Motivation via Natural Differentiation in Mathematics

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INTRODUCTION

As indicated in both the Lisbon strategy ("Education and training 2010 – diverse system, shared goals") and National Strategies, the importance of lifelong learning and continued teacher professionalization is increasing. For example, the National Programme for the Development of Education in the Czech Republic (White Paper 2001) stresses the "internal differentiation and profiling of schools in accordance with pupils' needs" and "teachers' autonomy", while German curricula in all states require the "inner differentiation" and "individual support" for all learners. Teachers in general should be able to develop and improve their teaching of mathematics.

It is expected that all learners should have opportunities to realize their potential, but at the same time the requirements of national curricula have to be fulfilled. Meeting the needs of *all* students in learning mathematics is one of the big challenges in today's teaching practice – especially with respect to a growing heterogeneity in classrooms (cf. PISA). Therefore, suitable kinds of differentiation are indispensable. What can be observed so far is that differentiation is limited to mainly organizational features, neglecting e.g. the prominent role of the content itself.

The present volume discusses a 'new' kind of differentiation starting in the first school years (and even in kindergarten). The various approaches are expected to contribute to a deeper understanding of what constitutes mathematics learning, by considering the learners' individual personalities and the advantage of learning in groups, as opposed to minimizing the individual differences among the students. By taking into account both the learner's individuality and the demands of the subject matter an increased and sustainable motivation can be expected, planned and eventually realised.

The innovative aspect of this approach is connected with a conceptual change in the minds of both teachers and pupils, which should lead to a change in classroom culture and a sound attitude towards the nature of mathematics.

Motivation in teaching and learning mathematics

Part 1

NATURAL DIFFERENTIATION IN MATHEMATICS (NADIMA)
THEORETICAL BACKGROUNDS & SELECTED ARITHMETICAL LEARNING ENVIRONMENTS

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1. HETEROGENEITY & DIFFERENTIATION

Differentiation in mathematics education is not a new problem teachers have to cope with. Since the early 70s of the last century, in Germany many publications can be found (cf. Bönsch, 1976; Geppert, 1981; Klafki, Stöcker, 1976; Winkeler, 1976), and those references are still cited in actual publications. Though it is an old problem, it can be stated that the extent of heterogeneity, the range between slower and faster learning students, low achievers and bright children, has been expanded (Reh 2005; Tillmann 2004). In one classroom there may be students whose range of proficiency can spread over three grades.

Besides this, *differentiation* has become a modern term in pedagogy and educational policy, if not to say a label which is expected to serve as an all-purpose weapon for optimizing the output of achievement at school. Differentiation and individualization are two of those pedagogical terms which are so universal and by this so ambiguous that it makes sense to have a closer look at opportunities and forms of differentiation, because it remains a convincing leitmotiv to create the most favourable learning conditions for each individual student (cf. Wielpütz 1998, p. 42).

1.1. Classical Ways for Coping with Heterogeneity

Analyzing the literature and the common practice of teacher education[1], it is obvious that till today nearly the same solutions as 20-30 years ago are offered for coping with heterogeneity (cf. Bönsch, 2004; Paradies, Linser, 2005; Vollstädt, 1997). Heterogeneity is mostly seen as a *problem* because of didactical traditions in teachers' heads which still may be oriented to an ideal of a homogeneous learning group (cf. Beutel, Ruberg, 2010). This habit in itself already can be questioned, what is part of our project as well, but will be addressed later. In our view – and our project experiences confirm this view – heterogeneity, under certain conditions, can be a source of productivity and a chance and great *advantage* for teaching and learning.

The suggested (classical) solutions in literature are the following:

[1] In this article we all along face and refer to the German situation and tradition for primary education.

- social differentiation (individual work, partner work, group work)
- differentiation via teaching methods (projects, courses, …)
- differentiation via media (textbooks, worksheets, manipulatives, …)
- quantitative differentiation (same amount of time for different amount of content, or different amount of time for same amount of content)
- qualitative differentiation (different aims resp. levels of difficulty)

This listing could have been taken from a current catalogue of pre-service and in-service courses. But it dates back to a publication from Winkeler (1976).

That is not to say that those mentioned recommendations *per se* are to be valuated in a negative way, nor do we say they are ineffective. But as we think, there are several indications

- that they are not always implemented in the intended way,
- that they are necessary, but not at all sufficient.

In our view, at least four main problems of classical inner differentiation can be identified:

How can the teacher decide what is an easy or difficult task?

In trying that, (s)he is faced with at least three problems:

(a) The felt level of difficulty not only varies between children, but also with the same child, on different times, and even with the same task (Selter, Spiegel 1997). ‘Difficulty’ is a quite *subjective* issue.

(b) A (subjectively) felt difficulty depends on diverse factors: the complexity of calculation (kind and size of numbers); the operations involved (addition/subtraction is often said to be easier than multiplication/division); the demand for cogitation, strategic understanding, process-related competencies; the understanding of the task (language demands); the amount of the (oral or written) text production; etc.

(c) In addition to this, the level of a task’s difficulty cannot be measured just by the formal-syntactic steps the solution requires. All these aspects clearly relativise the sometimes stated claim that a learning offer would individualize the process of learning according to levels of difficulty.

Individualization & social learning

Often, one can get the impression that the postulate of individualization neglects the postulate of social learning or tends to lead to a rather reduced understanding of the term, e. g. in the sense of rules and rituals which are arranged within and for lessons. In cases like that, individualization may be understood as totally independent from the subject matter content, and it can be aspired when a child deals with its own, individually diverse tasks or even contents, and when all children potentially work on something different. That is why some kind of 'open teaching' directly leads to the elimination of social learning. It is connoted that social learning is not a question of content.

But how can common argumentation and communication about experiences with shared contents emerge or even become plausible if they do not exist? Learning from and with each other by dealing with a shared content (communication of minds) inevitably stays apart. Some teachers even take pride in abolishing common plenum phases – for the benefit of, as they say, a 'consequent individualization'. And so one can observe multiple situations where the individuality of the learner becomes absolute and 'action' takes room where 'activity' should be postulated (see also 4.3).

Risk of arbitrariness and wasted substance

'Open teaching' and 'free work' sometimes is meant in the sense that students themselves should choose the contents they like to deal with. In our opinion, this fortifies the danger of arbitrariness. The choice of contents, their didactical design and having in mind far-reaching objective targets requires specific professional competencies and cannot just handed over to the students. Even a very autonomously learning and high-performing child is in need of a sound support when (s)he arrives at the zone of the proximal development (Wygotsky). The teacher, on the one hand, is responsible for leading the child to its individual limits. On the other hand, he must offer sound impulses for the child in order to push those limits more and more forwards. Invading into mathematical structures of the learning contents does not happen automatically or because a child feels like that.

We do not deny the requirement to gradually qualify children for autonomy and self-reliance regarding their own learning process. But a sound reflection is needed about where and when and with which prerequisites these degrees of freedom are meaningful, important, and rational for the child.

What about mathematics?

The theoretical and conceptual discussion concerning heterogeneity and differentiation is mostly guided by organizational and methodical questions. Secondly, if one looks at the publications, the discussion about conceptual forms of differentiation is dominated by pedagogy. This neglects the essential importance of the subject matter, here mathematics, and its specifics.

Although, by all means, there *are* several proposals for learning environments in mathematics education where desirable forms of differentiation can take effect – because, in a sense, it is implemented in the topic *itself* (Hengartner et al. 2006; Hirt, Wälti, 2009). But what is missing, is a more comprehensive contribution that deals with a concept of natural differentiation from a theoretical point of view and with the perspective of mathematics education. It is merely mentioned on a half page in a teachers' manual of the German textbook 'Zahlenbuch' (Wittmann, Müller, 2004, p. 15).

1.2. Natural Differentiation

The following attributes (cf. Wittmann, 2001a; Krauthausen, Scherer, 2007, p. 228 f.) are constituent for natural differentiation (in the following: ND):

- All students get the same offer. So there is no need for a vast number of additional worksheets or materials.
- This offer must be holistic (referring to the content, and, here, not meant in the sense of 'head, heart & hand'), and it may not fall below a specific amount of complexity and mathematical substance (complexity is not necessarily the same as complicated). This kind of challenging and complex learning environments (in contrast to common isolated tasks) are absolutely not just an advantage for better learning students (cf. Scherer, 1999).
- Holistic contexts in that sense *by nature* contain various levels of demands which must not be determined in advance. The level of that spectrum which is actually worked on is no longer assigned by the teacher, but by the student him- or herself. This not only subserves the support of increasingly realistic self-assessment, but the student him- or herself can rate his or her specific abilities better than the teacher who is not able to look inside the student's head. Though, students have to learn that kind of self-assessment, and that does not happen by simply tossing them again and again into such situations. Rather they need situations, consciously planned and organized by the teacher, where they can talk and reflect about demands and their criteria on a meta-cognitive level (cf. also Treffers, 1991, p. 25).
- In addition to the level the students decide to work on, they can freely make their own decisions concerning: the ways of solution, use of manipulatives and facilities, kinds of notation, and even the problems they decide to solve (problem solving also includes problem posing).
- The postulate of social learning from and with each other is fulfilled in a natural way as well, because it makes sense by the content itself: It is obvious to exchange various approaches, adaptations and solutions. In doing so, insight and understanding can grow up or be deepened

  All students will be confronted with alternative ways of thinking, different techniques, variable conceptions, independent from their individual cognitive level. Rigid inner differentiation more likely will just complicate this opportunity. […] So, the various, individually organized ways of solution also have an impact on affective, emotional areas. They leave a cognitive margin to students what can facilitate their identification with the learning demands. In this way, the direct experience of autonomy can lead to motivation and interest« (Neubrand, Neubrand, 1999, p. 155, translated GKr/PS; also cf. Freudenthal, 1974, p. 66 ff.).

These attributes have to be described in more detail (e. g. what means ‘the same offer’) and have to be put in concrete terms for teaching and learning processes. This is the aim of the project that will be described in the following.

2. THE PROJECT *NADIMA* (NATURAL DIFFERENTIATION IN MATHEMATICS)

2.1. Project Group

The European project NaDiMa ‘Motivation via Natural Motivation in Mathematics’ is supported by the Comenius-program ‘LifeLongLearning’ (LLL; duration: 01.10.08 – 30.09.10). The official scientific project partners are from Poland (Ewa Swoboda), Czech Republic (Alena Hospesova, Filip Roubicek, Marie Ticha), the Netherlands (Maarten Dolk) and Germany (Günter Krauthausen, Petra Scherer). Moreover, this international cooperational project includes partner schools and associated schools (for more details see www.nadima.eu).

2.2. Aims and Objectives

The project NaDiMa stresses the necessity of a differentiation which amplifies the nature of the subject matter. That is not the same as to make the formalism of the mathematics science absolute. Differentiation which lies within the subject, can be given complete expression under the premise that the topic is not step by step anatomized and supplied to the students bit by bit, but as a holistic, sufficiently complex learning environment (see 1.3).

We explicitly take the *teacher* responsible for the sound identification, selection and framing of the problems to work on in classroom – that means: the first two of the constituent attributes of ND mentioned above. Very helpful for that can be the four characteristics which Wittmann (2001a) has established as a definition of a so called ‘substantial learning environment’ (in the following: SLE; also cf. Krauthausen, Scherer, 2007, p. 197 ff.):

(1) It represents central objectives, contents and principles of teaching mathematics at a certain level.

(2) It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities.

(3) It is flexible and can be adapted to the special conditions of a classroom.

(4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research (Wittmann 2001a, p. 2).

Criteria like these, as well as others which are postulated for ‘good problems’, can be stated as necessary prerequisites, but do not guarantee effective teaching and learning by themselves (cf. Griffin, 2009).

In NaDiMa the concept of ND has been investigated. Our hypothesis is that for the learners this concept will contribute

- to a deeper mathematical understanding, as well as
- to the development of general learning strategies, and
- to higher (intrinsic) motivation. Although the relations between motivation (resp. its various components) and achievement in mathematics are not consistent in diverse studies (for an overview see Moser Opitz, 2009), we see intrinsic motivation as a general aim in mathematics education.

In addition to this, the theoretical concept and the term of ND needs to be sharpened.

2.3. Design and Methods

Together with selected teachers, different learning environments were designed and tried out in primary school (in Germany grade 1-4). The lessons (for pilot study and field test 1 completely and for field test 2 partly video-taped) are analyzed and evaluated with respect to the realization of ND. By a cyclic process of evaluation (pilot study, field test 1 & 2) with reflections of the participating teachers and an extension of classes and teachers the tested material has been tried out, evaluated and optimized (see tab. 1). In addition to this, the materials were and will be prepared for pre-service as well as for in-service courses.

pilot study (grade 2 & 4; one class)	March/April 2009	• 3 lessons SLE 'number chains' • individual interviews about motivation (5 to 6 per class)
field test 1 (grade 2 & 4; one class)	May/June 2009	• pre-test SELLMO-S (grade 4) • 8 lessons SLE 'number triangles' • post-test SELLMO-S (grade 4) • adapted TIMSS-items on motivation in mathematics resp. specific SLE • individual interviews about motivation (5 to 6 per class) • teacher interview
field test 2 (12 classes; different grades from 3 schools)	October 2009 – January 2010	• implementation of different SLEs • adapted TIMSS-items on motivation in mathematics resp. specific SLE • if applicable, individual interviews about motivation (5 to 6 per class) • if applicable, teacher interview or questionnaire

Table 1: overview on content, time schedule and methods for NaDiMa (Germany)

For measuring motivation, different instruments and methods have been chosen dependent on the children's age and on the phase of the project (see tab. 1): As standardized methods the TIMSS-items (see Walther et al., 2008) as well as the SELLMO-S (see Spinath et al., 2002) were chosen. For a qualitative analysis guided interviews took place (in all project phases video-taped) with selected children of each class. Beyond other aspects, the children were asked about their favourite subjects and their attitude towards mathematics, about the level of difficulty of mathematics and the specific learning environment they had experienced in the study.

Whereas the other participating countries tested context-embedded or geometrical learning environments, in Germany we focused on arithmetical SLEs for different grades. For the SLE ‘number chains’ used in the pilot study a series of three lessons was tried out. Field test 1 was extended to the following teaching unit, using the SLE ‘number triangles’:

- 1st lesson: introduction/revision of the format; own number triangles (cf. 3.1)
- 2nd – 4th lesson: investigating operative variations
- 5th – 8th lesson: diverse investigations, e. g. number triangles with numbers from the times-table, reaching even/odd exterior fields (cf. 3.2), number triangles with three given exterior fields etc.

With respect to pilot study and field test 1, teachers and researchers met several times for discussing the learning environments and the teachers were equipped with didactical literature, suggested lesson plans and worksheets. These materials should serve as a framework, and the teachers were free in selecting specific topics of the series and modify the given plans.

For field test 2, we offered a preceding in-service course for schools to clarify the concept of ND as well as illustrating the concept by experiences and results of the pilot study and field test 1. Participants of this course volunteered for engaging in field test 2. For this project phase, besides number chains and number triangles, the teachers for field test 2 could choose the SLEs ‘time-plus-houses’ (cf. Verboom 2002; Valls-Busch 2004), ‘minus walls’ (cf. Krauthausen, 2006; 2009) and ‘bars and angles on the field of 20’ (cf. Hirt, Wälti, 2009). For these SLEs didactical literature, partly including concrete proposals for teaching and concrete worksheets was delivered, and again various types of problems (operative and problem structured as well as open tasks) were proposed.

Field test 2	grade 1	grade 2	grade 3	grade 4	total
Number chains	0	2	0	2	4
Number triangles	0	0	3	2	5
Minus walls	0	0	0	1	1
Times-plus-houses	0	0	2	0	2
Bars and angles on the field of 20	0	0	0	0	0
total	0	2	5	5	12

Table 2: overview on chosen SLEs, according to grades, for field test 2

Table 2 shows that none of the 1st grade teachers volunteered for participating. As shown before, the phase for field test 2 was planned between fall holidays and Christmas. The in-service course was offered directly after summer holidays what means directly after school start for 1st grade. Due to the fact that mathematical topics just started, the 1st grade teachers might be hesitating if a complex learning environment could work.

It was also found that the participating teachers had a tendency to prefer the ready-made material (the SLE number chains was prepared for the pilot study, number triangles for field test 1). Moreover, students' examples of number triangles and reflections on students' strategies and ideas was a focus of the in-service course. Beyond the interpretations that teachers felt more secure with detailed material the reason could also be that number triangles are well-known and can be found in many textbooks for primary schools. This is not the case for minus walls, times-plus-houses, bars and angles on the field of 20 or number chains. Moreover, the presented and discussed examples during the in-service course had motivated the teachers to try it out with their own class. Exemplary statements:

> During the in-service course we were given an understanding of the number triangles. By that, I became motivated to find out something by myself (T11).
>
> I knew the number triangle as a format for practicing skills, already. The intensive analysis with the variety of possible task during the in-service course stimulated me to try them out 'differently'. With number triangles you can work on different levels and the offer per se an inner differentiation (T13).

For us, these utterances provide valuable information for the implementation of ND.

3. SELECTED RESULTS (FIELD TEST 1 & 2)

In the following, two different types of problems will be illustrated showing the opportunities for ND and discussing the different outcomes. The two examples refer to findings from field test 1 (SLE: number triangles) that was also used by several teachers in field test 2.

3.1. Natural Differentiation with Open Problems

For field test 1 we chose the SLE 'number triangles' (cf. Scherer 1997; Wittmann 2001b), and the format also went into action in field test 2. The rule for this format is the following (cf. fig. 1): The sum of two adjacent interior fields is written down in the exterior field.

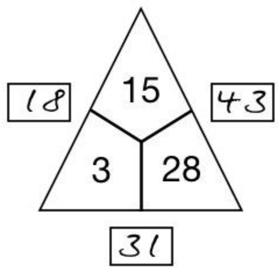


Figure 1: number triangles

For many classes that format was new, and after a short introduction at the blackboard the students completed a worksheet with given examples to ensure the correct application of the rule. Besides that, the students had a second worksheet with empty formats at their disposal.

Especially with open problems and without any specific demands, the students have the opportunity to work on their own level (cf. also Treffers, 1991, p. 25). The question arose, if and to what extent the students actually make use of this openness. In the pilot study, with the format number chains, the students should find own number chains and the results differed with respect to the quantity of tasks and worksheets as well as with respect to the chosen numbers. Anyhow, the quantity does not say anything about the chosen level of difficulty. The concrete tasks indicated a wide spectrum which was not only true for this format but also for others as experienced in field test 2 (cf. Scherer, Krauthausen, 2010). The children also made relations between numbers (e. g. equal or successive numbers) and relations between different number chains.

In field test 1 we wanted to ensure that the students made their choices with a greater (also meta-cognitive) awareness, and so we changed the corresponding worksheet for open problems for the SLE ‘number triangles’ in the way that children should differentiate between what they find ‘easy’, ‘difficult’ and ‘special’ number triangles (see also Van den Heuvel-Panhuizen, 1996, p. 145 f.).

Especially the ‘special’ number triangles showed interesting results: Students found number triangles with pure tens, hundreds, …, with zeroes (see fig. 2), with multiples of 11, 111, 1111 for the interior field etc. Moreover, relations between numbers became obvious as they chose equal numbers, successive numbers (see fig. 4, middle), getting pure tens, hundreds etc. for the exterior fields. Relations could also be seen between the triangles (fig. 3). Lili (2nd grade) explained “*Well, these two belong together ... that’s a tossing problem ... There are the numbers from the results! And then, you can continue with the next one!”*

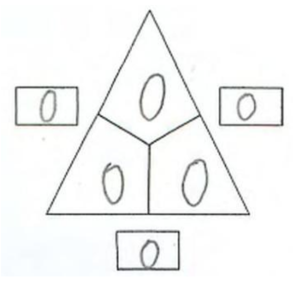


Figure 2: special number triangle with zeroes

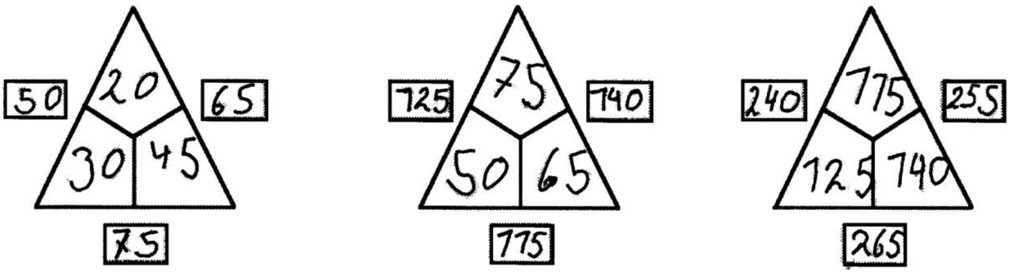


Figure 3: Lili's special number triangles

These examples more or less represent well known knowledge. But with this category 'special' we could see that a lot of children went a step further: They went into 'unknown' number spaces, like decimal numbers (fig. 4, left) or negative numbers (fig. 5) which – understandably – could not be solved correctly in every case. One child decided to put terms into the interior fields – not only numbers (fig. 6).

Some of the 'special' number triangles did not look special at a first sight, but they were explored by their authors as special by revealing personal meanings for specific numbers (e. g. ages of the family members). This also for some children may be an effective factor for motivation.

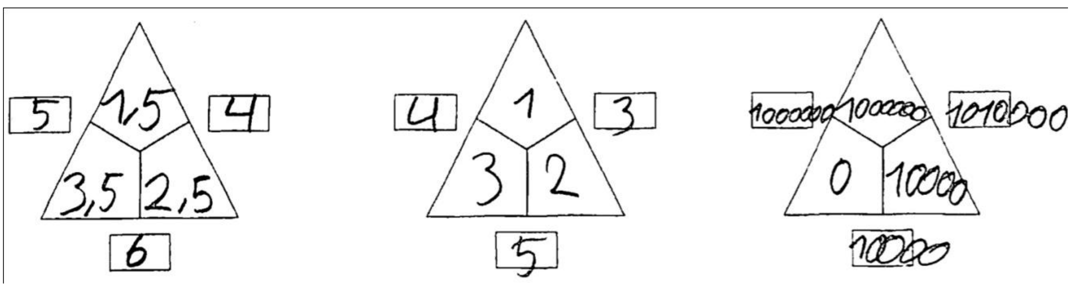


Figure 4: special number triangles with decimal numbers, successive numbers and big numbers

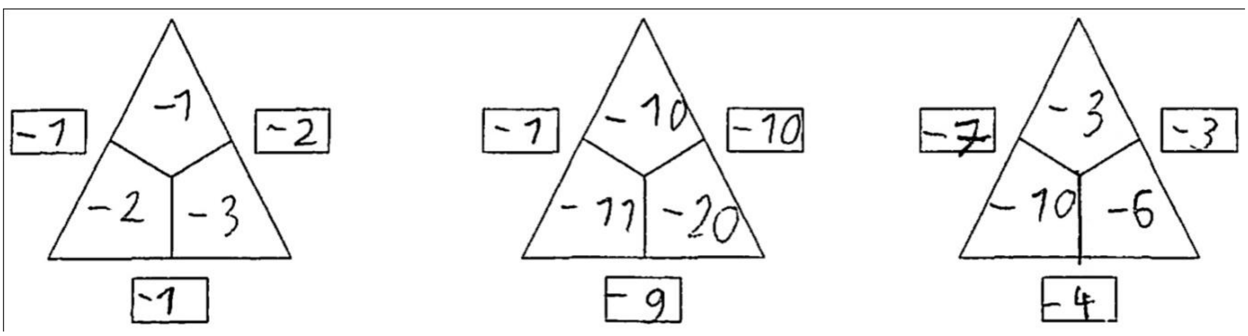


Figure 5: special number triangles with negative numbers

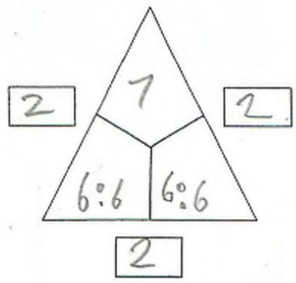


Figure 6: special number triangle, not only with numbers but with terms

Summarizing the results for this type of problems, the self-chosen tasks represented a wide spectrum according to different criteria and with a wide-spread size of numbers. At first sight, one could object, that the students did not choose the size of numbers appropriate to their achievement level (e.g. the 4th graders actually calculated with numbers up to one million). But one has to take into account, that this is allowed when dealing with an open problem. Moreover, the teacher could get helpful information with respect to the achievement level or more general for the work with open problems.

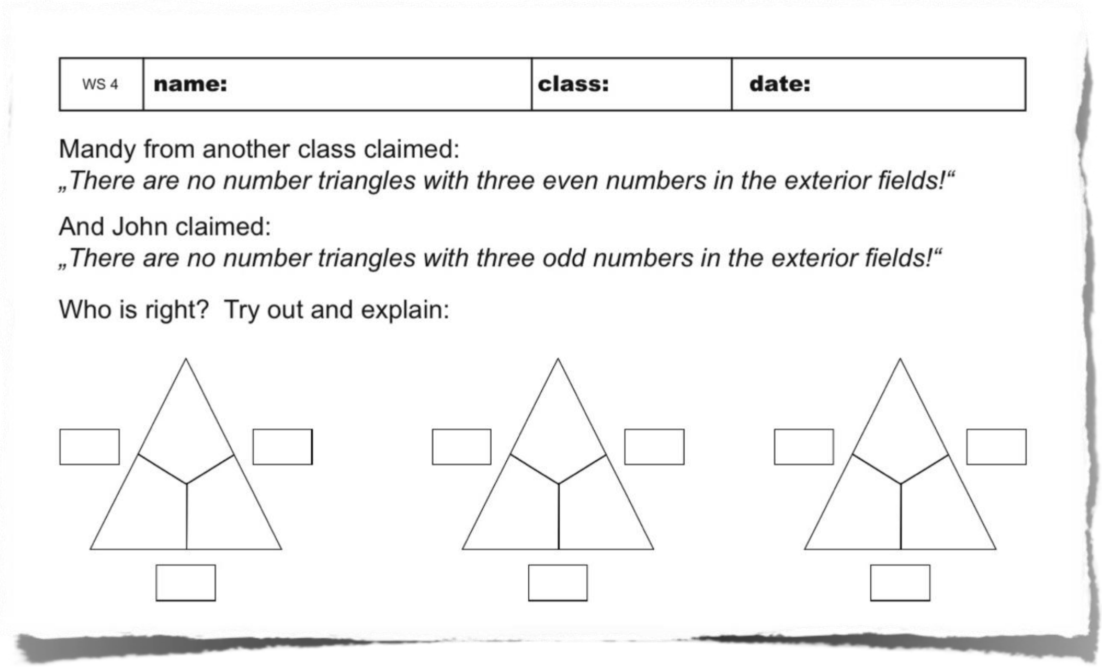
On the other hand, the above mentioned examples could show that special numbers, e. g. in the sense of operative variations, must not inevitably be done with bigger numbers. Perhaps they consciously shouldn't even be done with bigger numbers because then the calculation demands absorb too much cognitive energy. This energy should have priority for describing, explaining and justifying the patterns.

For us, a sound phase of plenary reflection at the end of the lesson seems very important (cf. 4.3). In such phases students should present and commonly reflect their individual learning products (see also Treffers, 1991, p. 25) and the individual reasons can be quite different. Students may present examples they judge as very easy or very difficult, examples which show patterns or a specific structure etc. Mathematical discoveries can happen the more likely the more the substance of the learning environment can foster that – a theoretical assumption which could also be affirmed practically in our project.

3.2. Natural Differentiation with Problem Structured Tasks

3.2.1. The Problem

For primary grades we (among others) designed the following problem for investigation (fig. 7) which was dealt with in the 6th lesson in the series of number triangles (see paragraph 2.3).



WS 4	**name:**	**class:**	**date:**

Mandy from another class claimed:
„There are no number triangles with three even numbers in the exterior fields!“

And John claimed:
„There are no number triangles with three odd numbers in the exterior fields!“

Who is right? Try out and explain:

Figure 7: worksheet for pupils

This format can be challenging for students and they might "happily play the role of sceptic, looking for falsifying examples" (Burton, 1987, p. 11). Checking, if the statements are true or false, can be done more or less extensive and on different levels.

Obviously, Mandy's statement is false and can be disproved by just *one* counter example (some children did so). Moreover, the situation can be investigated in a systematic way with systematic reasoning (three even or three odd interior fields). This could be found with several students: They had a look at several examples and different cases (three even interior fields with three equal or different numbers; or analogous for odd interior fields; fig. 8a and 8b). Others used case discriminations in order to explore what will happen with a combination of even and odd interior fields.

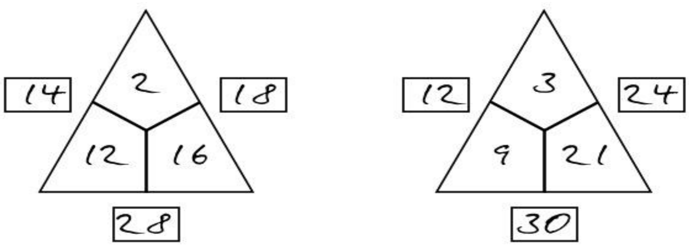


Figure 8a and 8b: even exterior fields with three even or three odd interior fields

The exploration of John's statement is more demanding and can be illustrated on a pre-algebraic level (*e* stands for even and *o* for odd numbers): If you want to reach three odd exterior fields, you have to choose for two adjacent interior field one odd and one even number (fig. 9a, right side). To get an odd exterior field also in the bottom field, you have to add an even number to the already given odd interior field. With this, the numbers for the interior field on the left side are fixed and, by force, make the exterior field an even one (fig. 9b). Therefore, you will never get a number triangle with three odd exterior fields if natural numbers are used.

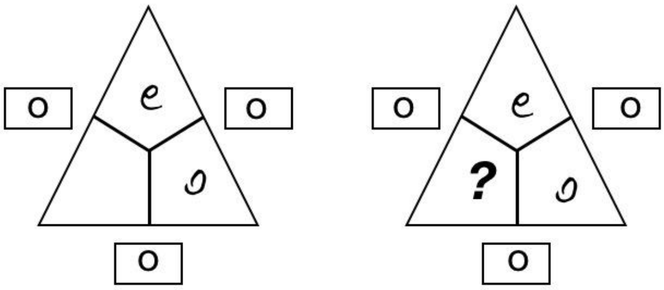


Figure 9a and 9b: trial to get three odd exterior fields

According to the intention of this investigation, we chose a true and a false statement (for primary students) on purpose, to analyze the potentially different argumentations. Unsolvable tasks may challenge arguments and reasoning in a special way, but at the same time demand adequate mathematical competences. Beyond content-related objectives, here, process-related objectives are required: the development of problem solving strategies, argumentations and reasoning (cf. Scherer 2007). These process-related competencies can be shown on different levels that will be illustrated in the following for John's statement.

3.2.2. Levels of Argumentation

As shown before, the students dealt with number triangles during six lessons (introduction, open problems, operative-structured activities; see paragraph 2.3). For this specific problem solving activity, the students should not only calculate but be encouraged to verbalize and write down their discoveries. Those verbalizations and notations could and should be done with the help of number examples and drawings/sketches, so that different representation levels can be integrated in a natural way (see also Steinbring, 2009).

based on concrete numbers

For primary grades a suitable level could be 'based on (arithmetical) examples' (see also for Mandy's statement, fig. 8a, 8b). Here is an example from a 4th grade lesson where Leon argues as follows (cf. fig. 10a-10c):

Leon:	So, it's always the case …, it's … you can only make an odd number, if you have one even, for example 6, and together with an odd one, the 3 [writes 6 and 3 below the number triangle, cf. fig. 10a].
Student:	Yes, even plus even makes even.
Leon:	Plus, makes 9. [writes down plus sign, equals sign and result]
Leon:	So. And differently, if you have 3 plus 3, then you have an even 6 [writes 3+3 = 6; cf. fig 10a], if you have two even, 6 plus 6 makes 12 [writes 6+6 = 12; cf. fig. 10a] ...

Student: Even as well.

Leon: Even as well. But, only this [points at the calculation 6+3 = 9] leads to something odd.

Student: Yes.

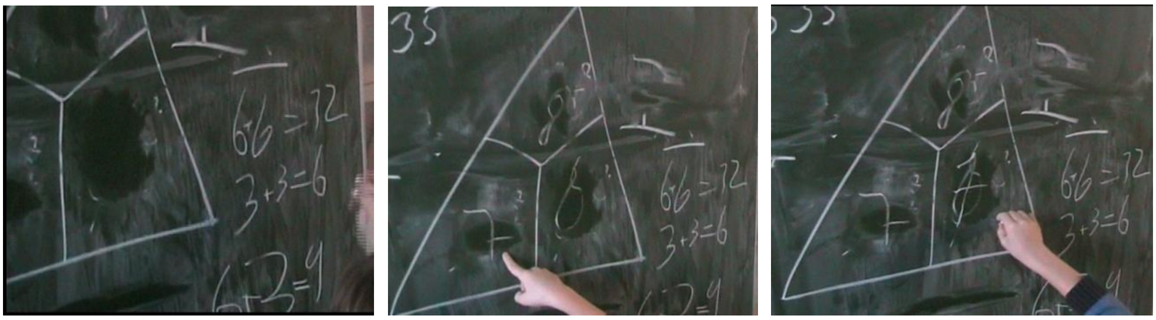


Figure 10a, 10b, 10c: argumentation, based on concrete numbers

Leon: And because there are three [points at the three interior fields of the number triangle], you can, you can now 8, 7 [writes clockwise 8, 7 and 8 into the interior fields; cf. fig. 10b]. You can put together two 8s and a 7, then …

Leon: … this is odd [points at the interior fields 7 and 8 at the bottom; cf. fig. 10b], that's odd [points at the interior fields 8 and 7 on the left], but this is even [points at the interior fields 8 and 8 on the right].

Student: Yes.

Leon: You can also take 7, 7, 8 [writes 7 above the 8 of the bottom right interior field; cf. fig. 10c]. But then, this is even [points at the interior fields 7 and 7 on the bottom].

Though Leon considers all three possibilities (even + odd, odd + odd, even + even), he demonstrates his idea at the blackboard (outside the number triangle) with concrete numbers (6+3, 3+3, 6+6). Then he fills the number triangle with concrete values as well (8, 7, 8 resp. 8, 7, 7). Nevertheless, his explanation includes verbal indications for generalization: "it's *always* the case…" and "if you have *one even*, *for example* 6, …". With the second example (3+3) Leon (verbally!) remains concrete, but regarding the third example (6+6) he again says: "if you have *two even*, 6 plus 6 …". The teacher must be sensible for verbal indications like these which can occur on the fly, irregularly and inconsistently. And it is the teacher's responsibility to take up these opportunities in order to foster verbal as well as mathematical precision and awareness.

(pre)algebraic level

As the problem aims at developing algebraic thinking – not on a formalistic level, but pre-algebraically – the children could argue with even and odd numbers. They can use the complete words (fig. 11 or as Leon did before) or the abbreviations (fig. 9a, 9b). In 2nd grade, this was emphasized, with a distinction by different colours.

The formal ‘algebraic’ representation (as *2n* for even numbers and *2n+1* for odd numbers) is not intended for primary level, but for secondary students (and of course for teacher students and teachers).

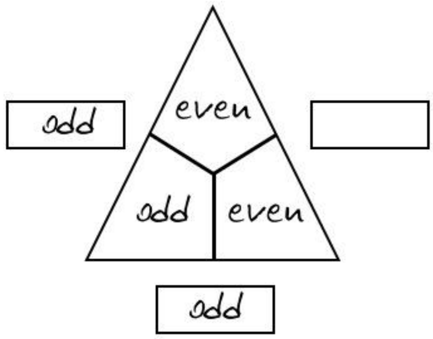


Figure 11: odd exterior fields on a pre-algebraic level

switching between levels

When verbalizing their arguments children could switch between the example-based and the more general (pre-algebraic) level, depending on one’s own understanding. This could also facilitate the access for other students which will be illustrated in the following scene.

Student A: For example, if I put here a 3 [writes 3 into the right interior field at the bottom] and I make …, here the same [writes 4 into the upper interior field], then it would work here – here you get a 7 [writes 7 into the right exterior field; cf. fig. 12a] – but here it doesn’t work [points at the left side of the number triangle].

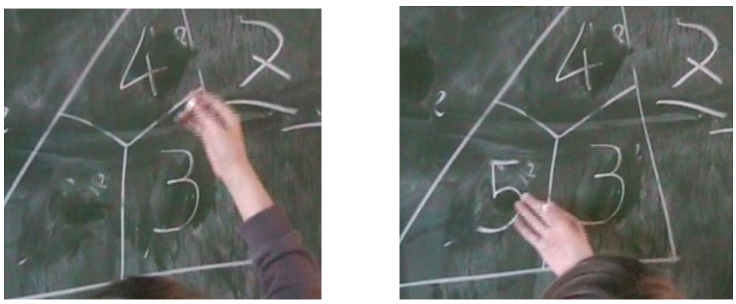


Figure 12a and 12b: concrete numbers and general argumentation

Student B: But there are only three.

Student: Yes.

Student A: If I put here, for example, a 5 … [writes 5 into the left interior field at the bottom; cf. fig 12b]

Student C: Oooh, you can explain that easier.

Student A: … then it makes 8 here.

Teacher: And if you take something different?

Student B: If even and odd, if you take even and odd, then it makes an odd.

Class: No … please show it … show it.

Student A: Please, be quiet. If an odd and an even …, then you get an odd. [points at the interior fields and the exterior field on the left]

Student A: … and if two odd come together, then you get an even. And if two even come together, you get an even, too. [points at different fields in the number triangle]

Teacher: Ok. Then it is impossible.

At first he explains based on concrete numbers, but then he takes up the comment of another student and alternates to a more general diction, using ‘even/odd’, to justify that the case of three odd exterior fields is impossible.

making the unsolvable solvable

For primary students the given problem is an unsolvable one, as in grade 1 to 4 the students (officially) only deal with natural numbers. Many students could not bear not finding a solution and looked for alternative conditions like Jeremi (4th grade) in the following scene. The teacher should be alert to such extensions and variations which might exceed the regular maths stuff:

Student: With decimals it would work.

Sev. stud.: Yes! ... Yes! ...

Teacher: Yes, please let Jeremi do, Jakob should explain later on, and Jeremi explains now, was what he had in mind with the decimal.

Jeremi: It will work with decimal, because if you take for example...

Student: 0.5

Jeremi: ... 0.5 [writes 0.5 into the left interior field at the bottom; the notation of the decimal comma is rather large and looks like a perpendicular mark between the numbers] …

Student: How do you write that?! Please, let me do it.

Student: Oh, he can (not understandable) ...

Sev. stud.: Oh, that’s not a zero. [alike, Jeremi writes 0.5 into the right interior field at the bottom; cf. fig. 13] That’s, häh, what number should that be?!

Figure 13: Jeremi’s solution with decimal numbers

Teacher: Oh guys, he has in mind 0.5 … please, do not pay so much attention to the style. [Jeremi writes 0.5 into the upper interior field]

Jeremi: Here 1. [writes 1 into the right exterior field]

Student: You don’t get a decimal.

Jeremi: 1 and 1. [writes 1 and 1 into the other exterior fields]

Damian (2nd grade, fig. 14) found a number triangle with fractions. Understandably, problems with this new symbolic notation occur: Verbally he explained *“Two and two fourth”*, so, in his writing he mixed up numerator and denominator.

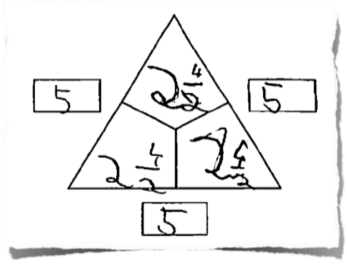


Figure 14: Damian's (2nd grade) number triangles with fractions

In other classes students tried a different operation (multiplication instead of addition), or they made changes to the format itself: A grade 4 class turned the triangle in other polygons (cf. also Schmidt 2009) and proved the rules with a number square. Systematically, they went on with a pentagon and proved that five odd exterior fields did not work:

Student: If you...

Teacher: For an even number of interior fields it will work...

Student A: We will now make a pentagon!

Teacher: … if you …

Student: It will work with every polygon that you can divide by 2!

Then they showed that the hexagon would work (cf. fig. 15).

Figure 15: proving a number hexagon

They even made the generalization for polygons with odd and even numbers of edges, represented by the (theoretical) example of a 100 edges polygon:

Student B: Yes, because you, then you can always an odd, because then one, two numbers together.

Students: Yes … yes.

