plus 1. She has to add "the address number plus 1" to "the address number"

5. Teacher Good! Translate it into mathematical language

Nicola +5+5+1

Nicola, Beatrice and some others enrich the blackboard with a new representation: each arrow of the previous representation is splitted in two, the first arrow appears to be a variable operator depending on the place number and the second arrow appears to be the invariant operator '+1'.

6. Teacher Then if the star starts from 78, what will be its new place?

Beatrice: 78+78+1=157

7. Teacher: (shaking hands with Beatrice. Then, addressing the class) Have you understood?

Giulio Then it has to go to number 157… I have written only the process: 78+78+1

8. Teacher: Would it be possible to write the same thing in different ways?

Alex, Enrico, Nicola e Giulio give these writings.

78+78+1=157; 78+79=157; 78×2+1; 78+(78+1)

9. Teacher Very good. There is a new challenge for you: The star Filippa is at place n. 100; where does it move to?

Alessia 100×2+1=201

10. Teacher Ok. The star Maria is at n. 300; where does it move to?

Alex 300+300 is 600, plus 1 that equals 601

Beatrice Or rather you can multiply its value times 2 and then plus 1

11. Teacher You have been very smart! We have not got to the generalization yet, but we are near

Some days after the class restarts the activity.

12. Teacher Go back to where we had stopped: which rule does the star Valeria follow to move to the new coral?

Some pupils: 78+78+1=157. One of them rewrites this expression on the blackboard

13. Teacher Someone has said 78×2+1=157, do you remember? Now tell me: if a little star starts from n. 15 where is it when it arrives?

Pupils 15×2+1!

14. Teacher Ok. And if it starts from 103?

chorus 103×2+1!

The teacher picks other starting numbers: 598; 3654; 92045; she writes in column the pupils' sentences, purposely leaving a space between the number of the starting coral and the chain of the operators acting on it: 78 ×2+1; 15 ×2+1; 103 ×2+1; 598 ×2+1; 3674 ×2+1; 92045 ×2+1

Chorus Times 2 plus 1, it remains the same!!!

15. Teacher Excellent! '×2+1'remains constant. Now try to express in Italian the rule of this moving. We have to write the "Regulation of the sea stars movings". Imagine that the star Carlotta arrives at the colony for the first time. When there is the new moon it notices that all the sea stars move and change their place, she does not understand

anything and she asksher neighbour star what she has to do. In your opinion which help can the neighbour star give her?

Alex She has to do the number of its coral times 2 plus 1.

16. Teacher How can you say it in another way?

Costanza From the number of her house you have to go forward times 2 plus 1

Piero I shall say: if you are in the coral house number 50, you have to move to… you have to go… yet 50 house more and plus another

17. Teacher Meanwhile the little star starved … .Listen to me, we need to assign some names; how do we call these numbers? (she indicates the first term of each sentence)

Costanza Number of the house

18. Teacher Both of them are numbers of house

Lucia Number of the coral

Chorus Starting number

19. Teacher How do we call these in a competition?

Chorus Start! Arrival!

(The teacher writes on the blackboard, respectively on the left-hand side and on the right-hand side of the sentences: "number of the starting coral"; "number of the arrival coral")

20. Teacher I suggest you to begin from the number that is after the equal sign. (She says) "The number of the arrival coral is equal …

Enrico … to the starting number times 2 plus 1

Alessia the number of the arrival coral is equal to twice the starting number plus 1

21. Teacher We can take away "of the coral". Dictate it to me

pupils: The arrival number is twice the starting number plus 1

(The teacher writes the rule on the blackboard and reads it.)

22. Teacher Do you know how to translate it for Brioshi?[21]

Matteo Times 2

23. Teacher Only so? In your opinion Does Brioshi understand?

Mattia 78...

24. Teacher Then does it hold true only for 78?

Enrico It holds true for any starting number

25. Teacher The idea is excellent, but in mathematics, after several studies, it has been decided to call 'any number' only with a letter

Mattia I had said it!

[21] Brioshi is a virtual Japanese student who exchanges messages in mathematical language with pupils. His acknowledged skill in this area, encourages pupils to check the correctness of the mathematical expressions to be sent out to him.

26 Teacher What do we choose as starting number?
Chorus s
27. Teacher And as arrival number?
Chorus a

Anna gives the rule in formal terms: s×2+1=a. The class writes the relationship to be sent to Brioshi: s×2+1=a

Comment

At a first reading of this discussion, the teacher's behaviour can appear good. But in actual fact she does not act well. She speaks only with few pupils, she does not promote any interaction in the class, and above all she does not relaunch the validation of the pupils' proposals to the classmates. She does not take into account pupils' contributions which offer elements of discussion and of comparison (see Alex's proposal and Alessia's proposal). She expresses judgments through exclamations or emphatic gestures (intervention 2, intervention 7). She immediately directs the class towards the solution she had foreseen, as soon as it appears (intervention 5-7)[22]. She disregards to enhance important contributions, even expressed in general terms, as the one by Nicola, which facilitates the emergence of the link between the initial and final coral-house of a sea star. Yet, she does not re-examine with the class the reasonings developed for sharing, pinpointing and consolidating them, but she limits herself to ask "did you understand?". She does not pose herself in a reflective way in front of the pupils, trying to help them overcome the procedural vision induced by the arrows representation, for instance discussing with the class about which coral-house they have to speak, the 'regulations' they have to write, so that the pupils can understand they have to write a verbal sentence related to the number of the final coral-house. She disregards the opportunity offered by Alex's intervention to clarify that a rule cannot be a simple procedure but it has to be a sentence with a complete meaning, forcing in this way a verbal representation of the sought rule. Trying to solve the question of the verbal representation of the relationship at stake, she poses a vague question (intervention 16) which does not allow pupils to face this delicate step, impossible to be done without a careful mediation of the teacher, where they have to shift from the number of the starting coral-house to the number of the final one. Yet, she does not bring the pupils to make explicit in the various numerical cases what the starting and final numbers represent, fact that prevents the pupils from formulating verbally a rule

[22] After the first intervention by Beatrice the teacher should have relaunched to the class the validation of the girl's reasoning, or at least she should have asked Beatrice to better explain why in her opinion 79 had to be added to 78, helping the class focus their attention on the extension of the regularity detected by the girl and trying to force her to express the relationship between the two numbers at stake (the number to be added to the number of the first coral-house is its successive), fact which allows to easily identify the relationship between the numbers of the two coral-houses of the sea stars.

through the interpretation of the arithmetical sentences, rule in fact suggested by her (intervention 20). Moreover she does not face in a constructive way the question of how to introduce the letters as representation of the variables "number of the starting coral-house", "number of the final coral-house", but only suggests their possible use. So, even if the algebraic representation of the rule is made in the class, this discussion does not allow the pupils to consciously understand the meaning of the algebraic expression.

Globally the discussion shows a bigger tension of the teacher for the attainment of her goal in a short time than for the care to appropriately address the pupils, educating their ways of seeing and facilitating the interaction among them; such a tension brings her to assume a procedural behaviour and to give scarce attention to the meanings associated with the actions in the various steps of a generalization process.

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Early generalization

Part 2

YOUNG CHILDREN SOLVING ADDITIVE STRUCTURE PROBLEMS

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This paper describes a study to analyse how 4-6-year-olds (N=45) children solve different types of additive reasoning problems. Individual interviews were conducted on kindergarten children when solving the problems. Their performance as well as their explanations were analysed when solving additive reasoning problems. The additive reasoning problems comprised simple, inverse and comparative problems. Results suggested that Portuguese kindergarten children have some informal knowledge that allowed them to solve additive structure problems with understanding. Children performed better in the simple additive problems and found the comparative problems more difficult.

INTRODUCTION

In mathematics children are expected to be able to attribute a number to a quantity, which is measuring (Nunes & Bryant, 2010a), but they also are expected to be able to quantify relations. When quantities are measured, they have a numerical value, but it is possible to reason about the quantities without measure them. In agreement with Nunes, Bryant and Watson (2010), it is crucial for children to learn to make both connections and distinctions between number and quantity. Quantitative reasoning results from a quantifying relations and manipulate them (Nunes & Bryant, 2010a), making relationships between quantities valuable (Thompson, 1994). For Nunes and Bryant (2010a), quantifying relations can be done by additive or multiplicative reasoning. Quoting the authors "[…] Additive reasoning tells us about the difference between quantities; multiplicative reasoning tells us about the ratio between quantities." (p.8). In the literature additive reasoning is associated to addition and subtraction (see Vergnaud, 1983) and multiplicative reasoning is associated to multiplication and division problems (see Steffe, 1994; Vergnaud, 1983).

Children can use their informal knowledge to analyse and solve simple addition and subtraction problems before they receive any formal instruction on addition and subtraction operations (Nunes & Bryant, 1996).

ABOUT THE ADDITVE REASONING

Piaget (1952) argued that children's understanding of arithmetical operations arises from their *schema*. A 'schema' is a representation of an action in which only the essential aspects of the action are evident. He identified three schemas related to additive reasoning: joint, separate and one-to-one correspondence. The author pointed out that children are able to master addition and subtraction only when they understand the inverse relation between these operations, which is achieved by the 7-year-olds. More recently, Nunes and Bryant (1996) referred that kindergarten children of 5-6-year-olds can relate their understanding of number as a measure of set size to their conception of addition / subtraction as an increase / decrease in quantities. This can help children to begin to understand that one operation is the inverse of the other. The schema from which children begin to understand addition and subtraction are representations of the act of joint and separate, respectively (Nunes, Campos, Magina & Bryant, 2005). These schemas allow 5-year-olds children to solve a problem such as: ”Anna has 3 candies. Her mother gave her 2 more candies. How many candies does Anna have now?”.

Additive reasoning problems involve one variable and they tell us about the difference between quantities. The part-whole relation is the invariant of the additive reasoning. The whole equals the sum of the parts. Nunes, Bryant and Watson (2010) argue that additive relations are used in one variable problems when quantities of the same kind are put together, separated or compared.

Carpenter and Moser (1982, 1984) presented a classification of addition and subtraction problem that does not characterize all the types of word problems involving additive reasoning, but those who are appropriate for primary age children. They distinguished four categories of addition and subtraction problems: change, combine, compare and equalize (see Carpenter & Moser, 1982, 1984).

Carpenter and Moser (1984) conducted a research on primary school children to analyse their solution strategies according to the type of problem presented. The authors argue that the processes that children use to solve addition and subtraction problems are intrinsically related to the structure of the problem. This idea that addition and subtraction word problems differ both in semantic relations used to describe a particular problem situation and in the identity of the quantity that is left unknown is also supported by other researchers (see De Corte & Verschaffel, 1987; Carpenter & Moser, 1982; Riley, Greeno & Heller, 1983; Fuson & Willis, 1986), who argue that addition and subtraction problem types are related to fairly systematic differences in children’s performance at various grade levels.

According to Nunes et al. (2005), children’s ability to solve problems involving an additive structure develops in three phases: first children can solve simple problems; then they can solve the inverse problems; and finally they can solve static problems. The addition and subtractions simple problems are those in which children are asked to transform one quantity by adding to it or subtracting from it (e.g., Joe had 5 marbles. Then he gave 3 to Tom. How many marbles does he have now?). These types of problems involve relations between the whole and its parts. The inverse problems are those in which the situation presented in the problem relates to a schema, but the correct resolution demands the inverse schema. For example, in the problem "Joe had some marbles. Then he won 2 more marbles in a game. Now Joe has 6 marbles. How many marbles did Joe have in the beginning?" (Nunes & Bryant, 2010a), subtraction appears as the inverse of addition; the quantity increased and the final one are given, and the initial quantity is unknown. The addition and subtraction static problems are those in which children are asked to quantify comparisons. For example, "Joe has 8 marbles and Tom has 5. Who has more marbles? (an easy question) How many more marbles does Joe have than Tom?" (a difficult question) (Nunes & Bryant, 1996; Nunes et al., 2005).

For Nunes and Bryant (1996) the difficulty of the problem is determined not only by the situation but also by the invariants of addition and subtraction that have to be understood by the children in order to solve a particular problem, and these invariants change according to the unknown parts of the problem. Nunes and Bryant (1996) also point out that the success in addition and subtraction tasks for young children is also determined by the resources that children are using to implement computational procedures, the system of signs. For the authors problems that involve relations are more difficult than those that involve quantities. The literature about additive reasoning has been giving evidence that compare problems, which involve relations between quantities, are more difficult than those that involve combining sets or transformations. Carpenter and Moser (1984) refer that many children do not seem to know what to do when asked to solve a compare problem.

Nunes et al. (2005) conducted a research with primary school Brazilian children, from grades 1 to 4, to analyse their performance when solving problems of additive reasoning. Their results indicate levels of success above 70% for the children of all grades when solving simple problems of part-whole relations involving addition and subtraction. When children were asked to solve inverse problems only 60% of the first graders and more than 80% of the 4^{th}-graders succeeded in a problem such as: ”Kate had some candies. She won 2 more in a game. Now she has 12 candies. How many candies did Kate have in the beginning?”. Their study also analysed comparative problems, such as: "In a classroom there are 9 pupils and 6 chairs. Are there more chairs or pupils? How many pupils are there more?". The authors reported around 50% of success for the second question, and almost 90% among the 4^{th}-graders. These results support the idea that the development of children's additive reasoning is progressive, but also suggest that children are able to solve many of these problems before they receive any formal instruction on addition and subtraction.

Literature gives evidence that kindergarten children are able to solve some addition and subtraction problems (see Fuson, 1992; Nunes & Bryant, 1996), but that does not mean that they understand all the relations in the context of additive reasoning problems. The children's understanding of addition a subtraction is progressive and develops over a long period of time.

To understand more about the children's additive reasoning, it becomes relevant to analyze younger children's ideas of addition and subtraction. Following previous research of Nunes et al. (2005), it was conducted a study with young children, from 4 to 6 years of age, concerning these issues. The study was developed to examine children's understanding of additive reasoning problems. For that two questions were addressed: a) how do children perform when solving additive reasoning problems?; and b) what explanations do they present when solving these problems?

METHODS

Individual interviews were conducted to 45 kindergarten children (4- to 6-year-olds), from Viseu, Portugal. There were 15 children from each age level. In these interviews children were challenged to solve 12 additive reasoning problems (4 direct problems, 4 inverse problems, 4 comparative problems). The interviews were conducted always by the same researcher.

The problems presented to the children were an adaptation of the problems previously documented in the literature by Nunes et al. (2005). Table 1 gives some examples of additive problems presented to children.

Type of problem	Example
Direct	Kate's mum gave her 4 pencils. Later she gave her 2 more. How many pencils does she have now?
	Ben had 7 candies and he gave 5 to his sister. How many candies does he have now?
Inverse	Anna had some candies. She gave 3 to her sister. Anna has 2 candies now. How many candies did she have in the beginning?
	Mark had 5 chocolate candies, he ate some and now he has 3 candies. How many chocolate drops did he eat?
Comparative	In a classroom there are 6 pupils and 4 chairs. Are there more pupils or chairs? How many more?
	Mary has 3 flowers. She has 2 more flowers than Betty. How many flowers does Betty have?

Table 1: Examples of additive reasoning problems.

All the problems were presented to the children by the means of a story problem and material was available to represent the problems.

No feedback was given to any child when solving the problems. All the children were asked “Why do you think so?” after his/her resolution in order to know children’s arguments. In the comparative problems, it was expected that some children could requested help to understand the problem. In some cases the interviewer had to repeat the problem to the child or to put a second question, transforming a static question into a dynamic one, in order to facilitate their understanding of the problem. For example, instead of “how many cars are there more than planes?” – a static question – the child would then be asked “How many planes should we give to Mark for him to have as many toys has Ben?” – a dynamic question.

For all these problems, the assessment of children’s performance was 0 for an incorrect response, and 1 for a correct one.

Data collection took place by means of video record and interviewer’s field notes.

Results

A descriptive analysis of children’s performance when solving additive reasoning problems was conducted. Table 2 summarizes this information for each type of additive structure problem according to the age level.

Additive reasoning problems			
	Mean (s.d.)		
Type of problem	4-year-olds (n=15)	5-year-olds (n=15)	6-year-olds (n=15)
Direct	2.13 (1.25)	3.75 (1.36)	3.53 (0.83)
Inverse	1.47 (1.30)	1.80 (1.27)	2.53 (1.25)
Comparative	0.80 (0.78)	2.33 (1.23)	2.33 (1.29)

Table 2: Mean and (standard deviation) of correct responses when solving the additive structure problems by age level.

It is remarkable the children’s success levels when solving additive reasoning problems. Even the 4-year-olds were able to solve successfully some of these problems. The inverse problems and the comparative problems seemed to be more difficult for children than the direct ones, but even in those 5- and 6-year-olds children presented a correct resolution. The comparative problems were the most difficult for the children. Very often the interviewer had to repeat the problem to the child or to ask a second question in the same problem in order to facilitate children's understanding of the problem, moving from a static question to a dynamic one, as referred before. Thus, the number of cases in which the interviewer had to transform a static problem into a dynamic one was registered producing two categories: without transformation, in which the child solved the problem with no changes; and with transformation in which the child need the interviewer to transform the problem. In any of these cases, the assessment was 0/1 for incorrect/correct responses.

Table 3 summarizes the number of correct responses given by the children when solving the comparative problems according to the need of changes in the presentation of the problem. As each child solved 4 comparative problems, 60 resolutions for each age group were produced.

	Correct responses in comparative problems		
Difficulty level	4-year-olds (n=15)	5-year-olds (n=15)	6-year-olds (n=15)
Without Transformation	2	14	19
With Transformation	10	21	16
Total correct responses	12	35	35

Table 3: Number of correct resolutions in the comparative problems, with the transformation and without it, according to the age.

Figures 1 to 3 present the distributions of the total of correct responses for the three types of additive reasoning problems, according to the age level.

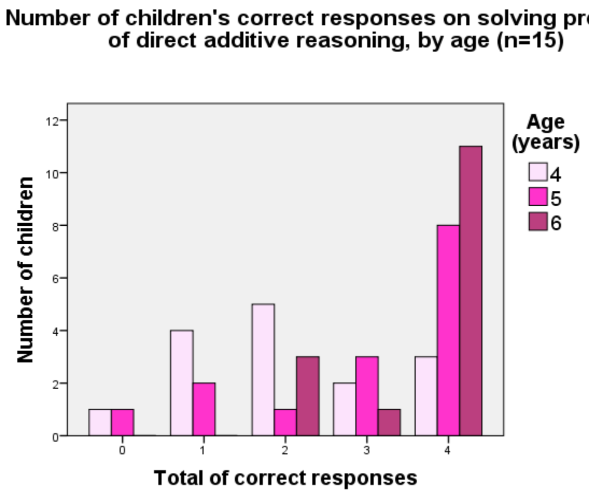

Figure 1: Number of correct responses for direct problems by age level.

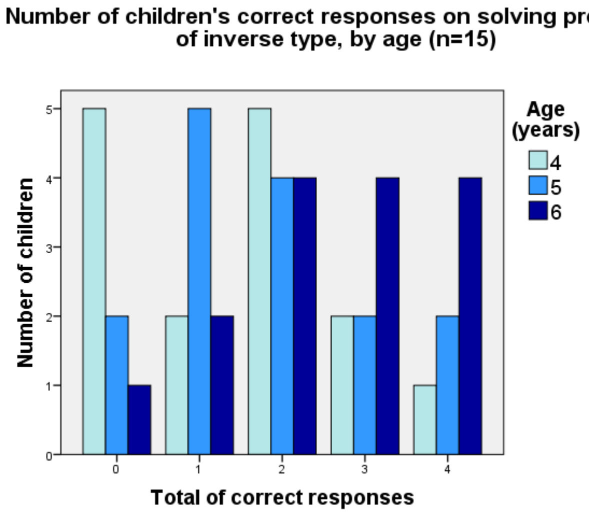

Figure 2: Number of correct responses for inverse problems by age level.

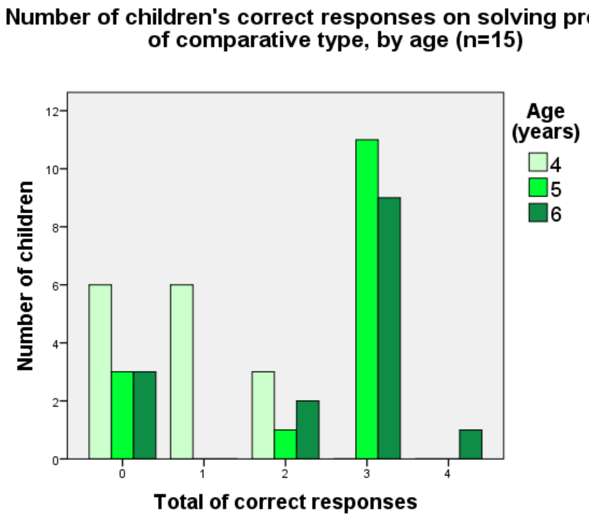

Figure 3: Number of correct responses for comparative problems by age level.

In order to analyse the effect of the age on children's performance solving the different types of additive problems a one-way Analysis of Variance (ANOVA) was conducted with performance in the type of problem (direct, inverse, comparative) as dependent list and age (4-, 5- and 6-year-olds) as a factor. There were no significant effects of the age on the direct problems neither on the inverse problems, but there is a significant effect of age on comparative problems ($F(2,42)=9.3$, $p< .001$) indicating that older children performed on this problems than the 4-year-olds. Bonferroni post-hoc tests indicate that children of 5- and 6-year-olds performed better than the 4-year-olds, but no significant differences were found on children's performance of 5- and 6-year-olds. Thus, in direct and inverse type of problems there was no age effect; the comparative problems were easier for older children than for the younger ones.

To know more about children's reasoning when solving these problems, their arguments were analysed for each type of problem. Four categories of children's arguments were considered in this analysis. The valid arguments comprise the justifications in which children consider all the quantities involved in the problem correctly; the incomplete category comprises children's arguments that refers only to one part of the quantities involved in the problem; the invalid arguments are those in which children do not articulate the quantities involved in the problems; and the no argument category that comprises all the cases of absence of argument.

Table 4 presents the number of arguments of each type that were used by children when solving additive reasoning problems correctly, according to the age.

Additive reasoning problems									
	Type of problem								
	direct			inverse			comparative		
Type of argument	4yrs	5yrs	6yrs	4 yrs	5yrs	6yrs	4 yrs	5yrs	6yrs
Valid	17	19	38	12	17	28	8	22	22
Incomplete	1	9	-	-	2	1	-	-	6
Invalid	3	8	4	7	2	7	3	9	4
No argument	11	9	11	3	6	2	1	4	3
Total correct resp.	32	45	53	22	27	38	12	35	35

Table 4: Number of arguments of each type given when solving the additive structure problems by age level.

Four categories of children's arguments were considered in this analysis. The valid arguments comprise the justifications in which children consider all the quantities involved in the problem correctly; the incomplete category comprises children's arguments that refers only to one part of the quantities involved in the problem; the invalid arguments are those in which children do not articulate the quantities involved in the problems; and the no argument category that comprises all the cases of absence of argument. Table 4 presents the number of arguments of each type that were used by children when solving additive reasoning problems correctly, according to the age.

Children of all age levels presented valid arguments were associated to correct resolutions. This suggests that the results obtained from children's performance are associated to an understanding of the problems presented to them. Around 53% of the 4-year-olds could solve correctly the simple problems presenting valid justifications; these percentage increases to almost 72% for the group of 6-year-olds children. Valid arguments were also presented in 54.5% of the correct answers given by the 4-year-olds children when solving the inverse problems, and in 66.7% of the correct resolutions of the comparative problems. In all type of problems there were children who were able to solve them correctly, but were unable to present a valid argument.

The use of an incomplete argument can be understood as child difficulty to articulate verbally a logic explanation that was carried on. Also children who solved correctly the problems presented no argument, as it happen with 34.4% of the 4-year-olds that solved correctly the simple problems.

DISCUSSION AND CONCLUSION

Children's informal knowledge is supposed to be the starting point for the formal instruction. Thus, it makes sense to know better what do children can and cannot do before being taught about arithmetic operations in primary school. The results presented here suggest that Portuguese kindergarten children are able to solve some problems involving additive structures with understanding, in particular conditions.

These results converge with those presented by Nunes et al. (2005) who analysed 5-8-year-olds children's performance when solving additive reasoning problems. These authors also reported that additive comparative problems were more difficult to young children than the direct and inverse ones. Our study extended these findings about children's additive reasoning as it gives evidence that 4-year-olds children can succeed in solving direct, inverse and also comparative problems. Their procedures do not vary from those used by the 5- and 6-year-olds relying on the schema of the act of join and separate for the direct and inverse problems previously identified in the literature (see Nunes & Bryant, 1996; Nunes et al., 2005).

The children's arguments were also analysed in order to get an insight on their reasoning when solving the additive structure problems. These arguments give evidence that children as young as 4 years of age can establish a correct reasoning and solve this type of problems. This suggests that their correct answers were not achieved by chance. If there are children of 4-year-olds able to solve some additive structure problems with understanding, relying in their informal knowledge, perhaps kindergarten could stimulate their early ideas about addition and subtraction. More research is needed to analyse these issues and to find out what sort of problems, if there are any, should be presented to kindergarten children in order to help them to develop their reasoning.

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DESIGNING TALES FOR INTRODUCING THE MULTIPLICATIVE STRUCTURE AT KINDERGARTEN

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We present a design study to introduce multiplicative thinking at Kindergarten level with an algebraic perspective. Starting from some theoretical assumptions about the psychological roots of multiplication and about the use of narration in Math Education, we build a suitable narrative context in order to promote children's actions consistent with such roots. We analyze the development of this path and its management, emphasizing the special role played by the dialectics between actions upon objects and graphic representations.

INTRODUCTION

The discovery in human beings of very early, if not innate, mathematical competencies, achieved by recent neuroscientific studies, induce to deepen the study of cognitive strategies recognisable as roots of mathematical structures and procedures, and to design learning environments to drive their evolution. This enterprise is not new, as it can be traced back to Piaget's studies about action schemata, from which a wide literature, in particular about the origins of arithmetical structures, has been produced. The common starting point is that action is at the root of any abstract thinking and in particular of the comprehension of arithmetical structures. This idea has been developed within different perspectives, also due to the increasing information we are gaining in these last years about our brain functioning (see e. g. Gallese & Lakoff, 2005).

In this field, our research group has been working for several years at the design and development of prototypes of long-term paths for primary schools aimed to promote in pupils arithmetical competencies as well as linguistic ones, in order to express and communicate their achievements. We are aware of the basic difference between actions upon objects and mathematical operations, but also of the neurophysiology discovery that the same neural circuits are deputed both to actions and to abstract thinking, therefore we think that to carefully identify the action schemata is fundamental in order to exploit them as roots: since these actions and the related mathematical operations will constitute the true base for the whole disciplinary structure.

In this paper we present a design study realized by our team in collaboration with an expert teacher: where a path is developed to introduce multiplicative thinking at a Kindergarten level with an algebraic perspective. A suitable narrative context was created in order to induce actions consistent with the theoretical roots of multiplication, identified according with some theoretical assumptions. We present these references in the next section, after which we briefly clarify the methodological equipment that has informed our experimentation; then, in the widest section we describe and analyze the main parts of the experimental path, and finally we draw some conclusive remarks from our research experience.

THEORETICAL BACKGROUND

In the last decades many studies have been developed about the cognitive roots of arithmetical structures. Without pretending to be exhaustive, we can distinguish two trends: to look for a correspondence between a given arithmetical operation (or arithmetical structure, i.e. the operation with its inverse) and an action scheme, as in Piaget or in (Davydov, 1992); or to classify the different situations in which the use of the operations is needed (e.g. Vergnaud 1983, Greer 1992, Steffe & Cobb 1998). The second kind of studies seems very useful especially for detecting cognitive problems that might underly a given recurrent mistake, whereas the first approach is more fruitful for planning class activities, particularly when arithmetic is addressed since the very beginning in an algebraic perspective, as in our approach (Iannece *et al.*, 2010).

In particular, we refer to Davydov's suggestion (1992) that rather than viewing different correspondences between each mathematical operation and an action schema, links the whole multiplicative structure to a specific psychological need. In his vision, indeed, the psychological root of multiplication is identified in the change of measure unit, when some magnitude has to be measured:

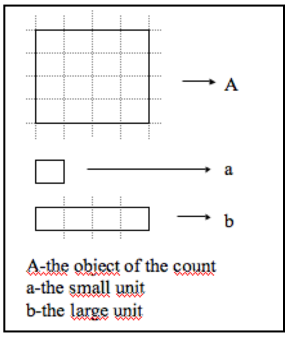

Figure 1

If the magnitude of an object is depicted by A, the small unit of count by a, the large unit by b, then the system of operation, carried out by determining the numerical value of A **indirectly** through a, can be expressed by the following formula: $\frac{b}{a} \times \frac{A}{b} = \frac{A}{a}$ (Davydov 1992, p. 11, see fig. 1).

According to Davydov's studies, we think that children can explore since kindergarten the arithmetical structures in an algebraic perspective by exploiting their cognitive strategies and using their languages. In this direction the graphic representation plays a special role since it can be viewed both as a perceptive metaphor of paradigmatic/structural aspects and as a cognitive support for generalization (Stetsenko, 1995). In Vygotsky's sociocultural vision of learning, in graphic representations sign and meaning arise together, then the integrated use of graphic, verbal and symbolic representations lets the concepts as well as the expressive tools develop. The functional role of drawing in children's cognitive and emotional development and its intertwinement with other communication tools have been explored in the sociocultural perspective. In particular it has been observed that

> young children do not radically differentiate between drawings and writing. At least part of this confusion must be due to the fact that children view both drawing and writing primarily as ways of communicating with others. (Stetsenko, 1995, p. 50)

In other words the intertwined development of drawing and of written and oral language in early childhood can be related to children's need to gradually grasp adults' means of communication. In this study we will show how this knot can be exploited and driven toward "paradigmatic" aspects of language, in particular by promoting the *array* as an effective representation of the multiplicative structure.

Our theoretical background includes also design-oriented studies about the role of tales to build mathematical meanings, in this case multiplicative ones. In the 1970s Donaldson has already observed how the child is particular sensitive to contexts where human intentionality can be recognized and how he uses this key to interpret and give meaning (Donaldson, 1978). To understand people's stories, reasons and feelings is linked to what Bruner calls "narrative" thought, juxtaposed to "paradigmatic" or "logic-scientific" thought. The complementarity of the two kinds of thoughts is put in evidence in several contexts of Math Education, as in problem solving activities (see e. g. Mellone & Grasso, 2008). About this, Zan (2011) observes how a word problem is both perceived as description of a 'human' situation, and analyzed for its paradigmatic features with the goal of solving a question. For this reason the mathematical information in a word problem has to be consistent with the narrated story and viceversa, in order to get resonance between the narrative thought and the paradigmatic one. Othewise the risk is to produce a

> 'narrative rupture' in the text of the problem, i.e. the question and the information needed for the solution are not consistent from the point of view of the narrated story. (Zan, 2011, p. 341)

As we will show in the sequel, we have tried to take this need into account in building the tale for our educational path, by describing characters who are moved by understandable feelings and goals, and by linking feelings and goals with the mathematical questions. Also the teacher's management of the activity has been careful in connecting and balancing the human and paradigmatic aspects of the story.

METHODOLOGY

The experience we are going to analyze comes from a wider research project carried out for several years in Naples by some researchers in Math Education and a group of Kindergarten, Primary School and Lower Secondary School teachers. This group has been working at building and validating prototypes of long-term paths for the teaching/learning of arithmetical structures in an algebraic frame. Common feature of these activities is the assumption of a Vygotskian perspective about learning, in particular on the role of signs in the semiotic mediation process. The research group has been working for several years about the use of the *array* as support for multiplicative thinking; in this study we explore the possibility of using such representation with 5-6 year-old children. To introduce multiplicative thinking in an algebraic perspective, we have built, in collaboration with a kindergarten teacher-researcher, a path that starts with the telling of a story. However, our goal was not just to validate in a class activity a path packed in advance, but rather to be able, starting from an initial plan, to repeatedly modify the path itself, according to classroom events and interactions, following in this a typical design study methodology (Cobb *et al.*, 2003). Consequently, the theoretical issues listed in the above section have not been transferred into action along a rigid sequence, but have been intertwined, in order to obtain effective outcomes for children.

In the next section we will illustrate the main parts of the design and of its three months development. Our collection of data includes children's drawings, transcripts from class discussions, photos, audio and video recording.

THE TALE OF THE GLUTTONOUS KING AND THE DIDACTICAL PATH

The story that opens the path has been invented in order to merge a change of measure unit in a narrative context. The story tells the adventures of a king's servant who has to do several trips through a tangled wood in order to reach a bakery and to buy cakes for the royal family, composed by four members. The cakes are carried 'two at a time' (first change of the measure unit) since the oven takes out only two cakes, one chocolate and one strawberry cake, each time, and each royal member wants to taste both. At the end of the story the teacher asks children to help the servant to pay the bill, knowing the total amount of the cakes bought (here, notice the care for consistency between narrative and paradigmatic aspects). As usual for the teacher, the story is enriched by every sort of details, concerning the different characters and the sequence of events; moreover the verbal language is accompanied by the mimic-gestural one, the exigence of a mime show and a dramatization naturally arise. In the first phase the tale is used to reflect upon the words meaning: for this purpose the children are invited to repeat the story and to discuss about the situation and the characters. The teacher suggests also to make a sort of proto-analysis of the text.

Afterwards the teacher asks children to represent the story with a drawing. In this way she wants to analyse which things have impressed more the children, in order to orient the didactic mediation toward the children's needs and her goals. In this phase the children draw only the passages of the story that turn out to be more meaningful or simpler to be represented. The "paradigmatic" aspects are left apart, certainly also because the previous work about the characters has favoured the narrative thought (see e. g. fig. 2).

Figure 2

Figure 3

Figure 4

Figure 5

The teacher decides for a bodily work, as a premise for reflecting on actions, and also for reaching more paradigmatic representations useful for catching the mathematical meanings of the story. After all, if we recognize action schemata at the roots of comprehension, then we have to make actions. A motoric activity is organized to reproduce the path covered by the servant from castle to bakery: six traffic cones and a cloth tunnel represent the wood, so a gymkhana has to be made to reach the bakery (the class kitchenette), that contains two tiles as the cakes (fig. 3-5). Each child performs his/her own servant's path in order to interiorize the trip as a meaningful experience. This means to carry a plate, to reach the bakery and to buy two cakes, one chocolate and one strawberry cake, as many times as needed to satisfy all family members.

Finally, the children are invited to represent the trips made and the cakes taken each time. This time, all the children try to answer the numerical question: nobody feels inadequate, everybody is involved. This guarantees children's self-esteem and confirms the effectiveness of the teaching methodology employed, which includes a careful balance between the exigencies that all the pupils live successful experiences and that nontrivial disciplinary contents are addressed.

In children's drawings a major attention to the paradigmatic aspects of the story arises, maybe supported by the motory activity and, in particular, by the iteration of trips. In all the drawings we can "see" the multiplicative structure expressed by the grouping: the cakes are linked to the trips and drawn as rhythms of repeated plates (fig. 7), some children represent the trips as lines, (perhaps recalling the feature of the path, see fig. 6) or, in most cases, as half-circles, that recall the cloth tunnel. Only two children (one less than 5 years old) use a person-marker (fig. 9), while only Maria Giovanna outlines a sort of array (fig. 8). Ivana traces also the numerals 2 and 4, although as simple drawing ornaments (fig. 9). We have already observed in the theoretical section how fuzzy is the boundary between drawing and writing at this age, both abilities being linked to children's attempts to appropriate adults' means of communication.

Figure 6

Figure 7

Figure 8

Figure 9

The day after the teacher orchestrates a mathematical discussion about the different representations. This is a crucial part of the teaching mediation based on children’s reflections upon their own and their fellows’ behaviour. After the drawings of the previous day are distributed to the pupils, their comments rapidly focus on the effectiveness of the representations in order to share the best symbols used. Everyone illustrates the way he/she has represented trips and quantities, then everyone is invited to redraw his/her symbols on the blackboard. In this way all symbols are under the eyes of everybody, and thus, after an analysis and a comparison of their features, the children choose the most effective among them (fig. 10). The half-circle is selected as the best representative of a trip, against teacher’s expectation, who hoped children would have chosen, since this phase, the array as a powerful sign to represent trips and quantities of cakes at the same time.

Figure 10

The next meeting between the teacher and the research group is devoted to understand why the children have not chosen the array, even though it appears in one of the initial representations (fig. 8), and why the collective discussion and the teacher’s guide didn’t induce this choice: maybe the two dimensions “trips” and “cakes at a trip” are not so meaningful till that moment, to deserve a special attention and a form of distinction. Therefore, and according to a Vygotskian approach, we decide to introduce an artefact, as a semiotic mediator for the two dimensions of the multiplicative structure: a rectangular tray divided in two-times-four boxes, into which the children can arrange the cakes during the dramatization. The teacher tells a further part of the tale of the Gluttonous King, in which the number of cakes for each trip is inverted with the number of trips, in order to suggest a two-dimensional representation, as well as to evoke a new change of measure unit (from “two at a time” to “four at a time”):[1]

> The Gluttonous King wants to organize a party for his family, where everybody will get a chocolate cake and a strawberry cake. Knowing that the baker has now a larger oven where four cakes at a time can be cooked, what has the servant to do?

> Maria Giovanna: He must cut the cakes into small pieces.

[1] Some fragments of this activity have been already presented and discussed in (De Blasio, Grasso & Spadea, 2008).

Teacher: But the King doesn't like small pieces since he is gluttonous!

Martina: Otherwise they need a still larger oven.

Mattia: No, the servant must do several trips, carrying two cakes for every trip.

Teacher:: Look, I have prepared a tray for arranging the cakes. So, what has the servant to do?

Sara: He must go to the bakery, buy the cakes, and put them on the tray.

The above transcript shows how the teacher mediation tends to justify the resort to the artifact-tray. It is also interesting to notice how she refrains from directly intervening on Mattia's difficulty, who has not caught the change of measure unit from the first part of the story. Instead, she prefers to address the whole class, using a different strategy. Thus she encourages the children to a new dramatization, making the same path but using this time the special tray to carry the cakes. And when Mattia, at the end of his path, arranges the cakes grouping them by two, as in fig. 11, the teacher stops the play and lets all the children look at the tray.

Ciro: In this way it looks like the servant is gone twice and has got two cakes each time.

Mattia [resentful]: No, I went only once [Mattia changes the cakes arrangement on the tray, putting them in a unique row].

Figure 11

Figure 12

Figure 13

Obviously, what is really crucial is not the artifact-tray as itself, but the teacher's mediation that, by promoting a shared action schema, helps children to catch, from the collective discussion, the analogies and the differences between the two parts of the story. The social interaction works very well at this moment: Ciro's remark, which is in better accordance with the use of the tray, immediately produces Mattia's reaction. Teacher's suggestion to reason upon his actions and not only upon the narrated story turns out effective, indeed if Mattia gets angry for doing something different from what he thought (or for being misunderstood), from the other side he is ready to conform himself to the rules of the game. Finally the teacher invites a child to figure the cakes on the blackboard, promoting in this way another step from the representation of the experience through the object-tray toward a representation through signs on the blackboard. Martina goes to the blackboard and draws a first row with four circles, then she begins a second row, as in fig. 12. So, Martina's way of reporting the 'mathematical story' is a sort of rhythm, already implicit in some previous drawings, where the circles displayed in two rows clearly prefigure a typical array.

Teacher: Let's look at Martina's drawing. What does it suggest to you?

Mattia (and others): That he's gone two times… and has got four cakes.

Teacher: Do you agree that now we understand what the servant has done? [She takes two equal 'two times four' trays and puts them close to each other, but differently oriented, see fig. 13] What has changed?

Antonio: Now the oven is larger and cooks four cakes at a time.

Teacher: But is the number of cakes the only thing that's changed? How many times the servant comes from the bakery with his full tray to satisfy everybody?

M. Giovanna: Twice.

Chiara [pointing at the columns of the array]: One and two, one and two.

Mattia: I don't see any change!

Teacher: Are you sure? I see a difference….

Chiara [her hand traces a turning in the air]: They become equal just if we turn them.

Martina: Of course, since in this case the cakes are four and the times are two, while in the other case the trips were four and the cakes were two.

Ivana: But they are eight, anyway.

The use of the tray in the action simulation has well oriented Martina to appreciate the value of the array in representing the performed action. However, the teacher prefers to go back to the material representation via the tray, to promote an effective sinergy between syntactital and semantical aspects of the story. This helps the children to focus on what stays and what changes between the two situations, in order to discover the commutativity of multiplication. Moreover, teacher's pressing requests of precision stimulates a refinement of children's linguistic expressions, supported by reference to the concrete experience or, as well, by representation tools like the arrays. For example, for Martina it is important to drive attention to the concrete meaning of what they did, while Ivana's statement goes exactly in the direction of the multiplicative operation, overlooking the details of the two situations: anyway, they both obtain the same result of 8 cakes[2].

In the rest of the year the teacher has proposed many variants of the story, in which the numbers of trips and cakes varied, but with the usual care for the above discussed consistency between narrative and paradigmatic aspects. We have observed that not all the children used the array to represent the different situations. Our goal wasn't clearly to impose the array, that is to train them to adopt a mechanical automatism, rather our goal was, in Vygotskian words, to

2 Similar behaviors have been observed in grade 3 children (see Mellone & Pezzia, 2008).

promote a “cultural” imitation, that is to drive children to repeat by their own a strategy after having experienced its effectiveness.

For this purpose, at a certain point the teacher decided to change the experience context and to work with rhythms of sounds. The children were invited to record the sound patterns, by recognising a group of notes repeated many times. As usual, they worked sharing, representing, and discussing. But this time the children naturally chose to represent each pattern of symbols, corresponding to a sequence of repeated sounds, one under the other instead of sideways, as in Martina’s ingenious drawing (fig. 12). In this way the children build an array, putting in evidence the role of multiplication and favouring an exploration of its properties. The possibility to experience the efficacy of the array in a new context allows children to recognize the structural analogy between two different situations. Finally, we can report that, during some further variations on the theme, children did make autonomous use of the array.

SOME CONCLUSIVE REMARKS

The design study presented above shows how the action schemata, evoked by telling a story in which the consistency between narrative and paradigmatic aspects is cared, can create resonance (in the sense of Iannece & Tortora, 2008) between children’s strategies and formal mathematical stuctures. In our opinion our study also confirms Davydov’s suggestion (1992) about the essential role played by the change of measure unit in giving sense to the multiplicative structure. Indeed, the story context allows to explore two semantical dimensions (trips and cakes for trip) and, at the same time, the peculiar syntactic properties of multiplication (as the commutative property). The dramatization lets the paradigmatical aspects arise and, on the other hand, the use of the *array* as a semiotic mediator leads the children to start using a genuine mathematical language to ‘put things in order’ (note the emerging of refined multiplicative expressions in Martina’s words “in this case the cakes are four and the times are two, while in the other case the trips were four and the cakes were two”). Finally, the analysis of the path shows a great difference between working with a representation proposed by others (the array at the beginning of the experience) and managing the same ‘linguistic’ tool autonomously (Chiara’s action on the array to recognize the commutative property). In this sense the adults’ cultural mediation in providing the array has to be very careful, due to the foreseeable children’s difficulties of interiorization.

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EXPLORING PARTITIVE DIVISION WITH YOUNG CHILDREN

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This paper focuses on a study with 4- and 5-year-olds children understanding of partitive division when discrete quantities are involved. The study analyse how young children understand the inverse divisor-quotient relationship when the dividend is the same. The participants were 30 kindergarten children from Braga, Portugal. Individual interviews were conducted when solving tasks involving the division of 12 and 24 discrete quantities by 2, 3 and 4 recipients. Results showed that 4- and 5-year-olds children have some ideas of division, can estimate for the quotient when the divisor varies and the dividend is constant, and can justify their answers. Educational implications of these results are discussed for kindergarten activities.

FRAMEWORK

Children learn a considerable amount about mathematical reasoning outside school known as informal knowledge. Literature refers that kindergarten children possess an informal knowledge relevant for many mathematical concepts (see Nunes, 1992; Nunes & Bryant, 1997). This informal knowledge should provide the building of formal mathematical concepts. Concerning the division, several authors suggest that young children can divide discrete quantities successfully (see Frydman & Bryant, 1998; Pepper & Hunting, 1998; Kornilaki & Nunes, 2005; Squire & Bryant, 2002), arguing that these children possess some type of informal knowledge related to the division of quantities, understanding the inverse relation between the divisor and the quotient when the dividend is the same.

Correa, Nunes and Bryant (1998) argue that sharing activities can be relevant in the understating of the inverse relation between the divisor and the quotient. Also Kornilaki and Nunes (2005) argue that understanding the sharing activity helps children to understand the logical relations involved in the division of quantities, i.e., the relation between the dividend, the divisor and the quotient.

When considering the division of discrete quantities it becomes relevant to distinguish the partitive and the quotitive division. In partitive division problem a set of objects is given to be divided among recipients, and the share that each recipient has received is the unknown part. (e.g., there is a set of 10 candies to be shared among 5 children. How many candies does each child get?). In a partitive division problem, the divisor is the number of recipients and the quotient is the share they receive. In quotitive division, there is an initial quantity to be share into a known number of parts. The size of the parts is the unknown (e.g., Mary has 12 candies and wants to give 3 candies to each of her friends. How many friends are receiving the candies?). In quotitive division problems, the divisor is the share to be given to each recipient and the quotient is the number of recipients. Concerning these types of divisions Kornilaki and Nunes (2005) argued that children understand more easily the partitive division than the quotitive division.

Research presents several results of young children procedures when solving division tasks involving discrete quantities (see Piaget & Szieminska, 1971; Desforges & Desforges, 1980; Frydman & Bryant, 1998; Squire & Bryant, 2002). Particularly, Correa, Nunes and Bryant (1998) when investigating the development of the concept of division in young children, examined whether children who could share would be able to understand the inverse divisor-quotient relationship in partitive division tasks when asked to judge the relative size of 2 shared sets. The participants were 20 children of 5-year-olds, 20 of 6-year-olds and 21 of 7-year-olds from Oxford, England. The authors investigated the children's understanding of the three-term quantity relationship in division when the dividend was constant and the divisor varies. In their experiment the experimenter shared a given amount (12 in some trials, 24 in others) of red and blue sweets between two groups of rabbits, one red and one blue, putting the sweets in the boxes attached to the rabbits' backs; the experimenter pointed to one blue rabbit and one red rabbit and each child was asked whether they had the same quantity of sweets or whether one of them received more sweets, and why did the child think so. The authors argued that "if the children succeed in tasks where the dividend is constant and the quotient is inversely related to the divisor, we can be confident that their success indicates some understanding of core relations in a division situations." (p. 322). Results showed that 9 of the 20 5-year-olds performed significantly above chance and about 30% were able to verbalize this inverse relation in their justifications and 11 out of 20 of the 6-year-olds scored above chance and verbalized the inverse relation between the divisor and the quotient in the partitive tasks. The authors also report age improvements between 5 and 7 years. Correa, Nunes and Bryant (1998) also analysed children's justifications according to children's age. Most of the 5-year-olds were not able to give a mathematical justification for their choices and did not mention facts relevant to the solution of the task. The 6-year-olds presented justifications that revealed a progress from some comprehension of sharing and numerical equivalence to the understanding of the inverse divisor-quotient relationship. The majority of the justifications presented by the 7-year-olds showed a logicomathematical approach, referring the inverse divisor-quotient relationship.

More recently, Kornilaki and Nunes (2005) investigated whether the children could transfer their understanding of logical relations from discrete to continuous quantities. Among other things, the authors analysed 32 five-year-olds, 32 six-year-olds and 32 seven-year-olds solving partitive division tasks involving discrete quantities. In this type of problems the number of recipients varied to produce two conditions: 1) in the same divisors condition, the size of the divisor was the same; 2) in the different divisors condition, the number of recipients varied. The results showed that the different divisors condition was clearly more difficult than the same divisors condition. Thus, the authors argued that the inverse relation between the divisor and the quotient is understood later than the equivalence principle of division. The authors also pointed out that in partitive division tasks, one-third of the 5- and 6-year-olds justified their responses as "the more recipients, the more they get", but this response decreased markedly with age as only slightly more 10% of the 7-year-olds used this incorrect reasoning.

The studies of Correa, Nunes and Bryant (1998) and Kornilaki and Nunes (2005) give evidence that, at age of 6 and 7, children have an insight into relations between the division terms, long before they are introduced to this operation at school. If previous research reports some success with 5-year-olds children, how would children of 4-year-olds would perform? Besides, it becomes relevant to get a better insight on young Portuguese children's informal knowledge of division.

This paper focuses on young Portuguese children understanding of division of discrete quantities, when solving partitive division problems. For that we tried to address three questions: 1) How do children estimate the quotient in a partitive division in which the divisor varies and the dividend is kept constant? 2) How do children perform the partitive division tasks involving discrete quantities? 3) What procedures do they use in this process?

METHODS

A study focused on young children's ideas of partitive division was conducted to address these questions. The participants were 15 four-year-olds (11 boys and 4 girls, mean age 4 years and 6 months) and 15 five-year-olds (7 boys and 8 girls, mean age 5 years and 6 months) from Braga, Portugal.

The participants were interviewed individually by one of the researchers when solving the problems. Each problem was presented to each child using a story and manipulatives representing the items involved in each story were available.

Each child was presented to 6 problems: 3 involving the division of 12 units (carrots) by 2, 3 and 4 recipients (rabbits), respectively; and 3 problems involving the division of 24 units (cabbage) by 2, 3 and 4 recipients (rabbits).

In the interview, first children were invited to estimate the effects on the quotient of increasing the divisor keeping the dividend constant. Then they were asked why they thought so. The idea was to have an insight on children's understanding of the inverse divisor-quotient relationship when the dividend is constant. Then children were asked to carry out the division. In this process, their ability to perform the division was assessed as well as the procedures used by them.

The story presented to the children involved a context in which a white little rabbit had 12 carrots. Then he had to share them fairly with his friend, the brown rabbit. At this moment the child was asked: “Do you think that the white rabbit would be with more or less carrots? Why?”. Them the child was invited to accomplish the division between the two rabbits. Them the child was asked: ”Do you think that both rabbits are happy with this division of the carrots? Why?”, “How many carrots did each received?”. Then a little grey rabbit came around and they had to put all the carrots together again and share them among the three rabbits. “Do you think that each rabbit is going to have more or fewer carrots now?”; “Can you help the rabbits to share the carrots?”; “Do you think that all the rabbits are happy with this division? Why?”. The story continues to include the black rabbit. The same questions were asked. In the very end, when the last rabbit came, the children were asked: “Do you think that all the rabbits are happy with this division? Why? Do you want to check it by counting?”.

When the 24 units were involved, an analogous story was presented to them but now involving the 2, 3 and 4 rabbits and 24 cabbages.

Each child took approximately 20 minutes to solve all the problems, in spite of having no limit for it.

RESULTS

In order to understand children’s ability to estimate the quotient in a partitive division in which the divisor varies and the dividend is kept constant, their correct responses and justifications were analysed. Table 1 resumes the percentage of correct estimates and valid justifications for the division of 12 and 24 units, according to the age. A valid justification is an argument in which a child expresses some ideas of the inverse divisor-quotient relationship, such as “because there are more rabbits and each one get fewer carrots.” or “they will have fewer carrots because now there is the X rabbit”.

	4-year-olds		5-year-olds	
	Correct resp.	Valid argum.	Correct resp.	Valid argum.
12 units	67%	43%	72%	67%
24 units	71%	52%	78%	83%

Table 1: Percentage of correct responses and valid arguments when estimating for the quotient with the dividends of 12 and 24 units, respectively.

It is interesting to note that children's performance in the estimating tasks improved from the first part of the problems (involving 12 units) to the second one (involving 24 units), in spite of the sizes of the initial sets. Perhaps this is due to the fact that when the problems involving the 24 units were presented to the children, they were not a novelty anymore.

Another remarkable point is the success observed among the 4-year-olds when asked to estimate and justify their judgement. Almost half of the children presented a valid justification for their correct answer when dividing the 12 units; when they were asked to divide the 24 units, their valid justifications increased slightly above 50%. These results suggest that children of 4-year-olds may have some ideas about the inverse divisor-quotient relationship presented in these conditions.

Children performance was analysed solving division tasks involving 12 and 24 units by 2, 3 and 4 recipients, respectively. Tables 2 and 3 resume the percentage of children's correct responses by age level, in these problems.

	12 units	
	4-year-olds (n=15)	5-year-olds (n=15)
Division by 2	87%	87%
Division by 3	67%	80%
Division by 4	67%	80%

Table 2: Percentage of correct responses by age level when solving the division of 12 units by 2, 3 and 4 recipients.

	24 units	
	4-year-olds (n=15)	5-year-olds (n=15)
Division by 2	60%	80%
Division by 3	86%	74%
Division by 4	67%	80%

Table 3: Percentage of correct responses by age level when solving the division of 24 units by 2, 3 and 4 recipients.

The results suggest that for young children it becomes more difficult to accomplish the division of 24 units than the division of the 12 units set, possibly due to the magnitude of the set.

As the children's performance was not normally distributed a Mann-Whitney U Test was conducted in order to analyse children's performance dividing 12 and 24 units according to the age level. The results show no significant differences on children's performance when dividing 12 units according to the age levels (age 4, Mdn=3, age 5, Mdn=2, U=149, n.s.) and when dividing 24 units according to the age levels (age 4, Mdn=3, age 5, Mdn=3, U=128, n.s.). Thus, results give evidence that there is no difference of 4- and 5-year-old children's performance in this division tasks.

Trying to explain these results, children's procedures were analysed when dividing 12 and 24 units by 2, 3 and 4 recipients, respectively. The same procedures were observed when children were dividing 12 and 24 units. The procedure I comprises the sharing procedures relying on the correspondence one-to-one by the recipients; the procedure II comprises the counting procedures; procedure III comprises sharing activity based on perceptual influence ignoring the size of the shares; and procedure IV comprises sharing activity combined with counting to produce equal shares.

Tables 4 and 5 resume the observed procedures used by the children of both age groups when solving the division problems of 12 and 24 units, respectively.

	12 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
Type of procedure	I	II	III	IV	I	II	III	IV
Division by 2	10	0	3	2	8	2	1	4
Division by 3	9	0	5	1	8	2	3	2
Division by 4	9	1	3	2	8	2	4	1
Total (Max.=45)	28	1	11	5	24	6	8	7

Table 4: Children's procedures solving the division of 12 units, by age level.

	24 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
Type of procedure	I	II	III	IV	I	II	III	IV
Division by 2	7	0	6	2	9	2	4	0
Division by 3	9	0	5	1	6	2	4	3
Division by 4	9	1	4	1	6	3	4	2
Total (Max.=45)	25	1	15	4	21	7	12	5

Table 5: Children's procedures solving the division of 24 units, by age level.

The procedures used by children did not change much according to the magnitude of the set to divide. Tables 4 and 5 suggest that sharing assumes an important role on children's performance when solving division problems, with discrete quantities. The sharing activity developed by each child and the type of shares produced give us an insight of children's ideas of fare share. Many 4-year-olds children used sharing activity without recognizing the need of producing fare shares, either when 12 or 24 units were involved (24% and 33%, respectively). This phenomenon was also observed in some 5-years-old children when 12 and 24 units were involved (17.8% and 26.7%, respectively). Nevertheless, the majority of the children of both age groups involved in this study recognized the importance of producing fare shares in the division tasks presented to them.

The procedure mostly used by both age groups of children was correspondence one-to-one. This procedure conducted children to correct resolutions, producing fare shares. The procedures using sharing activity based on perceptual influence ignoring the size of the shares were also popular among children of both age groups.

After carry out the division of the items by the recipients, the children were asked if they were happy with the division made through the question "Do you think that all of the rabbits are happy with this division? Why?". They were also challenged to verify their results by counting - "Do you want to check it by counting?" - to deepen the understanding of children's ideas of fare sharing by giving them an opportunity to correct themselves. Their reactions were analysed and allowed us to distinguished the following categories: CcE comprises children's verifications in which it was observed Correct counting of the items in each recipient when there are already equal shares; CcNon-NE comprises children's verifications in which it was observed Correct counting of the items in each recipient, but without equal shares; NnC comprises children's reactions in which they refuse to verify because they are sure about it and it is correct; NvNE comprise their reactions in which they do not recognise the need to verify and equal shares were not produced; NC comprise children's reaction in which the correct counting of the items was not accomplished.

Tables 6 and 7 resume children's reactions, by age group, when solving the division tasks of 12 and 24 units, respectively. The majority of the children of both age groups used the opportunity to verify their shares, correcting their distributions when necessary. This was observed by 60% of the 4-year-olds and 73.3% of the 5-year-olds when 12 units were involved; and by 51.1% and 62.2% of the 4- and 5-year-olds, respectively, for the 24 units. These results suggest that equal share is a concept understood by young children of 4-year-olds. In most of the problems presented to them, these young children recognised the importance of fair shares when accomplishing a sharing activity in a division of discrete quantities.

	12 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
	Division				Division			
	by 2	by 3	by 4	Total	by 2	by 3	by 4	Total
CcE	9	10	8	27	11	12	11	33
CcNon-NE	2	3	4	9	3	1	3	6
NnC	0	0	0	0	1	1	1	3
NvNE	2	1	1	4	2	1	0	3
NC	2	1	2	5	0	0	0	0

Table 6: Children's reactions to the produced shares after dividing 12 units, by age level.

	24 units							
	4-year-olds (n=15)				5-year-olds (n=15)			
	Division				Division			
	by 2	by 3	by 4	Total	by 2	by 3	by 4	Total
CcE	9	10	8	27	11	12	11	33
CcNon-NE	2	3	4	9	3	1	3	6
NnC	0	0	0	0	1	1	1	3
NvNE	2	1	1	4	2	1	0	3
NC	2	1	2	5	0	0	0	0

Table 7: Children's reactions to the produced shares after dividing 24 units, by age level.

It was also possible to observe a few children who did not need to verify their resolutions that were correct, being sure about their procedures and solutions obtained. A groups of children of both ages did not recognised the need of produce equal shares, in spite of using counting properly when verifying their results (20% and 13.3% of the 4- and 5-year-olds, respectively, when dividing 12 units; and 20% and 35.5% of the 4- and 5-year-olds, respectively, when dividing 24 units).

DISCUSSION AND CONCLUSIONS

The results presented here give some insights of young children ideas of division of discrete quantities but also their ideas of fair sharing. The findings of the study reported here suggest that young children of 4- and 5-year-olds possess some ideas related to the division of quantities, understanding the inverse relation between the divisor and the quotient when the dividend is the same. The analysis conducted here give evidence that children of 4-year-olds reveal some understanding of the effect of increasing the number of recipients when the amount to share is constant. These children were able to estimate the result of division. This suggests that children also have some ideas of the inverse divisor-quotient relationship in partitive division tasks, when asked to judge the relative size of shared sets. This idea is in agreement with Frydman and Bryant (1998), Correa, Nunes and Bryant (1998) and Kornilaki and Nunes (2005).

The study reported here has some similarities with some presented previously in the literature (see Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) but also offers some original contributions. Correa, Nunes and Bryant (1998) investigated 5- to 7-year-olds children's understanding of inverse divisor-quotient relationship, when partitive division was involved. Their findings give evidence that 5-year-olds children can succeed in these tasks. Also Kornilaki and Nunes (2005) give evidence of 5-year-olds children success when solving this type of tasks. In our study we analysed how children of 4- and 5-year-olds behave when dealing with this type of problems. Some positive signs arise from this investigation. Four-year-olds children are also able to understand some ideas of divisor-quotient relations in particular conditions.

The procedures used by the children of this study suggest that correspondence can play an important role on children's sharing activity and on their accomplishment of division. Some authors argue that sharing activities can be relevant in the understating of the inverse relation between the divisor and the quotient (see Correa, Nunes & Bryant, 1998) and that understanding the sharing activity helps children to understand the relation between the dividend, the divisor and the quotient (see Kornilaki & Nunes, 2005). In agreement with these ideas, one-to-one correspondence sustaining the sharing activity seems to allow young children to understand the logical relations involved in the division of quantities. This study also shows that equal share is a concept understood by some 4-yaer-olds children and recognized by them as an important issue of the division of discrete quantities. Nevertheless, fair sharing does not seem to be only concept for understanding the division of these quantities, as many young children were able to estimate the effects of increasing the divisor in the quotient, for the same dividend, before carry out the division.

These findings suggest that kindergarten activities could stimulate children's early ideas of division, relying of their informal knowledge. These activities could comprise the use of share and the production of equal shares, but also activities to promote the understanding of the logic relations involved in the division, when the dividend is kept constant. These ideas are crucial to understand some complex mathematical concepts such as fractions, later on in the formal traditional school.

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THE APPEARANCE OF EARLY GENERALIZATION IN A PLAY[3]

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The paper shows the appearance of generalization and its fundamental role in a didactical activity based on a play with rules, proposed to pupils 5-7 years old. Every play requires and promotes different competences, in particular logical and mathematical. The study of pupils' behaviours in front of the task furnishes some examples that prove the possibility of an early mathematical activity of generalization.

THEORETICAL FRAMEWORK

Usually the word 'generalization' is related to algebraic procedures and reasoning, but it is possible to observe the use of generalizations also in other mathematical activities. Generalization is often cited as typical form of mathematical thinking, but without using a definition or specify its meaning. Moreover generalization is often associated with abstraction, since the boundary between them is very thin.

In an Italian book for teachers, we can read this definition of 'generalization':

> the capability to *free oneself from particular*, to find *solutions more amply valid* to achieve a given aim. … capability that allows to distinguish the essential from the particular, "what it *needs* make in given situations" from the various "way in which it *can* be made". (Altieri Biagi & Speranza, 1981, p. 178)

In her analysis of the act of understanding, Sierpinska considers four basic mental operations: identification, discrimination, generalization and synthesis. Her definition of generalization is the following that completes the previous:

> Generalization is understood here as that operation of the mind in which a given situation (which is the object of understanding) is thought as a particular case of another situation. The term 'situation' is used here in a broad sense, from a class of objects (material or mental) to a class of events (phenomena) to problems, theorems or statements and theories. (Sierpinska, 1994, p. 58)

In his theory of 'universal model' Hejny (2004) distinguishes six different stages: motivation, isolated (mental) models, generalisation, universal (mental) model(s), abstraction, abstract knowledge. In particular, concerning the 'Stage of generalisation' he writes:

[3]Work done in the sphere of Italian National Research Project Prin 2008PBBWNT at the Local Research Unit into Mathematics Education, Parma University, Italy.

> The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme generalising all these models is discovered. The stage of generalisation does not change the level of the abstraction of thinking. (Hejny, 2004, p.2)

Hejny (2004) writes also:

> The generalisation of isolated models (experiences and pieces of knowledge) is determined by finding connections between some of isolated models. This web is the most important product of the stage of the isolated models. (Hejny, 2004, p.5)

In this paper the author presents and studies an example of generalization that appears during a play. It is well known that the play can promote logical and mathematical competences. Schuler (2011, p. 1912) highlights that:

> […] *play* and *relationship of playing and learning* have to be explored more closely when talking about mathematics for the early years.

Starting from the consideration of emotional, social and cognitive role of the play, she writes:

> […] play in early childhood is the motor of development and hence associated with learning. Consequently the underlying question seems not to be "Can children learn while playing?" but rather "How can learning while playing be modeled?" and "Can children learn mathematics while playing? (Schuler, 2011, p. 1913)

After an analysis of some theoretical models, she emphases "the central role of the educator and the quality of materials, games and activities". In fact, sometimes it is difficult to adopt a good equilibrium between a free and spontaneous play and a guided play. In other words, "Play is not enough. […] children need adult guidance to reach their full potential" (Balfanz et al., 2003), but when the teacher proposes a play finalised to promote particular abilities, he risks to force in some way the child and to impose directions of work connected with the play finality. In particular, Schuler (2011) studied situational conditions of learning while playing and she highlights three main blocks: affordance, liability and conversational management:

> […] rules can offer mathematical activities beyond a material's intuitive affordance and thus create liability. Intuitive affordance of materials is replaced in games by (*the affordance of*) keeping the rules and winning *the game*. (Schuler, 2011, p. 1919)

In the play utilised in the present research, an important role is done to row-column arrangements. Rożek & Urbanska (1999) studied in depth this topic:

> The children have a different awareness of the rows and columns arrangement. Some of them prefer rows, some of them columns. It appears that it was difficult to see both rows and columns, especially for young children.

In particular, Rożek in her researches about SCFL (Series-Columns Figures Layout) Rożek (1997, 1998) analyses children's behaviours, in terms of two activities constructing and drawing SCFL. She studies also verbal descriptions of SCFL and she organises the protocols in base of three different features: following the features of structures, following visual perception, using language. In the first, she observes the distinction between geometrical aspects as rows and columns or numerical aspects. In the second, she classifies the vision as global or analytical. In the third, the focus is on the language that can be referred to real world or in comparison with mathematical language. In our research, there is a part related to 'construction' and a second part based on 'lecture' of villages (2D) or palaces (3D), that can be analysed and organized following Rożek theory.

RESEARCH QUESTIONS

The present research is placed in the theoretical framework of early mathematical education by play, in particular it deals with children's development of reasoning in playing with rules. The initial hypothesis is that a suitable play can promote an early and spontaneous use of generalization. Our aim is to give answers to the following questions:

1. Is it possible to develop in children the construction of metacognitive instrument of generalization in the context of a guided play?
2. Under what conditions we can obtain learning of generalization, using a game that can promote it?

THE EXPERIMENT AND ITS METHODOLOGY

In this paper we present a research focused on a part of a wider study based on a play with rules, the 'Play of coloured houses', showed and analysed in a working seminar presented from the author in a CME conference (Vighi, 2010b). The main research aims were to study spontaneous reasoning made from children, playing with 'the play of coloured houses', to analyse their behaviors in front of row-column arrangements tasks and the possible recourse to metacognitive processes of symbolization and formalization. In this paper we refer only the part related to the appearance of generalization during the play and its crucial role. The experiment took place in the last year of kindergarten in which pupils (5-6 years old) worked in groups of seven or eight and in the first year of primary school (pupils 6-7 years old) with work in pairs. Pupils involved were 20 in kindergarten and 26 in primary school. The activities took place in every day context. In kindergarten they were conducted from the teacher[4] in presence of a researcher (the author of the present paper). Teacher presented the play and she conducted the works, promoting and fostering the viewpoints of

[4] I wish to thank the teacher Palma Rosa Micheli (Scuola dell'Infanzia Statale "Lodesana", Fidenza (PR), Italy), for her collaboration and helpfulness.

children, without force their thinking, but waiting to listen their ideas and observing their behaviours. Researcher observed, recording on video, later she analyzed and transcribed dialogues, making also written observations. In primary school the activities were conducted in part from the teacher[5] and in part from the author who worked with children in pairs.

THE "PLAY OF COLOURED HOUSES"

The "Play of coloured houses" is a play without winner, based on a disposition of houses with three different colours (red, yellow, green) in a grid 3x3, respecting the following rule: in each row and in each column it needs to have houses of three different colours. We report here some examples:

The play remembers Sudoku, in fact it can be seen as a simplified version of Sudoku with a grid 3x3 (instead of 9x9) and only three 'symbols' (it is possible to use digits 1, 2, 3 in place of colours). From the mathematical point of view, it is a 'Latin Square', i. e. a square in which "each element appears only one time in each column and only one time in each row" (Quattrocchi, Pellegrino, 1980).

The play requires the contemporaneous management of rows, columns and colours. It can be executed by means of 'method of attempts and errors' or using rules discovered during the play: "It is impossible to have a red house here", or "Here it must be a yellow house" etc. When a pupil plays, he makes argumentations, and also hypothetic-deductive reasoning: "If I put here a green house, then …" and so on.

G	R	Y
Y	G	R
R	Y	G

a

Y	R	G
G	Y	R
R	G	Y

b

Y	R	G
R	G	Y
G	Y	R

c

Figure 1: Examples of villages

THE 'SCALETTA THEOREM'

In scholastic year 2009/10 the "Play of coloured houses" was presented in kindergarten in a context of motor activity, after pupils played with coloured tiles and a support for tiles organized in three rows and three columns (Fig. 2). We drew a house on each tile with the aim to give an orientation that allows to distinguish clearly the built villages (in this way it is possible to have 12 different villages).

[5] I wish to thank also the teacher Ines Tommasini (Scuola Primaria Vicofertile (PR), Italy).

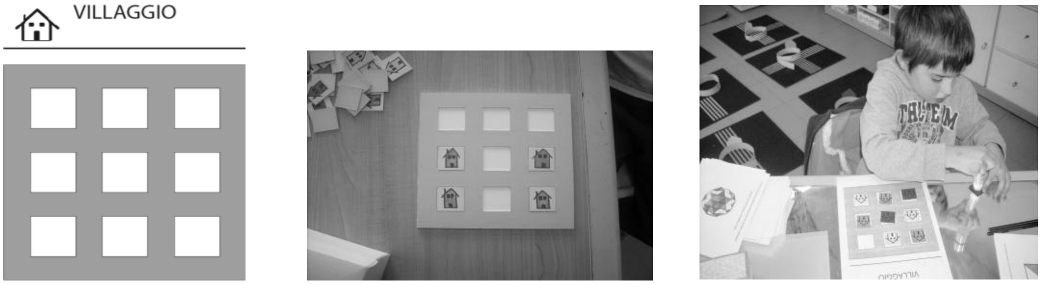

Figure 2

In scholastic year 2011/12 we presented the same activity in Primary School (pupils 6-7 years old). Here we refer only on comparison of villages constructed from pupils, suggested from the teacher. It is well known that the activity of comparison is fundamental in mathematics, to construct concepts: thinking about analogies and differences can promote the formation of a concept. It is also documented that comparison it is not spontaneous in young children; they start using intuition, but it is insufficient, so it compels the use of the language. After a lot of activities based on the play, teacher submitted couples of villages and she solicited their comparison starting from a couple of 'equal villages', and continuing with couples of villages with 'the same structure' etc. An important observation is about the different ways of seeing the SCFL (Rożek, 1997) that children showed: use of a local way of seeing, observing only some couples of tiles with the same colours, placed in the same places ("In the first village there is a green house here, in the second also"); observation of the disposition of all the tiles with the same colour and use of a words of natural language to describe their disposition ("It seems letter C"); recognition of rhythms or cycles ("red, yellow, green, red, yellow, green, …"); individuation of symmetric villages (for instance, villages *a* and *c* in Fig. 1); only observation of rows (or columns) and their exchanges (in Fig. 1, "The second row in village *b* is equal to third row in village *c* and vice versa"); observation of different orientations of diagonals (in Fig. 1, referring to *a* and *c* villages: " … but one go down, the other go up"); description of features of diagonals ("In one diagonal there is the same colour" and "in the other diagonal there are three different colours").

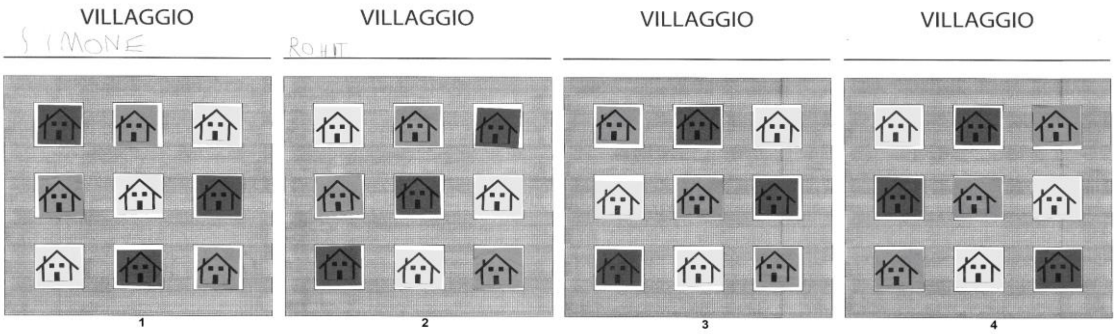

Figure 3

This last aspect suggested to the author of the present paper to put attention and to focus this topic: the visual perception of colour leads some pupils to move their attention from rows and columns, explicitly mentioned from the rules of the play, to diagonals that present a particularity, all tiles have the same colour. Our hypothesis is that it could be a starting point to investigate if children use or not ‘diagonal rule’ to make generalizations.

In the first experimentation, pupils of kindergarten school used the name ‘scaletta’ (in Italian language it means “little ladder”) to indicate this monochromatic diagonal; in fact, the disposition of tiles suggested the steps of a small ladder. The observation of ‘scaletta’ was developed in the following context: firstly each pupil constructed his village, gluing tiles on a sheet of paper expressly prepared for the use (Fig. 2); in a second moment teacher put some villages on a wall of the classroom and she asked observations from the pupils. In particular, they told: “The yellows are in single line” and “They are in angle”, “They are in little ladder”, “In a bandy row” (diagonal), “In a bandy row there are three equal colors, in the other bandy row there are three different colors”. It happens since teacher promoted the passage from micro-space to meso-space (Brousseau, 1983): micro-space is near to the subject and accessible to manipulation and vision, meso-space is accessible to a global and simultaneous vision (macro-space is accessible only for local visions). In fact, the first work proposed to the pupils took place in the space of the desk (micro-space), the second in the space of the classroom (meso-space). It changed the point of view in village’s observation: from rows and columns to diagonals. So, the “connection between some of isolated models” (Hejny, 2004) creates a web that produced generalization.

So, we observed an unexpected fact: pupils found and formulated a theorem that is a consequence of the play’s rule. We call it, the “Theorem of little ladder”: “In all villages there is a little ladder with only one colour”. It is an example of generalization in the meaning of Altieri Biagi & Speranza (1981): from particular to the essential.

Sometimes pupils used this theorem in their following constructions of villages that started from a diagonal monochromatic. In this way they adopted a strategy of village’s construction that involved new rules, different from these suggested from the play. It is a generalization as ‘capability to find solutions more amply valid’ (Altieri Biagi & Speranza, 1981), and also in sense of ‘a given situation is thought as a particular case of another situation’ (Sierpinska, 1994), but also in which the structure appears as generalizing isolated models in sense of Hejny (2004). But … the use of the theorem doesn’t guarantee success. It is evident in Chiara strategy (Fig. 4).

Y			Y	R		Y	R	G	Y	R	G	Y	R	G	Y	R	G	Y	R	G
	Y			Y			Y			Y		R	Y		R	Y	R	R	Y	R
		Y			Y			Y	G		Y	G		Y	G		Y	G	G	Y

Figure 4

Chiara started with a yellow diagonal, she continued with two correct passages, after she makes an error that leads to have at the end a ‘wrong village’.

PASSAGE FROM 2D TO 3D PLAY

In the present school year, we decided to submit to the pupils of kindergarten (5-6 years old) a new version of the play, in three dimensions: it consists in the construction of a ‘palace’ of three floors (a cube 3x3x3), with similar rules: “In each wall face it needs to have three different colours in each row and in each column”[6]. The play can be considered a three-dimensional (3D) version of the two-dimensional (2D) play of coloured houses. Sometimes in mathematics we observe the use of the locution ‘generalization’ also for the passage from 2D to 3D.

Pupils worked in groups following the indications suggested from the teacher. She arrived in classroom with two big boxes and she created a condition of waiting about their contents. After, slowly she opened the boxes extracting cubes (27 wooden coloured cubes, 9 red, 9 yellow, 9 blue), their wooden support (Fig. 5), named from children “palace” or “house with a lot of floors”, and a wooden rotating disk to facilitate gestures and the observation.

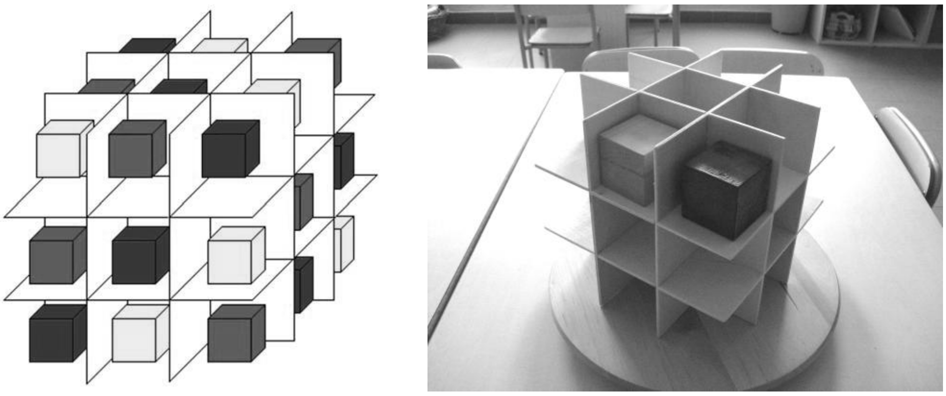

Figure 5

Firstly teacher suggested different free plays with cubes, after she invited each child to put a coloured cube on the support, promoting the construction of a building respecting rules; in a second time, she removed the support and she putted the cubes one near to the other (Fig. 6).

This choice promoted an important breakthrough, since, as Rożek (1997) write, in a row-column arrangement of figures the distance between objects influence in depth the observation. We choose to report here the development of the work in a group, named G2, but we could observe similar behaviours in other groups, of course not in all. In G2, a child observed the yellow diagonal present in the “roof of the palace” (Fig. 6), suggested from the colour and also from idea of “straight line” and he said that there was a mistake in the constructed palace.

[6] A similar problem was studied from M. Gardner (1980) that found only one solution for the final cube (excluding rotations, reflections or permutations of colour).

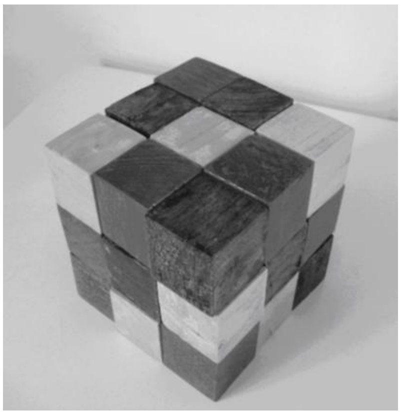

Figure 6

Teacher suggested that in fact all rows and columns respected rules and the child replies that “Yellow cubes are in point, as point of knife”. Immediately pupils find ‘points’ (‘scalette’ in the previous experience) in the other faces of the cube: “There are three points blue and three points red”. In fact, after the construction it is possible verify that the rule is respected also in the ‘horizontal floors’: in a floor there is a diagonal red, in another blue, in another yellow. So, they conclude that “This cube is magic!”.

Another breakthrough happens when a child observed that the other diagonal on the roof presented three different colours: he indicates it with his hand accompanying with gesture and sound: “here, blue, red, yellow, pum, pum, pum” and he repeated it for each face visible of the cube. He added: “A ‘point’ entirely yellow, another of three colours, it is an X”. We name it the “Theorem of two diagonals”. In other words, the disposition of diagonals in each face of the cube suggested the mental image of letter X, that produced a passage from isolated models to a general model in the meaning of Hejny (2004): children changed their cube construction way, they started from a face, putting cubes following the ‘X disposition’ (Fig. 7) and completing the remaining parts. Using the two diagonal’s theorem, the play becomes easier: the construction of a coloured village changes a lot, since starting from diagonals, the placement of the other houses is obliged.

Y			Y		B	Y	R	B
	Y			Y		B	Y	R
		Y	R		Y	R	B	Y

Figure 7

In other words, the finding of two diagonal’s theorem caused the passage from ‘the various way to make something to what it needs make’ in sense of Altieri Biagi & Speranza (1981) and also it produced the discovery of a common structure in the villages (Hejny, 2004).

Afterwards pupils found also that on the lateral surface of the cube there are three points (blue or red) that make a continuous and close paths. This was the input for another play, named ‘Cricket play’ (we prefer do not present it here), that conduced to find a ‘new’ theorem: “In the cube there is an “internal diagonal” with only one colour and the other diagonals of cube are of three different colours” (Fig. 8). In this way the analogy with the 2D play in the village emerged and the “small ladder’s theorem” reappears… Is it generalization?

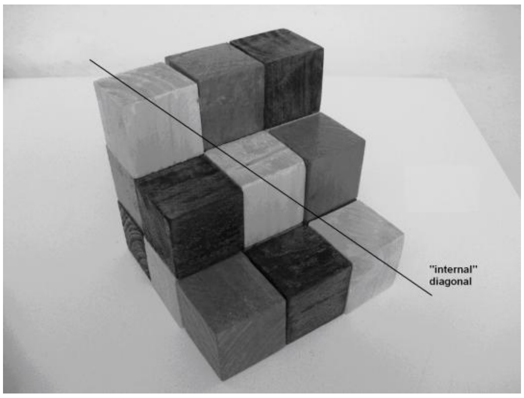

Figure 8: ‘Internal’ diagonal of cube.

CONCLUSIONS

We think that our experiment realized a good equilibrium between playing and learning, in particular we understood that play furnishes the opportunity to observe mathematical reasoning’s development in young pupils.

In reference to our first research question, we can reply affirmatively, concluding that in some kindergarten groups we observed the spontaneous appearance of the metacognitive instrument of generalization, motivated by play and also by context. So, that confirms our initial hypothesis about the early use of generalization. In literature we haven’t found similar researches and results with so young pupils.

In fact, in relation to the use of generalisation, we had better results in kindergarten than in primary school. We pose a possible explanation: in kindergarten the play was entirely conducted from the teacher with the presence of researcher as observer, whereas in primary school the work was conducted from both, teacher and researcher. In the first case, the observer had the possibility to “peek and catch” some observations made from children, while the teacher was involved in the action. That allowed to take advantage of these suggestions and to use them in the following activities and conversational managements. In primary school, may be that working with a researcher, an unfamiliar person, influenced negatively the performances of pupils. So, the answers to the second research question, according to Schuler (2011), could be: “Potentially suitable materials and games need a competent educator with regard to didactical and conversational aspects". In other words, the role of the teacher and a conversational management appeared determinant.

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THE GENERALIZATION OF THE MEASUREMENT CONCEPT IN KINDERGARTENS THROUGH THE BARTER MARKET

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This paper shows a proposal research that tries to describe how five-year-old children can learn the measurement process. The focus is on how everyday life experience can help children build mathematical concepts, especially the process of measuring, and how children learn to use a special scientific language.

INTRODUCTION

Kindergarten in Italy has now become an integrated system in evolution, characterized by the fundamental right to education. Therefore, the final goal of kindergarten education is to promote the development of independence, skills and good citizenship in children. All this is reflected in daily experiences when a child recognizes and communicates an understanding of fundamental activities and manages transactions with others. Moreover, the child learns to appreciate other points of view and to recognize rights and duties (NCTM, 2000; NRC, 1989; INC, 2007). This research tries to find whether measurement-related concepts can be introduced in kindergartens by letting children prepare food and drinks whose ingredients need to be measured in several ways. Our objective is also to see if children can seize the underlying differences and similarities between the use of different measuring instruments and units of measurement. The school undertook to send school materials to a school in India. To obtain these materials, the children prepared, packaged and “traded” food products. The experience of preparing food and beverages for this project taught them the concepts of weight, volume and length (preparing pasta, juices, blended drinks, pastry cream and chocolate rolls). Finally, in assigning a value to these products in order to exchange them for the school material children learned about numbers in relation to pricing.

CULTURAL REFERENCES

In the field of experience, specifically “speech and words”, the National Curriculum Guidelines indicate among its goals the development of specific skills including that of “communicating to others your own reasoning and thoughts through verbal language, used in an appropriate way in different activities” (INC, 2007). We have wondered what is the relationship between everyday language and scientific language at this particular stage of a child’s cognitive development. Exploration, observation and comparison in scientific activities can be used to support the development of language among children and between children and adults. Therefore, the problem of mathematical communication “depends at least as much on what we see as on other types of less abstract speech”. The question then concerns the “effectiveness of communication” and its mediators: semiotics, artefacts and visual (Sfard, 2009). “Equally important to the acquisition of mathematical ideas is the neural system that governs body movements” (Lakoff, Nunez, 2005). Some research shows that body movements can express the perception of objects and spatial orientation and therefore crucial elements of mathematical reasoning. Dealing with the problem of measurement in kindergarten leads to particularly complex experiences and language. Moreover, Vygotskij’s development theory entrusts schools with the task of “stimulating” the movement from spontaneous to scientific concepts; on the one hand, this “stimulation” provides for the maximum development of the scientific concept acquisition stage, while on the other hand it exploits spontaneous concepts in order to promote the highest levels of cognitive development (Vygotskij, 1984). We can see, therefore, that “measurement can constitute an area of near development in which experiences, although not completely understood by a child, can successively be integrated into a network of conceptualization” (Bartolini Bussi, 2008). Moreover, it seems important, once again, to affirm that the learning objective in kindergarten is to enter the world of adults by following the “who, what, where, how, why” method in order to make a concept clear and to explain the meaning of a process (Ginsburg, Pappas, Seo, 2001). This objective can be realized by resorting to well-defined mathematical concepts, such as the ability to invent and plan, make similarities and relationships, as well as to analyse the different forms of natural language that are the starting point of every activity of formalization. It seems to us that we have followed the guidelines related to everyday activities, knowledge of personal history, time rhythms and cycles, space orientation and exploration of nature. It also seems to us very relevant to point out the importance of gathering, arranging, counting and measuring by resorting to more or less methodical ways of comparing and arranging, in relation to different properties, quantities and events through the invention and use of objects or sequences or symbols to record and remember some simple measuring instruments and, finally, by making quantification, numeration, comparisons (Geary, 1994; Ginsburg, Seo, 2004; Clements, 2004; Copple, 2004).

METHODOLOGY

The didactic methodology uses the inquiry approach, a model based on assumptions of knowledge, learning and teaching derived from criticisms of the traditional method of transmission. Through the inquiry approach, it is possible to: encourage students to explore; help students to verbalise their mathematical ideas; bring students to understand that many mathematical questions have more than one answer; make students aware that they are capable of learning mathematics; and, teach students, through experience, the importance of logical reasoning. In other words, we try to enable students to develop the mathematical capabilities necessary to pose and solve mathematical problems, to reason and communicate mathematical concepts and to appreciate the validity and the potential of mathematical applications (Borasi, Siegel, 1994). This has been recommended in numerous important American and Italian studies on reforming the teaching of mathematics (NCTM, 2000; INC, 2007).

Several researchers who have studied the learning of mathematics have found that students must actively demonstrate a personal understanding of mathematical concepts and techniques. Only in this way can they reach a level of significant understanding (Ginsburg, 1983; Steffe, von Glaserfeld et all, 1983; Baroody, Ginsburg, 1990). This position is reflected in constructivism. The influence of constructivism on mathematics teaching can be seen in requests for teaching environments that encourage students to actively participate in developing their knowledge rather than receiving it from teachers or books. In these classes, the roles are reversed. Instead of passively listening, the students assume responsibility for their learning. The teachers, on the other hand, speak considerably less and listen a great deal more to the students' reasoning in order to help them understand what they have deduced (Confrey, 1991). In other words, to be good students, children today must be researchers ("inquirers"). Therefore, only doubt and uncertainty can motivate the search for new knowledge (Skagestad, 1991). Our experience was based on the inquiry approach model, which allowed us to alternate problem posing with problem solving. It showed children solving problems which arise and for which no one has the answer rather than solving problems prepared by the teacher. For example, when they have to assign a price to one of their products, they decide on the basis of their different personal daily experiences. We have then chosen to get children to make some types of food such as pasta, cream, fruit juices and chocolate roll; in this way they can form their own opinion about the best way to measure things, not to mention the experience they have already gained from their everyday life.

This model led us to use the problem posing method in which the children's answers, their questions and the data they used are analysed. In other words, with this methodology the children can make observations, ask questions and formulate proposals. Moreover, they can compare an external investigation with an internal one. It is also possible to compare and contrast exact and approximate investigations, using the strategy of "and what if…" to generate new hypotheses. It has especially been important to see how children know special terms and the two main aspects connected with the measurement process: i.e. comparison and order. That's the reason why it was useful to analyse the clinical-like conversation not with a view to verifying the correctness of the answers but rather to gain an understanding of the social and cultural motivations behind them. It was an extremely important method for forming, informing and maintaining the teacher's "intermediary inventive mind" (James, 1958).

OUR RESEARCH, ITS RESULTS AND THEIR ANALYSIS

Our research has been carried out in two classes of two different kindergartens. In the first class there were 16 children and in the second 19; all in all, the project lasted 35 hours. One of the kindergartens was twinned with a kindergarten in India. The children saw films of this school and with the teachers decided to send school materials to the students there. From this came the idea to organize a "market" whereby the children traded the food and beverages they had prepared for pens, exercise books, etc. to send to India. The aim of the research was to give the student an enjoyable experience in which to experiment with measurement and then to relate it to their primary needs ("the right to food") and their childish pleasures. This situation turned out to have a great influence on scientific learning; in particular, it allowed children to become familiar with the concepts of weight, volume and length. This establishes a connection between children and the "who, what, where, how, why" method (Ginsburg, Pappas, Seo, 2001) and leads them towards the scientific conceptualization of the measurement process (Bartolini Bussi, 2008).

Through the presentation of some objects (a stick, an orange, a piece of chalk, a pencil, a coloured ribbon, some coins, a sheet, a bottle, a glass) we have tried to understand what children know of the size, weight and volume of these objects.

Children have then been spurred to have a clinical-like talk like the following:

> Teacher : Is the pencil longer than the chalk? Is the pot higher than the orange? Is the pot larger than the bottle? Is the orange heavier than the sheet?

After looking at the objects put on the desk children have started to express their opinion as follows:

> Mattia: The pencil is longer if I put it this way, while if I turn it the pencil is short!
>
> Federica: The bottle contains more milk than the glass!
>
> Giovanni: The orange is heavier than coins.
>
> Mattia: I'm taller than the stick, but Federica is shorter than me!
>
> Teacher: Which are the longest things you know? Which are the widest ones? Let's try to find the longest, widest, highest and thinnest things in this classroom.
>
> Mattia: The door is tall! ... and the teacher too, because she's taller than me!
>
> Giovanni: On the contrary, the window is wide.

The distribution of strips of paper having different length to each child has allowed us to make some inquiries about their previous intuitive knowledge of comparisons and orders. In particular we have asked children to find in the classroom some objects as long as their strip of paper.

> Mattia: My strip is as long as Luisa's case on the zip side.
>
> Federica: On the contrary, my strip is as long as the poster which leaves are stuck onto… it is very long!
>
> Teacher: This means that the poster which leaves are stuck onto is longer or shorter than Luisa's case?
>
> Giovanni: I think that Luisa's case is shorter than the poster which leaves are stuck onto because the strip of Mattia is shorter than the strip of Federica.

Then we have made accurate inquiries about the order concept by asking children to find the longest and the shortest strips so as to arrange them in length order, from the shortest to the longest. It's at this stage that we can infer how visual and artefact semiotic mediators become an important instrument for their "effectiveness of communication" (Sfard, 2009). After the talk stage the activity carried out at school concerning the above-mentioned objectives developed in three further stages: an initial observation and exploration stage of the actions and movements of an "expert" adult in the preparation of sweets; the second stage in which the children become cooks and, handling the ingredients, they formulate and verify hypotheses, because they have to reconstruct the previously observed procedures, going through the recipes and proving their validity; the last stage in which the attention is focused on the possibility to set up a trade fair as a problem solving exercise concerning the "value" of the prepared products and the meaning of fair exchange, identifying the objects to trade and their value. All the activities performed show how the inquiry approach is carried out in real terms and draws attention in particular to the formation of concepts according to the constructivism theory in the teaching of mathematics (Steffe, 2004).

In particular, in the first stage, three adult experts were brought in to prepare single products: a grandmother for the preparation of an ear-shaped pasta (orecchiette) typical of their region; a mother to make a cream pastry and a blended drink; and, a professional pastry chef to prepare a chocolate roll. After watching the experts prepare the products in class, based on typical housewife measurements such as "a handful of sth", "a pinch of sth" and "a spoonful of sth" there was a fruitful discussion on what they had observed. Problems relating to weight emerged when trying to interpret recipe indications given by a grandmother, such as "a handful", and the additional problem of the different quantities of flour contained in a child's hand and an adult's hand. Children of the two schools have solved the problem in one or more ways also thanks to the use of different instruments. A scale with two plates was used in one school; the following discussion ensued:

Teacher: “What is happening?”

Denise: The amount of flour in my hand is smaller and the plate stays up but Grandma’s handful is heavier.

Teacher: Could we put the plates at the same height?

Giovanni: Let’s put some other handfuls of flour on the plate to make it go up.

Teacher: Ok, but how much flour do we have to add?

Giovanni: As many handfuls as the two plates are at the same height [and he shows the height with his hands].

In the other school Mattia realizes that the amount of flour hold in each handful is different and says:

Mattia: … but the amount of flour is different, … I mean, it’s more than my Grandma’s handful, yes but my handful is smaller.

Mattia tries to convince his friends of the truthfulness of his statement and says:

Mattia: Let’s take two sheets and let’s put my Grandma’s handful of flour on one sheet and my handful of flour on the other one. Look, it’s more! Look!

Federica: Yes, it’s true, you’re right!

It is possible to infer from what children have said two main aspects of the measurement concept at intuitive level, i.e. comparison and the additive principle between homogeneous quantities. The inquiry method is also reflected in this conversation, as there are a lot of solutions to the same problem and also the desire to support their opinions.

The importance of the linguistic aspects in the relationship between natural language communication and mathematical communication became as clearly evident as did the problem of learning mathematical concepts through body movements (Sfard, 2008; Lakatoff- Nunez, 2005).

When preparing the blended drinks and the pastry cream, the “expert” indicated the necessary quantities of ingredients but the children had to choose the proper instruments to measure the liquids and solids. For example: a big glass indicated a greater quantity of milk than a small glass which the children discovered contained exactly half the amount; a soup spoon rather than a teaspoon was used to put more sugar in a drink; a ladle contained even more than a soup spoon. The practical experience of preparing pastry cream and blended drinks involved the children in a discussion of volume-related units of measurement. With the expert, the children decided which utensils (soup spoons, teaspoons, ladles, big glasses, small glasses) should be used to measure the ingredients.

Mum and cook: Right, let’s see children which utensil is better according to you? Take a look at these utensils (the mum shows the spoon, the teaspoon, the ladle, the glass, the cup, and so on).

Vincenzo: Let’s measure the flour with a ladle because it holds more than a spoon which can be used to measure sugar.

This started a discussion on the quantity of liquid already prepared which, according to Federica, would not be sufficient for everybody once it became cream.

> Federica says: But we haven't got enough cream for everybody!
>
> Mum: But why? How can you say it?

When she was asked how she could be sure of this, she suggested dividing the cream among all the students. Upon verifying that there was only enough cream for half the students, she suggested adding double the amount of ingredients to the mixture. When they finished preparing the cream, they started looking for ladles to pour the cream into glasses and decided to pour four ladles into the big glass and two into the small glass. The children were able to see the change in volume between a glass of a substance before being blended and after. During the preparation of the blended drinks the children first invented and produced the recipes discovering the changes in volume between the quantity in a glass before and after it was blended. They filled a big glass with pieces of fruit, milk, orange juice and sugar but once it was blended the volume increased producing enough liquid to fill a big glass and a little glass. Another interesting aspect emerged during the preparation of the chocolate roll. This product was chosen to study a series of questions related to the concept of length which was dealt with in a natural way by the children during the activity. The ensuing discussion allowed the children to come to a common understanding. Then, the natural desire to eat the chocolate roll led to find a way of dividing the roll in equal parts. The teachers had equipped the classroom with "good" instruments for measuring and the children, looking around the classroom for something to help them measure, were able to identify instruments long enough for this purpose. Next, the children chose a strip of paper as the best tool for measuring and then they developed a way of folding the paper in equal parts. This folded strip was then used to cut the chocolate roll into enough equal parts for all the children. The direct experience of preparing the chocolate roll was planned as a problem-solving activity concerning the concept of length. The children managed to devise and execute a system for dividing the roll in equal parts for everyone thereby learning the concept of multiples. Mattia and Pietro try to compare the lengths of different sheets of paper scattered on the table to the length of the chocolate roll. When he finds one that was just slightly longer than the roll, Federica says, "Let's cut off the extra bit and write "The Length of the Chocolate Roll" so that we know which is the right piece!" Mattia suggests using the strip as if it were a ruler by putting measuring marks on it but the idea proves to be difficult to apply because the marks do not allow for cutting equal pieces. Mattia has another idea. He suggests folding the strip in two but the chocolate roll is still longer and bigger and, if cut in this way, there would only be enough for two children. In fact, Mattia measures the folded paper against the chocolate roll to see if it is exactly half the length and verifies that it is. Then Federica suggests, “Let’s fold the strip in half again” but it still is not enough for everyone. The children continue folding the paper strip until there are enough pieces for everyone. This discovery gave rise to interesting games on the meaning of double and half using other materials.

This way we have made inquiries about the possibility that children can develop the ability to face situations of problem solving and problem posing. Moreover, it is obvious that children have been able to take an indirect measure using the instrument of the semiotic mediation (i.e. the strip of paper) and so they have found a way of making an effective unconventional “metre” (base 2) having submultiples, too (principle of Eudosso – Archimedes).

In the second stage, when they personally prepared the baked goods and had to deal with measurements, the children had the opportunity to experiment with the concepts of weight, length, and volume. During the preparation stage we have observed how children can get the main concepts of the measurement process, even if at an intuitive level. As for the preparation of orecchiette, for example, children have to make a measurement roughly and at the same time more and more accurate, which doesn’t mean that this measurement does not follow a definite plan or pattern. Preparing the cream leads children to make comparisons thus choosing the suitable measuring instruments; and as for the preparation of the chocolate roll it is necessary to use adequate units of measurement.

In the third and final stage, the children organized and operated a barter market where they “exchanged” their goods for school materials on the basis of “price lists” which they had developed and agreed on previously. The expression “Barter-Exchange” was introduced at the beginning of the project, during the preparation of the orecchiette. The teachers bring to the children’s attention that the grandmother had worked hard to make the orecchiette and should be compensated for this. Our goal was to get the children to barter in exchange for school materials to send to the Indian school twinned with ours. This experience led to the organization of the barter market and the development of the “price” list. In establishing the “prices”, it is important to emphasize the process by which the children attributed value to their products. For example, “*if a complete chocolate roll was worth a package of ten exercise books, then how much was one piece of chocolate roll worth?”*. In choosing which products to exchange, it was necessary to use the concept of double. For example, the children agreed among themselves that a small glass of pastry cream was equal in value to exactly half that of a big glass and a big plate of orecchiette was worth double the amount of a small plate. To determine the value of the blended drinks, the children took into account the preparation time and the change in volume and therefore the need to ask for more school materials in exchange.

In order to exhibit the price list to the public, some kind of poster was necessary. The children solved the problem by designing one with drawings of all the instruments used to measure the various products: glasses, espresso cups, different kinds of plates and the short and long strips of paper. During the fair, each child bartered their products with the adults and, at the same time, explained how the products were prepared and, above all, how they arrived at assigning a value ("price") to the products. In this way, it was possible to verify that the child had acquired a full understanding of both the concepts related to measurement and the value attributed to the products. The observations relative to the price of the products are equally interesting. Initially, the children were reluctant to barter because of their personal feelings for the objects they had made, a behaviour that is typical in this age group. This strong personal attachment to the products was further highlighted in the barter stage when there was a request for a "big" piece of cake, for example. It was observed that often the value of an object was closely tied to its size. During the barter market another concept linked to the measurement process was examined. When children had fixed the price of each product the equivalence between different units of measurement, as well as the main concepts of the equivalence, have come out once again in an intuitive way. The drawings made by the children in the price list are evidence of how children have acquired the above-mentioned concepts of measurement. The entire project proved to have embraced all the fields of experience included in the Italian curriculum guidelines, not only the specific one related to mathematics and "knowledge of the world".

CONCLUSION

This experience allowed us to confirm the idea that it is possible to talk about a child's scientific knowledge, as long as we give this sentence the right connotation. To avoid making an "intellectual mistake", we must talk about "correct knowledge". This is what our research in kindergarten is generally devoted to: having the child's first experiences and reasoning follow a "correct" formulation, always respecting the development of the child, who must not be thought of as an "adult". Moreover, what we continue to observe in our research, and what stimulates and supports us, is the children's sincerity when facing different situations, that spiritual condition which prevents them from wanting to distort the observed reality, their capacity to ask questions without feeling judged, as well as their ability to change their mind. These are all typical of children's behaviour (something which adults no longer have) but they are also essential requirements when talking about "scientific nature".

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Developing different aspects of generalization in the first grades

Part 3

MENTAL REPRESENTATIONS OF MATHEMATICAL OBJECTS AND RELATIONS IN THE FIRST GRADES

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The importance of adequate external and internal (mental) representations for mathematical understanding as well as for generalization is shown with examples taken from early mathematics in pre-school education and from primary mathematics in schools. Regarding relations between numbers and special aspects of addition and subtraction, in the main part it is discussed whether or to which extent referring to actions with concrete materials and to children's every-day life experience might be a learning obstacle and not helpful for children's insight and for their ability to generalize mathematical concepts. In addition, alternative ways for classroom practice are discussed.

CONCRETE AND ACTION-ORIENTED THINKING IN PRIMARY SCHOOL MATHEMATICS

It seems to be common sense not only in primary mathematics education that to proceed "from the concrete to the abstract" is the best – or even: the only – option, i.e. to invite learners to carry out actions with concrete objects or to present them situations taken real life in such a way that they can grasp the "intended" mathematical concepts and procedures. In this article it will be discussed which alternatives do exist, and it will be shown that – especially with children who do not belong to the high achievers – in some aspects and situations another way is more promising, namely to focus their attention to the meaning of mathematical symbols and to help them to get insight in the way how mathematics is done by using words and signs.

An explanation for the dominance of the way "from the concrete to the abstract" might be found in Piaget's work and his – in fact – very important findings and ideas that might be summarized by his terms "abstraction à partir de l'action" or "abstraction réflèchissante" (see, cf., Aebli, 1980, p. 217). It should be remarked that in this abstraction process Piaget focusses on the *reflection* of actions, and not on the actions or on the real objects used to carry out these action. One of the most convincing ideas how to promote students' learning processes in using real situations was created by researchers of the Freudenthal Institute when they established the concept "realistisch rekenonderwijs" (realistic mathematics; see, e.g., Treffers, 1987, or van den Brink, 1989, who, however, also referred to the van Hiele levels). The introduction of mathematical concepts based on situations the students are familiar with from every-day life might also be founded on

Greeno's situated perspective on cognition and learning and his discussion of generative knowledge (see, e.g., Greeno, 1989, Stern, 1998, or Caluori, 2004, pp. 86ff).

There is no doubt that for a lot of learners and for a lot of mathematical concepts it is very useful and important to start with situations in which mathematical concepts and procedure can be applied. Nevertheless, it should also be taken into account that this method also includes risks, especially regarding children who have problems with the abstraction process which has to be passed inevitably to grasp the intended concepts and procedures as useful and universal *mental* tools. In their study on the numerical concepts with primary school children, for example Gray, Pitta and Tall underlined: "It is our contention that different perceptions of these objects, whether mental or physical, are the heart of different cognitive styles that lead to success and failure in elementary arithmetic" (1997, p. 117). Rowlands and Carson put it even more strongly from their "critical review of ethnomathematics" (2002, p. 98): "Independent of good intentions, ethnomathematics runs the risk of attempting to equalise everything down to the poverty of the 'builders and well-diggers and shack-raisers in the slums'.

In short: Action oriented mathematical thinking might be sufficient in many aspects of primary school, but it is not in higher grades.

EMPIRICAL FINDINGS FROM PRE-SCHOOL AND PRIMARY MATHEMATICS

Mathematical thinking obviously does not start with formal instructions in school, and therefore Early Mathematics has become an important field in research. Regarding pre-school and early primary school education, the last 15 years in the focus of our research there were four fields:

1. The development of the "Osnabrücker Test zur Zahlbegriffsentwicklung" (Early Numeracy Test), a diagnostic tool for children aged 4 ½ to 7, based on the Utrecht Getalbegrieps Toets" (van Luit, van de Rijt & Pennings, 1994; van Luit, van de Rijt & Hasemann, 2000; van Luit, van de Rijt & Hasemann, 2001).
2. Interviews on individual differences in mathematical thinking of children before and at the very beginning of formal instructions in school (Hasemann, 2006; see also Hasemann, 2007).
3. The relation between early structure sense and mathematical development in primary school (Lüken, 2010, 2011, 2012).
4. Work with mathematically gifted children aged 5 to 8 (Hasemann, Leonhardt & Szambien, 2006; Hasemann, 2007).

In addition, we will discuss some findings from to a teaching experiment in grade 2 and from interviews in grade 3 on "word problems and mathematical understanding" which were carried out together with E. Stern (Hasemann & Stern, 2002, Hasemann, 2005).

Findings from pre-school education

About 70 children in their last year of kindergarten in interviews the item in figure 1 was presented. Nearly all the children could to solve the problem, but the time needed to complete the task was extremely different: Some counted all the dots and needed minutes to find the correct square; others were ready in seconds as they had realized immediately that in this square there are six dots (arranged like those on a die) plus one dot, or they saw two times three dots and one dot (for more details and further items see Hasemann, 2006, pp. 74f; 2007, pp. 34ff).

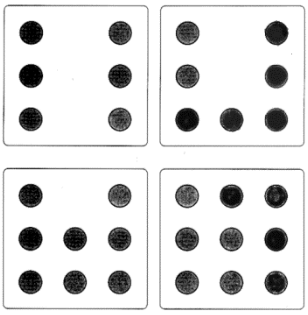

Figure 1: Point to the square with seven dots.

Even kids in kindergarten show extremely big differences in their kind of thinking: Some recognize visually presented pattern and structures and are able to use them flexibly to solve mathematical problems, others have to their disposal only counting procedures they are familiar with. Referring to Linchevski & Livneh (1999) and Mulligan and Mitchelmore (2009) Lüken (2010, p 241) called this ability to recognize pattern and structures "early structure sense", and she indicates with this term the "ability to see any predictable regularity or ordered entity and the relationships between parts in such a pattern". In a longitudinal study Lüken found out that there is a correlation between children's ability to recognize visual pattern and structure at the very beginning of school and their mathematical competences at the end of grade 2 (2010, p. 246): Children who have this structure sense already at the beginning of school are very likely to be the higher achievers at the end of grade 2, and vice versa, those how have no such sense tend to be the lower achievers.

In addition, Lüken discussed the question what the cognitive milestones in the development of an early structure sense are (2011, p. 2). From an analysis of video-taped interviews with children just starting school she concluded that the lower achievers, for example, do interpret a pattern of dots that are arranged as the die-five as *one* number (namely "*the* five") whereas the high achievers are able to interpret this pattern *in addition* as a partition of this number (4+1 or 2+2+1): "High achievers have an awareness of the spatial structure and function of particular configurations" (Lüken, 2011, p. 5). It follows that "a learner has to organize the perception of things in a particular, mathematical way, for instance learn to relate geometric clues to numerical matters, … flexibly decompose and related substructures" and "intentionally reframe the structures of a pattern". Most learners cannot do this process by themselves, "they have to be instructionally supported with" (p. 7).

Observations and findings in the first grades of school

The question is how to support the learners. In a teaching experiment in grade 2, Hasemann & Stern (2002) tried to find out which arrangements in the classroom might be more likely to support weaker students' ability to grasp numerical relationships (for details see the next section). As a starting point interviews on word problems were conducted at the beginning of grade 3. The following transcript is taken from an interview with an eight-year-old girl who was asked to solve this problem:

> Jan has got 7 rabbits. He has got 4 rabbits more than Thomas. How many rabbits do both boys have together?

I: Please, read the text.

S: (reads the text) … 7 + 4.

I: How did you do that?

S: Because there is 'how many rabbits do both boys have together' … 7 + 4 equals 11.

I: Why is it 7 + 4? ... (16 sec) ... How many rabbits has Jan got?

S: ... 7.

I: And how many has Thomas got?

S: 4.

I: 4? (The girl nods her head). Where is that in the text?

S Points to the text.

I: Please, read the text.

S: He has got 4 rabbits more than Thomas.

I: ... Who is 'he'?

S: Jan.

I: Fine. This means, Jan has got 4 rabbits more than Thomas ... and how many rabbits has got Thomas?

S: 4.

I: 4?

S: Nods.

This kind of dealing with the numbers in a word problem is widespread, and it might be interpreted in different ways: For example, it might be concluded that problems like this were ignored in the class yet; or the girl didn't pay enough attention or has bad understanding of this special kind of problems; or she doesn't like mathematics at all. Even if these conclusions were more or less correct (this girl was seen by her teacher as a rather bright learner in language, but not so good in mathematics), they do not reach to the heart of the matter.

Riley and Greeno (1988, see also Hasemann, 2007, pp. 196f) in a study with children from kindergarten to grade 3 found 14 types of word problems with extremely different levels of difficulty. Most items of the "compare" type (as for instance: "Mary has got 4 marbles. She has got 3 marbles more than John. How many marbles has John got?") are rather difficult for younger children. The level of difficulty of an item, above all, depends on the difficulty children have to transfer the real situation given in the word problem into the mathematical language, i.e. it depends on the fact how easy or how difficult it is to represent the situation in mind, to connect this situation with available knowledge, and to deduce adequate calculations.

Following the path "abstraction from realistic situations" sequences of symbols like "4 = □ + 3" or "□ + 4 = 7" only make sense for children if they have learnt to connect these sequences with different situations in such a way that they can transfer it also to new situations. A step in this mental process from situations to sequences of symbols (and back from symbols to situations) might be diagrams if they do not just reproduce the situations but represent the relevant mathematical relations without irrelevant details. As an example we take the task

There are 9 children on the red bus. There are 6 children more on the green bus than in the red bus. How many children are on the green bus?

The pictures in figure 2 were drawn by a student in grade 3 who reproduced the situation with a lot of (irrelevant) details whereas the diagram in figure 3 (which is taken from the work in the classroom [see the next section]) represents quantities and the relevant relationship between these quantities:

Figure 2: The red bus and the green bus

Figure 3: The red bus and the green bus

Action-related thinking becomes inadequate when the situations cannot directly be simulated by actions. In fact, a word problem becomes nearly insoluble for a lot of children when a relation between quantities has to be recognised; the girl in the interview mentioned above had this problem: She ignored the relation in the relevant statement (“he has got 4 rabbits *more than* Thomas”), but referred to a cardinal number (in the sense of “he has got 4 rabbits”) and did an obvious calculation (7+4 = 11).

Most lower achievers in mathematics are not able to detach their thinking from concrete objects and real actions: “The properties by which the physical objects are described and classified need to be ignored; and attention is focused on the actions on the objects which have the potential to create an ’object of the mind’, which has new properties associated with new classifications and new relationships. For some there may be a cognitive shift from concrete to abstract in which the concept of number becomes conceived as a construct that can be manipulated in the mind. For others, however, meaning remains at an enactive level; elementary arithmetic remains a matter of performing or representing actions” (Gray, Pitta and Tall, 1997, pp. 115). These authors’ evidence is based on responses to a range of elementary context-free addition and subtraction problems given by children at ages from 7 to 11: “’Low achievers’ tended to highlight the descriptive qualities of the items in strongly personalised terms, ... there was a tendency to associate these items with a story in the sense that they were seen as pictures that required colour, detail and a realistic content. In contrast, ’high achievers’ concentrated on the more abstract qualities within (the) series of items. Though they initially focused on core concepts, they could traverse at will a hierarchical network of knowledge from which they abstracted these notions or representational features” (p. 123).