Student B: Always two numbers together. [points at every two adjacent interior fields]

Student A: Yes, then an odd always can next to an even.

Student: Now, we even might make a polygon with 100 edges, that would also work.

As shown, for this unsolvable situation we found a variety of students' discoveries in our empirical study. Among several productive ideas and solutions, some students ascribed the fact that they could not solve the problem due to their own incapability or due to the fact, that they did not try enough. This is a crucial point for the plenary reflection, to work out the potential reasons for not finding a solution (see also Scherer, 2007).

4. INTERIM CONCLUSIONS AND PERSPECTIVES

4.1. Substantial Learning Environments and Opportunities for Natural Differentiation

Our first results could show, that the students really made use of the containing substance that allows working on different levels. Moreover, it could be stated that the different types of tasks (open problems, open explorations, more purposeful problems) have different effects. Beyond this, it became clear that the size of the numbers is an important and influencing factor and has to be reconsidered with respect to the underlying aims and objectives.

Absolutely essential for the use of substantial problems like these is the opportunity to both foster and demand the general, process-related competencies, besides the content-related ones. The implementation and the fulfilment of just these objectives has become more and more valued and stressed by education policy (NCTM 2000; KMK 2005). The concrete opportunities of ND became obvious with respect to the

Selection of numbers

The widespread selection was presented with the open tasks (see 3.1). With other problems (e. g. reaching a specific target number for number chains) this freedom was not given, but we could state that some students first tried out ‘easy’ numbers (pure tens or multiples of 5). The target number 50 consciously was chosen as an easy one and the importance of such a selection became clear.

With the number triangles (reasoning about the given statements; see paragraph 3.2), the children were completely free to choose the numbers. Many children chose numbers up to 10 or 20 as a maximum, which is quite sensible for such an investigation, as you may better concentrate on the pattern. This is a crucial point for the common reflection with respect to the development of more general problem solving strategies. If the focus is on investigation and explanation of structural relationships, then it is clever to reduce the own calculation demands because the size of the numbers does not play any role for the structure.

The children also used special numbers like equal numbers or successive numbers or multiples of 11. During the plenary reflection on John’s statement (looking for three odd exterior fields), a fourth grader commented *“I tried it out with a zero”* and another *“I guessed, perhaps it will work with prime numbers”*. This makes clear that the children do not choose the numbers accidentally but on the basis of their individual mathematical experiences and knowledge.

Number of examples

The problems also differentiate according to the numbers of examples (which is not independent from the former point; see also above: Here quantitative differentiation takes place).

Problem solving strategies

The second example focused on problem solving and as problem solving strategies we could find argumentations with respect to concrete numbers or in a more general manner. The students should gain experiences with these different levels and we found a wide spectrum. We could also find students who considered particular cases, distinction of cases, up to completeness of a proof. Such types of investigation could challenge a general requirement of reasoning and proofs (Winter 1983). For example, some of our children thought you just have to examine ‘enough’ examples in order to have a proof.

Searching for alternatives to turn a problem into a solvable one

For our second statement (John), it turned out that this was difficult for many students, mostly because they had only few experiences with such an activity (cf. Scherer 2007). But this type nevertheless lead some children to further investigations. The proposals could be amusing like *“If 6 and 6 would make 13, then it could work”*. Other ideas showed complex mathematical thinking processes, lead to fractions or variations of the format as shown before: Johanna (4th grade) suggested to change the operation of number triangles *“If you would multiply, then it would work”*.

One advantage of SLEs is the fact that the levels of demand are not determined in advance, but develop quite naturally within a substantial research problem with floating transitions. This facilitates ND for the students, combined with an economical expenditure for the teacher who therewith wins free zones a) for diagnostic work and b) for framing the plenary phase at the end of a lesson.

We consciously used the benefit of the mixture between individual approaches, partner or small group work, but also with common phases of reflection and integration. In doing so, the important social learning is strongly taken into consideration, and real deepening mathematical knowledge can take place (cf. also Treffers, 1991).

4.2. Findings Concerning Learners’ Motivation

The first results based on the interviews as well as on the TIMSS-items have shown a rather high motivation of the students with respect to mathematics education and mathematics learning in general. These results are in accordance with the findings of large scale studies for this age (cf. Walther et al., 2008, p. 78). In our study this positive attitude is not only true for the bright students but for nearly all students (Exemplary statements for mathematics in the interviews: “*It’s interesting”; “It’s tricky”; “Because it is easy”; “I like difficult* *problems"; "Because there exist easy and difficult problems"; "You have to do it with your brain and are not required to write so much stuff like in German language"*).

Some of the students commented on the given SLE, that they liked those activities more than the regular mathematics lessons because they were more successful than usually (Exemplary statements: *"I like this format very much. Because every time you could do something new with it"; "I was successful with the number chains"; "When I made sense of it, I liked it very much"*).

In the ongoing project we will scan and analyze different aspects of motivation in more detail (see Krauthausen, Scherer, 2011). Our examples certify that intrinsic motivation is not imperatively bound to a realistic, everyday context situation: Pure inner-mathematical problems and contents can be motivating for all children as well. More important than the question of contexts versus inner-mathematical structures is the amount of the enclosed *substance*.

4.3. Consequences for Teacher's Role and for Teacher Education

The crucial demands for the teacher and the teacher's role are related to various areas (see also Scherer, Steinbring, 2006; Speer, Wagner, 2009). In the following, we want to focus to some of these mentioned aspects in more detail.

Assuring sound mathematical framings for the learning processes

Especially the first two definition attributes for ND (cf. 1.2) clearly indicate: It is the *teacher's* responsibility (and only *(s)he* is able) to assure a sound mathematical framing of the learning environment. Within that framing, the students' degrees of freedom can become true – in a transparent, oriented way, and not accidently. Therefore, the attributes of a so called Substantial Learning Environment (SLE) as Wittmann (2001a/b) has put it can be rather helpful for teachers in the sense that the *fundamental ideas of mathematics* provide decision support for identifying, selecting and implementation of SLEs. By that, ND which lies inside the subject matter itself can evolve, as well as the learners' motivation can be supported. Because for students it is by no means prior or even indispensable (as it is often supposed) to wrap a task in a kind of 'framework story'. Rather the *mathematical substance* is crucial. If that is missing, we can understand each child feeling bored (inspite of a real world reference of the task).

Aspiring sustainable mathematical knowledge by the teachers themselves

Orientation to fundamental ideas, sound implementation of a learning environment, giving fostering initiations to students on-the-fly during the lesson, taking up students' contributions spontaneously – all that by nature requires mathematical competencies for the teacher. Some of them may feel overburdened, especially if they experienced no or no adequate *mathematics* courses during their teacher education program.

For successful realization of ND in the frame of SLEs, we want to stress the following demands:

- own *mathematical exploration* of the posed problems in order to be able to classify and value different strategies and solutions
- *anticipated* reflection of possible strategies and levels for students as well as
- analysis of *real* student documents
- reflection on *integration* of different strategies, solutions and argumentations and
- sound capability to moderate plenary discussions (including e. g. posing questions, initiations, even irritations if needed).

> The responsibility for ensuring that these mathematical conversations are productive *ultimately lies with the teacher*. To organize, manage, and support such classroom dialogue, teachers must model appropriate discursive practices for students, present meaningful tasks, and actively monitor students' interactions. Teachers should listen to and help guide the conversations by requiring explanations and clarification of ideas as well as provide feedback about students' ways of thinking about the problem and their solutions. Students should also have sufficient time to work on tasks, grapple with mathematical ideas, consider alternative approaches, and formulate reasonable explanations to support their solutions (Kilic et al., 2010, p. 352; emphasis GKr/PS).

Mathematical expertise also protects the teacher from restricting him- or herself to merely methodical variations independent from the mathematical content. Methods are important, but they are not an end in themselves. The method depends on the momentum of the content itself, and the substance of the content should have precedence.

Rehabilitation of plenum discussions

Common plenum discussions after an extensive phase of activities (individual, partner, or group work) are of great importance for ND. As clarified before, all children work on the same task or same problem. If everybody would have his or hers own problem to work on (as it often happens with classical inner differentiation), plenum discussions indeed would be very difficult, if not impossible. Because, firstly, it requests a lot of flexibility and knowledge for a student to *spontaneously* concentrate on his classmate’s task which is quite different from the own one. Secondly, the classmate will tell what (s)he has done from the perspective of a ‘knower’, and that presumably may include rather shortened statements, appropriately not even very well verbalized. These are reasons enough for a listening student to become inconsiderate and less interested.

But it is *just these discussions* which not only foster and support social learning – *if* all students have worked on the same problems before, though on their own levels. In this case, everybody knows what the others are talking about.

Discussions as well allow students to learn from each other with respect to the content: Low achievers can gather ideas from their brighter classmates they would not have found on their own yet. And also the high achievers may get ideas about other ways of solution, or they at least could profit from several examples which were produced by those who were still restricted to calculation without having an idea of deeper structures. Teachers may tend to raise the following two oppositions to these arguments:

(a) The weaker students do not understand what the brighter ones are talking about because it is not their actual level of understanding yet.

If heterogeneity is reality in everyday schooling, we have to deal with students of various capabilities. The teacher then principally has two alternatives: (S)he can spare the low attainers discussions like that, leaving them with their reduced demands, 'protecting' them from enhanced requests, and providing them with their own specific tasks. But this would evoke fixed levels, and the danger of determining the students' potentials. Where should they get their chance to proceed?

On the other side, the teacher could *regularly* offer discussions to *all* students (after they had worked on the *same* problem before; cf. 1.2), *especially* to the low achievers, in order to raise the likeliness (that's principally all (s)he can do, learning cannot be determined!) that these children some day will catch a glimpse of new ideas which were not within their reach so far. And who could really say when this will happen? Why to be pessimistic and think they will not understand? Remember little preschoolers listening to their parents' talks: Most of us know the situation where you had to ask yourself: "Where, for heaven's sake, did (s)he picked that up?!"

(b) At the end of the lesson there were some students who still did not find solutions for a problem or gather the patterns.

This opposition in our view includes a classical error in reasoning, because that statement does not describe a problem of the students, but rather an illusory expectation: If heterogeneity is true, then it is *normal* that not all students have understood on the same level and everything at the end of the lesson or a sequence. It is not a problem – it is at the most a problem to call that a problem. How can we start a lesson, taking care of heterogeneity, and then expecting equal levels of understanding at the end of that lesson?

There is no didactical conception at all around the world which can guarantee and determine that learning takes place. All the teacher can do is to *enhance the likeliness* that sound learning processes will happen. That sounds few, but (s)he can do a lot for that. And ND for us seems to be a rather promising way in that direction.

Establishing an appropriate classroom culture

In our view, a subserving classroom culture is of major importance (see also Griffin 2009): It is a prominent task for teachers to provide their students with a sound image of mathematics as a science. Inherent to this is to consciously value the central attributes of doing mathematics. And that cannot be successful by just giving isolated information, but with offering a) a continuous, b) explicit model by the teacher him- or herself, and c) habitually initiating and maintaining of meta-cognition.

Possibly, deeply internalized underlying pre-experiences have to be discussed. As traditional classroom culture more likely emphasized the *content-related* competencies and neglected the *process-related* ones, students internalized implicit valuations and units of value – unconsciously or caused by that kind of teaching and learning: Students are rather proud if they can deal successfully with difficult operations, unusual number systems (fractions or negative numbers) or big numbers. But they still do have less role models for cherishing process-related competencies. And that is also true for attributes of desirable attitudes of sound working (endurance, persistence, patience).

In addition to this: Concerning the level of difficulty in mathematics tasks students have various criteria for rating levels which could be related to …

- the complexity of calculation (kind and size of numbers)
- the operations involved (addition/subtraction is often said to be easier than multiplication/division)
- the demand of thinking and reflection, strategic understanding, process-related competencies
- the understanding of the given tasks (language understanding)
- the amount of necessary (written) text production

Concerning a sound culture of teaching and learning, the teacher therefore has to make it plausible for the students that the amount of reflections, the flexibility of strategic ways of solution, an attitude of questioning, and the individual request for proving (cf. Winter, 1983) at least are good reasons to feel proud of one's own capability. That works the easier the more a steady *culture of meta-cognition* is implemented in the classroom, so that everybody can talk about that in a natural and habitual way.

Students *by themselves* will offer reasoning and proves for solutions quite naturally if the teacher before has ever and ever asked "*why*" (in the beginning: in substitution for the child). »As teachers we get what we ask for. If we ask only for simple numerical answers, children will value only procedures and computational tasks. But if we ask for discussion, explanation, and elaboration – and if we reward these kinds of answers – then children will value understanding and meaning« (Higgins 1988, p. 2).

In the same way, with the role model of the teacher, the students' sensibility, diversity, and valuations of the aspects mentioned above will increase and establish as an attitude. This points to an educational assignment of mathematics education which leads far beyond the subject and the subject matter.

4.4. Final remarks

The project NaDiMa approaches to its end and we could gain important experiences. At the centre of the project are the teachers as well as the concrete design of mathematics lessons. We also suggest to keep in mind that the described concept of ND within SLEs is not meant as an exclusive one for the whole range of mathematical learning and teaching (though rather likely for its major part). Training the basic facts e. g. or introducing a specific procedure, may require other practices. Above all, successful and motivating learning processes should be possible and are desirable for as many students as possible.

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REFERENCES

Beutel, S.-I., Ruberg, C.: 2010, Differenzierte Leistungsbeurteilung in heterogenen Lerngruppen, SchulVerwaltung NRW(5), pp. 142-143.

Bönsch, M.: 1976, *Differenzierung des Unterrichts: Methodische Aspekte*, München: Ehrenwirth.

Bönsch, M.: 2004, *Differenzierung in Schule und Unterricht*, München: Oldenbourg.

Burton, L.: 1987, *Thinking Things Through*, Oxford: Basil Blackwell.

Freudenthal, H.: 1974, Die Stufen im Lernprozeß und die heterogene Lerngruppe im Hinblick auf die Middenschool, *Neue Sammlung*, **14**, pp. 161-172.

Geppert, K.: 1981, *Differenzierender Unterricht – konkret, Analyse, Planung und Gestaltung, Ein Modell zur Reform des Primarbereichs*, Bad Heilbrunn: Klinkhardt.

Griffin, P.: 2009, What makes a rich task? *Mathematics Teaching*, **212**, pp. 32-34.

Hengartner, E. et al.: 2006, *Lernumgebungen für Rechenschwache bis Hochbegabte, Natürliche Differenzierung im Mathematikunterricht*, Zug: Klett & Balmer.

Higgins, J. L.: 1988, One Point of View: We Get What We Ask For, *Arithmetic Teacher*, **35**(5), 2.

Hirt, U., Wälti, B.: 2009, *Lernumgebungen im Mathematikunterricht, Natürliche Differenzierung für Rechenschwache bis Hochbegabte*, Seelze: Kallmeyer.

Kilic, H. et al.: 2010, Techniques for small-group discourse, *Teaching Children Mathematics*, **16**(6), pp. 350-357.

Klafki, W, Stöcker, H.: 1976, Innere Differenzierung des Unterrichts, *Zeitschrift für Pädagogik*, **4**, pp. 497-523.

KMK – Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland (2005, ed.), *Bildungsstandards im Fach Mathematik für den Primarbereich Beschluss vom 15.10.2004*, München: Wolters Kluwer.

Krauthausen, G.: 2006, Minusmauern – ein variiertes Aufgabenformat, *Praxis Grundschule*, **1**, pp. 6-8.

Krauthausen, G.: 2009, Kinder machen mathematische Entdeckungen mit Zahlenmauern, in: Leuders, T. et al. (eds.), *Mathemagische Momente*, Berlin: Cornelsen Scriptor, pp. 86-101.

Krauthausen, G, Scherer, P.: 2007, *Einführung in die Mathematikdidaktik, 3rd Edition*, Heidelberg: Spektrum.

Krauthausen, G, Scherer, P.: 2011 (in preparation), Natürliche Differenzierung – Motivation, Einstellung und Selbstkonzept im Mathematikunterricht.

Moser Opitz, E.: 2009, Ich liebe es einfach, mit Zahlen und Brüchen zu jonglieren, Begründungsmuster zur Einstellung zum Fach Mathematik in Klasse 5 und 8, *Journal für Mathematik-Didaktik*, **30**(3/4), pp. 181-205.

NCTM – National Council of Teachers of Mathematics, Commission on Standards of School Mathematics: 2000, *Principles and Standards for School Mathematics*, Reston: NCTM.

Neubrand, J., Neubrand, M.: 1999, Effekte multipler Lösungsmöglichkeiten: Beispiele aus einer japanischen Mathematikstunde, In: Selter, C, Walther, G. (eds.), *Mathematikdidaktik als design science*, Leipzig: Klett Grundschulverlag, pp. 148-158.

Paradies, L., Linser, H. J.: 2005, *Differenzieren im Unterricht*, Berlin: Cornelsen Scriptor.

Reh, S.: 2005, Warum fällt es Lehrerinnen und Lehrern so schwer, mit Heterogenität umzugehen? Historische und empirische Deutungen, in: *Die Deutsche Schule*, **97**(1), pp. 76-86.

Scherer, P.: 1997, Substantielle Aufgabenformate – jahrgangsübergreifende Beispiele für den Mathematikunterricht, Teil III, *Grundschulunterricht*, **6**, pp. 54-56.

Scherer, P.: 2007, Unschaffbar, Unlösbare Aufgaben im Mathematikunterricht der Grundschule, *Grundschulunterricht*, 2, 20-23.

Scherer, P., Krauthausen, G.: 2010, (in press), Natural Differentiation in Mathematics – The project NaDiMa, To appear in M. van Zanten et al. (eds.), *Proceedings of 28th Panama Conference*, Utrecht: Freudenthal Institute.

Scherer, P., Steinbring, H.: 2006, Noticing Children's Learning Processes – Teachers Jointly Reflect Their Own Classroom Interaction for Improving Mathematics Teaching, *Journal for Mathematics Teacher Education*, **9**(2), pp. 157-185.

Schmidt, S.: 2009, Rechen-n-Ecke als Lernumgebungen für mathematisch besonders befähigte Kinder in der Primarstufe und darüber hinaus, in: Peter-Koop, A. et al. (ed.). *Lernumgebungen – Ein Weg zum kompetenzorientierten Mathematikunterricht in der Grundschule*, Offenburg: Mildenberger, pp. 226-234.

Selter, C., Spiegel, H.: 1997, *Wie Kinder rechnen*, Leipzig: Ernst Klett Grundschulverlag.

Speer, N. M., Wagner, J. F.: 2009, Knowledge Needed by a Teacher to Provide Analytic Scaffolding During Undergraduate Mathematics Classroom Discussions, *Journal for Research in Mathematics Education*, **40**(5), pp. 530-562.

Spinath, B. et al.: 2002, *SELLMO – Skalen zur Erfassung der Lern- und Leistungsmotivation, Manual*, Göttingen: Hogrefe.

Steinbring, H.: 2009, Ist es möglich mathematische Bedeutungen zu kommunizieren? – Epistemologische Analyse interaktiver Wissenskonstruktionen, in: *Beiträge zum Mathematikunterricht* 2009, Hildesheim: Franzbecker, pp. 107-109.

Tillmann, K. J.: 2004, System jagt Fiktion, Die homogene Lerngruppe, Friedrich Jahresheft XXII, Heterogenität. Unterschiede nutzen - Gemeinsamkeiten stärken, pp. 6-10.

Treffers, A.: 1991, Didactical backround of a mathematics program for primary education, in: Streefland, L (ed.), *Realistic Mathematics Education in Primary School, On the occasion of the opening of the Freudenthal Institute*, Utrecht: Freudenthal Institute, pp. 21-56

Valls-Busch, B.: 2004, Rechnen und entdecken am Mal-Plus-Haus, *Die Grundschulzeitschrift*, **177**, pp. 22–23.

Van den Heuvel-Panhuizen, M.: 1996, *Assessment and realistic mathematics education*, Utrecht: Freudenthal Institute.

Verboom, L.: 2002, Aufgabenformate zum multiplikativen Rechnen, Entdecken und Beschreiben von Auffälligkeiten und Lösungsstrategien, *Praxis Grundschule*, **2**, pp. 14-25.

Vollstädt, W.: 1997, Differenzierung im Unterricht, Oder: Prüfungswissen kompakt, *Pädagogik*, 12, 36-40.

Walther, G. et al.: 2008, Mathematische Kompetenz im internationalen Vergleich: Testkonzeption und Ergebnisse, in: Bos, W. et al. (eds.), *TIMSS 2007, Mathematische und naturwissenschaftliche Kompetenzen von Grundschulkindern in Deutschland im internationalen Vergleich*, Münster: Waxmann, pp. 49-85.

Wielpütz, H.: 1998, Das besondere Kind im Mathematikunterricht – Anmerkungen aus der Sicht einer reflektierten Praxis, Beobachtung und Beratung, in: Peter-Koop, A (ed.), *Das besondere Kind im Mathematikunterricht der Grundschule*, Offenburg: Mildenberger, pp. 41-58.

Winkeler, R.: 1976, *Differenzierung: Funktionen, Formen und Probleme*, Ravensburg: Maier.

Winter, H.: 1983, Zur Problematik des Beweisbedürfnisses, *Journal für Mathematik-Didaktik*, **1**, pp. 59-95.

Wittmann, E. C.: 2001a, Developing mathematics education in a systemic process, *Educational Studies in Mathematics*, **48**(1), pp. 1-20.

Wittmann, E. C.: 2001b, Mathematics in Designing Substantial Learning Environments, in: Van den Heuvel-Panhuizen, M. (ed.), *PME 25, Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1), Utrecht: Freudenthal Institute, pp. 193-197.

Wittmann, E. Ch., Müller, G. N.: 2004, *Das Zahlenbuch, Mathematik im 4, Schuljahr, Lehrerband Neubearbeitung*, Leipzig: Klett Grundschulverlag.

DEVOLUTION AS A MOTIVATING FACTOR IN TEACHING MATHEMATICS

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The paper presents a part of a broader research developed in the frame of the Theory of didactical situations in mathematics and focusing on how to understand, study and improve students' school culture in case of problem solving in mathematics education. It studies the influence of the process of devolution and problem posing on students' approach to problem solving in mathematics and on their motivation to transform their traditional approaches to it into more active and more "mathematical" ones. The five-stage organization of the developed didactical situation is described with respect to the role of devolution in each of the stages.

INTRODUCTION

In lessons of mathematics with students of various levels of mathematical knowledge and different attitudes towards mathematics, many teachers prefer to use the frontal teaching method. However, the potential of this teaching strategy with respect to meeting the particular needs of individual students is limited. This contradicts the present effort of many teachers to create a learning environment that would take into account the individual differences among the students.

In school mathematics, problems are often used as an instrument of checking if and what the students have learned. Very rarely are they perceived as an opportunity to learn mathematics. Consequently, students understand problems as a tool used for assessment (grading), not as a tool helping them to learn something applicable in different contexts – to the detriment of mathematics learning. Errors are considered to be manifestations of students' lack of knowledge, or signals of failures. Problems are seen as tools to distinguish those who are successful from those who fail.

This traditional point of view is different from the natural working habits of a mathematician who has not only to solve problems and ask questions, but also to pose questions and problems (Silver, 1994). With respect to the theme of the conference, we focus on natural differentiation of the work in mathematics classroom: The division of working tasks is not as strict as usually and the students gain some of the responsibilities of the teachers, which, we think, is closer to the natural differentiation of the tasks in mathematics.

In our research, we operate in the opposite direction than is traditional. We re-centre teacher's and students' activity towards problems themselves. We look for tools suitable for development of a "culture of problems", for tools that would change learners' relationship to problems by giving them the opportunity to look at problems as something they can build and work on. Our main aim is to find out to which extent this approach can be beneficial for the student's ability to solve problems and to learn.

THEORETICAL BACKGROUND

In this article, we show an example of a set of situations where word problems are involved and where the tasks which are traditionally carried out by a teacher become "students' tasks". We particularly focus on the phenomena of *devolution* from the Theory of didactical situations in mathematics (further referred to as TDSM) (Brousseau, 1997) and their impact on teaching and learning of mathematics.

Let us begin by offering an outline of the concepts of devolution, a priori analysis and problem posing.

Devolution[2]

In TDSM, devolution is a process in which the teacher gives a part of his/her responsibilities for teaching to the student and puts the student in the position of an actor in an a-didactical situation[3].

> "Devolution. The process by which the teacher manages in a didactical situation to put the student in the position of being a simple actor in an a-didactical situation (of a non-didactical model). In doing so, he tries to set things up so that the actions of the student are produced and justified entirely by the necessities of the *milieu* and by her knowledge, and not by the interpretation of the didactical procedures of the teacher. For the teacher, devolution consists not only of proposing to the student a situation which should provoke in her an activity not previously agreed to, but also of seeing to it that she feels responsible for obtaining the proposed result, and that she accepts the idea that the solution depends only on the exercise of knowledge which she already has." (Brousseau, Sarrazy, 2002)

A priori analysis and devolution

According to Brousseau (1997), the a priori analysis is one of the tools that teacher can use when planning his/her lessons. Based on a lesson description

[2] Devolution was an act by which the king, by divine right, gave up power in order to confer it on a Chamber.

[3] "The student knows very well that the problem was chosen to help her acquire a new piece of knowledge, but she must also know that this knowledge is entirely justified by the internal logic of the situation and that she can construct it without appealing to didactical reasoning. Not only *can* she do it, but she *must* do it because she will have truly acquired this knowledge only when she is able to put it to use by herself in situations which she will come across outside any teaching context and in the absence of any intentional direction. Such a situation is called an *a-didactical situation*." (Brousseau, 1997, p. 64)

he/she tries to predict its progression/course: to propose lesson stages, to think about the pupils' as well as the teacher's possible reactions and behaviour (obstacles, errors, mistakes, their possible correction), to think about the solving strategies that the children will use (the correct as well as the wrong ones), to figure out what previous knowledge is necessary for each strategy. The a priori analysis serves as a substantial source of information for the teacher; it points at the possible difficulties in the course of the lesson and when solving a particular problem.

A good a priori analysis is a necessary condition for successful devolution and consequently for creation of an a-didactical situation. As we mention in the previous paragraph, the analysis gives the teacher an insight into the course of the lesson, children's reactions, possible difficulties etc. Therefore it will help the teacher to prepare better an a-didactical situation, a situation where children get the knowledge on their own.

However, restricting considerations only to a priori analysis limits understanding of events of the lesson. Charnay (2003) emphasizes the connection of a priori analysis with the a posteriori analysis that enables e.g. the interpretation of unexpected strategies and children's arguments, revelation of errors and mistakes they have made etc.

Problem posing

Two stages of the research were based on different problem-posing activities. Problem posing was identified as an important component of mathematical education (e.g. Silver, 1994). Problem posing activities can enhance students' problem solving skills (Lavy, Shriki, 2007). Bonotto (2006) affirms that during the process of problem posing, students analyze the data of the problem in order to distinguish the data which are important to solve the problem from other data; and they must discover the relations between the data as well. Situations of problem posing reduce students' dependence on teachers and textbooks. The autonomy of student work and the possibility to work on their own or in groups without any direct intervention of a teacher are likely to be favourable conditions for construction of their own knowledge. Moreover, problem posing situation makes students much more involved in the education process and this alternative position can motivate students to learn mathematics (e.g. Christou et al., 2005). Creative activities and students' self-expression are motivating factors in learning mathematics (Petty, 1993).

It is its motivational potential that makes problem posing a good opportunity to devolve some of the teacher's responsibilities on students, namely the choice or creation of problems that will be used in the education process. The organisation of the research described in the following part of the text is responsible for the fact that the posed problems are not used merely as a goal of the activities, but also as a tool for development of understanding of the structure of word problems and, consequently for enhancement of students' mathematical culture. In our research, two types of problem posing are used: creation of a problem based on a solution to a given problem (its mathematical model) and creation of a problem based on a real-life topic.

OUR RESEARCH

Research questions

The presented research is a part of a broader research focusing mainly on the question "What are the conditions that allow re-routing the teacher's and students' activity towards the living culture of taught problems and mathematical notions?" (Novotná, 2009).

In this article, we focus on organization of didactical situations which may change the attitudes of both students and teachers.

The main questions concerning students are: Are students able to pose problem assignments on a given topic? Are they able to look systematically for similar or different problems? Would these activities be an opportunity to develop epistemological considerations and heuristics, useful for students and teachers?

The main question concerning teachers is: What preparation and organization of didactical situations will have of positive influence on students' participation in the lessons and their motivation to learn mathematics?

Research design

The experiments were carried out several times, during three school years, in Czech and Slovak schools with 12-14-year old students. The implementation as well as the analysis and evaluation of the experiment were carried out in cooperation with secondary school teachers from Prague, the Czech Republic. The minimum of classes participating in the experiments in each year was three.

The experiments were prepared in the following steps: Designing didactical situations that would change learners' approaches to solving problems; implementation of the proposed didactical situations; analysis of the implementation and modifications of the project design based on the experiment results. Although the goals of our arrangements were defined clearly, the real organization of our experiment had to take into account several predictable obstacles linked with the didactical effects that are likely to appear

The final organization consisted of the following five stages.

The central question of **Stage 1** is to find out whether there is or is not a certain common accord among students from the same class about several non-mathematical criteria related to mathematical problems. Students evaluated five problems assigned by the teacher according to the following criteria: *Length of the text*, *Difficulty*, *Attraction*, *Usefulness*, *Comprehensibility*, and *Length of the resolution*.

Devolution in this stage is present when students become responsible evaluation of some problems. Usually, when a teacher assigns a problem to his/her students, they are asked to solve it, not to think about it in a non-mathematical perspective. When asking students to evaluate problems with respect to the six criteria, the teacher "officially" delegates his/her responsibility of selecting suitable problems to students. This new role in the analysis of problems may become a factor in changing students' attitude to them.

The aim of **Stage 2** is to draw attention of students to mathematical models of a concept. Simultaneously, students should gradually realize that it is possible to solve some problems "in the same way". The teacher prepares a problem with a simple mathematical model which is solved by the whole class. Students then work in groups in order to pose problems that can be solved in the same way as the problem just solved. In a whole class discussion, each group presents the problem they posed and other groups evaluate if this problem can be solved "in the same way" as the original problem or not.

The chance for students to pose their own problem is one of the manifestations of devolution. When posing problems, students take the typical role of the teacher and look for or create a problem which is suitable for a given situation or a teaching goal. There is a difference in the position of the teacher who knows his/her reasons for assigning the first problem (he/she chooses a problem suitable in the given situation) and the position of students whose task is to realize analogies in solving similar problems while posing a problem with the same mathematical model. An analysis of the problem structure carried out while posing a problem can help students to understand how a problem may be created and how the relations among the given data function. This new insight into structure of a problem can enhance students' attitude towards problem solving.

Stage 3 is designed to prepare an environment where the notion of mathematical model of a problem comes out in a natural and clear way. The teacher prepares several type-problems with various mathematical models and fifteen word problems, some of which belong to the mathematical model of one of the "prototypes" and some of which belong to none. Students are asked to group the fifteen word problems around the type-problems according to their mathematical similarity. This part of the activity is followed by a whole class discussion where groups present their groupings and justify their solution. In the discussion, similarity based on non-mathematical items (the same context, the same words etc.) should be rejected.

During this stage, students are guided to becoming familiar with the notion of mathematical model of a problem that emerge naturally from the situation and that is not supplied by the teacher. The teacher delegates to them not only the power to group the word problems as they wish. He/she also does not intervene in the final discussion of similarities and differences of word problems in the background of grouping around the type-problems. It is the students who are the arbiters, not the teacher.

The goal of **Stage 4** is to foster an environment favourable for reflection on mathematical solving procedures of problems, to allow students to discover problems of the same type, and to let them realize how understanding of these problems and their similarities can help when solving other problems. The teacher prepares six word problems from a mathematical domain that the students are familiar with. The students have to solve them and in groups they look for similarities among these six problems. The criteria of similarity are not given. In the following whole-class activity, they write down their groupings on the blackboard and for each grouping explain the used criteria. Then, the teacher selects one of the problems and asks the groups to choose from the remaining five problems such a problem that they would offer as an aid to somebody who does not know how to solve the problem selected by the teacher. The choices made by the groups are subject to a follow-up whole-class discussion. This concluding part of Stage 4 helps students to move their attention from general similarities to mathematical similarities of the given problems.

In this stage, devolution of the teacher's responsibilities is not as striking as in the other stages. Nevertheless, the learning environment can help the students modify their approach to word problems by using a different way of their analysis. Students move from general characteristics of problems that are not important for their solution to mathematical characteristics that can help them see the problems in a different light and consequently solve them. The teacher does not introduce the notion of mathematical model of a word problem, students gradually discover it themselves. The teacher limits himself/herself to pointing out certain criteria without commenting on their relevance and validity. The experiences gained during the previous stages are a valuable hint here.

Stage 5 is designed to summarize and precise the knowledge acquired by students during the previous stages. It is conceived as a "competition of originality". Students are asked to pose word problems with a given non-mathematical topic (problems from a supermarket etc.). Each group submits one of the word problems that the members have posed to the whole-class "competition for an original problem". The activity is divided into two lessons. During the first lesson, students in groups compare the set of submitted problems and look for similarities in their solving procedures. The goal is to decide if their problem is useful for the solution of one or more other problems from the set. If this is the case, they can pose a new problem and replace their original one.

During the second lesson of this stage, each group presents their problem and its solving procedure to the whole class and gives evidence that their problem cannot help to solve any another originally submitted problem. Other students may discuss it and react to it.

As in Stage 2, students have to pose problems but this time in a different environment. In Stage 2, there were very strict rules on how to pose the problems (problems with the same mathematical model as the given problem). The situation in Stage 5 is much more open – the only constraint given by the teacher is the topic, other restrictions arise from the situation and from the actions of other students.

Devolution as factor enhancing students' motivation for learning mathematics

In the above description of the individual stages we have also pointed out the moments and activities of devolution. What follows now is a summary of the key moments of devolution of the teacher's responsibilities during the stages:

Problem posing by the students in order to analyse their mathematical model and search for (mathematical) similarities and differences among the problems

evaluation of problems created or prepared by the teacher and especially of those posed by other students

grouping of problems according to students' own criteria or criteria given by the teacher

participation on creation of a learning environment via problem posing

These manifestations of devolution can enhance students' motivation for learning mathematics. In (Lokšová, Lokša, 1999) the following methods for developing students' motivation are mentioned: creation of a learning environment based on students' activity, variability of teaching methods, possibility for students to explain their opinions, respect of student's personality, and responsibility of students for the learning environment and results of their work. The analysis of the five stages presented in 3.2 allows us to conclude that these methods were used in the classroom activities realised during the research.

Success of devolution in the classroom depends on several didactical conditions. One of these conditions is that the teacher must make a good analysis a priori in order to predict possible consequences of devolving some of his or her duties on students. However, even a thorough analysis a priori is sometimes not sufficient for successful devolution with expected results. An example from our research is the concept of *mathematical model of a word problem*: Although we prepared a classification of word problems posed by the students based on the criteria of similarities/differences of their mathematical model, students showed that our analysis was not sufficient; they gave the model other meaning by simplification of problem analysis to mere question of presence/absence of basic arithmetic operations.

CONCLUDING REMARKS

In order to get the feedback from students, they were given an evaluation questionnaire. Reactions of students from these questionnaires show that the set of activities described above was appreciated. Students enjoyed the possibility to look at word problems in a different way, to pose their own problems, to analyze word problems posed by other students and to discuss the problems, their characteristics and methods of solution.

Devolution of responsibility for creation of the learning environment seems to be a good tool for student motivation to active participation during the lesson and to use of prerequisite knowledge when asked to solve a new problem. Moreover, the possibility to participate in the discussion, to evaluate problems posed by other groups and to explain their own opinions on the problems had positive influence on the performance of some students who usually do not actively participate in "normal" lessons of mathematics.

The teachers who participated in our experiments confirm that the devolution phenomena presented above make the students more engaged in their mathematical education, contribute to an increase of their independence during the lessons and make them move from passively copying algorithms presented by the teacher to an active search for a suitable mathematical model for the assigned problem.

The research raises further questions about devolution and its effects on learning mathematics: What are the didactical conditions that safeguard positive impact of devolution on learning mathematics? Which responsibilities lend themselves for devolution on students? Under which circumstances?

REFERENCES

Bonotto, C.: 2006, Extending students' understanding of decimal numbers via realistic mathematical modeling and problem posing, in: Novotná, J., Moraová, H., Krátká, M., & Stehlíková, N. (eds.), *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2.), Prague, pp. 33-40.

Brousseau, G.: 1997, *Theory of Didactical situations in mathematics 1970-1990,* Kluwer Academic Publishers, Dordrecht.

Brousseau, G., & Sarrazy, B.: 2002, *Glossaire de quelques concepts de la théorie des situations didactiques en mathématiques,* DAEST, Universit Bordeaux 2, Bordeaux, (English translation by V. Warfield).

Bureš, J., & Hrabáková, H.: 2008, Création d'énoncés de problèmes par les élèves, in: *Actes du XXXVe colloque COPIRELEM*, Bordeaux.

Charnay, R. : 2003, L'analyse a priori, un outil pour l'enseignant, in : L. Grugnetti, F. Jaquet, P. Medici, & M. G. Rinaldi (Eds.), Actes des journées d'étude sur le

Rally mathématique transalpin, RMT: potentialités pour la classe et la formation, ARMT, Dip. di Mat. Università di Parma, Dip. Mat. Università di Cagliari, pp. 199-213.

Lavy, I., & Shriki, A.: 2007, Problem Posing as a Means for developing mathematical knowledge of prospective teachers, in: J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo, (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3), Seoul, pp. 129-136.

Lokšová, I., & Lokša, J.: 1999, *Pozornost, motivace, relaxace a tvořivost dětí ve škole*, Portál, Praha. [Children's attention, motivation, relaxation and creativity in school] (In Czech).

Novotná, J. : 2009, Contribution à l'étude de la culture scolaire. Cas de la résolution de problèmes dans l'enseignement des mathématiques, in: L. Poirier (Ed.), *Proceedings CIEAEM 61*, (in print).

Petty, G.: 1993, *Teaching today*, Stanley Thornes (Publishers) ltd. Cheltenham. Czech translation: Petty, G.: 2002, *Moderní vyučování*, Portál, Praha.

Silver, E.A.: 1994, On Mathematical Problem Posing, *For the Learning of Mathematics*, Vol. 14, no. 1, pp. 19-27.

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TEACHERS' BEST PRACTICES USING DIFFERENTIATION[4]

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Some theoretical reflections about differentiation originate from the analysis of protocols realized during an experiment involving a treatment of arithmetic in grade 1, on the basis of a semantic environment designed by Hejný et al. (2006). This environment together with the teacher's practice drives naturally towards differentiation by outcome and its acceptance by the pupils as a motivating tool.

DIFFERENCES – DIFFERENTIATION IN ITALY

A grade 1 primary school class reflects Italian society as a whole in that many kinds of differences are present which result from children's previous attendance at kindergarten as well as their family background and culture. Nowadays the presence of pupils coming from abroad without knowledge of the Italian language makes the situation much more complex. Therefore, each school topic assumes a double role: it builds up the child's personal knowledge and also creates a social space in which all the children can learn. Another feature of the Italian elementary school system that is significant is that the teacher remains with his or her class from grade 1 to grade 5.

Although grades 1 and 2 are mostly devoted to the development of this common classroom culture, Italian official documents about primary school disregard this feature. In other European countries, more attention is instead devoted to differentiation in order to enhance motivation and to support the development of a common classroom culture.

For the practitioner of grades 1 and 2, this task is very demanding: it is not simple to facilitate children in developing a common understanding of ways to work together as a group and at the same time help them to develop new mathematical knowledge. Mathematics teachers in these grades can be tempted to focus on giving children one idea of numbers, operations and other mathematical concepts. In this way they fulfil their aim of introducing the children to the necessary mathematical knowledge. However, the individuality and cognitive style of each child is not accounted for. The present Italian primary 'curriculum' [5] might suggest a similar attitude, which is frequently

[4] Work done in the sphere of Italian National Research Project Prin 2008PBBWNT at the Local Research Unit into Mathematics Education, Parma University, Italy

[5] At the present time Italian primary schools do not have an official curriculum, but only some directions, MPI (2007), since for both the school and the teacher the leitmotiv is 'autonomy'.

utilized by newly qualified teachers. This is also evident in the literature for schools: the possibility and value of differentiation or taking account of individual children's needs and prior knowledge are not mentioned in textbooks and handbooks for teachers. The younger practitioner is indirectly driven towards an emphasis on delivering the content of the curriculum in one way. The risk of such a one-dimensional treatment of mathematics, whose aim is to minimize individual differences in learning and understanding, is developing in children an early dislike of mathematics. This can be aggravated when the children's own beliefs contrast with the teacher's approach to the subject.

To overcome these difficulties requires an approach based on social constructivism. In this approach, individual differences are seen as a resource since they give the opportunity to the teacher of debating the pupils' various interpretations of a same task. This helps to elucidate from the children different interpretations of the meanings of the tasks and enables the teacher to celebrate differences between pupils and teach according to pupils' needs, providing them with individual support.

We would suggest that more attention to differentiation and supporting the individual needs of learners should begin in the education of practitioners and continue in the professional development provided for in-service teachers. This paper aims to convince Italian teachers that differentiation is possible and will provide a way of enhancing pupils' enjoyment of and attitudes to mathematics.

A THEORETICAL SCENARIO

A suitable theoretical scenario is required to demonstrate that differentiation is valuable in mathematics teaching. In this paper we present a wider interpretation of differentiation, which takes into account the pupil's point of view. Differentiation may be seen from the teacher's point of view as a suitable way of enhancing pupils' motivation towards mathematics by using the same rich starting point for the whole class. This starting point must be suitable for all the pupils whilst at the same time offering opportunities for individual children to respond at their own level.

The search for such starting points is difficult and time consuming and could reduce the time available for the practitioner's reflection on her/his own practice. There is evidence (Schön, 1983 and 1987; Lerman, 1990; Zan, 2003; Hošpesová & Tichá, 2004), that reflection about practice is a way of improving teaching. On the basis of the statements of some authors (Zan, 2000; D'Amore, 2008; Marchini & Cockburn, 2008) the teacher's attitude is a first and necessary step for taking care of pupils' meta-cognitive development: without reflection by the teacher on his or her practice the pupils will not develop the capacity to reflect on their own learning.

The suggested emphasis on considering the different responses that children may make to a problem can be resisted by the children: in every classroom we can find children who hurry to get 'the' answer and who do exercises in a great hurry. So we need to show children the value of paying attention to differences in the ways of thinking of their classmates.

The possibility of adopting different points of view about the same mathematical topic is a resource for differentiation but in itself is not enough for producing good effects in the classroom. Pupils must develop their personal knowledge and be also aware of their results. At the same time, each pupil must accept that others in the class can have a dissimilar approach to the same topic. Such an environment calls for personal reflection, as a piece of hidden curriculum (Silver, 1987), which can be a difficult part of the learning process. Thus each child should have opportunities to enhance her/his potential by means of a substantive comparison with the work of peers as a way of elaborating her/his own personal ideas about mathematical topics. In our opinion, the task of differentiation from the pupil's point of view is to allow pupils the opportunity to express their personal ideas about mathematics.

Word	**Picture**	**Icon**	**Letter**
Topo			**T**
Gatto			**G**
Oca			**O**
cane			**C**

Table 1

Hejný's (2008) scheme-oriented strategy with its arithmetical environments semantically anchored (Hejný *et al.*, 2009) seems a good theoretical starting point for helping learners in producing their personal paths for developing their beliefs about mathematical concepts. Possibilities for differentiation are evident in the *Father Woodland* proposal, a semantic environment for arithmetic presented in Hejný *et al.* (2006), which is a substantial learning environment (Wittmann, 2001). At the same time differentiation, in the wider meaning outlined above, is a good setting for approaching mathematical concepts through a cognitive transversal competence namely the conversion of representations from one semiotic register to another (Duval, 1993).

THE EXPERIMENT

During school year 2007 – 2008, in collaboration with a team of three teacher-researchers from the primary school of Viadana (Italy) (Guastalla, Previdi and Santelli) we organised an experimental activity based on *Father Woodland*, the semantic environment proposed by Hejný *et al*. (2006) (Table 1 is reproduced from it, translating animal names into Italian).

A feature of this environment is that the same object (an animal) can be represented in many semiotic registers (Duval, 1993): with its name, with a drawing, with an icon and with a letter. Thus pupils have many possible different ways for representing the same arithmetic facts. In our experiment they added spontaneously two other new ways to these: numbers and rods.

The whole teaching experiment in which we engaged aimed at verifying whether this learning environment was suitable for developing the mathematical concepts and competences intended (Hejný *et al*. 2006). The goals of the project were for the children to develop:

1. early number sense
2. understanding of the difference between a quantity (expressed in units) and a number (expressed in pieces)
3. pre-concept of equations
4. pre-concept of divisibility, the lowest common multiple and greatest common divisor
5. conceptual thinking beyond the elementary level
6. methods for solving linear equations.

The whole activity and its realization was inspired by a socio-constructivist approach, which showed great respect of each pupil's way of thinking and the opportunity for them to share their results and discuss them in the process of agreeing a shared answer. The experiment took place in two grade 1 classes (A and B, with 21 + 18 pupils) and lasted the second semester of the school year 2008 – 2009. Pupils faced this environment after the first semester in which reading and writing was their main commitment. They were also introduced to arithmetic with different representations of the natural numbers from 0 to 10.

From evidence gained from our analysis of a wide collection of protocols, we are convinced that our experiment succeeded for the goals 1 – 3 and 5; some of the other goals are being investigated during the school year 2009-2010 in grade 2. However this is not the focus of this paper.

PRESENCE OF DIFFERENTIATION IN THE EXPERIMENT

In this paper we present here a very small segment of the data collected: a question sheet (Sheet 1) which was given in class A to the whole class-group. We also offer an analysis of some pupils' answers to Sheet 1, and a protocol from class B, relative to another task. These documents were singled out since they show both the teacher's practice and the presence of differentiation from the pupils' points of view. They serve to illustrate the potential benefit of differentiation by outcome through the use of a rich learning environment.

IF YOU WANT TO PLAY WITH THEM
YOU MUST FIND HOW MUCH THEY EAT.
YOU MUST WORK AS A DETECTIVE
BUT…I CAN DROP YOU A HINT!

- EXAMINE CAREFULLY THE ORDERED SEQUENCE OF ANIMALS.

- FILL THE FORM WITH ANIMAL NAME:

EATS AS MUCH AS 2 **TOPI** (mice)

EATS AS MUCH AS 2 **GATTI** (cats)

EATS AS MUCH AS 4 ………

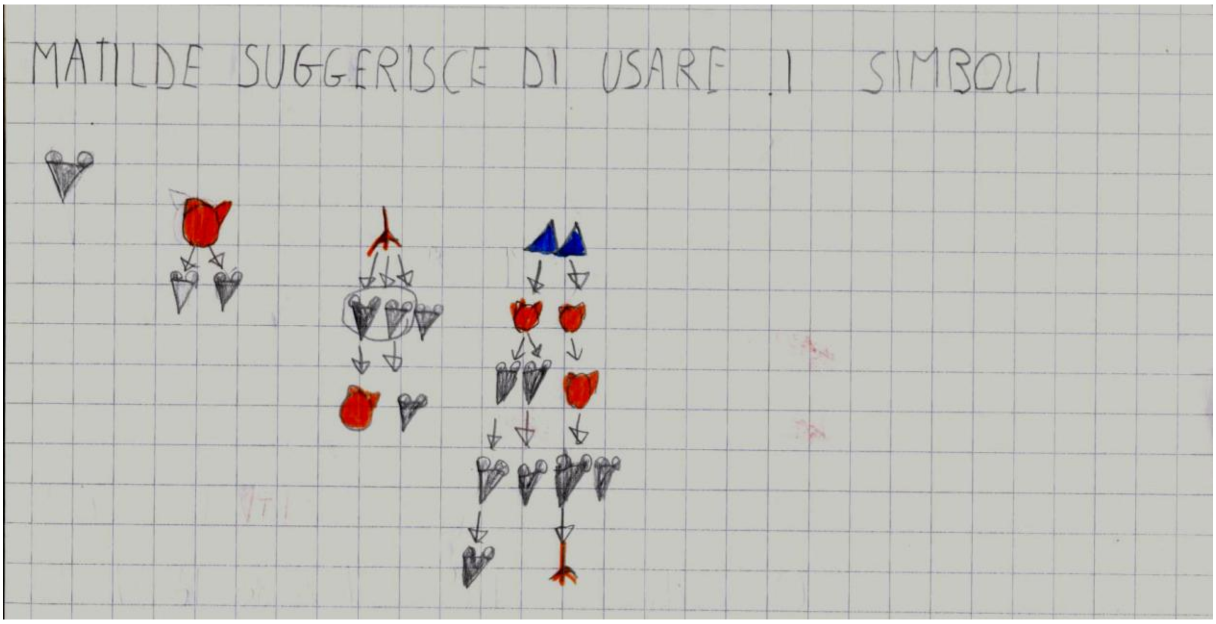
EATS AS MUCH AS 1 ……….AND 1 ………..

EATS AS MUCH AS 3………………..………..

EATS AS MUCH AS 1…… AND 1 …....….

Sheet 1

Sheet 1 is an original problem proposed by Guastalla and freely inspired from Hejnỳ *et al.* (2006). We present it here in a 'half-translated' form in order to support the reader in developing a better understanding of pupils' answers (in Italian) presented below. Pupils discussed this sheet a lot and solved it. Class discussion was necessary since, from a mathematical point of view, the explicit (algebraic) data of Sheet 1 are not sufficient for solving the task. The first question regarding dog was solved in simple ways; the second cannot be solved immediately, since its solution depends on solutions to the goose's questions. The first requirements regarding the goose can be fulfilled in different ways. The second one depends on the solution of the first. During the phase of whole class discussion, pupils found a possible solution to all the questions, exploiting the information that animals are given in a ordered sequence. The teacher asked them how to represent the problem solutions and all the children copied the proposal showed on the blackboard into their notebooks (Protocols 1- 3). On another occasion the teacher also asked them to represent individually the solution as they preferred (protocol 4), and after that there was another general

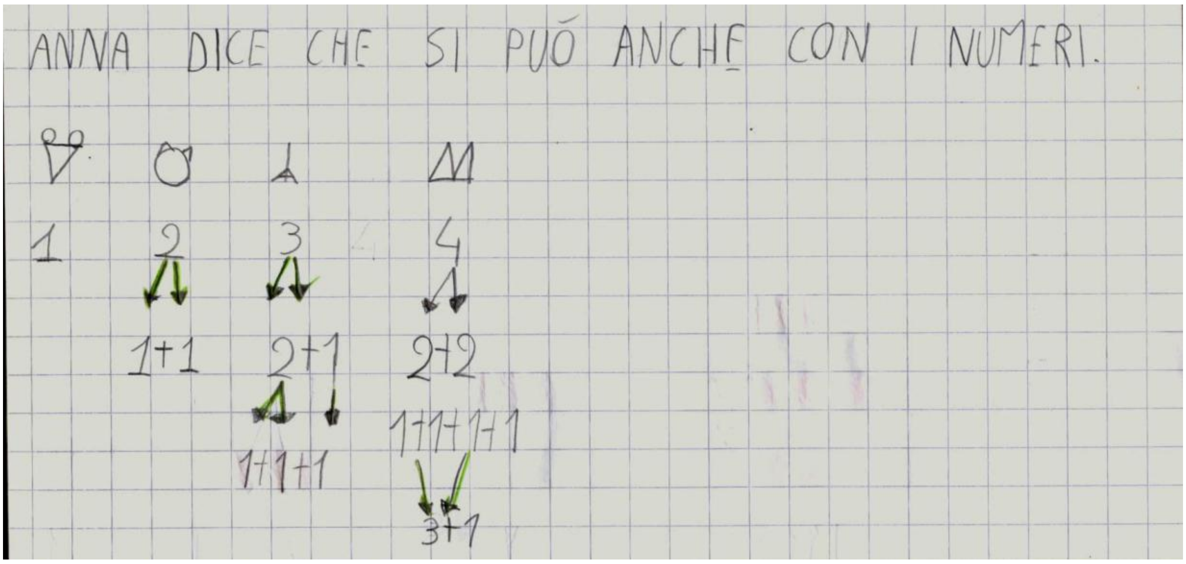


Protocol 1: Matilde suggests to use symbols (icons)

discussion.

Protocol 1 shows Matilde presenting only icons. In it the small circle around two icons of mouse and the arrow connecting this circle with the icon of cat is a trace of an early awareness of the process of substitution. The unique arrow connecting one icon of cat with two of mouse and three icons of mouse and one icon on goose can be interpreted in the same way. The relevance of these substitutions for the learning are suggested in Marchini (2002) and Marchini & Kaslova (2003). The expression of icons with greater 'value' in many ways has a combinatorial aspect which illustrates the child's understanding of the relations involved between the mouse, cat, goose and dog.

Protocol 2 is made with icons, numbers and addition. Anna adopted a semiotic register different from the ones which the environment provides, and the first step is a conversion of icons to numbers. The second is the introduction of addition for expressing numbers by additive decomposition, but it is also a conversion (symbolization) of the conjunction 'and' of the linguistic register in

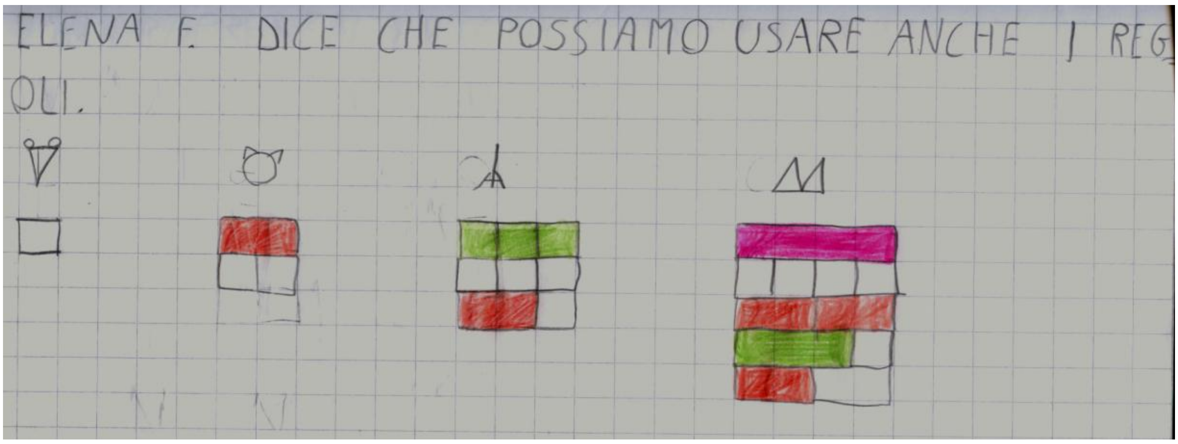


Protocol 2: Anna says that we can use numbers

addition. This competence is essential for solving word problems. In this case the arrangement of the arrows suggests the associative property of addition.

Protocol 3 presents another unexpected representation tool. The combinatorial aspect here is strongly evident.

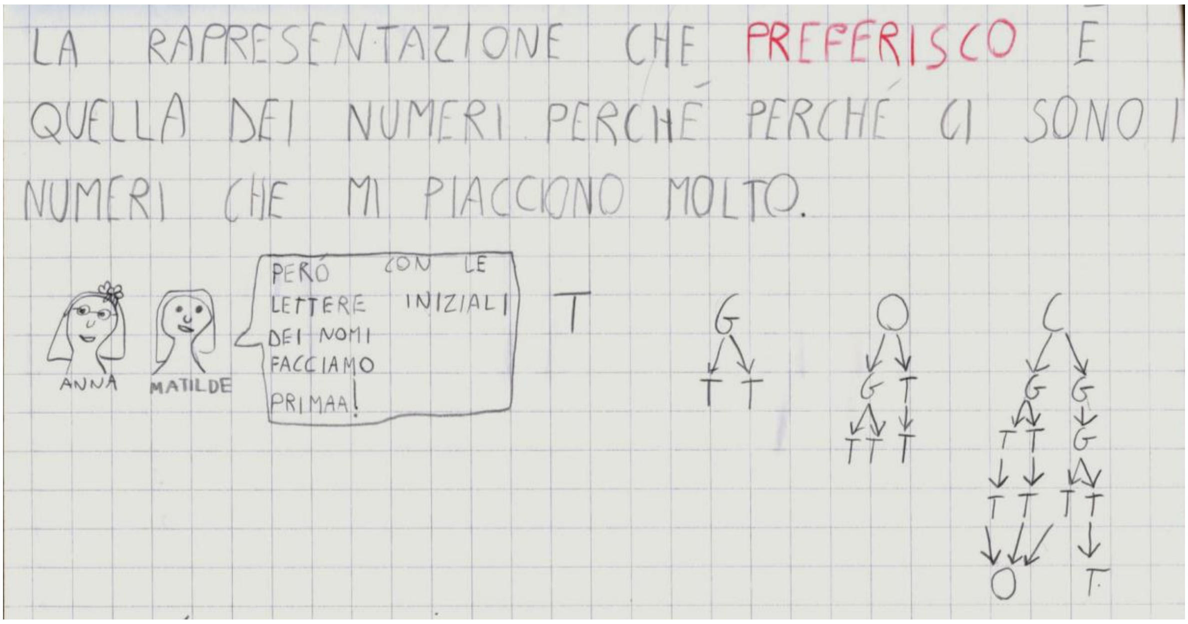
The first part of protocol 4, regarding the preferred representation, is made only by this child, but the second with the balloon was the same for all children in the class as the teacher decided that it was a worthwhile statement. Therefore the protocol presents two conversions from the original semiotic register (drawing) used in Sheet 1 to the registers of numbers and of letters. Protocol 4 is an example of a child's personal conclusion about the kind of representation s/he preferred. Each pupil wrote her/his preference in the notebook.



Protocol 3: Elena F. says that we can use (Gattegno-Cuisenaire)

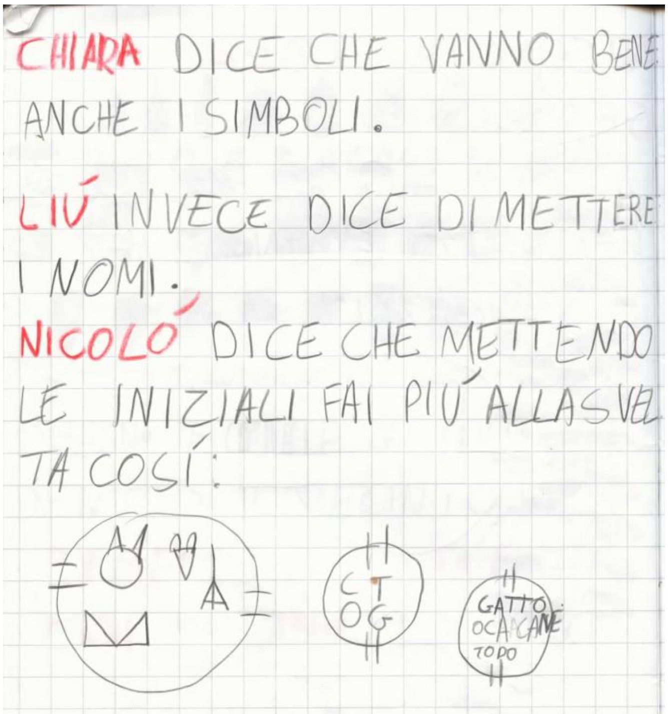
The results of class A for the task exemplified in Protocol 4 are:

Number of pupils	20		
Use of icons	4	Use of rods	4
Use of (only) numbers	1	Unintelligible	3
Use of numbers and addition	8		



Protocol 4: My preferred representation is made by numbers since numbers please me the most. In the balloon, Matilde states: But with initials we do it faster.

The same pupil might adopt two diverse approaches: Matilde in the class discussion remarked that the use of letters makes the task shorter (Protocol 4), but in her protocol she stated her preference for icons.



Protocol 5: Chiara says that symbols work well. Liù instead puts in nouns. Nicolò affirms that putting in initials speeds you up.

These protocols illustrate the presence in the same class at the same time of different ways of looking at the same subject and these ways are rooted in an affective ground since pupils expressed their preference with verbs as ‘to like’, ‘to please’, ‘to enjoy’, ‘to be easy’ or by quoting ‘colours’, ‘animals’, denoting a positive attitude towards mathematics (Zan, 2000).

In Class B (18 pupils) Sheet 1 did not generate a similar discussion, but the possibility of using different representations was given by other tasks. Protocol 5, of Lucrezia, is an example from class B, showing the result of discussion followed by the girl’s personal ‘symbolic’ interpretation of her schoolmates’ statements.

After having pointed out these particular instances of difference, the teacher assisted pupils in accepting these differences. The speculative background is supplied by the semiotic registers theory of Duval (1993). He suggests that there cannot be understanding of mathematical concepts without mastering their representations in different semiotic registers. The many ways for representing the same mathematical concept together with *treatment* inside a register and *conversion* between registers facilitate pupils' understanding and the construction of the concepts (Duval 1993).The *Father Woodland* environment drives naturally towards these *treatment* and *conversions* and one of the teachers drew attention to the multiple outcomes that this learning environment elicited from her pupils. The pupils felt the teacher's respect for their intuitions and this gave them the possibility of showing unexpected deep and relevant anticipations of the main mathematical concepts and procedures. In a less open learning environment this freedom of invention is forbidden and so opportunities for differentiation are limited.

CONCLUSION

Differentiation asks for a new teaching practice to support the development of children's learning competences by creating the opportunity for students to experience success in mathematics. In our paper we have outlined a model of good practice that builds on differentiation. We suggest the attention paid by the teacher to the meta-cognitive development of children should be based on the practitioner's personal reflection about her teaching practice, but that this needs to be coupled with the choice of suitable activities. These activities must drive naturally to exploit the richness of differentiation by offering opportunities for treating mathematical concepts within a given register as well as converting them between registers, along the lines of the theory espoused by Duval. At the same time the mathematical background of the activity must allow a suitable development of mathematical thinking, so that attention can be paid to connections between crucial mathematical ideas.

The value given by the teacher to differences exhibited in pupils' protocols enables pupils to accept their own and their classmates' points of view as natural ways of expressing mathematical facts and enriches their understandings of the concepts. This teacher's attitude assures each child of her/his possibility to express something interesting in the mathematics and so enhances their motivation as well as their enjoyment of the subject. Our experiment conducted in school year 2008 – 2009 showed that this approach to differentiation was both possible and valuable.

REFERENCES

D'Amore B.: 2008, Epistemology, didactics of mathematics and teaching practices, *Mediterranean Journal for Research in Mathematics Education,* vol. 7, 1, pp. 1–22.

Duval, R.: 1993, Registres de représentation sémiotique et fonctionnement cognitif de la pensée, *Annales de Sciences Cognitives et Didactique*, vol. 5, IREM de Strasbourg.

Hejný, M.: 2008, Scheme-oriented educational strategy in mathematics, in: B. Maj, M. Pytlak, E. Swoboda (Eds.), *Supporting independent thinking through mathematical education*, Warsaw: Nowa Era, pp. 40 - 48.

Hejný, M., Jirotková, D., Kratochvílová, J.: 2006, Early conceptual thinking, in: J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings 30th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3), Prague: Program Committee, pp. 289-296.

Hejný, M., Jirotková D., Marchini, C., Vighi, P., Kmetič S. & Harminc. M.: 2009, Arithmetical Environments Semantically Anchored, in: *SEMT '09. Praga, 23 - 28 Agosto 2009*, Prague: Charles University - Faculty of Education, pp. 273-274.

Hošpesová, A. & Tichá, M.: 2004, Self reflection and improvement of IALS International Adult Literacy Survey 2000, *Literacy in the Information Age. Final Report of the International Adult Literacy Survey*, Canada: Statistics Canada.

Lerman, S.: 1990, The role of research in the practice of mathematics education, *For the Learning of Mathematics*, **10**, 2, pp. 25-28.

Marchini, C.: 2002, Instruments to detect variables in primary school, *Proc. CERME 2*, vol. 1, pp. 47-57.

Marchini, C. & Cockburn, A.: 2008, Teaching practices revealed through arithmetic misconceptions, in: B. Maj, M. Pytlak, E. Swoboda (Eds.), *Supporting independent thinking through mathematical education*, Warsaw, Nowa Era, pp. 145-151.

Marchini, C., Kaslová, M.: 2003, Substitutions and variables in primary school, A comparative study on pre-conceptions, Proc. SEMT ’03, pp.113 – 117.

MPI: 2007, *Indicazioni per il Curricolo per la scuola dell’infanzia e per il primo ciclo d’istruzione*, Roma: Ministero della Pubblica Istruzione.

Schön, D.: 1983, *The Reflexive Practitioner*, New York: Basic Books.

Schön, D.: 1987, *Educating the Reflexive Practitioner*, Oxford: Jossey-Bass.

Silver, E.: 1987, Foundations of Cognitive Theory and Research for Mathematics Problem Solving Instruction, in: Schoenfeld, A. (Ed.) *Cognitive Science and Mathematic Education*. Hillsdale: Lawrence Erlbaum Associates.

Wittmann, E.: 2001, Developing Mathematics Education in a systemic Process, *Educational Studies in Mathematics*, **48**, pp. 1 – 20.

Zan, R.: 2000, Atteggiamenti e difficoltà in matematica, *L'Insegnamento della Matematica e delle Scienze Integrate*, v. 23A, n. 5, pp. 441-465.

Zan, R.: 2003, Formazione insegnanti e ricerca didattica, *La Matematica e la sua Didattica*, **4**, pp. 541-570.

CONDUCT OF MALE TEACHERS OF MATHEMATICS IN THE PERCEPTION OF FEMALE AND MALE PUPILS A LOWER SECONDARY SCHOOL PERSPECTIVE

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The purpose of this article is to determine the behaviour of male teachers of mathematics as perceived by Polish male and female pupils in lower secondary school. The results obtained suggest that female pupils, relatively to male students, assess input and output in a less favourable manner.

Female lower secondary school pupils taught by male teachers give a lower grade to the actual course of the lesson. In the perception of female pupils (as compared to male pupils), a lesson of mathematics becomes less effective and less frequently used to enhancing abilities of all pupils. Maybe the belief that "mathematics is a male domain" works both as described for stereotype (a teaching point of view) and autostereotype (point of view of female pupils).

AIM OF THE STUDY

This report is part of extensive research aimed at diagnosing the conduct of teachers of mathematics in the perception of male and female pupils. At the core of this research lie gender-related stereotypes present in education. The stereotype under analysis takes the form of a commonly accepted thesis whereby mathematics is the domain of males. Such an assumption may give rise to different expectations of teachers of mathematics towards pupils of the opposite gender. These expectations may, in turn, diversify the conduct of teachers, in line with the self-fulfilling prophecy in education.

In Poland the matter in question has not as yet been analysed within the terms of reference of research psychology in a manner proposed in this paper, i.e. a student perception analysis. The authors' studies, have been conducted for over a year, and the results obtained so far with respect to secondary school students indicate that the record of educational experience during mathematics lessons varies depending on student gender. Interestingly, this phenomenon occurs irrespective of the gender of the teacher. The following paper presents results of studies on lower secondary school pupils who assessed male teachers, since the stereotype in question, as relevant literature confirms, is strongly manifested among males.

REFERENCES TO RELEVANT PUBLICATIONS

A thesis whereby boys are superior to girls in terms of mathematical skills appeared in a seminal article by Anastasi (1958) and was duly confirmed in the studies conducted by Tyler (1965) and Maccoby and Jacklin (1974). Numerous attempts have been made to explain this phenomenon, which may be attributed to two main explanatory positions: biological and psycho-social. The supporters of the first school of thought emphasise a structural distinction between the mind of females and of males (Halpern, 1986; Moir and Jessel, 1989). Nevertheless, biology-oriented researchers have failed to account for the fact that the explicit superiority of mathematical achievement of boys relevant to girls, as disclosed in the 1960s, continue to disappear gradually during the following decades (Hyde, Fennema, Lamon, 1990). This is a strong argument in the hands of the supporters of the psycho-social approach, who point to the underlining role of stereotypes concerning learning abilities and the nature of educational activities subordinated to these stereotypes (Li, 1999) in contributing to the gender differences in mathematics achievement. One simply cannot ignore a glut of articles suggesting in unison that higher achievements of males in mathematical tests do not appear until after a few years of regular schooling peaking on the high school level (Hyde, Fennema and Lamon, 1990). Arguably, the educational system has a certain gloomy flaw which, rather than eradicating gender-related stereotypes concerning mental capacities of men and women, does exactly the opposite.

It is a popular stereotype in western countries that mathematics is a male domain (Gavin and Reis, 2003; Leder at al., 1996; Tiedemann, 2002), and this stereotype is common knowledge to all participants of the educational process: teachers, pupils, and their parents alike. This stereotype determines the perception of children’s competence by their parents attributing success to daughters’ efforts and to sons’ mathematical abilities (Eccles, 1993) and, by extension, shaping a more favourable perception of their competence by average male pupils than female pupils.

More importantly though, this stereotype has an effect on the gender-related expectations of mathematics teachers with respect to their pupils (Leder et al., 1996; Sadker et al., 1991). These expectations are the root cause of different approaches employed by teachers, which has been the subject of numerous analyses (Kimball, 1989; Jussim and Eccles, 1992; Li, 1999). Besides the diagnosis of the “educational state of affairs”, these studies fostered greater awareness among teachers of the discriminatory nature of this stereotype towards female pupils (Gavin and Reis, 2003).

Summarising 30 years of research on teaching mathematics, Leder et al. (1996) claim that the stereotype is most visible among English-speaking nations. The effect of the stereotypic assumptions of German teachers of mathematics on the perception of male and female pupils is presented by Tiedemann (2002).

Gibbons (2000) asserts that the evidence in support of pan-cultural similarities as regards gender stereotypes far outweighs cultural diversity. Therefore, it can be safely argued that specific Polish educational activities are also effected by the gender of those to whom they are addressed.

This assumption is reinforced by the contents of school handbooks, notably those used in primary schools. If there are problems with mathematics, they are invariably experience by a girl who turns for help to her older brother or father (Mazurkiewicz, 2006). Non government organisations such as (Partners Poland or PREMA POLSKA) sound a warning about Polish teachers failing to promote a conviction among their students that the achievements in sciences and in technology should be credited to scientists of both genders. Apparently, the only female scientist these teachers are aware of is Marie Curie. Treated more like a monument than anything, Marie Curie cannot function as a role model for female students (Iłowiecka-Tańska, 2008).

Interestingly enough, Polish empirical psychology does not report such situations at all. The only psychological studies conducted in Poland refer to the effects of gender stereotypes that are different from the way in which pupil perception is shaped. Such studies focus on the effects of the stereotype on the level of performance in mathematical tests or, in a broader sense, in tasks which require logical thinking. Inspired by the classic experiment by Steele and Aronson (1995), the studies shows that Polish female pupils have more problems – relative to male pupils – in solving mathematical tests when the instructions given constitute a threat of gender stereotyping ("The study, which constitutes a part of extensive international research, focuses on the ability to think logically. The results obtained so far indicate that males score better than females" Bedyńska, 2009; Babiuch-Hall, 2007).

It seems that with the activation of the gender stereotype cognitive capacity required to process difficult tasks is reduced. Such a situation can be explained by a hypothesis which assumes that by suppressing the stereotypic content female cognition acquires an extra burden to deal with. Consequently, cognitive capacity to handle a given task proves insufficient. Following up on the suggestion made by Steele and Aronson (1995), this is how females pay for avoiding to be described in the terms which are invariably linked with the stereotypic perception of their gender. The results of the studies described indicate in the first place that the stereotype whereby mathematics is a male domain is strongly rooted in Poland and is common knowledge, also among females, against whom it is addressed. Thus, these explorations refer to the level of the diagnosis and to the symptoms of the phenomenon under examination. Necessary as they are, they are nevertheless insufficient.

Mathematics seen as a "critical filter" (Sells, 1973) holds sway at prestigious technical universities (mechanical engineering, and electronics) where women account for approximately 10 per cent of all students (Tymowski, 2008). "Girls for Universities of Technology" campaigns and similar (modelled after the German *Maedchenzukunftstag*), will remain futile unless a decent diagnosis of gender gap generating educational activities in mathematics is performed and a program of counteracting measures is implemented. This is how the goal of our studies presented in this article is defined.

The self-fulfilling prophecy in education (Rosenthal and Jacobson, 1968) constitutes a theoretical basis for the studies presented.

Following up on the above, it was assumed that gender dependent stereotypes which determined teacher expectations activate the Pygmalion effects (the effect of a conviction that we deal with a good pupil) towards male students and Golem effects (the effect of a conviction that we deal with a poor pupil) towards female pupils (Madon, Jussim and Eccles, 1997). In order to analyse the mechanism of the effects of different expectations on genuine pupil achievements, Good's six-stage model (1980) was employed:

1. The teacher formulates different expectations concerning educational achievement of individual pupil.
2. Different convictions of the teacher leads to a diverse course of action, albeit in compliance with the teacher's own expectations, towards individual pupil.
3. Such a diverse approach of the teacher communicates to all pupils what type of achievement and/or conduct the teacher expect of them.
4. The teacher's conduct begins to affect the pupil' self-assessment, degree of their motivation, their level of aspirations, commitment in class and interaction with the teacher.
5. The said aspects and the aptitudes of pupils are generally in compliance with teacher expectations, which are subsequently reinforced.
6. In the last analysis disparate teacher expectations cause different, though consistent with these expectations, pupil achievements and ways of behaviour only to perpetuate the self-fulfilling prophecy in education.

For the purpose of this article, the two initial stages of the model are of particular importance. It was assumed that the essence of the first stage is described by the Pygmalion and Golem effects. The contents of the second stage become more precise subject to the operationalisation of the diverse conduct of the teacher in class. To this purpose the Four–Factor Theory by Rosenthal (1973) has been evoked. These factors determine:

- atmosphere during lessons created by means of verbal and non-verbal communication (*climate*),
- quality of feedback offered to pupils, the way teachers address questions and queries raised (*feedback*),

- degree of difficulty of the tasks provided and characteristics of gratification (*input*),
- degree of activating pupils in class (*output*).

The studies conducted also accounted for the gender of the teacher of mathematics as a potential modifier of behaviour towards male and female pupils. Relevant literature indicates that it is men rather than women who cling to the stereotypic perception of mathematics as their own domain (Tiedemann, 2002). It is therefore right to assume that the behaviour of male teachers increases the chances to sustain the largely distorting Pygmalion and Golem effects.

Numerous studies have been conducted with due respect for the fundamental premise of educational psychology, i.e. the relation between teachers' expectations and their behaviour towards pupils during mathematics lessons. This paradigm came down to measuring the degree of Rosenthal's factors in teachers' behaviour by outside observers (Kimball, 1989; Mazurkiewicz, 2006 – the only Polish study). To this end, a coding procedure developed by Brophy and Good (1969) was employed along with its modified versions. One such study confirmed that male teachers of mathematics spend more time on explaining and instructing male pupils in a mixed gender class than female pupils. During the entire school year the difference totals six hours (Leinhardt et al., 1979).

Apart from the fact that the coding procedure usually puts an emphasis on some of the four factors of teacher conduct (e.g. *input* as in the case mentioned above), it simply ignores another fundamental premise of educational psychology. Evidently, for any human being objectively confirmed external influences are as important as the subjective perception of such influences. In other words, demonstrating significant differences in the way in which teachers of mathematics behave towards male and female pupils will not provide information whether female pupils actually notice such practices and whether they perceive them as discriminatory in nature. Only such an interpretation ("My teacher does not expect me to succeed") will allow for the activation of the third and fourth step in Good's model (1980) of the self-fulfilling prophecy in education. The research procedure employed by the author of this paper focused on determining all four factors of teachers' behaviour as perceived by their pupils.

The following question was posed:

> What are the differences, if any, in the conduct of male teachers of mathematics in the perception of male and female pupils from lower secondary schools?

METHODOLOGY

It was assumed that that the studied variable, i.e. the conduct of the teacher, in accordance with Rosenthal's theory, comprises four elements (*climate, feedback, input,* and *output*). To measure the variable, a Teacher Conduct Scale (experimental version) developed by D. Turska was employed. The whole scale comprises 24 items where each element of a teacher's behaviour is described by six statements. The person examined is to assess the degree of accuracy with which a given statement describes the behaviour of a teacher of mathematics (in the scale from 4 – "totally true" to 1 – "totally untrue"). Theoretically, the raw result of each subscale ranges from 6 to 24 points.

The reliability of the Teacher Conduct Scale is high, as borne out by a high Cronbach's alpha coefficient at 0.893 for the entire tool, and for individual statements ranging from 0.884 and 0.900.

Participants

The study was conducted on 554 pupils from randomly selected a lower secondary schools in Lublin, Poland. Assessments of 232 pupils taught by male teachers of mathematics were taken into account. The average age of the participants was 14 years and 8 months. The research was carried out in May and June 2009.

THE PRESENTATION AND ANALYSIS OF RESULTS OBTAINED

In order to answer the question posed, i.e. what differences, if any, are in the conduct of male teachers of mathematics in the perception of male and female pupils, a test of significant differences for independent samples was made with respect to the results obtained from the four subscales and the global result of the Teacher Conduct Scale. Statistical conclusions are presented in Table 1.

Subscale / Persons examined	*Climate*	*Feedback*	*Input*	*Output*	*Total conduct*
Female pupils	14,67	16,00	16,25	14,91	61,84
Male pupils	15,68	16,75	17,80	16,97	67,21
Value t	-1,38	-0,95	-2,30	-2,56	-2,12
Significance t	non significant (tendency)	non significant	p=0,02	p=0,01	p=0,03

Table 1: Conduct of teachers of mathematics in the perception of male pupils (N=136) and female pupils (N=96).

The data presented in Table 1 indicate that significant differences in the assessment of male and female pupils refer to *Input* and O*utput s*ubscales. This difference also refers to the global result of the Teacher Conduct Scale with the discrimination threat somewhat weakened (due to the absence of significant differences in the *Climate a*nd *Feedback* subscales). A detailed analysis of results obtained in the listed subscales was also made in order to separate statements with a significant discriminatory threat. Table 2 shows the results of this analysis.

Statements	*t* Value and Significance	Evaluation of female students	Evaluation of male students
OUTPUT			
He gives each pupil a chance to solve a problem “at the blackboard”.	-2.33; $p<0.05$	rather untrue	quite true
He encourages each pupil to be active believing that “practice makes everyone perfect”.	-1.96; $p<0.05$	rather untrue	quite true
During the class he works with those few who seem to know “what it is all about”.	2.11; $p<0.05$	quite true	rather untrue
He tries to motivate all pupils to develop an interest in mathematics pointing out that the knowledge gained will come in handy in life	-2.24; $p<0.05$	rather untrue	quite true
INPUT			
When introducing new concepts, he explains them clearly in detail.	-3.28; $p<0.001$	rather untrue	quite true
He analyses problems encountered by pupils while learning the new material with their active participation.	-2.18; $p<0.05$	rather untrue	quite true

Table 2: Statements from the Teacher Conduct Scale concerning the significance of discrimination in the perception of male and female pupils.

The data presented (Table 2) indicate that female pupils give a lower grading in 4 out of 6 statements under *Output* with respect to male teachers. Ostensibly marginal, these statements disclose significant discrimination between the relations and grades given by male pupils. Female students perceive that they are not as motivated as male pupils are to optimise class time. With respect to *Input*, only two statements demonstrate a distinction in the assessment of pupils of the opposite sex. The biggest difference compared to analyses conducted so far is reported in the statement concerning the introduction of new concepts

(-3.28; $p<0.001$).

DISCUSSION

The study discussed is generally treated as a pilot one paving the way for more comprehensive explorations which, in our view, are essential. We are fully aware that the results obtained should be treated with caution given the small number of teachers under evaluation and the experimental nature of the tool employed. However, the data collected at this stage of research clearly suggest that the record of educational experience during mathematics classes differs depending on the gender of the participating pupils.