The next examples are taken from our work with mathematically gifted children, aged 5 to 8. Confronted with the item

> To finish a special work 4 machines need 25 days. Unfortunately, after 7 days one machine breaks down and the work is finished with only 3 machines. How many days the work is delayed?

a boy produced as an answer the diagram in figure 4 (the reader is hearty invited to find out why the boy – rightly – regarded it as a solution of the problem):

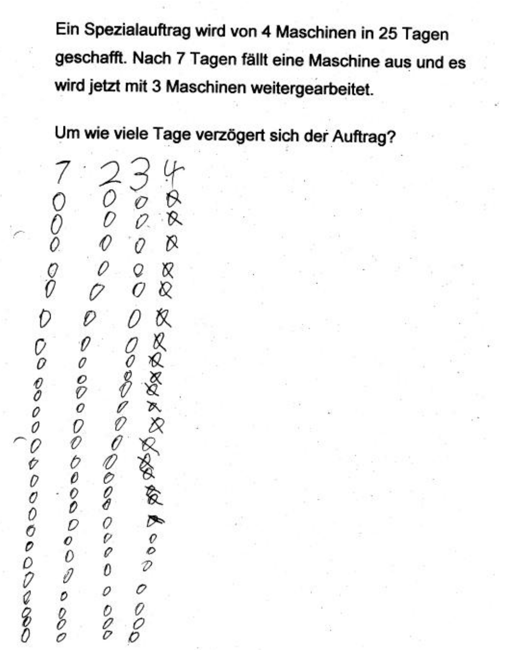

Figure 4

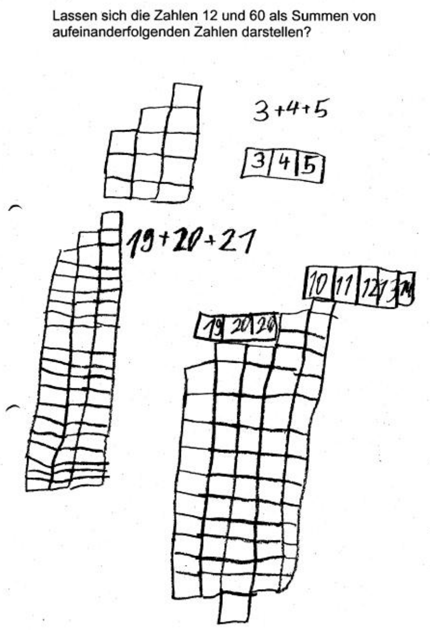

Figure 5

The diagrams in figure 5 were produced by the same boy some days later to this item

Is it possible to write 12 and 60, resp., as sums of consecutive numbers?

These diagrams, especially that one which is related to 60, highlight to the role and the importance of external and internal representation in the process of generalization.

A teaching experiment in grade 2

Having in mind the behaviour of students as presented in the interview in the previous section, Hasemann & Stern carried out a 12 lessons intervention study on word problems in nine classes at the end of grade 2 in the Hannover area. At that time word problems were well-known to the children. Two different additional training programmes for the solution of word problems were developed, each of which was tested in three classes; in addition, there were three control classes. One of the programmes focussed on students' real-life action-related behaviour. In this programme the teachers' instructions followed the scheme "from the concrete to the abstract" (and were guided by the idea of "ongoing schematisation" developed in the Utrecht project mentioned above) while the other programme was based on abstract and symbolic activities.

The "abstract-symbolic" training-programme was conducted in three classes. The mathematical relations and structures that are particular difficult for children were made explicit in these lessons, and specific help to overcome the obstacles were provided. This programme wasn't "abstract" in the sense that just formal calculations were carried out, instead this programme was also action-related and included a lot of "games" appropriate for children at grade 2.

However, as media to visualise relationships between numbers mathematically structured representation tools were used as, for instance, the 100 square and the number line (figure 6 and 7).

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

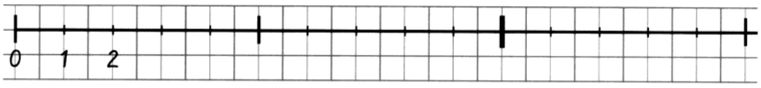

Figure 6: 100 square Figure 7: The beginning of the number line

Exercises with the 100 square: The children sat in a semicircle in front of a big 100 square-poster and followed a route on it given by the teacher:

1. At the beginning I stood on the 7.
2. Then I walked a step downwards.
3. After that I walked 26 steps forwards.
4. Then I walked 3 steps upwards and one to the left.
5. Where I am?

After some “Where I am”-games the students were encouraged to follow the route with blindfolded eyes.

At the number line a game called “Mister X” was played: An empty number line was drawn on the board and the teacher (or a student) wrote a number (“Mister X”) between 0 and 100 on the back of the board. The players tried to guess this number by narrowing down the numeral range, it was only announced whether the number was too small or too big; the players had 10 attempts at most.

The children also played “brain-games” like: “I imagine two numbers. One is bigger by 5 than the other. Which numbers could it be?” In the training in these classes mainly relations between numbers were emphasised, and then more and more used by the students to solve word problems.

Before and after the training-programme a test was carried out in the classes taking part in one of the two training programmes mentioned above and also in the control classes. During the evaluation a considerable improvement was becoming obvious especially with the low-achieving children, not only in the correct solution of word problems but also regarding their ability to solve arithmetical problems. An increase of efficiency was to be expected because there is always a correlation between time of lessons and progress in learning. The main surprise was however that the programme which focussed on pupils’ real-life action-related behaviour had the lowest success, while the “abstract-symbolic" programme achieved the most increase of efficiency with the low-achieving children (Hasemann & Stern, 2002, pp. 235ff; Hasemann, 2005).

FACIT

This finding is not really surprising. It's even plausible that especially the less competent children are best aided by helping them to recognise relations, patterns and structures which they – in contrast to the more competent children – are not able to find by themselves in the concrete and obvious. This recognition evidently stands in contradiction to a popular way of acting (cf., e.g., Gellert, 1999, pp. 114/131); most of the teachers seem to believe that especially with the less competent children the only way of acting is "from the concrete to the abstract", or – the worst – the only way of teaching is to come down just to the obvious and concrete.

The difficulties of numerous children with mathematics, not only in primary but also in secondary schools, are partly due to the use of numbers exclusively as cardinal numbers (quantities) and in rather simple arithmetic. This leads to a restricted mathematical understanding and makes generalization difficult (or even impossible). In the first grades it is possible to solve most arithmetical tasks only by the conception of concrete actions. This thinking is insufficient in higher classes (and – among other problems – leads to the well-known difficulties with fractional arithmetic), children should learn to shape relations between numbers already in primary school. In addition, the procedure "from concrete to abstract" is not sufficient enough to help low-achievers to detach themselves from the concrete and obvious and to recognise the relations and structures in the actual situation. As a matter of course it is necessary in mathematical lessons to start with concrete actions and a practical context which is directly comprehensible for children; however, it is important to go carefully directed (and not only implicit) into relations and structures. If they are not misunderstood as counting-tools, materials like the 100 square and the number line (with their pre-forms abacus, 20 number grid and calculation chain) are excellent fields of experience and practise especially for less competent children to create mental models of situations where mathematical relations are represented. The study showed that it is possible to encourage low-achieving primary school children through carefully directed abstract-symbolic activities to insights in mathematical relations. Materials for instruction and methodological suggestions for such lessons are available for a long time past.

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ON EVOKING CREATIVE MATHEMATICAL ACTIVITIES RELATING TO GENERALIZATION AND SPECIFICATION IN EARLY GRADE PUPILS

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This study regards issues relating to evoking and developing creative mathematical activities in early grade pupils. Both theoretical observations will be expressed, regarding working with pupils who are interested in mathematics and concepts of working with such pupils will be discussed. The main part of the study comprises presentation and analysis of children's works created during after-school meetings of the Young Mathematician's Club. Diversity of the task-solving strategies applied by pupils will be shown, together with those activities which may become the basis for shaping generalisation and specification skills.

INTRODUCTION

The studies run in Poland on mathematical skills in small children show that

> one may see some signs of mathematical skills in nursery school pupils and early primary school pupils, and the number of children gifted with those skills is impressive. (Gruszczyk - Kolczyńska, 2011a)

It stems from the studies that such children are willing

> to participate in games requiring a considerable intellectual effort and combinatorial reasoning (…). At the same time, they demonstrate astonishing cognitive inquisitiveness (…). They are also able to focus for a longer period of time on complex tasks; what is more, they find them on their own, thus manifesting astounding inventiveness. (Gruszczyk - Kolczyńska, 2011a)

Therefore,

> the necessity of supporting mathematically talented children already at the level of nursery schools and in the first grades of school education is emphasised. (Gruszczyk -Kolczyńska, 2011b)

First years of school education are significant for further education. This is when the child develops a conviction about his/her abilities, which is so motivating. In this period of early education, skills of different kinds of reasoning are shaped, and we know “that assistance in creating opportunities for developing thinking is a much more important investment in child's cognitive development than lots of knowledge”. In such shaping of thinking one should use a natural child's inclination to games and create opportunities of experiencing success. Success and joy of action affect intellectual development positively, “and by achieving success we frequently want to repeat it and enjoy an emotional feeling that despite possible failure we constantly win" (Chmielewska - Łuczak D, 2011).

A natural children's inclination to intellectual effort gives a teacher a possibility to develop children's skills. Those pupils who like mathematical tasks should be surrounded with special educational care. Currently, as emphasised by E. Gruszczyk - Kolczyńska (2011b), "within the scope of pre-school and early-school education there are no classes preparing for supporting development of talented children, including those with mathematical skills". The author believes it is necessary to create an additional educational path. That path aimed at developing mathematical activities and skills could include contents and skills to be taught both during lessons and after-school classes.

YOUNG MATHEMATICIAN'S CLUB

One of possible concepts of after-school classes, which may support intellectual development of early school pupils in the area of mathematical creativity, has been presented in a manual for early - school teachers titled *Young Mathematician's Club*[1]. The exercises presented in it are addressed to pupils who are interested in numbers, geometric world, mathematical relations and who enjoy creation. The range of topics of a series of the *Club* meetings is loosely connected with the curriculum of the first stage of education and refers to the situations well known to children; the topics cover selected mathematical activities the beginnings of which can be shaped in pupils who are willing and interested in mathematics. Games, exercises and tasks are arranged in such a way that pupils have a number of opportunities to do manipulation exercises, repeat them and discover their own strategies of conducting and solving mathematical problems. The manual includes, apart from presenting a series of classes, *Characteristics of classes* and *Comments to tasks* with detailed tips and suggestions for a teacher.

All classes presented in the manual have been conducted after school with third-grade pupils from primary school[2]. In each *Club* meeting, the pupils solved exercises and tasks, which constituted a thematic series. In accordance with the concept presented in the manual, the pattern of each series of classes was the same and it consisted of three stages: a *Starter, Manipulation Classes* and a *Work Sheet.* The *Starter* introduced the pupil into a situational context. At that stage, the teacher agreed with the pupils the language of communication and understanding of the meaning of the proposed manipulation material and graphic presentations. *Manipulation Classes* were a form of playing games for children. The pupils could experiment and discuss their ideas how to solve particular tasks

[1] Rożek B., Urbańska E.; *Klubik Młodego Matematyka*. A manual developed within the frameworks of the project titled: *Development and implementation of a complex system of work with talented pupils,* co-financed by the European Union under the European Social Fund

[2] The classes within the *Club's* frameworks were conducted by a teacher representing the first stage of education, B. Jachymczak, M.A., working in the Public Primary School run by Salesian Sisters in Kraków

and exercises. At the third stage of classes, the pupils solved tasks included in the *Work Sheet* on their own. Those tasks referred to the same mathematical activities as in the *Starter* and *Manipulation Exercises,* but they often had a different real context.

The manual describes pupils' works, enriched with the scans of authentic children's solutions. It constitutes an illustration of different pupils' approaches to tasks and shows children's creative skills.

STRATEGY AS A GENERALISATION TOOL

The issues relating to the task solution process are central to many psychologists and pedagogues. The word *strategy,* borrowed from other fields of science is often used to describe that process. M. Ciosek (2005) writes about it in her monograph and she quotes different definitions used to describe that word, coming from various psychologists, for instance:

> strategy is a regularity in taking action;
>
> strategy is a certain systematic way of solving problems;
>
> strategy is a certain detailed plan of action.

The application of the term *strategy* in the context of solving mathematical tasks emphasises that aspect of task solving which is related to planning and consistent implementation of that plan. Yet, it is worth noting that in order to be able to build a strategy one needs to become clearly aware of what a given task is about and find significant relations between data. Next, it is good to analyse a given situation in several particular cases in order to notice some common features, which will allow for discovering a general principle. Such analysis of particular cases can be done randomly, but it can be also conducted systematically in order to make generalisation easier. Thus, we have to do with specification and generalisation processes, which play a crucial role in discovering a strategy and help the pupil to achieve success in solving mathematical tasks. In A. Z. Krygowska's (1977) opinion, the analysis of several special cases and looking for a common pattern for them form elements of an inductive generalisation process. Its further stages constitute verbal creation of a common idea, its expression in a symbolic language, and last but not least, checking whether the generalisation we have achieved is proper. It seems important that while teaching mathematics teachers should shape in children those important creative activities, namely specification and generalisation. It is also good to realise that the

> generalisation process occurs individually. The pupil "grows into” the generalisation process depending on his/her psycho-physical development and mathematical experience. Here is a great role of a teacher to arrange such situations for the pupil at the right time, not to impose the final effect and to support the pupil discretely in looking for generalisation. (Legutko, 2011)

PUPILS' WORKS ILLUSTRATING APPLICATION OF DIFFERENT STRATEGIES

Below, we present selected pupils' solutions of tasks included in the *Work Sheets* and we analyse them in terms of activities which may be the basis for shaping generalisation and specification skills. Diversity of the applied strategies and consistency in their application is striking in children's solutions of the tasks. Pupils' inventiveness both in geometrical and arithmetical tasks is illustrated with examples of solutions of the selected tasks.

Task 1

> Jacek has blocks in 4 colours: pink, green, blue and yellow. He builds towers by putting blocks one on top of the other. The tower is built from 3 blocks, and each block has a different colour. Draw as many Jacek's towers as possible.

In the solution of the task, the pupils tried to create as many towers meeting the specified criteria as possible. They did not have to draw all twenty four towers. Thus, some of them draw only a few towers. We can say that they specified task criteria in a random way. One pupil, for instance, did the following:

Most probably he made towers randomly using three blocks with the specified colours, and he focused on fulfilling the task conditions while making subsequent towers; therefore, his towers had different colours and each of them was different from the previous ones.

Interestingly, one pupil fulfilled a partial strategy of making towers:

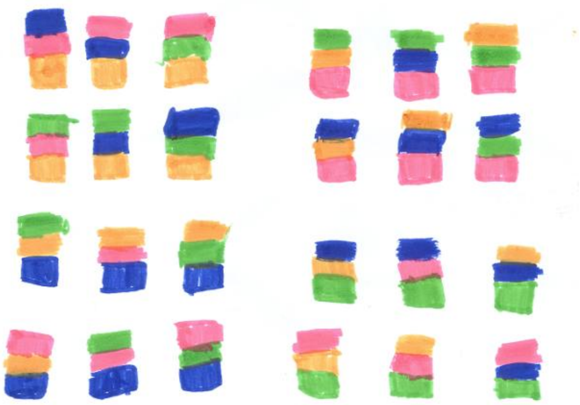
It can be seen that a protagonist of the strategy here is a yellow block, changing its position. One by one, the pupil draws possible towers in which this one is at the bottom, later on in the middle and finally on top. In this way, 18 towers have been built. Creation of another six towers would require supplementing of the applied strategy with another step: towers without a yellow block.

There were also pupils who, while applying their strategies in a systematic way, created all possible towers. Such a way of selecting examples points to the way of reasoning, which may constitute the basis for creating generalisations.

One of the pupils applied successfully a strategy consisting in building, one by one, possible towers, which start with a block of the same colour:

She started drawing with six possible towers, in which the first block is yellow. After setting the first block, she was building possible towers where the second block was the same. Next, she was drawing possible towers changing the colour of the initial block. It is worth noting that those six towers in each group were built by applying the same strategy once again. After deciding, for instance, that the first block was yellow, possible towers were now created in which the second block was green, and later on such towers in which the second was one blue and finally all towers in which the second block was pink.

Similarly, starting with a block of the same colour the pupil below created her towers:



The original arrangement of towers in a regular system of rows and columns should be noted here, which could help in generating all six elements of a given group.

Task 2

Kasia arranged with cards all two-digit numbers, where the sum of digits equals 9. Try to write down as many such numbers as possible. What do you think, have you managed to find all Kasia's numbers?

The question at the end of the task was to encourage the pupils to try to write down all possible numbers meeting the task criteria.

Some of the pupils gave randomly only several numbers meeting the criteria, such as the pupil below:

54 , 63 , 81, 09, 72

There were also such pupils who listed all numbers in this random way:

18, 54, 27, 45, 81, 63, 72, 90, 36,

There were also pupils who applied a partial strategy here. For instance, they gave some two-digit numbers with the sum of digits equal 9, and next a number with transposed digits. Subsequent pairs were totally random; it was enough that the sum of digits of the written number meets the task conditions. They continued to do so until they were unable to think of a new number meeting the requirements. For example, the pupil below acted like that:

36, 63, 72, 27, 81 18, 54, 45, 92

As can be seen, while applying his strategy he "lost" only number 90. After number 45, the pupil tried to write down some example, which may confirm the fact that he was not sure to have listed all cases.

Another strategy applied here was to build numbers by putting tens of subsequent digits as digits and adding a matching single digit. Some pupils did it correctly:

18, 27, 36, 45. 54, 63, 72, 81, 90

Others had problems with number 90 here:

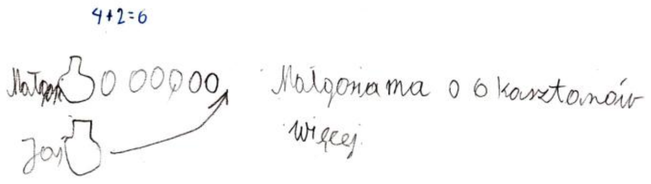
18, 27, 36, 45, 54, 63, 72, 81, 0

Task 3

> Jaś and Małgosia received sweets. They counted them and it turned out that Małgosia had 4 sweets more than Jaś. Jaś gave Małgosia 2 sweets. Who has more sweets now and how many more?

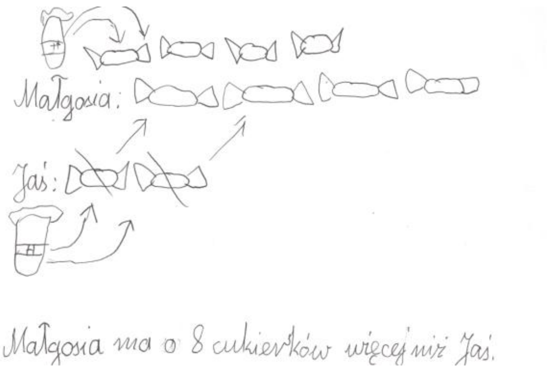
That task required that pupils compared the size of sets, although the number of elements in each set was unknown. The pupils had to conclude from the fact that Jaś gave Małgosia 2 sweets that the difference increased by 4, so Małgosia has now 8 sweets more.

Some pupils made typical errors in their solutions:



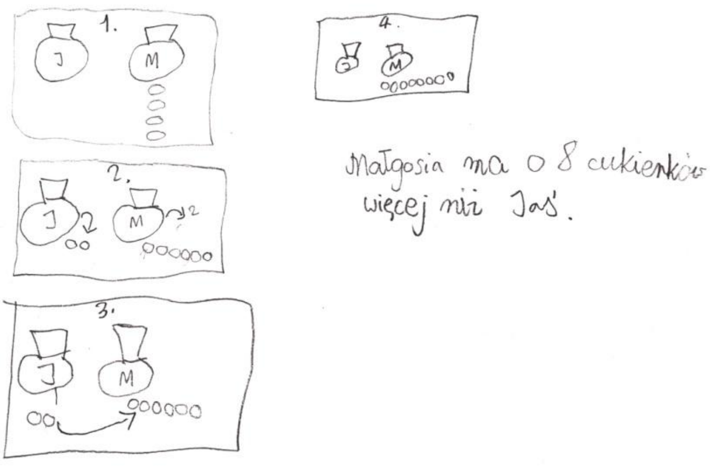
As can be seen, from the fact that Małgosia received two sweets, and already had 4 sweets more than Jaś, the pupil concluded erroneously that Małgosia will have 6 sweets more.

Some pupils presented the situation given in the task in the form of an activity. Such a strategy of the solution may constitute the basis of a general point of view and discovering the way of solving such kind of tasks. For instance, one of the pupils did the following drawing:



He presented the initial situation: two bags with sweets and 4 additional sweets next to Małgosia. Later on, by means of arrows he showed that he took out 2 sweets from each bag and those from Jaś he deleted and drew next to Małgosia's bag.

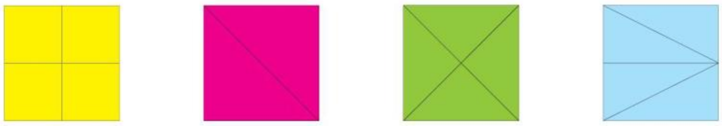
Another pupil presents her reasoning by drawing as if shots from a film:



First, we can see the illustration of the initial situation (1), then two sweets are taken out from each bag (2), afterwards we can see the activity of giving the two sweets (3) and then presentation of the final situation (4) in which the answer can be read.

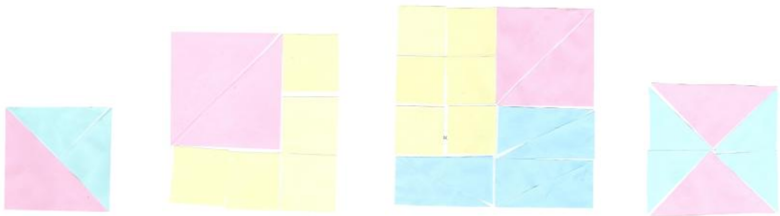
Task 4

Each square was cut in a different way:



Arrange a colourful square from the created elements, being puzzles.

The strategy of solving that task requires finding equalities of adjacent sections and interdependencies between angles of the figures. The pupils presented the squares which were created as a result of manipulation with concrete material. Equality of sides of the puzzles from which squares were built was evaluated by the pupils visually. Frequently, such evaluation, despite the fact that the resulting square looked a bit ”crooked,” was accurate from the mathematical point of view. It can be seen, for instance, in the following solutions:



While looking for a task solution empirically, specific difficulty appeared with applying of the strategy regarding comparing the lengths of sections. That difficulty related to visual comparisons. In some solutions, the quadrangle built visually resembled a square, but its sides were not equal, so contrary to the pupil's intentions and conviction it was not a square. The pupils fell in a trap of putting sections, which differed in length only slightly, one next to another as equal. It can be seen below:

The upper side of the quadrangle built by the pupil is one and a half times longer than the diagonal of the initial square, since three legs of the green triangle are adjacent to it. The lower side of that figure is twice as long as the side of the initial square, because it was built from four yellow squares. Thus, the figure built in this way cannot be a square, since the upper and the lower side are not of the same length, although visually they may look equal. That apparent equality of sections stems from the size of the square from which the puzzles were cut. The initial side of the square was 8 cm long, so its diagonal was $8\sqrt{2}$cm long. Thus the pupil built a quadrangle whose lower side was 16 cm long, whereas the upper side was less than 17cm long. The difference between lengths of those sections on the puzzle could be unnoticeable for the pupil. In order to become convinced that the sections created in this way are not equal, at this stage of education one cannot refer to relations between numbers. An accurate application of the strategy relating to the equality of sections requires advanced knowledge about disproportionate sections. However, while making do with the visual evaluation, it is worthwhile showing apparent equality of the sections to the pupils. One may present such erroneous pupils' arrangements using puzzles created on the basis of a bigger square. For instance, if we make puzzles from a square with a 20 cm long side, the difference between the lengths of sections under discussion will be over 2 cm and it will be clearly noticeable for each pupil.

GENERALISATION OF THE TASK SOLVING METHOD

In children's works, one could see their fascination with a task solving strategy discovered by them. It could be observed in applying it carefully to solving different tasks, frequently with different topics. Undoubtedly, one can notice here manifestation of a generalisation method. For instance, it can be seen when solving tasks for which a mathematical model is similar. For example, it regards a series of combinatorial tasks below.

Task 5

> An ice-cream vendor sells chocolate, strawberry, blueberry and vanilla ice cream. Jacek wants to buy 3 scoops of ice cream with different flavours. Mark chocolate ice cream with a brown colour, strawberry with red, blueberry with purple and vanilla with yellow. Draw as many different ice creams that Jacek can buy as possible.

Task 6

> Zosia has round biscuits with jelly, each with some sauce. Sauces are in five flavours and colours: brown, purple, red, green and yellow. She decided to put two biscuits on each plate in such a way that each cake is of a different colour. Draw as many biscuit sets as Zosia may arrange.

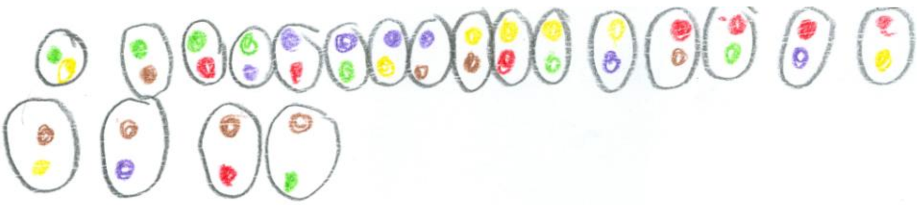
Task 7

> Wojtek has sweets in five flavours: chocolate (brown), raspberry (red), blueberry (purple), gooseberry (green) and lemon (yellow). He puts three sweets of each different flavour into one bag. Draw as many bags as Wojtek may prepare.

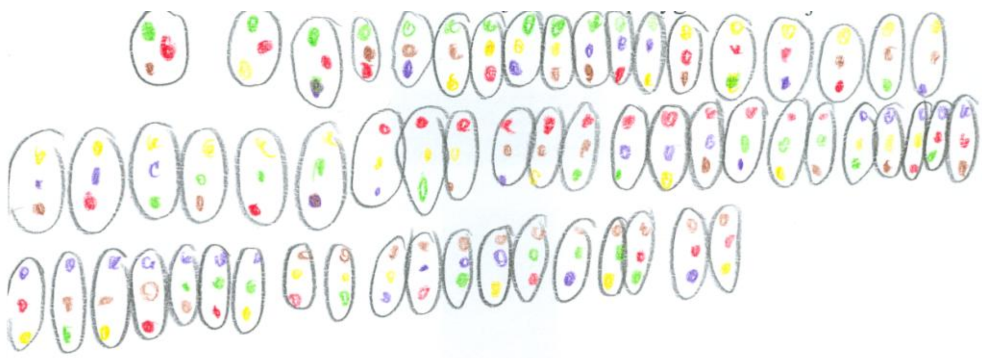
Most of the pupils, in accordance with the intention of the task authors, came to conclusion that the order of occurrence of particular elements is not important in the groups being created. Taking such interpretation into account, in Task 5 one can create only four different ice cream portions, in Task 6 ten plates with biscuits and in Task 7 also ten bags with sweets. However, for some pupils it was difficult to disregard, especially in the solution of Task 5, the condition which is important for them in a real situation. For those pupils, the order of putting scoops in the portions being created was important.

It is worthwhile analysing solutions of the three above tasks given by one of the pupils, who transferred the method of solving Task 5 he discovered to the other ones. In Task 5, he considered the order of putting scoops necessary and based on this interpretation he created all possible 24 ice cream sets:

As can be seen, he used the following strategy consistently: *I am making all possible ice cream portions one by one; they start with a scoop of the agreed flavour, and later I change the first scoop*. When solving subsequent tasks, perhaps fascinated with the regularity of the discovered method, the pupil transferred the interpretation: "the order of the elements is important”, to the second and third task. He did not pay attention to the fact that in a real situation of making portions of biscuits or bags with sweets the order of elements is not significant. When using his strategy, he received all possible 20 plates with biscuits:



and as many as 60 bags with sweets:



His solution of Task 7 is impressive. The pupil was able to apply properly and consistently until the very end the strategy invented to solve Task 5 in order to create such sets of sweets in which the order is significant. He created all possible sets in which the first element is a sweet of the agreed colour. At the beginning, he drew twelve different bags, in which the first one is a green sweet. Next, he drew other possible bags, changing the colour of the first sweet into yellow, red, purple, and finally brown. In this way, he received five groups with twelve bags in each.

FINAL OBSERVATIONS

The third-grade pupils, where the *Club* classes were tested, participated in them with great pleasure. They were happy to do manipulation classes, solved all the tasks and frequently designed such tasks on their own. It should be emphasised that by solving tasks included in *Work Sheets* they focused on finding solutions of even quite complex tasks, which constituted an intellectual challenge for them. Looking at the works description, it can be seen that task solutions were original and clever. This illustrates a thesis which is important for the development of thinking, namely that it is not good to impose on children one's own ways of task interpretation or task solving methods too early. Such children's "different views” on the same reality, if properly developed, lead to independent reasoning and action, and they support the development of creative mathematical activities of a small pupil.

The tasks and exercises offered in the manual turned out to be available to all pupils taking part in the *Club* classes. They stimulated pupils to look for solutions adjusted to the abilities of each of them and to create their own strategies. Most of them were able to give examples of objects meeting the task criteria. Some selected examples randomly, which often did not allow them to obtain all possible solutions. However, we can say that those pupils performed specification of task conditions. This is an important stage of development of children's mathematical reasoning. The next stage will be related to the ability to perceive some regularity in the examples being created.

Many pupils discovered partial strategies of task solutions. They perceived certain regularities and presented, frequently in accordance with the instruction to a given task, as many cases meeting the specified conditions as possible. Although it did not generate all possible solutions, pupils' thinking was clearly directed towards looking for some regularity.

It was striking that several participants of the *Club* classes were able to find all possible solutions meeting task requirements, even if the task contained only a suggestion to find as many objects meeting the specified conditions. Most frequently, all those possible examples meeting task criteria were given by those pupils who had discovered and applied consistently their solution strategies, which were often remarkable. One may draw the conclusion that pupils' experiences acquired in building their own strategies and effective application of those strategies should influence handling of mathematical tasks by them later on.

It is worth adding that the pupils who applied their strategies surprised with carefulness and consistency, when creating their solutions in a systematic way. Such regularity of example selection allowed them to notice general patterns. Finding regularities is an important feature of mathematical thinking. It manifests a creative mathematical activity, which may become the basis of the generalisation process.

In further development of creative activities relating to the generalisation process it will be important for pupils to be able to express, both verbally and symbolically, general regularities observed by them. At another stage of developing those activities, it will be important to remember that the formulated generalisations are certain hypotheses which need to be further studied and supported with evidence.

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TEACHER'S BEST PRACTICE FOR THEORETICAL THINKING – THE CASE OF COMMUTATIVITY[3]

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We present the results of a long-term teaching practice aimed at favouring theoretical thinking in primary school pupils. Our research question is to assess whether this practice achieves pupils' good results in this way of thinking. We focus on the operations commutative property as a detector of forms of generalization. The minutes of a 4th graders arithmetic activity show that teacher's methodology gave children the opportunity to express their different points of view about generalization.

THEORETICAL FRAMEWORK

Literature about generalization offers several ways of interpreting this mental activity. The generalization can be considered differently depending on the features of the mathematical topics. Here we consider the realm of arithmetic for primary school, with its peculiarity. We consider generalization a product of theoretical thinking, owing this position to Douek (2006, p. 823).

[a] The same researcher presents the following remark:

> A change of semiotic system […] can be a means for the considered processes [cognitive processes towards generalization] to take place, as well as a sign that they are taking place. (Douek, 2006, p. 824)

[b] We can find (Moss & Beatty, 2006; Geraniou *et al.*, 2010) that generalization is produced by the individuation of a pattern. The 'exploration' in arithmetic occurring in primary school can give rise to some children's intuitions about generalization, via patterns. The appearance of a pattern from a repeated investigation can be in itself a sort of generalization, but the fact is clearer when variables are used instead of specific numbers. This is a first step in the origin of algebra, but we can delineate other ways towards generalization.

[c] Generalization has an important cognitive role for economizing information needed in the construction of knowledge. But for being useful it is necessary that generalization it is paired with competence of implementing the generalized statement in particular cases, together with the identification that this peculiar situation is deduced from the general statement. Therefore in the phase of constructing/recognizing a generalization pupils are faced with direct (i.e. from

[3] Work done in the sphere of Italian National Research Project Prin 2008PBBWNT at the Local Research Unit into Mathematics Education, Parma University, Italy.

general to particular) and inverse problems (i.e. from particular to general) (Marchini, 2002).

[d] The 'natural' and historical evolution from arithmetic to algebra presented the passage from rhetorical (syncopated) treatment to the symbolic one. This is mainly due to Viète (1591) with his transition from numbers to 'species'. Some researchers consider the introduction of symbols in algebra and the consequent formalisation of thoughts as cornerstone for the presence of generalization.

> For example, the generalization of a rule or procedure that would hardly be understood through a single listing of numerical cases can be expressed in a literal code. (Malara & Iaderosa, 1999, p. 167)

[e] But Rogers (2002, p. 578-579) distinguishes between icons and symbols in algebraic thinking:

> […] archaeological evidence suggests that over time we have created icons, used indexes and developed symbols which first replace and later become the objects of thought […]. In our case mathematical objects are represented by icons, indexes and symbols which we use as tools to develop processes whereby we describe and manipulate the world. The distinction between icons, indexes and symbols is a subtle one. On one level, an icon can be taken to represent the object itself. […] The interpretive process that generates iconic reference is what we call recognition. The word "re-cognition" means thinking about something again, and "re-presentation" is to present something again. Iconic relationships are the most basic means by which things can be "re-presented", and hence "re-cognised".

Therefore, the presence of a 'representation' of something with letters as icons could be considered a generalization, but not the formalization of a thought.

[f] Another point of view is presented by Hejný (2004, p.2):

> 3. **Stage of generalization**. The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme generalizing all these models is discovered. The process of generalization does not change the level of the abstraction of thinking.
>
> 4. **Stage of universal (mental) model(s).** A general overview of the already existing isolated models develops. It gives the first insight into the community of models. At the same time, it is a tool for dealing with new, more demanding isolated models. If stage 2 is the collecting of new experiences, stages 3 and 4 mean organising this set into a structure. The role of such a generalizing scheme is frequently played by one of the isolated models (e.g. fingers serve as a universal model for a simple counting).

In his paper the author presented an example of generalization realized by a three year old girl with her finger, without formalization, as it is usually meant. This theory of generic model has been presented recently by Hejný (2008). We adopt Hejný's proposal as the main theoretical framework for our research, since it is independent from formalization and the examples of children's statements often set aside the formal aspects. Moreover the stage 3 can be applied also to generalization 'by extension', e.g. from addition to multiplication and from natural numbers to rational numbers.

[g] In order to explain our didactical experiment, it will be useful the so called (Pirie and Kieren, 1989) 'onion' model[4] which is a recognized tool for looking at growing understanding as it is happening. In it, generalization does not appear explicitly, but it is present differently mainly in the more 'external' stages. In our theoretical framework generalization overlaps some of these stages, but does not coincide with them (e.g. we have examples of generalization without formalisation).

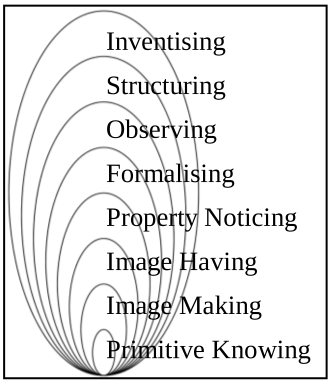

Figure 1- The 'onion' model

DIDACTIC AND LOGIC OF THE COMMUTATIVE PROPERTY

Didactical analysis

The addition and multiplication commutative properties are often proposed as a 'fact', in the sense that in all the examples aiming at facilitating the learning of these operations the environment justifies the use of commutative properties. With addition, this fact happens in a dynamic situation of adding to or in a static situation (putting together) or adding in combining disjoint sets (Tsamir *et al.* 2008, p. 57). The same happens when multiplication is presented by arrays.

Therefore, the property assumes the role of an 'en act' knowledge and the reflection about it can be considered a superfluous remark and an unnecessary terminology ('commutative'). But the comparison between addition and subtraction is enough for casting light upon the necessity of a name for a property holding always for addition and hardly ever for subtraction. Commutativity of addition is useful in case of 'counting on' when the first addend has cardinal value lesser than the second addend one (CAL strategy of Baroody and Gannon (1984), p. 322). Later on, when an explicit algorithm for addition in a column is used, commutative property is used to check the result. This practice is more useful in the case of multiplication. These procedures attach importance to commutative properties.

Didactical attention in primary school to the commutative property as a fundamental peculiarity of operation is rare.

Logical analysis

We can consider commutativity as an axiom stating a peculiarity of a binary

[4] Reproduction difficulty suggested us to slightly modify the graphical aspect of the original drawing of Pirie and Kieren (1989, p.8).

'operation' in a suitable structure [5]. Therefore this property is intrinsic part of the definition of that operation as a specific two-argument functional symbols. Therefore this property is a 'brick' which is essential for the construction of that suitable structure. To state correctly commutativity for 'addition' in a first order logical language, we must use a specific name for the binary operation, two indeterminates and two universal quantifier on them:

(1) $$\forall x \forall y (x+y = y+x),$$

The sentence (1) is the result of a generalization in the sense of a statement which resumes many cases[6]. In fact, in the 'standard' arithmetic interpretation of the logical structure the two indeterminates should be interpreted as variables on the set of natural (integers) numbers; therefore, they can assume the numerical value you want. The statement $24 + 35 = 35 + 24$ is an example of (1) in which we interpreted x as 24 and y as 35. This is the 'direct' problem we considered in [c]: from the statement to the examples. The 'standard' arithmetic interpretation is not the possible unique one, since we can consider different 'abelian' structures, all of them having the commutative property as an axiom.

This structural – syntactic point of view is a final point of a reflexion about the concept of 'structure' and it is not the way in which pupils can act. They know some aspects of arithmetic and they can generalize their semantic experience with numbers in order to obtain something similar to the statement (1). In this case we can speak of a generalization by induction from the everyday experience with arithmetic. It is the 'inverse' problem in [c]. Hejný's approach to generalization [f] is close to this.

In the statement of commutative property there is also a morphologic aspect which can be considered as a pattern [b]. The open formula $x + y = y + x$ is an equation (in logical terms), i.e. it is an equality of two terms. The evident morphological aspect is that the term to the second member of equality is obtained from the first member by exchanging the indeterminates.

THE TEACHER'S PRACTICE

The first author participates from a long time to research activities of the Local Unit of Research in Mathematics Education of Parma University managed by the second author. The possibility of teaching mathematics to the same pupils, following them in all the grades of primary school, favours her choice of a long-term didactical project. Now (2012) she is teaching in grade 4. In grade 1 she followed a teaching project borrowed from Hejný *et al*. (2006), based on the semantic environment 'Father Woodland'.

The Czech authors suggest that this environment is useful for the first step

[5] The structures for arithmetic are abelian monoid with addition and with multiplication.
[6] The statement (1) formalizes the previous quotation of Malara and Iaderosa (1999) [d].

towards algebraic topics such as pre-concept of equations, conceptual thinking in pupils not only at the elementary level, solving methods of linear equations, solving of Diophantine equation. Some results of such a practice have been presented in Hejný *et al.* (2009), Marchini & Back (2010), Jirotková *et al.* (2011). In Rossella's today experience, 4th graders recall their learning produced by the means of this semantic environment. In this environment, commutative property of the addition is an 'en act' theorem since it is (obviously) true in it.

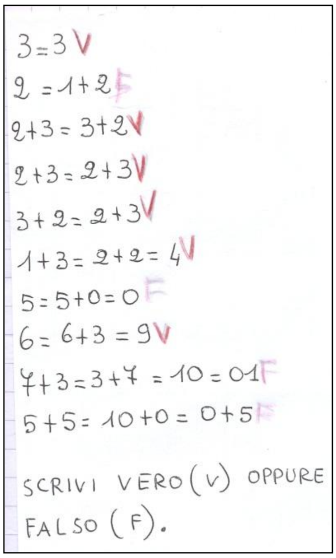

Figure 2 A protocol of a 1st grader in a true – false task

The structural properties of arithmetic and relational thinking in the meaning of Molina *et al.* (2007) were always present in Rossella's teaching. An instance of that is evident in Figure 2, in which there are examples and counterexamples of transitive and commutative properties of addition, together with neutral element and reflexive property of equality. Therefore, her pupils are always 'exposed' to the commutative property of operations. In the previous grades she presented the verbal statement expressing this property, commenting suitable equalities.

THE RESEARCH: AIMS AND METHODOLOGY

This research aims to assess whether a constant care of the theoretical thinking favours the process of generalization in arithmetic. More specifically, we want to detect if pupils not only remember the statement, but are able to handle generalization and in which form. In particular we are interested to the pupils' use of a suitable language and to their management 'direct' and 'inverse' problems related with generalization.

We can define the teacher's actions as 'yeast' methodology. Rossella very often presents open questions to pupils leaving them to discuss freely the topics. In particular, for this research she asked what they know or remember about the commutative property of addition. The discussion was presented in different days and the pupils' contributions were recorded by writing each pupil's statement on a poster (110 cm × 70 cm) put up on the wall, waiting for the 'rising' of the topic. This 'yeast' methodology increases the class learning since the intuitions of brightest children (the yeast) are shared among all and each motivated pupil can learn with her/his time of attention.

This didactical methodology is suitable for the research, since these written posters are valid tools for understanding the overall dynamics of the activity, even if the diachronic dimension lacks.

THE MINUTES

This non-'ordinary' presentation of the research results can help the reader to follow up on the appearance of different way of generalization taking part in the class dynamics. In the minutes we tried to reproduce the children's speaking style; in them we can 'listen' echoes of the previous teaching/learning activities, but we focus our comments only on the commutative property. The children's names are not the real ones, but their statements are reported accurately. We comment, in round brackets, on some statements, by presenting our interpretation in italic; letters in square brackets refer to the theoretical framework.

The teacher' question: what the commutative property of addition is?

1 Max: 24+35 = 35+24 – Max is not able to express this property verbally, but he is able to do that (*'en act' knowledge or Primitive Knowing of*[g]*: possibly* [b], *but also the use of numbers as icons* [e]) .

2 Omer: Changing the order of addends the sum doesn't change (*verbal generalization in rhetoric style, or simply, remembrance of a teacher's statement*).

3 Fabio: Addends can be two or more. (*verbal generalization in rhetoric style by extending the property from two to many addends* [f]).

The Fabio's statement diverted the children's attention to an unexpected combinatorial problem. Teacher did not intervene: it is an occasion, a new 'yeast', to catch what can be retaken in another time.

4 Gino: Commutativity is also for multiplication. (*verbal generalization in rhetoric style by extending the property to another operation* [f]; *'en act' knowledge or possible individuation of a common pattern* [b] *and Property noticing of* [g]).

5 Axel: Addends or factors must be different can be two or more. (*This statement, connecting addition and multiplication, looks strange and wrong, but it is motivated by a linguistic reflection focusing on the fact that the commutative property with equal addends or factors cannot be distinguished from reflexive property of equality, violating the pattern* [b] *on the basis of a didactical contract that different writings imply different things – a morphologic point of view*).

6 Carla: Commutativity can be made also with letters: A+B = B+A; C+D+E = E+D+C = D+E+C = E+C+D =…(*verbal generalization with the use of letters. We can consider the girl's assertion as an example of generalization by a formalization having sprung from a pattern* [b] *or an Observing of* [g]. *Our feeling is that letters are not variables, but they are icons* [e] *and the girl wants to introduce an arithmetic among them. In every case this generalization goes from numbers to letters as an 'inverse' problem* [c]. *The second statement could be the formalization, or a translation of* (3), *by extending the property from two to many addends. We think that she is able to produce formalization* [g]. *The last equality sign looks like a thinking pause. Carla's combinatorial thinking does not follow a unique pattern: the*

first writing is in alphabetic ordering. The second ‘E+D+C’ *is obtained from the first by ‘specular reflection’, the same pattern as B+A from A+B* [b]. *The other two expressions are obtained applying other strategies*).

7 Dante: There are six ways since with each letter I have two combinations = C+E+D = D+C+E (*Dante’s statement concludes the combinatorial thinking of Carla in the case of three letters, even if the translation of his statement should be e.g.* C+D+E = C+E+D, *fixing one letter and exchanging the remaining two. It is a case of Structuring of* [g]. *The term ‘combination’ instead of ‘permutation’ is not mathematically correct. The self-confidence of Dante in the individuation of the exact number of permutation is remarkable in grade 4. These expressions are obtained by following a combinatorial scheme* [b] *and by detecting of the lacking cases in Carla’s statements*).

8 Luce: 3×2 = 6 (*Luce resumes in a numeric formula the previous proposals of her classmates. Her presentation is not merely a list or a computation of the number of permutation, but she shows a combinatorial intuition, i.e. Inventising of* [g]. *Her change of language is relevant: from description – counting from a list of Carla and Davide, to a normative language with the change of operation. The presence of a multiplication instead of an enumeration can be considered a change of semiotic register which is coupled with a generalization as in* [a]).

9 Lia: As many letters there are, as many conbinations there are. (*Property Noticing of* [g]. *She writes with spelling mistakes noticing that the number of permutations is an increasing function of the number of letters involved in permutations*).

10 Fabio: It is enough to discover how many combinations we have with the letter A: A+B+C+D; A+B+D+C; A+C+B+D; A+C+D+B; A+D+B+C; A+D+C+B (*Fabio discovers the pattern* [b], *extending to four letters* [f]. *Here he presents only one case, the permutation starting with A, but his statement alludes that this isolated model, in reality, is in Hejný’s stage 4* [f], *presenting the whole generalization suggested by his ordered thinking, via a pattern* [b], *showing a Structuring stage of* [g]. *Nevertheless, he needs the counting from a list procedure*).

11 Luce: Since there are 6 it is enough to make 6×4 = 24. (*Luce recognizes the pattern of the solving procedure for the permutation problem, by using the ‘normative’ language as in* (6) [a], *without justification or proof. We could think that she is formalising Fabio’s suggestion. He did not state explicitly that there are six permutations holding fixed the first letter, but he hints at it. Luce is ready to translate the suggestion in a numerical statement. The result of this exchange looks like the generic model* [f] *for the number of permutations problem*).

a. Rossella: What happens with multiplication? (*Teacher grasps Gino’s suggestion* (4) *for two reasons. The first is to lead again the discussion to the commutative property, leaving the combinatorial setting. The second is for asking pupils to generalize the commutativity to another structure and in this way to allow another* *interpretation of the same condition expressed by* (1) [f]. *Her question mark is rhetoric and by it she mobilizes newly children's attention to the main focus of the research*).

12 Kira: It is in the same way as with addition, but we make multiplication $A \cdot B = B \cdot A$; I use dot for avoiding confusion between the sign '×' with a letter, as it appears on the pocket calculator (*Kira shows an 'en act' knowledge the Primitive Knowledge of* [g], *but she does not return to a numerical example, since she expresses* (4) *directly in a formal way. It is a generalization by extension of the property to multiplication* [f]. *She faces the writing problem of the possible ambiguity of the sign '×' even if the letter 'x' is not usual in Italian words*).

13 Omer: Dot is used in middle school.

14 Luce: Factors can be two or more (*This statement is a generalization by extension to multiplication parallel to* (3)).

15 Gino: Letters can be any number (*This statement resumes the role of generalization by a first formalization* [d] *and direct problems of* [c]. *The presence of the linguistic universal quantifier 'any' is relevant. But Gino uses this quantifier in a semantic interpretation, not in a formal way*).

16 Luce: The letter can be a one-digit number or a many-digit number; e.g.: (Figure 3) (*This example shows what Luce means. The letters are 'templates' for whatever number, direct problems of* [c]. *In this way she is able to grasp the role of generic model* [f] *of the literal writing by giving examples of semantic interpretation. The arrow she uses denotes the 'production' from the formal writing to its interpretations. Compare with* (6) *in which Carla use the inverse direction: from numbers to letters*).

$$
\begin{array}{ccccccc}
200 & \times & 30 & = & 30 & \times & 200 \\
\uparrow & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow \\
A & \times & B & = & B & \times & A \\
\downarrow & \downarrow & \downarrow & & & & \\
1.405 & \times & 500 & = & 500 & \times & 1.405
\end{array}
$$

Figure 3 – Luce's statement

$$
\begin{array}{ccc}
AB & \times & BA \\
\downarrow & & \\
25 & \times & 52
\end{array}
$$

Figure 4 – Dante's statement

17 Dante: I think that behind any letter there is a digit whether there are two letters, there is a two digits number. Equal letter, equal digit (Figure 4). (*Dante makes a wrong generalization and he misunderstands the commutative property of multiplication. The reason could be that the pupils made simple activities of cryptography for the learning of substitutions. He understood the morphological aspect of the commutative property, i.e. the pattern* [b]; *from that he produced a formalization on this base, but he is not aware of the relational aspect of the equality whose presence produces a statement. He shows mastery of variable treatment (*Marchini, 2002*), but the role of equality in the commutative property is completely misunderstood*).

18 Dante: With the numbers, commutativity gives you the result, but with the letters you don't have the result nor the value of numbers, thence you cannot calculate. (*This second statement shows the abridged passage from isolated model given from experience of computation to generic model* [f] *given by formalization* [d]. *Dante is feeling a strong necessity to connect his experience with numbers. We can consider him as a semantic thinker in the Property Noticing stage of* [g]).

19 Luce: I don't take interest in the number hidden behind A, but I am interested in the procedure. (*Luce is in another developmental state: her interventions, generally, show a structural understanding* [g]. *She is speaking of a 'procedure', but without numbers, there is not a procedure to be performed neither a computation algorithm; her interest is in the structure of the arithmetic* [f]. *She is anticipating a more ripe structural thinking*).

20 Carla: If I must explain the commutative property I would take interest in what I am performing (*Carla in* (6) *introduced letters, but by comparing interventions* (20) *and* (6) *it is evident that she is able to formalize [g], but not to generalize*).

21 Omer: The use of letters is senseless, because you have not the result (*Omer resists the change. He knows generalization by heart* (2), *but he is not able to put his statement into a formal writing*).

22 Luce: The use of letters is meaningful every time when I am concerned with the procedure; whenever my concern is about the letter value I must use number as if we have to solve a problem and we call *x* a number. (*Luce states clearly the difference between the use of letters as unknown in problem solving and the use of letters as indeterminates in the definition of structures* [f]. *As in* (19), *'procedure' is structure. From logical point of view she looks sensitive to the difference between an existential quantifier – the existence of solution in a problem – and the universal quantifiers - involved in the statements of a formal property of addition and multiplication. In this case she works syntactically*).

b) Rossella: What do you mean with 'any number'? (*Teacher retakes* (17))

23 Luce: All the numbers. (*This statement could be misunderstood: small or big number or other kind of numbers, i.e.* [f]? *The girl's classmates interpret her thinking*).

24 Gino: With natural numbers 10+20 = 20 + 10 : 10×20 = 20×10.

25 Dante: With integer numbers -1 + (-14) = -14 + (-1) ; -1×(-14) = (-14)×(-1) (*This statement extends the arithmetical structure to integers number in a form of generalization of monoidal structure of arithmetic to the ring structure of* $\mathbb{Z}$. *It is worth noticing that at the moment in which this discussion took place, pupils have known relative integer numbers as magnitudes - the winter temperature – and only addition was introduced among them. The 'force' of the structure suggested them to consider also multiplication,* [f] *and* [g]).

26 Fabio: With numbers with comma 5,06+7,03+10,05 = 10,05+5,06+7,03 6,7×3,5 = 3,5×6,7 (*In this case, addition and multiplication were well known. Therefore, we can consider this statement as an*

example of 'en act' knowledge or Property Noticing of [g]*. Remark that in Italy the 'comma' is used instead of Anglo-american 'point' for separating the integer part from the decimal one*).

27 Dante: With squared numbers 16 + 25 + 36 = 36 + 16 + 25.

28 Luce: With number with powers $1^2 + 4^2 + 10^2 = 10^2 + 4^2 + 1^2$ $2^2+3^4+5^0+7^3=3^4+7^3+5^0+2^2$ $7^3 \times 4^4 = 4^4 \times 7^3$.

29 Fabio: With fractions 3/5 + 4/10 = 4/10 + 3/5 2/5 × 7/10 = 7/10 × 2/5 (31/40 × ¾ = 93/) (*Fabio presents an attempt of computation*) .

These proposals (23 – 29) can be interpreted as the children's search of models for the 'abelian' theory, among the interpretation they know. In a certain sense it is a 'validity' proof of the commutative property in logical meaning.

30 Dante and Kira: For explaining commutative property it is sufficient to say $A+B = B+A$ $A\times B = B\times A$. But for using commutative or for facilitating computation or for checking computation WE USE NUMBERS. (*Dante is repeating* (18)*; Kira shows the same cognitive style as her classmate*).

31 Dante: Instead of letters we can use symbols. In this case $A\times C = C\times A$ is equal to §×* = *×§. (*In this intervention Dante suggests to use icons* [e] *instead of variables*).

CONCLUSION

With this work we tried to show that most pupils involved in the class debate attained generalization of the several meanings presented in the theoretical framework. Some of these pupils were able to handle this mental activity using a suitable language and direct and inverse procedures related with generalization. Therefore, we can positively assess the long-term teacher's activity.

The developmental levels were not the same for all pupils. One of the last levels in the 'onion model' of [g] and the Stage of universal model of [f] were approached with high similarity by Luce's interventions, although with some exceptions. Pupils whose arguments presented aspects of generalization showed that they accomplished this mental activity in different ways with some awkwardness.

Less than half of the class participated in this debate. Is this a success? Can we assume that the long-term teaching activity gave pupils sensitivity towards theoretical thinking? We are convinced that generalization and the consequent management of the general concepts are possible in grade 4, but not that every pupil is ready for this important step. There are relevant intuitions that can be followed and exploited by the teacher for improving children's understanding. The long-term teaching project can be evaluated as fruitful if we consider important that a fairly high number of children has approached such a complex topic. In fact, a teacher's task should be to avoid mortification of clever pupils.

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ASPECTS OF GENERALIZATION IN EARLY ALGEBRA

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In this paper we will present some studies we have recently developed in our research project in the theoretical framework of early algebra. We will illustrate a first "inventory" of those conditions which might foster the construction of significant basis to support young students' gradual approach to generalization from different points of view (linguistic, perceptive, social and mathematical).

INTRODUCTION: GENERALIZATION AND EARLY ALGEBRA

Traditionally, most curricula separate the study of arithmetic, mainly taught in primary school, from the study of algebra, considered to be suitable for secondary school students. However, many researches have shown the negative effects of a too quick transition from arithmetic to symbolic manipulation. Warren et al. (2006), for example, suggest that algebraic activity can occur at an earlier age and that this kind of experiences, proposed through appropriate teacher actions, could assist students in this complex transition. Blanton and Kaput (2011), too, stressed the importance of giving children opportunities to begin using symbolic representations as early as first grade in order to make them acquire those basic concepts which can allow them easily explore more complex concepts in later grades. These ideas have brought to the rise of early algebra, which now has the characteristics of a real new discipline (Kaput et Al. 2007). This is the frame in which we have developed our *ArAl Project* (Malara & Navarra, 2003) [7].

The hypothesis of early algebra is that the common "arithmetic to algebra" framework is too limiting and narrow (Smith & Thompson, 2007) and that therefore it should be reformulated in order to give students the opportunity to develop algebraic thought when they start carrying out the first activities in arithmetic. This approach does not require to bring the algebraic curriculum in primary school, but to revise the way in which arithmetic is conceived and taught in order to promote a shift from a procedural conception of arithmetic to a relational and structural one. We believe that it is also necessary to clarify what is the meaning of promoting the development of algebraic thinking at this level. We agree with Radford (2011), according to whom the use of notations is neither a necessary nor a sufficient condition for thinking algebraically and that algebraic thinking is characterised by the specific manner in which it attends to

[7] ArAl Project (Arithmetic pathways towards favouring pre-algebraic thinking) is a National Project developed by the GREM (Group for Research in Mathematics Education) directed by N. A. Malara (professor in the Mathematics Department of Modena and Reggio Emilia University) and coordinated by G. Navarra.

the objects of discourse. The author suggests that algebraic thinking is about dealing with indeterminate quantities conceived of in analytic ways (i.e. considering the indeterminate quantities as if they were known and carry out calculations with them as with known numbers).

Fostering the teaching of early algebra means, for teachers, giving their students the opportunity to activate different modes of thinking such as: analyzing relationships between quantities, predicting, generalizing, exploring stimulating situations, modelling, justifying, proving.

Generalization is considered to be an important determiner of growth in algebraic thinking and a fundamental preparation for later learning of algebra (Cooper and Warren, 2011). A rich context from the point of view of the different meanings that could be conveyed through it, and therefore potentially suitable to stimulating generalization processes, is represented by activities related to the research of regularities (see paragraph D2). During this kind of activities students have the possibility to experiment a crucial aspect in the generalization processes: seeing a generality through the particular and seeing the particular in the general (Mason, 1996). Cooper and Warren (2011) suggest that, during these activities, a step towards full generalization in natural language and algebraic notation is *quasi-generalization,* in where students are able to express the generalisation in terms of specific numbers and can apply a generalisation to many numbers, and even to an example of 'any number'.