The data presented in the study indicate that female lower secondary school pupils give a lower assessment to the conduct of male teachers on both *Input* and *Output* scales. These results are quite consistent (since the perception of females is less favourable), but, at the same time, different from those obtained from secondary school students. Secondary school female students gave a lower grade to the *Climate* of the lesson and *Feedback* (Turska and Bernacka, in print) to both make and female teachers. They pointed out a lower degree of patience and a greater sense of emotional detachment of the teacher in individual contacts with female students. Interestingly enough, female lower secondary school pupils taught by male teachers give a lower grade to the actual course of the lesson. In the perception of female pupils (as compared to male pupils), a lesson of mathematics becomes less effective and less frequently used to enhancing abilities. Female pupils see that teachers activate only certain pupils whereas male pupils tend to admit that teachers attempts to get each pupil involved. The data obtained are therefore congruent with the results obtained by Mazurkiewicz (2006) in which – by way of coding and from the standpoint of an outside observer – greater activation of male students during science classes was confirmed. The authors' observations lead to a conclusion that such a varied stimulation is perceived not only by an outside observer but also by female pupils present in the classroom! Only when this fact is confirmed, is it possible to move on to the third and fourth stage of Good's model of self-fulfilling prophecy in education. It appears that this is the right context for the interpretation of a discrepancy between the perceptions of the teacher conduct on the *Input* scale. After all, the way of reasoning recommended, the quality of teaching or the way in which classwork is reviewed are addressed by the teacher to the whole class. A significantly lower assessment of female pupils with respect to the statement "When introducing new concepts, he explains them clearly in detail" cannot be attributed to a lower level of intelligence among females. In fact, there are no known studies which would confirm such a hypothesis. Some explanation may come from the knowledge of styles in communication, a gender distinguishing feature which plays a significant role in a teaching profession. According to Cross and Markus (2002), females and males use different narrative forms and verbal styles to communicate. Females usually adopt a social oriented emphatic style while males' chief feature of communication is based on competition. Perhaps establishing a rapport between a teacher and a male student is therefore easier than between a teacher and a female student. Assuming that such a general statement holds true, the system of education should be organised accordingly so that pupils of one gender stay in contact with a teacher of the same gender. The problem is that such a procedure is not designed to prepare young people to life which, after all, is highly coeducational! This, one is bound to resort to the stereotype threat hypothesis, verified in the experiments carried out by Steele and Aronson (1995), and by Polish scholars, i.e. Bedyńska (2009) and Babiuch-Hall (2007). If one assumes that the commonly held stereotype whereby mathematics is a male domain is also known to females, then, the very phrase "a lesson of mathematics" and, in addition, conducted by a male functions as a living reminder of this stereotype. This, in turn, gives rise to a feeling of uncertainty about one's capacity to handle such a lesson. This feeling of uncertainty may seriously hamper cognitive capacity, especially that which is described as higher order processes including integrating assumptions, drawing conclusions, combining remote ideas, and processing data. The application of cognitive processes faces a particular hurdle in the cases of problems defined as difficult. These are mathematical problems connected with the newly introduced material. It is worth bearing in mind that the statements which differentiate male and female assessments on the *Input* scale refer to situations connected with the mastering of the new material. The less favourable assessment of the teacher conduct by females seems to account for their somewhat blocked reasoning.

The results obtained at this stage of research clearly indicate that the stereotype whereby mathematics is a male domain has an effect on the Polish practice in education. This effect – on the a lower secondary school level and in the case of male teachers – manifests itself in strictly content-related situations.

In the light of the authors' research, such a situation, highly unfavourable for females, can become even more acute with the reinstatement in 2010 of mathematics as an obligatory subject of the secondary school leaving examination. For over two decades mathematics was treated in the Polish educational system, as an optional subject. A change of its status into an obligatory one was announced three years before the due date (i.e. in 2007). It follows that all students are forced to include the final examination in mathematics in their educational plans. Failing this examination will practically deprive them of any chances of higher education, including humanities. We are deeply convinced that it is high time to undertake efforts with a view to establishing equal educational opportunities for both boys and girls with respect to teaching mathematics (the results of our studies prove no such equality is in place). As reported by Hyde and Linn (2006), the activities undertaken in the US over the last 20 years are now starting to bear fruit in the form of similar educational achievements by both male and female students.

CONCLUSION

In the present study shows that female pupils in lower secondary school (as compared to male pupils) give a lower grade to the actual course of the math lesson conducted by male teachers. On the secondary school level, lessons of mathematics invariably prove to be a rather unfavourable experience for female students (Turska, Bernacka, in print). The results of a pilot study fully justify a need for more comprehensive explorations, both in terms of the number of students and teachers examined and in terms of the adequacy of ecological exploration. Relevant studies – which go beyond the communities of large towns – are already conducted by the authors of this paper.

REFERENCES

Anastasi, A.: 1958, Heredity, environment and the question how, *Psychological Review*, **65**, pp.197-208.

Babiuch-Hall, M.: 2007, Czy matematyka jest wciąż tylko dla mężczyzn?, in: M. Czerwińska-Jasiewicz, E. Dryll (Eds.), *Rozważania o rozwoju i wychowaniu*. Wydawnictwo Instytutu Psychologii Polskiej Akademii Nauk, Warszawa, pp. 67-83.

Bedyńska, S.: 2009, Disabling the able: stereotype threat and women's deductive reasonin, in: S. Popek & E.R. Bernacka & C.W. Domański & B. Gawda & D. Turska & A. Zawadzka (Eds.), *Psychologia twórczości - nowe horyzonty. (Psychology of Creativity-New Approaches).* Wydawnictwo Uniwersytetu Marii Curie-Skłodowskiej, Lublin, pp. 327-335.

Brophy, J. E. & Good, T.: 1969, *Teacher–Child Dyadic Interaction: a Manual for Coding Classroom Behavior* (report series no 27), The University of Texas, Research and Development Center for Teacher Education, Austin.

Cross, S. & Markus, H.: 2002, Płeć w myśleniu, przekonaniach i działaniu: podejście poznawcze, in: B. Wojciszke (Ed.), *Kobiety i mężczyźni: odmienne spojrzenia na różnice.* Gdańskie Wydawnictwo Psychologiczne, Gdańsk, pp. 48-115.

Eccles, J.: 1993, School and family effects on the ontogeny of children's interests, self-perceptions, and activity choice, in: J. E. Jacobs (ed.), *Nebraska Symposium on Motivation*. University of Nebraska Press, Lincoln, pp. 145-218.

Gavin, M. K. & Reis, S. M.: 2003, Helping teachers to encourage talented girls in mathematics, *Gifted Child Today*, **1**, pp. 32-44.

Gibbons, J. L.: 2000, Gender development in cross–cultural perspective, in T. Eckes & H. M. Trautner (Eds.), *The Developmental Social Psychology of Gender*. Lawrence Erlbaum Associates, Mahwah, pp. 389-415.

Good, T.: 1980, Classroom expectations: teacher–pupil interactions, in J. McMillan (Ed.),*The Social Psychology of School Learning*.Academic Press,New York, pp. 57-72.

Halpern**,** D.: 1986, *Sex Differences in Cognitive Abilities*, NJ: Erlbaum, Hillsdale.

Hyde, J. & Linn, M.: 2006, Gender similarities in mathematics and science, *Science,* **314**, pp.: 599-600.

Hyde, J. & Fennema, E. & Lamon, S.: 1990, Gender differences in mathematics performance: a meta–analysis, *Psychological Bulletin,* **107**, pp.: 139-155.

Iłowicka-Tańska, I.: 2008, *Jak mówić o przedmiotach ścisłych uczennicom*.11.04. 2010, http:// www.fpp.org.pl

Jussim, L. & Eccles, J.: 1992, Teacher expectations: II. Construction and reflection of student achievement, *Journal of Personality and Social Psychology*, **63**, pp. 947-961.

Kimball, M. M.: 1989, A new perspective on women's math achievement, *Psychological Bulletin,* **105**, pp. 198-214.

Leder, G. C. & Forgasz, H. J. & Solar, C.: 1996, Research and intervention programs in mathematics education: a gendered issue, in A. Bishop & K. Clements & C. Keitel & J. Kilpatrick & C. Laborde (Eds.), *International Handbook of Mathematics Education*, part two. Kluwer, Dordrecht, pp. 932-961.

Leinhardt, G. & Seewald, A. & Engel, M.: 1979, Learning what's taught: sex differences in instruction, *Journal of Educational Psychology,* **71**, pp. 432-439.

Li, Q.: 1999,Teachers' beliefs and gender differences in mathematics: a review, *Educational Research,* **1,** pp. 63-76.

Maccoby, E. & Jacklin, C.: 1974, *The Psychology of Sex Differences,* Stanford University Press, Stanford.

Madon, S. & Jussim, L. & Eccles, J.: 1997, In search of the powerful self-fulfilling prophecy, *Journal of Personality and Social Psychology*, **72**, pp. 791-809.

Mazurkiewicz, G.: 2006, *Kształcenie chłopców i dziewcząt. Naturalny porządek, nierówność czy dyskryminacja*? Centrum Edukacji Obywatelskiej, Warszawa.

Moir, A. & Jessell, D.: 1989, *Brain Sex,* Mandarin Books, London.

Rosenthal, R. & Jacobson, L.: 1968, *Pygmalion in the Classroom,* Holt, Rinehart and Winston, New York.

Rosenthal, R.: 1973, *On the Social Psychology of the Self-Fulfilling Prophecy: Further Evidence for Pygmalion Effects and Their Mediating Mechanisms*, AA MSS Modular Publication Module 53, New York.

Sadker, M. & Sadker, D. & Klein, S.: 1991, The issue of gender in elementary and secondary education, *Review of Research in Education,* **17**, pp. 269-334.

Sells, L.: 1973, *High School Mathematics as the Critical Filter in the Job Market. Proceedings of the Conference on Minority Graduate Education*, University of California, Berkeley.

Steele, C. & Aronson, J.: 1995, Stereotype threat and the intellectual test performance of African American, *Journal of Personality and Social Psychology,* **69,** pp.797-811.

Tiedemann, J.: 2002, Teachers' gender stereotypes as a determinants of teacher perceptions in elementary school mathematics', *Educational Studies in Mathematics,* **50**, pp. 49-62.

Turska, D. & Bernacka, R. E.: (in print), Lekcje matematyki – czy mają płeć? *Psychologia. Edukacja i Społeczeństwo.*

Tyler, L.: 1965, *The Psychology of Human Differences,* Appleton-Century Crofts/Prentice-Hall, New York.

Tymowski, W.: 2008 March, *Kobiety! Co wy wiecie o Politechnice*? 30.06.2009, http:// miasta.gazeta.pl/Warszawa.

MOTIVATION OF ACQUIRING THE CONTENT OF MATHEMATICS, DOMESTIC SCIENCE AND TECHNOLOGIES IN INTEGRATED LEARNING AT PRIMARY SCHOOL

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The sustainability oriented education process at primary school puts an emphasis on the learners' conscious and motivated learning activity. Learner's personality development and value acknowledgement create the necessity for acquiring the content of learning as a whole. Integrated learning is one of the means that make it possible.

The present article regards the theoretical prerequisites of framing the content of learning in mathematics, domestic science, and technologies and the pilot research data testifying to the possibility of holistic learning of these subjects. The conceptual standpoints provide the basis for the teaching material "Practical Mathematics for Form 1"(2008) by the authors of the present article.

INTRODUCTION

Contemporary learning process is oriented at motivated learners who organize their cognitive action and acquire certain competences, education being a priority value in their personality growth.

In primary school it is vital to consider the learner's natural striving of cognition. Learning as a holistic process is supposed to stimulate learners' enthusiasm for cognition and arouse emotional experience in what they investigate and discover. Practical observation and the analysis of historical experience suggested turning to integrated learning during which the acquisition of the content of learning mathematics, domestic science, and technologies proceeds simultaneously. Diverse activities provide motivation for learners revealing the surrounding world in its wholeness.

MOTIVATION FOR ACTION

The notions of motive and motivation have been considered in a different way even in research (Geidžs, Berliners, 1999, Frīdmans, 1988, Markova, Zelmenis, 2000). Motive is treated as the true and deeper, often even unconscious cause and urge of action, while motivation – as a conscious one. And finally, the notion of motivating (Plotnieks, 1976) denotes the true or altered motivation of action expressed in words.

The inner motives are as follows: self-development in the process of learning; acting with others and for others; learning the new and the unknown.

Desire for learning is consolidated by the “I” awareness – “I can”, “I am capable of”, “I managed”. Integrated learning is one of the ways of its practical implementation. One lesson a week for domestic science in the syllabus is too little for the child’s practical activity. Thus, integrating subjects of domestic science and mathematics is a mutual benefit.

CONDITIONS OF ACQUIRING SKILLS IN INTEGRATED LEARNING OF MATHEMATICS, DOMESTIC SCIENCE, AND TECHNOLOGIES

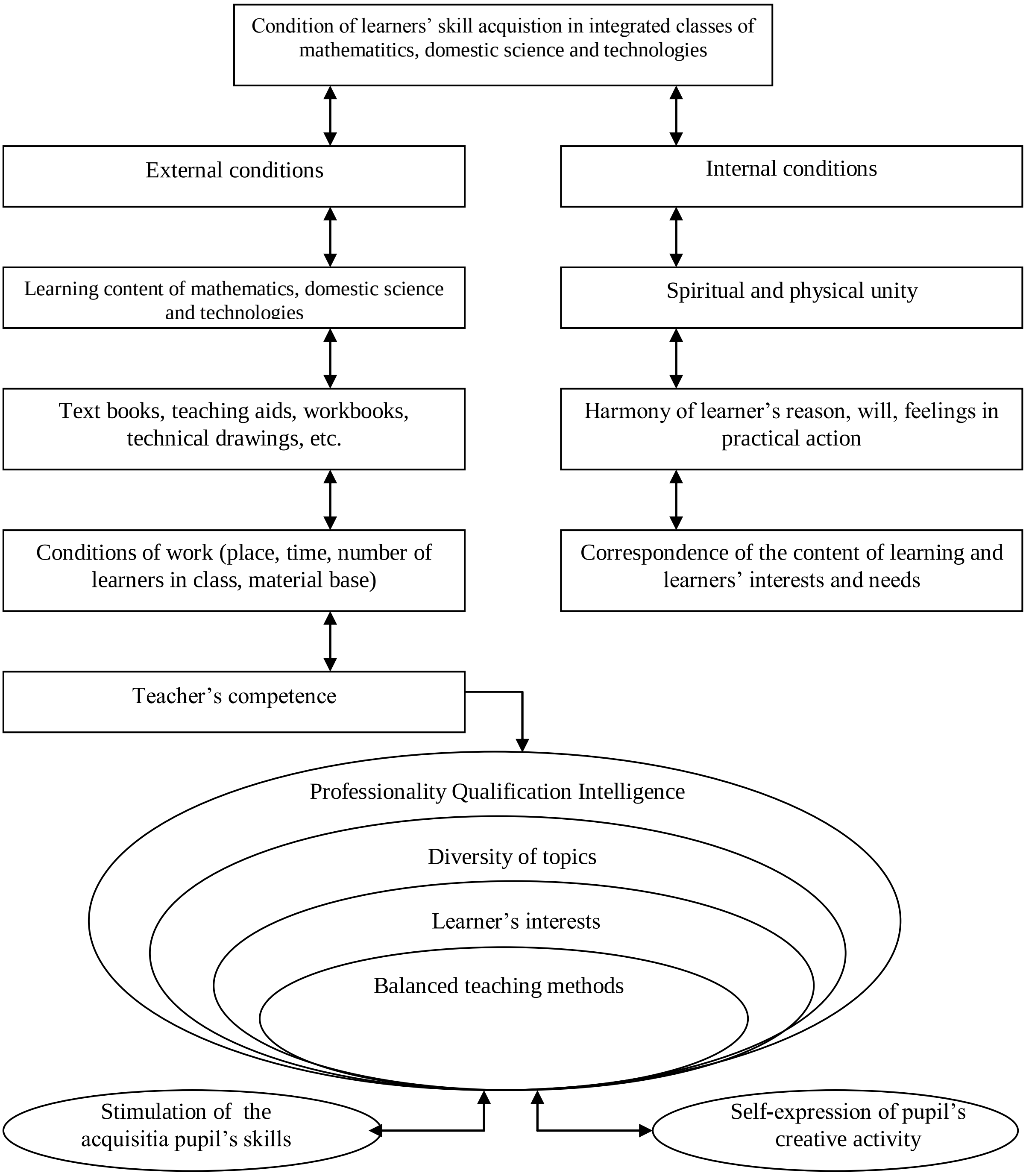

Figure 1: Conditions of skill acquisition in integrated classes of mathematics, domestic science, and technologies

In the development of integrated content of learning mathematics, domestic science, and technologies, diverse external and internal conditions are to be taken into consideration for the acquisition of learners' skills in these subjects. The major conditions are structured in the following scheme (Figure 1).

Articles published by teachers in the period of the independent Republic of Latvia (Panteļejevs, 1936, Bīlmane, 1924) on the conditions of acquiring skills in integrated learning remain topical till the present.

A. Panteļejevs considers that, in the process of learning, importance is attributed to the formation of skills of work, the work result as well as awareness that the work has been done well (Panteļejevs, 1936, p. 3). According to him, during the classes of domestic science and technologies, it is not only learners' independent work that is important but also their ability of using the knowledge, skills, and competences acquired in other subjects. One should note the author's suggestion to pay special attention to the ability of making items that are similar in their composition and significance to the ones produced before, thus relating to one's prior experience of work.

M. Bīlmane's "Šķēru griezumi" [Scissor cutting] (1924) is the first methodical aid in which the author suggests to facilitate the bond with other subjects, especially mathematics. The author suggests to acquire skills of scissor cutting and to use the cut strips for learning the notions "bigger, smaller, shorter, longer, wider, narrower" as well as learn the directions, e.g. "horizontal, vertical, slanting", etc. M. Bīlmane suggests using the strips to learn counting, fraction numbers by dividing strips (1/2; 1/3, etc.) as well as notions "so many times longer, shorter", etc. For learning geometrical figures, the author suggests to use not only strips (to form geometrical figures) but also cut out quadrangles that resemble familiar things – books, board, window, etc. In the further activities, the cut out quadrangles of different size are grouped according to size, folded diagonally thus obtaining triangle (Bīlmane, 1924, p. 16). Another suggestion is making different pictures from geometrical figures that is the initial stage of application so widely used nowadays. Round and oval forms are the hardest elements for scissor cutting. These are suggested to make later after having acquired the skill of cutting straight lines, using them to make a clock, moon, sun, egg, etc. Such suggestions are included also in the textbook and methodical aids for teachers.

Interestingly enough, M. Bīlmane's work was published at the same time as A. Dauge's "Skolas ideja un tautas audzināšanas uzdevumi" [The idea of school and tasks of people's education] (1924) where the author gives his suggestions for creating primary school syllabus by grouping subjects around the central ones that would interrelate and supplement one another (Dauge, 1924). M. Bīlmane, in turn, shows in her work concrete patterns how academic subjects may be related to one another. This is also exemplified by her suggestion to envisage lessons specially for cutting paper just at the very beginning, but later practice doing it at lessons of mathematics and other subjects where the cut forms might be also used in the process of learning. It is noteworthy that M. Bīlmane suggests obligatory use of scissor cutting in the first three years of learning mathematics (Bīlmane, 1924, p. 19). Before Christmas, pupils prepare decorations that may be counted and their geometrical forms analyzed.

Evidently the aforementioned works manifest the pedagogical ideas of that time as well as the educating power and significance of handicraft (domestic science and technologies), practical work, mathematics, and art. M. Bīlmane in her works implements the suggestion by A. Dauge – to use elements of art at school for developing and cultivating learners' taste, disciplining their fantasy, and enriching their feelings (Dauge, 1924, p. 25). Pupils make decorations with symmetrical figures, broken line ornaments, model parquet, etc.

CONDITIONS OF CREATING THE INTEGRATED LEARNING CONTENT OF MATHEMATICS, DOMESTIC SCIENCE, AND TECHNOLOGIES

It is important to create an integrated content of mathematics, domestic science, and technologies that would make it possible for learners to realize, see and relate the skills, competences, and knowledge acquired in learning mathematics to the skills and knowledge used for handicraft technologies and to real life situations. This kind of integrated content of mathematics, domestic science, and technologies should provide and coordinate the learners' development tendencies, needs, and interests implementing the holistic approach in learners' development.

The ideas of K. Cīrulis bring out the conditions for the creation of integrated content of learning mathematics, domestic science, and technologies:

- the significance of the balance between physical work and brainwork for children's all-rounded development;
- children develop enthusiasm for work in the process of working, therefore they must be trained to work since young age by selecting tasks according to their abilities;
- work affects the development of child's character, will-power, moral traits, skills, abilities, talent and the acquisition of work habits (Cīrulis,1894,2-5);
- to preserve child's interest for handicraft, it is advisable to change the types of work;
- to keep up child's enthusiasm for work and facilitate the development of work aptitude that lies at the basis of intellectual activity – skills of comparing, draw conclusions and arguments, it is necessary to facilitate the cognitive action and meet the child's striving for creativity (Cīrulis, 1894, 76);

The author's ideas are important and applicable nowadays creating an integrated content of learning mathematics, domestic science, and technologies. K. Cīrulis' (1894) suggestion to relate learning handicraft (domestic science and technologies) to learning other subjects, especially geometry, technical drawing, drawing, geography, natural science, and chemistry, remains topical.

In the course of the formation of the content of learning mathematics, domestic science, and technology, it is necessary to observe the gradation and succession of content, thus revealing and executing gradual and interrelated acquisition of required skills in mathematics, domestic science, and technologies. Textbook for form 1 entails united acquisition of the content of learning. Thus, the notion 'equivalent' is related to producing equivalent figures by copying, cutting out, tearing out, folding. By analyzing the form of objects, e.g. quadrangle, learners learn to make a quadrangle sheet, make an aircraft by folding paper, etc. The contingent measuring is related to practical measuring of length by inch, foot; volume is measured by vessels of diverse size and form. Equation of number is learned by pupils while laying the table for a picnic breakfast. Learning the notion of number, pupils learn to weave a bookmark from one, two, or three threads. Action and the result of action are manifested not only in mathematics by calculating the sum and the difference but also in domestic science – by making fruit salad. Preparing to celebrate Mārtiņš festival (harvesting festival in Latvia), pupils make masks but in mathematics they learn to guess numbers hidden in equation under some figures. Making finger dolls, pupils make a dialogue with their deskmates "What would I buy if I won 10 lats?" Hence, K. Cīrulis' ideas may be transformed and applied as criteria for the formation of an integrated content of mathematics, domestic science, and technologies.

RESEARCH STUDY OF THE LEARNERS' ATTITUDE TO PRACTICAL WORK

During our research at schools where Riga Education and Management Academy students take their pedagogical practice, we focused on the opportunities of implementation of the reformed learning content. The study covered 24 schools in Latvia. We investigated 3rd form learners' attitude to their home and duties at home. Reading their essays "My Home" was a pleasant surprise for us. Learners had expressed a positive attitude to their duties at home. They had decorated their essays with artistic details (applications, collage, drawings, both printed and handwritten texts). A number of learners had characterized their home as very significant and dear to them, associated with warmth, family, close and dear people. Home for them is a place of living. It is a place where they are welcome, loved and experience their home as their treasure. Learners had depicted their home inner and outer environment in drawings bringing out their vision of home. These drawings brilliantly reveal learners' psychological peculiarities, problems, and their perception of life showing how they feel in the world. Summing up the information provided by the learners, we may draw the following conclusions:

- some learners have a computer at home with which they, especially boys, spend much time. Those who do not have a computer would like to have it.
- It is positive that some learners have pets at home which they take care of.
- The learners like having a room of their own and their duty is to keep it tidy. Some of the learners admit having no duties at home. Some learners would like to have a cleaner coming to their place and making their life easier.
- Learners living in multi-storeyed houses wish to have there a bigger and cleaner elevator.
- Only some learners would like to make changes at their home themselves. Hence, one boy writes, “When I grow a little older, I will renovate our hall to make room for furniture and balls and roller skates.” Another learner, Ieva writes, “If I could, I would like to renovate our house and put on it a new green roof.”

Polling learners we found out that 93% of them help parents with kitchen duties. The majority of these – 62% help with washing dishes, while 31% help with cooking. These findings have been summed up for better visuality in Figure 2.

During our discussions with learners, we found out that some of them help in the kitchen by cleaning the sink, mopping the floor, clearing away after breakfast, keeping order in the fridge. In the study we found out other duties of learners at home as well.

Practical assignments in domestic science are related to the consolidation of mathematical terms and calculating skills. Practical tasks are related to measuring and using measures in calculation. Hence, learners develop an appropriate attitude towards mathematics that is widely applicable in life.

According to the results provided in Figure 2, 98% of the polled learners participate in cleaning the house and only 2% of them do not. The main task performed by 32% of learners is mopping the floor. 22% of learners help with sweeping the floor. There are learners whose duty is cleaning carpets (17%), watering potted plants (12%), tidying just their room (14%). It should be noted that the acquired results invite second thought, especially after talking with the learners’ parents who admitted that their imparted duties were not always regularly performed.

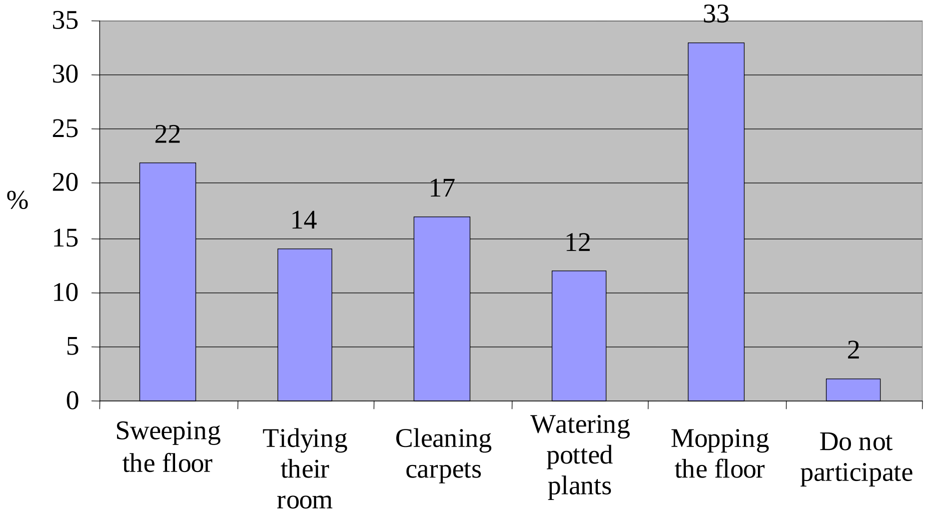

Figure 2: Duties of 3^{rd} form learners in cleaning the house

According to R. Kutkovska, learners' attitude to their academic work is formed by way of cooperation between the teacher and the learner as well as a result of family upbringing (Kutkovska R., 1998).The exemplary syllabus of academic subjects and lessons schedules for domestic science and technologies **1 hour a week**. To perform the planned work load **regularly** working within a limited time period (40 minutes), learners *are hurried* to work faster. This results in carelessness being trained as a character trait. Thus, J. A. Komenski's idea that withdrawing children from classes gives rise to negative consequence is completely ignored. How can learners cope with the tasks set out in the standard of the subject including "ability of keeping in order their working place" in 40 minutes that include also the time needed to arrange the working place before and after work as well as self-analysis of the work performed? This explains what one often witnesses in pedagogical practice that more and more school learners, students, and young teachers are careless with their work, lacking precision and responsibility for their performance. By integrating subjects, the time may be scheduled in a different way.

According to the practice, even if everybody works very fast, the time for a well-rounded analysis is always too short and learners' self-analysis is completely lacking. In diet studies, 40 minutes are insufficient to prepare for work, acquire the skills for cooking the dish, then set in order the place of work, to say nothing of the assessment of the performance. Assessment, especially learners' self-assessment, is necessary to develop at primary school as it is determined in the standard of the subject. (*Noteikumi par valsts standartu pamatizglītībā un pamatizglītības mācību priekšmetu standartiem,* 2007). It becomes possible if learners have developed reflection abilities and have been trained to assess their own and others' work and action and to project consequences.

A. Tūna points out that everybody must learn to lead and plan their own lives in conditions of the 21st century. This is also pointed out in the *objective* of **National basic education standard** (1998, p. 7): to facilitate learners' responsible attitude to themselves, their family, people around, their nation, native land and higher moral values, society, environment, state. However, there are no tasks **orienting learners towards work** as the sole morally accepted source of making living and welfare, as indicated in the Children's Right Protection Act, paragraph 4.2. (2007).

The standard of domestic science and technologies states that learner must attribute at least 75% of the allotted time of the learning process to *gaining skills* and *building the experience of practical action* (from 1 lesson a week). Within the united process of learning mathematics, pupils have an opportunity to change modes of activity and exercise in doing mental arithmetic.

One must add that the theoretical issues of the content of learning domestic science and technologies at primary school are better implemented in subjects of social science and the Latvian language by selecting corresponding topics and appropriate texts. In turn, in mathematics, by doing sums, it is possible to learn saving the usable resources and understand the basic principles of shopping. In integrated learning class, learners play the applied game "Doing shopping". It has been observed in practice that the majority of learners gain diverse skills **only by practical work**.

Nowadays A. Panteļejevs' idea that items made in handicraft (domestic science and technologies) classes are not the aim but the means of personality development is still topical (Panteļejevs, 1936). In this respect, the formulation of the aim of the subject: "for everyone to be able to independently **plan and organize** one's personal household" invites second thought. It is **organize**, not get engaged in the work by oneself. This testifies to the fact that, in changing socio-economic conditions, the role of this subject has also changed and its significance for learners' development is ignored.

Domestic science and technologies the same as mathematics involve several additional experiences of seriousness, hardship and joy; therefore learning this subject is important for the personality value orientation.

The conditions of working and the respect to the learner are important as well. According to National basic education standard, it is important to create the learning environment which would make it possible for the learner to feel his or her abilities and trust in them as well as develop the sense that his or her work is significant (1998). If a learner is no good at arithmetics, s/he may find satisfaction in doing practical assignments or vice versa.

According to DU teaching placement poll of 100 pupils, 42% like mathematics lessons. Using the method of unfinished sentences, we found out learners' joys and sorrows in learning domestic science and technologies at primary school. By grouping the received replies, we learned that 4% of learners expressed joy at working in group, 19% expressed joy at their performance, while 4% expressed joy for a nice working place. 58% of learners expressed joy at working, while 15% expressed joy at everything that goes on during domestic science and technology classes. At one of schools in Riga, 3rd form learners expressed a wish for more frequent classes of domestic science and technologies; some even wished for them to be all day long because they thought that time at these classes passed very quickly. See Figure 3.

1 – joy at working in group, 2 – joy at everything that goes on during handicraft classes, 3 – joy at one's own performance, 4 – joy at a nice working place, 5 – joy at working

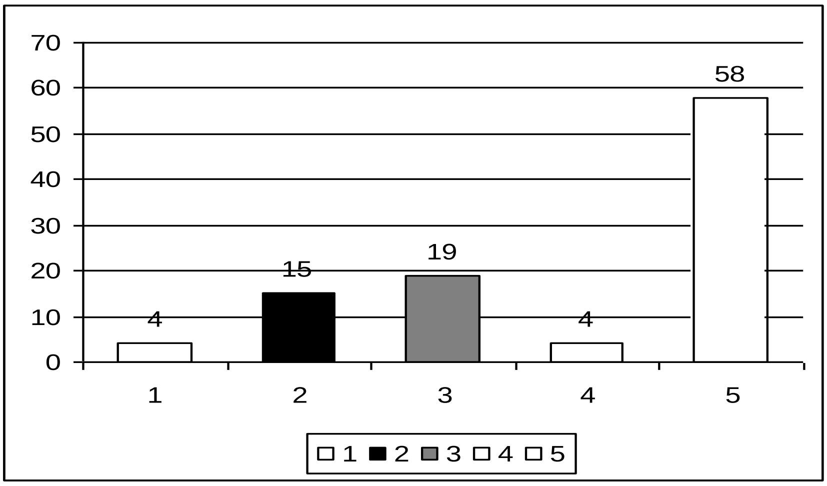

Figure 3. Learners' joys at domestic science and technology classes

It was also interesting to read 3rd form learners' essays on the topic "My Joys and Sorrows at Domestic Science and Technology Classes". A boy writes:

> I like domestic science and technology classes because they are interesting. Sometimes one lesson is not enough. But sometimes I cannot wait for it to end, if I have forgotten to bring something to the class and cannot work, but that is my own fault. I like most of all those lessons that are taught by young practicing teachers because then they are even more interesting.

Another boy writes that his joy at the domestic science and technology class is caused by such a task that he can manage and that he likes. But even more he likes it if these classes do not take place. He does not like that the teacher makes learners clean the classroom. It must be noted that dislike of cleaning the classroom has been mentioned by 9% of the learners. The majority of primary school learners at domestic science and technology classes fear of making mistakes. Some of learners admitted being afraid of not managing everything in time and being ridiculed for poor performance. Some learners expressed sorrow of spoiling something, poor performance, having not brought with them the needed materials. A boy writes, "I am very glad if I understand the task and can manage to do it, then I am asked **to help the teacher**." It must be stated that searching for new forms of learning organization, learners have an opportunity of applying their acquired skills in diverse activities related to real life and social situations that, in turn, orientate learners for work. According to J. Stabiņš, the teacher's mission is professional organization of the learning environment as a process of action in which learners develop their aptitudes for acting in real life (Stabiņš, 2001).

According to observations, learners do not have a special interest for mathematical tasks that have no practical application. Their attitude changes when the teacher offers them creative tasks. Those learners feel especially lucky whose teachers sense and can accept the *characteristic trends of the epoch* in domestic science and technologies and implement them respecting the learners' needs. Necessity for knowledge of dealing with practical tasks in domestic science motivates learners to better acquire mathematics.

At the stage of primary school education, the basis is laid for the learner's self-assessment that is formed each lesson since form 1. We noticed this in one boy's essay where he openly writes, "I don't like working with glue and make my hands dirty at domestic science and technology classes. One needs great patience for domestic science but I don't have patience." Patience is developed also by doing sums. E. Erikson admits that at this age either the sense of competence or inferiority is formed. The teacher has a great role in helping learners form healthy self-assertion and reveal their talents and individual abilities (Erikson, 1968).

CONCLUSIONS

Integrated learning of mathematics, domestic science, and technologies facilitates learners' positive attitude to their academic work. Taking into account the aforementioned ideas, the advantages of integrated learning of mathematics, domestic science, and technologies are as follows:

- learners gain knowledge holistically as a result of partial experimentation, search, and creative action; learners are interested and motivated to participate in the process of learning because they see the application of their knowledge and skills in real life situations;
- integrated learning secures the coordination of learners' experience – learning of the notions of mathematics, domestic science, and technologies proceeds in the social environment, and this provides for the acquisition of knowledge and skills in mathematics, domestic science, and technologies necessary for the individual and social life;
- developing of friendly relations among learners; learners do not develop inferiority complex;
- learners learn to express their opinion and listen to others and respect others' opinions;

- the mutual relatedness of academic programmes extends the common skills and knowledge of the content of mathematics, domestic science, and technologies, thus revealing the vitality of these subjects in dealing with everyday life issues and the possibilities of learning in one's practical life.
- Processually integrated learning needs conceptual coordination of teaching and methodical aids.

REFERENCES

Bērnu tiesību aizsardzības likums: 2007, http://www.likumi.lv/doc.

Bīlmane, M.: 1924, Šķēru griezumi. Rīga, Kultūras balss izdevums.

Dauge, A.: 1924, Skolas ideja un tautas audzināšanas uzdevumi. Rīga, 29 lpp.

Fulans, M.: 1999, Pārmaiņu spēki. Rīga, Zvaigzne ABC.

Geidžs, N., Berliners, D.: 1999, Pedagoģiskā psiholoģija. Rīga, Zvaigzne ABC.

Krastiņa, E., Limanoviča, E., Volāne, E., Drelinga, E.: 2008, Praktiskā matemātika 1.klasei, Rīga, Zvaigzne ABC.

Kutkovska, R.: 1998, Skolēnu mācīšanās stimulēšana, sagatavošana apzinātai darbībai //Mājturības pedagoģiskās aktualitātes. Latvijas Mājturības institūta dibināšanas 60.gadadienai veltītas starptautiskas konferences referāti. Jelgava, LLU.

LR IZM ISEC Valsts pamatizglītības standarts: 1998, Lielvārde, Lielvārds.

Markova, A.: 1986, Mācīšanās motivācijas veidošana skolēniem. Rīga, Zvaigzne.

Panteļejevs, A.: 1936, Zēnu rokdarbi. Lauku un pilsētu pamatskolām. Rīga, Balodis.

Noteikumi par valsts standartu pamatizglītībā un pamatizglītības mācību priekšmetu standartiem: 2007, http://isec.gov.lv/normdok/mk061027.htm.

Plotnieks, I.: 1976, Emocijas un jūtas. Rīga, LVU.

Rubana, I.M.: 2000, Mācīties darot. Rīga, RaKa.

Stabiņš, J.: 2001, Skolotāja misija. Rīga, RaKa.

Zelmenis,V.: 2000, Pedagoģijas pamati. Rīga, RaKa.

Цируль, К.: 1894, Ручной труд в общеобразовательной школе. СПБ.

Erikson, E.: 1968, H. Identity: Youth and Crisis. New York.

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"WITHOUT MATHS WE WOULDN'T BE ALIVE": CHILDREN'S MOTIVATION TOWARDS LEARNING MATHEMATICS IN THE PRIMARY YEARS

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This paper presents data on the beliefs, values, and attitudes of 9- to 11-year-olds towards their mathematics learning. The children were very positive, optimistic, persistent, and highly motivated to learn mathematics. These findings are considerably more positive than those found with children five years earlier. Possible reason for the shift is that a decade of reform in mathematics education has had time to impact on classroom practices. Another possibility is that the present cohort differed by including more children from high SES communities who were average to high achievers in mathematics for their year level.

INTRODUCTION

The connections between motivation and achievement have been of interest to researchers for many decades. It is argued that human beings have an inherent need to feel competent and to achieve (Elliot & Dweck, 2005). An important distinction can be made between positive motivation (*approach* towards desired activities) and negative motivation (*avoidance* of aversive experiences). The term "competence motivation" (formerly called achievement motivation) refers to the domain in which motivation and competence/achievement are considered together. According to Elliot and Dweck (2005), competence motivation is part of everyday life and has a major impact on people's emotion and wellbeing.

Research literature argues for the importance of taking into account the affective as well as the cognitive domain in investigations of mathematics learning. The affective domain of mathematics learning has been the focus of considerable research. Writers have distinguished between different aspects of the domain, including beliefs, values, attitudes, and emotions/feelings (Leder & Grootenboer, 2005). These can be conceptualised as being on a continuum, with beliefs at one end (characterised by increased cognition and decreased affectivity) and emotions/feelings at the other (characterised by increased affectivity and decreased cognition). Values, thought to overlap with beliefs, and attitudes (overlapping with both beliefs & values) fall between beliefs and emotions/feelings (Leder & Grootenboer, 2005).

According to Burns (1998), the majority of adults (in the US, at least) have a deep fear of mathematics. This "math anxiety" or "math phobia phenomenon" (Bahr & de Garcia, 2010), can impact adversely on students' mathematics learning. Jo Boaler's (2008) recent book, *What's math got to do with it? Helping children learn to love their most hated subject --- and why it's important for America,* provides a powerful argument for the urgent need to address this issue at the primary/elementary school level. Whereas in the past, mathematics was reserved for the academic elite, nowadays the expectation is that mathematics is "for all" (Gates & Vistro-Yu, 2003). Monitoring how primary/elementary students feel about mathematics is important to ensure that students get the most out of their learning opportunities in mathematics.

As part of educational reforms over recent decades, many Western education systems have made literacy and numeracy/mathematics their highest priority (e.g., Bobis et al, 2005; Department for Education and Employment, 1999; National Council of Teachers of Mathematics, 2000; New Zealand Ministry of Education, 2001). Mathematics education reforms have stressed the importance of conceptual understanding over procedural knowledge. Several education systems have developed programmes that include a framework outlining progressions in numeracy learning, diagnostic assessment tools, and professional development for teachers (Bobis et al, 2005). Recent approaches to mathematics teaching acknowledge the social and cultural nature of mathematics thinking within a community of learners.

Recently, there has been substantial work on the value of listening to students' views of learning (e.g., Boaler, 2008; Cook-Sather, 2002). McCallum, Hargreaves and Gipps (2000) have argued that pupils' voice is important in understanding schools and schooling. The UN Declaration on Human Rights states explicitly that children should be given a voice on matters that have an impact on them (New Zealand Ministry of Foreign Affairs & Trade, 1997).

Research has focussed on students' views of mathematics learning (Boaler, 2008; Young-Loveridge, 2005). It is interesting to note that despite reforms calling for greater communication in mathematics, students' responses have tended to reflect the belief that mathematics is a private, solitary activity where talking to others is not acceptable. Students are more positive about the value of explaining their own thinking to other people than about knowing how their peers solved mathematics problems. Many children are concerned that having an interest in the solution strategies of others might be misconstrued as "cheating" (Young-Loveridge, Taylor & Hawera, 2005). However, in schools where communication of mathematical thinking is highly valued by teachers, students are very articulate in explaining the reasons for the importance of mathematical communication (Young-Loveridge, 2005). Students who were asked, "*What do you think maths is all about?"* talked about mathematical content, learning in general, problem solving in particular, and the usefulness of mathematics for everyday life (Young-Loveridge, Taylor, Sharma, & Hawera, 2006). Much of this research was undertaken early on in the reform process.

The purpose of the present study was to explore students' views about mathematics learning. This was part of a larger study that investigated the impact of instruction on students' additive thinking.

METHOD

Participants

The sample consisted of 64 students (31 girls and 33 boys) in Years 5/6 (9- to 11-year-olds) attending 4 schools. Two schools served high socio-economic status (SES) rural communities (deciles[6] 9 & 10), and two served low socio-economic small-town communities (deciles 2 & 3). Just over half (55%) the sample was European, approximately one quarter (27%) was Maori (the indigenous people of New Zealand), and the remainder (19%) were of Pacific Islands descent. A group of students was selected by each of the nine teachers as being at about the expected achievement level in mathematics for their year level (stages 5-6 on the New Zealand number framework). Students began the school year (6 months prior) ranging from Stage 4 (they used "counting on" to solve addition/subtraction problems) to stage 6-7 (they used a wide range of partitioning and recombining strategies to solve addition/subtraction and multiplication/division problems mentally).

Procedure

Students were taken out of class and given a paper-and-pencil questionnaire, including 21 statements each with a 4-point Likert scale, ranging from "strongly disagree" to "strongly agree". Ratings at each end of the scale were aggregated ("Strongly disagree" with "disagree" and "Strongly agree" with "agree"). Students were also asked to write their responses to two questions: *"What is mathematics?"* and *"If a new child started in your class and wanted to be a success at maths, what advice would you give to them?*" The children's writing was edited to ease its readability but original spelling was preserved as far as possible. Content analysis was undertaken to identify common themes within the written data.

RESULTS

Table 1 presents the percentage of students who agreed or disagreed with each statement for the sample overall, and the percentages of students who agreed as a function of gender. The highest levels of agreement were on statements about: being good at maths (Q1: 94%), the acceptability of mistakes in maths (Q9: 98%), persisting even when the maths becomes difficult (Q10: 95%), working with another student whose answer differs until it is clear which is the correct answer and the reasons (Q11: 98%), solving problems in one's own way (Q13: 91%), listening to the solution strategies of others (Q14: 91%), and the answer

[6] Each school in New Zealand is assigned a decile ranking between 1 (low) and 10 (high), based on the latest census information about the education and income levels of the adults living in the households of students who attend that school.

making sense (Q21: 95%).

Statement	Overall SD	S	A	SA	A or SA Girls n=31	Boys n=33	Diff
1. I am good at maths		6	77	17	94	94	*0*
2. Learning maths is mostly remembering facts and rules	3	47	39	11	42	58	*16*
3. I learn more by working with other children	6	17	45	31	84	70	*14*
4. I ask questions in class about maths	2	13	53	33	84	88	*4*
5. It is important to be able to explain how I solved a problem to other children in my class	6	5	36	53	100	79	*21*
6. Knowing why an answer is correct in maths is just as important as getting the right answer	8	28	42	22	61	67	*5*
7. When two children don't agree on an answer in maths they need to ask the teacher to see who is correct	17	30	41	13	52	55	*3*
8. I talk about my ideas in maths in a group or with a partner	3	13	59	25	84	85	*1*
9. It is okay to make mistakes in maths		2	25	73	100	97	*3*
10. When my work in maths is hard I don't give up	3	2	45	50	97	94	*3*
11. When two children don't agree on an answer to a maths problem they can usually think through the problem together until they work out who is right and why		2	59	39	97	100	*3*
12. You can be good at maths without understanding it	45	34	17	3	19	21	*2*
13. I can come up with my own ways to solve maths problems	5	5	50	41	94	88	*6*
14. It is important for me to listen to how other children in my class solved a problem in maths	2	8	30	61	97	85	*12*
15. Adding, subtracting, multiplying and dividing are only a small part of maths	14	30	36	20	48	64	*15*
16. It is important to get the answer right in maths	25	48	23	3	19	33	*14*
17. Learning maths involves more thinking than remembering	2	22	53	23	81	73	*8*
18. Maths is difficult for me	41	41	14	5	19	18	*1*
19. Maths is interesting	6	8	36	50	90	82	*9*
20. Asking questions in maths means you didn't listen to the teacher well enough	39	38	16	8	19	27	*8*
21. When working on maths problems it is important that your answer makes sense to you	3	2	28	67	100	91	*9*

Table 1: Percentages (rounded to whole numbers) of students who agreed with each statement as a function of gender (SD: Strongly Disagree, D: Disagree, A: Agree, SA: Strongly Agree; *the difference is shown in italics*)

Gender

Overall, there was considerable similarity in the responses of girls and boys to the questions. However, the most notable difference was that, whereas all girls thought it *important to be able to explain their solution strategies to other children* (Q5), just over three-quarters (79%) of the boys had that view. Less than half (42%) of the girls thought that *learning maths is mostly remembering facts and rules* (Q2), compared to 58 percent of the boys. More boys (64%) than girls (48%) thought that the four operations *are only a small part of maths* (Q15). More boys (33%) than girls (19%) thought that *it is important to get the answer right in maths* (Q16). More girls (84%) than boys (70%) thought that they *learn more by working with other children* (Q3). Similarly more girls (97%) than boys (85%) thought that *it is important for them to listen to how other children in their class solve problems in maths* (Q14).

Enjoyment of Mathematics (Q19)

Although the students were not asked directly whether they enjoyed maths, the closest question was about whether maths is interesting. A substantial majority of students (86%) thought that maths is interesting. Two-thirds of the students who did not find maths interesting (n=9) were boys, and two-thirds were average to high achievers (stage 6 on the number framework), raising the possibility that they were not being challenged at an appropriate level.

An analysis explored possible differences between those students who did not find maths interesting (n=9) and those who did (n=55). More than half of students who did not find maths interesting (56%) thought it was important to get the answer right in maths, whereas this was the case for less than one quarter of those who were interested in maths (22%). Another notable difference was on the question about maths being mostly remembering facts and rules. Less than one quarter of those who did not find maths interesting (22%) agreed with that idea compared to more than half the others (55%). Virtually all those who thought maths was interesting (98%) thought that it important that answers to maths problems make sense, whereas 78 percent of those who did not find maths interesting responded that way. Two third of those who did not find maths interesting perceived maths as difficult for them (despite virtually all of them being at stage 6), compared with only 16 percent of other students. Students who thought maths was not interesting were also less likely to think it important to listen to the solution strategies of other children (78% versus 93%), or to think it acceptable to come up with their own strategies (78% versus 93%). More of them thought that it is possible to be good at maths without understanding it (33% versus 18%).

Difficulties with Mathematics (Q18)

Most of the students (81%) did not find maths difficult. An analysis was done to explore possible differences between students who experienced difficulties with mathematics (n=12) and those who did not (n=52). Half the students who found maths difficult thought that asking questions in maths meant that you didn't listen to the teacher well enough, compared to only 17 percent of other children (Note that 75% did ask questions in class about maths, compared to 89% of the others). Three quarters of those who found maths difficult thought they were good at maths, compared with almost all (98%) the other children. Students who found maths difficult were less likely to think that learning maths involves more thinking than remembering (58% versus 81%), or that knowing why an answer is correct in maths is just as important as getting the right answer (50% versus 67%). They were also less likely to ask questions in class about maths (75% versus 89%), or to find maths interesting (75% versus 89%).

Understanding is a Pre-requisite for Mathematics Proficiency (Q12)

The majority of students (80%) disagreed with the idea that you could be good at maths without understanding it. An analysis was done to explore possible differences between students who thought you could be good at mathematics without understanding it (n=13) and those who did not (n=51). The most notable difference between the two groups was that the majority of students (85%) who did not view understanding as important for maths proficiency thought that when children disagreed about an answer, they needed to ask the teacher who is correct, whereas only 45 percent of other students took this view.

Willingness to ask Questions in Class about Mathematics (Q4)

Most (86%) students were willing to ask questions in class about mathematics. An analysis was done to explore differences between students who were not willing to ask questions in class about mathematics (n=9) and those who were (n=55). A possible reason for being unwilling to ask questions in class was evident in responses to Question 20. More than half (56%) of the students unwilling to ask questions thought that it meant that they hadn't listened to the teacher well enough, whereas as only 18 percent of other students held this view. Only one third of students unwilling to ask questions thought that knowing why an answer in correct in maths is just as important as getting the right answer, compared with 69 percent of other students. Only two thirds of those who were reluctant to ask questions thought they were good at maths, compared with virtually all (98%) of the other students. Students who were reluctant to ask questions were more likely to believe that learning maths involves more thinking than remembering (100% vs. 73%), and less likely to think that the four operations are only a small part of maths (33% vs. 60%).

Learning Mathematics is a Social Process (Q8)

The majority of students (84%) reported that they talk about their ideas in maths in a group or with a partner. An analysis was done to explore differences between students who did not talk about their ideas in maths with others (n=10) and those who did (n=54). The most notable difference between the two groups was that those who did not share their ideas with peers were more likely than the others to want to give up when the work in maths was hard (30% vs. 0%). These students were less likely than others to come up with their own ways of solving problems (70% vs. 94%), or to believe that they could learn more by working with other children (60% vs. 80%). They were more likely to think that the teacher should arbitrate if two children disagree about an answer in maths (70% vs. 50%). One interesting difference that was hard to explain was that none of the students who were unwilling to share their ideas with peers thought that asking questions in maths meant that you didn't listen to the teacher well enough, compared to 28 percent of the other students.

Students' Written Responses showing their Views of Mathematics

Students' responses to the question: *"What is mathematics?"* were organised according to common themes. Only one student was unable to answer the question and wrote: *"I don't know?"* The most frequent idea mentioned by students was about the *utility* of mathematics, both now and in the future.

> Mathematics is a big thing in life because counting money is using maths and subtracting money, because when you go to the shop and you give the person more money than what it cost, you need to know how much change you get. A2

> Mathematics is a skill to learning, you need to know this skill because you might be in a supermarket one day and need to know how much it costs. A4

> Mathematics is way of working things out. Without maths we wouldn't know how much things weigh, how many there are altogether and we'd be very dumb. A7

> Mathematics is in our everyday life, and we use it every day. Eg, counting money, making a house, finding the primeter [perimeter] of something, finding the area of something. A8

> Maths helps you when you are needing to go to a shop and you want to buy something. And if you wanted to be a builder you would have to learn your maths. A19

> Maths is a helpful task in life, you will always need maths even if your [you're] not very good. You need to learn as much as you can, maths is one of the most used subjects, you'll need it for your house, car and even shopping. Your life will be much easier knowing maths. A21

Quite a number of children mentioned *problem-solving* in their response.

> Mathematics, working out problems with numbers and measurement, algebra and geometry. It's also adding, subtraction, multulcation, [multiplication] and division. A25

> Maths is when you learn how to work out all sorts of things such as times, division, plus and subtraction. Maths is how you learn skills and good ways to work out an answer. Maths is also the way how you can get a job. There are more things in maths that can learn such as desimils [decimals] and fractions and measurement. A26

> Mathematics is an interesting way to work out a problem like 50 x 50. There are lots of words in mathematics like divesion, times tables, adding and sibtracting. A10

> Mathmatics are about solving problings and it helps you with your maths. A14

Some children's responses reflected an emphasis on *learning*.

> Mathematics is equasions [equations] for kids to learn. The come in four types like addition, subtraction, multiplication, and divide a by. It takes a lot of practice to be a maths genius. A15

> Maths is a part of learning at school. A28

> I think mathematics is a fun and easy way to learn, sometimes you get it right the first time but never stop trying, and also maths could help you later on in life because maybe just maybe you could [win] the one million dollar question. A29

> I think math is a great part of learning. I think that mathmatics is something that you use everyday. People use math in there [their] work job because if you are a builder you would have to tell all of the measurements and where you are going to put the stuff you are using. I rekon [reckon] that math is a great subject to learn. A50

> Mathematics is about learning new stratergy [strategy] and how to multiply big and small numbers. You can use them by times tables, Plus-takeaway and dividibys. You can also use your family of facts to work out the answer to take away and Plus also dividing + times tables. And you just work through the answer properly. A63

Students' Advice to a New Student wanting to Succeed

Students' responses to this question revealed a great deal about their values and beliefs. The most frequent advice given focussed on effort and persistence - to try hard, listen to the teacher, ask questions, and never give up.

> I would tell them to just try as hard as he or she can. I would tell them that you should listen to the teacher and if you don't get the question you should ask the teacher and listn [listen] carefully with what the teacher explains. I would also say that you don't have to be the most brainiest mathematition {mathematican] but you can be smart if you try hard and your best. A1

> Use your own stragie [strategy] to work out problems and it's ok to have a maths mistake, you'll only learn from it, don't sit with people you know you can't work with, If you stronly [strongly] believe in a answer, don't back down. If you struggle with a maths problem, don't be afraid to tell the teacher, he/she will only help you. A16

> Lison [listen] to the teacher and try to under stand it and you will keep going fords [forwards].

Two students advised not to cheat or copy the work of other students:

> Listen to all people's ideas and find the easiest strategy to find out the answer. Try not to copy anyone's work because you wont learn anything. And ask questions if you not sure of things. A22

> Try your hardest at maths and don't cheat because it makes things worse. Ask questions if you don't understand. Give it a go, it doesn't matter if you get some wrong. A27

Several children mentioned that making mistakes provides further opportunities for learning:

> You don't have to get all of the answers right you can get right or rong [wrong] But you can still Be a champion, all it is is a miner [minor] set Back. A49

> Never give up. Listen really well so you don't miss a thing. Always try your best. Don't think about the big number. It's okay to make mistake in maths and when you do, learn from them. Good Luck. A61

DISCUSSION

On the whole, the students responded very positively to mathematics. They seemed to be more committed to the idea of mathematics learning being a social process than were the students in the study by Young-Loveridge et al. (2005). The majority of the students agreed that they learned more by working with other children (77%), they believed it to be important to explain their solution strategies to others in the class (89%), to solve differences in their answers by working through those differences (98%), and to listen to how others in the class solved maths problems (91%). These percentages are approximately double those found five years earlier in the study by Young-Loveridge et al. (2005). In the earlier study, only about half of the students thought that sharing solution strategies with others in the class was important, and only about one third of the students thought it was important to know about the solution strategies of others. These findings are consistent with the idea that significant reform in mathematics education takes considerable time (Lamon, 2007). It is also possible that the differences were a function of markedly different cohorts - virtually all of the schools in the earlier study were from low socio-economic areas (only one school was middle SES and none were high SES). The students in the earlier study were at lower stages on the number framework, including many who were still using counting strategies. It may be that until students are able to decompose and recompose numbers mentally, the benefits of sharing strategies and exploring alternative ways of solving problems are not evident to the students. Like those in the earlier study (Young-Loveridge et al, 2006), the students in this study commented on the utility of mathematics, mentioned problem-solving, and referred to number more than to other aspects of mathematics. On the whole, the students were very positive about mathematics learning and acknowledged the value of communication as part of that learning.

The last word on this topic goes to a 10-year-old girl whose advice to a new child starting the class who wanted to be a success at maths reflects the high levels of positive motivation evident for many of the children:

> It's a very huge part of life. It's your bills, your gardens, you need to find out how much seeds to buy. Without maths we wouldn't be alive. Maths is our future in +, ÷, x, = and - so learn it now! A29

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REFERENCES

Bahr, D. L. & de Garcia, L. A.: 2010, Elementary mathematics is anything but elementary: Content and methods from a developmental perspective, Belmont, CA: Wadsworth Cengage Learning.

Boaler, J.: 2008, What's math got to do with it? Helping children learn to love their most hated subject --- and why it's important for America, New York, NY: Viking.

Bobis, J., Clarke, B., Clarke, D., Thomas, G., Wright, R., Young-Loveridge, J., & Gould, P.: 2005, Supporting teachers in the development of young children's mathematical thinking: Three large scale cases, Mathematics Education Research Journal, **16** (3), pp. 27-57.

Burns, M.: 1998, Math: Facing an American phobia, Sausalito, CA: Math Solutions Publications.

Department for Education and Employment: 1999, The National Numeracy Strategy: Framework for teaching mathematics from reception to year 6, London, UK: Author.

Cook-Sather, A.: 2002, Authorizing students' perspectives: Towards trust, dialogue, and change in education, Educational Researcher, **31** (4), pp. 3-14.

Elliot, A. J. & Dweck, C. S.: 2005, Competence and motivation: Competence as the core of achievement motivation, in A. J. Elliot & C. S. Dweck (eds.), Handbook of competence and motivation, New York, NY: Guilford Press, pp. 3-12.

Gates, P. & Vistro-Yu, C. P.: 2003, Is mathematics for all? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (eds.), Second international handbook of mathematics education, Dordrecht, The Netherlands: Kluwer, pp. 31-73.

Lamon, S.: 2007, Rational numbers and proportional reasoning: Towards a theoretical framework for research, in F. Lester (ed.), Second handbook of research on mathematics teaching and learning, Charlotte, NC: Information Age, pp. 629-667.

Leder, G. & Grootenboer, P.: 2005, Editorial: Affect and mathematics education, Mathematics Education Research Journal, **17** (2), 1-8.

McCallum, B., Hargreaves, E., & Gipps, C.: 2000, Learning: The pupil's voice, Cambridge Journal of Education, **30**, 275-289.

National Council of Teachers of Mathematics: 2000, Principles and standards, Reston, VA: Author.

New Zealand Ministry of Education: 2001, Curriculum Update 45: The numeracy story, Wellington, NZ: Author.

New Zealand Ministry of Foreign Affairs & Trade: 1997, Convention on the rights of the child: Presentation of the initial report of the government of New Zealand, Wellington, NZ: Author.

Young-Loveridge, J.: 2005, Students' views about mathematics learning: A case study of one school involved in the Great Expectations project, in J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge, Findings from the New Zealand Numeracy Development Project 2004, Wellington: Ministry of Education, pp. 107-114.

Young-Loveridge, J., Taylor, M., & Hawera, N.: 2005, Going public: Students' views about the importance of communicating their mathematical thinking and solution strategies, in J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge, Findings from the New Zealand Numeracy Development Project 2004, Wellington: Ministry of Education, pp. 97-106.

Young-Loveridge, J., Taylor, M., Sharma, S., & Hawera, N.: 2006, Students' perspectives on the nature of mathematics, in F. Ell, J. Higgins, K. C. Irwin, G. Thomas, T. Trinick, & J. Young-Loveridge, Findings from the New Zealand Numeracy Development Project 2005, Wellington: Ministry of Education, pp. 55-64.

Differentiation in building mathematical knowledge

I. Algebra-related knowledge

Part 2

HOW TO HELP A STUDENT WHO ERRS WHILE KNOWING

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The paper concerns the use of algebra by lower secondary school students. The way in which a student tried to solve a few word problems will be shown in detail. The way was long and not free from mistakes, but ended with a success. An analysis of the student's behaviour reveals, that what did cause his difficulties was not transforming verbal conditions of a problem into a system of equations – as we could expect – but ... something else.

INTRODUCTION

A common opinion on difficulties that students of junior secondary level meet while solving word problems is that they are caused by the lack of knowledge and/or algebraic skill. It is not always so.

The course of work of a student that I will present here seems to indicate another possible etiology of the difficulty. The student was observed by a university student preparing her diploma dissertation. It happened a couple of years ago. But only when the idea of cognitive caching became known to me I decided to come back to that evidence and analyze it again.

Below is the evidence. It concerns David, a student of the second grade of junior secondary school (14 years old). He eagerly accepted to be observed because, as he said, he liked solving mathematical problems.

OBSERVATION 1 - DAVID'S WORK ON PROBLEM 1

Problem 1.

For each correctly solved problem a student gets 10 points, but he loses 5 points for each bad solution. After having solved 20 problems a student collected 80 points. How many problems he solved correctly and in how many ones he committed errors.

First attempt

After having read the problem the student designates:

x – the number of correct solutions

y – the number of wrong solutions

Then he writes the following equations:

$$\begin{cases} x + y = 20 \\ 80 = x + 10 + y - 5 \end{cases}$$

He applies the opposite coefficients method and arrives at

0 – 0 = 55

Looking at this equation he says: “It’s wrong. Neither x nor y came out”.

Second attempt

In a while Davit reads the problem again, looks at the equations and says: “I know already where the error is. Instead of multiplying the number of points by the number of problems I added them up”. He changes the second equation getting the following ones:

$$\begin{cases} x + y = \cancel{85}\ 20 & \cancel{/}\cdot(-1) \\ \cancel{x + 10 + y - 5 = 80} \\ 80 = x \cdot 10 + y \cdot 5 \end{cases}$$

$$\begin{cases} x + y = 20 \\ 80 = x \cdot 10 + y \cdot 5 \end{cases}$$

After applying the opposite coefficients method David arrives at:

$$\begin{cases} y = 24 \\ x = -4 \end{cases}$$

Frustrated, he says: “*Oh no! Again something is wrong. There must be somewhere an error”.* He checks the calculations without finding an error. He says he must have wrongly put the equations.

Third attempt

David reads the problem for the third time. He remarks: “I have added all those problems; yet for a wrong solution I should have subtracted points. I know; it will be so”:

$$\begin{cases} x + y = 20 & /\cdot(-1) \\ 80 = x \cdot 10 - y \cdot 5 \end{cases}$$
$$+\begin{cases} -x - y = -20 \\ 80 = x \cdot 10 - y \cdot 5 \end{cases}$$

$$\begin{cases} x + y = 20 \\ 80 = x \cdot 10 - y \cdot 5 \end{cases}$$

He solves the system as previously getting

$$\begin{cases} y = 8 \\ x = 12 \end{cases}$$ and ends.

COMMENT (PROBLEM 1)

It is clear that

1. The student has certain difficulties with formalizing the conditions of the problem: one equation in two attempts does not correctly reflect the conditions.

2. He skilfully solves a system of two linear equations with two unknowns.
3. He is not helpless, neither in a situation when he finds that the system of equations composed has no solution, nor when he sees that his solution does not fulfill the conditions of the problem. In both cases, when the attempt does not bring a success he reads the problem checking if his equations correspond with the conditions. After deciding that it's not so he builds a new equation. He looks for errors and corrects it without any intervention of the observer.

Where is the problem then? For the student it might be dramatic. If it was an examination problem, one of a few, if each would require several attempts to solve, the student would fail, despite the obvious fact that he knows what he is required to. I'm seriously afraid that this happens to not so few students, otherwise good. How could we help them?

A closer examination of the protocol suggests that the second equation in the first attempt may have come out of an understanding of the problem different from the author's intention. The student started writing equations immediately after one passing through the text. He did not undertake any verbal attempt of concretization of the given information (Mason, 2005) or getting to understand the problem's content (Polya, 1970). David might have been thinking that the solver got as many points as problems solved and, additionally, 10 points for a correct solution and –5 points for the fact that some were solved incorrectly. Only after realizing that his simultaneous equations have no solution he read again the text and, may be, noticed the word *each*. Then he changed the second equation so that it corresponded with his new understanding of the problem. He solved the system and realized again that the numbers received couldn't be correct as one of them was negative. He read then the problem for the third time, this time noticing the phrase 'he loses 5 points for each bad solution'. He wrote a new equation, now confident that it was correct.

It is very possible that David's hardship in formalizing the problem's conditions results from a quick, careless reading the text. This explanation was confirmed by the fact that David acted in a very similar way when solving several other word problems. His strategy can be generalized as follows:

- the student reads the problem,
- defines and denotes unknowns with letters,
- puts down the conditions using the letters, according to his first idea, thus receiving an equation or simultaneous equations,
- solves (skilfully) the equation or simultaneous equations,
- realizes that the solution is unrealistic,
- reads the problem again, puts down new equations, and solves,

- repeats the cycle until results do not contradict evidently the problem's conditions (but does not check if they fulfill them!).

Here is another example of applying this strategy.

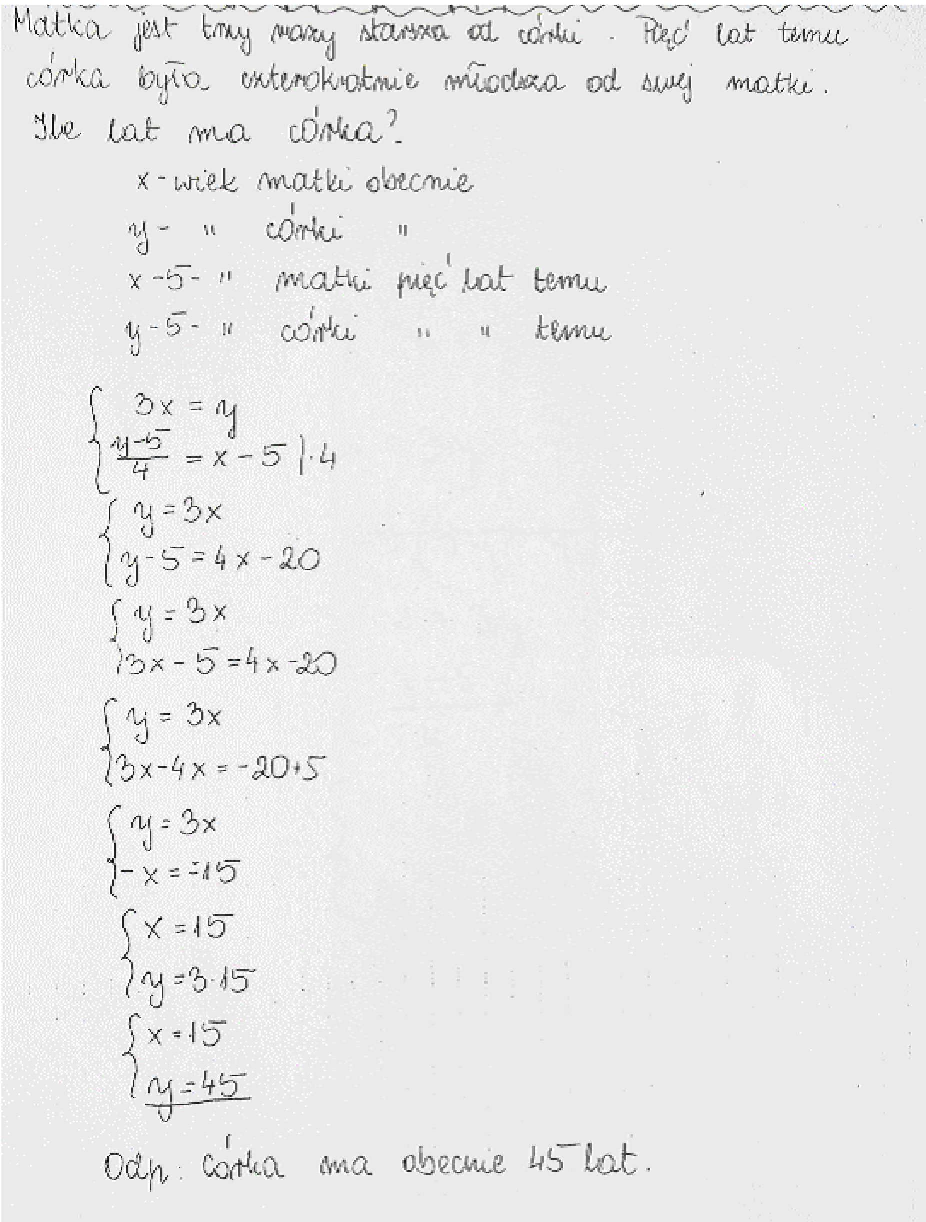
OBSERVATION 2 - DAVID'S WORK ON PROBLEM 2

Problem 2.

Mother is three times as old as her daughter. Five years ago the daughter was four times younger than her mother. How old is the daughter?

David was handed a sheet with the problem and another's student solution (figure 1). His tasks were

1. solving the problem,
2. validating the other student's solution.



Matka jest trzy razy starsza od córki. Pięć lat temu córka była czterokrotnie młodsza od swej matki. Ile lat ma córka?

x - wiek matki obecnie
y - " córki "
x-5 - " matki pięć lat temu
y-5 - " córki " " temu

$$\begin{cases} 3x = y \\ \frac{y-5}{4} = x-5 \quad |\cdot 4 \end{cases}$$

$$\begin{cases} y = 3x \\ y-5 = 4x-20 \end{cases}$$

$$\begin{cases} y = 3x \\ 3x-5 = 4x-20 \end{cases}$$

$$\begin{cases} y = 3x \\ 3x-4x = -20+5 \end{cases}$$

$$\begin{cases} y = 3x \\ -x = -15 \end{cases}$$

$$\begin{cases} x = 15 \\ y = 3\cdot 15 \end{cases}$$

$$\begin{cases} x = 15 \\ \underline{y = 45} \end{cases}$$

Odp: córka ma obecnie 45 lat.

Figure 1.

Stage 1.

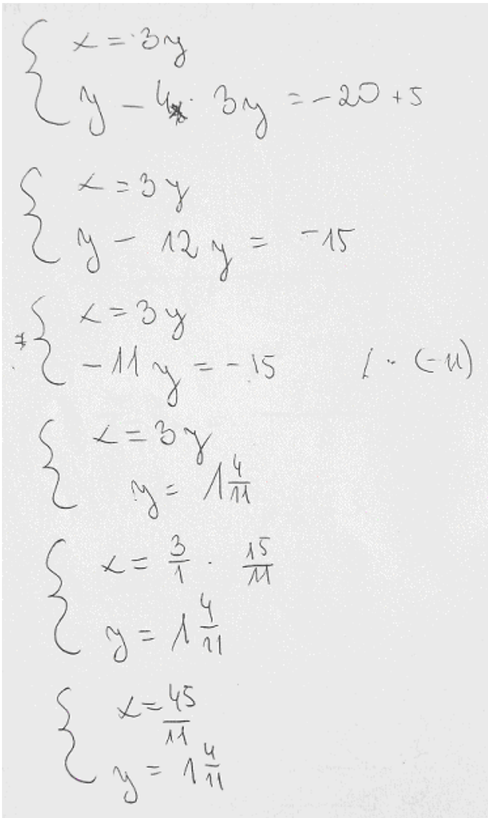
David quickly looked at the problem, the solution “x = 15, y = 45”, and the answer “Daughter is 45 years old”. He said that the solution was correct. But seeing no confirmation from the observer he changed his mind. He said: “This problem is solved badly”.

Stage 2.

David returned to the text, read it again, thought a while, and decided that the first equation is wrong, as it should be x = 3y. He put down the equations:

$$\begin{cases} x = 3y \\ \dfrac{y-5}{4} = x - 5 \end{cases}$$

then solving it quickly received:



$$\begin{cases} x = 3y \\ y - 4 \cdot 3y = -20 + 5 \end{cases}$$

$$\begin{cases} x = 3y \\ y - 12y = -15 \end{cases}$$

$$\begin{cases} x = 3y \\ -11y = -15 \quad / \cdot (-11) \end{cases}$$

$$\begin{cases} x = 3y \\ y = 1\frac{4}{11} \end{cases}$$

$$\begin{cases} x = \frac{3}{1} \cdot \frac{15}{11} \\ y = 1\frac{4}{11} \end{cases}$$

$$\begin{cases} x = \frac{45}{11} \\ y = 1\frac{4}{11} \end{cases}$$

$$\begin{cases} x = \dfrac{45}{11} \\ y = 1\dfrac{4}{11} \end{cases}$$

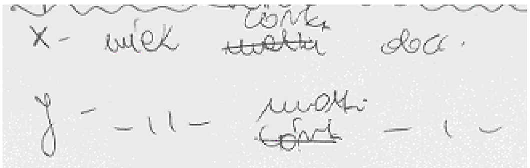
A bit perplexed, he said: “The age cannot be in such fractions, though”.

Stage 3.

David returned to the solution shown to him at the beginning and solved himself the simultaneous equations. He got x = 45, y = 15 and said: “It’s wrong as according to this solution the daughter is 45 and mother only 15”.

Stage 4.

In a while David is struck by an idea: “It should be the other way round. Letter y should denote the mother’s age, x the daughter’s age”. Thus he ended.



x- the age of the daughter

y - the age of the mother

COMMENT (PROBLEM 2)

The second task David must have perceived as untypical as it is not usual in the classroom to assign both finding a solution and validating some other's solution. Problem 2 is specific because it concerns ratio comparison which students have troubles with. The condition "number a is 3 times more than b" happens to be coded as 3a = b instead of 3b = a. Indeed, this error had been made by person U whose solution David was to validate. Did he the same error? This cannot be excluded. But it is plausible, though, that David, like in his solution of problem 1, failed to read carefully the problem prior to starting the work. He just looked at the conditions (equations) put down by U. He may have noticed that it correctly codes the fact that one person is 3 times older than another one, stated in the problem. So he decided that the condition was coded correctly. He then focused on the calculations – solution of simultaneous equations. Finding there no errors David admitted the solution. He did not verify if the solution fulfilled the problem's conditions as it did not – evidently for him – contradict with what he knows: the age was expressed by integers. The lack of any sign of approval on the part of the observer made David to think that he should come back to the problem and read it more carefully. He did, and realized that the first condition is coded wrongly with the assumed denotation of letters. He then corrected it, but failed to check if the second condition is coded rightly. He rushed to solve the simultaneous equations. The solution found he regarded as obviously contradictory with common sense (age expressed with such strange fractions?!) What did he do in this uneasy situation? He came back to the equations put down by person U, may be, thinking: "I might have overlooked some calculation error". He independently carried out the calculation, received the same results, and asked himself how old had to be the mother. Realizing that she was 15, he was struck by the absurdity of that answer. Only now he saw that swapping the meaning of letters eliminates this absurdity.

How long the way was that David had gone through to arrive at this conclusion! Mind that he arrived at it by himself, without any support from the observer. How could we help students like David so that their road to the successful end would be less dramatic?

Leron and Hazzan (1997) distinguish two kinds of approaching the problem of students' errors: those that mainly analyze cognitive aspects of students' thinking, and those that address the affect and the social environment as the *subject* of the research. The first kind of research looks at the processes "from the outside" (from the researcher's perspective), the other try to take the student's view by looking "from within". My analysis of David's behavior falls within the second type, as it attempts to understand all that happens in his mind, not only mathematical thinking, but also the coping perspective.

HOW COULD WE HELP DAWID ?

I think that the answer to this question can be found in the cognitive coaching method, whose designers Costa and Garmston (1998) define as:

> a set of strategies, a way of thinking and a way of working that invite self and others to shape and reshape their thinking and problem solving capacities. In other words cognitive coaching enables people to modify their capacity to modified themselves. [...] It is a process, which supports individuals and organizations in becoming self-directed and self-renewing. A coach is a mediator, one who figuratively stands between a person and his thinking to help him become more aware of what is going on inside his head. A large part of the role of a mediator is based on trust and rapport with the person being coached. In cognitive coaching, the person being coached, not the coach, evaluates what is good or poor, appropriate or inappropriate, effective or ineffective about his/her work.

David would need individual coaching, where the teacher (trainer), e.g. during the weekly office hour, would analyze with him the course of his work on the problem. She would remind him his actions taken in consecutive stages (based on his sheet), then ask what, in his opinion, was the reason of slowing down, what enabled him to finish sooner, more skillfully, how many times he was coming back to the problem, what would he like to change in his way of working on word problems, when he fully got to understand the problem etc. The "trainer" might point to him the need of careful reading a problem and asking himself questions that facilitate good understanding the matter of fact (questions suggested by Polya, 1970).

Then David might practice – in the presence of the teacher – action leading to good understanding and then solving selected problems.

Would it help? I leave the question open.

REFERENCES

Cielniak, J.: 2001, Postawa ucznia gimnazjum wobec zadań tekstowych. Studium przypadku. Diploma dissertation, unpublished, Pedagogical University, Kraków.

Cognitive Coaching, http://www.funderstanding.com/content/cognitive-coaching.

Costa, A., Garmston, R.: 1998, Cognitive Coaching: Mediating Growth Toward Holonomy, in: J. Strunk , J. Edwards, S. Rogers, & S. Swords, *The Pleasant View Experience*, Golden, CO: Jefferson County P.S., pp. 52-60.

Krygowska, Z.: 1989, Zrozumieć błąd w matematyce, *Dydaktyka Matematyki*, **10**, pp. 141-147.

Leron, U., Hazzan, O.: 1997, The world according to Johnny: A Coping Perspective in Mathematics Education, *Educational Studies in Mathematics*, **32**, pp. 265-292.

Mason, J. Burton, L., Stacey, K.: 2005, Matematyczne myślenie [Thinking Mathematically], WSiP, Warszawa.

Polya, G.: 1970, *Jak to rozwiązać? [How to solve it?]*, PWN Warszawa.

Schwank, I., Nowinska, E.: 2007, Zur Vorbereitung algebraischen Denkens, Beitraege zum Mathematikunterricht, Hildesbeim Franzbecker.

Turnau, S.: 1990, *Wykłady o nauczaniu matematyki*, PWN Warszawa.

Turnau, S.: 2007, Problem szkolnej algebry 1, *Matematyka*, **10**, pp. 588-594.

Turnau, S.: 2008, Problem szkolnej algebry 2, *Matematyka*, **5**, pp. 269-271.

DEVELOPING EARLY ALGEBRAIC REASONING THROUGH EXPLORATION OF ODD AND EVEN NUMBERS

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Student transition from arithmetic to algebraic reasoning is recognised as an important but complex process. An essential element of the transition is the opportunity for students to make conjectures, justify, and generalise mathematical ideas concerning number properties. Drawing on findings from a classroom-based study, this paper outlines how odd and even numbers provided an appropriate context for young students to learn to make conjectures and generalisations. Tasks, concrete materials and specific pedagogical actions were important factors in students' development of algebraic reasoning.

INTRODUCTION

For those students who complete their schooling with inadequate algebraic understandings access to further education and employment opportunities is limited. The ongoing concern, both in New Zealand and internationally, with the number of students in this position has resulted in increased research and curricula attention of the teaching and learning of algebraic reasoning. One response to addressing the problem has been to integrate the teaching and learning of arithmetic and algebra as a unified curriculum strand in policy documents (e.g., Ministry of Education, 2007; National Council of Teachers of Mathematics, 2000). Within the unification of arithmetic and algebra, students' intuitive knowledge of patterns and numerical reasoning are used to provide a foundation for transition to early algebraic thinking (Carpenter, Franke, & Levi, 2003). An essential element of this approach is the provision of opportunities for students to make conjectures, justify, and generalise their mathematical reasoning about the properties of numbers. As Carpenter and his colleagues explain, deep conceptual algebraic reasoning is reached when students engage in "generating [mathematical] ideas, deciding how to express them …justifying that they are true" (p. 6).

We know, however, from classroom studies, that currently many primary age students have limited classroom experiences in exploring the properties of numbers (Anthony & Walshaw, 2002; Warren, 2001). The more typical dominance of experiences with arithmetic as a procedural process works as a cognitive obstacle for students when later they need to abstract the properties of numbers and operations (Warren, 2001). Drawing on a national sample of Year 4 and 8 New Zealand students, Anthony and Walshaw described how many were not able to use materials to model conjectures related to arithmetic properties. Nor were they able to provide warrants to their responses which referred to generalisations of number properties. These researchers concluded that very few students in their study were able to draw upon learning experiences which bridged number and algebra.

However, studies involving teaching experiments provide clear evidence that young children are capable of reasoning in general terms. They can learn to construct and justify generalisations about the fundamental structure and properties of numbers and arithmetic. Importantly, the classroom based research studies of Blanton and Kaput (2003) and Carpenter and his colleagues (2003) demonstrate that when instruction is targeted to build on students' numerical reasoning, students can successfully construct and test mathematical conjectures with appropriate generalisations and justifications.

THEORETICAL FRAMEWORK

The theoretical framework of this study draws on the emergent perspective promoted by Cobb (1995). From this socio-constructivist learning perspective, Piagetian and Vygotskian notions of cognitive development connect the person, cultural, and social factors. Therefore, the learning of mathematics is considered as both an individual constructive process and also a social process involving the social negotiation of meaning.

I draw also on the body of research that suggests that making conjectures, generalising, and justifying are fundamental to the development of algebraic reasoning (Kaput, 1999). For young children the development of early algebraic thinking needs to go beyond simply making conjectures. Children need to gain experience in using mathematical reasoning to make explicit justifications and generalisations (Carpenter et al., 2005). Within the classroom opportunities for public/social argumentation allows exploration of conjectures and assists students to develop understanding of what comprises a suitable explanation or justification (Walshaw & Anthony, 2008).