In the approach to early algebra teachers play a crucial role in identifying the best activities to be performed and in promoting those processes which foster generalization. Obviously their way of proposing these activities in their classes is strictly connected to their deep beliefs, which have been highlighted thanks to our analysis of the numerous transcripts (about 4500, collected from 2004 and 2011) of the activities performed in our project. These transcripts were object of a joint reflection carried out by teachers and researchers through the *Multicommented Transcripts Methodology (MTM)*[8.] Some reflections on methodological aspects recur independently of the age of students (from 5 to 15); therefore they can be considered mirrors of the most widespread behaviours of teachers. The high number of reflections referred to generalization from different points of view suggested us to identify a tentative but enough detailed "inventory" that we will present in the second part of the paper. Before proposing this inventory it is necessary to introduce some theoretical aspects which constitute our framework for the approach to early algebra, with particular reference to the aspects related to generalization processes.

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[8] The MTM, developed in the ArAl Project, is based on the critical analysis of transcripts of the audio-recordings of whole-class discussions, carried out by the teachers involved in the Project, through the intervention of different actors: the class teacher, his/her E-tutor, other teachers, teachers-researchers and university researchers. The commented transcripts are shared through E-mail and during periodical meetings for a critical exchange.

OUR APPROACH TO EARLY ALGEBRA

Our perspective in the approach to early algebra is *a linguistic and metacognitive one* and is based on the hypothesis that there is a strong analogy between modalities of learning natural language and algebraic language (Cusi, Malara and Navarra 2011). In order to explain this point of view, we make use of the metaphor of *algebraic babbling.*

This metaphor represents the process through which the student acquires first a semantic, then a syntactic control of the mathematical language in a way similar to the one he/she learns natural language. This learning is first characterized by an initial discovery of meanings and a gradual, creative appropriation of rules and by a subsequent deeper knowledge, developed during the school years, when the student is able to reflect upon the structure of the language.

Fostering this process requires to build up an environment able to stimulate the autonomous elaboration of formal codings, to be negotiated through class discussions, and a gradual experimental appropriation of algebra as a new language. The rules of this language are then located into a didactical contract, which tolerates initial moments of syntactic ‘promiscuousness’.

Another fundamental aspect in our approach to early algebra is therefore recognizing the potential role played by the relationship between argumentation and generalization in the social construction of knowledge. Only when argumentation becomes a shared cultural instrument in the class this relationship can be made explicit and the students can understand the role played by verbalization in the development of their capability of reflecting upon what they are saying. Moreover, comparing particular cases help students recognize their similarities, gradually highlighting their connecting thread.

Another crucial aspect in this approach to early algebra is helping students recognize and interpret *canonical and non-canonical representations* of numbers[9] in order to make them build up the semantic basis for the understanding of algebraic expressions. Non canonical representations can be considered “semantical ferries” towards generalization (see paragraph A2).

Because of the central role played by verbalization in supporting the achievement of symbolic notation, another critical aspect is making students understand the importance of respecting the rules of algebraic language.

While students start soon interiorizing the importance of respecting the natural language’s rules in order to facilitate communication, it is difficult to make them develop a similar awareness in relation to algebraic language. It is therefore

[9] Among the possible representations of a number, one (for instance 12) is its name, called canonical form, all the others (3×4, (2+2)×3, 36/3, 10+2, …) are its non-canonical forms, and each of them will make sense in relation to the context and the underlying process.

necessary to help them understand that algebraic language, too, is a finite set of arbitrary symbols which can be combined according to specific rules to be respected. This kind of conception could be fostered through the creation of linguistic mediators which force the respect of rules in communicating even advanced concepts by means of algebraic language, in a perspective which foster generalization[10].

FACTORS WHICH CONTRIBUTE IN STUDENTS' CONSTRUCTION OF THE SEMANTICAL BASIS FOR GENERALIZATION

As researchers who develop their studies in the field of early algebra, having to face the theme of this volume (*Generalization in mathematics at all educational levels*) made us try to identify what kind of situations, methodologies and attitudes could foster, in young students, the construction of the significant premises for a gradual approach to generalization in order to help them overcome the difficulties they will have to face in later grades. In the following we will present a first 'inventory' of the situations we have identified, subdivided according to the ambits they refer to: linguistic, perceptive, social, mathematical.

A1. Generalization and language: the role of argumentation

The students of a class (11 years old), who are used to argumentation, are exploring a growing pattern, whose components are called 'pyramids', with the aim of identifying general laws to connect the characteristics of every pyramid (the total number of triangles it contains, the number of rows, the number of white triangles…) with its position in the pattern.

1 2 3 4 …

When the class is working to find a general law to determine the number of black triangles in the row which constituted the base of every pyramid, a student (Y.) observes: "*On the line where the pyramids lie … for example, in the fourth pyramid the black triangles are four and the white are three … my pyramid of six floors has six black triangles and five white triangles on its base… The white (triangles) are always one less than the blak ones. Maybe a pyramid with any number of floors has a number of black triangles on its base which is equal to the number of floors and as many white triangles as the black ones minus one*". The teacher of the class proposed this reflection as a comment to the transcript: "Before her intervention, Y. wasn't aware of her conclusions but, as she was verbalizing, she started deducing and expressing the general rule".

[10] In the ArAl Project, as a linguistic mediator, we use Brioshi, a virtual Japanese student who doesn't speak the Italian language but knows how to express himself using a correct mathematical language. Brioshi is an algebraic pen friend with whom students communicate using mathematical sentences which should be written through a correct application of syntactical rules in order to be understandable (Malara e Navarra, 2001).

This example highlights the fundamental role played by the relationship between argumentation and generalization in the social construction of knowledge. This relationship can be made explicit only when argumentation becomes a shared tool for the teacher and the students: every component of the class has to get involved in this process and has to relate him/herself with the ways in which the other components get involved. This means that the students must take the responsibility for their learning and that the teacher must take the responsibility for fostering students' social construction of their knowledge.
We could say that the power of argumentation is related to the fact that those who start developing it are not completely aware of their ideas before they try to express them. As argumentation becomes an habit, the student understands its value and becomes aware of its role in comparing facts and in making their similarities gradually emerge, together with their connecting thread.

A2. Generalization and language: the *potential general*

Through the activity called 'pyramids of numbers' (the sum of every couple of numbers written on two adjacent bricks is equal to the number on the brick over them), the teacher guides students toward the identification of the law which expresses how to determine, without any calculation, the number written on the brick at the top of a three-floors pyramid as a function of the numbers written on the three bricks on the basis of the same pyramid.

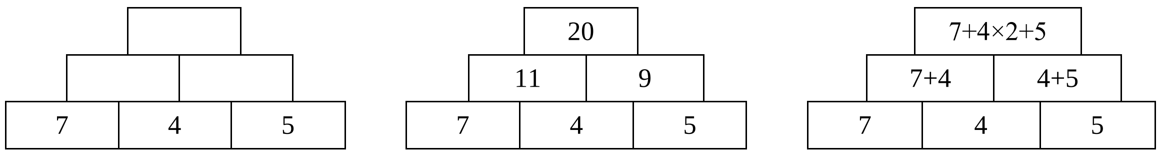


Fig.2a Fig.2b Fig.2c

The classical method of completion (Fig.2b) is not enough in order to determine the required law because it leads to an 'inexpressive' result (in this case 20). The non-canonical representations (Fig.2c), instead, allow the construction of what we call a *relational-ontological* representation of the number at the top of the pyramid, i.e. the representation which constitutes the best explicitation of the general law "The number at the top is the sum of the two side numbers and the double of the middle one". The next step to be carried out is the translation of the equality $20=7+4\times2+5$ into natural language. The final step for students is becoming aware that this sentence, expressed in natural language, constitutes a *potential general* through which it is possible to carry out a further conversion into algebraic language: $n=a+2b+c$. We think that the first, epistemological, source of difficulties associated with the use of letters in mathematics, is related to the capability of *conceiving a letter as a number*. This aspect could represent an insurmountable barrier to algebraic language and generalization.

The concept of *potential general* could be related to the notions of quasi-variable (Fuji and Stephens 2001) and *quasi-generalization* (Cooper and Warren, 2011) as possible bridges between arithmetic and algebra for students from 6 to 14 years old. This observation leads to the introduction of another theoretical construct, essential in the construction of the necessary conceptual and methodological premises in an effective approach to generalization.

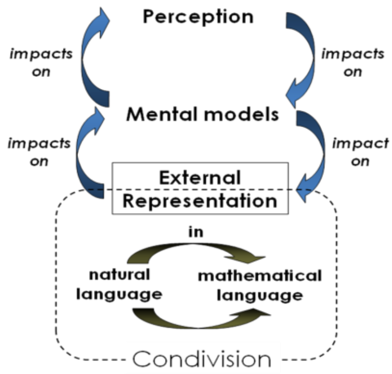
A3. Generalization and language: the pupil as *thought producer*

The ‘law’ identified in the previous example of the bricks pyramid is: “The number on the top is the sum of the two side numbers and the double of the middle one”. This conclusion represents an important moment of condensation in the evolution of algebraic babbling. The pupils have been guided towards the collective construction of a general, though improvable, definition and have formulated its explicitation. They were protagonists as *producers of ‘original’ mathematical thought*: it means that they were able to express with a clear and synthetic language what they have understood and what they have said in public. Traditionally, however, the teacher is the one who mediates between the topical moments of institutional mathematical thinking (principles, theorems, properties, etc.) and their application; in these cases the pupils are mainly *re-producers* of a theory, to the organisation of which they are basically strangers. On the contrary, it is very important that pupils are educated – through forms of collective exploration of thought-provoking problematic situations – in *producing, in the natural language, general conclusions to be shared with the classmates and the teacher,* organising them in a coherent and communicable way, as an intermediate step towards a later *translation* into mathematical language.

B. Generalization and perception

Perception, i.e. the psychic process operating a synthesis of sensory data into meaningful forms, developed in a socio-costructivistic context, allows to create meaningful premises to the approach to generalization. If, for example, one is asked to express his/her calculation strategies in order to find out the number of pearls contained in this necklace:

two different perceptions arise, which lead to two different representations of the counting strategies (on this aspect, see also paragraph D1): (a) visualising the black and the white pearls separately leads to the representation $2\times9+3\times9$; (b) ‘concentrating’ on the pattern leads to $(2+3)\times9$. We interpret the dynamics of the situation in the classroom through the following model:



If an (a) or a (b) pupil were alone, he/she would limit him/herself to his/her personal mental model and to its consequent external representation, because he would not be motivated towards searching for other interpretations, and therefore counting modes. A didactic contract based onto collective argumentation, on the contrary, promotes the sharing of knowledge: each pupil compares his/her representation with the other one and discovers that his/her way of 'seeing' the necklace is not the only one. The result is therefore a feedback that influences the internal representations and the new way in which the necklace structure can be perceived. The social construction of knowledge promotes the evolution of thought towards a shared conquering of new meanings. Overcoming the initial difficulty of integrating the other's vision is the first step towards the understanding of the equivalence of the representations: $2\times9+3\times9=(2+3)\times9$. This shall lead to the development of the general meaning of the equality $a\times c+b\times c=(a+b)\times c$ and therefore to the understanding of the distributive property (Malara & Navarra, 2009).

C. Generalisation and conceptualisation: the conceptual condensation

The class (10-years-old) is exploring the behaviour of a scales, seen as a metaphor of first grade equations at one unknown quantity.

Teacher: Let's describe the situation.

Jacopo: On the right hand side there was baking soda and 100 grams. On the left hand side there were three glasses of baking soda.

Teacher: And what are we aiming at?

Jacopo: We want to find out how much a glass of baking soda weights.

Teacher: Ok. So what have we done, Matteo?

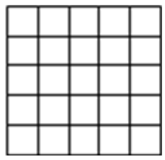
Matteo: We have removed a glass from both sides, then we have divided by two the content of both dishes. So now we have a glass of baking soda on the left and 50 grams on the right. A glass weights 50 grams.

We refer to the transition from the *dynamic* phase of concrete, generative activities, which characterize the pupils' educational path particularly in the first eight years of schooling, to a phase in which the teacher promotes the condensation into knowledge of the mathematical concepts underlying the activities. The one in the example is meant to promote the need to spot out the principles of equivalence as tools to represent the experiences carried out. These new concepts shall then be linked to knowledge concerning operations on natural and relative numbers, to the properties, to the use of letters, to the meaning of ‘equal to’. By reflecting onto the experiences carried out, the pupils are guided towards the identification of general principles that allow to solve other, structurally similar situations. A weak leading in this transition phase does not allow – and sometimes inhibits – the progressive approach to generalization, since the pupils shall keep operating at a concrete level, without working out any theory.

D1. Generalisation and foundational mathematical aspects: the evolution of counting strategies

During our cooperation between Italian classes of the ArAl project and English classes (pupils aged 9 to 15) we presented the following situation:

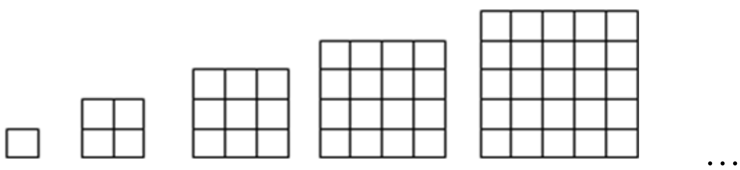
> This drawing represents a structure made of toothpicks.
> Count the number of toothpicks and explain in the mathematical language your counting strategy. It doesn’t matter to determine the number of toothpicks.



With the Italian pupils (13 years old) we discussed the strategies produced by the English pupils (15 years old): (i) 5+5×11; (ii) 3×(3×5+1)+6+6; (iii) 5×4+5×4×2. We asked them to interpret these strategies so as to make clear the meaning of these expressions. The evident result was that each counting strategy reflected the way in which the groups had perceived the structure of the construction (see paragraph B). For instance: the Italian pupils explained that the members of the group (i) had seen the five pillars as ‘combs’, and they had then added the last five vertical toothpicks. Free to count, the pupils discovered many alternative strategies, some of which were more ‘economical’ than others. When they were guided in comparing the expressions, they found out equivalences through proves, e.g. for (i) and (iii):

$$5+5\times11=5\times4+5\times4\times2 \rightarrow 5\times1+5\times11=5\times4+5\times8 \rightarrow 5\times(1+11)=5\times(4+8) \rightarrow 5\times12=5\times12$$

Starting from this activity, generalization arises as soon as the *static* situation is transformed into a *dynamic* one, that is in the moment in which students begin to explore how the counting strategies change in relation to the changing of the square’s dimensions, and they are asked to say if it is possible to find out a ‘law’ that allows to determine the number of toothpicks that are necessary to build a given shape. The pupils discover that it is better to organize an in-order research, for instance through a display of drawings of the following kind:



1 2 3 4 5

The pupils are guided to activate a common counting strategy which express the interrelation between the number of toothpicks and the number of the place of the corresponding square and which can be expressed through a formal representation of the number of toothpicks of a construction, at the generic place n. In this way, they can identify the structures that allow to express the relations connecting the numbers in play in a given problematic situation, i.e. its *structure*. In this case (if n is the number expressing the position and s is the corresponding number of toothpicks) they write, for instance, $s=2n(n+1)$. If the teacher concentrates mainly on the calculus processes, neglecting the reflection on them, she prevents the pupils from going through the experience that is necessary to the process of generalization and to the conceptualisation of arithmetical structures.

D2. Generalization and foundational mathematical aspects: the progressive achievement of the concept of structural analogy

Rosa (kindergarden - 5 years old) is comparing cardboard 'trains', the carriages of which contain objects set in a precise order. She is concentrating on two of them.

Teacher: Why are you looking at those two particular trains? What do they contain?

Rosa: Here is a red, a red and a yellow.

Teacher: Yes, they are Duplo bricks. And what have you got in this one?

Rosa: A walnut, a walnut, a sunflower and it goes on so.

Teacher: So what?

Rosa: They are almost the same.

In this example, Rosa is doing algebra, since she finds out in a naive way the *structural analogy* between the two trains. Right from kindergarten or primary school, pupils can be allowed to recognize relationships between the elements of a sequence and their place number. They discover analogies (in this case, between two train structures), describe them with words and represent them with a code (e.g.: AAB), thus approaching a germ of formalised language, and therefore generalization. The common construction of the code, developed at the stage allowed by the pupils age, hence represents the *collective* result of a *relational* reading of the situation, in which the attention is concentrated not on its elements, but rather on the relationships that connect them. Being able to spot out such correspondences between different situations allows the development of *analogical* thought. Kindergarten constitutes the first step of this process, within a logic of continuity with primary school, where these germs of thought shall gradually ripen along the following school grades, through the exploration of a kind of arithmetic built up in the perspective of the development of algebraic thinking, hence towards a more mature generalization and a more advanced kind of abstraction.

CONCLUSION

What we have described shows educational aspects that we believe should be constantly strengthened, since they support the process towards generalization, promoting in the pupils *metalinguistic* and *metacognitive* aspects, and consequently reflection: (A) on *language*: the ability to construct argumentations, to translate from natural into algebraic language, to produce original thought; (B) on the relationships between *perception* and the social construction of shared knowledge; (C) on passing from *concrete generative situations* to the construction of concepts (conceptual condensation); (D) on some foundational mathematical aspects: the evolution of counting strategies and the progressive attainment of the concept of structural analogy.

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COGNITIVE OSMOSIS IN CLASS AND YOUNG PUPILS' COGNITIVE PROCESSES IN GEOMETRY

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This article has two objectives: first, to present some preliminary results of our research in which we focus on mapping young pupils' cognitive processes in geometry and on the process of knowledge transfer from an individual pupil to a group, i.e. on cognitive osmosis. The second objective is to demonstrate through own practice and experience how conducting experiments and engaging in their deep analysis can be an effective tool for the development of a teacher's professional competences. The tool selected for the experiment is a set of geometrical problems from the learning environment of Cube buildings, involving 3D shapes and their 2D representations.

INTRODUCTION

The educational process is determined by various factors. In one of the universally known models, the relationship between a) the pupil, their learning and their development, b) the teacher and their professional endowment and c) the subject content is expressed by the model of a didactic triangle. The importance that has been assigned to the different parts of this didactic triangle, has varied throughout the past.

T. Janík in (2009) writes that "There are numerous didactic transformations, transpositions or reconstructions taking place between the vertices of the didactic triangle. These are the subject of specific subject didactics.... It has become apparent that teacher's didactic content knowledge is a prerequisite for a well mastered didactical transformation." Lately, the notion of teacher's pedagogical beliefs and their role in the individual teaching style of a teacher has become a frequent subject in mathematical education research. The teacher's belief develops and grows more mature the longer he/she teaches. It is formed by his/her education, life style, demands of the society, it reflects the opportunities of life-long education and many other circumstances (Hejný, Kuřina, 2009).

Important prerequisites for a change in pedagogical belief system and for the shift towards a development of a constructivism-oriented teaching style are a need for self-improvement and the availability of adequate tools.

Based on our experience, conducting classroom experiments with pupils or students, reflecting on such experiments and analyzing them not only from the perspective of an independent researcher but also that of a participant, form a very effective tool for teacher self-education. For that reason, one of the authors of this paper took up an offer to teach mathematics at the first year of primary school to complement her teaching contract at the faculty of education in primary school education. Thus a longitudinal action research was started that has been initially set for the period of five years, with the possibility of a 4 year extension. This presents an opportunity to collect a significant amount of valuable data. The outcomes of day to day analysis and reflection on the observed classroom situations are being regularly used to modify the subsequent teaching plans and strategies with the ultimate goal of centering the class activities around students. The data collected in experiments prompted by unforeseen situations is being collected and in the future will be subjected to thorough analysis. This work will form the basis for the dissertation of the first of the authors.

METHODOLOGY

The first author in the role of a teacher-researcher started teaching mathematics 4-5 times a week in the first grade of a primary school in Prague on the 1st September 2010. The teaching content is given by the School Educational Program (SEP). A rough draft of a lesson plan is usually prepared by the teacher-researcher for the whole week and a detailed lesson scenario is done for each upcoming lesson. Each lesson is videotaped and the elaboration of a detailed scenario for the following lesson is based on reflection upon viewing the video recording. In the scenario much attention is paid to differentiated approach to pupils.

The participants in this research are all pupils attending the class. There was no initial selection. There were 25 pupils in this class on the 1st September, out of whom 18 currently attend classes on a regular basis, there are 10 girls and 8 boys.

The following research documents are being collected and compiled: framework weekly program, detailed updated protocol for every lesson, video recording of each lesson, transcripts of selected video recordings, pupils' written production, including individual work, pair work and whole class work. A teaching journal is kept to record the first reflection based purely on the teacher-researcher own daily observations. Once a week a second reflection is done based on the week's video recordings. In this reflection stage, some interesting phenomena are identified, and relevant samples from the video recordings are transcribed, formatted and archived. The reflection is also guided by feedback from colleagues and students who observed the lesson. This "external reflection" is also documented.

Whenever necessary a further analysis of selected video segments is conducted in cooperation with one of my more experienced colleagues – experts. The theoretical framework for this analysis is specifically Hejný's Theory of generic models (Hejný, 2011a). This analysis usually results in an elaboration of a further partial experiment or a series of experiments with the potential to further reveal a particular phenomenon.

The teaching is guided by the principles of constructivist approach to teaching and focuses on the building of schemata as understood in the didactic framework of scheme-oriented education (Hejný, 2007; Hejný, 2011 a, b). This approach is supported by the use of the textbook authored by M. Hejný et al. The pupils work in many different learning environments, both arithmetic and geometric, e.g. Stepping (Slezáková, 2007; Jirotková, 2011), Bus (Hejný, Jirotková 2009a, b), Additive Triangles, Neighbours (Hejný 2007), Wooden Sticks, Paper folding**,** Parquets (Hejný, Jirotková, 2010), Cube Nets (Hejný, Jirotková, 2007; Jirotková, 2010), ...

One of the learning environments significantly contributing to development of spatial imagination is the learning environment Cube Buildings (Jirotková, 2010). Work with a set of cubes has been incorporated to class work on a regular basis since the beginning of the first year. The concept *cube building* is not explicitly defined for pupils but by many different activities (see Fig. 1. Task 1, 2, 3, 4). This concept is pre-concept of *geometrical solid*. At the same time, it is included as a topic in various mandatory courses for students in primary education programs at the Faculty of education.

THE FIRST EXPERIMENT

In March 2011 an interesting phenomenon was observed in the pupil Vena (all pupils' names have been changed). We decided to study this phenomenon and the subsequent lesson scenario was prepared in great detail. As a result, the first experiment here presented was carried out in two consecutive lessons. The experiment was conducted by the first author and so the I-statements here refer to her.

In the introductory stage of the first lesson the pupils were working in groups of five and six. Each group had at their disposal an unlimited number of cubes and a square grid. The size of the squares in the grid corresponded to the size of the face of the cube. The pupils were given these oral instructions:

Task 1.

> Construct a cube building using exactly four cubes and draw its plan into the square grid. Carry on with this activity and try to construct as many buildings as possible.

The groups could work at their own pace.

What is didactically important at this task? The task requires manipulative activity. Each cube building is a geometrical object, however, the whole set of solutions is a combinatorial object. The task thus connects two mathematical areas – geometry and combinatorics. When we look at the set of solutions as a combinatorial object two questions are elicited: 1) Do I have all of them? 2) Are not two of them congruent? The second question brings our mind back to geometry.

Vena (Ve) was working in a group with three girls. He took up the role of a coordinator. The girls were engaged in the building process and Vena was deciding which cube building should be recorded. After a moment, he reported that they were finished with the task. The following discussion (transcript min 38:05 – 40:53) took place then: (Te means teacher)

Te01: There's still some time left. Look for other buildings.

Ve01: But we have all of them.

Te02: How many do you have?

Sa01: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve. And this one was constructed by me (pointing at one building).

Ve02: Well, we have twelve of them and that's all.

Te03: I think that some other may still be built, what do you think?

Vi01: I'll try (constructing some buildings again but in a different position).

Ve03: No, not this one. We already have it. It's simply impossible.

Te04: But your friends don't believe you, look, perhaps they'll manage.

(Others take cubes into their hands and try to construct new buildings, Ve takes them out of their hands.)

Ve04: But it's really impossible! Really.

Te05: Try to persuade them that it's impossible.

Ve05: Well, if I take from the tall one (pointing at the tower building a) the top cube, I must place it here next to it. And we already have that here (pointing at b). And if I take the top one here, then we can place it (still pointing at the building b) here (c), here (d), here (e), here (f) or here (g). And we already have all these here (pointing at c – g). And it's the same in case of the lower ones (pointing at the rest of the buildings h – l).

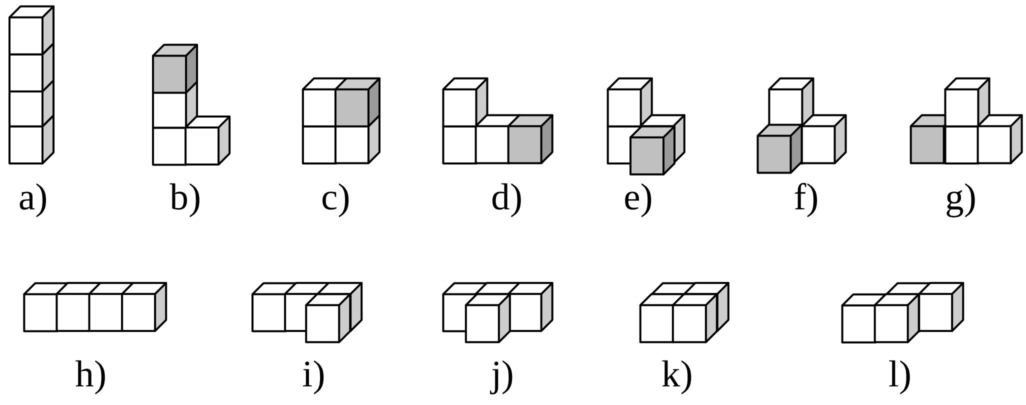


Figure 1: Cube buildings constructed out of four cubes.

Te06: So, what do you think? Is Vena right?

St01: I don't understand him at all.

Sa01: (Only watching Vena up to this point, holding one cube in her hand and moving it in space according to Vena's explanation.) Yes, we won't find any more.

Te07: So, you'll tell the rest of the class in the end of the lesson what you've just discovered, OK?

The teacher-researcher tried to dissuade Vena from his conclusive statement (Te01, Te02, Te03). In the end she was able to guide him towards reasoning and justifying his conjecture. In (Ve05) the pupil described a strategy for constructions of all cube buildings out of four cubes. He spoke very quickly and he tried to tackle the difficulties of using geometrical terminology by frequent use of gestures and demonstrative pronouns. It was apparent that the rest of the pupils did not completely comprehend his explanation (St01). They tried to verify his conjecture by searching for new buildings. Perhaps only Sara (Sa01) was ready to accept Vena's construct, i.e. his generic model (Hejný, 2011a) of strategy for constructing all cube buildings out of four cubes, and to take ownership of it.

In the final stages of the lesson, each group presented their findings and the rest of the class was checking them. They found five, eight and twelve (ten, after the others corrected the result) buildings. Vena was the last one to be given the opportunity to speak and present his group's findings. At the exact moment when he said: "We have built 12 constructions and there are no more possible," the bell rang. Due to the noise, it was impossible to record precisely his explanation (*why* it is impossible). From his gestures we could infer that he was showing a way to exhaust all possibilities by sorting out the buildings by their height and constructing all buildings of the same height. We believe that Vena presented his strategy for constructing all cube buildings out of four cubes.

This fact inspired further experimenting. The idea was to investigate the persistence of the generic model (the strategy) in Vena's mind, his ability to modify it for another context and its transfer to other pupils; we call this latter process cognitive osmosis (Hejný, 2011a). In the next class, then, all children, with the exception of Vena, were given this task:

Task 2.

Take out exactly four cubes, not more than that. Construct a cube building from all of these cubes and draw its dotted plan into a square grid. Using the same cubes, construct another building and also draw it. Try to find as many buildings as possible.

By *dotted plan* of a cube building we mean 2D representation of a cube building where the cube is represented by a square and e.g. tower of three cubes is represented by three dots in a square (see Fig. 2.).

Special attention was paid to the three girls who had been members of Vena's group the previous day. These pupils all used the result of their previous work and arrived at 12 different constructions. Vena's problem-solving strategy, however, was not used, due to the fact that there hadn't been time for him to share it with his classmates the previous day. The exception to this phenomenon was Sara who organized her plans in a way that made it easy to see that all possibilities were exhausted. We suggest that Sara picked up on Vena's strategy and now was applying it in her own way, in the ordering of her pictures. In other words, cognitive osmosis took place within this particular group.

Vena worked entirely on his own, but he got a slightly modified assignment:

Task 3.

> Take all your cubes. Construct as many buildings as possible but only those that are made of exactly five cubes and which have not more than 3 cubes on the first floor. Draw their dotted plans into the square grid.

We expected Vena to use the same strategy that he had used when justifying his hypothesis of completeness of his solution. Yet, this expectation was not met. Clearly, Vena approached this new situation as a completely new problem. He was asked again to come up with the number of all possibilities but this time the restriction placed on the number of first floor cubes made the task more difficult. Vena repeated his process of discovery through creating isolated models. This time, though, the process took him less time and using only a few isolated models, Vena was able to create a sequence of these models: five-floor buildings, four-floor buildings, three-floor buildings with two cubes in the first floor and so on. These generic models became isolated models of a higher level and based on them Vena constructed a new generic model. It is apparent in the video recording that he was showing this strategy with his hand movement. As any change of language was not present in Vena's communication we still speak about generic model, not a piece of abstract knowledge.

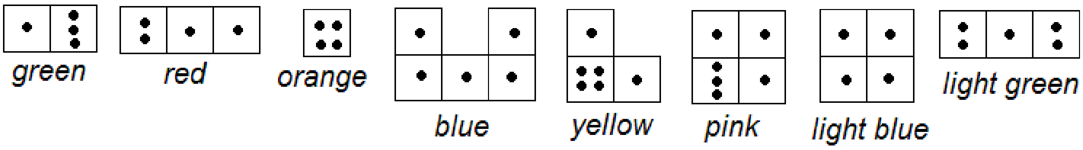
THE SECOND EXPERIMENT

The second experiment took place in the second year in November 2011, i.e. there was a lag of six months between the experiments. In those six months children were regularly given problems about Cube Buildings including those that called for examining and describing different types of buildings based on a given plan. In November 2011 the pupils were presented with this task:

Task 4.

> Construct buildings based on the given plans and record the number of cubes in each floor for each building in a table.

The worksheets available to the pupils contained the dotted plans of eight different buildings built from four, five or six cubes of different colors. There was also a table designed for recording the number of cubes in each floor (Fig. 2).



Building colour	green	red	orange	blue	yellow	pink	light blue	light green
Number of cubes in first floor								
….								

Figure 2: Second experiment worksheet 1.

After the problem-solving stage of the lesson, the following discussion took place:

Ka01: Teacher, I noticed something interesting, but I'm not sure, it may be wrong.

Te01: Try to explain it to us and we'll try to decide whether it's wrong or not.

Ka02: Here in this red building (points at her worksheet), we have three cubes in the first floor and one in each of the rest of the floors. We have done this before but the building looked different, sort of like a corner.

Te02: You mean that the plan had a different shape?

Ka03: Well, it was drawn in a different way (nods) but it was the same in the table.

Pe01: It was a different building so it should be drawn differently. And there can be the same number of cubes in the (gestures repeatedly a horizontal line) floors.

Ka04: And so it's not wrong? (looks around the class)

Si01: Teacher, can I show them?

Te03: Of course.

Si02: Our red building (goes to the board and draws a plan of the red building) has three cubes in the first floor and one in the second. But if the top cube was here (draws a second plan with the same shape but with the second floor cube in the middle) it would be a different building and described in the same way. Or the three cubes could be in the first [floor] like a corner but then they could be two different buildings, either the top cube would be at the end or in the corner (draws plans of two new buildings).

Ja01: (runs to the board) Or the top cube could be on the other end (points at the other end cube in the L-shape building).

Pe02: No, it couldn't. Then it would be the same as Simon's, only turned around.

Te04: So, Kaja, did the boys explain to you if your idea was correct?

Ka05: I think so.

Kaja noticed that one of the buildings is made of the same number of cubes as in a previous class (Ka01-04). She was concerned that there may be a mistake in the formulation of the problem. Simon (Si02) argued using the set of buildings that have the same values in the table but are distinct. The teacher-researcher did not interfere in the pupils' discussion and let the pupils explain to Kaja in their own way.

The next problem in the experiment was:

Task 5.

Build coloured cube buildings based on the table. Then draw a dotted plan for each of your buildings. Find all possible solutions.

Children worked in groups of three, each group had one of the pupils who participated in the above described discussion.

Colour	red	blue	yellow	green	black
Number of cubes in 1st floor	3	3	2	2	1
Number of cubes in 2nd floor	2	1	2	1	1
Number of cubes in 3rd floor	0	1	1	1	1
Number of cubes in 4th floor	0	0	0	1	1
Total number of cubes	0	0	0	0	1

Table 1. Second experiment worksheet 2.

The maximum of six and minimum of two cube buildings that corresponded to the problem's statement were found from the total amount of eleven (or thirteen, including two symmetrical cases) buildings. This number is relatively small, most likely due to time constraints experienced in this problem solving activity. Ten days later this activity was picked up again in class and groups worked separately to complete their assignment including the description of their strategy.

The results of this activity will be shown and discussed in detail during the planned presentation.

CONCLUSION

We described in the article the findings derived from two experiments conducted as part of the first author's action research. These two experiments were specifically focused on the cognitive processes of pupils working with cube buildings (constructed from four cubes). The data yielded were analysed and revealed the presence of the following four cognitive phenomena:

1. The discovery of a generic model (by Vena in Experiment 1).
2. The modification of the generic model attained through previous experience (Experiment 1).

3. The cognitive osmosis within one working group (Experiment 1).
4. Discussion as a tool for the discovery of the relationship between the table representation of a problem situation and the actual cube building. (Experiment 2, Ka 01, Ka 02)

This paper points out certain interesting cognitive phenomena observed among second year pupils in classroom experiments. Naturally, experiments will be continued in the future. We will continue to focus on mapping young pupils' cognitive processes in geometry and on the process of knowledge transfer from an individual pupil to a group, i.e. on cognitive osmosis (Hejný, 2011a) as well as on the role the teacher can play to make the process effective.

Meticulous record-keeping of class activities will enable us to observe any developments in the teacher-researcher's strategies. Since the beginning of the study, we have been able to detect shifts in at least the following four directions:

1. The voice dominance of the teacher is receding, pupils are given more and more space in discussing their problem-solutions, ideas and opinions.
2. The teacher does not interfere even when there is an incorrect construct.
3. The work climate in class has been improving, pupils often work in groups and their work has been progressing from working together to collaboration. Mistakes and misconceptions are dealt with by pupils.
4. The teacher is more aware of differentiating between her pupils. She chooses tasks so that the pupil is both capable of solving the problem and appropriately challenged by it at the same time.

A significant part of the material and documentation collected in this action research has been used in class work preparing student-teachers at the department. Student solution samples (written or video recorded) are used in the course of Didactics of Mathematics for further analysis. This authentic material – which is based on pupils known sometimes to the student-teachers from their teaching practice – proves to be more effective than artificially compiled material in this type of teacher education.

Student-teachers at our faculty come across Cube Building problems at different levels. First, in an introductory mathematical course, they encounter the subject of different geometric languages used in describing 3D objects. Later they solve problems that combine geometry and combinatorics in the course in Mathematical Problem Solving Methods, all the while they are encouraged to try the subject out within their student-teaching activities. They tend to guide their pupils within the limits of their own one strategy or solution, and assess the difficulty level based on their own experience.

In conclusion we would like to emphasize the fact that each experiment represents a valuable enrichment in terms of not only our experience with teaching children but also working with teacher-students at the primary school level. We find such experiments consistently to be an effective tool in an individual's professional development.

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Creating learning situations that stimulate generalization

Part 4

THE ROLE OF TILING, CUTTING AND REARRANGING IN THE FORMATION OF THE CONCEPT OF AREA

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At the beginning of the teaching of the concept of area we focused on various activities. The tiling is emphasized particularly with congruent patterns. We make a distinction between tiling in plane and tiling polygons of a given area. Special attention has been paid to the development of the skill of estimation during the covering of the tiles, namely what kind of patterns are suitable for tiling the polygons on the one hand, and how many tiles from the selected patterns are needed to cover the polygons on the other. Another activity is cutting and rearranging polygons or tiles. These activities were carried out in classes 4, 5 and 6.

INTRODUCTION

In this work we want to show an experiment in grades 4, 5, 6, where we used the tiling to introduce the concept of area.

Research in this field of mathematical education often reveals poor understanding of the processes used for area measurement of plane figures. However, it is not only students who have difficulties in understanding the concepts of area and measurement; it is also student teachers (Simon, 1995; Baturo & Nason, 1996; Zacharos, 2006; Murphy, 2009). Some of our students have this kind of difficulties. Probably these difficulties are mainly attributed to the emphasis placed on the use of formulas, starting from the very first steps of introducing students to this subject. Though it is generally accepted that mathematics should be taught through understanding but in the topic of area it would seem that children often rely on the use of formulae with little understanding of the mathematical concepts involved (Dickson, Boyd & Davis, 1990). In our opinion the topic of measurement is very useful to develop problem solving, spatial sense, estimation, and concept of numbers.

THEORETICAL BACKGROUND

The area of a shape or object can be defined in everyday words as the ″amount of stuff″ needed to cover the shape. Common uses of the concept of area are finding the amount of tile needed to cover a floor, the amount of wallpaper needed to cover a wall etc. Measuring area is based on the notion of ′space filling′, i.e. tessellation. Tessellations can be very useful in education and in teaching mathematics. They can be used from kindergarten to high school. Since tessellations have patterns made from small sets of tiles they could be used for different counting activities. Two or more tiles usually make some other shape. Tiles can be used to teach students that area is a measure of covering. Tessellation patterns can produce many lines of symmetry. With young children you could have them learn about tessellations with pattern block pieces of geometric figures such the square, triangle, rectangle etc.

We applied these patterns to measure areas.

The starting point for the creation and development of geometry is the process of measurement, which presupposes comparison between quantities. Comparing different figures: Sometimes areas of different objects can be compared directly, without measurement. We can compare areas of two different objects that one of them divided into parts which, if appropriately recomposed, would form the other figure. We can compare areas of two different objects with tessellation, i.e. both of them are covered with congruent geometric figures (tiles) without gaps or overlaps. Activities related to each of three comparisons and evaluation of areas is carried out throughout the teaching process. At this point, either the logic of analysis and reconstruction or that of overlapping is introduced. In the case of overlapping, different shapes such as rectangles, triangles, squares, hexagon and trapeziums as well as irregular geometric shapes are used as measurement units. The teaching process provides an introduction to the concept of area and its measurement.

Measuring length can be realized by using a ruler, measurements of areas is more complex, since length is directly measured by a ruler, while area is indirectly measured through the lengths appearing in the formula for calculating it (Zacharos 2006; Murphy, 2009; Vighi & Marchetti 2011), but it can be made using other artefact, e.g. tiles which can be different geometrical figures. According to Santi & Sbaragli (2007) the use of ruler brings ′unavoidable misconceptions′, i.e. a misconception that ″does not depend directly on the teacher′s didactic transposition″. In early grades we assist to the blind application of formulae and according to our observations, the early use of formulas in area measurement has been criticized on the grounds that it generates misconceptions about area measurement.

In Hungarian schools comparison between areas is generally reduced to evaluating areas and to comparing numbers. Teachers tend to determine equivalence of the magnitude of area of two figures by means of measurement. But ″transferring the comparison to the numerical field, we are in fact working with numerical order which doesn’t consider the criterion of quantity of magnitude″ (Chamorro, 2001).

Estimation is very important in measuring area, too. Children can estimate which patterns are suitable for tiling the geometrical forms and how many tiles are need to cover the figure. Estimation is very important in real life for checking measurement and spatial ability.

To establish the concept of area tiling is a fairly important activity, but not the only one. When a polygon cannot be covered totally with a given pattern, the need to cut and rearrange the pattern into pieces obviously emerges. The aim of cutting and rearranging is to achieve the most appropriate covering of the polygon. This also happens in real life when the floor is tiled for instance. The two basic types of cutting and rearranging are the cutting of the pattern of and the cutting of the polygon. The first activity is closely related to the actual tiling, it is a more sophisticated version, whereas the latter one serves as a comparison of the areas of the polygons thus contributing to the preparation of the measurement of the area. In this way both activities contribute to the establishment of the concept of area.

RESEARCH QUESTIONS

1. What sort of activities based on tiling, cutting and rearranging contribute to the establishment of the stable concept of area?

2. To what extent are activities required in classes 5-6?

METHODOLOGY

A teaching experiment was carried out in classes 4, 5 and 6 of the demo primary school of the teacher training college. The teaching material and the teaching aids were compiled by the research team and the sessions were conducted by the class teacher in accordance with our guidelines. The tasks were done in groups of four, whereas setting the tasks and the discussion of the experience took place in the whole class. The sessions were recorded and photos were taken of the works produced and also the kids carrying out the tasks. Two 45 minute sessions were designed for all three years based mainly on the tiling activity, whereas another 45 minute session was planned for classes 5 and 6 relying on the activities of cutting and rearranging. Classes 5 and 6 had the same teacher and class 4 another one. Prior to the sessions none of the classes were involved with measurement of area. Learners in class 4 were not familiar with the concept of area at all, whereas learners in class 5 were introduced to the concepts of square and rectangle at the end of the previous term during some lessons. Learners in class 6 did tasks related to the area of rectangle and square two months before. They made use of the formulas of area and the SI measurement units of area were also introduced.

Session 1. Tiling the plane with various patterns. Tiling the rectangle with various patterns.

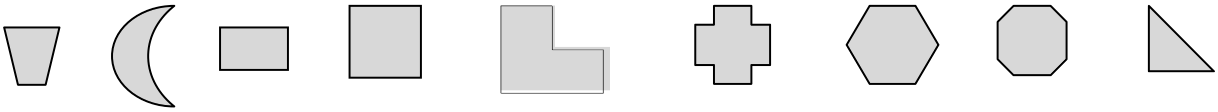
Session 2. Tiling various polygons with patterns selected appropriately and estimating the number of patterns required to cover the polygons.

Session 3. Cutting and rearranging patterns in order to cover a given polygon. Comparing the area of the two polygons by means of cutting and rearranging the polygons into each other.

DESCRIPTION OF THE SESSIONS AND RESULTS

Session 1

There were nine groups of four children. All the groups were given one kind of tiles from the set below.



Every group got several tiles of the same kind so that the rectangle could be covered.

Teacher: Try and cover as economically as you can the rectangle. Make use of the most of them but without overlapping.

The works were put on the board and the experience gained was discussed. We were wondering in what ways learners were able to put the coverings into groups.

Observations

Figure 1: The works on the board (Class 6)

In all three grades the sessions took place very much in the same way. Children were required the same amount of time to do the tasks and raising the problems and the interest in the topic was roughly similar.

Perceiving the difference in covering rectangle and plane: In class 4 learners did not perceive the concept of plane. They could not make a difference between coverings with gaps (such as octagon) and coverings without gaps and coverings as well as coverings deficient on the margins (for instance hexagon). In classes 5 and 6 the plane was illustrated by the tabletop and the geographical notion of the steppe (e.g. Hortobágy in Hungary). Children put the nine coverings into three groups according to the arrangement of the tiles: The plane cannot be covered with them without gaps (octagon, crescent) (set 1). The plane could be covered but not the rectangle (hexagon, trapezium, cross, L-shape) (set 2). Both the plane and the rectangle can be covered (rectangle, right angle triangle, square) (set 3).

In all three classes the demand for cutting emerged. During the discussions the idea came up in that case when the paper rectangles could not be covered:

S1: If one of the them could have been cut then it would have been put …(the trapezium on the L-shape) (Class 4)

S2: In set 2 the margins should be cut. (by margin they meant the uncovered parts) (Class 6)

Children have made an effort to create regular patterns of tiles.

S3: We created squares from triangles, at first we did it at random, but it was not really appropriate. (Class 4)

There were some hints at the types and the size of the tiles, as well as the connection between size and the possibility of covering:

S4: It was possible to make square from rectangles and triangles. (Class 4)

S5: It is impossible to cover, because the elements are too big. (hexagon, trapezium, octagon) (Class 4, 5)

Teacher: How many tiles do you need for set 3?

S6: The larger the plane figure, the fewer are needed. (the plane figure was meant to be a tile by the child) (Class 6)

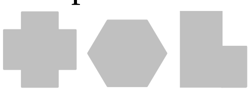
Teacher: What kind of tile would you like to plan?

S7: The trapezium would cover it, but it is ugly.

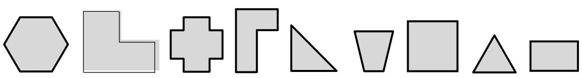
S8: The rectangle is rather thin that would reach as far as it is. (Class 6)

Session 2

We put three large polygons cut from large sheets of papers on the board:



At the bottom of the board the following patterns of tiles were seen:



Teacher: Which plane figures do you think could cover the shapes cut out of sheets of paper without gaps?

Children answer some plane figures, and then the teacher asks who agrees with the answer. The votes are counted.

Then children were put into nine groups of four, and they were given 3-3-3 large polygons together with a bag of tiles of the same pattern. Children were told to tile the large polygon with the tiles they were given. In case the large sheet could not be covered without gaps, then fewer tiles should be used so that they would not be overlapping. The idea was to make use of as many tiles as possible so that the large sheet could be covered totally. The nine tiling patterns children produced were put on the board.

Observations

It is shown in the Table 1 which tiles were selected by most of the children to tile some of the polygons.

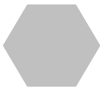
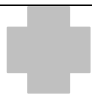
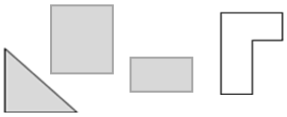
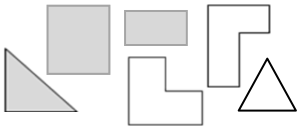
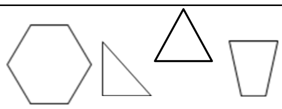
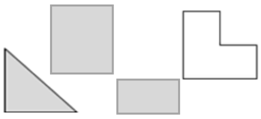
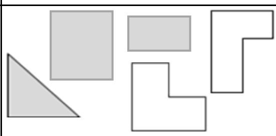
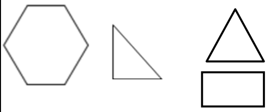
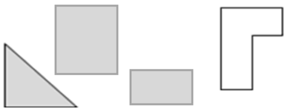
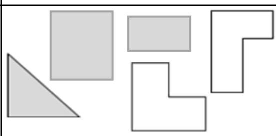
			
Class 4			
Class 5			
Class 6			

Table 1: Selected tiles

Right angle triangle was represented in every case.

Right angle triangle, square and rectangle can be found together except for one case; presumably they have recognised the relationship between them. A square can be covered by two rectangles or two right angle triangles.

S9: If the rectangle is OK, then so is the square, because two rectangles cover exactly as much as a square and we could have counted by two. (Class 4)

S10: I insist on the triangle, because if it can be covered with the square, then with this one too. With 24, because we said 12 squares” (Class 6)

S11: It seems that similar polygons can be covered with similar ones, hexagons with hexagons, L-shape with L-shape, irrespective of its shape. (Class 6)

To select the right pattern of tile for the given polygon is not that easy (Figure 2).

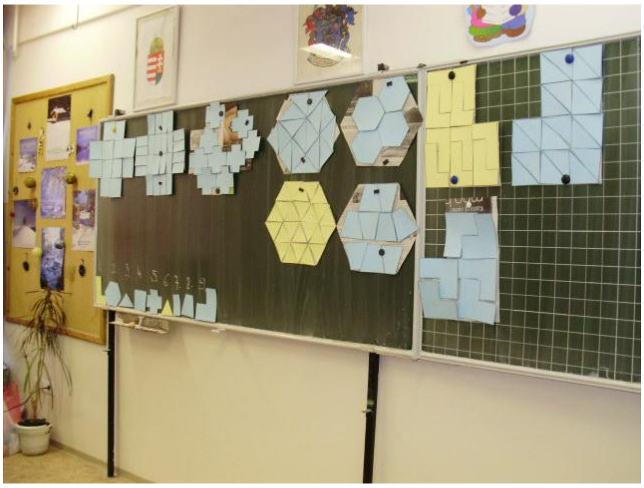


Figure 2: The works on the board (Class 6)

Covering similar shapes with similar ones does not always work.

S12: I was disappointed with the hexagon, because when I figured out my idea, I did not realize that it will be wider in the middle. (class 6)

S13: L-shape can be covered with the yellow L-shape, but not with the blue one. (He has another try with the blue one.) It can be done with another type of covering, no, it does not work...” (Class 6)

The difference in the size of the angles can be seen for them when they fit the tile to some of the vertexes of the polygon.

S14: I thought hexagon can be covered with right angle triangle, it can be done. But no, it cannot be done, because of the angles... (Class 6)

S15: I meant the trapezium in a slanting direction. The trouble is with the angle here as well.(Class 5)

Out of the two triangles and two L-shape polygons only one of them is appropriate for covering hexagon and L-shape respectively. Thus the fact that the names are identical, it is still not enough.

S16: We tried to turn around the right angle triangle to the hexagon at random, but it did not work. It cannot be done with this triangle, but it can be done with the other one, because its shape is different. (Class 5)

S17: The yellow L-shape could have been OK, one of its branches is not thick. (Class 5)

When estimating the tiles for covering children are able to make use of the relationships they have discovered between areas most of the time, but the position of the tile also matters.

The estimation of the number of tiles required for covering is useful and children are keen on it.

Session 3

To recall the previous lesson by making use of the photos of the coverings. For instance it can be seen what kind of tiles were used when they tried to cover the hexagon, and how many of them were needed.

Teacher: How many hexagons or right angle triangles will be needed for covering if the tiles are cut into pieces?

During this session children were put into groups of four.

After the first task, children were asked to compare the area of two polygons of large sheets of paper by means of cutting and rearranging. This time it was the tiles but the polygons that had to be cut and rearranged.

Observations

Children were happy to do the tasks and they were also delighted to have access to the scissors. They made an effort to cover the large sheets of paper economically, however as the teacher did not point out that they should cut the tiles only it is required. Thus they did not really paid attention to how many of the original tiles they cut into pieces. When finally the pieces pasted to the sheets were counted the results were rather various such as 19, 23, 20, 15 pieces right angle triangles to cover the hexagon. Of course it was not really conducive to the establishment of the concept of the area measurement unit, but it contributed to the covering the polygons without gaps and overlapping.

Two strategies could be observed: in the first case they aimed at systematic arrangement, whereas in the other case they tried and made use of every little pieces of cutting (Figure 3)

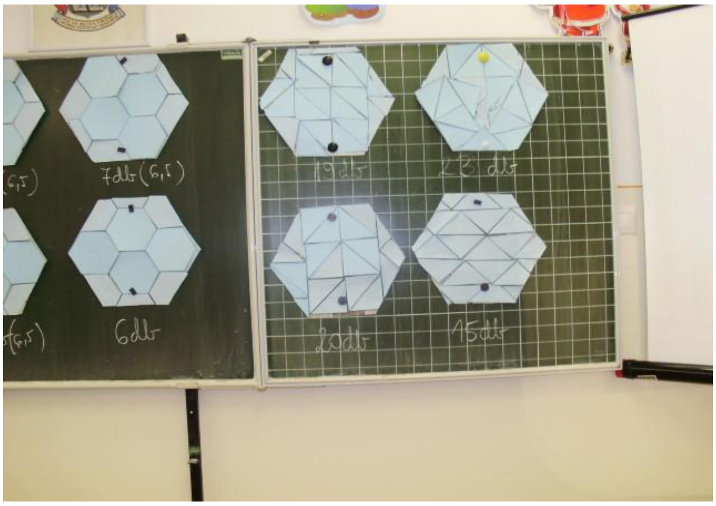


Figure 3: Hexagons on the board (Class 6)

This first task meant to make a connection between tiling and cutting and rearranging, although cutting the tiles into pieces not necessarily leads to the concept of the area measurement unit, still in everyday life cutting the tiles into pieces is quite often used in tiling the bathroom for instance.

Children of Class 6 applied three different strategies in the second task, in the comparison of the hexagon and the L-shape, the hexagon was cut into L-shape (Figure 4), the L-shape was cut into hexagon, and they tried to turn both polygons into rectangle (Figure 5)

Figure 4: Hexagon into L-shape

Figure 5: Both polygons into rectangle

In Class 5 children did not manage to turn polygons into rectangle.

Cutting and rearranging one polygon into another seems to be useful to compare their areas.

Rectangle has emerged as a kind of transmission polygon for the comparison of areas.

The task has actually focused on the main thing in comparison: if one of the polygons can be cut and rearranged in a way that it covers the other one, then the two areas are equal, but if there is gap, the areas are not equal.

CONCLUSIONS

The two approaches to cover the subject matter, the frontal classroom and group work were appropriate.

After the first two sessions both the children and the teachers came up with the idea to continue the experiment. Cutting seemed to be a useful way to solve the problems, however we realized that in class 4 more activities of tiling are required prior to cutting and rearranging.

Activities are enjoyable in class 5 and 6 and they are not boring at all. Learners were highly interested and creative.

In Class 6 during the activities learners did not rely on their knowledge about rectangles and squares gained earlier. For instance they did not want to tile the rectangle only with squares, or to use the number of squares for the estimation. For them the tasks of cutting and rearranging were as much as novelty as for younger learners.

The teachers, who were involved in the experiment, came to realize the complexity of the problem and also the benefits of the extended elaboration of the topic.

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STUDENTS WORKING ON REGULARITIES: A CASE STUDY FROM POLAND

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The paper presents a research focused on the ability of recognising regularities. It was carried out on grade 5 pupils from Polish primary schools and focuses on the ability of recognizing regularities. The results led us to three different types of students' reasoning, depending on whether they used the arithmetic, the geometric or both aspects of the task.

INTRODUCTION AND THEORETICAL FRAMEWORK

Regularities constitute a basic topic of Mathematics. They can be found at almost every branch of mathematics, for instance, sequence classification and the principle of mathematic induction are based on them.

Discovering and recognizing regularities is an issue that constitute a very important problem, present in international learning trends (Zazkis, Liljedahl, 2002a, 2002b; Littler & Benson, 2005; Carraher, Martinez & Schliemann, 2008). The problem of rhythm and regularities have drawn much attention in mathematics education in Germany. To set an example we can look at the MATHE2000 programme (Wittman, 2001) created for students and teachers in Dortmund. Studies that were conducted within this programme have shown that even 6-7-year-old children can sort out the issue of regularities without bigger problems, if they are taught in the "rhythm and regularities' spirit".

Teaching how to spot and make use of regularities is based on introducing some bases principle that works in mathematics, namely readiness to action and activity. Regularities stimulate the kind of reasoning that goes beyond particular cases, the one that enable students to think about general rules. This kind of attitude is realized not only in Germany but as well in a lot of other countries such as Italy (Malara & Navarra 2003), the Netherlands or Czech Republic (Hejný & Littler, 2002). This trend was as well taken into consideration in International Students Knowledge and Abilities Study PISA (Białecki, Blumsztajn & Cyngot 2003). Mathematical content that occurs in PISA tasks was divided into four areas among which a group called "change and connection" was distinguished. Tasks from this group made up 26% in total and to solve them students had to recognize and use some regularities of recurrence type.

A lot of research results that encouraged us to immerse in the role of regularities in children mathematical education can be found in Poland. According to Siwek

(1985) the ability of recognising regularities and rules in simple mathematical contexts is a key factor in the proper mental child development. Other studies (Gruszczyk-Kolczyńska, 2001; Urbańska, 2003) have shown that the child's ability to recognise the rule is the source of satisfaction. On the other hand, there is also the point of view according to which in teaching mathematics:

> …there are not enough opportunities to enable students to spontaneously derive pleasure from discovering and experiencing the fact of finding out something new. Students are rather indifferent to mathematical problems, and a human being that is indifferent to something cannot be creative. (Dyrszlag from Skurzyński, 1992, p. 34).