Whilst students' propensity to offer justifications can be encouraged by classroom norms that reinforce the expectation that justifications are required, providing adequate mathematical explanations requires appropriate scaffolding, modelling and teacher intervention (Carpenter et al., 2005). Studies (e.g., Carpenter et al.; Lannin, 2005) which have examined the forms of arguments that elementary students use to justify generalisations classify students' justification as either empirical or generic examples. In the first instance, most students view specific examples, or trying a number of cases, as valid justification. These and other studies (e.g., Kaput, 1999) have shown that using concrete material can support young students to develop their justification skills. A further study by Kaput and Blanton (2005) illustrated how the introduction of large numbers as quasi-variables focused student attention on the structural features of odd and even numbers. Students were then able to draw on these structural features to connect arithmetic concepts in algebraic ways to justify their conjectures. Therefore the purpose of this paper is to report on how an examination of odd and even numbers offered young students a valuable context in which to learn how to make conjectures and construct generalisations. A particular focus is placed on the role of mathematical inquiry, concrete materials, and teacher interventions which scaffold the students to use arithmetic understandings as a basis for early algebraic reasoning. The specific question addressed in the paper asks: How can the exploration of the properties of odd and even numbers support students to use arithmetic understandings as a basis for early algebraic reasoning?

METHOD

This research reports on episodes drawn from a larger study involving a 3-month classroom teaching experiment (Cobb, 2000). The primary aim of the larger study focused on developing younger students' early algebraic reasoning through building on their numerical understandings. The study was conducted at a New Zealand urban primary school and involved 25 students aged 9-11 years. The students were from predominantly middle socio-economic home environments and represented a range of ethnic backgrounds. The teacher was an experienced teacher who was interested in strengthening her ability to support the development of early algebraic reasoning within her classroom. Each lesson followed a similar format. They began with a short whole class discussion, then the students worked in pairs or small groups and the lesson concluded with a lengthy whole class discussion.

At the beginning of the study student data on their existing early algebraic understanding was used to develop a hypothetical learning trajectory. Instructional tasks were collaboratively designed and closely monitored on the trajectory. The trajectory was designed to develop and extend the students' numerical knowledge as a foundation for them developing early algebraic understandings. This paper reports on the tasks on a section of the trajectory which built on student understanding of odd and even numbers as a context to support their algebraic reasoning. Data was generated and collected through pre and post interviews, classroom artefacts, participant observations, reflective interviews with the teacher participant and video recorded observations.

The findings of the classroom case study were developed through on-going and retrospective collaborative teacher-researcher data analysis. In the first instance, data analysis was used to examine the students' responses to the mathematical activity, and shape and modify the instructional sequence within the learning trajectory. At completion of the classroom observations the video records were wholly transcribed and through iterative viewing using a grounded approach, patterns, and themes were identified. The developing algebraic reasoning of individuals and small groups of students was analysed in direct relationship to their responses to the classroom mathematical activity. These included the use of concrete materials, the classroom climate of inquiry, and the pedagogical actions of the teacher.

RESULTS AND DISCUSSION

I begin by explaining the starting point for the section of the trajectory related to the properties of odd and even numbers. The initial starting point for classroom activity is outlined and I explain how this was used to press student reasoning towards richer understandings using concrete representations. Explanations are then offered of how the press toward deeper student reasoning was maintained through the introduction of quasi-variables. I conclude with evidence of the effect of the classroom activities using observational data from latter lessons.

The starting point on the trajectory

In order to focus student attention on the properties of odd and even numbers, a dice game was introduced during which students worked in pairs. Prompted by observations of recorded patterns the students formulated initial conjectures about the odd and even numbers:

Rani: That's odd, even, odd and then there is even, even, even, and even, odd, odd.

Matthew: There is no odd, odd, odd.

Early in the instructional sequence of tasks, students were observed to mostly use specific numerical examples to justify their conjectures:

Rani: Odd plus odd equals even… because five and seven are both odd numbers and they equal twelve which is an even number.

While continuing to justify their conjectures using numerical examples, there was evidence that students shifted from using examples involving smaller numbers to larger ones. For example, in the following exchange Heath provides a challenge to use larger numbers:

Heath: What about we could make them a bit harder? What about one hundred and eleven plus one hundred and two?

Numerical examples were also used by students as counter-claims to disprove erroneous conjectures. In the following instance the teacher's press for a student to validate a counter-claim involved the provision of a specific counter-example:

Susan: When you add to an odd number you always get an odd number.

Matthew: I am not really convinced by that.

Teacher: Why not?

Matthew: I don't think it is true because when you add like three and three you get a six which is an even number and when you go five plus five you get an even number so I don't really think that is true.

Although the students had begun to justify their reasoning using additional examples it was evident to the teacher and I that they needed to extend and deepen their reasoning. This was particularly so if they were to learn to use richer forms of justification.

Shifting students to justify using representational material

Collaborative discussion and a review of the trajectory led to the introduction of further mathematical activities. It was evident to us that the students did not have access to representations on which they could base their mathematical explanations of the properties of odd and even numbers. Therefore, we placed an explicit focus on the use of a range of different equipment. This offered the students ways to justify their conjectures, and shift their arguments into more generalised terms.

The students working in pairs used the conjectures about the addition of odd and even numbers which they had formulated in the previous lesson. Equipment (popsicle sticks, grid paper, and counters) was introduced. The students were required to develop explanations but also to represent and justify their conjectures using the concrete materials.

Many students demonstrated initial difficulties modelling their conjectures using materials. Their common initial response was to justify a conjecture by making numbers and symbols with the material:

Ruby: What we could do is… we could draw them [begins to make numbers and the plus sign from the popsicle sticks].

From these responses, it was evident that the students required further scaffolding to support their understanding of how the representational material could be used to model and justify the conjectures. The teacher intervened:

Teacher: You need to think about how you are going to prove it.

She then modelled an initial pattern of odd and even numbers (see figure 1) in order to support the students to further develop justification of their conjectures using concrete materials.

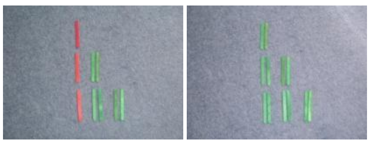

Figure 1: A model of the pattern of odd and even numbers

This visual representation of the pattern of odd and even numbers became a useful tool which the students were able to draw upon to deepen their understanding. In one instance, a student developed an explanation of the structure of even numbers through her observations of the visual pattern:

Ruby: So if it's an even number, that's basically like a twos number…adding an even number on say four which is a two's number, and so if you have got a two's number like four, six, eight or ten…then that's how you know it is an even number because if you counted up in two's that's how you would know. Eleven is not one because two, four, six, eight, ten.

In another example, a student developed a verbal definition and explanation of the structure of odd numbers through use of the model.

Rachel: That's the first odd [puts out popsicle stick I] and the second odd is three [puts out popsicle sticks I II] and you can't really put it in a group because there will always be one left over.

The model was also used in subsequent instances by students to develop concrete justifications for their conjectures. For example, during the whole class discussion a student justified her conjecture that adding two odd numbers made an even number:

Hannah: You could move the three [moves single sticks to make pair] or you could move the five [moves single sticks to make pair].

Another pair of students justified their conjecture about adding even numbers:

Rani: For an even plus even equals even and for every number you could have pairs like twos, two and two and two [places the popsicle sticks II II II]

Matthew: So every time you add an even number onto an even number it equals an even number.

The material provided an important thinking tool which focused student attention on the structure of the numbers. Teacher modelling and the requirement that the students justify their reasoning using numeric and visual strategies supported their transition toward the use of more sophisticated justification strategies.

Deepening the reasoning using quasi-variables

In line with the progression along the trajectory we now analysed that the students were ready for the introduction of equations with large numbers as quasi-variables. These were introduced to extend the student understanding of the properties and structure of odd and even numbers. The students' initial discussions focused on the structure of the odd and even numbers. Through extended discussion they explored their understandings, readily drawing on mathematical argumentation which deepened their concepts of number properties. This is illustrated in paired work when a student began by examining the quasi-variables digit by digit:

Rani: In the six one [reference to the number 6398] it goes even number, odd number, odd number and even number.

Her partner noticing that the oddness or evenness of a number depends on the final digit disagrees and validates his reasoning through analysing the end digit:

Matthew: No on the first one they are both even…

Rani: But the five and the six aren't even.

Matthew: But it is the number at the end that matters.

Again in the following whole class discussion the structural aspects of the numbers are noted. Michelle states that the sum of two numbers are odd after looking at the two first numerals in the numbers 6398 + 5296, Gareth challenges, and provides backing for his argument:

Michelle: Because six plus five equals eleven and eleven is odd.

Gareth: How do you know that that works to make that question odd because it is just using the six thousand and the five thousand?

The teacher revoices the statement and presses the students to consider which digits in the number determine whether it is odd or even:

Teacher: When you add odd numbers and even numbers you don't necessarily look at the first number. Which part do you think we should look at?

Caitlin: The numbers at the end.

Following this episode, the teacher asked the students to reflect on what they had learnt and how their thinking had changed:

Caitlin: Because me and Gareth we looked at all the numbers.

The teacher revoiced the statement as a question.

Teacher: Oh okay you looked at all the numbers in here?

At this the student further clarifies her developing understanding of which digits to look at when judging whether a number is odd or even.

Caitlin: Not just at the two at the end.

Teacher: Now you know you can just look at the end one.

In the reflective discussion which followed this lesson many students stated that this episode had been significant in shifting their thinking. For example, Michelle referred to her growing understanding of how to validate whether a number was odd or even as:

Michelle: Because at first I thought you look at the front numbers and now I think you look at the back.

In subsequent lessons the teacher continued to prompt students to examine the structural features of numbers, employing instructional strategies that revoiced and pressed the students to use their previously justified conjectures to determine whether the sum would be odd or even:

Teacher: You don't have to add the numbers though. What do you notice about five and seven Hannah?

Hannah: They are both odd.

Teacher: They are both odd. What do two odd numbers make together?

Hannah: An even number.

Students were able to utilise and model this argument in latter instances. For example this is illustrated during paired work solving the equation (e.g., 189 197 + 36 455):

Matthew: The end numbers are seven and five and they are both odd so you would go odd plus odd equals even.

Later in a large group discussion the students developed an explanation for a solution to an equation (e.g., 192 197 + 124 364) using similar justification:

Sarah: Because the end numbers are an odd and an even.

Teacher: An odd and an even and what do you know about an odd number and an even number Sarah?

Sarah: When you add an odd number and an even number you get an odd number

These examples illustrate that the combination of extended exploration, the requirement that students explain and justify their reasoning with materials and the use of quasi-variables, supported the students to deepen their understanding of the properties of odd and even numbers and provide more proficient explanations.

CONCLUSIONS AND IMPLICATIONS

This study sought to explore how student exploration of the patterns and properties of odd and even numbers supported the students' construction of conjectures, justification and generalisations. The importance of mathematical discussion and argumentation was illustrated. Importantly the teacher's role was central to shifting the students towards using justification and age appropriate proof. Similar to the findings of Carpenter and his colleagues (2005), these students initially viewed multiple numerical examples as justification for their conjectures and generalisations. The introduction of equipment led to student modelling of conjectures and concrete forms of justification although as Anthony and Walshaw (2002) described, many students initially had difficulties modelling conjectures with materials. Teacher intervention and modelling was required to scaffold student use of concrete forms of justification. The visual representation of the pattern of the numbers also drew student attention to their structure.

Tasks using large numbers as quasi-variables, combined with expectations and practices involving extended discourse, deepened student understanding of the properties of odd and even numbers. This study confirmed Kaput and Blanton's (2005) claim that through using large numbers students "had to examine the structural features to reason whether a sum would be even or odd, and were led to focus on the properties of evenness and oddness and (implicitly) to treat the numbers as abstract placeholders" (p. 114). Thus, links were made across arithmetic and early algebraic reasoning.

Findings of this study affirm that the context of odd and even numbers can provide students with effective opportunities to make conjectures, justify and generalise. Argumentation and teacher intervention supported students to model their conjectures on material and use concrete justification strategies. However, further research is required to validate the findings of this study due to the small number of participants involved.

REFERENCES

Anthony, G., & Walshaw, M.: 2002, *Students' conjectures and justifications: Report of a probe study carried for the National Education Monitoring Project*, Palmerston North, Massey University.

Blanton, M., & Kaput, J.: 2003, Developing elementary teachers "algebra eyes and ears", *Teaching Children Mathematics*, **10**, pp. 70-77.

Carpenter, T., Franke, M., & Levi, L.: 2003, *Thinking mathematically: Integrating arithmetic and algebra in elementary school*, Portsmouth, Heinemann.

Carpenter, T., Levi, L., Berman, P., & Pligge, M.: 2005, Developing algebraic reasoning in elementary school, in: *Understanding mathematics and science matters,* Mahwah, NJ, Lawrence Erlbaum, pp. 81-98.

Cobb, P.: 1995, Cultural tools and mathematical learning: A case study, *Journal for Research in Mathematics Education*, **26**, pp. 362-385.

Cobb, P.: 2000, Conducting teaching experiments in collaboration with teachers, in: *Handbook of research design in mathematics and science education*, Mahwah, NJ, Lawrence Erlbaum, pp. 307-333.

Kaput, J.: 1999, Teaching and learning a new algebra, in: *Mathematics classrooms that promote understanding,* Mahwah, NJ, Lawrence Erlbaum, pp. 133-155.

Lannin, J.: 2005, Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities, *Mathematical Thinking and Learning*, **7**, pp. 231-258.

Ministry of Education.: 2007, *The New Zealand curriculum*, Wellington, Learning Media.

National Council of Teachers of Mathematics.: 2000, *Principles and standards for school mathematics*, Reston, VA, Author.

Walshaw, M., & Anthony, G.: 2008, The role of pedagogy in classroom discourse: A review of recent research into mathematics, *Review of Educational Research*, **78**, pp. 516-551.

Warren, E.: 2001, Algebraic understanding and the importance of number sense, in M. van den Heuvel-Panhuizen (Ed.), Proceedings of the 25th annual conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 399-406). Utrecht: PME, pp. 399 – 406.

THE EFFECT OF INTELLIGENCE TYPE ON LEARNING ALGEBRA

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The author applies the theory of multiple intelligences (MI) while teaching algebra to below-average 14-year-old students (scoring between the 25th and 45th percentiles on the Illinois Standards Achievement Test), hypothesizing that using didactic methods based on dominant intelligence types would enhance students' learning. The author administered 2 MI tests to students, supplemented by a questionnaire for parents and 4 student assessments to identify learning success. Students showed improvement in assessments, particularly when the dominant intelligence type was linguistic. Thus, mathematics education seems to depend on students' problem-solving predispositions and how the teacher organizes instruction and selects mathematical tasks.

INTRODUCTION AND STATEMENT OF THE PROBLEM

This research was inspired by the author's personal experiences in didactic work as a mathematics instructor at Thomas Kelly High School in Chicago. Starting in 1991, I taught mathematics at different levels, including Pre-Algebra, Algebra I 10X, (regular) Algebra I, Geometry, and Algebra II with Trigonometry.

I based my study on Howard Gardner's concept of multiple intelligences (Gardner, 1983, 1993, 1999; Gardner & Hatch, 1989) and Zofia Krygowska's approach to didactic methods for teaching mathematics (Krygowska, 1957, 1977). I also used Benjamin Bloom's taxonomy (Bloom, 1986) and concepts from the writings of Richard Skemp (1971, 1979). My research question explored whether classifying students' mathematical activity according to the intelligence type characterizing a particular group influences how well they learn mathematics. The dominant intelligence types considered were the eight proposed by Gardner (1999): logical-mathematical, linguistic, kinesthetic, visual-spatial, intrapersonal, interpersonal, naturalistic, and musical. Students possessing different intelligence types are characterized by different learning strategies.

For my research, I hypothesized that pairing the didactic methods teachers use with the students' dominant intelligence types would improve students' learning. First, it was necessary to answer several preliminary questions, such as:

- What kinds of activities are best suited for persons with a particular dominant intelligence type?

- For which dominant intelligence types are textbooks and traditional teaching programs designed?

My research, including both the present study and studies conducted in many previous years, support the assumption that the most common curricula, presented in conjunction with matching textbooks and other didactic materials, are tailored primarily for people whose dominant intelligence type is logical-mathematical. In a sense, these curricula discriminate against people with different predispositions that manifest through other types of intelligence.

METHOD AND INSTRUMENTS

The research was carried out in September 2009 at the Thomas Kelly High School in Chicago. In conducting the didactical experiment, I used numerous research tools at different stages. I constructed all of these instruments myself, although in a few cases they were based on tests already used for similar diagnostic purposes. I also used commonly available online sources in developing my tests.

Intelligence Type

The first group of tests, based on Gardner's (1983) concept of multiple intelligences, was used to determine students' dominant and least developed intelligence types. The two multiple intelligences tests (given to students) were based on the commonly used Teele Inventory for Multiple Intelligences (TIMI) (Teele, 1992), which aims to assess learning styles and is currently used at many schools in the United States. Although Teele claims that the TIMI has adequate reliability, this conclusion has been questioned (McMahon, Rose, & Parks, 2004).

Given that properly identifying the dominant type of intelligence was of key importance for the study, I decided to use two multiple intelligences tests spaced across three days, which enabled me to determine the intelligence types with greater validity. The two tests (given to students) were supplemented with a questionnaire distributed to students' parents that asked them about their children's predispositions for particular learning styles.

Learning Algebra

The next group of tools addressed the didactic situation. I modified the methods proposed by Bellman (2009) in the textbook I used for teaching Algebra I. The main tool consisted of extensive (seven block) lesson scripts intended to help students of a particular intelligence type, based on the students' profiles. To understand specific topics in algebra, tasks based on Krygowska's (1977) approach to didactic methods for teaching mathematics and incorporating the essence of each intelligence type in the chosen classroom strategy were designed and administered to students. According to students' recognized common dominant intelligence types, the lesson scripts included analytic, introspective, and interactive tasks to facilitate interconnections between/among intelligences. To help students become multi-sensory learners (tapping into multiple intelligences), lessons were presented in a variety of ways, incorporating problem-solving, visual and movement activities, and print-rich activities as well as time for reflection and group activities.

To assess the effectiveness of the adopted methodology, I used four key tools: a preliminary test, an ex post facto test, quizzes, and a student survey. The preliminary test and the ex post facto test consisted of word and computation problems. Although the two types of problems were similar in terms of content, they were presented in a different order from topics presented in class. Moreover, during each of the classes the students took quizzes measuring how well they had learned the material covered in the preceding class. Each quiz contained one word problem and one computational problem. In each class, the students also filled out a questionnaire aimed at collecting information on how well they believed they could comprehend the class material. Thus, the tools were designed to be appropriate for the particular types of students tested.

PARTICIPANTS

The sample included 24 students from the Algebra I 10X course, which is for students who score between the 25^{th} and 45^{th} percentiles on the Illinois Standards Achievement Test (ISAT). The students were required to take two class-hours in mathematics as well as two in English to improve their reading and writing skills. Based on the questionnaires I distributed in September to both parents and students, 22 of the 24 students came from non-English-speaking homes. These students were bilingual, with Spanish as their first language.

Four of the 24 students were eliminated from the study because they did not attend all of the lesson blocks. One student in a special education program was eliminated because she had been assessed as functioning at the fourth grade level of intellectual efficiency; including her could have distorted the study's findings. Finally, two students were eliminated because they were the only representatives of the dominant or least developed intelligence type in the sample (one had musical-rhythmical as the dominant intelligence type and the other had kinesthetic as the least developed type). Thus, the final sample consisted of 18 students for the analysis of dominant intelligence and 17 students for the analysis of least developed intelligence.

Data about the dominant and least developed intelligence type were made available the students in both groups, respectively. The following dominant types of intelligence were found: linguistic ($n = 4$), linguistic plus intrapersonal ($n = 3$), intrapersonal ($n = 5$), interpersonal ($n = 4$), and kinesthetic ($n = 2$). For least developed intelligence, the breakdown was as follows: linguistic ($n = 3$), interpersonal ($n = 2$), and logical-mathematical ($n = 12$). The large number of students in the latter category is noteworthy given the nature of the textbooks used, as previously mentioned.

RESULTS

Only the most important results are presented below; results included herein do not include data from the student surveys. The data for the dominant and least developed intelligence type were analyzed separately. Table 1 presents the results for the quizzes, preliminary tests, and ex post facto tests for the dominant intelligence types. Table 2 presents the results for the quizzes, preliminary tests, and ex post facto tests for the least developed intelligence types.

Dominant Intelligence Type	Incorrect Answers (%)															
	L1				L2				L3				L4			
	Q1	T1	T2	Tx	Q2	T1	T2	Tx	Q3	T1	T2	Tx	Q4	T1	T2	Tx
Interpersonal	63	86	46	40	63	82	46	36	50	75	46	29	50	64	43	21
Kinesthetic	50	79	50	29	50	71	29	42	25	50	14	36	25	64	43	21
Intrapersonal	67	66	37	29	44	66	23	43	31	66	23	43	19	61	23	38
Linguistic	50	73	33	40	36	69	16	53	21	67	31	36	14	61	27	34
Linguistic, Intrapersonal	33	71	38	33	33	71	14	57	33	76	24	52	0	57	24	33
Dominant Intelligence Type	**L5**				**L6**				**L7**				**Total change of Tx for L1–L7**			
	Q5	T1	T2	Tx	Q6	T1	T2	Tx	Q7	T1	T2	Tx				
Interpersonal	50	75	54	21	50	68	50	18	50	79	43	36	201			
Kinesthetic	25	57	29	28	50	79	43	36	0	50	30	20	212			
Intrapersonal	25	64	25	39	13	64	18	46	6	57	27	30	268			
Linguistic	8	73	20	53	14	53	20	33	7	67	20	47	296			
Linguistic, Intrapersonal	0	76	10	66	0	57	24	33	0	67	24	42	317			

Table 1: Mean percentages of incorrect test answers as a function of dominant intelligence type.

Note. L1–L7: numbers of lesson blocks. Q1–Q7: numbers of lessons that the quizzes refer to. T1: preliminary test. T2: ex post facto test. T1 (under L1): average of incorrect answers on the preliminary test on topics covered during the first lesson block. Tx: average improvement (T1-T2)

Least Developed Intelligence Type	Incorrect answers (%)															
	L1				L2				L3				L4			
	Q1	T1	T2	Tx	Q2	T1	T2	Tx	Q3	T1	T2	Tx	Q4	T1	T2	Tx
Interpersonal	50	86	43	43	50	71	29	42	25	50	14	36	25	50	14	36
Linguistic	50	57	29	28	50	67	29	38	33	62	23	39	33	62	23	39
Logical-mathematical	67	77	38	39	54	73	27	46	38	69	35	34	33	69	34	35

Least Developed Intelligence Type	**L5**				**L6**				**L7**				**Total change of Tx for L1–L7**
	Q5	T1	T2	Tx	Q6	T1	T2	Tx	Q7	T1	T2	Tx	
Interpersonal	25	57	29	28	50	79	43	36	0	50	36	14	235
Linguistic	33	57	33	24	17	71	10	61	0	52	29	23	252
Logical-mathematical	27	74	30	44	29	60	20	40	42	70	30	40	278

Table 2: Mean percentages of incorrect test answers as a function of least developed intelligence type.

Note. Symbols used in Table 2 are identical to those used in Table 1.

DISCUSSION

Noticeable improvement occurred in the results of all tests over the course of the study. Students were able to absorb the issues related to one problem, which enabled them to better understand the subsequently presented problems and acquire new skills more quickly. This noticeable improvement positively influenced students' motivation, translating into greater engagement and more intellectual activity.

Dominant Intelligence Types

Participants whose dominant intelligence type was linguistic demonstrated a great improvement not only in quiz scores, but also in their scores on the ex post facto test. These students made fewer mistakes on the quizzes than other groups of students. Given the previous discussion of traditional teaching methods and textbooks, it might be tempting to conclude that these students—although less able than their peers—still coped much better than students for whom the didactic methods used in their regular junior high school classes were inadequate for their type of intelligence. Linguistic-type students, despite their shortcomings, acquired some skills during the previous course that they were able to use to some extent in subsequent courses. This finding explains the surprising improvement exhibited by students with two types of dominant intelligence.

At the same time, it is worth noting that students with the intrapersonal dominant intelligence type achieved great improvement in the quizzes and ex post facto tests. For such students, didactic procedures can play a decisive role in educational success. Many contemporary didactic fields emphasize teamwork, often implemented as group problem solving for a different purpose than the formal lessons. In such a situation, students who prefer individual problem solving most likely lose more than students who prefer an interpersonal approach but are forced to act individually. The latter students withdraw or try to impose their opinions on the group. Giving these students the option of independent problem solving could markedly increase their effectiveness while allowing them to realistically assess their potential.

Relatively less improvement was shown by students whose dominant intelligence was kinesthetic or interpersonal. This limited improvement could have resulted from difficulties in adapting the didactic methods used in teaching algebra to the specific needs of such students as well as from the contradiction that arises from applying the same methods to students with interpersonal and intrapersonal intellectual tendencies.

Least Developed Intelligence Types

An important issue facing mathematics teachers is dealing with students whose least developed intellectual skills are logical-mathematical or linguistic. Given that the didactic methods used in the standard (textbook) course were modified to better address the needs of students with other dominant types of intelligence, the significant improvement in the results of the students whose least developed intellectual skills were logical-mathematical and linguistic is impressive. This finding suggests that progress in mathematics education depends not only on individual predispositions for solving problems, but also on how the teacher organizes the instructional process and selects the mathematical tasks for students. Moreover, this finding calls into question the validity of the standard distinction between “scientific minds” and “art scholars”—a distinction applied to students at an early age, often perhaps in a “hidden curriculum” (Bourdieu, 1992). In our sample, improvement in learning is apparent not only from the results of ex post facto tests, but also from the quiz results. A systematic decrease occurred in the average percentage of incorrect answers on these tests, suggesting that students were enhancing their ability to combine individual skills.

Perhaps the most important factor explaining this improvement is motivation. Students’ awareness of the existence of multiple intelligences and their own dominant type of intelligence enabled them to attribute their poor achievement to an inability to properly organize lessons and select the optimal didactic tools. Thus, knowledge of their multiple intelligences strongly motivated them to try to learn the course material.

CONCLUSIONS

Analysis of the data led to a single set of conclusions. Educators should pay attention to the twofold role that tests of intelligence type play in the didactic process. These tests allow teachers to profile their classes on this dimension, thereby allowing them to teach more effectively. The tests also serve to motivate students, especially those commonly assessed as below-average but who in fact simply have different learning propensities and thus require a different educational approach than the standard (textbook) one. Knowing the dominant intelligence type of below-average students allows for a more effective selection of didactic tools. As shown in the present study, matching didactic methods with students’ learning tendencies, as defined by Gardner’s (1999) intelligence types, greatly improves their mathematical learning. Thus, it is reasonable to assume that a proper selection of didactic methods based on dominant intelligence types can accelerate the process of acquiring knowledge on a wider range of topics, thereby promoting the inclusion of students requiring additional help in regular academic courses.

Although further analysis of these results is warranted, it is still worth stressing at this point that this subject area can play a significant role in the didactics of mathematics at almost every level. Further research is needed to test new hypotheses suggested by the results presented here.

REFERENCES

American Commission on Excellence in Education: 1983, *A nation at risk: The imperatives for educational reform,* ACEE, Washington, DC.

Bellman, A., Bragg, S. C., Charles, R. I., Hall, B., Handlin, Sr., W. G., & Kennedy, D.: 2009, *Algebra 1*, Prentice-Hall, Upper Saddle River, NJ.

Bourdieu, P., & Wacquant, L. J. D.: 1992, *An invitation to reflexive sociology,* University of Chicago Press, Chicago.

Bloom, B.: 1986, *Mastery of learning evaluation: Comments on volume 1, no. 2,* University of California, Los Angeles.

Gardner, H.: 1983, *Frames of mind,* BasicBooks, New York.

Gardner, H.: 1993, *Multiple intelligences: The theory in practice,* Basic Books, New York.

Gardner, H.: 1999, *Intelligence reframed. Multiple intelligences for the 21st century,* Basic Books, New York.

Gardner, H., & Hatch, T.: 1989, Multiple intelligences go to school: Educational implications of the theory of multiple intelligences, *Educational Researcher,* **18**(8), pp. 4-9.

Krygowska, Z.: 1957, *Metodologiczne i psychologiczne podstawy czynnościowej metody nauczania matematyki*, WsiP, Warsaw, Poland.

Krygowska, Z.: 1977, *Zarys dydaktyki matematyki,* WsiP, Warsaw, Poland.

McMahon, S. D., Rose, D. S., & Parks, M.: 2004, Multiple intelligences and reading achievement: An examination of the Teele Inventory of Multiple Intelligences, *Journal of Experimental Education*, **73**, pp. 41-52.

Skemp, R. R.: 1971, *The psychology of learning mathematics,* Penguin Books, Middlesex, England.

Skemp, R. R.: 1979, *Intelligence, learning, and action: A foundation for theory and practice in education,* Wiley, Chichester, England.

Teele, S.: 1992, *Teele Inventory for Multiple Intelligences (TIMI),* Sue Teele and Associates, Redlands, CA.

THE SOCIAL CHARACTER OF LEARNING VIA BUILDING INDIVIDUAL COGNITIVE WEBS

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The social character of learning plays a vital role in the process of learning mathematics. It can support the process of building students' mathematical knowledge. This paper presents how students build their own webs of cognitive connections during working on tasks concerning generalization and how the social character of learning influences these individual webs.

THEORETICAL BACKGROUND

According to general theory developing and building child's mathematical knowledge is functioning in the frame of cognitive constructivism, The child learns/develops its mathematical knowledge through building its own cognitive structures, a web of interrelationships, mental "maps" (Hejný, 2004, Skemp, 1979). Accumulated experience enables children to create a so-called data set, used by them to build up their mathematical knowledge.

The inner structure of mathematical knowledge (internal mathematical structure - IMS, Hejný 2004) is:

> ... **dynamic web of connections** with many elements of knowledge, such as concepts, facts, relations, examples, strategy of solutions, algorithms, procedures, hypothesis, ..., creating nodes of this web. All these cause the existence of the IMS. The IMS is a web by itself, connecting these all elements. At the same time IMS is the way of organizing all these elements which create knowledge (Hejný 2004)

According to this definition the most crucial issue is that this cognitive web is dynamic, permanently changing. The development of knowledge is connected with dynamic, its changes and reorganization. Due to it, not only the new nodes of web are aroused and the new connections are created, but also those existed gain new meaning. The process of building the new inner structure of mathematical knowledge is firmly connected with restructuring the web already existed. To influence the cognitive web we need both inner and outer factors. The inner factors are student's own experiences, their own knowledge, independently created solutions of problems and their own interpretation of noticed facts and phenomenon. They create a very subjective web of cognitive connections.

The process of building new mathematical knowledge is shown by prof. Hejný in the TGM theory (Theory of Generic Model) (Hejný, 2005) According to this theory cognitive process is decomposed into two levels: generalization (understanding in a local sense) and abstraction. Connecting the point of both levels is a generic model – the pivot term of TGM theory. For Generalization level the generic model is the final stage and for Abstraction level – it is a starting point. The process of building the new knowledge begins with collecting experiences which are stored in our minds as isolated models connected with concrete situations.

If this set of experiences is extensive enough that among similar isolated models will start to create connections. This web of connections will be thicken slowly, the object represented wide group will appear. When one model representing general features of others, which means general model for particular situation appears instead of many isolated models, we can start to build our abstract mathematical knowledge. If a child is not able to create a general model for a particular situation she or he would not use abstract knowledge.

On the other hand, it is very essential that the outer factors have an influence on the development of individual thinking. Those which have an influence on the development of personal cognitive web are first of all: the environment of learning, the classroom, the teacher and other students. The way of thinking depends not only on the students' involvement in a process of solving the mathematical problem, but also on interactions between the teacher and the students or among the students, which appears during the work on a task or a problem (Steinbring, 2003).

The reflection on our experience is a perfect starting point for our own understanding of the world. The reflection appears when we have to manifest our ideas. Everyone creates their own 'rules' and mental models, which we try to apply in order to understand and use our experience of mastering the knowledge of our environment. While expressing our thoughts, we look for an appropriate form of words (Wygotski, 1989). Uttering our thoughts we dress them in the words. We move from an 'inner' speech (that is speech for oneself) to an 'outer speech' (that is expressing our thoughts, presented them for others). The language transformation follows, which relies on uttering thoughts in our own words (Wygotski, 1989). The verbal language plays very important role during analysing of a particular problem's solution. Its causes a change in reception of discussing text and stimulates creating new connections in the existed set of information. Reflection does not appear automatically among 7-11 children. Therefore, a conversation during a cooperation with students is an opportunity to recognize their mental processes while solving their tasks. Whereas, for students it is a tool for organizing their own lines of thinking (Consogno, 2005). It allows to observe the new possibility to solve a problem or to understand its issue.

Additionally the social character of learning mathematics requires many activities, inter alia comparing our own ideas and solutions with the other participants of the process of learning. At the first glance those outer factors interfere with building our own web of cognitive connections and they also demolish already existing subjective web. However it is a deceptive disturbing because in reality, due to those outer factors, a student is able to convert his or her inner and individual cognitive web to pick up the gaps and the wrong connections in it. Due to overlapping those two factors the student entirely develops his/her individual web of cognitive connections.

THE AIM OF THE RESEARCH

The research shown here is the part of wider research concerning developing students' algebraic thinking, building their personal web of cognitive connections during solving the task connected with discovering mathematical regularity.

The aim of my research was to give the answer to the following questions:

- How 9-10-years old students create their own web of cognitive connections during their work on the task concerning discovering regularity?
- In what way they 'think' about regularities and what is their thinking processes while solving tasks in which they have to discover and use noticing rules?
- Will they be able to cooperate while solving the task and how will this common work influence their or the ways of solving the task?

In this paper I will focus on an analysis of the following phenomena:

- Differences in individual cognitive webs among students working together.
- the social character of learning influences the building of individual cognitive webs.

METHODOLOGY

Presented research was carried out in November 2009 among students from the fourth grade of a primary school. Twenty (9-10 years old students) working in pairs took part in them. The research contained four following meetings, during which students were solving following tasks. All meetings were recorded by a video camera. After the research, the report was presented. The students were working in pairs. The researcher was talking with every group of students while they were solving the tasks.

The students had work sheets, matches (black sticks), ball pens and a calculator. Before students started their work, they had been informed that they could solve this task in any way they would recognize as suitable; their work would not be graded; teacher would be videotaping their work and that they could write everything on the work sheet which they recognize as important. The research material consists of work sheets filled by students, as well as the film recording their work and a stenographic record from it.

The research tool consisted of four sheets and each of them consisted of two tasks. The tasks were as following: the students make a match pattern consisting of geometrical figures – one time there are triangles and another time there are squares with a side length of one match. In the first two sheets the figures were arranged separately, in the second two – connected in one row. The next sheets concerned: (1) separated triangles, (2) separated squares, (3) connected squares and (4) connected triangles. In each of the sheet the problem was presented in a frame of two next tasks. They were constructed in such a way in order to inspire students to search and discover occurring rules.

In the first task the students had to give the number of matches needed to arrange one after the other from one to seven triangles or squares. The question was: How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 of such figures? The results should be written in the table. In the task two, there was a question about a number of matches which are needed to construct 10, 25 and 161 of such figures (Littler, 2006). In order to give an answer for these questions the students had to discover the rule occurring in the first task.

The choice of the tasks and the order of the sheets were not random. The problem was to check if the students will benefit from their earlier experience while solving the new tasks. As already elaborated, the strategy of solving the problem will be applicable while doing the next task. Accepting this kind of strategy will prove an appropriate construction of the research tool – that is which provokes enlarging already existed cognitive web towards building a generic model.

This task and the way of its presentation (four following sessions) were something new for students. So far during maths lessons they did not solve the tasks concerned with the perception of the appeared rules and generalization of noticed regularities.

GENERAL RESULTS OF ANALYSING THE STUDENTS' WORK

After analysing all students' papers we can differentiate the following characteristic way of acting in a particular sheet:
<u>The first sheet</u> – students often start from arranging one or a few triangles (collected physical experiences); they investigate the whole sheet as separate parts: the table was another kind of task than the task 2. Students usually discover two rules. For the task 1 (in the table) the discovered rule that is: add three to a previous value. For the task 2 used rule sounds: multiply the number of triangles by three. All answers and formulated rules were correct, students were able to make a generalization, they did not use any symbolic notation. Students move from isolated models to generic model.

The second sheet – this time students resigned with physical experiences. They noticed an analogy to the previous task (from the first sheet). Everyone applied the rule “multiply by four”. Solving this task lasted less time than in the case of the first sheet. Students treated the task as a whole and started from the generic model.

The third sheet was a challenge for students. At the beginning of their work, they were trying to transfer a solving method from previous sheets. However noticing that it was ineffective, they looked for another solution. They started to analyse contents of the task, next they arranged a fragment of a pattern using matches – for two, three squares. They discovered the rule: the first square made of four elements, each following made of three elements. Therefore, in order to give the number of needed matches, one should add three to the previous number. After completing the table, when students moved from the first task to the second, two ways of actions appeared: continuation of “adding numbers three” to ten squares or searching for “components” in the table, using previously obtained data. They discovered different rules for this task: “add 3 to the previous number” for the first task and for the second one: “multiply number of squares by 3 and add 4”, “Number of matches is the number of squares less one multiply by 3 and add 4” and “Number of matches is the number of squares multiply by 4 and then subtract number of squares less one”. Then an attempt to move from isolated models to a generic model appeared, which was connected with verbalisation of students’ gained experience that emerged at the end of the work on the second task

The fourth sheet - again the students referred to physical, manual experiences connected with arranging pattern. Here however, they used their own experience gained while working on the third sheet, so the solving process of the task progressed quite efficiently. The students treated the task as a whole however different rules appeared. They discovered rules: for the first task “add 3 to the previous number” and for the second task “multiply the number of triangles by 2 and add 1” or “multiply by 2 the number of triangles diminished by 1 and then add 3”.

At this point it would be interesting to take a closer look at the solution of tasks from all sheets, which was made by two boys: Adam and Damian.

THE RESULT OF THE ANALYSIS OF ADAM AND DAMIAN’S WORK

Setting about solving the task from the first sheet each of the boys arranged one of the triangle. It was their answer for the teacher’s question “Can you arrange a triangle?” For the task they used only one rule: “Multiply the number of triangles by 3”. They were able to utter the applied rule. They were also able to put down this rule in an elaborated descriptive form. Although the boys’ work seemed very similar, however the detailed analysis of a collected material (the film about the course of their work) showed that their ways of thinking were different. The following scheme (Figure 1) shows this idea:

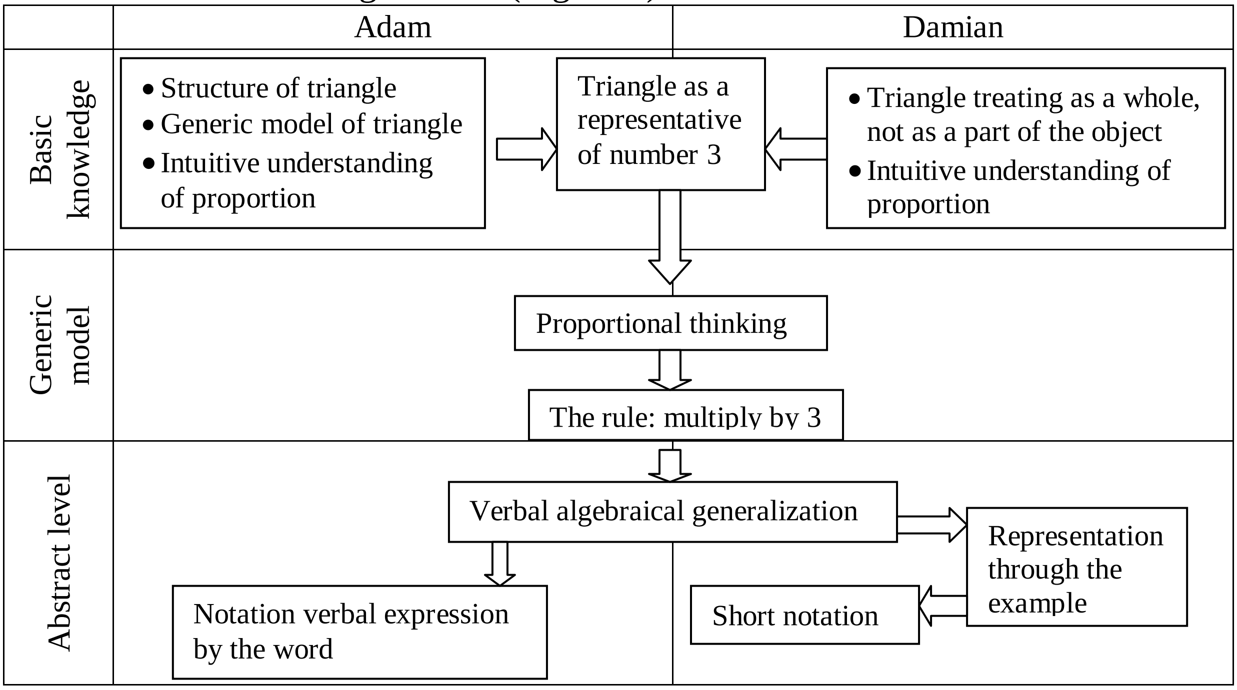


Figure 1

The work on the second sheet was very similar. This time Adam resigned from arranging the squares and he started to fill the table. At first Damian arranged one square and then he moved to solve the second task. In the whole task both students used the rule "Multiply number of squares by 4".