Within Polish school practice in teaching mathematics students are mainly exposed to tasks which aim at applying ready knowledge. In particular, students do not meet problems that involve the use of regularities. The restricted number of hours of mathematics classes and the tight schedule does not let the teacher introduce something that is not a part of the curriculum, which in fact may be relevant to the international mathematics teaching trends. Nevertheless, teachers are supposed to adjust the curriculum in a way it involves solving problems such as open tasks, looking for regularities, the ability of presenting the results as well as their justifying and assessing, finding examples and counter-examples. This is why the preparation to trace these kinds of issues calls for recognizing the children's natural strategies for solving these problems.

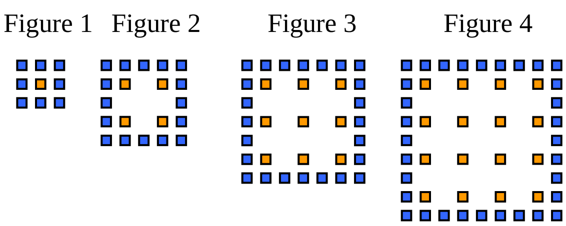
METHODOLOGY

Our research, which was conducted among fifth-class students from primary school, aimed to find the answer to the following questions:

- How do the pupils form primary schools deal with a task involving discovering and spotting regularities?
- Are they able to make generalizations within some noticed rules?

The task that served as our research tool is given below:

1. Bolek and Lolek thought up a new game: making figures from colour blocks. Bolek arranged the yellow blocks and Lolek arranged the blue ones. Their work looked like this:



1. Complete the table:

Number of figure	Number of yellow blocks	Number of blue blocks
1	1	8
2	4	
3		
4		
5		

2. If the boys wanted to arrange the seventh figure which blocks of both colours they would use?

3. Is a figure with the same number of yellow and blue blocks possible? Why or why not?

4. Bolek and Lolek decided to arrange a very big figure. Which blocks would they need more: blue or yellow?

This was an arithmetic-geometric task which allowed a lot of different interpretations. There was nothing mentioned about the fact that those pictures presented some scheme that should be followed. The aim of the task constructed in this way was to check whether the students are able to discover the regularities occurring in the task in order to use them in their later work, or if they are more likely to take their own free actions.

The research included 38 students at the age of 11 attending two different grade 5 classes. There were 20 girls and 18 boys in the research group. The research was conducted in school conditions, during normal classes. The students were sitting at their desks. It was about creating a situation that would be as similar to school everyday life as possible. The students were working in pairs. Each pair obtained one sheet that contained a task to solve and an extra paper where they can put down their calculations. The students were informed before starting their work that there were going to work in pairs, that their work would be recorded but they would not be assessed.

Arranging task to work in pairs was done on purpose, as the process of solving task by a joint effort created the opportunity of conversation among students. We wanted to reach some verbal statements as they would be a very helpful tool in trying to reconstruct the way of reasoning that the particular pair had been following. Additionally, during the work we had at least a short conversation about the tasks with each pair. In the analysis that follows we use both types of data, written students' work and the records of the conversation supplied with a video showing how the work was performed. The atomic analysis (Hejný, 2004) of students' work and the atomic analysis of the video mentioned above were the research method that we used. Some additional information was obtained through the conversation with the teacher who was present during the whole experiment. The conversation took place not only after the finished work but as well during performing the task by the students.

STUDENTS' WORK ANALYSIS

The activities in both classes were similar: students started their work by counting the elements located in the pictures, by depicting the figures and then they filled in the table with the results of their work. Filling the first four lines took them comparatively little time. At this stage of solving the task any trials of discovering regularities were spotted. They did not find any rules until they reached the fifth line of the table (to which no illustration was attached) which made them think about the task and looking for the proper solution. The quest followed different ways, and its analysis let us distinguish the following three types of reasoning (Pytlak, 2006):

- *geometric*: paying attention mainly to the pictures, spotting geometric connections; strong visual aspect;
- *arithmetic*: ignoring the pictures and paying attention to the numeric data taken from the table; finding and discovering arithmetic connections between number given in the table;
- *arithmetic-geometric*: both the table and the pictures were taken into account at the same extent, finding geometric connections and combining them with arithmetic ones; replacing geometric connections with arithmetic ones.

The types of reasoning given above are connected with the way of students' activity during their work at the whole task. The most essential issue in the task was the way students filled in the fifth line. The analysis of this particular part of the task enabled us to differentiate the following ways of the work that led to the discovery of what is going to be the next step in the process of finding the final solution:

1. Analysis of the table's columns filled with the numbers (up to this stage) - arithmetic reasoning:
   - Discovering the rule connected to the blue blocks and generalizing it so as to apply for the yellow blocks;
   - Discovering two different rules for the blue and the yellow blocks.
2. Analysis of the pictures presenting the figures and drawing a sketch depicting the 5th figure; following the scheme and drawing the next figures- geometric reasoning;
3. Analysis of the given pictures and on this basis filling in the tables- arithmetic-geometric reasoning.

Below two examples of students' work according to the type of their reasoning (1st and 3rd method) are presented.

Robert and Bartek's work as example of the 1st method

The two students started their work by counting the elements of each figure. Then they put their results to the proper places in the table. The four first lines of the table were filled in very quickly (number 16 in 2nd line, 9 and 24 accordingly in the 3rd and 4th lines). The problem started with the 5th line to which they did not have a picture enclosed. To find the solution they started analyzing the table, this time column by column. By looking deeper at the data, they found out that each number is 8 times bigger than the previous one. It led them to the conclusion that in the case of the blue blocks the difference between the following values is fixed and amounts 8. The rule seemed to be clear and obvious. By applying the rule that they had just discovered they filled in the 5th line in the 3rd column (32+8=40).

It was much more difficult to discover the rule the changes in the amount of the yellow blocks that was to be put in the 2nd column in the table. One suggestion was to fill the 5th line with number 24. The four previous numbers were directly read from the picture, this could not be done for the 5th line. The students knew that the number of yellow blocks also changes (is growing). They might even not have been analyzing the numbers given in the column "yellow blocks number". The most probable is that the students wanted to apply "first signal strategy" (Żeromska, 1998) while trying to fill in the 5th line. The conviction that if the yellow and blue blocks occur in mutual configuration, the rule that work for blue blocks should as well work for yellow ones, might took its toll. The fact of discovering a principle and realizing how obvious and clear it was might have led to coming to the conclusion that it is a generally working rule for the whole arrangement. They also felt more confident and as a result having already a rule they decided to use it once more, namely for the yellow blocks. There is also another possibility. The boys did not even have to count how big the difference is between 16 and 9 but they might have merely roughly estimated it, claiming that it is 8. On this basis they could have decided at last to apply the rule working for the blue blocks and add as well 8 yellow ones. Hence they obtained that 16+8=24 in case of 5th figure. The entry that they made in the 2nd column in the table created the sequence of numbers 9, 16, 24 and the numbers 9 and 16 were the results of the pictures' analysis.

The blank cell in the table where the number of yellow blocks should be written prefaces the one where the number of the blue blocks is to be put down. The order in which the boys gave their answers is the following: firstly the number of blue blocks and only then the number of the yellow ones. This procedure may imply how the boys analysed the table. At first, the rule that could be applied for the blue blocks was discovered and it was over-generalized to apply for the yellow blocks.

There is a gap between the figure number 5 and the one with number 7. The boys probably filled in this gap by using the rule that they had already discovered on their own. As the starting point the students took the number of blocks for the figure number 5. The numbers for the next- seventh- figure were figured out through following calculations: 40+8+8=56 for the blue blocks; and 24+8+8=40 for the yellow blocks. At this stage of the work the boys were not aware of receiving a wrong answer to the number of yellow blocks of which consisted figures number 5 and 7 (numbers given by the boys: 24 and 40). The mistake was due to this overgeneralization of the rule that was true only with blue blocks and not how the students thought for the yellow ones as well.

When filled in the whole table and gave the answer to the 2nd question, they had a little conversation with the teacher:

N1: How do you know that here will be fifty six blue blocks and forty yellow ones?

B1a: We count that here [He shows on the column “the number of blue blocks”] it is always add eight [reads and shows in the same time particular values from the table] 8, 16, 24, 32, 40, 48, 56 and then we obtain that results.

This piece of conversation demonstrates that the students’ strategy for the blue blocks was the following: “it’s growing at 8”. Initially, during the conversation with the teacher the students were only giving records from the work they had performed before. They were confined only to the work at the column “blue blocks number”. Bartek gave a tangible description of their work: he read the data from the column mentioned above and extended it with the next two blanks. The statements about the yellow block were rather succinct:

B1b: And there [He shows on the second column of the table] we also added.

T2: What is here?

R1: Here? The number of yellow blocks.

T3: How did you count the yellow blocks?

B2: The same like here [He shows on the 3rd column] ...we also added.

The teacher’s questions (T3, T4) show that the pupils had not analyzed the column for the yellow blocks but they assumed that the number of yellow blocks increased according to the same rule as it did for the blue blocks (reply B2 clearly infers it). Not until this point of conversation had Robert verified former hypothesis, according which the number that differs successive amount of blocks is 8. After realizing that something may be wrong he started checking differences among the next numbers. It led him to find out that the difference did not account for 8.

T4: Also did you add after eight?

R2: No, after five.

B3: This means that here is ... In the second figure we have …

R3: In the third nine.

B4: Do you have nine blue blocks?

R4: No

Boys: Yellow ones

B6: You have nine yellow ones.

R6: And twenty four blue.

The two first lines in the table had been already filled in, so the students started their work with putting down the number to the 3rd line. It is why Robert fixed number 4 (which was given in the 2nd line) and number 9 (with which he himself filled in the 3rd line) as a starting point for his consideration under the matter of his own strategy. The difference between those numbers was 5. It was quite different value that the one that the students took as a constant difference among the successive amounts of the yellow blocks. This situation caused some internal problem for the boys. They had already taken the rule they had discovered on their own as generally working when they realized that this same rule is not useful in fact.

The boys got into conversation with each other once more. As we can see right now, the empiric data that they filled in the table with is starting to live on its own.

T5: So what is it with yellow blocks?

B7: Should it be every five?

R7: I suppose so.

B8: But look [show on the line *2 and 3, where are numbers 4 and 9*]

T6: Every five?

R8: I got lost [He paints over the values *9, 16, 20 from the second column*]

B9: Me too.

The fact of discovering some stable situation (the one in which changes are fixed and easy to establish) turned out to be a very strong condition for the students. This situation is clearly depicted by the over-generalization of the rule for the blue blocks to apply it for the yellow ones. After finding that the number 8 is not in fact correct, the students went on looking for the principle that could have been stated as the "stable increase". Although the fact that the boys actually realized and spotted their mistake, the hypothesis stated by the teacher (T6) which was simply the verbalization of their former thoughts (R2, B7, R7, B8) did not convince them. They might have verified this hypothesis with a further sequence of numbers so as to have spotted that the two statements: "stable increase at 5" and sequence of number 6, 16, 20 did not go in agreement with each other. Robert deleted the previously written numbers (even the ones that were written down after counting the elements prominent in the pictures). He gave up his struggle to find any other solution ("I got lost").

At that moment, a change in the way of reasoning occurred. This time they analysed all numbers from the "yellow blocks number" column, not as it had been done before; they analysed only the numbers that they had filled in. It was the ground-breaking moment in the students' reasoning that resulted in discovering a new rule that applied for the yellow blocks.

T7: Look, between one and four there isn't five

R9: Three

T8: There is five between four and nine

B10: Odd we add!

R10: [Instead of deleted values he writes numbers 7, 10, 13]

T9: How did you add odd?

R11: For example seven

Robert did not keep up with his classmate thoughts. For all the time he was working according to the principle that the number grew with the stable value. He had already known that it was neither 8 nor 5. When the teacher paid attention to the first two values: 1 and 4, between which the difference amounted 3, Robert thought that the given number 3 would be the stable value with which the number of the block grew. This misconception made him replace the number that he had earlier printed out with the following ones: 7 (4+3), 10 (7+3) and 13 (10+3).

Bartek verified his new discovery and simultaneously filled in the table once more. They went on commenting on his conduct, having claimed that he applied the rule "add the successive odd number to the former number". He consequently used the past tense as if he wanted to underline the fact that he was strongly convinced that the "new" rule was correct.

B11: Because we add for ex ample like this: three [shows on the 1 and 4 in the second column], because three is odd. Why have you crossed it out?

R12: I think that it was wrong.

B12: Three, and then five is odd, and then we add ... [he turns to his friend] Was it sixteen here?

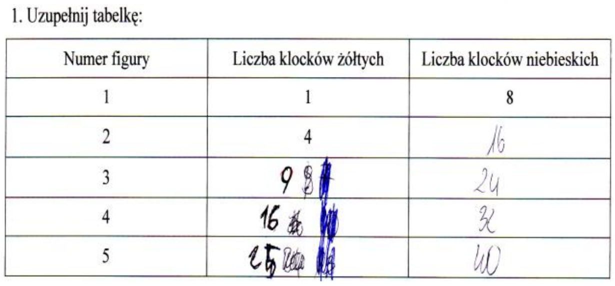
R13: Yes, sixteen

B13: To sixteen we add odd ...

R14: Eight

B14: No, to sixteen we added seven odd. And only odd we added

Bartek crossed out the record given by his classmate and put down the correct values: 9, 16, 25. From then on the group work was abandoned as Bartek continued his work on his own. He did not even try to explain his mate how he knew what to put down in the appropriate spaces. He was so deeply immersed in the task that he did not feel like wasting time to needless explanations. He took entirely the initiative. Robert did not keep up with Bartek's ideas. He also did not take over the conventional meaning of the past tense he was using. When he reached the 4th figure and number 16 appeared, he got into the conversation and gave the report of the performed work saying "the difference is 8, it is growing" (R14)


1. Uzupełnij tabelkę:

Numer figury	Liczba klocków żółtych	Liczba klocków niebieskich
1	1	8
2	4	16
3	9	24
4	16	32
5	25	40

Figure 1

Robert remained merely the observer of Bartek's actions during the process of solving further tasks. Meanwhile, Bartek was trying to apply the regularity he had discovered in the work. At the beginning, he counted the number of the yellow blocks presented in the picture number 7. He got the result 57. Next, he placed the paper with the written task on his side and replaced it with a clear sheet of paper where he started writing down the rest of the table's entities. The last data that was written down in the paper with the task involved figure number 7. Then Bartek extended the table so as to put there as well the number of blocks that the figure number 8 consisted of (Figure 1). However, he was not sure whether he acted in a proper way, so he found out his notes unimportant and did not take them into consideration while answering the 3rd and 4th questions.

In Picture 2 the left column corresponds to the blue blocks, and the right one to the yellow ones. It is clearly depicted that the students applied the rule that Bartek discovered. In the first column they were adding 8 to the former numbers and in the second column they were adding the succeeding odd number (15, 17, 19). As they had made a mistake before, while trying to count the number of blocks used to create the figure number 7 (they obtained 57 and not the correct number 49), they still did not reach the results that they could have used in the further work (even though they changed their way of reasoning and applied the correct rule).

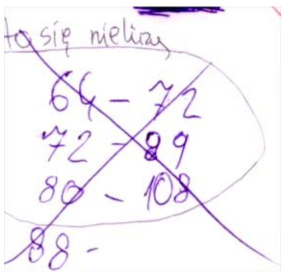


Figure 2

While putting on the paper the next values for the number of yellow and blue blocks, the boys could have spotted that they results had started to drift apart. If the results had met for the figure number n=8, they would have reached the answer for the 3rd question. Instead of that, they even did not know how based on the results they had reached should they explain that the figure that would consist of the same amount of yellow and blue blocks did not exist.

The fourth task called for going beyond the very general level they stuck in, and the boys did not succeed in answering the last question.

Ola and Karolina's work as example of the 3rd method

The two girls were working together. They divided the work so that one of them was counting the elements of each figure whereas the second one was filling in the table with the data given by her partner. During the whole work they were exchanging their findings about the task. The person who took the lead in the task was Ola. Karolina, in fact, was only performing the tasks that Ola asked her to perform. Nevertheless Karolina also tried to control whether they follow the proper way of reasoning and if Ola's ideas were incorrect, Karolina did not hesitate to put forward their own ones.

The girls launched their work by counting the elements and filling in the table with the data they obtained. In order to fill in the 5th line of the table, the girls returned to the pictures of the existing figures. They tried to find out any way in which the next figure was created through the analysis of the given figures. The analysis started by taking into consideration a blue frame and then they went on to look at a yellow interior. It led the girls to the following statements: the number of the blue blocks located on the one edge of each figure grew by the stable number-2, the number of the yellow blocs located on the one edge of each figure grew in turn with the stable number-1. The girls were able to recognize the geometric connection among the figures but on the other hand they were not able to establish the number of the blocks of the particular colour that they needed to create the 5th figure. They had to draw the picture of this figure by using the geometric connections they had already found out. They made the picture of the 5th figure in an additional piece of paper.

The girls started drawing their sketch with the frame consisting of circles that depicted the blue blocks. Then in the interior of the frame five rows, each consisting of five elements that were to represent the yellow blocks, were drawn. The issue how the yellow and blue blocks are located towards each other was not taken into consideration by the girls. They also did not maintain any connections between the blocks while sketching their picture. Neither the proportion between particular elements, nor the layout or the shape of these elements were properly depicted. As we can see in the Picture 3 the girls might have found it more convenient or suitable to represent the square shape of the blocks with the circles. The first two elements that we can spot in the interior of the frame might suggest the "square" way of the girls' reasoning. Nevertheless, the girls quite quickly came to the conclusion that the shape of the objects did not matter. They started to draw circles as it was less demanding task. They differentiated the colours of the figures but they did not correspond to the colours given in the task. This differentiation served only as a way of recognizing which object was which. The picture did not depict any particular figure; on the contrary it was drawn so as to illustrate some arithmetic values that corresponded to the particular figure. Not until finishing the sketch did the girls counted its elements: firstly the blue blocks simultaneously filling the circles that presented them appropriate numbers; secondly the yellow blocks. The process of counting the blue blocks was abandoned while reaching number 15. While counting the yellow blocks a quite different strategy was used. The girls were pointing in turns the elements from each row (the dots put on the each element imply this

Numer figury	Liczba klocków żółtych	Liczba klocków niebieskich
1	1	8
2	4	16
3	9	24
4	16	32
5	25	40

Figure 3

procedure) and simultaneously they together were counting them undertone (what was recorded on the video). The results that they obtained in this way were written down in the table.

When the girls embarked on the trial of finding an answer to the second question, the teacher came and got into the conversation:

O1: How many are together? Count. [She turns to her friend to count how many elements she has already drawn]

K1: [She is counting the elements of figure no. 7]

T1: What are you doing now?

O2: To this second question.

T2: And how do you know how picture has to look like?

O3: Just because as here it was 11 in the fifth figure in one row [she showing at first on fifth line of table and then on column of blue blocks in figure No. 4 and draws up 2 blocks], then for sixth one would be 12, and for seventh one will be 13.

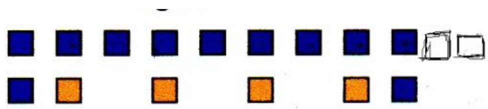


Figure 4

The girls once more divided the work among them. Ola by taking advantage of the picture of the 4th figure was counting how many elements they should have painted; meanwhile Karolina was drawing and counting up the elements that had been already sketched. Number 11 given by Ola did not ensue from the table. While answering teacher's question Ola actually firstly pointed the 5th line in the table but it was merely a reference to the number of figure that she was talking about. So far the girls had been only analyzing the connections between particular figures and not among the arithmetic data given in the table. The next step done by Ola shows also this fact, what she done was drawing two extra squares to the figure number 4.

T3: How do you know it will look like this?

O4: Because we noticed that [points all the figures, thinks] here it changes, about 2 blocks, so it will be 11,13,15 [turns to Karolina] so draw 15 blocks

K2: [continues drawing the frame for the figure no 7 consisting of blue blocks]

Thanks to the analysis of the pictures the girls were able to spot that in each next figure the number of the blue blocks increased at 2. At first, Ola while trying to answer the teacher's question (O3) made a mistake. She might have done it unconsciously as she started to enumerate the successive natural numbers 11, 12, 13. Nevertheless, the girl very quickly stated the rule that she had discovered and went on applying it in a proper way. The picture number 5 could have been very helpful although they did not refer to it during their conversation with the teacher within which the girls explained that discovered on their own regularity.

The picture was the result of the empiric work that was obtained thanks to applying the discovered regularity. The discovery of the rule according to which the next figures were created enabled the girls to draw the picture of the given figure without necessity of referring to the previous figure. This conclusions are based on some Ola's statements such as "here it the number changes at 2" – what might have be taken by the girls as some kind of certainty that could have be used- "it will be 11, 13, 15" – and here the girls gave the results of applying the successive adding of the number 2.

The way in which the picture presenting the 7th figure was created was quite characteristic. Firstly, the column consisting of the blue blocks was drawn, then the amount of the elements was counted and checked whether it was correct. Afterwards, the horizontal line was drawn in which the 15th element was treated as the first in a new row. It clearly suggests that the spotted strategy was applied for one edge. Hence it seems to be clear that the girls took four elements twice into consideration. However, it did not hinder the girls in their work at the task. The elements were treated in two different ways: when were perceived as the product of the strategy was treated as "two elements more in one edge than in the previous row", whereas when perceived as the component of the whole figure was treated in totally different way.

During the next stages of the conversation, the teacher came back to the first question and asked the girls about the way they had filled in the table. In the sheet of paper in which the task was written down there were no extra sketches added by the girls and in the additional piece of paper there was drawn only part of the 7th figure.

T4: How did you complete the table?

O4: Here? [Points to the last line in the table] we counted the squares [shows the fig. 1-4]

T5: You could count only these four figures, what about the fifth one?

K3: The fifth one is ...

O5: We added 2, because we noticed that they increase by every 2. We added this to the blocks [points to the perimeter of the 4th figure], here 2 [adds 2 squares to the left column of blue blocks] and here...[tries to draw squares on the left side of the top row, after a moment of hesitation] no, it can't be here

K4: Ola, it was sufficient to add eights here [points to the third column of the table], then it would be together 40.

O6: [looking at the numbers from the columns 'the number of blue blocks'] oh, that's true.

Teacher: So was it sufficient to add 8 in blue blocks?

Girls: Yes, it was.

When the teacher asked the girls how they had filled in the table, Ola introduced the rule that they had discovered and show how the rule could have been used for the 5th figure. That rule was based on increasing the length of each edge with two elements. In order to explain the strategy the girl referred to the 4th figure and using it she showed what should have been done in order to get the 5th figure. The elements that she added were standing out from the circumference which resulted in the fact that the newly created figure did not maintain the shape of the former figure. Ola did not show the picture that was previously drawn by Karolina in the additional sheet of paper that depicted the 5th figure. That shows that the picture itself was not the most important tool but the way in which it was created was in fact crucial factor. The girl wanted to present how the rule that had been discovered could have been working for the blue blocks. For all the time she was referring to the rule "2 elements more than in the previous figure". Based on the pictures 1-4 Ola was able to imagine the further sequence of edges for hypothetically constructed figures. In order to move on to the 7th figure, she started with the analysis of the 4th figure. She did not treat the figure as a whole but instead was looking only at the one edge (which then had 9 elements) and on that basis she was able to generate (count up) the amount of the elements of the further figures through controlling the number of undertaken steps (to obtain 7th figure from the 4th one you have to add to the each edge 2+2+2 (11=9+2, 13=11+2, 15=13+2).

During the conversation with the teacher the girls came up with a new idea. They spotted that instead of drawing the figures over and over and then counting their blue blocks, it was enough just to add 8 to the previous value. Karolina was the first girl who noticed that instead of adding new squares to the pictures and subsequently counting them is was easier to add 8 to the number of the blue blocks that the previous figure was built of. The discovery was made thanks to the analysis of data that had been put in the table. As soon as Karolina spotted this newly-found regularity she shared this idea with Ola who agreed with her friend without doubts.

We can see here how the geometric rule that was once spotted and applied was replaced with the new rule, the arithmetic one. These two regularities did not rule out each other but they were still working independently.

T7: And the yellow ones? How did you find out how many of them there are?

O8: Because we noticed, that if there is one in the first figure [points yellow blocks in the figure] here in the figure no 2 there are 2 [points to the first column of yellow blocks in the figure no 2], in the figure no 3 there are 3, and here 4 [saying this points to the first columns in each of the figures]

Firstly the only one rule for the yellow blocks that had been discovered by the girls was the one saying that there is a connection between the amount of the yellow blocks in one row with the number of the figure (O8). In order to make the sketch of the figure the girls were drawing consequently 5 rows with 5 elements in each for the figure number 5; 7 rows with 7 elements in each for the figure number 7 and so on. Ola tried to explain her point of view to the teacher by presenting the picture number 5 that had already been made.

K5: [points to the second column] here increased by 3, and here by 5…

T8: Did you draw the figure and then count the blocks? It is a very good idea.

During her conversation with the teacher, Ola was focused only on giving a report of the work she had done whereas Karolina took a more reflective attitude. It could have been influenced by the work that she had been supposed to do – merely drawing the figures. Actually, she did not have the opportunity to analyze the arithmetic data given in the table. Not until Karolina's conversation with the teacher, had Karolina had the chance to investigate the data and found out any connection between successive numbers. The discovery for the blue blocks that had been made and agreed by Ola encouraged Karolina to continue on her searching for existing connections.

T9: [turns to Karolina] Do you have any idea?

K6: That's right, here it increases by 8 [points to the third column], and here by 3 as we counted [points to the second column (1 and 3)], and here by 5...[stops for a while, a moment of hesitation]. Yes since there was 1 here [points yellow blocks in the figure no1], there were 4[points with circular movement to yellow blocks in the figure no2] and here 2 [points to the first column second line], here 9, and there 3 [points 9 and 3].

O9: Or maybe not...

K7: We counted in this way: here 1, here 2, here 3, here 4 [points to the following figures in one column], in the next one should be 5

O10: And in the sixth one 6 and in the seventh one 7.

T10: So there would be 5 yellow blocks in one row in the figure number 5. And how many altogether?

K8: We can easily count this [points to the picture of the figure no 5 made by her on the separate sheet of paper]

Karolina was trying to find any connection between the numbers of yellow blocks in the individual figures. She started analyzing the data written down in the column "yellow blocks number". She was looking at the differences between successive numbers. Her aim was to find a rule similar to the one that was working for the number of the blue blocks. She was looking for any constant number that while added to the previous number will give the next one. She spotted that the numbers grew in a quite regular way: To get the second figure you have to add 3 to the previous one, to get the next one you have to add 5, then the number you have to add is 7 and so on. Nevertheless, Ola was not sure if it was the right way. She continued her trial to connect somehow the number of the yellow blocks with the number of the appropriate figure. She might have reached the right conclusion if it had not been her friend interruption.

Ola interrupted her friend; Karolina's explanation was different from the points that the girls had discovered and used within their work. After Ola's interference in Karolina's matters, the second girl gave up her investigations about the issue of the number of the yellow blocks and went on to report the tangible course of their work. She showed what they had spotted, namely that the number of the yellow blocks in the figure were equal to the number of that particular figure.

O11: There will always be more blue blocks.

T11: Why do you think so?

O12: Because yellow as if on this basis, in the fifth there would be 5 each, in the sixth one 6, so in the fifth figure they double.

K9: It can't be more here, there can't be equal number of the figures [points to the blue and yellow row; points to the blue blocks in the corner]

O13: There can't be more yellow blocks than blue ones.

K10: If we counted the figures, these small squares, would be the same here. [points to the row of blue blocks and adjacent to it the row of yellow blocks as well as the two blue blocks being in the same row as yellow ones.]

O14: Nothing can be done, nothing, then it would be...

Girls: No. Nothing can be done.

Both girls agreed that there would always be more blue blocks. They took into consideration not only the pictures that they had been given at the very beginning but also the ones they had drawn on their own. While they were looking at the figures they spotted that the blue blocks were located close to each other whereas the yellow one were in a more distant location to each other. Furthermore, the girls noticed that the yellow blocks were surrounded by those blue ones, which implied that in one row there would never be more yellow than the blue blocks (as there were always two extra blocks in a row consisted of the yellow blocks).

The last conversation on the strategy that the students followed eventually show that they were not analyzing the figures as a whole but they only took into consideration the extreme columns and rows. They applied the so-called "local visualization". If Karolina had been a leading person in the group or if she had been able to realize her points, they might have correctly solved the task.

Although both girls, Ola and Karolina, noticed some regularities, they were not able to state and report them clearly. They were working only within the real, tangible objects (in that case, that objects were mainly the pictures of the figures). They were not able to go beyond the data that the task presented to them. It shows that the girls are not able to work at the higher level, level of abstraction, although it is possible that if the teacher had helped them, the girls would have been able to succeed and draw some regularities.

CONCLUSIONS

The students who took part in the research were for the first time exposed to a task in which they had to find out and apply some regularities. While approaching the task they began with the action of counting the elements that had been already drawn. Based on that students' conduct we can draw the following conclusion: it is not enough to depict the task as a sequence of connections to provoke students into searching for rules and connections. The way in which students started their work shows that they are not used to "reasoning through regularities" and what is more they do not present the "connections searching " attitude. As Krygowska (1977) underlines, students tend to form their image of mathematics only on the basis of the tasks they are exposed to during normal day-to-day classes. In general, students solve the tasks in an exactly the same way as they were taught to do it during classes. In the task that we presented its contents (the table, the figures, the blue and the yellow blocks) had motivated students to undertake a practical attitude towards it. They counted the elements and wrote down in the table proper numbers. Besides that, they were partly able to spontaneously discover some regularities. Students' activity was focused on looking for the connections. When they succeeded in finding one they were encouraged to look for another. It should seem that the students' attitude presented within the research could be the foundation to making a hypothesis that looking for regularities is a human natural tendency.

The two different ways of reasoning that are presented in this paper depict three different ways of mathematic thinking. As the work at the task went on, been stuck to one particular way of reasoning brought different results. Some ways of thinking helped students in making generalization whereas others were disturbing. What happened with these methods at the further stages of solving the task? The answer to this question is presented through the elaborate descriptions of students' work.

Bartek and Robert were consequently applying the first method of work. From the moment when they spotted the arithmetic regularities the pictures started to be useless for them. They ignored them during the next stages of their work. Firstly they tried to over-generalize "their" rule for the blue blocks and they applied it for the yellow ones. The conversation they conducted with the teacher let them realized some mistakes in their way of reasoning: at the same time it was a stimulus for them for finding a new rule for the yellow blocks. To some extent, they were even able to generalize these rules. The method of the work at the first part of the task that was adopted by the boys turned out to be too stiff. They used the method over and over while trying to find a solution to the next questions and did not bother themselves to look for any other strategy. The one they had discovered restricted them to looking at the issue from only one point of view. Nevertheless, their method led them to the full and correct solution of the task. It is apparent that these boys in fact do have a big potential of reasoning through regularities, finding connections and, what is the most important, of both independent and active way of thinking. Their decision to analyze connections between figures confirms that.

They succeeded in going beyond the data given in the task and to move the rules that had discovered on the elements that they could neither touch nor see. The boys were able to discover some rules but their difficulty was to admit the generality of those rules.

The difference in the way in which boys try to solve the task is visible. Both of them presented different ways of reasoning. The trait that dominated at Bartek's attitude was the accommodation of his own reasoning. In the course of the work at the task Bartek modified the rules he had discovered and adjusted them to the already existing empirical data. In comparison, Robert was stuck in his first idea all time. The matter that was likely to be the most important for him was purely solving the task and getting some data that could have been taken as the correct one. Indeed, he did not seem to worry about the way in which the data had been obtained. It seems that Robert is good at real, tangible situations in which he can deal with the existing objects as opposed to the situations when he is forced to move on to the abstract level. Bartek takes care of both the result and the method that led to it. He derives satisfaction and joy from the same activity of discovering and creating something new. However also he finds it difficult to be in his element while dealing with abstract problems. He tried to solve the task to the very end but his own restrictions are stronger and too difficult to overcome.

Using the third method may suggest the students' mathematical maturity. However the student who is not sufficiently familiar with mathematics can suffer a failure while trying to adopt that method in solving the task as Ola and Karolina's struggle shows. Initially, the girls were only focused on the analysis of the figures' pictures that were enclosed to the task and on discovering some geometric regularities. After the discovery made by Karolina, who spotted some arithmetic connections, the girls were trying to use only this sort of regularity in their further work. Unfortunately, they kept too strongly to the geometric representations of particular figures. They focused on the way the blocks were located in the pictures (an analysis of particular rows, not the figure as a whole). We can point some duality in their conduct. On the one hand we have the arithmetic regularities spotted by the girls, on the other hand there are some geometric connections among the adjacent rows of the blue and yellow blocks. If the girls had been working independently, Karolina might have discovered, while using the third method, two different strategies: one for the blue blocks and another for the yellow ones. She then might have been able to make some generalizations as well. In spite of the fact that both girls are assessed as "equal" by their mathematics teacher, they differ at the intellectual level. During the research Karolina presented a more mature mathematical reasoning, while Ola needed particular elements in order to apply her mathematical thinking. We can see how crucial it is within mathematics education to expand the students' abilities of abstract thinking.

By participating in the research the students from grade 5 of primary school were able to find out some rules and regularities which occurred in the task. They were using them correctly up to some point as they were able neither to generalize them without any external help nor to move them on to the more distant figures. They in fact did state the rule correctly as well as did not make a mistake in explaining how it work but it turned out to be not enough so as to solve the task to the end. Not until the teacher's interference had they been able to apply the discovered rule for the comparatively big figures. This fact implies that the process of generalization is not a spontaneous skill and it does not occur automatically as the result of solving some sequence of tasks. It is the ability that should be stimulated by the teacher and consequently enhanced by him/her. The results of the research also indicate the specific way of comprehension of the "generality rule" by the students. It is an accessible generality which is always looked at consideration the specific values. The teacher sees this generality in a different way namely while discovering the rule he takes for granted its accuracy for any n. Students in turn having in front of them the few next elements is able to discover the rule through which they were created. Subsequently, pupils are able to use the rule properly for the next elements. Nevertheless, the students get lost in a situation which forces them to apply the rule for any n. Then even the proper rule stops working from the students' point of view. The students seem able to experience only the "local generality" (in the described task the rule was working for the first 8 elements) and this generality (at some stage of education) is sufficient for them.

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HOW TO MOTIVATE YOUR STUDENTS FOR MATH AND SCIENCE EDUCATION

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This paper considers the influence of the pedagogical/didactical role of the teacher in classroom on the choice of students for math- and science education. Classroom observation in relation to research findings for motivation, are carried out to find ten focus points. Those ten points of attention are presented to increase the influx in math and science formation for 14-15 year old pupils.

THE MAIN AIM OF THE RESEARCH

In this research the focus is to increase the influx of students for math and science studies. Several possibilities to motivate students for math and science are carried out by a variety of schools (Bouton & Fanselow, 1997; Vinke & Schokker, 2001). The main question is what can teachers do in their every day lessons to increase interest and motivation of low achievers in high school for 14-15 year old pupils to choose for math and science education.

METHODOLOGY

The research was carried out on thirteen Dutch Universum[1] schools in 'havo 3'[2]-classes. The research specifically targeted on the interest of low achievers in high school at the moment these students had the opportunity to skip math- and science-education in their choices for further formation. Out of pre-research, ten points of attention for teachers were found (Vermaas, 2007). In two different research groups (group I and II) math/science teachers were trained with these points of attention to study their effectiveness on the short and long term.

[1] Universum schools are high-schools with more students interested in math- and science formation. During five years, high-schools in the Netherlands had the opportunity to profile themselves with additional financial support from the national government, in order to get more students interested in math- and science education. 100-150 high-schools took the opportunity and developed many different kinds of activities in order to reach the target of more influx in math-and science education for 16-18 y.o. pupils. They were called 'universum'-schools.

[2] 'Havo 3' is the group of low achievers in high-shools in the age of 14/15 y.o. In the end of the school year those students have an opportunity to skip their math- and science education, or not.

The results as presented find direct application in the classroom to achieve a positive attitude of students in havo-3 towards math- and science-education. The research project was carried out by APS[3], during three years (2008-2010).

Pre-research

Out of all Dutch high schools, the best four schools were selected using two performance indicators:

- the high influx of havo-3 students who choose math- and science-education for further formation
- average or higher grades at the final math/science exams

Literature research about low performers in Dutch high schools combined with school visits to the four excellent schools have been conducted. Interviews with pupils, teachers and school leaders in connection to classroom visits were performed to determine the influence of the pedagogical/didactical role of the teacher in the math and science classroom on the two performance indicators. Especially for the group of students mentioned, ten focus points of attention for teachers were found:

1. keep personal contact
2. give positive feedback
3. challenge your students
4. be clear in the procedures to follow
5. provide accessibility to yourself
6. make use of teamwork
7. activate your students in classroom activities
8. connect curriculum with real life
9. structure your lessons
10. give clear explanations

Research design

In two research groups with math/science teachers were trained to take care of a self chosen number of the ten points of attention mentioned above. They focus on these points especially during their lessons in the research - havo-3 - class. Pre- and post-classroom visits, interviews with students, the teacher and the school leader were carried out at each school, to find which of the ten points of attention proved to be most effective.

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[3] APS is a non-profit company for educational consulting in the Netherlands and abroad. See http://www.apsinternational.nl APS also carry out research- and development projects for the governmental educational department.

The first research group (group I) consists of eight schools in the first year. In the second year, four of those schools were selected to study the sustainability of the changes in the didactic role of the teacher. The second group (group II) consists of five schools, who were offered a one-year consulting period, with the training and research as described above.

RESULTS

The focus points with the most important changes of attitude of the students were measured, as well as the percentage of pupils that make choices for further math/science formation

Results	most important focus points	less important focus points	percentage change	relative increase of percentage
Group I first year	2. positive feedback 5. accessibility 8. curriculum	1. contact 6. teamwork	35% > 45 %	29 %
Group I second year	4. procedure 8. curriculum	2. positive feedback	45% > 52 %	16 %
Group II	1. contact 4. procedure 5. accessibility	6. teamwork	37% > 44%	19 %

Table 1: results of group I and II in first and second year

Interviews carried out among pupils of the teachers in these research groups, gave indications for what reasons those results were achieved.

CONCLUSION

Among other indicators to influence the attitude of low achievers for math- and science education, the teachers' role is one of the most effective. Besides focus points mentioned above, the personal touch of a teacher influences the classroom atmosphere, the way students are motivated and feel good with the subject. An interesting research question is what actions for teachers are possible to train, in order to achieve higher influence to students' attitude towards math and science and the study of those subjects.

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ANALYSIS OF THE SOLUTION STRATEGIES OF ONE MATHEMATICAL PROBLEM

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The main aim of our article is to focus on analysis of the solution of one geometric application task for pupils of the 6 graders (from 11 to 12 years old). We are observing how they choose the possible strategy, their calculations, how they note it and success of the solution. The task contains the image and pupils are supposed to calculate the area of the shape placed in the square grid. We are using the statistic software CHIC to find the relationship between the didactic variables.

THEORETICAL BACKGROUND

The knowledge of geometry is basis to understanding the environment in which one lives. Geometry and shapes concepts are important for students to understand as they can be transferred to different subject across the curriculum. The idea of 2-dimensional and 3-dimensional shapes and solids can serve as stepping stone into the concepts of angle and measurement (Mulligan, A. et al.). When teaching how to calculate the areas of the basic geometrical shapes 2nd level of the primary school pupils we need to link to skills and experience already obtained. For pupils is very important to have the correct understanding of units of measurement before the introduction of the standard units. Children know geometrical shapes since their early childhood from daily activities. Young children move through levels in the composition and decomposition of 2-D figures. From lack competence in composing geometric combination of shapes, they gain abilities to combine shapes into pictures, they synthesize combination of shapes into new shapes (composite shapes), eventually anticipating making larger shapes out of smaller shapes and combining those composite shapes, which they then think of as new units, or shapes (Sarama and Clements,2008). In order to classify shapes children need the language of properties as well as the names of shapes and teachers need to plan activities which stimulate children discuss properties (Hopkins-Pope-Pepperell, 2004). Clemens claimed when teaching a measure of shapes, many curricula lead through a sequence of comparing objects directly, then measuring with nonstandard units as paper clips, then with standard units. Recent research suggests that following this sequence rigidly may not be best. Children benefit from using objects such as centimetre cubes and rulers to measure as their ideas and skills develop. Not only do children prefer using rulers, but also they can use them meaningfully and in combination with manipulate units to develop understanding of length measurement. Even if they do not understand rulers fully or use them accurately, they can use rulers along with manipulate units such as centimetre cubes and arbitrary units to develop their measurement skills (in Sarama and Clements, 2008). Waren and English (1995) posit that ability to recognize and manipulate plane shapes has been acknowledged by many educators as having a major intimate relationship between many aspects of mathematical learning such as the ability to visualize mathematically and the ability to conceptualize plane shapes. In the study of Hassan (2002) was found that there was significant relationship between visual perception of geometric shapes and achievement of secondary school students in geometry. Based on these results, it was recommended that students should be exposed to the geometry of the immediate surrounding since this will restructure and enhance their cognitive and affective abilities in the learning of geometry.

One of the possible strategies how to calculate the area of plane shape is to compose shapes together or to decompose them into elementary shapes. If working in the square grid then one square is one unit. When solving mathematical problems we could see different solving strategies as well as different ways to express the solutions. Larson and Chinnappan (2000) argued that, among other factors, the organizational quality of students' geometric knowledge is associated with better problem-solving performance. In their study, they reported finding on the extent to which content and connectedness indicators differentiated between groups of high –achieving and low-achieving 10 years old students undertaking geometry tasks. The challenge for mathematics educators and classroom teachers is to devise strategies for helping all students to improve the state of connectedness of their knowledge bases, but particularly to assist the less effective problem solvers to exploit more of the knowledge they have acquired. Rickard (1996) denotes that it has been acknowledged by the relevant research literature that when students are dealing for an adequate period with problem solving tasks of the same conceptual backdrop, they develop problem solving techniques. They also make connections among mathematical ideas using specific problem solving strategies (in Papadopoulos, 2008). In our case conceptual backdrop is the area of triangle and trapezoid and the techniques are decomposition shape given in the square grid and using that picture for relevant calculation of the area.

METHOLOGY OF THE RESEARCH

In our research we focused on the analysis of solutions of a mathematical problem, which was prepared for the numeracy test for 6 graders. We had chosen 281 solutions for deeper analysis excluding solutions rated zero (where pupils didn't solve or didn't know how to solve the task). This left us 88 possible solutions (31.3%). We didn't look for reasons of the failure in solving such tasks as it wasn't the aim of our research. The main purpose of this research was to analyse the link between

- solution strategies of the task given as a figure,
- the actual solving the problem (calculating the area),
- the notation of the calculation,
- the final result of the task.

We have investigated the geometric as well as numerical conceptions of the pupils.

We were creating task in accordance with mathematical competencies and competency clusters as per OECD PISA study:

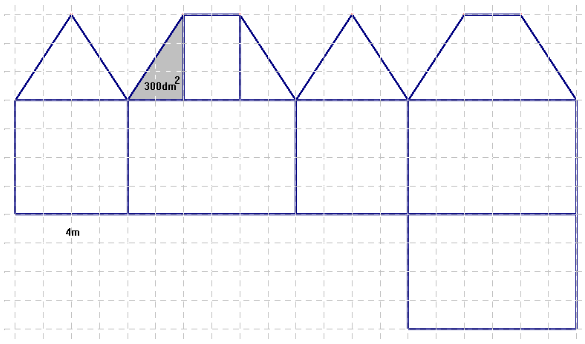
1. The tasks which are measuring the competencies at reproduction cluster.
2. The connections cluster competencies based on the reproduction cluster.
3. The competencies based on reflection cluster.

(http://www.oecd.org/dataoecd/63/35/37464175.pdf)

Our surveyed problem has been created on the reflection cluster.

Problem: Garden house.

Mr. Plum would like to build the garden house as per the following drawing.



Picture 1

How much will the new roof covering cost him? He will use the shingle type DIAMANT. The price list is in the table below.

Asfaltové šindle IKO	Názov výrobku	balenie	Cena (EUR
	SUPERGLASS pravouhlý trojtabuľový	$3{,}00m^2$/bal	7,44
	VICTORIAN - BIBER Bobrovka	$2{,}75m^2$/bal	10,66
	DIAMANT	$3{,}00m^2$/bal	11,07

Picture 2

(3.00 m^2/bal expression means that one package will cover 3m^2 of the roof)

Analysis of pupil’s solutions

When surveying pupil’s solutions we could observe the following:

1. Pupil has or hasn’t worked with the picture when calculating the area of the roof.
2. If pupil worked with the picture he divided the roof to various elementary shapes.
3. He used different strategies when calculating the area.
4. When calculating the price of the shingles and his choice from the table was correct, he used various notations of the solution. The result was either correct or incorrect.

When dividing the roof into basic shapes pupils were correctly targeting to the polygons where they were able to use the given area of the triangle 300dm^2 (which they knew is 3m^2). The codes are assigned to the shapes to show exact relationship between the used elementary shape and the area calculation. For example 2.(6C) means that pupil in his solution counted shape C (area of 6m^2) twice. So we could directly see that part of the area is 12m^2 (so the area of two rectangles).

We could see the following elementary shapes in their solutions:

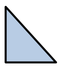
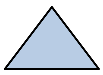
	Shape	The area of a shape	Code
A		3m^2	3A
B		6m^2	6B
C		6m^2	6C
D		12m^2	12D

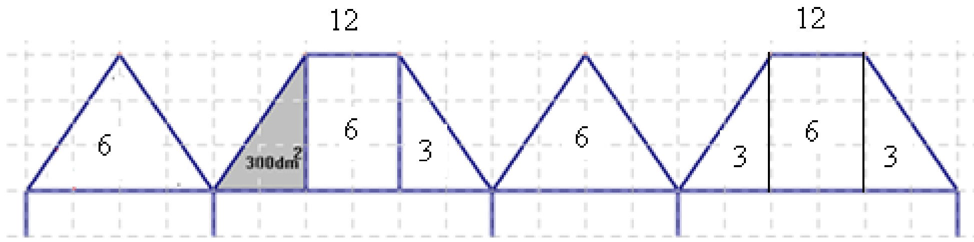
Table 1: Elementary shapes.

Based on the observed phenomena, we have identified the following didactic variables:

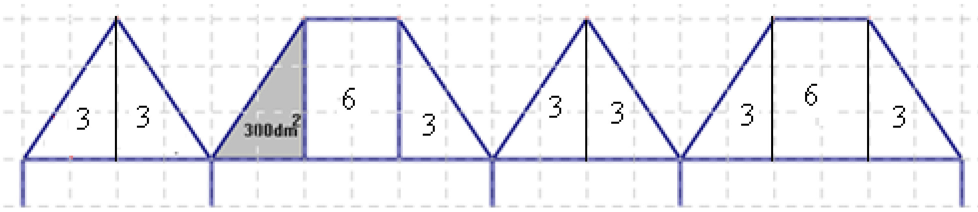
Type P – work with the picture

P0 – pupil didn’t worked with the picture

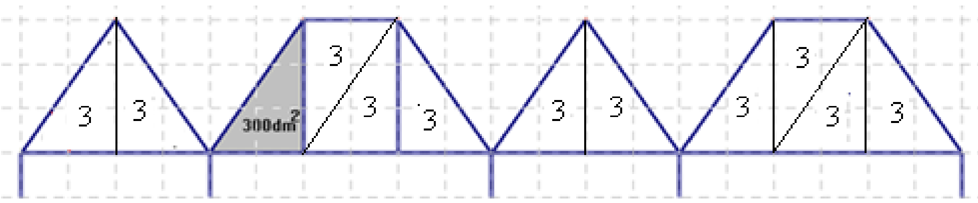
P1-P3 – pupil used the following division of the roof

P1


Picture 3

P2


Picture 4

P3


Picture 5

Type C – roof area calculation

C0 – pupil didn’t write any area calculation

C1- pupil calculated the area as 12.3 what means 12.(3A)

C2- pupil calculated the area as 8.3+2.6, what means 8.(3A)+2.(6C) or 8.3+6+6, so 8.(3A) +1.(6C)+1.(6C)

C3- pupil calculated the area as 6.6 what means 6.(6B,C)=4.(6B)+2.(6C)

C4- pupil calculated the area as 6+12+6+12 or (6+6)+(12+12) what means 1.(6B)+1.(12D)+1.(6B)+1.(12D)

C5- pupil calculated the area as 6+6+6+6+6+6, what means 1.(6B)+1.(6B)+1.(6C)+1.(6B)+1.(6B)+1.(6C)

Type F – calculation of the price for shingles

F1 – pupil wrote down sequences of the calculation of the price for shingles: 36(m^2):3(m^2) =12 packages; 12.11,07(€)=132,84(€)

F2 – pupil wrote down sequences of the calculation of the price for shingles: 36:3=12.11,07=132,84(€)

F3 – pupil wrote down sequences of the calculation of the price for shingles:

36:3.11,07=132,84(€)

F4 – pupil used the “rule of proportion” for the calculation: 11,07(€):3(m^2)=3,69(€); 3,69(€).36(m^2)=132,84(€)

F5 – pupil counted: 36(m^2).3(m^2) or 36(m^2). 11,07(€)

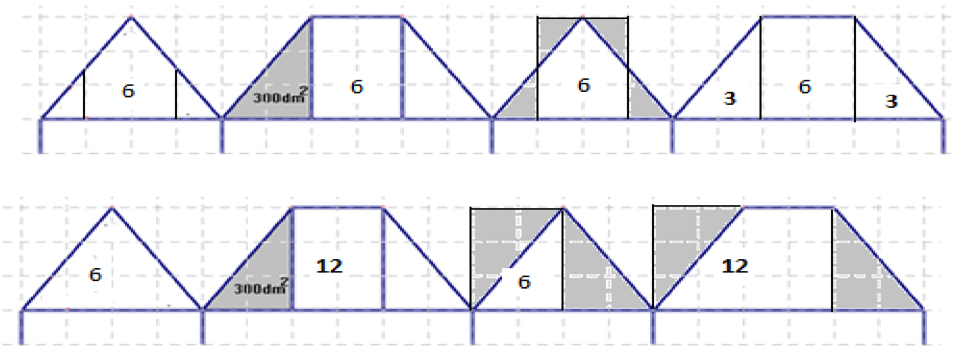
F6 – pupil counted: 11,07(€).3(m^2)

Type R – the solution

R1 – correct result

R2 – incorrect result

Except for the chosen strategies using the picture (shown as didactic variables type P) there were several other options to calculate the area of triangle and trapezium by using the rule of the same area of geometrical shapes and the area of rectangle (Picture 6)



Picture 6

EVALUATION OF THE RESEARCH

Each didactic variable was assigned either value 1 (the variable in the pupil’s solutions occurred) or 0 (the variable in the pupil’s did not occur). We investigated the occurrence of didactic variables but mainly the relationships and dependencies between them. We were interested in the relationship between the numerical and geometrical ideas of pupils that resulted in the use of the picture in calculating the area and expressing the solution. We used the statistical software CHIC *(Classification Hiérarchique Implicative et Cohésitive)* for a deeper analysis of dependencies and relationships, which was developed by Regin Gras at al. who works in Laboratoire Informatique de Nantes Atlantique (Gras, 2007).

C.H.I.C. is software which works with the frequencies of particular significant unites. It represents the connection between quantitative and qualitative analysis, it makes possible to compare the similarity of didactical variables present in research, suggest the relations of coherence between variables and describes also the probability of realized implication between variables by probability rate of their realization. Apart from the relations between particular didactic variables this software allows also the comparison of relations between whole classes of didactic variables in three type of graph (similarity tree, implicative graph and implicative tree) (Földesiová, 2003).

All three types of the diagrams were created to survey the relationship between didactic variables types P-C-F-R, P-C-R, P-C-F, P-R, P-C, C-R, F-R. We chose the following graphs to represent the examined relationships.

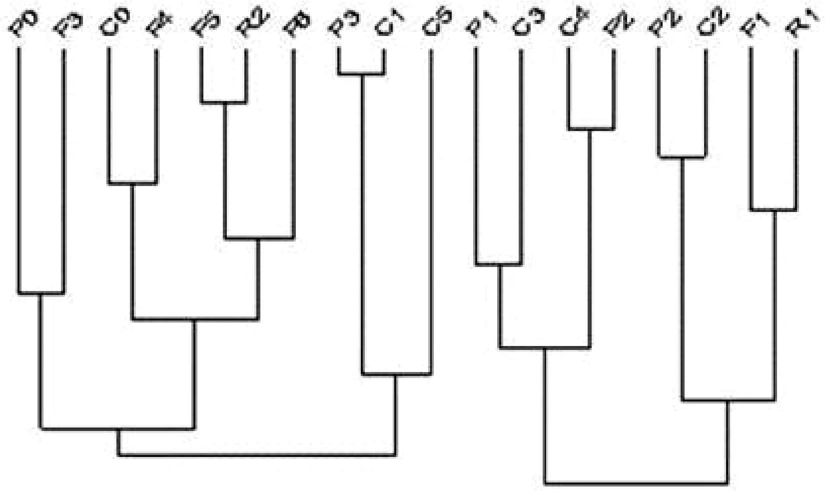


Diagram 1: Similarity tree for all variables.

Similarity tree define the similarity and intensity between two classes of defined variables. In the construction of the graph there are two variables with the most similar bases connected into one class (the highest level). Then there are added one or two variables with similar base and they create another but weaker level. Another variable with the similar bases are added this way. Only two highest levels are important for the evaluation of the experiment. The others are not significant from the statistical point of view.

From the graph we could see that all variables are connected on the certain level of similarity. The highest level of similarity is between variables P3 and C1 which is analyzed more deeply in diagram 2. The second level of similarity is between variables F5 and R2 which is described as implication in graph 3. In this graph is interesting similarity between two groups of variables: (P2, C2) and (F1, R1). It is at the lower level, however, here we see a similarity between solution strategies of the roof area task (division in the picture) and calculation of the price of the shingle leading to the correct result. Therefore we could say strategies P2, C2, F1 led to the correct result on the certain level.

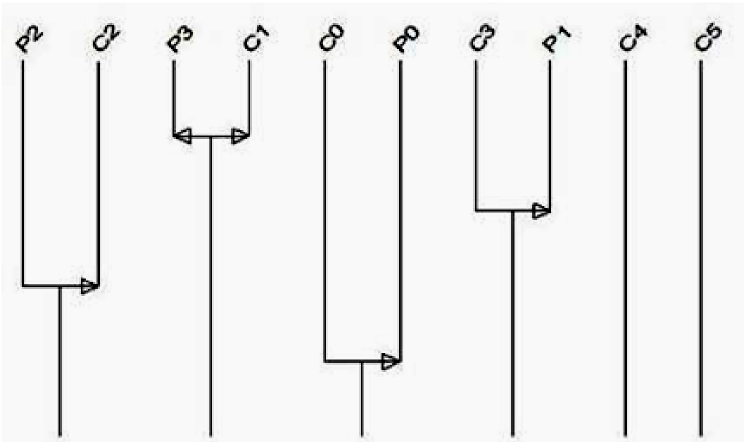


Diagram 2: Implicative tree for variables P and C

Implicative tree represents the implications or the equivalencies between some stated variables. By evaluation of experiment's results the most significant are the first two levels in the graph, the others are irrelevant.

To evaluate the relations using this type of graph we chose only variables type P and C. We were interested in bonding between the work with the picture and strategy of calculation the area of the roof. It means between numerical and geometrical conceptions of pupils. We could see strong equivalency between variables P3 and C1 from our graph. It means pupil would use division of the roof as per picture P3 if and only if he uses the strategy C1 to calculate the area. Looking through these strategies we could understand strong mental link of the calculation to the picture and vice versa. The second level of implication is between C3 and P1, so if pupil used the strategy C3 to calculate the area of the roof he divided the roof as per picture P1. Further relations between the picture and calculation are already at the lower semantic level between P2 and C2 as well as C0 and P0. Strategies C4 and C5 are not showing as significant in regarding the work with the picture.

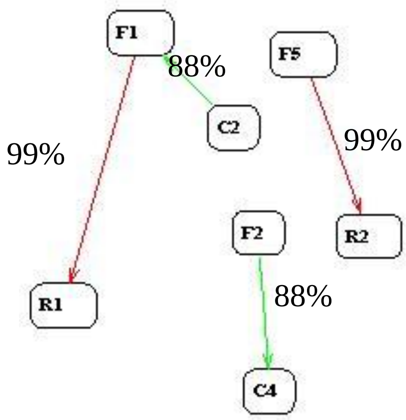


Diagram 3: Implicative graph for all variables.

Implicative graph is reflecting the possibilities how pupil could think or consider the strategies of the procedure how to solve the task. Arrows between variables in the graph are coloured. Different colours represent percentage intensity between variables or indicate percentage of the pupils with the same knowledge who gets to the next variable. Only relations between variables over 85% are interesting for the results of the experiment.

If we look at our graph we could see the software generated dependencies over 85% between the following didactic variables: R1, F1, C2; F2, F5, and C4, R2.Therefore we can say, that if pupil used the strategy C2 to calculate the area of the roof than for 88% he used the strategy F1 to calculate the price. In terms of method and accuracy of both calculations, we can say this group of pupils had the best calculation skills. A similar relationship exists between variables F2 and C4. Method F2 represents the correct way of thinking when calculating the price but incorrect way of writing the calculation. C4 is a strategy for calculating the area of the roof which uses adding areas of the triangles and trapeziums, so pupil is not using multiplication. We can say that this group of pupils is able to think correctly but has less calculating skills. Relation between variables F1-R1and F5-R2 is obvious.

CONCLUSIONS

When analysing pupil's solutions of the problem we were observing other interesting strategies additional to the introduced. We won't list them because of limited range. We can conclude quite a strong connection between numeric and geometric conceptions of pupils showed. Different strategies how to divide the roof to elementary shapes shows variety geometric ideas of pupils, different options how to work with geometric shapes what are not possible to fill by full square units. Creating own strategies of solutions is related to the fact, pupils didn't know formulas for calculation the areas of the shapes. If pupils did solve this task their results were correct in as much as 73%. Following our research we identify three different levels of student's solutions according to their approach to the problem. However the order of the levels isn't distinct. In one of the levels are pupils who have good calculations skills and functional thinking. These pupils were able to do correctly notations of mathematical calculations in link with geometrical interpretation of the task solving. On the other level are pupils who have also good calculations skills and functional thinking. However these pupils preferred mental calculation as well as mental manipulation with figures. In the second part of the task (F as per didactic variables) were more calculations using also decimal numbers, so notation of the calculation was needed. Here we could see their incapability to write correct notation of mathematical calculation. It is quite common even for mathematical talented pupils who underrate formal part of the solution. It would be interesting to investigate further which level pupils are globally having better mathematical skills and how important was influence of their teacher. The next level is the lowest from the mathematical skills point of view. Here are pupils whose calculation skills as well as functional thinking are lower. Their geometrical interpretation of the task wasn't clear. Generally we can say there is a strong mental relationship between the picture and the calculation. The results of this study confirm the fact that pupils of certain age, assuming the appropriate level of solving and mathematical skills and knowledge build and develop skills to solve mathematical problems by choosing their own solution strategies.

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SUPPORTING MATHEMATICAL CREATIVITY IN REALISTIC SURROUNDINGS

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In this paper we are interested in the work with 8-grade pupils from grammar school in Nitra, who were 14-15 years old. Our activity with these pupils consisted of looking for mathematics in any form while walking in the park and taking pictures of the objects that represented some form of mathematics. Then we asked them to try and create some concrete mathematical tasks, which would be connected with their pictures. We recorded the pupils´ pictures, their creativity and original thinking.

INTRODUCTION

One of the goals of education at any school level should be to stimulate pupils to think creatively, think logically and to be able to solve problems. Creative thinking can be developed by a creative teacher who would help to form creative situations, support pupils' initiative and give space to new and original ideas. (Blažková, Vaňurová, 2011).

According to Pavlovičová (2008) mathematics as a school subject informs pupils of abstract system of knowledge. It is useful to make this system closer to pupils by using actual objects and activities, with which they already have certain experience. Pupils gain the ability to use mathematics by being directly involved in the world, which surrounds them. Interrelating knowledge gained in the process of teaching with reality gives pupils greater motivation to continue learning and gaining information in this way.

THEORETICAL FRAMEWORK

Creativity is an essential feature of personality which each of uses in our everyday life, since it allows us flexibility when dealing with real life situations. Study of mathematics should be seen as one of the opportunities for its development, although creativity is not traditionally associated with maths.

Mathematics can be an interesting activity not only for the mathematicians but also for the teachers and students (Ponte, 2008). People who think independently feel the need to make sense of everything based on personal observation and experiences rather than on information they were given without questioning it. It is more tempting to concentrate just on the memorization of facts and practice algorithmic skills, than to work on creativity and independence in thinking (Hoffmann, 2008).

According to survey conducted by Blažková and Vaňurová (2011), creativity of children depends on great deal on teacher's approach. When pupils solve only classical tasks by always using the same methods, they have problems to change their learned way or create a task independently. Children can develop certain commodity in thinking, little initiative or even unwillingness to work.

Creativity is defined as a production of new and original ideas (Zelina, 1996). Research on creativity, at the group level, has highlighted the potential trade-off between social control and creativity (Nemeth & Staw, 1989).

Creativity in the mathematics classroom is not just about what pupils do but also about what we do as teachers. If we think creatively about mathematical experiences that we offer our pupils we can open up opportunities for them to be creative.

Mathematics is as much about problem posing as it is about problem solving. It is about noticing within a situation that there is a question waiting to be asked. At this point, creativity lies in noticing that there is something to be investigated. When setting up situations in the classroom we should make an effort to choose contexts that offer students opportunities to pose their own problems.

Creative teaching requires from teacher to create and exercise such tasks, which would enable pupils to use their acquired knowledge more freely, in new contexts and when solving new and unknown problems (Lokšová &Lokša, 1999). When talking about creative activity, just as about any school activities, what children produce depends not only on their own abilities but also teacher's skills. Teacher who has wide spectrum of interests and who wants to share them with children in the school and outside it supports creativity of pupils more than stereotypical teachers (Vidermanová & Melušová, 2011).

The results of research of Maj (2006) show that:

- Among the mathematics teachers the knowledge and skills regarding creative mathematical activities are insufficient.
- Among the mathematics teachers it is generally wrongly believed that the creative mathematical activities are developed by themselves during the mathematics lesson and do not require any special didactic endeavours, methods or tools to develop them.
- The mathematics teachers do not have experience and skills of undertaking these activities and what happens is that they cannot provoke these activities and they cannot include them in the work with students. (p. 138)

One of the main aims of mathematical education as such is preparing the students for dealing effectively with the real-life situations. The effect of activity in this area in Holland was the Realistic Math Education theory created by H. Freudental (in Heuvel-Panhuizen, 1998). We can find a lot of mathematics around us, in our surroundings. The world around us provides us with many opportunities to come up with mathematical tasks that would be based on everyday situations.

In our experiment we analysed one activity that teachers can use to increase creativity and motivation of pupils in mathematics.

When working with students we used the method of brainstorming, which also helps to develop pupils' creativity. The basic principles of this method say that in the first phase of solving a problem, we must produce the most various and original ideas, but also be critical about them. In the second phase of problem solving, evaluation and further work with the ideas follow (Zelina, 1996).

METHODOLOGY

By supporting mathematical creativity we can provide a combination of knowledge and real life situations. Solving mathematical problems can be done by supporting pupils to be active and creative. A very suitable thing to do would be to look for mathematics in all that surrounds us. Pupils should look for mathematical properties, patterns, geometric shapes or bodies in the city, at home, in nature and so on.

In our opinion, a very good activity appears to be finding mathematics in the environment of the park, which is a rich collection of such information. We can apply it at different age of pupils, but also in all the areas of mathematics.

Looking for various plane geometrical shapes or bodies in nature and creating tasks to given problems by the pupils themselves can lead to increase in pupils' motivation and their spatial imagining. It would be also useful if students were able to formulate and interpret their results. During these activities pupils can also apply all the knowledge they have acquired in geometry.

Our research questions are following:

- Do pupils know how to find geometry in the park?
- Can pupils come up with concrete mathematical tasks that would involve geometrical objects that were found in nature by the pupils (out of set of geometrical objects given by the teacher)?

In this paper we present one such activity which was conducted with 8-grade pupils from grammar school in Nitra, who were 14 -15 years old. The activity was divided into two phases. In the first one, we asked them to look for mathematics in any form while walking in the park. We told them to take pictures of objects of their interest. In the second phase, we asked them to try and create some concrete mathematical tasks, which would be connected with their pictures. This is how we gained rich collection of pictures. The activity was video recorded too.

The results of students were classified according to the common characteristics into several groups. We decided that the common characteristics in the pupils´ pictures will be:

- mathematics that can be easily seen in the picture (the first group),
- mathematics that could be seen only after teachers consulted pupils to help them see it. These pupils also subconsciously created mathematical tasks (the second group),
- mathematics that could be seen after pupils added yet another object into the one they found, so that the newly created object became geometrical figure (the third group).

CONCLUSION

This activity was interesting for both pupils and their teacher. Pupils very active and after a few minutes, their creativity was awaken to the fullest. Here is a description of both phases together with the pupils' pictures.

First phase

Pupils’ pictures were then divided into three previously described groups:

Easy-to-see mathematics – in this group we put those pictures in which it was easy to tell, what kind of mathematics was involved. Here pupils took pictures of geometrical shapes they found in the park.

Usage of mathematics – here we included pictures, in which connection with mathematics was not immediately clear. One could have even thought that they have nothing to do with mathematics. From the conversations that followed we could see that pupils were subconsciously creating mathematical tasks.

Transcript of the interview is as follows:

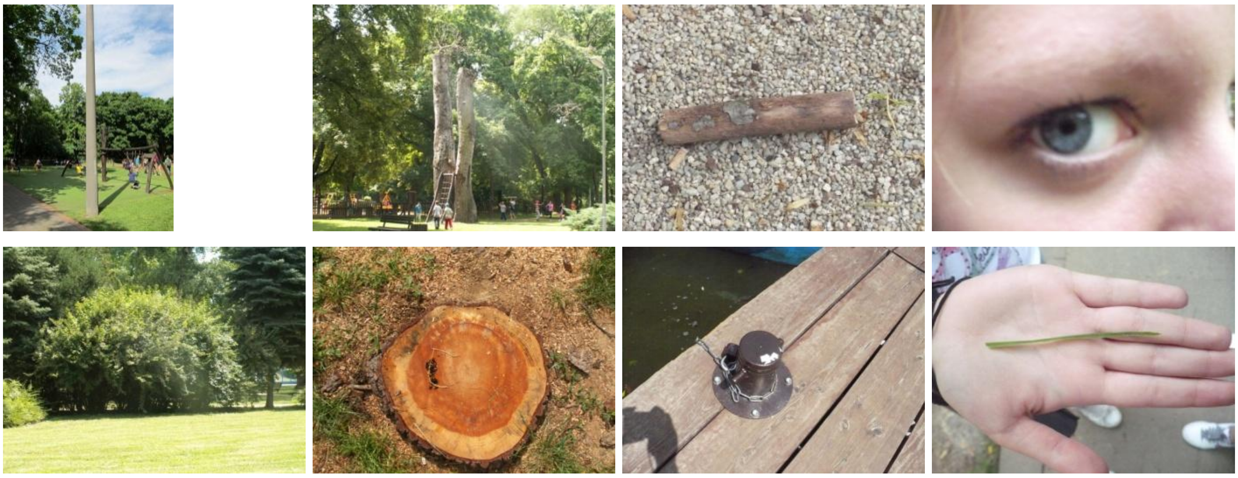
E: Where can you see mathematics in this picture? (Picture with face)

P: But how many percent of the face makes the eye?

E: So, you created tasks to go with the pictures?

P: We did, some yes. We should not have?

E: But yes. We will do this now for all your pictures. This will be your following activity.



"Completion" of mathematics

When pupils did not find appropriate objects with easy-to-see mathematics, they modified the surroundings in such way that mathematics became obvious. Difference between this and the first group is in greater creative and original thinking of pupils in this one.

Second phase

In the second phase of activity we asked students to create text which would represent problems to go with their pictures. This was the way in which we obtained a rich collection of photographs which presented our bases for creation of mathematical problems. There are no concrete measures related to geometrical objects in our problems, they are open problems which could be completed and solved by pupils.

Easy-to-see mathematics on photos of pupils

When looking at the pictures in this part we can see what kind of mathematics pupils discovered there. Pupils are usually looking for the objects that look as specific geometrical solid; they are looking for different 2D shapes. They would be also able to count contents and volumes of found objects; possibly they could detect the different symmetry of a given object. Therefore, in the following photos it is not difficult to see what exactly inspired pupils from mathematics in the park.

What symmetry can we find on the hubcap car?

Usage of mathematics

In this second part we have selected photos, which pupils already think that their knowledge of mathematics used to find mathematics in the park. Here are a few specific tasks, which the pupils came up with.

If we empty one-fifth of the bin the total weight will be 80 per cent of the full one. What is the weight of an empty bin?

How does the length of the rope depend on the length of the shadow? When is the length of shadow in its maximum and when in its minimum?

How many kg of red and white painting was necessary for these columns, if 1 kg of paint covers approximately 8 m^2 area?

What is ratio of the tail to body of the jumping squirrels in the park?

I have 6.50 euros. How many friends can I invite for a hot-dog and a small drink?

Completion of mathematics

From the photos in this section it is clear that pupils tried to involve their classmates into the experiment. They tried to complete existing objects in the park so that we would be able to create specific text of problem. Next, we mention interesting pupil’s text problem.

How many Peter´s feet can cover the top of this tree stump?

The following table shows the number of photographs, which were previously divided into three groups formed by the researchers.

Group of pictures	Quantity of pupils’ pictures
Easy-to-see mathematics	13
Usage of mathematics	15
Completion of mathematics	4

Table 1: Classification of pictures into groups

The table shows that pupils were interested in the activity. Most photographs ended up in the second group. This means that pupils were thinking about mathematics while taking pictures.

In the next part, we wanted to see what parts of mathematics were used by pupils when creating the tasks. Pupils classified their photos into units of mathematics which they knew from previous years at grammar school. These were then included into table 2.

Part of mathematics	Number of pupils' pictures
The planar shapes	13
The solids	14
The content and circumference of the shapes	11
The volume and surface of the solids	10
The ratio	9
The computational geometry (length of segments, magnitude of angle, …)	11
The arithmetic	1
The symmetry	4

Table 2: Classification by parts of mathematics

Creativity of pupils was visible while they were creating the tasks. Most tasks were made for parts "The planar shapes" and "The solids". This fact is also result of pupils' surroundings. Park was inspiring for them, since there were many objects with easy-to-see mathematics.

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UNDERSTANDING OF A MATHEMATICAL CONCEPT AT THE GENERALIZATION LEVEL VS. INDIVIDUAL STUDYING A DEFINITION BY STUDENTS

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In this paper we will use an approach to the term "to understand a mathematical concept" developed in Poland by Zygfryd Dyrszlag. Among his four levels of understanding we focus on the third one: understanding at the "level of generalization". The word "generalization" in this approach means rather not a student's mathematical activity, but a level of understanding. Based on the analysis of results of empirical research we propose answers to three questions: 1) What degree of understanding of a new mathematical concept can students reach as the effect of independent reading, on each of the levels: definition, local complication, generalization? 2) If the degree of understanding reached by the student can increase following heuristic directions given? 3) Is there a connection between the degree of understanding of a new mathematical concept on each of the levels and school grades?

INTRODUCTION

One of the main objectives of secondary school is to prepare young people to study. This is closely connected with a self-supplementing of knowledge by tapping into various sources of information and learning in this way new content and concepts. The need for self-educating does not finish at the end of one's study. On the contrary, is present at the next stage of life, which is work. Upgrading different 'job skills' has already become a mandatory standard for most employees, and is essential, not only for possible promotion, but often simply to maintain their position at work.

Most companies take steps to improve productivity while reducing employment. A very demanding job market and growing competition make the need for additional training, and sometimes even completely re-training a perspective, that any modern man has to take into account. Reading and analysis of mathematical texts is a skill that involves the student in taking a variety of mathematical activity and, therefore, directly implements the objectives of teaching mathematics as a subject.

It also directly affects the ability to read and analyze a mathematical texts (Konior, 2002), that is, to which each of us has to do in everyday life (tax return forms, maps, manuals of various kinds of automatic equipment, legal documents, leaflets on the operation and use of medicines, etc.). What's more, this ability is also an important factor in the whole process of self-education (Konior, 2002).

It is interesting to us, therefore, what is the efficiency in the assimilation of new mathematical concepts at this stage of learning through reading a mathematical text. This is related to our research questions.

THE AIM OF THE RESEARCH

We assume the definition of the term *levels of understanding of a concept* according to Dyrszlag (1972) and in this context we define a new term: *degree of understanding of a concept at the given level.*

In the light of the terminology assumed the aim of the empirical research done was:

1) a diagnosis of the degree of understanding of a new mathematical concept at the levels: definition, local complication and generalization that can be reached by students from the secondary school as the result of an independent reading of a mathematical text in the form the definition of a mathematical concept,
2) examining the possibility of an increase of the degree of understanding of a new concept at the given level of understanding as influenced by heuristic directions given to the student,
3) finding if the degree of understanding of a new mathematical concept at the particular levels correlates with the school grades that students receive.

THEORETICAL FRAMEWORK

Attempts of defining the term to understand a mathematical concept was and is being undertaken by many researchers during many years (e.g.: Vollrath, 1974; Bergeron&Herscovics, 1982; Locke, 1985; Dewey, 1988; Hoyles&Noss, 1986; Klakla M. at al., 1989; Sierpińska 1992, 1994). Here, we will bring forward a definition by Dyrszlag (1972, 1974), which is being used in the Polish mathematics education research and literature, but rarely mentioned in the English language literature (Watson&Mason, 1998; Mason&Watson, 2001; Vollrath, 2002).

The author distinguished *four levels of understanding of a mathematical concept: definition level* (UL1), *local complication level* (UL2), *generalization level* (UL3), *structural understanding level* (UL4). Below we give a concise outline of symptoms determining the levels according to Dyrszlag (1972). According to him (1974) a diagnose of understanding a concept is possible and consists in the analysis of the product of students' work on appropriately selected problems aimed at verifying the attainment of knowledge and skills assigned for the given school level.

LEVELS OF UNDERSTANDING (DYRSZLAG 1972):

UL1 - The definition level

Understanding at this level means the ability:

- to produce the formal definition of the concept, including all conditions that identify it,

- to apply the definition in a concrete example,

- to show or create (or tell how to) designata, and

- to recognize the designata and non-designata in a given collection of objects.

UL2 – The level of local complication

Understanding at this level means an extension of that at the UL1 by addition of specialized designata and non-designata and, at the same time, accounting for some extra conditions. Though, the complication is local only and embedded in concrete examples, not requiring a generalization.

> The various specialization of counter-examples may be followed by a partial abstraction of some features of the concept […]. It essentially lowers the degree of isolation of the concept from among other ones being either precedent (when counter-examples are specified) or subordinate (when examples are specified). (Dyrszlag 1971, p. 49).

But – the author stresses – the extra conditions shouldn't be too numerous and factors like age and mental development of the student should be accounted for.

Understanding UL2 manifests itself then through:

a) indicating examples and counter-examples of the concept considering additional simple conditions,
b) explaining why an indicated object is not a designatum of the concept,
c) checking which of the given defining conditions is satisfied, and which is not, by the indicated object,
d) introducing changes in the indicated example such that it becomes a counter-example.
e) deciding if a boundary example is designatum of the concept or not.

UL3 – The level of generalization

Understanding at this level is characterized by:

- thinking detached from concrete designata of the concept,

- disposing with an operative knowledge on the whole class of designata, and

- flexible use of symbolism linked with the concept.

At this level the possession of a fully formed mathematical abstract object is expected. It manifests itself through

a) the knowledge of relationships between the range of the given concept and ranges of synonymous concepts,
b) ability to point precedent and subordinate concepts,
c) ability to classify a precedent concept in a way that one class is the range of the given concept,
d) ability to avoid difficulties when solving problems with variable symbolism,
e) assertive answering to confusing suggestions concerning the concept,
f) ability to remove the so called „information noise" (unessential information) in problem situations,
g) ability to prove theorems concerning the concept.

UL4 – Structural level

Understanding at this level refers to structure and their models. It is characterized by spontaneous looking for analogy objects in considering models, which means seeing a common structure of different ones (e.g. Group, Vector space). This level is hard to reach (or at least evidence) in secondary school.

METHODOLOGY

Empirical research (Luty, 2010) was carried out among 18-19 year old students attending the 3d (last in upper secondary school) grade. They followed the extended-level curriculum in mathematics.

It was done in two stages. In the first stage understanding of the concept of the Cartesian product was examined, in the second – the concept of a complex number. Reasons for selecting those concepts were:

- they are not known to the students at this school level so recapturing remembered knowledge can be excluded,

- prerequisite knowledge possessed by Polish students at this school level is sufficient for the introduction of these concepts without a preparation,

- the linguistic and logical structures of the proposed definitions, questions, and problems are new, while at the same time the language and symbolism are legible for the students.

In both stages of the research written products of the subjects were analyzed, but the methodology in each stage was different. Figure 1 present the scheme of the research.

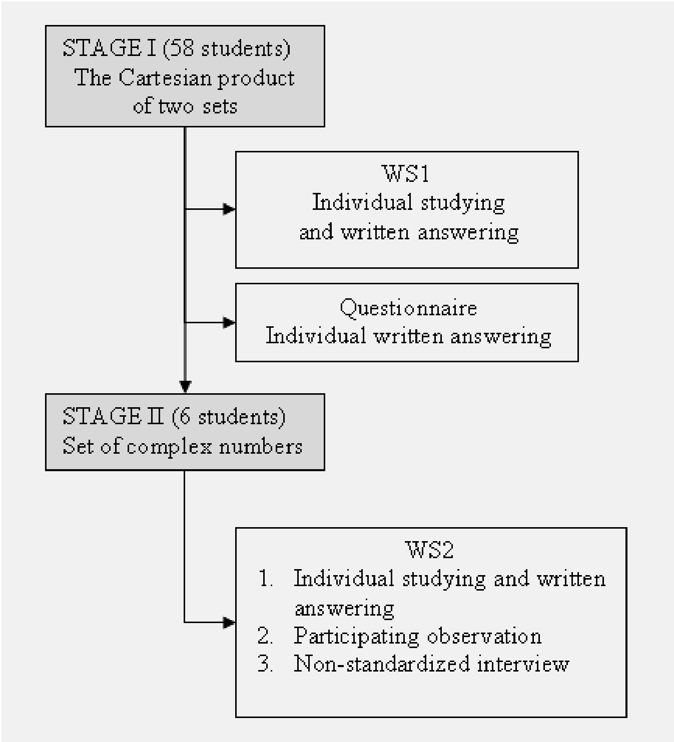

Figure 1. The scheme of the research

The basic research tool were specially designed work sheets (Luty, 2010) referred to as respectively Work Sheet 1 (WS1) and Work Sheet 2 (WS2). Both bared the same layout. It started with a short text proposing the definition of a concept – the Cartesian product of two sets in WS1, the set of complex numbers in WS2. We quote them below.

WS1:

> With symbol $(a;b)$ the *ordered pair* is denoted, in which the following order is determined: a is the first element, and b is the second element.
>
> Let A and B are arbitrary sets. The set of all ordered pairs $(a;b)$ such that a belongs to A and b belongs to B is called ***the Cartesian product of sets A and B*** and is denoted with symbol $A\times B$. Otherwise, $A\times B = \{(a,b)\colon a \in A \wedge b \in B\}$. The Cartesian product $A\times A$ is denoted shortly as A^2.

WS2:

> With symbol i the so called *imaginary unit* will be denoted possessing the property $i\cdot i = i^2 = -1$. Set $C=\{z\colon z= x+y\cdot i \ \wedge x \in R \wedge y \in R\}$ is called *the set of complex numbers*, and its elements are called ***complex numbers***.
>
> For each complex number $z= x+y\cdot i$ where $x, y \in R$ we distinguish the *real part* equal x and the *imaginary part* equal y, which is expressed respectively as Re $z=x$, Im $z=y$. An arbitrary complex number $z= x+y\cdot i$ where $x, y \in R$ can be interpreted as point in the plane with coordinates (x,y).

Each work sheet included besides **six problems** diagnosing the understanding of the concept at each Dyrszlag's level. We are aware, though, that the proposed assignment of problems is not unequivocal: a problem can examine the understanding at different *levels* at the same time. The diagnostic problems were designed according to directions concerning the quality of concept understanding's control at the given *level* (Dyrszlag 1974, p. 30). In particular, they were so chosen as not to require from the student carrying complicated reasoning, drawings or calculations. Also, questions transgressing the *definition level* did not admit casual answers as each required a justification of the answer.

An additional research tool at the first stage was a **questionnaire**. It served to obtain the information which problems brought most difficulties for the students, if they had any doubts while solving them, and how they evaluate the difficulty of the whole work sheet WS1. The questionnaire also offered the students an opportunity of a free expression on the work sheet and comments.

In the first stage 58 students participated. The **analysis of the products** and the **questionnaire** were here the only research methods.

For the second stage 6 students were selected out of those 58. The selection criterion was school grades in mathematics at the end of last semester: two students with an A or B, two with C, and two with D or E. With each of those the researcher **interviewed individually** (Luty, 2010).

The task of the student was to read the text and try to solve problems included in work sheet WS2. During his/her work the researcher was watching it and taking scrupulous notes. The objects of his observation were: the order of problems being solved, kind of corrections made, frequency of referring to the text, time devoted to each problem. In this stage then, as at the previous one, the method of analysis of students' written products was applied, but here it was complemented with the method of **participating observation**.

We were also interested if a common analysis of student's answers and offering him/her **general directions** would make him/her inclined to correct his wrong solutions. So the research was continued with the method of **non-standardized interview**. After the end of the student's independent work on WS2 the student was asked to justify his/her answers. In the case of a wrong solution the researcher initiated a discussion. It was carried on the base of previously constructed list of heuristic hints and helping questions that aimed at facilitating the discovery of a correct solution by the student.

All answers by the examined to problems in work sheets WS1 and WS2 were qualified according to correctness in four groups: **fully correct** (correct essentially and in the editorial layer), **with minor faults** (essentially correct, editorial faults), **wrong** (with many essential errors or lacking).

The analysis of answers to problems related to understanding at the *generalization level* indicated the need of introducing one more category: **incomplete answer**: essentially correct but missing some possible cases.

In the analysis we assume that the solutions belonging to the first two groups, that is fully correct solutions or solutions with minor editorial faults, are the manifestation of the student's understanding of the concept at a particular level.

We define the ***degree of understanding of a concept at the given level*** as the percentage of correct answers to all problems diagnosing understanding at this level.

THE PRESENTATION OF THE RESULTS AND THEIR ANALYSIS

The results and analysis of the first stage of the research

1) What *degree of understanding* of a mathematical concept *at the given level* can students reach as the effect of independent studying a definition?

By making quantitative analysis of the answers to problems from WS1, it turned out that the total *degree of understanding* of the concept at the given levels was respectively:

55% - understanding at the *definition level* (UL1),

48% - understanding at the *local complication level* (UL2),

18% - understanding at the *generalization level* (UL3).

Comparing the results obtained we observe that the *degree of understanding* of the concept of the Cartesian product in the surveyed students group at levels UL1 and UL2 proved to be very similar, because the difference in the results is only 7 percentage points.

On the other hand we can pay attention to a dramatic drop in the number of responses demonstrating the *generalization level of understanding* of the concept. The number of correct responses on tasks diagnosing UL3 was over two times smaller. The weaker students' results at this point do not seem to be surprising, as the *understanding* of the concept *at the generalization level* manifests itself in the difficult skill of location the concept in the structure of parent and child concepts and the network of their interconnections.

Furthermore, if the student has not received a high degree of understanding of the concept at the lower levels UL1 and UL2, a chance to solve any problems relating to the level UL3 is small.

Also interesting is the sheer scale of this phenomenon. Globally, the *degree of understanding* of the concept at UL3 is twice lower than the *degree of understanding* at UL1 and UL2.

2) If the *degree of understanding* of a new mathematical concept at the particular *level* correlates with the school grades usually received in mathematics by students?

To provide an answer to this question we have taken into account students' school grades in mathematics at the end of the last semester. They prove to be more objective assessment of students' mathematical ability and aptitude to learn the subject. Among all respondents had been identified three groups. The first was formed by 16 students with *good results* in mathematics (with grades A or B) we call them *good group*, the second *average group* represented those with the grade C (28 students), and the third *poor group* was formed by 14 students (with grades D or E). In this paper we will call a member of the *poor* or *good group* respectively as a *poor* or *good* student.

Analysis of the WS1 results shows that there is a close connection between the *degree of understanding* of the concept at a given *level* and the grades achieved in mathematics in school education. The *degree of understanding* of the concept at a given *level* increases with the semester grade in mathematics. For example, the *degree of understanding* of the Cartesian product at the *generalization level* was 27% among students from the *good group*, 15% in students from *average group*, and only 12% in students whose mathematics achievements are *poor*. In Figure 2 we present the summary of the results.

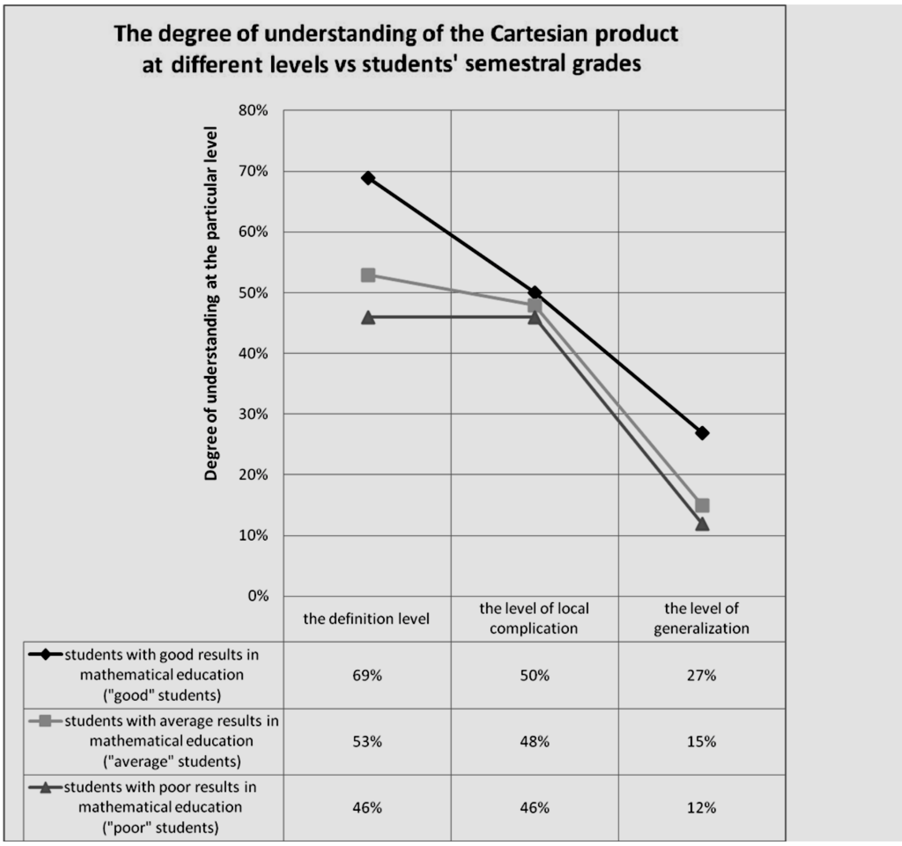

Figure 2. The *degree of understanding* of the concept at levels UL1, UL2, and UL3 released by WS1 among students with different learning outcomes in mathematics

Analyzing the students' results presented in the graph (Fig. 2), we would like to note three more facts. Firstly, the *degree of understanding* of the concept at levels UL1 and UL2 in students from *poor group* is identical and is 46%. Undoubtedly, it would be worth to examine whether such a tendency is more general. Another interesting observation is that in each group of students with different learning outcomes in mathematics the *degree of understanding* of the concept at the *local complication level* UL2 is the same and is about 50%. What is more, it seems to be in this case independent of *the degree of understanding* of the concept at UL1. Both observations encourage us to propose further research hypotheses. But we will devote more attention to the third observation. Interesting for us is that up to a half of correct answers to problems related to UL3 level are given by students from the *poor* and *average groups*.

The *degree of understanding of the concept* at UL3 disclosed by WS1 in students from the *good group* is indeed higher than the *degree of understanding* achieved by other examined. However, the fact that attempts to solve problems related to UL3 level have been taken by the students with *poor results* in mathematics and some were successful, fills us with optimism and has been recognized as interesting.

Comparative analysis of the first and second stage of the research

Comparing the results achieved by the students who take part in both stages of the research it appeared that in general all the students have achieved higher *degree of understanding* of the concept from WS2, at all *levels*. The numbers shown in the graph (Fig. 3) confirm it.

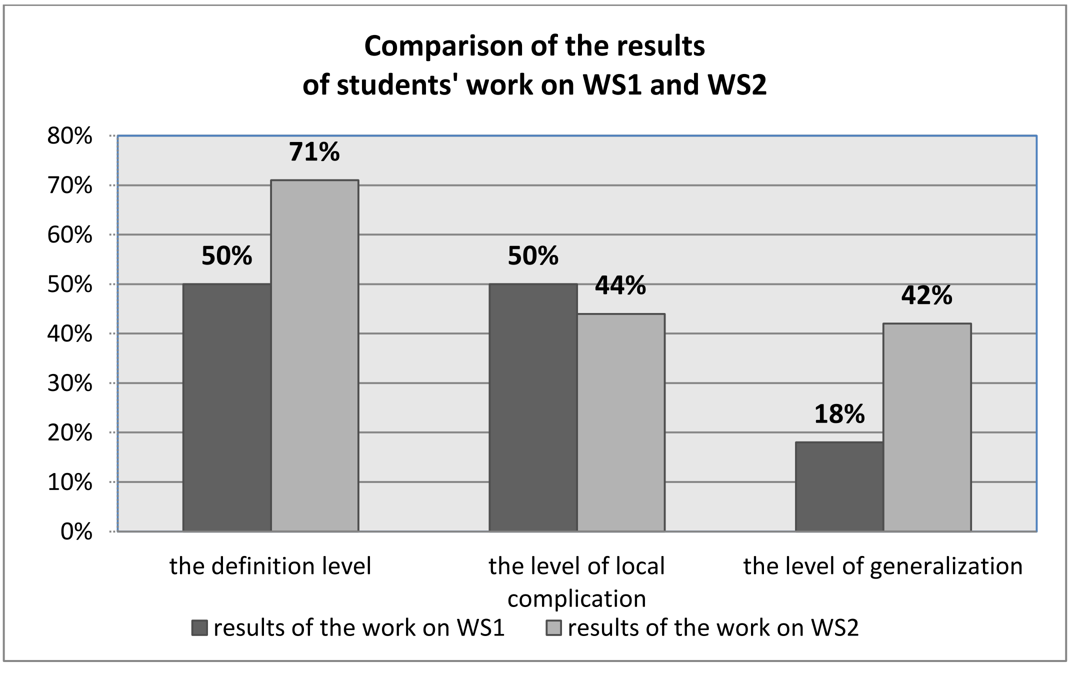

Figure 3. Comparison chart of the *degree of understanding* (vertical axis) of a new concept diagnosed in all the students working on WS1 and WS2

Among all the answers given by the students the total *degree of understanding* of concepts at different levels was as follows:

WS1 – 50% and WS2 – 71% (understanding at the *definition level*),

WS1 – 50% and WS2 – 44% (understanding at the level of *local complication*),

WS1 – 18% and WS2 – 42% (understanding at the level of *generalization*)

It is not difficult to notice a significant increase in the number of correct answers in the levels UL1 and UL3 (respectively 21 and 24 percentage points). The only decrease was recorded in the number of answers investigating the understanding at the level of *local complication*, however it was relatively small – only 6 percentage points.

Does this increase in the *degree of understanding* of the concept at the second stage of the research is only incidental, associated with the specific concepts being considered or with idiosyncratic approach of the research subjects?

It seems not. What is more, the students even stated during interviews that for them working on WS2 was more difficult than on WS1, and yet their results were better in their work on WS2. Therefore, it appears that since the second concept was more difficult for students, their better results were due to experience which they previously acquired working on a similar sheet of WS1. Thus, the *degree of understanding* of the concept of complex number was found to be generally higher than the the *degree of understanding* of Cartesian product.

We claim that more general and optimistic hypothesis can be made: regular work on similarly designed work sheets give the opportunity to improve students' performances.

Comparative analysis of the results of the second stage of the research – a student from *good* and *poor* group

We selected a one girl from the *good group* and a one boy from the *poor group* to provide the comparative analysis of the results achieved by students from the groups of different performances in mathematics education. Having analyzed the results of their individual work we notice that their answers to problems from WS2 are almost identical at the first stage of work on WS2, i.e. individual work without any help from the experimenter. There are only two differences to sub-tasks from UL2 and UL3 (student from the *good group* provided correct answers and the other wrong). We can summarize that both students reached the *definition level* of understanding of complex number, since the *degree of understanding* revealed during the work with WS2 was 100%. The results of both students revealed incomplete understanding of the complex number at the level of *local complication*: the degree of the *poor student* was 40% while of the *good* one – 60%. The first student as well as the second one made mistakes in deciding if the boundary examples were designatum of the concept or not. Understanding of the concept of complex number was revealed by these students only through the ability to determine whether a given example, which was imposed on simple additional conditions, is the designatum of the concept or not.

None of the students reached the *level of generalization*, however the *good student* achieved slightly better degree since he managed to cope with the geometric interpretation of complex number. Neither *good* nor *poor* student made the correct classification of real numbers as complex numbers. *Poor student* did not see any connection between these concepts and the *good* one identified a complex number with a real number.

May heuristic hints contribute to a greater degree of understanding of the concept at given level?

Answering this question, we analyze the correction of wrong answers made by the students after giving them heuristic hints. To better illustrate the process of making corrections by students the content of selected problems from the WS2 is given below. Both students as a result of independent work gave incorrect answers to the problems referred to understanding at U2 level, among others incorrectly classifying the boundary example in the problem 3 (subproblem e, f):

Problem 3.

Among the examples given below underline those that in your opinion are the complex numbers.

a) $-\pi+\sqrt{5}i$ b) $3-\frac{2}{3}i$ c) $\frac{i-4}{6}$ d) $-3i$ e) $\sqrt{7}$ f) 0

Justify your answer:

……………………………………………………………………………………….

Both students under the influence of hints given to them by en experimenter improved their answers to the two examples, therefore, properly considered the boundary example. Below we present a conversation with the *good student*:

Interviewer: Why did you decide that in the problem 3 the case “e” is not an example of a complex number?

Student: Because when I looked at other examples in further tasks there was no similar one. But now I am wondering…that “e” is also an example of complex number.

Interviewer: What is the reason of this change?

Student: As I can see it now this real number could equal $\sqrt{7}$ and that…imaginary number could equal 0.

Interviewer: How did you discover it?

Student: I haven’t noticed that before, but y can equal any number, so if we put here 0 (student indicates y) and here $\sqrt{7}$ (student indicates x) we will have this number. So, yes! This is a complex number.

Interviewer: Does it change anything in your opinion in the case of the “f” or “g”?

Student: Well, yes.. because now I can see that “f” will also be a complex number too. Only that here both x and y will equal 0.

The *poor student* similarly discovered that numbers presented in sub problem 3e and 3f are complex numbers. He concludes that if $y = 0$ then:

Student: Then we will have only x left, so this will be a complex number as well. Actually both of these examples (student points numbers exemplified in “e” and “f”) will be complex numbers!

The *poor student* expressed more emotions discovering the correct answer and we have seen his enthusiasm when he spoke the words: "Sure! I have not thought of that!".

In both cases students change their wrong answers to the problems related to UL2 level, which can be taken as a symptom of increased understanding of the concept of a complex number at the *local complication* level.

Extremely interesting is the fact, that the *poor student* just after making the corrections of his answers to problems referred to the UL2 spontaneously expressed the need for adjustment of subsequent answers provided by him to the problems diagnosing understanding at the *generalization level*. In certain cases he even gave ready and correct answers to problems, while in others realized the need for a change of the previously used incorrect methods of solving. Once again we are illustrating this with an example. We use here the problem 5, which required from the student to determine the relationship between real numbers and complex numbers:

Problem 5

Review correctness of the following statements (underline the correct evaluation of each of the sentences).

a)	Every complex number is also a real number.	TRUE / FALSE
b)	Every real number is also a complex number.	TRUE / FALSE
c)	Only some of the complex numbers are also real numbers.	TRUE / FALSE
d)	Only some of the real numbers are also complex numbers.	TRUE / FALSE
e)	There is no connection between the real and complex numbers.	TRUE / FALSE

Justify your answer:

……………………………………………………………………………………..

The *poor student* initially as a result of his independent work in each of the following points marked the answer "false" not recognizing thereby any connection between the real and complex numbers. However, the correction of the previously mentioned answers to the problems 3e and 3f resulted in the spontaneous reaction of the student:

Student: Here (student is talking about sentences from the task 5) I had a problem... I couldn't make my decision. Well now I probably would have changed it. If a moment ago I pointed out the roots … it can be here that some real numbers can be complex ... and basically all of them!

Interviewer: Could you tell me then, which of the answers would you change?

Student: That one! (student is changing his incorrect evaluation of the second sentence in problem 5). But in that case I would probably change these next sentences as well...

Interviewer: Which one?

Student: Well now I can see that y can equal 0 and then there is only x left and x is real. That means that sentence "c" is true as well.

The student not only noticed the fact that earlier had made a mistake in his reasoning, but also – almost immediately – he gave the correct answers to the next problems. We could observe a very similar reaction in the student after his having corrected the answer to the problem connected with the demonstration of the geometric interpretation of the complex number z=2+3i from the problem 6a (Fig. 4).

Problem 6.

Interpret all the complex numbers fulfilling the conditions given in the sub problems a - c in the following coordinate systems. Name the geometric figures received and label them under each of the coordinate systems.

a) $z = 2 + 3 \cdot i$

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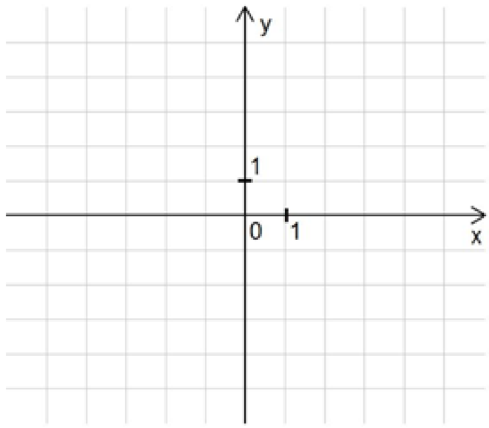

Figure 4. Problem 6a from WS

The student firstly interpreted this complex number drawing half line with its initial point (2, 3) and parallel to the y-axis. After having reviewed his response and giving the correct answer, he spontaneously saw his error in the answer to a further embodiment of this problem 6b (Fig. 5).

b) $z = x + 5 \cdot i$; where $x \in R$

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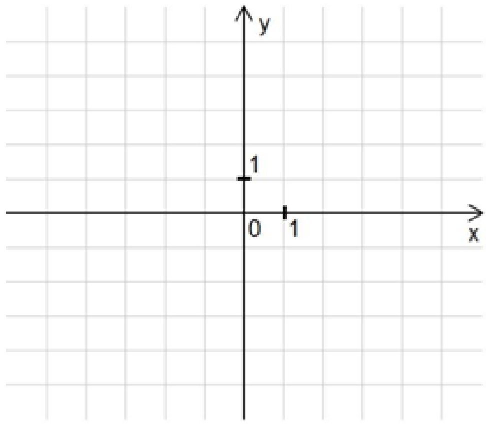

Figure 5. Problem 6b from WS2

Despite the fact that in this case he was not immediately able to provide the correct solution, he was aware of the error in his answer. The following part of the conversation shows this:

Student: Probably case „b” is also wrong. I just do not know how to correct it.

Interviewer: Try to do the same operation as you did a moment ago, that means try to introduce this number in the geometric form. Write it in a different way.

Student: That will be the set of points?

Interviewer: And what properties would this set of points have?

Student: There will be x, and y will equal 5. But I do not know how to write it..

Interviewer: Well, just like you said. You suspect that it is a set of points. What coordinates would these points have?

Student: y for sure will equal 5..

E: And what do we know about first coordinate?

Student: Uhm..

Interviewer: Which number can x equal?

Student: We know that x is contained in a set of real numbers (the student writes down the point which coordinates are equal x and 5).

Interviewer: In that case try to draw a few points that have such coordinates.

Student: That...that, and that! (the student drew three points).

Interviewer: Are these three points the only ones?

Student: No, no. There will be infinite number of them.

Interviewer: If so, than try to draw a few others (the student shows with his pen a few other points). So can you tell me now what kind of figure all these points will form?

Student: A straight line (student draws a straight line).

Interestingly, the *good student's* adjustment of the responses to the problem related to UL2 level and given heuristic hints or helping questions had no effect on an improvement her replies to the problems at UL3 level.

In summary, it is *poor student* who expanded his understanding of the concept of complex number. He had reached high enough degree of understanding (83%) at the *level of generalization*, making only one mistake. His error was not to identify just one example correctly.

The role of heuristic hints or helping questions on problems from the sheet WS2 in the case study of selected students was threefold. These questions:

1. resulted in providing improved answers to those problems for which they were targeted and caused spontaneously done improvement of the response to subsequent problems,
2. resulted in improved answers to those tasks for which they were directed and in student's awareness that other tasks were solved by him wrong, even though the student did not yet know how to solve them properly,
3. resulted in correction of the selected answer but did not cause significant changes in the understanding of the concept.

CONCLUSIONS

1. The research revealed that understanding of a concept at the generalization level UL3 is qualitatively very different than understanding at two lower levels UL1 and UL2 for the following reasons.

(a) It is the most difficult level for students. We observe a drastic decrease in students' performances at this level. The revealed by WS1 degrees of understanding in students at the UL1 and UL2 levels were comparable, but the degree of understanding at UL3 was twice lower.

(b) The students while solving the problems referred to this level make qualitatively different mistakes. The analysis of the replies to such kind of problems related to UL3 level indicated the need of introducing additional category of answer evaluation: incomplete answer, which is factually correct but does not take into account all possible cases.

(c) Students often commit two types of errors: the first one of replacing whole class of designata of a concept by a part of it, the second one the opposite effect, i.e. skipping some (more or less important) conditions of a definition, which results in classifying a concept to a wider class of objects. (see also Dyrszlag, 1972).

2. On the basis of the questionnaire from the first part of the research the conclusion can be drawn that the overwhelming number of students are aware

that understanding of a concept at the generalization level is difficult for them (the opinion was expressed in total 62% of all students working on WS1).

3. The answer to the question: if the degree of understanding of a mathematical concept at the level of generalization UL3 is associated with assessments of mathematics received by students at school – is interesting, complex and ambiguous at the same time.

(a) The results of the first stage of the research showed a general trend of increased degree of understanding of a concept at a given level along with the better outcomes achieved in school education.

(b) The degree of understanding of the concept at UL3 in students achieving good results in mathematics is indeed higher (in the case of WS1 up to twice higher) than the level reached by the other subjects, however, there are poor students, who solved the problems related to UL3 and good students, who were unable to solve them.

(c) At the same time qualitative analysis of the second stage of the research showed the disturbance of the tendency described in paragraph (a). It turned out that students who have obtained comparable results in the first part of the study – under the influence of the heuristic hints and questions asked – significantly changed their answers. The good student did not come out beyond the UL2 level at all, only reinforcing her understanding at this level, while the poor student has achieved a high degree of understanding of the concept at the generalization level. It can therefore hypothesize that the answer to this question also depends on the student way of working, and of course on student way of thinking.

4. Sets of problems in WS1 and WS2 sheets turned out to be not only the diagnostic tool, but also a didactical one. The students achieved significantly better results in the work of WS2, although they considered the notion of a complex number difficult. In our opinion, the optimistic hypothesis can be formulated: allowing students to work on the similarly constructed sheets based on independent work on definitions of new concepts has a positive effect and gives the opportunity to improve students’ performance and achieve a higher level of understanding of other new concepts introduced in this way. This is the conclusion of significant educational implications.

5. The results of our empirical research indicate the need for change in the theoretical conception formulated by Dyrszlag (1972). It is difficult to answer the question: what does it mean to achieve a certain level of understanding? It seems to us contradictory and awakening many inconsistencies statement: "Student X has reached the understanding of the concept at the Y level". Our empirical research suggests that the answer to this question depends on considered level of understanding. While achieving a UL1 level (and even UL2) can reasonably be conditioned by the solution of all problems related to these levels from the diagnostic set, the big questions arise in the context of the level

of generalization UL3. For example, if a student is able to place the concept on the background of parent and child concepts, which is resulted in solving at least one problem related to UL3 level, it should be recognized that his/her understanding of the concept has gone beyond the UL2 level, even though the UL3 is not fully, or 100% achieved. We therefore consider necessary to introduce an additional term: the degree of understanding of a concept at the given level which we define as the percentage of correct answers to all problems diagnosing understanding at this level.

6. Another methodological question concerns the finding that achieving the next level of understanding of each concept is conditional on reaching the level of the previous one. The same doubt was also mentioned by Dyrszlag:

> Is it possible to obtain by a student the third level of understanding at once, without a thorough check on whether he obtained the lower levels? This case has not been investigated and is difficult to say about that. (…) I am inclined to suspect that both the L1 and L2 level cannot be entirely disregarded, if we want our average student to understand a concept at the generalization level. (Dyrszlag, 1972, pp. 62-63.)

The results of our empirical research show that a lot of correct solutions to problems related to UL3 level were provided by those who had not reached 100% degree of understanding of the concept at levels U1 and U2. This generates the negative answer to the mentioned question as to whether the achievement of each subsequent level of understanding of a concept conditioned by reaching the previous one.

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GENERALIZATION
DO THE IT TOOLS ENABLE PROVOKING AND DEVELOPING STUDENTS' MATHEMATICAL ACTIVITIES?

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This paper presents an attempt to answer the question whether new technologies such as graphing calculators and computer programs allow the provocation and the formation of students' creative specific activity that is generalization. The paper contains a description of the pieces of research conducted for several years at various levels of education (gymnasium school, high school, teachers' studies on mathematician's direction) to examine the student's way of working with graphing calculators Texas Instruments and Casio, and free computer programs such as GeoGebra and WinPlot.

> In the course of a professional career every person solving a task develops their personal style and individual methods. The strive for perfection in systematic use of those methods is often a slow and painful process, which ripens within years if it ever ripens. (Alan Schoenfeld)

INTRODUCTION

What are the expectations of a modern school of a Maths teacher? What is required from a student gaining knowledge of that subject? The above questions are essential when referred to curriculum and organizational change, which has been introduced in Polish schools for a few years. The idea of mass education, both at medium and higher levels[1], a many year period of the lack of obligatory matura Maths examination, which results in the decrease of the number of students interested in studying Science, as well as at universities and polytechnic schools, all have caused the introduction of the new curriculum foundation and new work schedules for schools at all educational levels.

Mathematics is a work tool for many professionals, it is a language describing almost all the phenomena surrounding us, it is the language which it is impossible to function without in everyday reality. General requirements, formulated in the new curriculum foundation, establish the main teaching goals for each subject to be taught. The ones referring to Maths focus mainly on teaching mathematical thinking and shaping activities peculiar to Maths.

[1] At present, 80% of the same year students attend schools ending with a matura examination (just a few years ago the number was about 50%), and the number of higher education students has increased five times (Marciniak 2011).

What should that mathematical activity rely on? Should it be the same at any stage of a student's intellectual and mathematical development? The answer to these questions is indispensible for the proper organization of the Maths teaching process by a teacher.

Zofia Krygowska in one of her books writes:

> …are students active when they diligently learn by heart the formulas, definitions, proofs and schemes of solving typical problems they are given? That is definitely necessary to some extent. However, modern didactics exposes another kind of activity, creative activity, a conscious and active learner's contribution to discovering ideas, notions, formulas, theorems, proofs, to schematizing situations, to mathematizing them, generally, to solving extremely differentiated problems, covering the whole of the teaching material. (Krygowska, 1981)

Later the author in the same work describes mathematical activity of a learner. Among six categories of activities typical of a Maths learner, she mentions a specifically creative activity, which consists of, among others, detecting and formulating problems, discovering, formulating and proving theorems, **generalizing** and specifying, problem solving in untypical situations, etc. The theory is still up-to-date and vividly supported by Maths didacticians, although it was created almost 30 years ago.

Maths didacticians and school teachers are constantly looking for and mastering such mathematical content presentations, such work methods, such didactic tools, communication forms adequate to students, attitudes and exertions, that the most perfect and effective mathematical activities mentioned above are provoked and shaped in the students.

In the article the attention is going to be focused on one of the specifically creative activity, that is on generalization, as well as on certain didactic means which enable provoking and developing that activity. The conclusions and remarks described below are the result of the author's 10-year-long research, the research aiming at, among others, observations of the work of students at gimnazjums, high schools as well as the students at teacher's colleges at Maths departments, while solving a group of untypical mathematical problems with the use of new didactic tools such as graphic calculators, emulators of graphic calculators and free software such as GeoGebra or WinPlot.

GENERALIZING

To achieve a student's full mathematical development, it is essential to activate and form mathematical activity, referring both to mastering basic knowledge and also gaining the skills indispensable for solving mathematical problems and evoking behavior typical of Mathematics. None of them performs by itself in the course of gaining basic information and skills. The latter is accessible only to the students who are particularly skilled and interested in Maths.

G. Polya , when answering the question: "How to solve it?" in his book under the same title, emphasizes how much the ability to generalize can be helpful in problem solving; he also claims that „… The transition from a problem "with numbers" to a problem "with letters" offers us new opportunities…" (Polya, 2009, p. 205). He characterizes

> generalization as a transition from analyzing one object to analyzing a set including that object, or a transition from analyzing a narrower set to analyzing a wider set, including our narrower. (Polya, 2009, p.203)

Generalizing theorems is a process so typical of mathematical activity, that it is worth looking for didactic situations in which we will provoke that process in the student's mind (Krygowska$_{3,}$ 1977; p.113). Generalizing theorems may be performed in different way, however, it is essential to apply such didactic efforts that they will activate a student's mathematical thinking, also in different. Z. Krygowska describes, among others, **generalizing through noticing the law of recurrence** (when we are able to perform reasoning and state that the condition is fulfilled for a specific case, for an initial element and it is also fulfilled between any two succeeding elements belonging to the same set), **generalization through unification** (when we unite formulas obtained as a result of reasoning performed for different cases in one, more convenient to apply, or arranging hitherto existing knowledge), **generalizing theorems through generalizing reasoning** (when reasoning performed in a particular case can be conducted for every other specific case and it will continue in the same way or only with little change, then a certain object can be substituted for with any other object from a given set) and **generalization of an inductive type** (based on the analysis of a few specific case and looking for a common scheme for them, particular statements are substituted for with a general theorem including all the cases, whose rightness has to be tested and proven) (after: Legutko, 2012). In the literature we find two suggestions how to perform the process of generalization of notions: as a discovery of a superiority relation between two known ideas or as constructing a superior notion for a well known idea. Not only statements are the objects to undergo generalization in mathematics. Ideas, problems, methods, hypotheses, ways of reasoning and proofs can undergo that process, too.

The activity of generalizing can be accessible to every student, the good organization of the teaching process may make it easier and more highly valued.

GOALS, ORGANIZATION AND METHODOLOGY OF THE RESEARCH

Below there is a presentation of a part of research whose general aim was to examine and describe the ways of applying new technologies by students of different educational levels and university students during the process of solving a certain group of problems.

Because of a limited content of the article I am going to focus on the research that was performed on gymnasium students and I will only mention the research referring to students of high schools and universities.

Gymnasium

In studies conducted at the third education level I have been looking especially for the answers to the following questions:

What do the students use a graphic calculator for when they solve a problem?

What are the students' strategies[2] in the course of solving tasks?

What mathematical activities does the graphic calculator provoke?

In the article I am going to concentrate on the answer to the third question, with an emphasis on the manifestations of provoking generalization activity by IT tools.