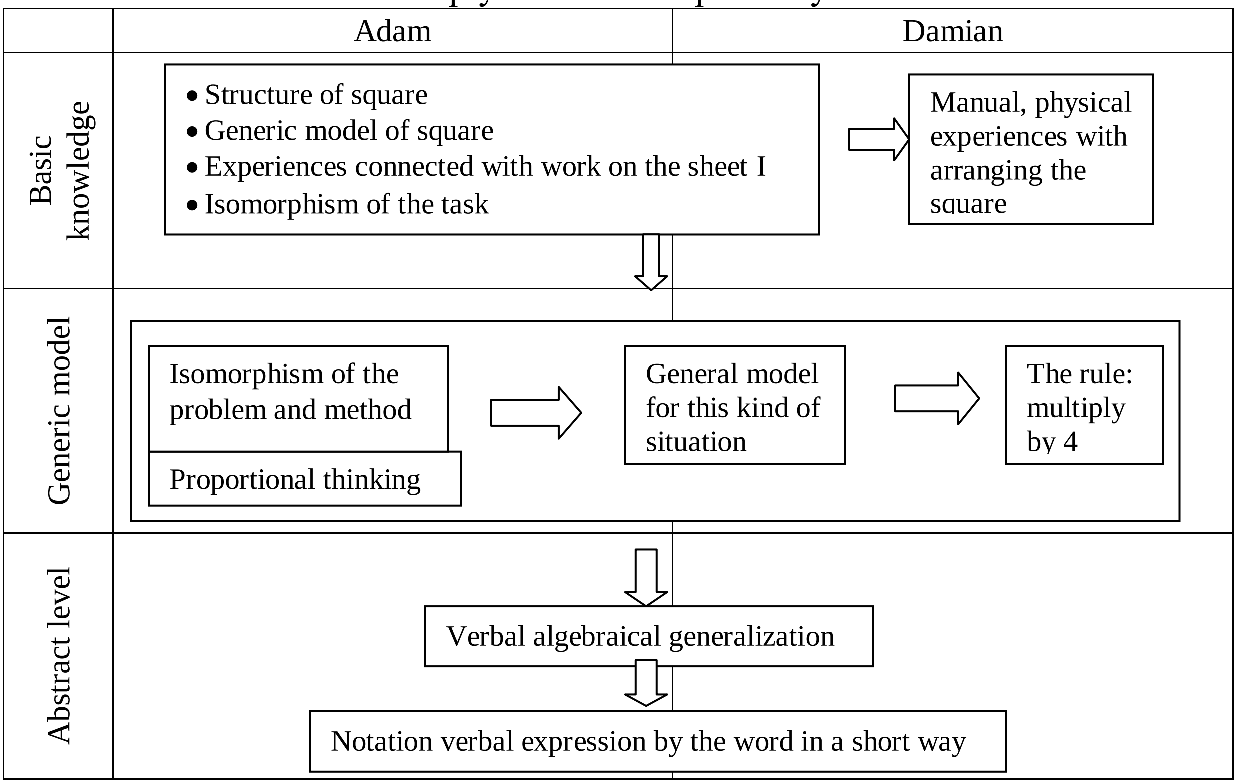
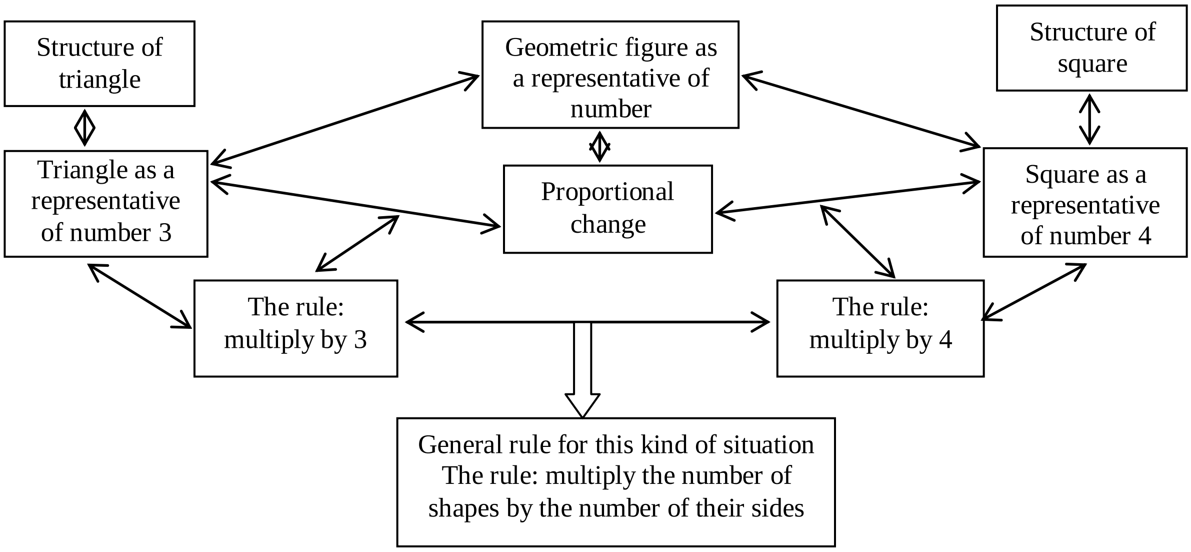


Figure 2

They were able to use the previous experience connected with the work on the first sheet and throughout an isomorphism of the problem and an isomorphism of the method they were able to acquire the generic model for this kind of situation. For the students it was not a problem how to solve this task but only which multiplier they should use. Here the process of thinking applied by both boys was almost the same, what can be showed on the Figure 2.

Boys were exceptionally agreeable in their own convictions. In spite of the very early differences in their basic knowledge, both of them achieved the same results. They were successful in gaining a generic model for all series, a set of problems. The web of connections for both boys was very similar and we can present it as following:



It seems that the common solving of the task supported their inner conviction about the rightness of the undertaking steps. They noticed that they thought in a very similar way and appearing differences they treated as irrelevantly.

Starting the work on the third sheet students expected further generalizations. Before they received the task they thought that it would be applicable to diversify the appearing figures appearing in the pattern (twenty-sides, twelve – sides). After that, during reading the task they interpreted it towards searching the dependences. We can observe this in the following dialogue which took place when students received the sheet:

1. D: I know what it will be today – twelve-sides.
2. A: Yes, or twenty-sides.
3. A+D: [*they are reading the task quietly*]
4. A: So, we are arranging [*he is arranging the square*]
5. D: Yes, and for it two the same like this.
6. A: And here this one will be the one side of the square
7. D: Oh, yes
8. A: It should be subtract… (…)

After getting to know the context of the task, the boys returned to the strategy from the first sheet. At first they arranged the first three elements of considered pattern and then they filled the table using the rule “Add 3 to the previous number”. For Adam that rule was connected with the occurring in the pattern recurrence: the first element composes of four components, each next with 3. In this moment Damian did not have his own idea, therefore he accepted Adam’s one. He assumed that the base of the new rule was the repeated addiction. Consequently for Damian, Adam’s generated rule was static and isolated arithmetic structure, which was not connected with the way of arranging the pattern. This various attitude to that rule resulted in two different strategies of solving the task 2. On the base of an early discovered recurrence, Adam acquired the new rule which was closely connected with arranging the pattern: “multiply by 3 the number of squares with the one less and then add 4”. This rule was very clear for him and he consequently used it in the further work on the task. In the meantime he made an encapsulation of repeated addition of the same component and replaced this process with multiplication. At the very beginning Damian used the strategy of adding next 3 and in this way he received the number of matches for the 10 squares. Then for 25 squares he used “not entitled proportion” showed 25 as a 2x10+5, and after that he used the obtained results for 10 and 5 squares. Comparing his result with the result of his friend he observed his mistake. Another analysis of the task allowed him to gain his own rule that is “multiply the number of squares by 3 and add 1”.

None of the boys wanted to resign from the general rule each one had created before to benefit his friend’s one. In order to emphasis the significance of their own discovery each of them wrote down their own created general rule.

1. Najpierw ~~X~~ piszemy 4 i dodajemy iloczyn liczby kwadratów i 3 ale od kwad. odejmujemy 1.

2. Podaną liczbę mnożymy przez 3 i dodajemy 1

Figure 4

During solving the task from the third sheet, boys’ ways of thinking were very different. The following diagram (Figure 5) shows it in a very simple manner.

The essential moment of the process of solving this task was that one when students received two different results in their calculations. Adam was sure of his solution. Damian started to analyse his way of thinking. In conclusion it resulted in finding the mistake. It was the mistake arising from an improper interpretation of the rule given by his friend. In this way he not only understood the Adam’s rule but also converted it and changed into his own – completely different. The emotions during this discovery were so strong that Damian did not want to subjugate his thinking to the rules discovered by Adam. He knew that his rule was correct. The boys checked their results received by means of different rules, and that they were glad that they had ‘the same’.

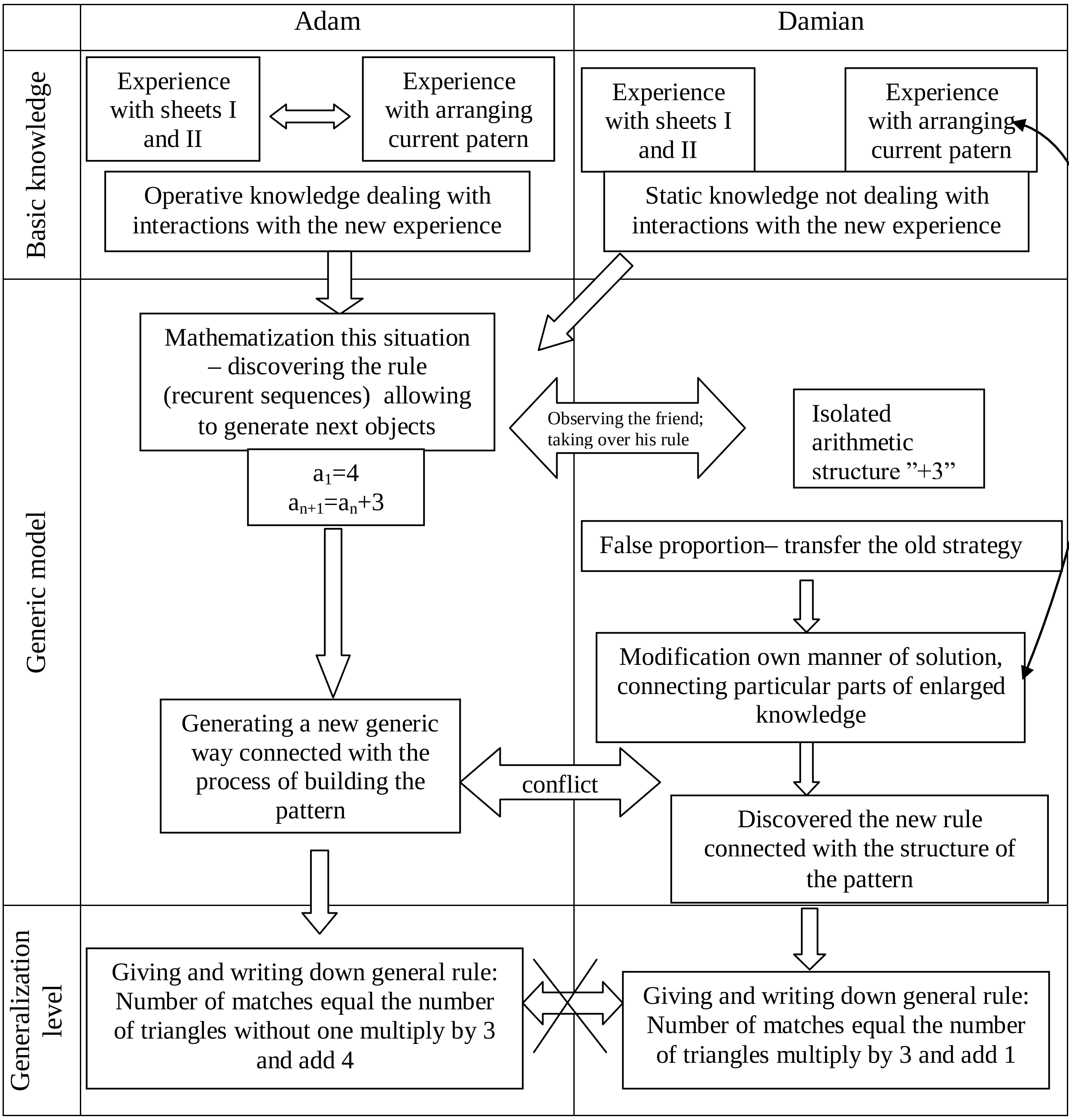


Figure 5

Setting to work on the tasks from the fourth sheet boys once again arranged the first three elements of the investigated pattern. They filled the table together used the rule “add 2 to the previous number”. However while moving to the task 2 they decided about an individual solving the next part of the task.

1. A: So, here we will go in the same way …
2. D: ...like previous. Ok., I will do it in my way, and he does it in his.

3. A: Ok. So I will be do three … nine times two equal … eighteen … twenty one. Twenty five triangles… three plus twenty four times two equal … forty … forty eight… plus three… fifty one. One hundred sixty one… [*he is accounting quietly*] three hundred and twenty three.
4. D: Ten times two plus one equal… twenty one. Two times twenty five plus one equal … fifty one. (…) Two times one hundred sixty one plus one … [*he is accounting quietly*] three hundred and twenty three.

None of the boys resigned from the work according to his own rule. Adam noticed that Damian's method was faster and calculations made were easier. However it was not a sufficient argument in order to take it over as his own. The students received the same final results and it was visible that this individual competition made a sense for them, and due to it, they had the possibility to compare their own effects of work.

While solving the tasks from this sheet, the boys started from the same level but during the further work, their ways of thinking were completely separate.

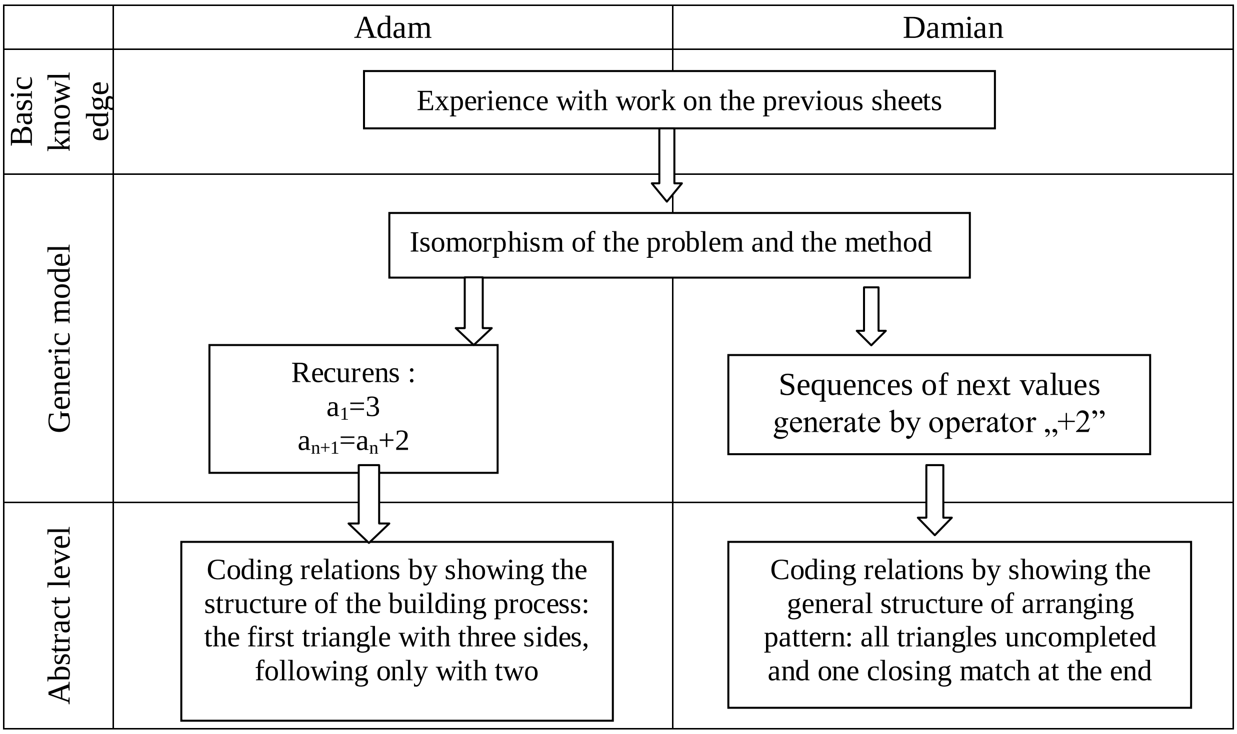


Figure 6

The differences in both attitudes to the task and using ways of solving, had a crucial influence on development of the webs of cognitive connections, which were created for each of the students. Contrary to the first, two sheets of both of webs were considerably different this time. The following diagram (Figure 7) describes it (the solid line stands for Adam's web, the dashed line stands for Damian's web).

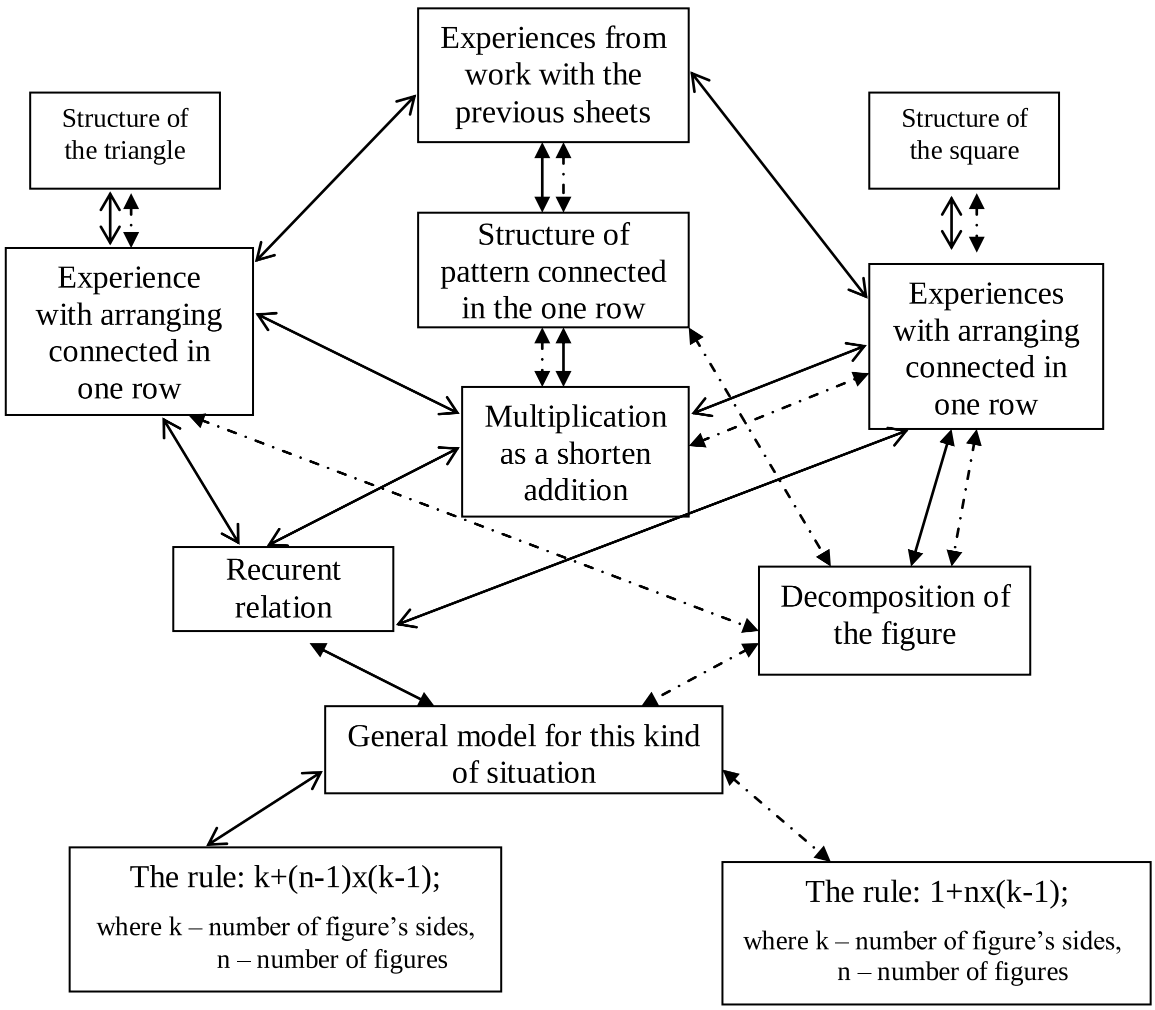


Figure 7

SUMMARY

The common work on the next tasks had a considerable influence on both the process of solving the task, and the way of creating an individual web of cognitive connections for each of the students. It allowed them to acquire two various rules connected with the same situation. Those rules existed independently but also they did not exclude each other. This common work enabled them to control the correctness of using their own created rule. The boys were forced to prove their own views, to discuss and verify their own beliefs. Due to this common work they were sure about the value of their individual way of thinking and about the conviction that "I can do that". They realized that different ways led them to the final solution of the task. They did not confine to one dimensional thinking.

It is very important that while learning of mathematics students can compare their own ideas. It is considerably easier to take a criticism from a friend. Trying to convince him to our own ideas we can discover the appearing mistake in our understanding. Although the teacher's opinion is very important for students it has an authoritarian meaning. If a teacher says to a student that in his understanding there is a mistake, the student starts to correct all the task. When the student's way of solving the task is different than this showed by the teacher, very often this student resigns from his or her own way to the benefit of the teacher's one. In this situation when students exchange their own opinions they are on the same level. Due to it, they together have to make a compromise on a contentious issue. This results in new discoveries and developing their own knowledge and abilities.

Therefore, it will be indispensable that during the maths lesson students will have more opportunities to work together, to share their own ideas and to discuss about their own ideas openly in the classroom.

REFERENCES

Consogno, V.: 2005, The semantic-transformational function of verbal language, *Proc. of CERME4*, Feb. 17-21, Sant Feliu de Guixols, Spain, http://cerme4.crm.es/.

Hejny, M., Littler, G.: 2002, The Beginnings of Algebraic Thinking, in: Bergssten, C., Grevholm, B. (eds), *Challenges in Mathematics Education*.

Hejný, M.: 2004, Mechanizmus poznávací procesu, *in*: Hejný, M., Novotna, J., Stehlikova, N., (Ed.), Dvadcet pět kapitol z didaktiky matematiky, Univerzita Karlova w Praze, Pedagogicka faculta, Praha, p. 23-42.

Hejný, M., Kratochvílová, J.: 2005, From experience through generic model to abstract knowledge, *Proc. of CERME4*, Feb. 17-21, Sant Feliu de Guixols, Spain, http://cerme4.crm.es/.

Radford, L., Bardini, C., Sabena, C.: 2009, Perceptual semiosis and the microgenesis of algebraic generalizations, *Proc. of CERME6.*

Skemp, R.: 1979, *Intelligence. Learning and Action: A New Model for Theory and Practice in Education*, Chichester, Wiley &Sons, Ltd.

Steinbring, H.: 2003, *The construction of New Mathematical Knowledge in Classroom Interactions, an Epistemological Perspective*, Manuscript Dortmund, Universitat Dortmund.

Wygotski, L.: 1989, *Myślenie i mowa*, Warszawa.

Differentiation in building mathematical knowledge

II. Geometry-related knowledge

ABILITY TO SEE IN GEOMETRY OR A GEOMETRIC EYE[1]

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The article deals with a special type of mathematical problems which we call (geometric) problems effectively solvable by insight (or PSI problems). The PSI problems have a quick geometric solution, nevertheless, pupils often attempt a much more complicated or even impossible algebraic solution. The text presents a theoretical framework (seeing in geometry, geometric eye, insight, schema) and results of related research. The methodology of our research is described and preliminary results of the pilot study discussed.

INTRODUCTION

Let us start with a problem (Fig. 1). Given a circle with centre *S* and radius *r*. Determine the length of *x*.

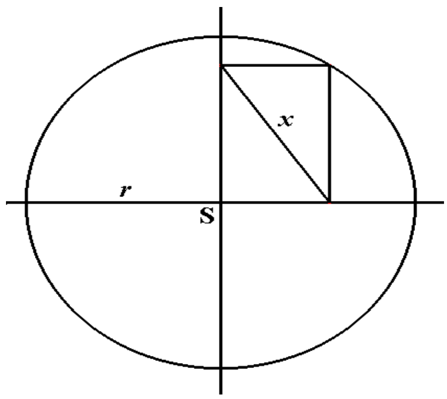

Fig. 1

The solution is easy – the line segment *x* is also a diagonal of the rectangle inscribed in the top right quarter of the circle and thus it must equal radius *r*. A kind of geometric insight is needed in order to see this solving strategy. However, according to our experience, a substantial number of pupils regardless of age will attempt an algebraic solution trying to use Pythagoras' theorem or introduce a coordination system, etc. Why is that? Cannot they see the rectangle in the picture? Do they start looking for an analytic solution without first considering the picture? Is there any relationship between the strategies used and the age of pupils? Is it possible that the knowledge of algebraic procedures functions as an obstacle, so that a pupil starts on them automatically and does not even consider the geometric solution? These questions led us to considering problems of a similar kind which can be solved via a geometric insight but are often looked at by pupils from an arithmetic or algebraic point of view. Our search of mathematics education literature on research on geometry did not reveal any special name for these types of problems, thus we called them *(geometric) problems effectively solvable with insight* (or PSI problems).

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In our understanding, a PSI problem is a problem which complies with the following requirements:

1. It is given via a picture or in words in such a way that it can easily be re-formulated with a picture.
2. It includes some known parameters, numbers or letters, or its assignment creates an impression that these parameters could be found (e.g., the problem is given on grid paper).
3. It asks for a numerical or algebraic expression of the unknown value (length, perimeter, area, etc.)
4. It has a geometric solution or a solution via an insight.[2]

Some problems which we consider to be of the PSI type are given below.

THEORETICAL FRAMEWORK

First, let us consider the notion of seeing in geometry or of insight in geometry.

Kuřina (2002) distinguishes so called 'arts'[3] which played a key role in the development of mathematics and, according to him, should play such a role in its teaching, too; it is the 'art' to count, to see, to construct, to prove, and to abstract. According to Kuřina, the *'art' to see* includes using geometric terms, geometric insight and developing intuition for solving problems. Similarly, in foreign literature we find the notion of *geometric eye*. It was probably first introduced by Godfrey (1910, quoted in Fujita, Jones, 2002) who characterised it as "the power of seeing geometric properties detach themselves from a figure". This should happen for a problem in Fig. 1 – we need to see the rectangle as detached from the whole figure.

For geometric problems, we sometimes see a solution all of a sudden; we usually say that we get an *insight*. Insight could be understood as "an original and seemingly sudden [immediate] understanding of a problem or strategy which helps to solve it" (Sternberg, 2002). However, many authors say that it is by no means sudden, that there is a lot of knowledge previously acquired behind it. For example, Minsky (1988) puts it like that: "[...] it is bad psychology to assume that what seems "obvious" is therefore simple or self-evident. Many such things are done for us by huge, silent systems in our mind, built over long forgotten years of childhood." Brown and Wheatley (1997) add that an important part of insight is the pupil's ability to divide a visual picture into simpler parts and to connect them again in a new whole. With the word 'insight' which we use in the name of the PSI problem we therefore mean a geometric solution which seems to be sudden but behind which lies a lot of experience and knowledge which, as we assume, can be developed.

[2] It can be solved with algebraic means too, but the algebraic solution is more complicated and can sometimes be outside the knowledge scope of pupils of certain ages.

[3] In Czech 'umění'; this word is difficult to translate into English in this particular meaning.

The pupil's success when solving a problem depends on his/her ability to evoke *schemas* and on the quality of schemas. In geometry, the concept of schemas was used as a means of organising geometric knowledge (Chinnappan, 1998): "[A schema is] a cluster of knowledge that contains information about core concepts, the relations between these concepts and knowledge about how and when to use these concepts. As organised knowledge structures, schemas guide both information acceptance and retrieval, and their subsequent use."

Similarly Hejný (2007), developing Gerrig's ideas, characterises a schema as a mental structure which includes clusters of information relevant to understanding, and claims that isolated models are important for the origination of schemas. He emphasises that a schema is a dynamic organisation of different elements, that is, both elements and their organisation dynamically change.

In geometry, Chinnappan (1998) proposes to speak about schemas organised around a shape. So, e.g., there is a schema of a right triangle whose parts are all knowledge and skills connected to it (Pythagoras' theorem, etc.). In the example above, the schema of a rectangle must be evoked which, among others, includes the knowledge of the existence of two diagonals and their congruent lengths.

Three types of research are pertinent to our work; research focusing on

1. solving strategies for problems of the PSI type,
2. interpretation of a picture as a part of the problem,[4]
3. visualisation of a problem (in the sense of changing a problem into a picture).

Ad 1. Chinnappan (1998) investigated what types of schemas pupils use for the following problem: *AE* is a tangent to the circle, centre *C*. *AC* is perpendicular to *CE*, and the angle *DCE* has a measure of 30°. The radius of the circle is equal to 5 cm. Find *AB*. (Fig. 2.)

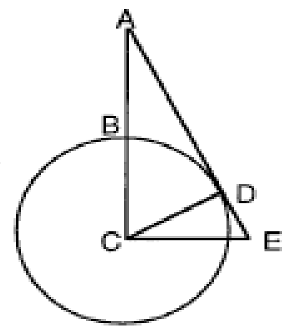

Fig. 2

In an interview with 30 pupils from Grade 10, Chinnappan found that they activated 17 schemas and that high achievers activated four times more schemas than low achievers. He identified 4 solving strategies, one of which was a geometric one but it was not used by pupils at all. Most pupils used a strategy based on trigonometry relations.

Kuřina (2006) describes various solving strategies of student teachers and practising teachers for the task to construct a regular 12-gon inscribed in a circle with radius *r* and to determine its area. He presents a simple solution based on the fact that a 12-gon consists of 12 isosceles triangles whose sides make an

[4] Given the scope of our work, we will not deal with research which focuses on understanding pictures when teaching a new content or on understanding a 2D representation of a 3D object.

angle of 360°/12 = 30° (Fig. 3). As $v = |AP| = 1/2 \cdot |AC| = r/2$, for the area of a regular 12-gon, it holds $S = 12 \cdot 1/2\ r \cdot r/2 = 3\ r^2$.

Kuřina presents some solutions of his research participants (Fig. 4) and concludes that the wider range of mathematical knowledge (of mathematical formulas) does not necessarily mean that a more elegant solution will be used.

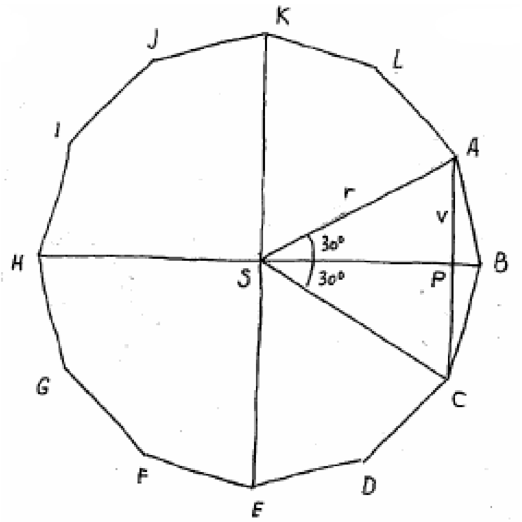

Fig. 3

$$S = 6r^2 \sin 30°,$$
$$S = 12\, r^2 \sin 15° \cos 15°$$
$$S = 12\, r^2 \sin 75° \cos 75°$$
$$S = 6\, r^2 \sqrt{2-\sqrt{3}}\ \sin 75°,$$
$$S = \frac{12r^2 \sin 30° \cos 15°}{2 \sin 75°},$$
$$S = 12\, \frac{r^2}{4} \sqrt{2-\sqrt{3}}\ \sqrt{2+\sqrt{3}},$$
$$S = 3r^2\, \frac{1}{\sin 75°} \sqrt{1-\frac{1}{16 \sin^2 75°}},$$
$$S = 12 \,.\, \sqrt{\frac{r^4\sqrt{2-\sqrt{3}}.\sqrt{2-\sqrt{3}}.(2+\sqrt{2+\sqrt{3}}(2-\sqrt{2-\sqrt{3}})}{16}},$$
$$S = 6r^2 \,.\, \frac{\sin 30°}{\sin 75°} \cdot \sqrt{1-\frac{\sin^2 30°}{4 \sin^2 75°}}.$$

Fig. 4

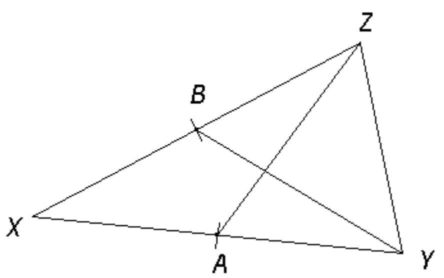

Fig. 5

Ad 2. Fujita and Jones (2002) quote their previous work in which they gave 87 Japanese pupils aged 14 and 15 this task: Let A and B be midpoints of congruent sides XY and XZ of an isosceles triangle. Prove that $|AZ| = |BY|$ (Fig. 5). Nine pupils were not able to see any congruent triangles and four of them did not find the correct pair of congruent triangles. They conclude that for some it is difficult to see a particular part of a picture as detached.

Mesquita (1998) divides pictures according to their character into *illustrations* and *objects*. He gave about 300 14-year-old pupils the task in which the picture had a role of illustration (Fig. 6): “Suppose that shape 1 is an equilateral triangle, 2 is a rectangle, 3 and 4 are squares and the shape consisting of shapes 3, 4 and 5 is a square. Prove that line segments AC, LF and FG are congruent.”

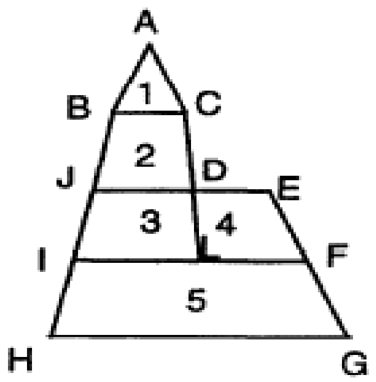

Fig. 6

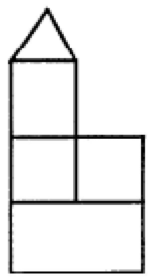

Fig. 7

The equality $|LF| = |AC|$ was proved by 64% of pupils and $|FG| = |AC|$ by 50% only. 51% of pupils used information both from the description and picture while 24% of pupils solved the problem using the picture only and tried to measure distances and looked for ratios. The main problem was the character of the picture as an illustration. Fig. 7 shows the picture to the same problem which has a character of an object. Some pupils in the research in question made such a drawing spontaneously. One of the conclusions is that a descriptive picture which has a character of an illustration rather than an object can become an obstacle for some pupils.

Ad 3. The most cited references in this area seem to be those by Eisenberg and Dreyfus (1986, 1990, 1991) who reached a conclusion that pupils are often reluctant to use visualisation when solving a problem and prefer an analytical elaboration and that they often do not know how to use the picture they themselves drew. The authors suggest two main causes. First, it is the teacher's tendency to say or imply that an analytical approach is superior to the visual one. Second, the visual elaboration of information represents a higher level of mental activity than analytical reasoning. The authors compare it to the distinction between analytic and pictorial presentation of topics. The analytic one is sequential, pieces of information go one by one; we can follow them and miss none. Any relations among them are expressed separately. A pictorial representation is simultaneous, both information and relations among them are presented at the same time.

However, the situation in this respect seems to be changing. Stylianou (2001) points to some more recent work (around 1998) which shows that pupils are increasingly willing to use visualisation when solving problems. It is attributed to the fact that in many countries, the use of pictures and diagrams in teaching has increased.

METHODOLOGY

In our work, we focus on two **research questions**:

1. What are the solving strategies of pupils for PSI problems (with a particular emphasis on the interplay between geometric and algebraic strategies)?
2. Are these strategies age related (in other words, do the strategies change with age)? What are the differences?

Research participants are pupils of various ages, from Grade 6 of a primary school (12 years old) to Grade 4 of a secondary school (19 years old).

The **methods** used in our research will be twofold. First, a written test will be administered to pupils during their normal school time. They will be encouraged to justify their solutions and explain their answers. The test will consist of 3–4 PSI problems which will be chosen from a larger set of problems in a pilot study (see below). The data from the test will be complemented with interviews conducted with pupils whose solutions are interesting from the point of view of our research questions (for example, why an algebraic or arithmetic solution was used, will the pupil see the geometric solution when given a hint, etc.).

PILOT STUDY AND ITS RESULTS

The main goal of the pilot study was to trial some PSI problems whether they are suitable for our purposes, that is, whether pupils use different solving strategies – both geometric and algebraic, and which ones.

From the set of PSI problems which we had assembled after a search of relevant literature on geometry we chose 5 which seemed to be promising. The test was assigned in October 2009 by the mathematics teacher, the first author, in a mathematics lesson. There was a time limit of 45 minutes, the pupils had not been informed about the test beforehand and there was no special preparation for the test. The participants were 23 pupils from Grade 1 (15–16 years old) and 12 pupils from Grade 3 (17–18 years old) of a secondary grammar school.

The data, that is, the pupils' written solutions were analysed in a qualitative way (using procedures based on grounded theory). The solutions to individual problems were coded and indications of different strategies were looked for. For each problem, a table with the number of pupils using the appropriate strategy will be given and briefly commented.

Problem 1. *There is a strip of grass 4.5 m wide around a swimming pool of 25 m times 12 m. Find the area of the grass strip.* (Fig. 8.)

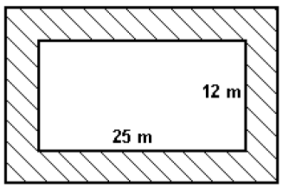

Fig. 8

	Strategy 1		Strategy 2	
	correct	incorrect	correct	incorrect
Grade 1	16	4	3	0
Grade 3	7	1	2	2

<u>St1:</u> The geometric strategy consists of realising that there are two rectangles and that the grass strip is, in fact, a complement of the small rectangle to the big one. If a pupil used this strategy, he/she subtracted the area of the small rectangle from the area of the big one. Some mistakes appeared, too.[5]

[5] The mistakes will not be given here; naturally, in the main study, they will be our focus, too.

<u>S2:</u> Another type of geometric strategy consists of dividing the grass area into 4 (or 8) rectangles (or 4 squares and 4 rectangles) and calculating individual areas one by one. Some pupils did not see that some parts are congruent.

Problem 2. *Rectangle SABC is inscribed in a circle. Find the length of AC, given the lengths of SA and AD. Justify your answer.* (Fig. 9)

	Strategy 1		Strategy 2		Strategy 3		Other strategy	
	correct	inc.	correct	inc.	correct	inc.	correct	inc.
Grade 1	0	2	5	0	11	1	0	4
Grade 3	0	1	1	0	10	0	0	0

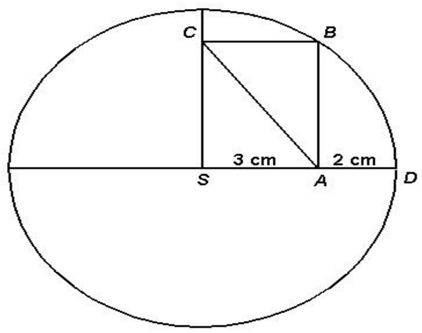

Fig. 9

<u>St1:</u> Three pupils drew point symmetric shapes into the circle, trying to calculate the lengths of various line segments.