The study was based on the students of grade one (14 years old) of a gymnasium in Bielsko-Biała, where mathematics was taught on the basis of the course book and syllabus called "Mathematics in gymnasium with a graphic calculator and a computer" („Matematyka w gimnazjum z kalkulatorem graficznym i komputerem") (Kąkol; Wołodźko, 2002). The study was performed on the subject of "Functions". Four students solved problems during weekly 45-minute, extra-schedule sessions, for four months. At every session the students were given the content of the task and a graphic calculator TI 83 Plus. The course of the session was recorded by the author with a cassette recorder. Every meeting finished with a conversation with a student about the method and the purpose of the student's use of the calculator.

The main research tool was a unique at that time calculator program[3] enabling the work performed on the calculator to be recorded. The program made it possible to record the work of each student participating in the individual meetings. It is essential that the recording program enables succeeding screen views to be shown in a form of an accelerated movie, as well as it enables to scan the list of the buttons pressed by the student in the course of their work. The program allowed me to have an insight into each student's thinking process and learn the exact sequence of their work stages on the task, not narrowing the analysis to merely the final record of the solution on the paper, or, as it happens most often, limiting it to the result itself.

The tasks given to the students come from a course book and a collection of practice exercises called "Mathematics with a graphic calculator and a computer" („Matematyka w gymnasium z kalkulatorem graficznym i komputerem"), from a part devoted to Functions (Kąkol; Wołodźko, 2002).

[2] By strategy I understand the way of a student's performance with the use of a graphic calculator leading to the solution of the problem.
[3] prof. John Berry from the Plymouth University is the author of the program.

The curriculum of teaching mathematics with the use of the graphic calculator and the computer in grade one of the gymnasium assumed the performance of the following issues: the Cartesian coordinate system, geometrical figures in the coordinate system, the notion of function, the graph of a function, simple proportionality, the properties of the function f(x) = ax, linear function and its properties, linear equations with one unknown and inequalities with one unknown. They were adapted to their skills connected with the use of a graphic calculator, as well as to their current mathematical knowledge. A few tasks "slightly outdistanced" their knowledge and skills in mathematics, however, they fitted in the "range of their nearest future development", which was mentioned by Wygotski. The students solved 13 tasks during individual sessions.

The group consisted of: Dorota (D), Monika (M), Janek (J) and Szymon (S). The survey they had filled in at the beginning of the study reveals that all of them had a positive attitude towards the graphic calculator and they were not afraid of using it. Those students acquainted with the tool for the first time at their math classes when they were in grade one of the gymnasium. For the initial six months of their work with a graphic calculator Dorota and Monika used it for readymade programs and for making auxiliary calculations, while Janek and Szymon wrote their programs in addition.

All the students achieved good or very good results. Moreover, it can be assumed that Janek is the best one in the group, while Dorota is the weakest one.

The analysis of the research results revealed the following mathematical activities of the students, which had been provoked by the use of the calculator:

- empirical conclusions,
- symbolic language usage,
- generalization,
- hypothesis formulation and verification,
- formulation of new problems,
- deduction.

I am going to present below the content of the task in which using the calculator caused a particular mathematical activity and a copy of the student's notes accompanied by a short commentary to the solution with his copy of the notes.

Task
For which "a" values the graphs of the f(x)=ax function will be perpendicular?

<u>The description of a student's work</u>

1) He activates the Zsquare option from the Zoom menu.

2) In the "Y=" editor of the function formula he introduces the pairs of function formulae, next he draws their graphs: y=x*1 and y=x*3, y=x*1 and y=x*-1. *(The student draws on the calculator screen any pair of graphs of functions of the form y = ax. After this attempt he draws straight lines which are perpendicular. The student admits in an interview that this was an example that appeared once in math class. Stating only by watching the student verbally states a hypothesis "the graphs will be mutually perpendicular, when the coefficient a of one of these functions is a number opposite to the coefficient a of the second function".)*

3) *(The student verifies the hypothesis)*
In the editor "Y=" he introduces the formulae of three pairs of functions:
y=x*1 and y=x*-1, y=x*8 and y=x*-8, y=x*2 and y=x*-2,
and he draws their graphs in the Zdecimal window.

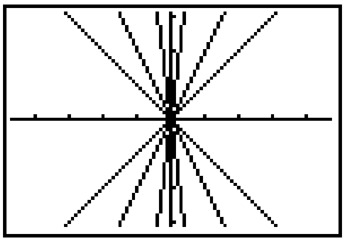


Figure 1 (for all the function pairs)

(The student notices, that the rule does not work for other pairs of functions of the form y = ax and y =-ax, and that no matter how you format the display window. The student visually wagered just refuting the hypothesis. He knows, and informed me verbally that a greater number of cases is no need to change as a result of the verification process. He undertakes another way of solving the task.)

4) He draws the graphs of the functions y=x*1 and y=x*-1 again.
(The student returns to the example satisfying the conditions of the problem. Very long looks at the graphs of par of functions, he looking for some patterns, which enable him to find examples of other pairs of functions satisfying the conditions of the problem.)

5) He draws the graph of the function y=x*2.

6) He moves the free cursor over the screen.

7) He stops it at the point with the co-ordinates (1,2).

8) He moves the free cursor over the screen.

9) Next he stops the cursor at the point (-2,1) and sets a point there.

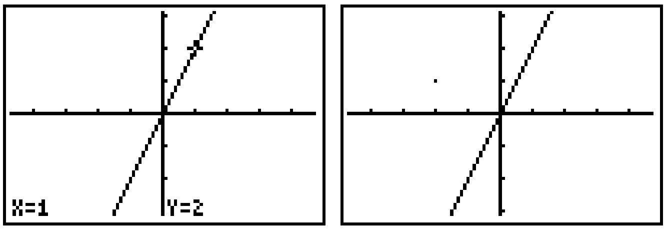


Figure 2

10) *(He looks for the straight line crossing the point with the coefficients (-2,1) and the origin of the coordinates.)*
He draws the graphs of the pairs of straight lines:
y = x*2 and y = x*1, y = x*2 and y = x*4, y = x*2 and y = x*-.5,
(At the beginning of a randomly selected value of a coefficient of the second function, visually check whether the graphs of these functions are orthogonal. When this method of working is not effective, and states that after the second attempt, runs other mathematical tools. On a sheet of paper[4] *performs the appropriate transformations, computes the value coefficient "a" as a function y=ax with the coordinates of the point (-2,1).)*

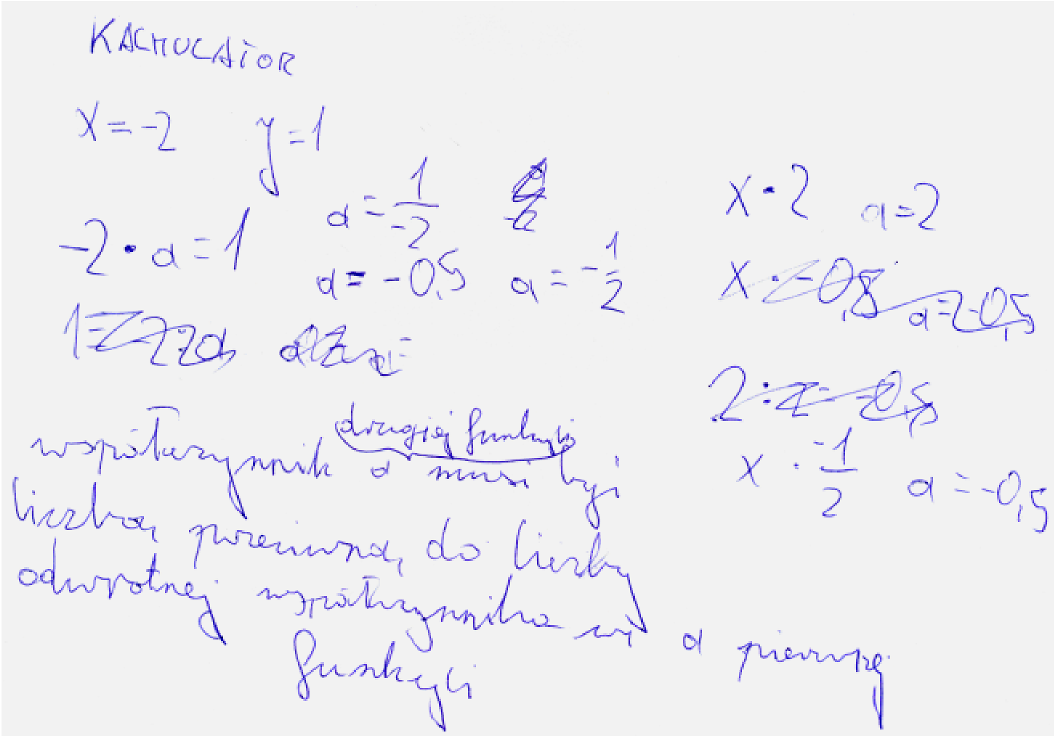
In summary stage the student's reasoning contained in steps 4-10:

The starting point to his reasoning was a graph of mutually perpendicular lines y = x and y =-x. The student noticed that the lines form four angles of the same measure, which means right angles. That enabled him to obtain the graph of one of the straight lines as the result of the rotation of another one by an angle of $90°$ regarding the origin of coordinates. The method, consequently, allowed him to find the image of a point belonging to the straight line on the line perpendicular to the first one. The student finished the reasoning with comparing the coordinates of the chosen point and its image.

11) He draws the graphs of the succeeding pairs of straight lines:
y=x*8 and y=x* -4, y=x*6 and y=x* -1/4, y=x*6 and y=x* -6/4, y=x*6 and y=x* -1/6.
(The student looking for relationships between the coefficients of pairs of functions orthogonal graphs, he has two examples 1 and -1, 2, and -0.5. Only the fourth test is the correct one. Student exposed orally formulated rule, and then wrote it on paper. He could not write a rule using symbols.)

4 See the copy of Janek's notes.

The copy of student's notes[5]



The a coefficient of the second function must be a number opposite to the converse number of the a coefficient of the first function.

Commentary

Working on that task Student has made 13 attempts. He found three examples of functions with graphs mutually perpendicular. He did not confine himself to merely give a few correct examples. He found a general rule in spite of the fact, that the instruction of the task does not directly say anything about formulating the rule (look at the copy of the notes) - the coefficient a of the second function has to be an opposite number to an inverse number of the coefficient of the first function. The student, concluding from a few examples whose correctness he estimated "by the looks" of reciprocal position of the graphs of the pairs of functions, formulated a hypothesis, which constituted a generalization of the obtained detailed results of his work on that task. The student felt a need to justify and prove his discovery not only in a so called "graphic way". He was not able to use the symbols properly but he perfectly managed to formulate the generalization in written words.

Moreover, after having finished that stage of working on the task, he was tempted to continue his effort by using the objects, which had not occurred during the lesson before. Namely, he wondered if the rule he had just

5 Note the student has been commented on in steps of 10 and 11 student's job description.

discovered, would work for linear functions intersecting at another point, different from the origin of the coordinates. The student stated, only verbally, a problem more general than the one which had been formulated in the task instruction.

Conclusions

The student, with the use of a graphic calculator drew many correct graphs of a linear function in the form y = ax in a very short time. The calculator enabled him to concentrate on the main task, that is on finding the linear function formulas meeting given requirements, without the necessity to leave the task in order to draw many graphs thoroughly and correctly and to make auxiliary calculations. In order to discover specific cases fulfilling the conditions of the task, the student took an advantage of the opportunity to follow the coordinates of the points belonging to the tested graphs, to make changes to the setting of the drawing window, and to define a new drawing window. The use of the graphic calculator by the student contributed to coming up with an idea to solve the problem, to find specific cases fulfilling the requirements of the task, to generalize, and consequently, to extend the task and to formulate a more general problem. The analysis and testing a sequence of pair of graphs of simple proportionality activated the student's mathematical curiosity.

Throughout the student's thought process we find elements of both inductive generalization, when a student requesting a number of examples that meet the conditions of the problem replaces them with a general statement, and a generalization of reasoning from example by verbal changing of the constants. Moving from words to the letter proved to be too difficult for a student starting first grade school, the general rule was formulated, both orally and written on a piece of paper.

It is worth mentioning that three out of four students participating in the study were able to formulate a general rule, only two of them in the form of verbal and one also with the use of symbols. Undoubtedly on this stage of development, in a situation of independent work on a task- a problem, uncontrolled by either a teacher or a classmate, achieving the success was possible thanks to the contribution of a tool, that is the graphic calculator.

Teacher study at mathematics department

The study performed at the Department of Mathematics and Computer Science of Adam Mickiewicz University of Poznań refer particularly to the methods of solving untypical Maths problems with the use of the graphic calculator Casio, the emulator to the calculator and free software GeoGebra or WinPlot, by the students of the teacher department with the reference to the dominating cognition structure, that is the functional or the predicative ones and to study the attitudes of Maths teachers-to-be towards those problems. The analysis focuses on the same elements which have been described for the study of the students of gimnazjum (the goals of tool application, strategies, mathematical activities).

For three years a group of nearly 10 students (on each year) who independently participate in the research program, has taken part in the study each year. Throughout the year they solve a set of a few tasks in two ways - a classical one, that is with the use of only "a pencil and a sheet of paper", and then with the use of modern technology (the students make their own choices as far as the tools go, depending on their needs). After the work on the problem has been completed, work evaluation takes place and the questions about the course of solving the problem are answered to. Similarly to the study of gymnasium students, here also a film recording of a student's work is made - if the student applies the emulator of the Casio calculator or the construction report in the case of working with the GeoGebra software. The students work unaided.

Because of the editorial limitations the article lacks a detailed description of the examples as well as an extended didactical commentary. The results of the study will be discussed by me systematically at local didactician conferences, among others at the Szkoła Dydaktyki Matematyki (The School of Mathematical Didactics). Here I would like to state shortly that like in the case of the gymnasium students, the university students while working with the help of IT were provoked by the wide range of opportunities that those tools offer - they generalize, which is not always observed during work with the classical method.

High school

In the last school year 2010/2011 a union project entitled The Śniadecki's Collegium – an innovative curriculum for teaching Science[6] was introduced in the area of the Wielkopolska Region. Its main goal was to increase high school students' interest in mathematical-scientific subjects through the inculcation of the teaching and in- advance learning method together with the intensive use of the educational platform into chosen high schools. One of the final results in the innovative work on Maths lessons should be provoking the students to active, unaided and effective work after school, as well as during Maths lessons. In the project we strive for shaping a mathematically aware high school student.

Being one of the authors of teaching materials for primary school students (grade one), referring to the development of the notion of function, I decided to apply my past research experience[7] concerning possible effects of students' work (at different ages) when solving mathematical problems, the untypical and difficult ones in particular, with the use of new technologies. When writing about the effects I mean provoking and developing mathematical activities,

[6] More information about the project can be found on the website http://kolegiumsniadeckich.pl

[7] More about it in the article which is going to be published in book IV of Współczesne Problemy Nauczania Matematyki; Edyta Juskowiak „Technologie informacyjne w kontekście innowacyjnej koncepcji nauczania wyprzedzającego - PROJEKT KOLEGIUM ŚNIADECKICH”.

including generalization, too. The tool that I suggested for the students to apply was a free and easy to use computer program – GeoGebra. School work, based on the aforementioned materials, started in the current school year 2011/2012. So at this stage of work it is difficult to present the results, the partial ones will be available in the months to come. It seems, however, that new technologies, if applied properly, one more time will enable provoking and developing the students' mathematically proper behaviour in relation to the problems they face.

CONCLUSIONS

Piere M. Van Hiele in his works underlines the fact that students should learn Mathematics through acting, and not be just informed about it (after: Turnau, 1990), as it happens in the course of the giving teaching. On the other hand, H. Freudenthal in his lectures and articles claimed that we are not allowed to teach children the things that they could discover by themselves, a child has to learn to discover mathematical structures (Kutzler, 2000; Freudenthal, 1976). New didactical tools and calculators and computers with mathematical software in particular, allow to introduce a change in the teaching methods, they grant it a problem character (Kąkol, 1991). They enable the students to experiment in many branches of Mathematics, allow to make numerous observations and to discover their "own Mathematics". As the research results show, IT enables to provoke and develop many mathematical activities, including also the specifically creative ones, and among them, also generalization. Skillful application of new technologies, as one of many didactic tools, both by the students and the teachers, as well as the ability to make the students at the right moment face the task which help them develop certain mathematical activities is extremely important in the process of developing proper mathematical thinking of a young learner.

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Increasing teachers' awareness and skills of generalization

Part 5

PROSPECTIVE TEACHERS' MATHEMATICAL KNOWLEDGE OF FRACTIONS AND THEIR INTERPRETATION OF THE PART-WHOLE REPRESENTATION

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Fractions are one of the more complex mathematical concepts children encounter in their schooling. While the majority of existing research addressing fractions has focused mainly on students, leaving aside the teachers' role and the importance of teachers' knowledge in and for teaching, we focus on early years' prospective teachers knowledge on fractions and the role of the whole and its (possible) impact in preventing pupils from achieving a full understanding of the topic.

INTRODUCTION/SOME MOTIVATIONAL WORDS

The International Summit on the Teaching Profession has addressed the challenge to equip all, instead of just some, teachers for effective learning in the 21st century (OECD, 2011, p. 5). This requires an emphasis on, among many aspects, "the kind of initial education recruits obtain before they start their job" (ibid). Several studies have documented that teachers have a greater impact than any other factor on student achievement (e.g. class size, school size, or school system) (e.g., Nye, Konstantopoulos & Hedges, 2004). There has been an increasing amount of attention and focus laid on teachers' knowledge, and on how gaps in such knowledge relate to limited treatment in subject courses prospective teachers' receive in their education. Studies have shown that an exclusive focus on content knowledge, by increasing requirements for more advanced mathematical courses, has no positive effect on student's achievements (Begle, 1972). Also, due to Shulman's (1986) distinction between subject matter knowledge (SMK) and pedagogical content knowledge (PCK), the importance of teachers' knowledge has received increasing attention.

In mathematics education, Shulman's ideas were developed further into a framework for teachers' mathematical knowledge for teaching (MKT) by a group of researchers lead by Deborah Ball at the University of Michigan (e.g., Ball, Lubienski & Newborn, 2001; Ball, Thames & Phelps, 2008). The Michigan group has identified a number of specific challenges related to teaching mathematics, and it is assumed that these challenges (tasks of teaching), are similar in different countries (Ball et al., 2008). Examples of tasks of teaching are recognizing what is involved in using a particular representation, and linking representations to underlying ideas and to other representations.

To improve practice and teacher training at all educational levels, teacher education has to focus more on teachers' knowledge, on the tasks involved in teaching, and on the mathematical critical situations and topics identified, which will contribute to a smoother transition of students between educational levels. One of these critical topics concerns fractions. Students struggle to understand both the mathematics embedded, and the different interpretations and representations fractions can assume, i.e. part-whole, quotients, measures, ratio, rate, and operators (Behr, Lesh, Post & Silver, 1983). For such an understanding it is of fundamental importance that a good understanding of the relationship of the parts and the whole, and the possible different "kinds of whole", be acquired. Students' limited understanding might be related to how their teachers' understand and interpret fractions, and such limitations may result from the fact that this topic is not addressed explicitly and does not have the focus that is needed in teachers' education.

Teachers' training has in certain respects been left behind in the research. We still know little about how (prospective) teachers' knowledge of fractions influences students' broader view of mathematics, and its connection and evolution within and along schooling. This has motivated our research. By calling attention to prospective teachers' training and the role it has on teachers' professional knowledge and development,[1] we hope to better understand the (possible) impact such knowledge has on their (future) practices, and on their students' achievement. With this in mind, we address the following research question:

> What kind of subject matter knowledge (in terms of MKT) is revealed about their interpretation of fractions and the role of the whole by early years prospective teachers', and how can we characterize such knowledge in order to specify critical aspects to focus on teachers' training?

THEORETICAL FRAMEWORK

Teachers' knowledge can be perceived from different perspectives. Grounded in Shulman's (1986) work, some new conceptualizations on mathematics teachers' knowledge have emerged (e.g., Rowland, Huckstep and Thwaites, 2005; Davis & Simmt, 2006; Hill, Rowan & Ball, 2005). In our focus on teachers' knowledge, we focus on the MKT conceptualization with its various sub-domains (Ball et al., 2008). One reason for favoring this conceptualization of knowledge is that we perceive the sub-domains of MKT (see Ball et al., 2008) as

[1] We assume teachers' professional development to start, explicitly, and in a formal way, in prospective teachers' training and thus this is (for excellence) the starting point for discussing, promoting and elaborating teachers' knowledge in order to allow them to teach with and for understanding.

a relevant starting point for designing tasks for the mathematical preparation of teachers, and for doing research on what inputs to teachers training shows effects on students and practices. Interestingly, the Michigan group has found a connection between teachers' MKT, as measured by their MKT items, and students' achievement in mathematics (e.g., Hill et al., 2005).

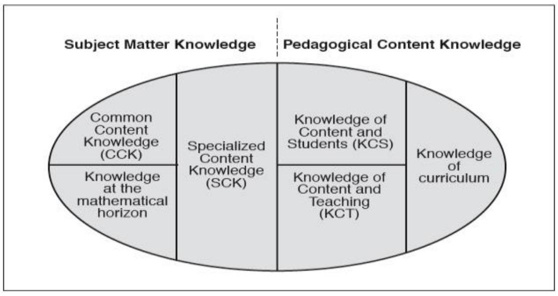

Figure 1: Domains of MKT (Ball et al., 2008, p. 403)

The MKT conceptualization of teacher knowledge comprises Shulman's domains (SMK and PCK) and considers each one of them as being composed of three sub-domains. We will here approach only the sub-domains concerning SMK. SMK comprises what is termed "common content knowledge" (CCK), "specialized content knowledge" (SCK), and "horizon content knowledge' (HCK). CCK is knowledge that is used in the work of teaching, but also commonly used in other professions that use mathematics. It can be seen as an individual's knowing the topic for themselves – e.g. knowing how to obtain the correct answer when multiplying fractions. Teachers (obviously) need to know how to do this, but it is also common knowledge within a variety of other professions. However, in order to give students' opportunities to achieve a deeper understanding of the topics (here fractions), besides knowing how to perform the calculations (find the correct result or identify incorrect answers), teachers' need to know the mathematical hows and whys behind such calculations. Such knowledge on the hows and whys related with fractions is a core knowledge in order to allow teachers' to (amongst others) being able to explain it to students', listen to their explanations, understand their work, and choose useful representations of fractions that can support students' learning. This is knowledge that requires additional mathematical insight and understanding (Ball, Hill & Bass, 2005), and is considered SCK. The last sub-domain is termed HCK, which is described as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al. 2008, p. 403), and is important for developing students' connectedness in mathematical understanding along the schooling.

Teachers' knowledge and what concerns the specificity of the topic being approached (mathematics) is inter-related, it influence and is influenced by a large span of dimensions and aspects. Examples of these dimensions and aspects are teachers' role, actions and goals (Ribeiro, Carrillo & Monteiro, 2009). Teacher's participation in professional development programs can contribute to an important part on their awareness of practice (Muñoz-Catalan, Carrillo & Climent, 2006). It also contributes to the development of their MKT and on their awareness of the role of teachers' professional knowledge dimensions in practice (Ribeiro et al., 2009). We assume that teachers' professional development starts, explicitly and in a formal way, in pre-service teachers' education, and thus, this is (should be)[2] the starting point for discussing, promoting and elaborating teachers' knowledge allowing them to teach with and for understanding.

Within the new Portuguese National Curriculum (Ponte et al., 2007), the understanding, representation and interpretation of fractions is transversal to all the first nine years of schooling. In this new curriculum, it is mentioned that the approach to rational numbers should start on the first two years of schooling, in an intuitive manner. Thereafter, one should progressively introduce the representation of fractions, using simple examples. In years three and four, the different interpretations of fractions should be deepen, starting from situations involving equitable sharing or measuring, refining the unit of measure – using discrete and continuous quantities.

Discussing the importance of the role of the whole is a core element in allowing for understanding of all the different interpretations and representations of fractions (Kieren, 1976), and is perceived as a "prerequisite" for such understanding (Ribeiro, in preparation).

Fractions are among the most complex mathematical concepts that children encounter in their years in primary education (Newstead & Murray, 1998). These difficulties can be originated from the fact that fractions comprise a multifaceted construct (e.g., Kieren, 1995) or they can be conceived as being grounded in the instructional approaches employed to teach fractions (Behr et al., 1993). These identified difficulties illustrate the importance of improving teachers' initial training. A consequence of such an improvement will be increase students' CCK concerning fractions, contributing to a new and better direction at all educational levels.

METHODOLOGY AND CONTEXT

This paper is grounded in data gathered from an exploratory study between sixty prospective early years' mathematics teacher in Portugal. By combining a qualitative methodology and an instrumental case study, we focus on these prospective teachers' MKT on fractions, and on their revealed understanding about the role of the whole.

[2] We consider that pre-service teachers' training should start to assume a broader and important role in teachers training, as it is the first stage and contact with most of the aspects referred to in literature as being problematic and in need of a change.

Data is from a sequence of tasks assigned to these prospective teachers in the context of a course focusing on the SMK sub domains of MKT (with 28 hours of classes, meaning 2,5 ECTS). Fractions were one of many topics approached in the course. Tasks used in the assignment were taken from Monteiro and Pinto (2007), and then modified for implementation in teachers' training[3] and aligned with the Portuguese National Curriculum for the first nine years of schooling.

Besides focusing on CCK, the aim was also to look into the different interpretations and representations of fractions, in particular the role of the whole. All tasks were discussed in groups of four or five prospective teachers, and at the end there was a large group discussion aiming to obtain a deeper understanding of their knowledge of fractions (SCK and HCK).

The assigned set of tasks was designed with a specific goal to promote the development of prospective teachers SMK (Ribeiro, in preparation) on fractions. They were specifically related to the work of teaching mathematics and they were grounded in tasks of teaching (Ball et al., 2008). In this paper, we only present part of the first task:

> Teacher Maria wants to explore with her year one students some notions concerning fractions. For such she has prepared a sequence of tasks involving 5 chocolate bars. What amount of chocolate would 6 children get if we share the 5 bars equally among them?

The prospective teachers were asked to solve the task with two different perspectives in mind: 1) as if they were year one students, and 2) giving their own answer as prospective teachers. In both answers they were supposed to describe and justify what they did and why they did it.

In the analysis we focus on prospective teachers' mathematical critical situations: their revealed gaps in knowledge, their different interpretations of fractions, and on the role of the whole. Our aim is to obtain a deeper understanding of the mathematical reasons why such gaps occur, in order to be able to design materials to improve teachers' training and the ways in which we, as teacher educators, approach such training.

SOME RESULTS AND DISCUSSION

Here we present, analyze and discuss answers from some of the prospective teachers. All groups presented at least one numerically correct answer, frequently found by using different ways of dividing the chocolate bars aiming to express the final result as a sum of different numerical fractions. They commonly had difficulty in explaining the sense of the answers.

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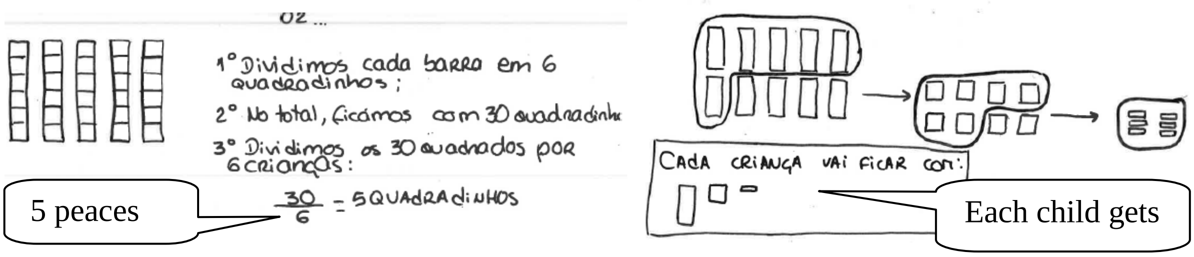
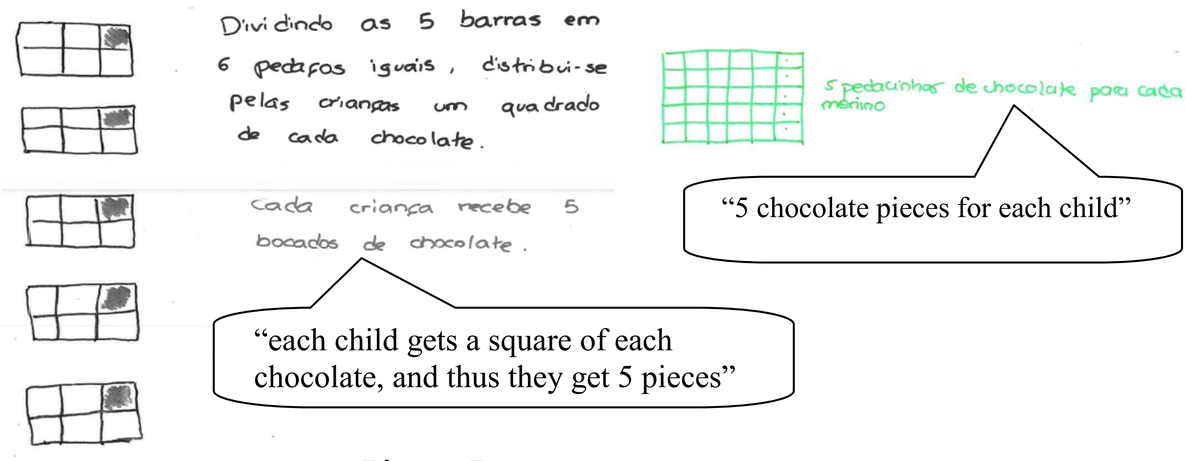
[3] In Ribeiro (in preparation) the nature of such tasks and of the specificities associated to the context in which they are aimed, are discussed. Part of such discussion concerns, also, the kind of necessary changes to be implemented to tasks prepared to be implemented with pupils/students, in order to contribute to develop teachers MKT in all its sub domains of SMK.

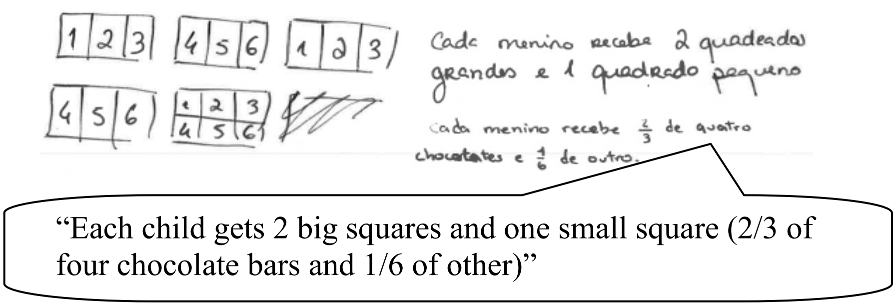
Many of these prospective teachers failed to consider the role of the whole when solving the task, they failed to consider how they would divide the chocolate bars (the whole) in order to “share the 5 bars equally”, and often they did not even consider the importance of finding the “exact” amount of chocolate each child would get. They used either exclusively pictorial representations or tried to represent in different ways (one of) the correct answers. When using exclusively pictorial representations the answers can be divided in two groups: (i) the whole is 5 chocolate bars isolated (simply pictorial answer); the prospective teachers’ then just draws the chocolate bars and divides each part they are obtaining (in each step) in halves and thirds (Pictures A and B); and (ii) the whole corresponds to a continuous unit compose by the 5 chocolate bars; the teacher then draws a representation of the 5 chocolate bars (as a whole) (Picture C).



Picture A



Picture B



Picture C

Figure 2: Examples of student work using exclusively pictorial representations

When they tried to represent (one of) the correct answers in different ways, prospective teacher considers the whole to be the set of 5 chocolate bars, but seen as discrete "subunits", and their focus was on obtaining the answer using different ways of dividing the 5 chocolate bars. Then, they tried to match the drawings with the numerical representation. For such, they tried to present the same answer throughout the matching of various hypothetical representations with formal fractions notations (Picture D).

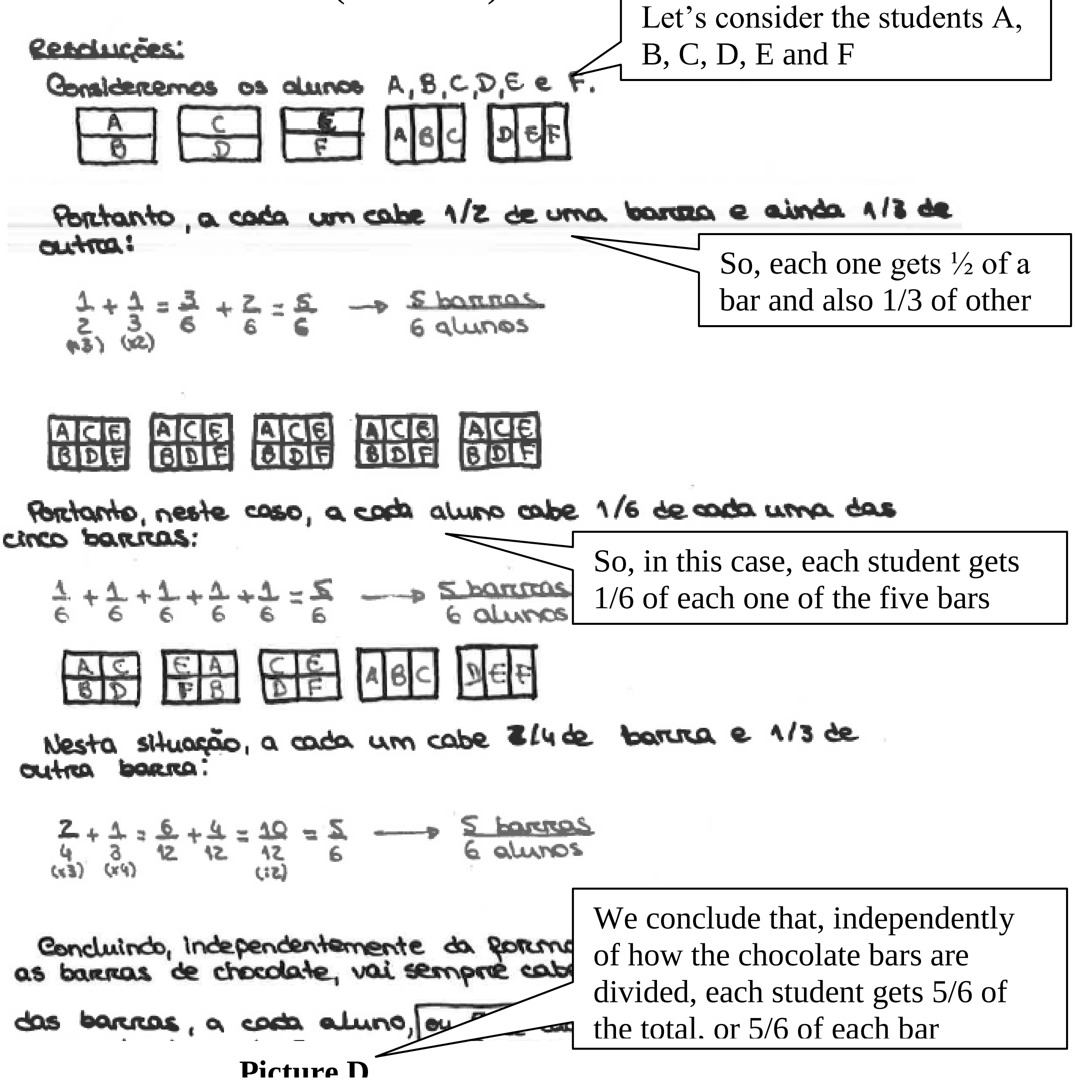


Figure 3: Examples of student work considering different ways of representing one of the correct answers

Although they consider different ways of representing one of the correct answers (5/6), visually and algebraically, they did not pay any attention to the different whole in this situation, nor the different notion the whole could take. Their answer would typically be something like: "*each student will get exactly 5/6 of the total amount of chocolate or 5/6 of each chocolate bar*".

The prospective teachers who presented different approaches to the answer (which occurred in more than half of the groups) frequently believed that the reasoning would necessarily be different whenever their way of dividing and representing the solutions algebraically was different. From another point of view, they consider it to be the same to say that the pupil will get: 5 pieces of chocolate; one bar of chocolate (when collecting 1/6 of each bar and transforming it in one other bar with 5/6); or 5/6 of each chocolate bar. Such difficulties in understanding the role of the whole impeded them from being able to interpret, afterwards, different representations and interpretations of fractions in the subsequent tasks.

SUMMARY AND IMPLICATIONS FOR MATHEMATICS TEACHERS' TRAINING

The subject matter knowledge revealed by these prospective teachers' is aligned with the knowledge revealed by early years' students (Monteiro, Pinto & Figueiredo, 2005). Their different ways of seeing a discrete whole, and giving answers involving fractions (and, necessarily, the impact of this on the interpretation of fractions) is problematic, because they show some of the same gaps in knowledge as the ones their (future) students are struggling with.

These gaps in knowledge, which may be admissible at an early stage at primary school level, would make it impossible for them (at least at the time) to develop a broader understanding on the interpretations and representations of fractions. This would limit the learning opportunities they are able to provide to their students, the nature and richness of the tasks they would propose, and these gaps in knowledge should thus become an explicit focus of training.

These results, in terms of the gaps in prospective teachers' knowledge and the way(s) they consider the whole and, consequently, the notion(s) of fractions, appear problematic to us because the large amount of research being done on fractions (focusing on the students) seems to have had no significant impact in teachers' training. This led us to problematize our own practice and the focus of the training we are offering, and we began to think differently about teachers' training, primarily in the direction of reinforcing the primary role of the SMK sub-domains for improving training. This will allow prospective teachers to approach the topics with and for understanding, and with a sense of the possibilities of conceptualizing all the possible levels of generalization. Only through such a change will it be possible to allow students to achieve a global view and understanding of the mathematical topics, and on the ways they relate and evolve along schooling and the different connections between each of them. It will also allow them to generalize with sense and effective knowledge.

Our final thoughts are introspective reflections as teachers and mathematics educators, informed with responsibility in teachers' training (at all its different stages). The fact that these prospective teachers reveal gaps in fundamental knowledge is also our fault, and we have to really reflect on this and change both the nature and focus of our training and of the tasks we use in teachers' training. Such a change would make sense if it really takes consideration of an effective approach between theory and practice, focusing on the specialized knowledge for the mathematical topics, assuming such knowledge to be something that can be effectively taught (Hill & Ball, 2004).

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COMPARISON OF COMPETENCES IN INDUCTIVE REASONING BETWEEN PRIMARY TEACHER STUDENTS AND MATHEMATICS TEACHER STUDENTS

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Inductive reasoning is part of the discovery process, whereby the observation of special cases leads one to suspect very strongly (though not know with absolute logical certainty) that some general principle is true. It is used as a strategy in teaching basic mathematical concepts, as well as in problem solving situations. In the paper the results of the study on primary teacher students' and mathematics teacher students' competences in inductive reasoning are presented. The students were posed a mathematical problem which enabled them to use inductive reasoning in order to reach the solution and make generalizations. Their results were analysed from the perspective of the problem solving depth and from the perspective of the applied strategies. We also analysed the relationship between the depth and the strategy of problem solving and established that not all strategies were equally effective at searching for problem generalizations.

INTRODUCTION

In many cases the researchers related the inductive reasoning process to the problem solving context (e. g. Christou & Papageorgiou, 2007; Küchemann & Hoyles, 2005; Stacey, 1989). These examinations pay attention to the cognitive process, as well as to the general strategies, that students use to solve the posed problems. Problem solving fosters in mathematics education various kinds of reasoning, more specifically, inductive reasoning.

In literature terminology of various kinds is used when addressing reasoning in mathematics: deductive reasoning, inductive reasoning, mathematical induction, inductive inferring, reasoning and proving. *Deductive reasoning* is unique in that it is the process of inferring conclusions from the known information (premises) based on formal logic rules, where conclusions are necessarily derived from the given information, and there is no need to validate them by experiments (Ayalon & Even, 2008). Although there are also other accepted forms of mathematical proving, a deductive proof is still considered as the preferred tool in the mathematics community for verifying mathematical statements and showing their universality (Hanna, 1990; Mariotti, 2006; Yackel & Hanna, 2003). On the other hand, *inductive reasoning* is also a very prominent manner of scientific thinking, providing for mathematically valid truths on the basis of concrete cases. Pólya (1967) indicates that inductive reasoning is a method of discovering properties from phenomena and of finding regularities in a logical way, whereby it is crucial to distinguish between inductive reasoning and mathematical induction. *Mathematical induction* (MI) is a formal method of proof based more on deductive than on inductive reasoning. Some processes of inductive reasoning are completed with MI, but this is not always the case (Canadas & Castro, 2007). Stylianides (2008, 2008a) uses the term *reasoning-and-proving* (RP) to describe the overarching activity that encompasses the following major activities that are frequently involved in the process of making sense of and in establishing mathematical knowledge: identifying patterns, making conjectures, providing non-proof arguments, and providing proofs. Given that RP is central to doing mathematics, many researchers and curriculum frameworks in different countries, especially in the United States, noted that a viable school mathematics curriculum should provide for the activities that comprise RP central to all students' mathematical experiences, across all grade levels and content areas (Ball & Bass, 2003; Schoenfeld, 1994; Yackel & Hanna, 2003).

INDUCTIVE REASONING

As our research shall be dedicated to inductive reasoning, this will be specified from the perspectives of various theories and practices. Glaser and Pellegrino (1982, p. 200) identified inductive reasoning, as follows: »All inductive reasoning tasks have the same basic form or generic property requiring that the individual induces a rule governing a set of elements.« Inductive reasoning tasks can be solved either by applying the *analytic strategy or the heuristics strategy (*Klauer & Phye, 2008*)*. The former enables one to solve every kind of an inductive reasoning problem. Its basic core would be the comparison procedure. The objects (or, in case of correlations, the pairs, triples, etc., of objects) would be checked systematically, predicate by predicate (attribute by attribute or relation by relation), in order to establish commonalities and/or diversities. However, the solution seekers generally tend to resort to the heuristics strategy, at which a participant starts with a more global task inspection and constructs a hypothesis, which can then be tested, so that the solution might be found more quickly, depending of the quality of the hypothesis. We believe that problem solving in mathematics is based on both strategies, with pupils, who learn mathematics, as well with scientists, who can reach new cognitions by applying either the analytic strategy or the heuristics one.

There are various theories as to the detailed identification of the stages of inductive reasoning. Pólya (1967) indicates four steps of the inductive reasoning process: observation of particular cases, conjecture formulation, based on previous particular cases, generalization and conjecture verification with new particular cases. Reid (2002) describes the following stages: observation of a pattern, the conjecturing (with doubt) that this pattern applies generally, the testing of the conjecture, and the generalization of the conjecture. Cañadas and

Castro (2007) consider seven stages of the inductive reasoning process: observation of particular cases, organization of particular cases, search and prediction of patterns, conjecture formulation, conjecture validation, conjecture generalization, general conjectures justification. There are some commonalities among the mentioned classifications: Reid (2002) believes the process to complete with generalization, Polya adds the stage of »conjecture verification«, as well as Cañadas and Castro (2007), who name the final stage "general conjectures justification". In their opinions general conjecture is not enough to justify the generalization. It is necessary to give reasons that explain the conjecture with the intent to convince another person that the generalization is justified. Cañadas and Castro (2007) divided the Polya's stage of conjecture formulation into two stages: search and prediction of patterns and conjecture formulation.

The above stages can be thought of as levels from particular cases to the general case beyond the inductive reasoning process. Not all these levels are necessarily present; there are a lot of factors involved in their reaching. Pólya also states that induction, analogy and generalization are very close to each other. By observing and investigating special cases we notice similarities, regularities based on analogy and finally we state that the observed, noticed regularity yields in general case too.

EMPIRICAL PART

Problem Definition and Methodology

In the empirical part of the study conducted with primary teacher students and mathematics teacher students the aim was to explore their competences in inductive reasoning. In the early school years inductive reasoning is often used as a strategy to teach the basic mathematical concepts, as well as to solve problem situations. In the very research the focus was on the use of inductive reasoning at solving a mathematical problem. We believe that in mathematics only teachers who have competences in problem solving can create and deal with the situations in the classroom which contribute to the development of those competences in children.

The empirical study was based on the descriptive, non-experimental method of pedagogical research.

Research Questions

The aim of the study was to answer the following research questions:

1. Do the students possess adequate knowledge to solve the problem by applying the inductive reasoning strategy?
2. How much do the students delve into problem solving, i.e. which step in the process of inductive reasoning do they manage to take?

3. Which strategies are used by the students at their search for problem generalizations?
4. Is there any difference in the achieved problem solving depth and in applied strategies between primary teacher students and mathematics teacher students?
5. Are all the applied strategies equally effective for making generalizations?

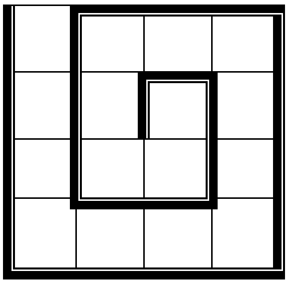
Sample Description

The study was conducted at the Faculty of Education, University of Ljubljana, Slovenia in May 2010. It encompassed 89 third-year students of the Primary Teacher Education and 72 first-year students of Mathematics Teacher Education programme.

Data Processing Procedure

The students were posed a mathematical problem which was provided for the use of inductive reasoning in order to reach a solution and make generalizations. The problem was, as follows:

On the picture below the shaping of the spiral in the square of 4x4 is presented. Explore the problem of the spiral length in squares of different dimensions.



The students were solving the problem individually, they were simultaneously noting down their deliberations and findings, they were also aided with a blank square paper sheet of, so they could delve into the problem by drawing new spirals.

The data gathered from solving the mathematical problem were statistically processed by employing descriptive statistical methods. The students' solutions were analysed from two different perspectives: from the perspective of the problem solving depth and from the perspective of the applied strategies. As some students tested various problem solving strategies, thus contributing more than one solution to the result analysis, the decision was made to use the number of the received solutions and not the number of the participating students as the basis for the analysis of the problem solving depth and of the strategies of solving. We received 95 solutions from primary teacher students and 76 solutions from mathematics teacher students. Six primary teacher students and four mathematics teacher students contributed two different approaches to the problem solving task.

Results and Interpretation

In continuation the results are shown, which are analysed as to various observation aspects.

a) *The problem solving depth*

The received solutions were classified into many levels, which were graded as to the achieved problem solving depth:

Level 1: the record contains only the pictures of the spirals,

Level 2: the record contains the drawn spirals and the corresponding calculations of the lengths of the spirals,

Level 3: the record contains structured records of the lengths of the spirals, but only for those cases, that are graphically presented,

Level 4: the record contains structured records of the lengths of the spirals and the prediction of the result for the case, which is not graphically presented,

Level 5: the record contains also the prediction for the general case.

As obvious the transformation of the problem from the geometric to the arithmetic one, and consequently operating with numbers and not only with pictures of the spirals is witnessed not until one has reached the level 2. Taking into account the stages in inductive reasoning (Polya, 1967, Reid, 2002, Canadas and Castro, 2007) we can also state that all the students at the levels from 1 to 5 reached the stage »observation of particular cases«, yet they were not equally successful in the process of searching and predicting of patterns. Mere drawings of spirals and calculations of their lengths (the levels 1 and 2) did not provide for a deeper insight into the nature of the problem and for making a generalization for the spiral of any dimension. The level 3 may be considered a transitional stage. These students already knew that mere calculations would not suffice, so they tried to structure them, i.e. they analysed the calculated numbers, and tried to define a certain pattern and a rule, respectively. However, they considered this to be enough and did not try to make a rule for the “n”-number of times-steps. In these cases students were deliberating on a possible pattern just for the cases they were observing. In comparison with them the level 4 students were already thinking about a possible pattern for a non-observing case, but they were still not thinking about applying their pattern to all cases. According to Reid (2002) the students at the level 4 reached the stage of conjecture (with doubt). They were convinced about the right of their conjecture for those specific cases, but not for other ones (see also Canadas and Castro, 2007). Only those students who achieved the level 5 can be considered to have reached the stage called »generalization of the conjecture« according to Reid

(2002). In the opinions of Canadas and Castro (2007) generalization is by no means the final stage in the inductive reasoning process. The final stage - general conjectures justification – includes a formal proof that guarantees the veracity of the conjecture, namely. Similar to the research conducted by Canadas and Castro (2007), also in our research none of the students recognised the necessity to justify the results. They interpreted the results as an evident consequence of particular cases, with no need of any additional justification to be convinced of its truth.

Table 1 shows the distribution of responses regarding the achieved problem solving depth. »Other« group comprises the responses of students who were eliminated from further analysis of the problem solving procedures due to their non-understanding of the instructions.

	Primary teacher students		Mathematics teacher students	
Depth	Number of responses	Responses in percentage	Number of responses	Responses in percentage
Level 1	6	6%	6	8%
Level 2	**19**	**20%**	**3**	**4%**
Level 3	24	25%	10	13%
Level 4	11	12%	5	7%
Level 5	**32**	**34%**	**48**	**63%**
Other	3	3%	4	5%
Total	95	100%	76	100%

Table1: Distribution of the responses regarding the achieved problem solving depth.

A closer comparison of primary teacher students' and mathematics teacher students' achievements shows some differences regarding the generalisation of the problem solving situations:

- there were more primary teacher students in comparison to mathematics teacher students who did not notice any structure among the collected data, which prevented them from further exploration (level 2);
- there were more mathematics teacher students (almost two thirds) in comparison to primary teacher students (one third) who achieved the highest level of generalisation (level 5). According to the presented results it could be concluded that the mathematics teacher students have better abilities to see the relations among the numbers, and have more knowledge for solving problems with inductive reasoning.

In addition, it is interesting to compare the percentages of the students who achieved the levels 3 and 4 (37 % of primary teacher students and 20 % of mathematics teacher students): they did notice the structure of the number pattern, but they were not able to develop the general form even it was explicitly noticeable. Most likely either they did not know how to write their findings in a general form or they did not feel the need to upgrade their concrete findings with a general form. Similar conclusion was made also by Cooper and Sakane (1986) who investigated 8th-grade students' methods of generalising quadratic problems where most of the students could not explicitly recognise that particular cases should be examined for the general rule; some of them claimed that a pattern of numbers was sufficient rule in and of itself. Nevertheless, we think that the percentage of the primary teacher students who reached level 3 or 4 is quite high, and may reflect the orientation of primary teacher education focusing on dealing with concrete situations.

b) *Problem solving strategies.*

The analysis of the modes of reasoning that the students applied at their search for generalizations revealed that it was possible to perceive the posed problem from various perspectives. Various problem perception modes are addressed as various solving strategies in continuation, out of which the ones that were encountered among the students' solutions are presented in Table 2.

Strategy	Strategy description	Generalization record
1 – »squares« strategy	It is observed that the values of the lengths are obtained by squaring the lengths of the consecutive square (e.g. 15 = 16 – 1)	$(n+1)^2 - 1$
2 – »product« strategy	It is observed that the length of the spiral is equal to the product of two numbers that differ for 2 (e.g. 15 = 5x3)	$n(n+2)$
3 – »binomial« strategy	It is observed that the length of the spiral is calculated by adding the double length to the square of the square length (e.g. 15 = 3x3 + 2x3)	$n^2 + 2n$
4 – »difference« strategy	When observing the differences among the lengths of the spirals, it is obvious that the result is the	The difference between the spiral in the square with nxn dimensions and the consecutive spiral is 2n + 1 or in a recursive

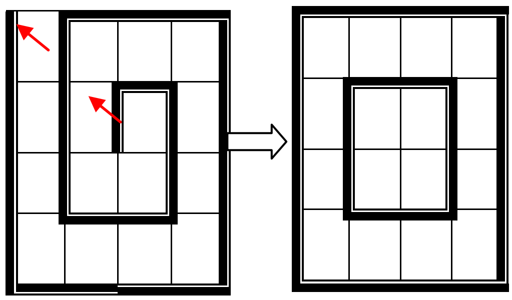
	sequence of odd numbers (e.g. from 1x1 square onwards the lengths of the spirals increase by 5, 7, 9, 11, 13, 15….	manner: $d_{nxn} = d_{(n-1)x(n-1)} + (d_{(n-1)x(n-1)} - d_{(n-2)x(n-2)} + 2)$, whereby the denotation d_{nxn} stands for the length of the spiral in the square with nxn dimensions.
5 – »sum« strategy	It is observed that the length of the spiral can be presented as the sum of individual even sections of the spiral (e.g. 15 = 1 + 1 + 2+ 2+ 3 + 3 + 3.	$3n + 2(n-1) – 2(n-2) \ldots + 2x2 + 2x1$
6 - »quadrilateral« strategy	It is observed that the length of the spiral equals four times the length of the square enlarged by the product of two numbers that differ for 2 (e.g. 15 = 4x3 + 1x3)	$4n + n(n-2)$
7 – »transformation strategy«	It is observed that in cases when the dimension of the square is an even number, spirals can be transformed in squares, the perimeters of which can be calculated.	$4n + 4(n-2) + 4(n-4) \ldots + 4x2$; $n=2k, k \in N$ 

Table 2: Description of the applied problem solving strategies

In continuation the students' selection of the strategies is presented. The strategy was evaluated only with the responses, achieving the depth of the levels 3, 4. or 5., i.e. of those students, who noted the length of the spiral in a structured record, as it was possible to define the applied strategy and the mode of reasoning, respectively, only with this record.

	Primary teacher students		Mathematics teacher students	
Strategy	Number of responses	Responses in percentage	Number of responses	Responses in percentage
1 – squares	2	2%	0	0%

2 – product	8	8%	3	4%
3 – binomial	12	14%	4	5%
4 – difference	28	29%	16	21%
5 – sum	16	17%	38	50%
6 – mixed	0	0%	1	1%
7 – transformation	1	1%	0	0%
Other	28	29%	14	19%
Total	95	100%	76	100%

Table 3: Distribution of the responses as regards the applied problem solving strategy

Let us have a closer look of the results presented in Table 3.

- Among the primary teacher students the strategy where the students focused on the difference between the lengths of the neighbouring spirals (29%) prevails whereas among the mathematics teacher students this was the sum strategy where students focused on adding the lengths of the individual even length sections of the spiral (50%).

- Among the primary teacher students the distribution of the used strategies is wider then among mathematics teacher students (or in other words: the distribution of the used strategies is more steady for primary teacher students in comparison to the mathematics teacher students). We can see that the »product« and »binomial« strategies are more often used among primary teacher students. Two primary teacher students also noticed that there was a correlation between the lengths of the spirals and the squares of the natural numbers which was not noticed among did the mathematics teacher students.

- In the »Other« column (Table 3) the responses were placed at which it was not possible to consider the selected strategy (all of the students who did not reach even the level 3).

The analysis of the problem solving strategies helps us to make conclusions about the effectiveness of a particular strategy for creating generalisation. It is very important to realise that all strategies are not equally effective for making generalisation and that the context of the problem might (not) support generalisation (Amit and Neria 2008). Therefore, further research question can be posed in analysing solving strategies, such as: were all the strategies equally effective when searching for generalizations?

The following table provides for the answer to this question, clarifying the relation between the selected strategy and the problem solving depth.

	Primary teacher students			Mathematics teacher students		
Strategy	Level 5	Total	Percentage of responses at the level 5	Level 5	Total	Percentage of responses at the level 5
1 – squares	2	2	100%	0	0	0%
2 – product	5	8	63%	3	3	100%
3 - binomial	10	12	83%	4	4	100%
4 – difference	4	28	14%	5	16	31%
5 – sum	11	16	69%	35	38	92%
6 - mixed	0	0	0%	1	1	100%
7– transformation	0	1	0%	0	0	0%

Table 4: Problem solving depths in relation to the problem solving strategy

The values in the last column for primary teacher students and in the last column for mathematics teacher students attest to the percentage of the responses pertaining to the selected strategy of those students who managed to reach the final level, i.e. the generalization.

According to the results one of the applied strategies was substantially less effective than the others for the both groups of students, i.e. the strategy 4 – ‘difference strategy’. Since it was most often used strategy among the primary education students (see table 3), a conclusion can be reached that the lower percentage of the achieved generalization among the primary teacher students compared to the mathematics teacher students was also due to the choice of the strategy. From this perspective some of the strategies (e.g. strategies 2, 3 and 5) were much more useful for creating general form than the other ones (strategy 4).

Let us have a more detailed examination of the strategy which was used by the most primary teacher students and gave the least correct generalisation – the ‘difference strategy’. The reason for choosing that strategy by a lot of students might be that searching for the difference between consecutive numbers is a very basic and well known strategy for making a generalisation. It is not difficult to obtain a generalisation if we get a constant difference between consecutive numbers at the first level of difference in a number pattern. On the other hand, the generalisation on the basis of the difference between consecutive numbers can be much more difficult if it demands the generalisation by function of higher order (not linear). In our case, the generalisation of the number pattern in the presented problem with spirals is expressed as quadratic function and this is in our opinion the main reason for a low ratio of the students who succeeded in creating generalisation on the basis of ‘difference strategy’ (see table 4).

In addition, it is also worth analysing the most used strategy among the mathematics teacher students, i. e. the ‘sum strategy’. This strategy was used by 50 % of them and proved to be very effective for forming generalisation. The closer look at those generalisations gave us 4 levels of quality difference among the achieved generalisations.

Level 1: Generalisation with an error. A student performs a generalisation in a recursive form as a sum of the even lengths of the spiral but does not determine the last article in a form (6 students)

Level 2: Generalisation in a recursive form as a sum of the even lengths of the spiral: (3n + 2(n-1) – 2(n-2) …+ 2x2 + 2x1) (24 students)

Level 3: Generalisation with the sum symbol: $3n+2\sum_{k=1}^{n-1}k$ (3 students)

Level 4: Simplifying the sum by transforming it into some of the records recognised in the strategies 1, 2or 3, i. e.: 3n + 2(n-1) – 2(n-2) …+ 2x2 + 2x1 = 3n + 2((n-1) +(n-2)+...+2 +1) = 3n + 2n(n-1)/2 = n^2 + 2n (2 students)

It is worth emphasising that all primary teacher students who used the ‘sum’ strategy and created generalisation (11 students) could be placed in level 2, i. e. generalisation in a recursive form as a sum of the even lengths of the spiral.

What can we learn from these results? According to Steele and Johanning (2004) we could learn that the different quality levels of forming generalisation are the result of different schemas of the learners. They found out that the students whose schemas were partially formed could not consistently or clearly articulate the generalizations and had more recursive unclosed forms of symbolic generalizations (e.g. n+(n-1)-(n-1)+(n-2) and not 4n-4). If we compare their results with ours it could be concluded that only a few students (5%) who have chosen the ‘sum’ strategy, achieved the level of well-connected schema.

SUMMARY

In the course of their studies at the Faculty of Education one of the important competences to be developed with primary teacher students and mathematics teacher students is to qualify them to solve mathematical problems. We are aware of the fact that this field of expertise is often neglected in our primary schools, mostly in favour of consolidating the learning contents by calculations and attending to classical word problems. We believe that students – future teachers are the ones, to whom we should start to bring about changes of this mindset, and introduce the role of the problem situations as an indispensable part of mathematics lessons in elementary schools. The presented research provided us with some important responses as to the qualification of students for problem solving by inductive reasoning. It was established that the majority of the students usually perceive the given situation as a problem, however, their abilities to delve into the problem are rather different: based on the stages of inductive reasoning according to Polya (1967), Reid (2002) and Castaneda and Castro (2007) it can be inferred that the students' responses were mainly pertaining to the following three stages: observation of particular cases, searching for pattern and prediction, as well as generalization. We find it important to establish that the stage an individual student manages to reach is largely influenced by his strategy selection. Some strategies in the process solving proved to be more effective than the other ones, from the perspective of making generalizations. Participating students approached the problem situation in a creative manner, as they applied seven strategies of different quality, and they were highly motivated to deal with such problem situations; both facts seem to be extremely encouraging from the perspective of their later role as teachers of mathematics to the youngest children.

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AN ANALYSIS OF PRE-SERVICE TEACHERS' PROBLEM SOLVING BY GENERALISATION: THE BILLIARD PROBLEM

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The paper presents an analysis of Greek primary school teachers' problem solving methods, with a focus on the type of generalisations produced. Our results show that although most working groups have formulated some partial conclusions, they did not manage to reach a higher level of generalisation by some kind of 'shift' in their attention. Moreover, their works have demonstrated their reluctance in the use of mathematical notation in the form of variables and formulas.

INTRODUCTION

Generalisation is considered one of the most important processes involved in mathematics. Whether it is viewed as part of a higher level process, like abstraction (Dreyfus, 1991b) or as the core process involved in a particular mathematics field, like algebra (Mason, 1996), there seems to be an agreement on its significant role in advanced mathematical thinking. Moreover, significant curriculum documents, like NCTM's *Principles and Standards for School Mathematics* (2000) state that:

> Students should enter the middle grades with the view that mathematics involves examining patterns and noting regularities, making conjectures about possible generalizations, and evaluating the conjectures. In grades 6-8 students should sharpen and extend their reasoning skills by deepening their evaluations of their assertions and conjectures and using inductive and deductive reasoning to formulate mathematical arguments. (p. 262)

In accordance with the above, recent research has shown that even young children may engage in forms of generalisation (Lins & Kaput, 2004). Accepting that such processes can be introduced at an early age, it is vital to consider teachers' education and how they should be prepared for initiating their students into algebraic reasoning. Firstly, one can make a distinction between secondary and primary school teachers, based on the premise that the former are expected to having been involved in advanced mathematical processes during their university studies. Indeed, Van Dooren, Verschaffel and Onghema (2003) have shown the different problem solving strategies followed by future primary and secondary school teachers in Flanders and the reluctance of some of the former to use algebraic methods. Our research stems from a similar need: in the context of Greek primary school teacher education, we aimed to analyse the student-teachers' problem solving methods, in order to examine the extent of their use of generalisation, which were expected as a solution to the problem posed. Particularly, our research questions were the following:

- How did the students interpret the task's request for a (general) relation?
- What were the basic characteristics of the solution processes followed by the students?
- What form of representations did the students use to solve the problem and to present their answer?

THEORETICAL FRAMEWORK

According to Kaput (1999) algebraic thinking consists of: (a) the use of arithmetic as a domain for expressing and formalizing generalizations; (b) generalizing numerical patterns to describe functional relationships; (c) modelling as a domain for expressing and formalizing generalizations; and (d) generalizing about mathematical systems abstracted from computations and relations. The strong bonds between generalisation and mathematics (especially algebra) are quoted by numerous other researchers. Lee (1996) states that:

> … it is possible to make a case for introducing algebra through functions, and through modeling, and through problem solving, quite as honestly as it is to make the case that *generalizing activities are the only way to initiate students into the algebraic culture*. (p. 102, our emphasis)

The various processes involved in generalisation have been identified by a number of researchers; Rivera and Becker (2008) in their literature review state that the initial stages in generalization involve: focusing on (or drawing attention to) a possible invariant property or relationship, 'grasping' a commonality or regularity and becoming aware of one's own actions in relation to the phenomenon undergoing generalization. Mason (1996) offers an interpretive overview of these phases by seeing them as forming a spiralling helix, which contains:

> - manipulation (whether of physical, mental, or symbolic objects) provides the basis for getting a sense of patterns, relationships, generalities, and so on;
> - the struggle to bring these to articulation is an on-going one, and that as articulation develops, sense-of also changes;
> - as you become articulate, your relationship with the ideas changes; you experience an actual shift in the way you see things, that is, a shift in the form and structure of your attention; what was previously abstract becomes increasingly, confidently manipulable. (pp. 81-82)

One of the basic tools that one has during generalising is visualization, i.e. a "process by which mental representations come into being" (Dreyfus, 1991b, p. 31); however, its use is not unproblematic for students who may be likely to create visual images but are unlikely to use them for analytical reasoning (Dreyfus, 1991a). Generally, from the point of view of students, coming to think algebraically is not an easy process. The ‘shift of attention’ mentioned by Mason (1996) is the activity that differentiates the professional mathematician from the novice. Thus, the transition from arithmetic to algebra is a challenging aim for teachers in the last classes of primary school and the first of secondary school; and ‘early algebra’ is now a commonly used term (Lins & Kaput, 2004), signifying the assumption that the initiation to algebraic thinking may start in primary school:

> … experiences in building and expressing mathematical generalizations – for us, the heart of algebra and algebraic thinking – should be a seamless process that begins at the start of formal schooling, not content for later grades for which elementary school children are “made ready”. (Blanton & Kaput, 2005, p. 35)

In order to clarify the teachers’ role in that initiation, and its consequent implications for teachers’ education, we could adopt a situated view of learning (Lave and Wenger, 1991), in which learning is seen as changing participation and formation of identities within relevant communities of practice. To put it simply, teachers should be initiated into the practices that they will initiate their students (Borko et al., 2005). Additionally, we are in line with Cobb (1994) who stresses that learning “should be viewed as both a process of active individual construction and a process of enculturation” (p. 13). In other words, we do not want to ignore the importance of engaging students in activities that are expected to promote the construction of meaningful knowledge. Bearing all these in mind we have designed a whole-semester teacher preparation programme, which forms the basis of the research presented in the paper.

CONTEXT OF THE STUDY AND METHODOLOGY

The teacher preparation programme in focus took place in the spring semester of 2011 at the Department of Primary Education of University of Ioannina in Greece. The participants of the course were 102 students in the third (out of four) year of their studies and both authors of the paper designed and realised the course. The course, entitled “Didactics of Mathematics I” is obligatory for all students and its intended aim is to provide the basic knowledge on contemporary theories for teaching and learning mathematics. Besides the lectures on the various approaches on mathematics education, the course included group-work activities, which aimed to improve our students’ basic mathematical competences (Niss, 2003), with a special focus on posing and solving mathematical problems and mathematical modelling (i.e. analysing and building models). Additionally, the students were initiated into a number of generalisation tasks, e.g. a variation of the ‘handshakes problem’ and the matchsticks problem (e.g. Mason, 1996).[4] The task presented here, taken from Dąbrowski (1993) aimed to further stimulate students' mathematical investigations (Ponte, 2001) and eventually lead them to a generalisation; the type of the task called for visualisation, but in a simple form. An important characteristic of that task is that it can be implemented in differently aged students, allowing them to reach different levels of generalisation by observing and grouping the data. For example, primary school students are not necessarily required to reach a general formula for all the possible dimensions of the table. Concerning the way of working on the task Dąbrowski (1993) suggests group work as the optimum way, since it allows students to simultaneously consider different cases of the table's dimensions.

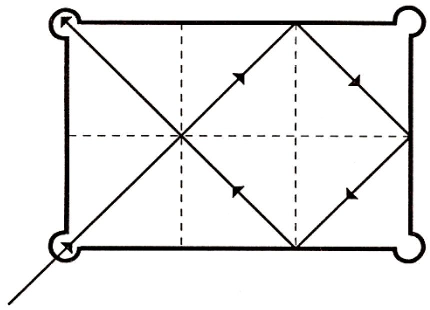

Figure 1. The billiard problem

The problem is the following: In the billiard table shown we hit the ball from the bottom left hole at an angle of 45 degrees. The ball hits the table walls three times before it ends up in the top left hole. Thus, when the table's dimensions are 3x2 the ball hits 3 times. What will happen if we hit the ball in the same way but in a table of different dimensions? Find the relationship between the table's dimensions and the number of the ball hits.

The task was presented in the class and the students were given the opportunity to ask for any clarifications. They were then asked to form groups of two to four and work on the problem for around an hour. Their working sheets were collected and they comprise our main source of data.

The analysis of our data was done according to our research aim, i.e. to examine the generalisations reached by the students in their problem solving. Particularly, our data led us to focus on four aspects of the solutions, namely visualisation (C1), considered cases (C2), conclusions (C3) and formulas (C4). From these aspects, visualisation (C1) is the first part of the process of manipulation that we mentioned before, while the remaining three were informed by Mason's (1996) view of generalisation as including variation, extension and pure generalisation. Particularly, the number and the type of considered cases (C2) were indicators

[4] It is important to note the big differences in our students' level of mathematical knowledge which was mainly due to the Greek examination system which allows for students from different school specializations to enter the education university departments.

of the variety and the extent of students' manipulations. The type of conclusions (C3) informed us on the extent of students' articulations, while the type of formulas (C4) refers to the 'shift of attention' which is related to pure generalisation. These four aspects led us to the establishment of the following categories:

Category C1: Visualisation. It refers to the type of visualisations used and contains the following subcategories:

A. Orderly visualisation: contains the works in which the drawings were in some order, e.g. 4x3, 4x4, 4x5.

B. Non-orderly visualisation: contains the works in which the drawings were not clearly related to each other, e.g. 3x2, 4x5.

C. Multi cases visualisation: contains the works in which the considered situations were drawn on a single drawing.

Category C2: Considered cases. It refers to the number and the type of the cased considered and contains the following subcategories:

A. No (explicitly related) cases.

B. One case considered, the one where the billiard has the same dimensions (it forms a square).

C. The cases in which one dimension is constant, while the second dimension is changing.

D. The cases in which both dimensions are changing, having a fixed relation between them (e.g. the one is twice the other or they are two consecutive numbers).

E. The cases in which one dimension is an even (or an odd) number and the second dimension is changing.

F. The cases in which both dimensions are even (or odd) numbers.

G. The cases in which one dimension is an even and the other is an odd number.

H. The cases including any dimensions.

Category C3: Conclusions. It refers to the type of conclusions reached and contains the following subcategories:

A. Lack of conclusions.

B. Conclusion for a single case (I. correct, II. incorrect).

C. Conclusions for some cases (I. correct, II. incorrect).

D. Conclusions for all possible cases; works in which a number of conclusions appeared which were related to all particular cases, e.g. if m, n are even then... (I. correct, II. incorrect).

E. A general conclusion with some assumptions concerning the dimensions; works in which a general relation concerning the number of hits was provided, e.g. if m, n are the dimensions of the billiard table then the number of hits is described by the formula $m+n-2$ (I. correct, II. incorrect).

Category C4: Formulas. It refers to the type of formulas reached and contains the following subcategories:

A. Lack of formulas.

B. Formulas for only some of the considered cases (I. correct, II. incorrect).

C. Different formulas for all the considered cases (I. correct, II. incorrect).

D. A general formula for all cases (I. correct, II. incorrect).

A more detailed description on the implementation of this analytical frame is given in the next section, where we present the results of this analysis.

RESULTS

Table 1 provides an overview of the way that our data were assigned to the categories described in the previous section. Additionally to the data shown below we examined the use of variables for the table's dimensions. The columns marked in grey represent the sub-categories which were not finally related to any data.

	C1			C2								C3									C4						
	A	B	C	A	B	C	D	E	F	G	H	A	B		C		D		E		A	B		C		D	
													I	II	I	II	I	II	I	II		I	II	I	II	I	II
1	✓		✓		✓	✓							✓								✓						
2	✓		✓			✓									✓						✓						
3	✓				✓		✓								✓						✓						
4			✓					✓								✓							✓				
5			✓			✓								✓							✓						
6			✓		✓	✓	✓								✓						✓						
7	✓			✓	✓	✓									✓						✓						
8	✓		✓		✓	✓	✓								✓						✓						
9			✓		✓	✓	✓								✓	✓					✓						
10	-	-	-			✓						✓									✓						
11	-	-	-	✓												✓							✓				
12	-	-	-			✓									✓						✓						
13			✓			✓	✓								✓						✓						
14		✓		✓										✓							✓						
15		✓			✓										✓						✓						
16			✓		✓	✓							✓														

17			✓	✓								✓									✓						
18			✓	✓								✓															
19	✓		✓			✓	✓								✓						✓						
20	✓			✓										✓							✓						
21	-	-	-	✓												✓					✓						
22	✓				✓	✓	✓								✓							✓					
23	✓						✓						✓								✓						
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30	-	-	-			✓									✓						✓						
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33			✓				✓						✓	✓													
34	✓		✓			✓			✓	✓								✓							✓		

Table 1. The initial data categorisation

Table 1 can be read in two ways, horizontally and vertically. By looking it horizontally one can follow a particular group's work, i.e. observe the processes of manipulation and articulation (C1, C2 and C3) and whether the group has reached the level of pure generalisation (C4). For example, Group #8 has provided orderly drawings of the billiard table and some of them were done in the same drawing. That group has considered the following cases: a) the billiard has the same dimensions (it forms a square), b) one dimension is constant (equal to 2 and then to 3), while the second is changing (1, 2, 3, 4, ...), c) both dimensions are changing, having a fixed relation between them (the one is twice the other and then three times the other). For these cases – which do not account for all possible cases – the group has provided some correct conclusions, without the use of any formula. By looking at Table 1 vertically one can observe the strategies chosen by the students (C1 and C2) as well as the number of groups that formulated (correct) conclusions (C3) and formulas (C4). For example, by looking at the columns related to C4 we can notice that 26 groups did not formulate any formula (C4-A), while only one group formulated a formula for all cases, which was incorrect (C4-DII).

Sample analyses

An interesting result on the use of visualisations was that five groups did not make a single drawing (or they did not put it on their working sheet). In two of these cases it was apparent that the students did not make any drawing and this resulted in their solution process. A characteristic example is the work shown in Image 1, where the students suggest the use of proportions in order to calculate the number of hits of the ball. Particularly, we can see that the students are initially calculating the area of the table in order to calculate the number of hits ("6 $m^2 \rightarrow 3$ hits"). They then work on two cases (4x2 and 5x2) and after calculating the relevant number of hits by the use of proportions they write their first conclusion: "when the length is changing, the number of hits is the same with the length". Then they examine the case of 3x3, which leads them to 4,5 (!) hits. The peculiarity of the non-natural number of hits does not prevent them from formulating their second conclusion that "the width is changing and defines the number of hits according to if it is even or odd number".

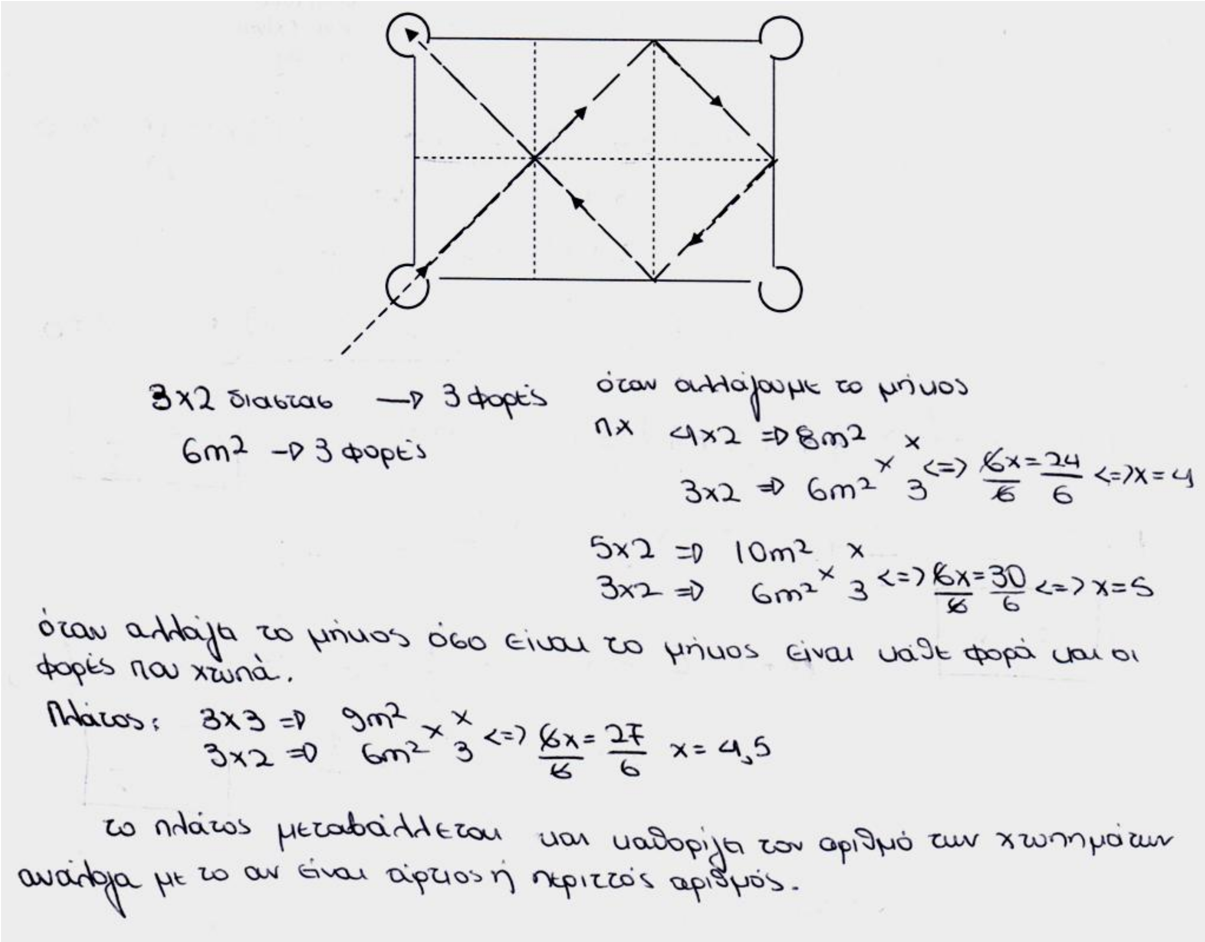

Image 1. Work #11

The majority of student groups (19) have considered the cases in which one dimension is constant and the other is changing (Category C2-C). With the exception of one group, this led them to a conclusion at least for one case (Category C3-B). A characteristic example of a work belonging only to the C2-C category is shown in Image 2:

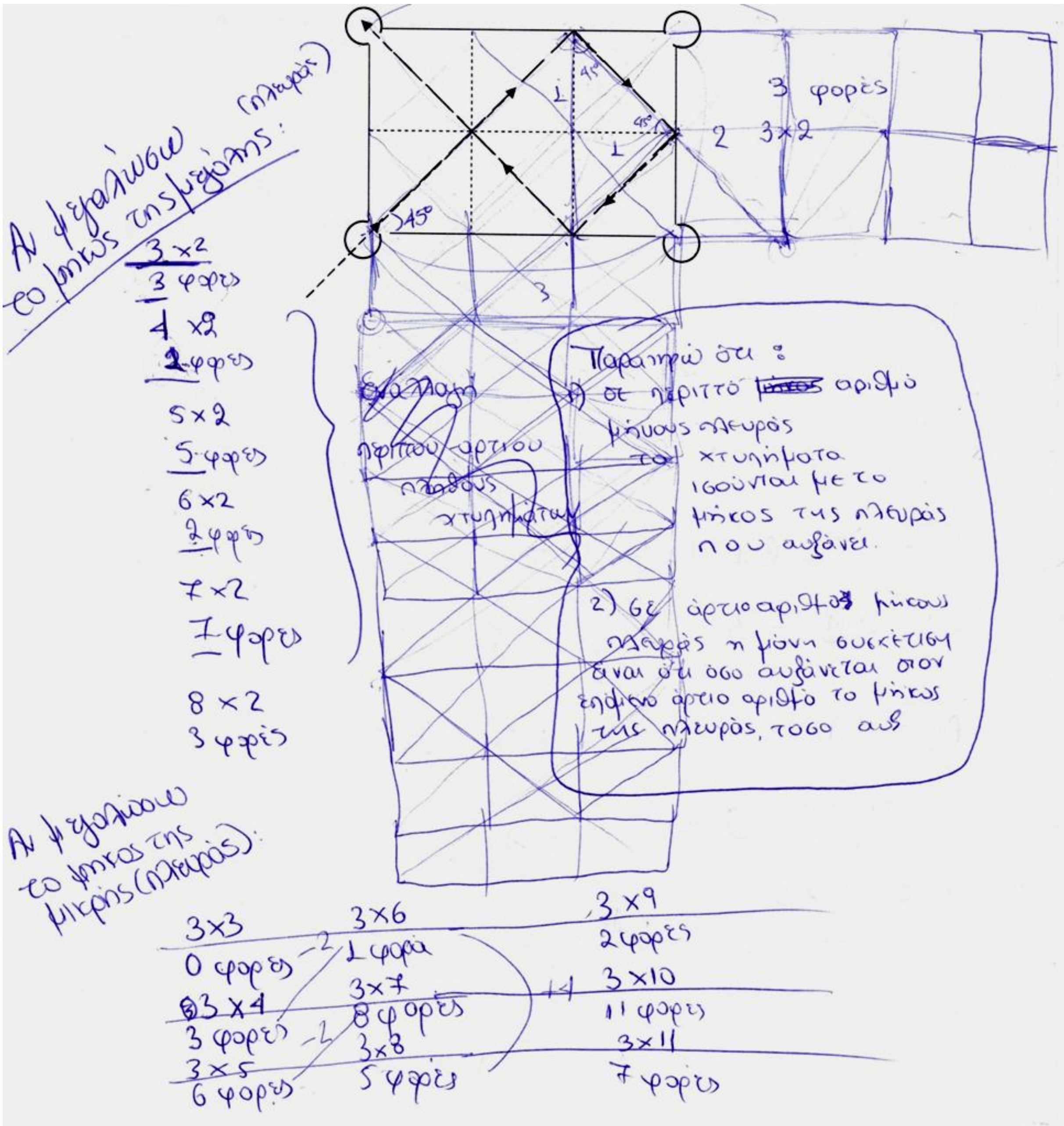

Image 2. Work #27

In the work above we see that the students have made their drawings in a single figure (C1-C) and they extended the table by one dimension each time. In their work we read: “If I increase the length of the large (side)” (and they consider the cases 3x2, 4x2, etc.) and then “If I increase the length of the small (side)” (and they consider the cases 3x3, 3x4, etc.). Their conclusions are written in the frame:

I observe that:

i) in an odd number for the length of the side the hits are equal to the length of the increasing side

ii) in an even number for the length of the side the only correlation is that the more the length of the side is being increased to the next even number, the more is incr... [*the sentence is unfinished*]

In the same category (C2) we can see that 10 groups have considered cases which were not explicitly related to each other (C2-A). Half of them did not manage to reach any conclusion; from the remaining five groups, four have reached incorrect conclusions (Works #11, 14, 20, 21). For example, in Work #14 the students have considered the cases 5x2, 4x2, 4x3 and 9x6 and their conclusion was that: "The more we increase the length of the billiard, the less the number of hits become".

Another quite common solution found in 11 groups consisted of cases in which both dimensions were changing, but having a fixed relation between them (Category C2-D). The most frequent relation considered was that the dimensions are two consecutive numbers, e.g. 2x3, 3x4, 4x5, etc. In most cases that consideration led the students to correct conclusions, like the ones provided in Work #3:

When one dimension is double from the other, it makes one (1) hit.

When the dimensions are the same, it makes zero (0) hits.

When the dimensions are even and their difference is 2 we have as many hits as the smaller dimension minus 1. For example, 4x2→1, 4x6→3, 6x8→5, 8x10→7

When they are odd and their difference is 2 we have as many hits as the smaller dimension times 2 (x2). For example, 5x7→10 (2x5=10), 3x5→6 (2x3=6)

When their difference is 1, i.e. when one is odd and the other even, the hits will be the double of the smaller dimension minus 1. For example, 3x2→3 ((2·2)-1=4-1=3), 4x3→5 ((3·2)-1=6-1=5), 7x8→13 (7·2)-1=14-1=13)

It is noteworthy that only two groups provided more general cases for the table's dimensions; particularly, one group (#34) considered all the possible cases for odd and even dimensions (Categories C2-F, C2-G) and another group (#29) considered the case for all possible numbers (Category C2-H). The work of the former of these groups is partially shown in Image 3. We have to note that apart from the worksheet with the printed version of the task, the particular group provided two more pages including more drawings and the relevant calculations which led them to their conclusions:

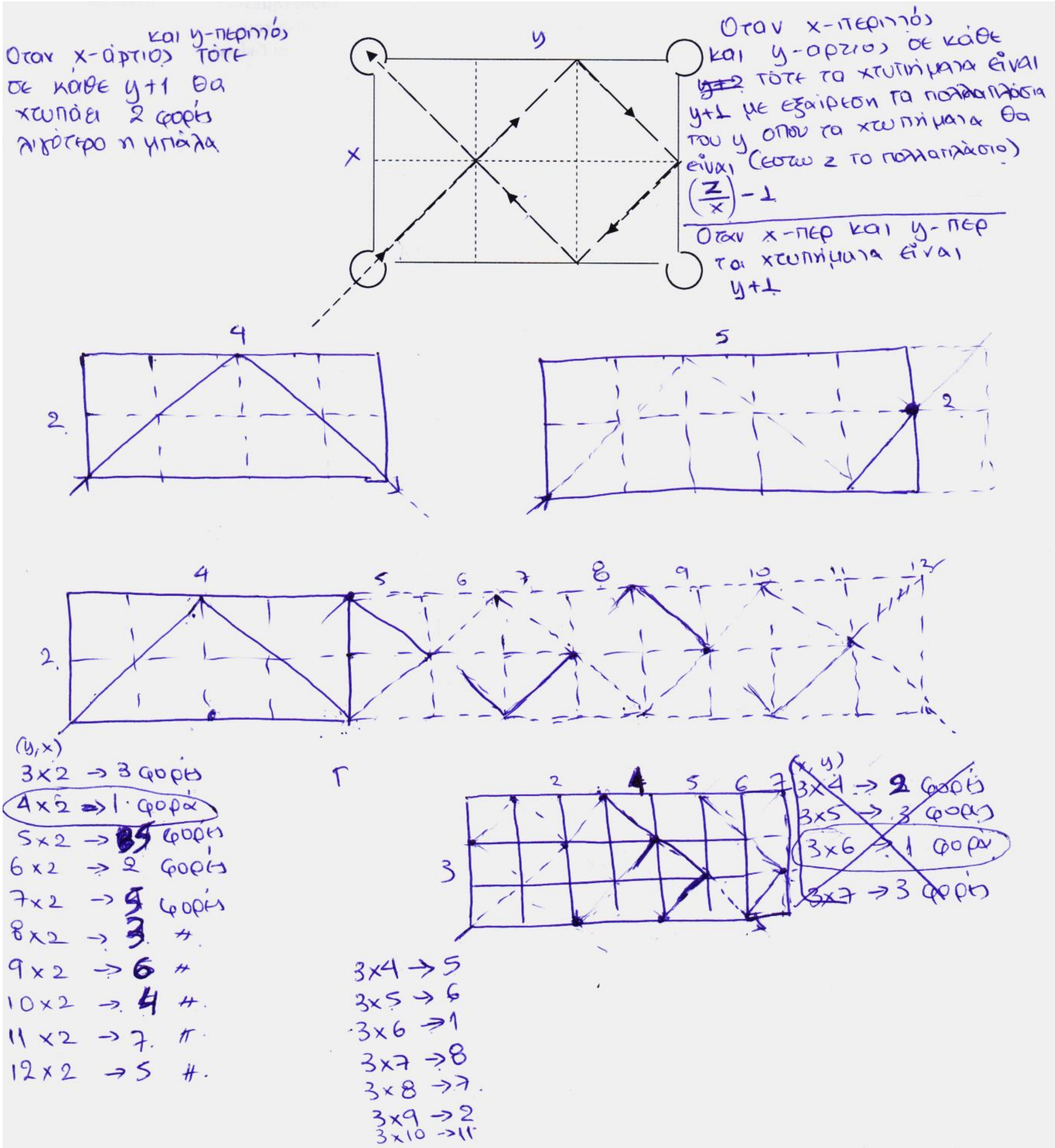

Image 3. Work #34

In the worksheet shown in Image 3 the conclusions are written in the top left and right sides and they quote that:

> When *x*-even and *y*-odd then in each *y*+1 the ball will hit 2 times less
>
> When *x*-odd and *y*-even in each ~~*y*+2~~ then the hits are *y*+1 with the exception of the multiples of *y* where the hits will be (assuming that z is the multiple) −
>
> When *x*-odd and *y*-even the hits are *y*+1

Although, as we already mentioned, the particular group has performed several calculations based on their drawings, their conclusions are all wrong; this could have been avoided by a process of verification.

The same was the situation concerning the use of variables and formulas (Category C4). Particularly, only eight groups used variables in their conclusions[5], only five of them created some kind of formula and among them only one formula was correct, namely the formula (n/2)-1 provided by Group #22 for one dimension equal to 2 and the second dimension being an even number. An interesting case was Work #29 (Image 4 shows the main page in a total of three pages), which contains a 'formula' with two 'variables', for the length and the width: "For any number > width, *hits* = *length*+(*width* -1)". Here the group miscalculated the actual number of hits by one, since the 'formula' *hits* = *length*+*width*-2 is true if the greatest common divisor of the length and the width equals to one.

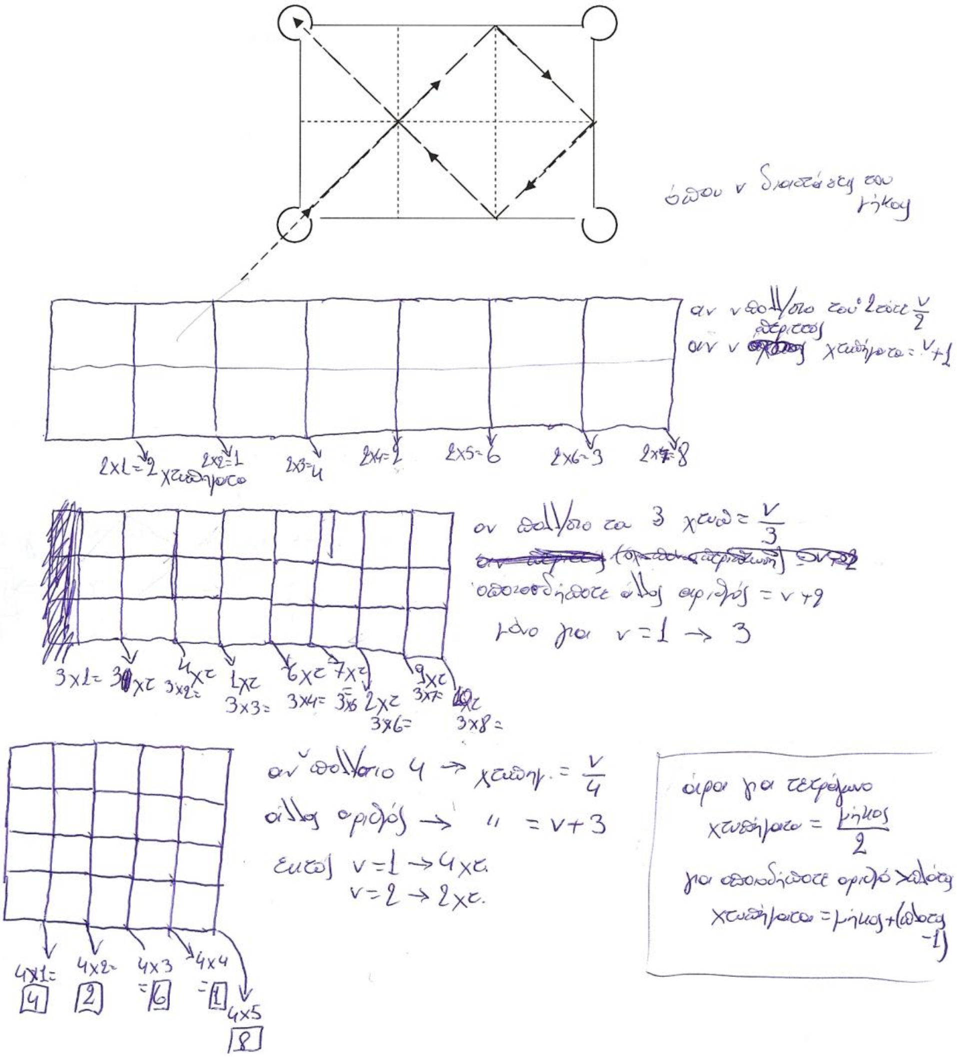

Image 4. Work #29

In the above work we read from the top to the bottom:

where ν the dimensions of the length

if *ν* multiple of 2 then *ν*/2

[5] Note that the use of variables is not presented in Table 1.

if v odd *hits* = v +1

if multiple of 3 *hits* = v/3

any other number = v +2

only for v =1 → 3

if multiple 4 → *hits* = v/4

other number → *hits* = v +3

except v =1 → 4 hits

v =2 → 2 hits

Inside the frame we read:

so for square *hits* = *length*/2

For any number > width

hits = *length*+(*width*-1)

Finally, we have to note that no group reached the formula which represents the number of the ball hits for *any* table dimensions, which is: $[(n+m)/GCD(n,m)] - 2$, where n, m stand for the table's dimensions and $GCD(n,m)$ is the Greatest Common Divisor of n, m.

CONCLUSIONS

The study presented and the chosen task aimed to examine our pre-service teachers' ability to generalise. Concerning our research questions, initially we may say that the students were much engaged in the task and all of them provided an answer to the question posed. During the process we had to explain to some of the students that they have to consider the number of hits for *any* dimensions. Finally, most groups provided the answer for the case of a square table; additionally, many groups provided a solution for one fixed and the second dimension of the table changing.

However, it seems that the expected *shift in the form and structure of attention* did not take place in most cases. This could be attributed to the students' interpretation of the task; in other words, for some groups their 'partial' solutions were adequate, since they handed their worksheets quite early. In Mason's (1996) words: "Generality is not a single notion, but rather is relative to an individual's domain of confidence and facility. What is symbolic or abstract to one may be concrete to another" (p. 74). Thus, our students were not fully able to *stress* the important aspects and *ignore* the unimportant aspects of their data; this in turn may be attributed to the nature of the task's data: in a first look the students were faced with a sequence of increasing and decreasing number of hits, not following an 'obvious' pattern.

Concerning the solution processes followed we may note that most of them were based on the following scheme: visualisation (drawings of related cases), observation and articulation of regularities, and, finally, articulation of a conclusion. It is noteworthy that we have not seen in any paper a table for gathering the data, thus making it easier to study. What was also missing – or not provided in the worksheets – was any process of verification. The students seemed rather 'easily' (i.e. after examining very few cases) convinced on the validity of their statements and this eventually led some of them to wrong conclusions.

Apart from the drawings made as part of the initial phase of the solution, the students showed a clear preference on written descriptions of their considered cases, usually accompanied by mathematical expressions. In their conclusions, as we already noted, there was a clear lack of mathematical notation in the form of variables and formulas. We consider this an indication of our students' mathematical background, which hindered them from the articulation of a 'pure' mathematical expression.

All the above call for a need for a more focused approach to generalisation in teachers' education, preferably in the form of tasks that require not only a variety of manipulations but also some decision making by the students on the handling and interpreting data.

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GENERALIZATIONS GEOMETRY IN ART ENVIRONMENT

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In this paper we describe activities leading to different types of generalizations of properties of polygons and symmetric patterns. Data shows that explorations and generalizations improve students' learning and knowledge retention as well as their overall attitudes towards mathematics.

INTRODUCTION

Many recent studies demonstrate the power of explorations that integrate art and elementary mathematics. Loeb's visual mathematics curriculum (Loeb, 1993), the "Escher World" project (Shaffer, 1997), and our earlier work (Grzegorczyk, Stylianou, 2005) show that mathematics learning was very effective in the context of arts-based lessons, and led to generalizations and abstraction on various levels. The National Council of Teachers of Mathematics (2000) also supports the introduction of extended projects, group work, and discussions to integrate mathematics across the curriculum.

In this study we presented three instructor-initiated explorations and discussion-based group activities leading to generalizations. Instead of starting with theoretical concepts, we introduced simple geometric examples to serve as a starting point to more complex, mathematical relationships. Students worked both individually and as a group and used art drawing and image-manipulation programs. The results of the study supported the main goal of this research, which was to show participants' understanding of the generalized concepts and their strong knowledge retention. Additionally, we have observed increased positive attitudes towards mathematics.

METHODOLOGY

This study was conducted during the *Mathematics and Fine Arts* course in the arts studio environment with 21 students. The mathematical content of the course included the generation and analysis of artistic patterns, and the properties of polygons. During this study, students participated in three one-hour activities conducted during three 2-hour class sessions (the remaining time was used for testing, surveys and other issues not related to this study). Most of the participants had high-school level knowledge of mathematics, hence familiarity with algebraic formulas and geometric figures. Since the coursework involved the creation of artistic designs and patterns, the majority of participants were interested in fine arts. Table 1 below summarizes the initial characteristics of the participants. Note that participants majoring in Liberal Studies were prospective elementary school teachers, while mathematics majors were prospective secondary school teachers.

Interested in	Number of Major	Liked Art	Liked Mathematics
Art	7	7	1
Liberal Studies	7	5	2
Mathematics	7	5	7
TOTAL	21	17	10

Table 1: Initial description of participants.

This study was done as an introduction to tilings, tessellations and crystallographic groups. The following three-questions, 15-minute pre-test was given at the beginning of the first session to assess students' initial knowledge.

P 1: What is the sum of the angles in a decagon?

P 2: How many diagonals does a decagon have? Justify your answer.

P 3: Draw a design that has a rotation by 120 degrees and at least one reflection.

Table 2 below summarizes the results of the pre-test for the group.

Question	Correct answer	Correct Picture	Justification
P 1 (21)	4	9	4
P 2 (21)	3	14	0
P 3 (21)	4	4	1

Table 2: Pre-test results.

Note that only math majors gave correct answers on the pre-test. All sketches drawn to answer question P1 used a regular decagon subdivided into identical (central) triangles, which were later used for angle calculations. In P2 all students simply counted the diagonals.

The first two activities are based on the properties of polygons included in a typical high-school geometry course. However, they are presented as shortcut formulas for calculations, rather than used for building generalizing skills in students (Grzegorczyk, 2000). The first activity stresses recursive thinking and the second generalizes a counting algorithm. The third activity requires transfer of geometric properties (symmetries in this case) to non-mathematical objects (artistic images).

Discussion-Explorations-Generalization structure was used in all three activities. Instructor kept students focused on the following tasks.

1. Introduction of an initial problem.
2. Group discussion of the problem and possible generalizations.
3. Individual explorations of slightly generalized cases.
4. Group discussion of exploration results, methodologies and cases.
5. Verbalization of further generalization of the initial problem.
6. Individual explorations of generalized problem.
7. Group discussion of the proposed solutions and testing.
8. Verbalizing of the final generalized statement and further testing.
9. Justifications (proofs) of the statement (theorem).

First Activity – The sum of the interior angles of a polygon.

1. Initial problem: *What is of the sum of angles in a triangle*? All students knew this sum is 180 degrees, but they could not justify. Instructor introduced the geometric proof represented graphically in Figure 2.

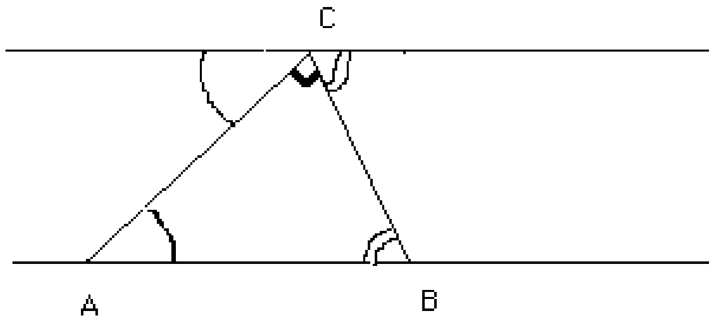

Figure 3. Angels in a triangle add up to a straight line.

2. Students discussed the proof and decided the argument would work for all triangles. Then the question was raised about quadrilaterals. Students agreed that the sum of the angles should be 360 degree based on their knowledge of rectangles. But they were not sure about other quadrilaterals.

3. First generalized problem: *What is of the sum of angles in a quadrilateral*? Students explored special cases: rhombus, parallelogram, and trapezoid. They generalized the triangle construction to quadrilaterals with two parallel sides.

4. Further discussion led to the idea of ‘cutting’ a quadrilateral into triangles and adding the angles, as shown in Figure 4. The case of non-convex quadrilaterals was raised and resolved. Hence, the solution to the first generalized question was established.

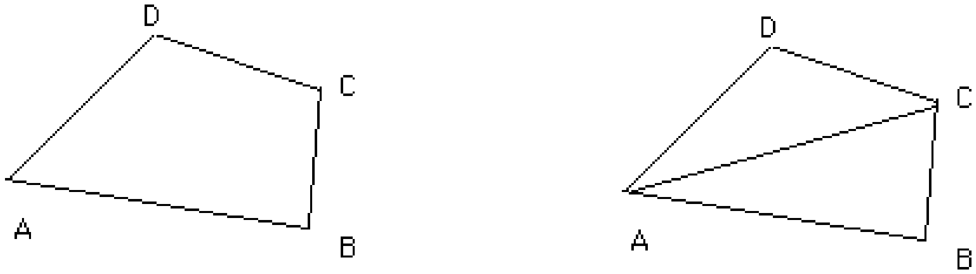

Figure 4. A quadrilateral can be divided into two triangles.

5. Further generalization: *What is of the sum of angles in a pentagon?*

6. Students tried to apply the idea of suitable cutting of polygons into triangles. Discussion led to systematic division of each pentagon into three triangles meeting at one vertex (that worked well for convex cases), see Figure 5. Suitable diagonal cuts always gave three triangles regardless of the shape of the pentagon. Students established the answer as 540 degrees.

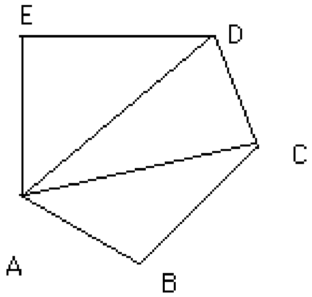

Figure 5. Systematic subdivision of a pentagon into triangles.

7. Discussion led to further generalizations: *What is of the sum of angles in a hexagon, heptagon, and octagon?* Students figured out the answers to be 720, 900, 1080 degrees respectively. Instructor summarized their results as follows.

Polygon	Number of sides	Sum of angles
Triangle	3	180
Quadrilateral	4	360
Pentagon	5	540
Hexagon	6	720
Heptagon	7	900
Octagon	8	1080

Table 6. Angle sums for polygons.

8. Students noticed that the sum increases by 180, as there is one more triangle in the next step. They compared the number of sides to the number of triangles in each polygon (in each case getting 2 less). Their generalized statement: *The sum of the angles in an n-gon is* 180(*n-2*). They tested on various cases (the sum of angles in 102-gon is 18,000!).

9. All students thought that the formula does not require proof (because 'construction shows it is true'). Instructor used *Mathematical Induction* to prove the statement. While all the students actively participated in the first 8 steps of this activity, only 9 (including all math majors) were interested in the proof.

Note that this activity requires *recursive thinking*. Students have discovered a universal truth (a theorem) about all polygons (even the ones that they did not consider in their explorations).

Second Activity – The number of diagonals in a polygon.

For simplicity students concentrated only on convex polygons in this activity.

1. Initial problem: *What is the number of diagonals in a (convex) hexagon*? All students could draw a hexagon and calculate diagonals as in Figure 7. Most of them colored them while counting.

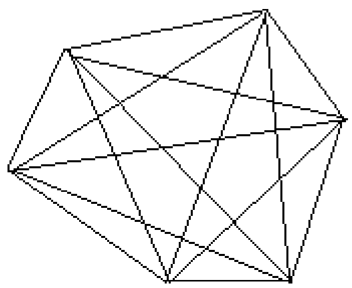

Figure 7. Diagonals in a hexagon.

2. First generalization: *Calculate diagonals in pentagons, quadrilaterals and triangles.*

3. Discussion led to general questions of heptagons and octagons. All students calculated 14 and 20 diagonals respectively.

4. Discussion of various counting methods led to a generalized question: *Is there a connection between the number of sides and the number of diagonals?*

5. The group decided that since diagonals connect vertices, the number of sides is important. They collected their results in Table 8 below. *Is there a connection between the number of vertices and the number of diagonals?*

Polygon	Number of sides	Number of vertices	Diagonals?
Triangle	3	3	0
Quadrilateral	4	4	2
Pentagon	5	5	5
Hexagon	6	6	9
Heptagon	7	7	14
Octagon	8	8	20

Table 8. Vertices and diagonals in an *n*-gon.

7. Students looked for patterns in the table, a systematic way to express the relationship between the numbers of sides, vertices and diagonals. They noticed that sides already connect some vertices; hence only *n-3* diagonals start at each vertex. They conjectured that there are *n*(*n-3*) diagonals. Testing showed that they were overestimating. They noticed double counting, as each diagonal was counted for two vertices.

8. General statement was formulated as: *n-gon has n*(*n-3*)*/2 diagonals.* Students tested it on previous results. They calculated the number of diagonals in a nonagon, decagon, and some other polygons.

9. Students justified their formula as follows: Each *n*-gon has *n* corners and there are (*n-3*) diagonals starting at each corner. Since each diagonal starts at two corners, it gets counted twice while we count corner by corner. Therefore there are *n*(*n-3*)*/2* diagonals.

Note that in this activity students had to *generalize their counting method*. The final statement was based on the invented systematic counting procedure.

Activity 3 – Classifying small artistic designs using symmetries.

This activity was conducted after students were familiar with reflections and rotations. They used software that generated images with various symmetries.

1. Initial Problem: *Describe symmetries (reflections and rotations) of a square.* Students sketched the picture representing the symmetries (see Figure 9).

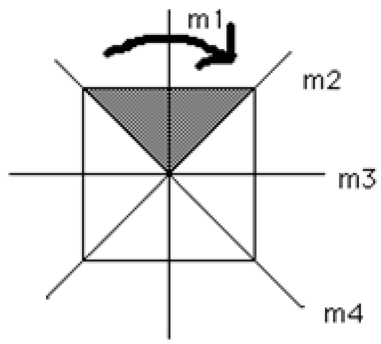

Figure 9. Symmetries of a square {m1, m2, m3, m4, r90, r180, r270, id}.

2. Slightly generalized problem: *Describe all symmetries an equilateral triangle, a regular hexagon and a regular octagon.*

3. Students worked on lists of symmetries for each figure and searched for patterns. They decided on two generalizations below.

4. Generalization 1: *n*-sided regular polygon has exactly *n* different reflections. Student tested a regular octagon and other polygons to confirm their claim.

5. Generalization 2: *n*-sided regular polygon has rotations generated by 360/*n* degrees and confirmed that by checking on sketched figures.

6. Student worked individually with a pattern generating software to analyze symmetries of images. Figure 10 below shows an example of an image that was analyzed.

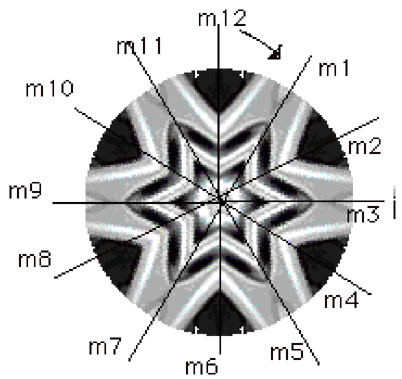

Figure 10. A pattern with 12 reflections and a rotation by 30 degrees.

7. Discussion led to grouping of the images with polygons that had similar symmetries.

8. Students decided that images could be classified by their symmetries. Generalization statement: *There are infinitely many types of (small) images depending on number of reflections D0, D1, D2, …Dn, and the polygons represent each type depending on number of reflections.* (Note these symmetries form dihedral groups in abstract algebra).

9. Instructor and the students tried to justify the classification statement.

In this activity students had to *generalize the classification system to include non-mathematical objects*. They applied the language of mathematics to describe properties of artistic designs.

DATA COLLECTION AND ANALYSIS OF RESULTS

A week after each activity (at the beginning of the next session), students were given two post-test questions. Their answers were evaluated for correctness of the response, properness of the images used, and the quality of their justification. The following tables display the questions as well as the number of credits given to each group of students for correct answers. Note that questions

1, 2, 3 correspond to pre-test questions P1, P2 and P3. Paired T-test for pre- and post-test questions 1, 2 and 3 showed (statistically) significant improvement of students' knowledge. Questions 1a, 2a, 3a were modified questions 1, 2, and 3. T-test comparison with corresponding pre-test questions shows significant improvement as well. Hence the activities used were an effective learning method.

Question 1: What is the sum of the angles in a regular twelve-sided polygon?

Major	Correct answer	Correct Picture	Justification
Art (7)	6	7	6
Lib. Studies (7)	7	6	5
Mathematics(7)	7	7	7
TOTAL (21)	20	20	18

All students used the formula from the first activity. One student miscalculated, one did not have a picture. Justifications explained derivation of the formula. Overall the group was very successful answering this question, ca 95% correct.

Question 2: Two identical regular pentagons were glued along one of the sides. What is the sum of the angles of this new polygon?

Major	Correct answer	Correct Picture	Justification
Art (7)	6	7	6
Lib. Studies (7)	6	7	5
Mathematics(7)	7	7	6
TOTAL (21)	19	21	18

All of the students could picture the situation, but some were confused by the fact that the octagon was not convex. They subdivided the figure into triangles, and most of them calculated angles of the triangles and added them rather than using the formula. Justifications explained this addition process.

Question 3: How many diagonals does a decagon have? Justify your answer.

Major	Correct answer	Correct Picture	Justification
Art (7)	5	6	3
Lib. Studies (7)	6	5	4
Mathematics(7)	7	6	7
TOTAL (21)	19	17	14

Four students did not provide pictures at all. All the math majors used the formula from Activity 2. Two Art and two Lib. Studies students counted the diagonals**,** but did not provide justification. All correct justifications explained the formula.

Question 4: All diagonals of a decagon were colored blue except for the all the diagonals starting at one given vertex that were colored red. How many diagonals were colored blue? Justify your answer.

Major	Correct Answer	Correct Picture	Justification
Art (7)	5	4	3
Lib. Studies (7)	5	5	4
Mathematics(7)	7	6	6
TOTAL (21)	17	15	13

Six students did not provide a correct picture and as a consequence had errors in their calculations. The majority of students used the formula to calculate all diagonals and subtracted the red ones. All justifications included the formula. Four math majors introduced an algebraic formula for calculations. Over 80% of students were correct and close to 60% could justify their answers.

Question 5: Use letter P to create a design that has 4 reflections. Explain how to create this type of design, D4.

Major	Correct Answer	Correct Picture	Justification
Art (7)	7	6	3
Lib. Studies (7)	5	5	3
Mathematics(7)	7	7	6
TOTAL (21)	19	18	12

Two students sketched wrong designs, but one more did not show the lines of reflections on the correct picture. 10 pictures included a square in the background. In justification 8 students noted that ‘lines of reflections are like in a square’ and 4 said ‘mirror lines have to intersect at 45-degrees’.

Question 6: Use letter P to create a design that has a rotation by 60 degrees and at least one reflection.

Major	Correct answer	Correct Picture	Justification
Art (7)	4	4	2
Lib. Studies (7)	3	3	2
Mathematics(7)	6	5	5
TOTAL (21)	13	12	9

This question confused many students, as some of them tried to create a design with one reflection and the 60-degree rotation, which is impossible. Justifications included statements like ‘the design has to be like hexagon’, ‘the design has to have 6 reflections’. 11 students did not justify, while one art major with a correct design just said ‘since the angles are 60 degrees – it works’.

Attitude evaluation

At the end of the study, students re-evaluated their attitudes towards mathematics. Below is a summary of their responses. Note that the positive attitude towards mathematics increased from less than 50% (see Table 1) to over 70%. Almost all students had a positive experience with the software and over 85% liked the explorations. Interestingly, one math major was not happy with the activities.

Do you like	Graphing software	Explorations	Mathematics ?
Art (7)	7	6	4
Lib. Studies (7)	7	6	4
Mathematics(7)	6	6	7
TOTAL (21)	20	18	15

CONCLUSIONS

The results of this study show that students of various interests and backgrounds can successfully be involved in mathematical explorations and generalizations. This particular group of students was able to think recursively, invent a counting method and transfer the classification criteria from simple geometric objects to an uncountable amount of designs. The numerical results show an improvement of students’ knowledge, more frequent use of formulas, and their ability to recover and verbalize the methods used to discover them. Additionally, almost the entire group liked the art-studio environment and graphics as a basis of learning. We observed an improvement of the general attitude towards mathematics among students with various interests. Students also commented that they enjoyed discussions, explorations and social ways of learning untypical in mathematics courses. Since the mathematical content of these activities is accessible to many younger students some modification or simplification of these explorations and generalizations activities may be successful in earlier grades.

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