<u>St2:</u> Six pupils realised that they knew a right angled triangle with a leg of 3, that is the triangle 3-4-5 and thus they reached the correct solution of 5.

<u>St3:</u> In total, 22 pupils used the above geometric solution and except for one, reached the correct answer.

<u>Other strategy.</u> For example, one pupil considered triangle *SAC* to be isosceles and calculated the hypotenuse *AC*.

Problem 3. *Find out what part of the area of rectangle ABCD is the area of triangle AED.* (Fig. 10)

	Strategy 1		Strategy 2		Strategy 3		Other strategy	
	correct	inc.	correct	inc.	correct	inc.	correct	inc.
Grade 1	7	0	7	0	4	3	0	2
Grade 3	7	0	3	0	2	0	0	0

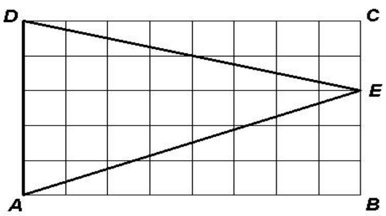

Fig. 10

<u>St1:</u> Fourteen pupils solved the problem via insight – they could see two rectangles *ABEX* and *XECD* where *X* is on line segment *AD* and it makes a horizontal line segment with *E*. They realised that sides *DE* and *AE* are diagonals of the two rectangles and divide the rectangles into two congruent triangles. It is then obvious that triangle *AED* takes a half of the area of rectangle *ABCD*.

<u>St2:</u> Ten pupils used a kind of algebraic solution in which the square grid was used. The pupils knew lengths of sides and the height of the triangle. They calculated the areas of the rectangle and of the triangle using formulas and compared them.

<ins>St3:</ins> Nine pupils used the 'counting squares' strategy in which they tried to put together parts of squares so that they could see how many whole squares originated (an example is in Fig. 11). Given the age of pupils, we were rather surprised by this strategy.

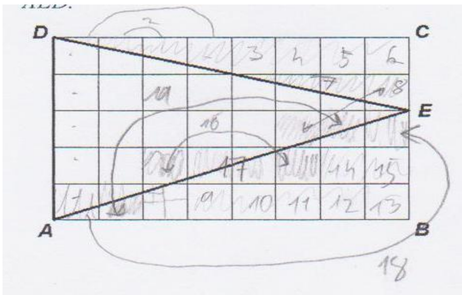

Fig. 11

Problem 4. *Given triangle ABC and points D, E which are in turn midpoints of sides AC, BC. Line segment DE divides triangle ABC into a triangle and a trapezium. Find the ratio of their areas. Explain your answer.* (Fig. 12)

	Strategy 1		Strategy 2		Strategy 3		Other strategy	
	correct	inc.	correct	inc.	correct	inc.	correct	inc.
Grade 1	10	0	0	7	0	2	0	4
Grade 3	5	2	0	1	0	1	0	2

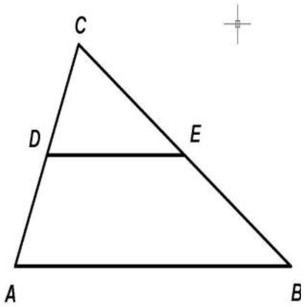

Fig. 12

<ins>St1:</ins> Seventeen pupils realised that *DE* is a midline and drew the remaining two midlines. The resulting picture led most of them to the correct solution – they saw 4 triangles which they considered to be congruent. Some justified it by theorems for congruent triangles.

A few strategies did not lead to any solution. E.g., the triangle was divided into two right triangles and one rectangle (<ins>St2</ins>), or the triangle was divided into several right triangles with which the pupils hoped to find the answer (<ins>St3</ins>).

Two pupils did not even attempt the problem and three labelled different parts of the figure and tried to use various formulas which did not lead to any solution.

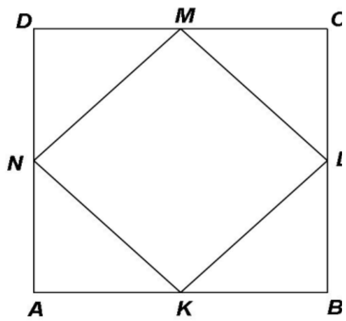

Fig. 13

Problem 5. *Square KLMN is inscribed into square ABCD in such a way that midpoints of the sides of square ABCD are connected. Determine the ratio of areas of squares ABCD and KLMN and explain your solution.* (Fig. 13.)

	Strategy 1		Strategy 2		Other strategy	
	correct	incorrect	correct	incorrect	correct	incorrect
Grade 1	17	0	3	0	0	1
Grade 3	7	2	0	1	0	2

St1: Nearly all pupils drew line segments *KM* and *LN*. Most of them spotted 4 congruent squares and 8 congruent triangles, 4 of which cover the smaller square *KLMN* and the remaining 4 complete its area into the area of square *ABCD*. Thus square *ABCD* must be double the square *KLMN*. One pupil justified this statement by saying that it was possible to fold triangles *AKN*, *KBL*, *LCM*, *MDN* to the centre of the square so that a small square originated.

St2 (algebraic solutions): E.g., a pupil labelled line segment *DM* as an unknown (Fig. 15). Using Pythagoras' theorem he found an expression for side *MN* and then used the formula for the area of square reaching the correct answer.

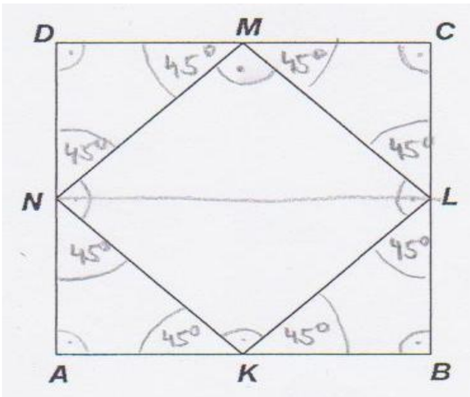

Fig. 14

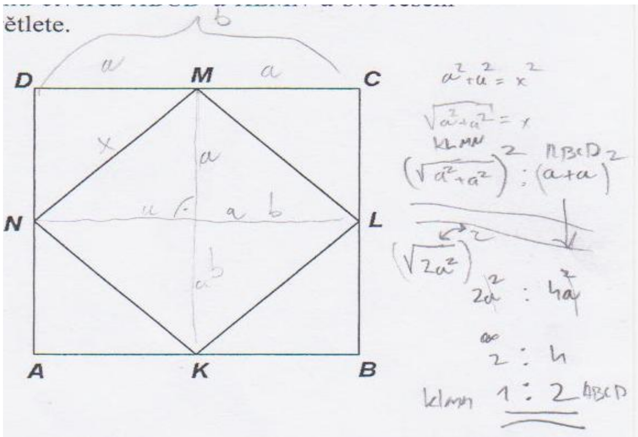

Fig. 15

Let us consider the differences in strategies and success rates for Grade 1 and 3 pupils. There is hardly any difference at problems 1–3. We find it particularly surprising for problem 3 as even Grade 3 pupils used 'counting squares' strategy. Younger pupils were more successful in solving problems 4 and 5; they used simpler strategies including a geometric insight. Due to the limited number of pupils in the pilot study, we will not speculate about possible reasons.

CONCLUSIONS

Thanks to the pilot study, we decided to omit problem 1 in the main study. Most pupils were able to use their geometric insight and not many different strategies appeared. There were not algebraic strategies at all. The problem might be seen as problematic by younger pupils only. Problem 4 might be too difficult for primary pupils so, prior to the main study, it will be used with one class to see how they will cope. For problem 2, a change of the assignment will be done. To prevent the 3-4-5 solution, decimal lengths of segments will be used instead of 3 cm and 2 cm. Problem 3 and 5 can be used as they are.

The test will be given to pupils of different ages at primary and secondary schools to an opportunistic sample. Some pupils will be chosen for an interview and asked to describe their solution in detail and if they used an inappropriate solving strategy, they will be given some hints so that we get an insight into what might have prevented them from the geometric strategy. The written solutions will be analysed in a qualitative way. The results from interviews and tests will be put together in order to answer the research questions.

REFERENCES

Brown, D. L., Wheatley, G.: 1997, Components of imagery and mathematical understanding, *Focus on Learning Problems in Mathematics*, **19**, 1, pp. 45–70.

Chinnappan, M.: 1998, Schemas and mental models in geometry problem solving, *Educational Studies in Mathematics*, **36**, pp. 201–217.

Dreyfus, T., Eisenberg, T.: 1986, On visual versus analytical thinking in mathematics, In, *Proceedings of PME10,* Univ. of London, vol. 1, pp. 152–158.

Dreyfus, T., Eisenberg, T.: 1990, On difficulties with diagrams: Theoretical issues, in G. Booker, P. Cobb & T. N. Mendicuti (Eds.), *Proceedings of PME14*, vol. 1, pp. 27–34.

Eisenberg, T., Dreyfus, T.: 1991, On the reluctance to visualize in mathematics, in: W. Zimmermann & S. Cunningham. (Eds.), *Visualization in teaching and learning mathematics* (MAA Notes, 19, pp. 26–37), MAA, Washington.

Fujita, T. Jones, K., 2002, The bridge between practical and deductive geometry: Developing the ‚geometrical eye‘, in A. D. Cockburn & E. Nardi (Eds), *Proceedings of PME26*, vol. 2, pp. 384–391.

Godfrey, C.: 1910, The board of education circular on the teaching of geometry, *Mathematical Gazette*, 5, pp. 195–200.

Hejny, M.: 2007, Budování matematických schémat, in A. Hošpesová, N. Stehlíková & M. Tichá (Eds.), *Cesty zdokonalování kultury vyučování matematice,* Jihočeská univerzita, České Budějovice, pp. 81–121.

Kuřina, F.: 2002, O matematice a jejím vyučování, *Obzory matematiky, fyziky a informatiky*, **31**, 1, pp. 1–8.

Kuřina, F.: 2006, Geometrie jako příležitost k rozvoji žákovských kompetencí, in *Podíl učitele matematiky ZŠ na tvorbě ŠVP : Studijní materiály k projektu,* JČMF, Praha. [CD ROM.]

Mesquita, A.L.: 1998, On conceptual obstacles linked with external representation in geometry, *Journal of Mathematical Behavior*, **17**(2), pp. 183–195.

Minsky, M.: 1988, *The Society of Mind,* Simon & Schuster, New York.

Sternberg, R. J.: 2002, *Kognitivní psychologie,* Portál, Praha.

Stylianou, D.: 2001, On the reluctance to visualise in mathematics: Is the picture changing? In M. van den Heuvel-Panhuizuen (Ed.), *Proceedings of PME25*, vol. 4, pp. 225–232.

PREPARATION FOR AND TEACHING OF THE CONCEPT OF AREA

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In this study the preparation and the teaching of the concept of area as well as the introduction of area standard units and problems related to the computation of area are presented. An experiment conducted in class 5 and 6 is shown, in which we focused on various activities, patterns of work and the application of area computation in everyday life. The efficiency of the experiment was measured on the basis of pre-tests and post-tests.

INTRODUCTION

The teaching of this topic offers several options for differentiation and cooperative activities. The use of various activities helps learners in the solution of real life problem situations and makes them more cooperative. Their success in shared work is also motivating for the solution of further tasks. A great deal of attention has been paid to the role of activity and motivation in education by our predecessors. First we would like to refer to Farkas Bolyai, the father and the teacher of the greatest Hungarian mathematician János Bolyai, who formulated several educational principles at the beginning of the 19th century, which are still valid these days. One of Farkas Bolyai's pedagogical principles is that teaching should not be started with suffering, but rather with autonomous activity based on the personality of the child; this activity is to be assisted also later on during systematic learning.

> regarding the drive to learn the best thing would be to win the learner's affection for the thing itself, the desire for knowledge is an inherent wish in the soul, and it only has been stimulated (Farkas Bolyai, 1830).

In the 21st century schools has to cope with new role. Learners have to be provided not only with basic knowledge and skills but also with thinking and communication skills at high level and sociable behaviour.

> Unfortunately we came to believe that we learn most when we are taught. But the thing is that we learn most when we are motivated and the conditions necessary for learning are provided (Spencer Kagan, 1994).

In the teaching of the topic the Bruner's representational levels can be observed, but the transition from one level to the other does not go smoothly. The symbolic level is reached too soon and putting down things with figures and letters seems to be a kind of magic trick for many learners. In the formation of the concept of area only few presentations and activities are concerned with measuring large areas outside the classroom and probably this can be the reason why learners are not able to compare and estimate large areas. Activities involving movements in a large area can be motivating for the learners and could contribute to a better understanding of the role of mathematics in science and everyday life. This is also a key issue in realistic mathematics teaching. It is an essential thing that learning should be more than the acquisition of facts. Learners should be able to construct and create their knowledge in an active way. In building up this knowledge children can help each other and shared activities and interactions are highly relevant.

THE HYPOTHESES OF THE RESEARCH

We assume that the preparation and the formation of the concept of area can be assisted by the following:

H1. Tiling various polygons with various patterns by means of fine movements and walking in the school-yard

H2. Comparing, estimating and determining the area of polygons by geometric transformations and rearrangement

H3. Presenting the use of the concept of area in everyday life and solving practical problems.

THE PLACE AND TIME OF THE RESEARCH

The developmental sessions were held in the Calvinist Primary School and Secondary school in Kisvárda.

Children filled in the pre-test containing five tasks on 3rd October 2008.

The developmental sessions were held in the afternoon by Mónika Dancs, a student of the teacher training college.

In the experiment two classes participated, class 5/b with 18 pupils and class 6/b with 15 pupils. 25-30 minute sessions were planned.

The post –test was filled by the children on 12th December 2008.

THE METHODOLOGY OF THE RESEARCH

The three stages of the research:

- Examination of the level of knowledge regarding the measurement and computation of perimeter and area in class 5 and 6. Two of the tasks will be evaluated together with the post-test. In both classes the knowledge gained in the previous school-year was taken into account, as geometry was not in the curriculum of the given term prior to the survey.
- Seven developmental sessions were held in class 5 and 6. The lessons were taped and photographs were also taken.
- The efficiency of the development was checked by means of a post –test.

DEVELOPMENTAL SESSIONS

The aim of session 1:

- To make children realize that various shapes can have the same area.
- To demonstrate the relation of size in different areas by means of rotation or cutting and rearrangement.
- To make children realize that the area of plane figures of identical perimeter is not necessarily the same.

The solution of one of the problems is summed up as follows:

Out of six various plane figures children were asked to select the ones of identical area. (Fig.1)

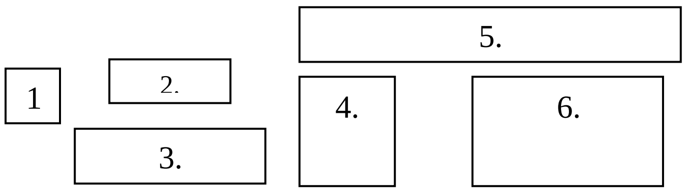

Figure 1

Polygon number 3 and number 5 can be transformed into number 4 and 6 respectively. The solution of the problem is illustrated by an excerpt of the lesson which was taped in class 6.

Teacher:	Is there a polygon of identical area with rectangle number 6 on the blackboard?
Cs.L.	Well that thin stripe (He meant rectangle number 3)
M.V:	It’s no way the same!
B.Cs:	It is only the same size if it covers the whole lot.
Teacher:	Try and find a polygon of the same area out of the ones here.
B.Cs:	There is no polygon of the same area; some of the smart ones should come here! (He meant the congruence)
V.L.:	But this is not of the same area!
M.F.	There are no identical ones!
G.O.:	But there is!
G.O.:	This is of the same area! (He rearranged polygon number 3 into number 4)
B.K.:	Sure, it won’t be OK
G.O.:	That’s it, I told you it’s OK (He demonstrated that is of the same area)

The aim of session 2

- To make children realize that all of the given polygons can be rearranged into rectangle.
- Sharing each other's ideas while working in pairs.

In one of the tasks of the session children rearranged rhombus, right-angled trapezoid, deltoid, symmetrical trapezoid and hexagon into rectangle while working in pairs. The pairs were given 3 polygons of the five types respectively in order that they could make use of the experience of cutting in the wrong way.

Children started rearrangement very creatively and some of them did it with one cut, whereas some of them cut the polygon into small pieces.

In both classes the rearrangement of the right-angled trapezoid was the most demanding. Almost every one noticed that one triangle ought to be cut and placed so that I would be rectangle, but the trouble was that they could not find the proper place of the cut.

In class 5 the rearrangement of the deltoid was also rather challenging, as they were not familiar with the characteristics of deltoid, which is why the majority of them cut it into several pieces. Pupils in class 6 however figured out what the right solution was (fig.2).

The rearrangement of symmetrical trapezoid and hexagon (fig.4) seemed to be easy for the children.

Some of them who were not able to rearrange the given polygon had the idea to cut the missing parts from an identical leaflet.

Some of the children did not really pay attention to continuous tiling (rearrangement of the rhombus: fig.3)

Children were happy to do these tasks of rearrangement, which is shown by the fact that they did not want to stop the sessions even after 45 minutes. They also came up with new ideas.

Figure 2

Figure 3

Figure 4

The aim of session 3

- To highlight problems related to the continuous tiling of single layer when a rectangle of a given area was tiled with leaves.

- To encourage cooperative activity in pair work.

First they tried to use leaves of almost the same area (fig 5) then they covered the leaflets with leaves of various type and size (fig.6).

Figure 5

Figure 6

During the activity of covering we observed that boys mostly selected smaller leaves than girls.

As continuous covering was difficult to carry out with leaves, they cut them into various shapes, they preferred rectangle (fig.7).

Children realised that smaller leaves are necessary to cover the plane figures than bigger one, which is why they used bigger leaves so that they should not glue a lot. As it can be seen in the picture they were rather keen on continuous tiling, the single layer was not important for them (fig.8).

Figure 7

Figure 8

The aim of session 4

- To make them realize that area of plane figures of the same perimeter is not necessarily the same.
- Using strings of given length forming various polygons and computing their area in the school-yard.
- Group work contributed to the co-operation of learners.

In the school-yard children worked in groups of six. The groups formed various polygons by using string of a given length (9.6m). The square that can be made from the string of 9.6 m could be covered exactly with the newspapers provided. At the vertex of the polygon one child was standing and another child measured the sides of the polygons, and another one covered the area by newspapers as a given unit.

Class 5:

- Only one group was able to determine the perimeter of the polygons they formed and to tile the square formed by them.
- The other group managed to compute the perimeter of the polygons, but they were not able to cover them exactly.
- The third group was not able to complete the task, as they were engaged in watching and following the activity of the other groups.

Class 6:

- The group of girls made a square first, then after determining the perimeter they were also smart to cover the area with newspapers cut into rectangles. (fig.9)
- The boys first stretched the string to make an isosceles triangle. While covering they folded the newspaper where it was necessary. During this session we observed that they made an effort to cover the area continuously. (fig.10)
- The girls formed the rectangle in a way that the feet of two children were the vertices, as they were standing astride at a short distance. Having seen this, the boys were more practical-minded and they were standing stride so that the stride should equal exactly the length of the newspaper (fig.11).

Figure 9

Figure 10

Figure 11

Pupils in class 6 were more dedicated to doing the tasks and they came up with several ideas, which was probably due to the difference between the age groups.

The aim of session 5

- The simultaneous use of the concept of perimeter and area,
- The comparison of the area of polygons by means rearrangement
- Determination of the perimeter of polygons by measuring

During the pair-work activity worksheets were used to facilitate the task.

In one of the tasks children determined the perimeter of four polygons and compared their area (fig.12).

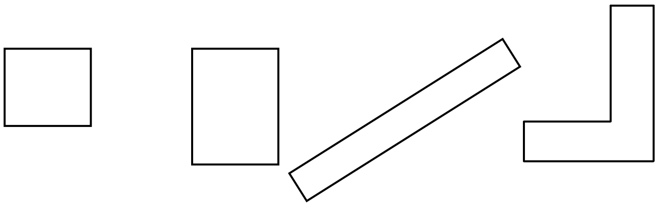

Figure 12

Children measured the sides of the polygons and their calculation of the perimeter was correct.

They soon realized that the area of number one and number three and number two and number four are identical. They demonstrated it by rearrangement.

Some of them also noticed that despite the fact that the area of rectangle 1 and 3 is identical, but their perimeter is different.

Figure 13

Figure 14

The aim of session 6

- The preparation of the formula of the area of the square and the rectangle
- Drawing plane figures of identical area on grid.

During the activity worksheets were used to facilitate the task.

They drew polygons by using right-angles triangles of 16 area units so that their area could be the same.

Children actually enjoyed drawing polygons of given area, some of their interesting ideas can be seen in the figures. In fig 15 a solution from class 5 and in fig 16 and 17 the more advanced level of class 6 can be seen.

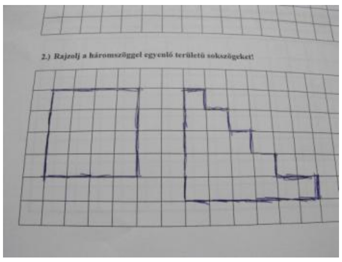

Figure 15

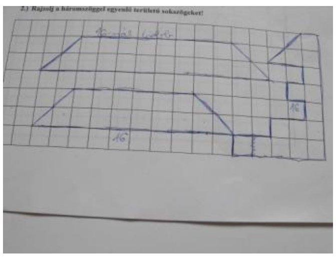

Figure 16

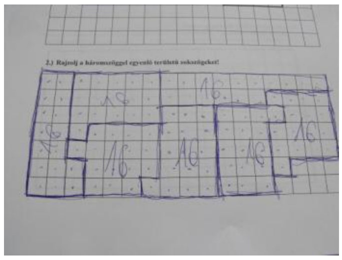

Figure 17

The aim of session 7

- Introduction to the standard units of area
- Definition of the area of rectangular and square by formula
- The application of the concept and the computation in real life problems.

Children covered their own desk with sheets of 1 dm^2 length. Having gained experience in drawing on grid, the children were aware of the fact that it is not necessary to cover to whole desk in order to compute the area. This is why they placed leaflets cut into 1 dm^2 length to the short and long side of the desk (fig.18). In this way children were able to calculate the area. In both classes there were some children who placed sheets of 1 dm^2 round the desk; however this was not really conducive to the formation of either the perimeter or the area (fig.19).

They did another task in the same way when they covered the sheet of 1 dm^2 with sheets of 1 cm^2. Children were rather impatient in placing the 1 cm^2 sheets so they did not realize how many small squares are needed to cover the sheet of 1 dm^2. In our opinion this kind of tiling could be useful in the more exact estimation with cm^2.

Figure 18

Figure 19

Determining the area by means of a formula was rather demanding for pupils in class 5. In order to find the solution they quite often had to rely on counting the squares on grids. Pupils in class 6 reached the level of abstraction where describing the area by letters was easy for them.

In the second part of the session children solved tasks with text in order to examine to what extent they were able to put their knowledge of the concept of area in practice. Children did two tasks together, and they tried to do two tasks on their own. However this proved to be too demanding for pupils in class 5. This failure can be due to their poor comprehension skills.

In the task below two data were provided for one side respectively. *The side of a square room is 30 footprints. One footprint equals 25 cm. How many* cm^2 *is the area of the room?*

As it turned out from the discussion the concept of area has been established, as they were able to determine correctly the area in simple tasks, but in this particular task the too large index-number of the side or the multiplication made them think that they do not need to calculate further on, and they completed the task and they were happy with finding out the length of the side.

Post-test

In the post test we examined

- the establishment of the concept of area during the developmental sessions,
- the reliable knowledge of the concept of perimeter and area, and
- the use of the concept of area in realistic problems.

The post-test, which was filled in by the children on 12th December 2008, contained similar tasks as the pre-test. They took much more delight in filling in the post-test than the pre-test.

Two tasks of the pre-test and the post-test are compared and evaluated below.

In one of the tasks the given rectangle was tiled with given patterns (fig.20). In the pre-test rotation was needed for tiling with pattern *b*, and in the post-test cutting and rearrangement of pattern *b* was necessary to cover the polygon.

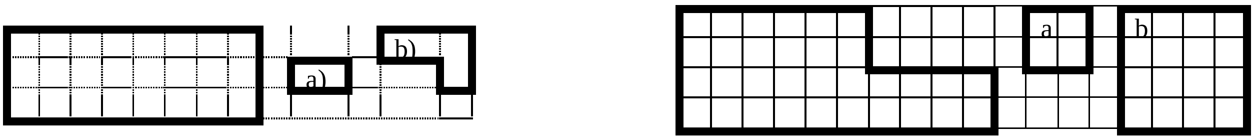

Figure 20

Children in class 5 were able to tile with squares without mistake in the post-test, but fewer of them were able to carry out the rearrangement. It might be due to the fact that the more time should have been spent with rearrangement in class 5. One of the pupils in the pre-test counted the squares in the polygons and calculated the number of tiles by division.

Pupils in class 6 improved a lot. Apparently the activities during the sessions contributed to the clarification of the concepts to a considerable degree (fig.21).

In the other task we examined how children were able to apply the concept of area in everyday life.

The task in the post-test was more demanding than the one in the pre-test, as it included several questions. In this case they calculated not only area but also perimeter, which they might have mixed up.

The task of the pre-test: *The width of a garden is 18m, its length is 42m and the buildings are situated on140 m^2. How much is left for other purposes?*

The task of the post-test: *The width of a rectangular lot is 25 m; its length is 34 m. The area of fruit trees covers 320m^2. How much is left for construction? How many meters of wires are necessary for the fence of the area, if 4 meters are left for the gates?*

In fig.21 and fig. 22 the percentage of those is shown who completed both parts of the tasks without mistake.

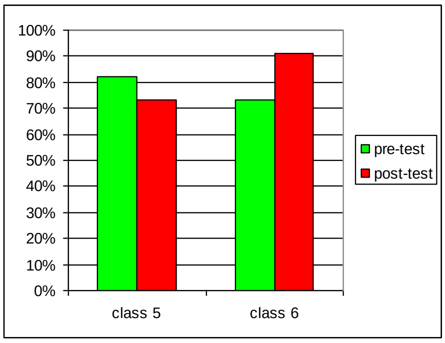

Figure 21

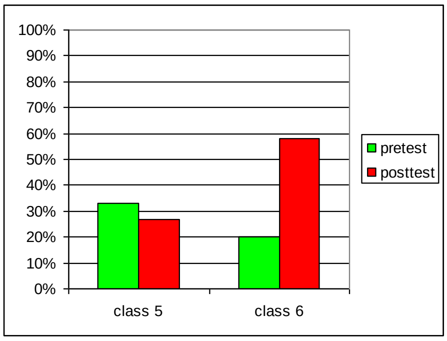

Figure 22

This task of the pre-test was not done by the majority of the pupils in class 6; some of them did not even start to solve it. In class 5 everyone except two pupils attempted to solve it, but they calculated perimeter instead of area most of the time.

In the pre-test very few children drew, whereas the post-test there was hardly anyone, who did not.

In class 5 the skill to calculate area seemingly weakened, which may be due to the poor level of comprehension. When it was only the area that they had to calculate in a task, they managed to do it very well, but the simultaneous application of the two concepts was confusing for them.

The majority of pupils in class 6 took the real life content of the problem into consideration, but the calculation of the perimeter was too demanding for them. During the development sessions we did not really paid much attention to it.

When doing the task pupils in class 5 were concerned only with figures and counting, they rarely discovered logical relationships.

CONCLUSION

We consider the developmental sessions to be efficient.

H.1 Considerable progress can be observed in class 6. They were enthusiastic from the beginning which can also be seen in their results. Pupils in class 5 were also enthusiastic at the beginning, but during the outdoor sessions in the school yard they were much more interested in games than in learning. Pupils in class 6 very often remembered school-yard sessions, where they moved a lot and gained experience in this way. Progress in the exact tiling can be mostly observed in class 6.

H.2 Children were happy to rearrange and cover the plane figures. Children in class 5 rushed to get their kit containing the scissors, glue and ruler when they were told to cut leaflets or newspapers into pieces. They enjoyed using these things, they liked the figures on them and they tried cut so that these figures would not be damaged. They made a progress in comparing areas, whereas they lagged behind in estimation.

H.3 According to the evaluation of the post-test it can be seen that children are still not able to perceive the size of real life areas and the relationships between them. This is also shown by the fact the in the task above several children distracted the area of the garden from the area of fruit trees.

After analysing the developmental sessions and the tests we came to the following conclusions. In the future more attention should be paid to the following:

- In class 5 the difference between the concept of perimeter and area should be highlighted by means of more developmental activity.
- Children should measure more both small and large objects in order that their estimations get near reality.
- More co-operative activities are needed in the upper primary, because these engaging activities can arouse children's interest, and the shared activities can facilitate the understanding of complicated relationships.

REFERENCES

Ambrus, A.: 1995, *Bevezetés a matematika didaktikába*, ELTE Eötvös, Budapest.

Bolyai, F.: 1830, *Az aritmetika eleje*, Marosvásárhely.

Kónya, E.: 2006, Difficulties of Using Words of Spatial Relations, University of West Bohemia, Plzeň, *CIEAEM 58*.

Holt, J.: 1991, *How Children Fail*, Gondolat, Budapest.

Kagan, S.: 2001, Cooperative Learning, Önkonet, Budapest.

Nahalka, I.: 2002, *Konstruktivizmus és pedagógia*, Nemzeti, Budapest.

Szendrei, J.: 2005, *Gondolod, hogy egyre megy?* Typotex, Budapest.

Tarcsi, M.: 2007/1 (p. 119-145.) Learning and Knowledge: The results, lessons and consequences of a development experiment on establishing the concept of length and perimeter, *Teaching Mathematics and Computer Science*.

STRUCTURES OF 2D ARRAYS IDENTIFIED IN CHILDREN'S DRAWINGS

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The research presented in this paper concerns the understanding of a certain type of a regular two-dimensional array of elements by pupils by 10-13 years old children. Two-dimensional arrays which are characterised by different structures there are for example: rectangular structure end a slant one. The objective of the study was to answer the following questions: Are children able to notice two different structures in the same array? Perceiving the features of a slant structure by children proved to be considerably more difficult than drawing a rectangular structure. The disproportion in the degree in which the drawings reflected these particular types of regular arrays is significant and important. It suggests that in the case of children of this age the structures of two-dimensional arrays are not formed yet but still undergo the process of being constructed.

INTRODUCTION

The extract from research presented below[6] deals with structuring a two-dimensional rectangular array of elements by children aged 10 to 13. The location of elements in the plane is determined by a two-dimensional network of points. Understanding the essence of the two-dimensional array, distinguishing various structures is vitally important to mathematical thinking – is plays a fundamental role in the process of moulding numerous mathematical notions and their properties (e.g. the coordinate system, tables).

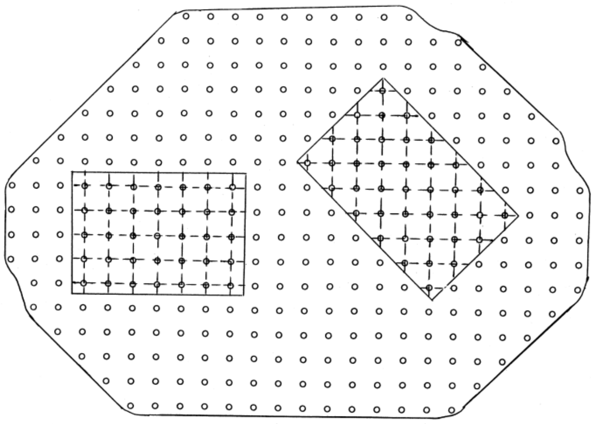


Figure 1

[6] *The remaining problems presented to the children in the course of the study were described in the paper (Rożek, 2005).*

In figure 1 the dots create an infinite two-dimensional array, in which two types of a rectangular-shaped arrangements are discernible. These rectangles define finite two-dimensional arrays which are characterised by different structures (structures is understood in the same way as Freudenthal says, 1991). In the first type **a rectangular structure** may be recognised, in which the indicated rows and columns intersect the sides of the rectangle at the right angle. The other type is characterised by **a slant structure**, in which the marked rows are positioned in a slanting direction to the sides of the rectangle.

THE STRUCTURE AND OBJECTIVE OF THE RESEARCH

The objective of the study was to answer the following questions: Are children able to notice two different structures in the same array? The research was conducted among pupils from forms V-VII (children aged 10-13). The total of 30 pupils participated in the study. Each pupil was working separately in the presence of the researcher.

> **Problem.** All the tablecloths shown in this chart (fig. 2) were made of the same paper fabric (fig. 3). Unfortunately, on the tablecloths the dots are missing. Your task is to draw the dots in such a way that the pattern of the tablecloth agrees with the pattern of the fabric.

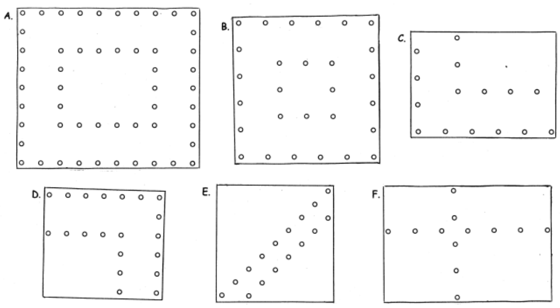


Figure 2

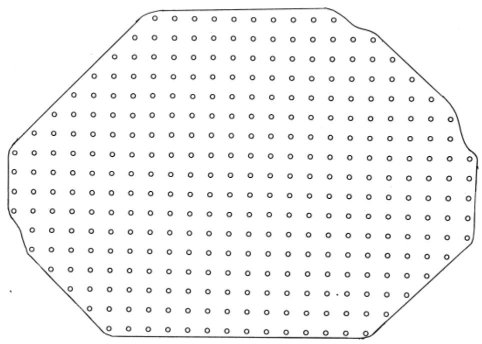


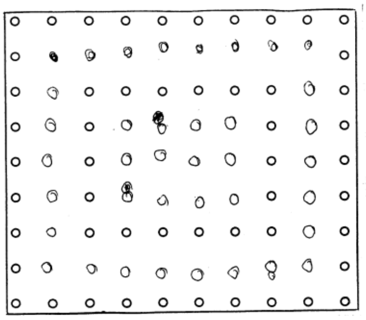
Figure 3

The chart shows two types of tablecloths, the ones with a rectangular pattern and the ones with a slant one. The aim of this problem was to verify if, basing on segments of an arrangement of dots, children are able to tell the tablecloths (the type of the structure) apart and if they can reconstruct the array.

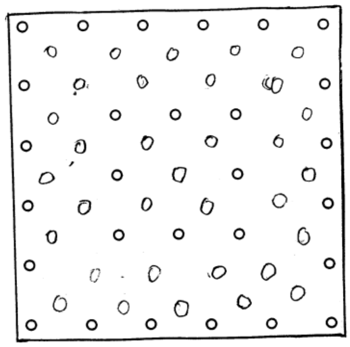
STRUCTURALISATION OF TWO-DIMENSIONAL ARRAYS IN CHILDREN'S DRAWINGS

The analysis of pupils' drawings illustrating the problem reveals several characteristic methods of drawing tablecloths:

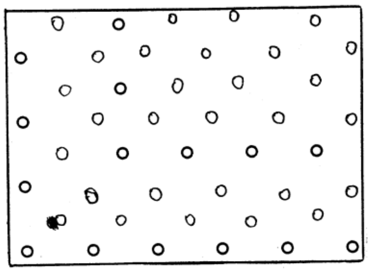
- **preservation of both the slant and the rectangular structures** (fig. 4)



Piotr - 13;9



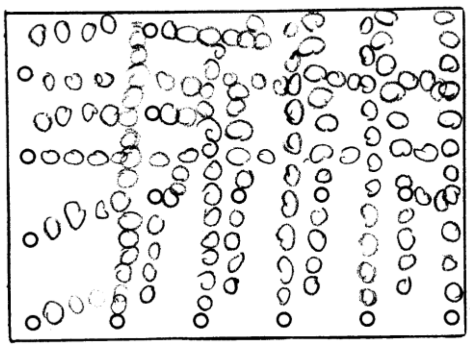
Łukasz - 10;5



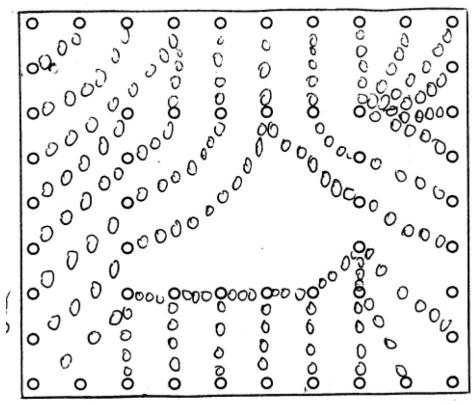
Kamila - 11;8

Figure 4

- **creating one's own structures** (fig. 5)**;** in some children's drawings neither the skew nor the rectangular structure is preserved. In most cases these children try to link the existing dots by 'the lines of dots' in an erratic way.



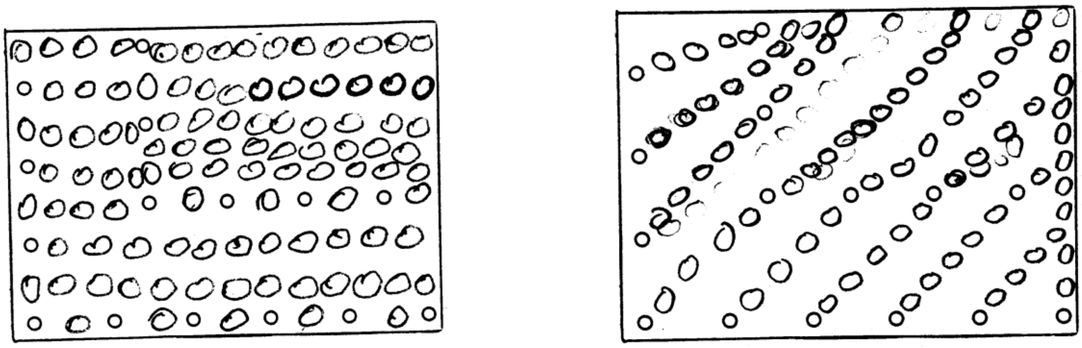
Kasia- 13;6



Marcin - 11;2

Figure 5

– **lack of coordination with the other direction** (fig 6)**;** while drawing parallel rows in one direction (horizontal, vertical or slant), children do not pay attention to coordinating it with the other direction. As a result, it does not lead to the reconstruction of a regular array in the drawing, but reflects merely one of the elements of the structure: only rows, only columns or only slopes. Similarly, attempts to draw rows in some sections and columns in others without proper coordination may lead to the lack of a row-column structure in the obtained array.



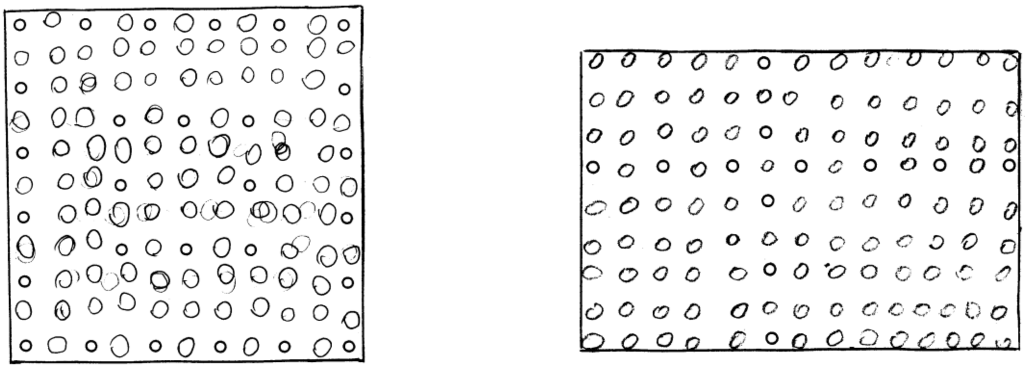
Piotr - 12;5 Rafał - 13;0

Figure 6

Transforming Skew Tablecloths Into Rectangular Ones

Reconstructing a rectangular structure proved to be a task noticeably easier for children than illustrating a slant structure. Therefore, it is not surprising that some children directed their effort at obtaining a rectangular structure, even in the case of a slant one. The analysis of children's worksheets enabled distinguishing four patterns of transforming slant tablecloths into rectangular ones:

– **thickening the net of dots up to a rectangular structure** (fig. 7)**;** some pupils try to draw the dots of each tablecloth so that the rectangular structure should emerge. Quite often the participants transform skew tablecloths into rectangular ones by thickening the net of dots.

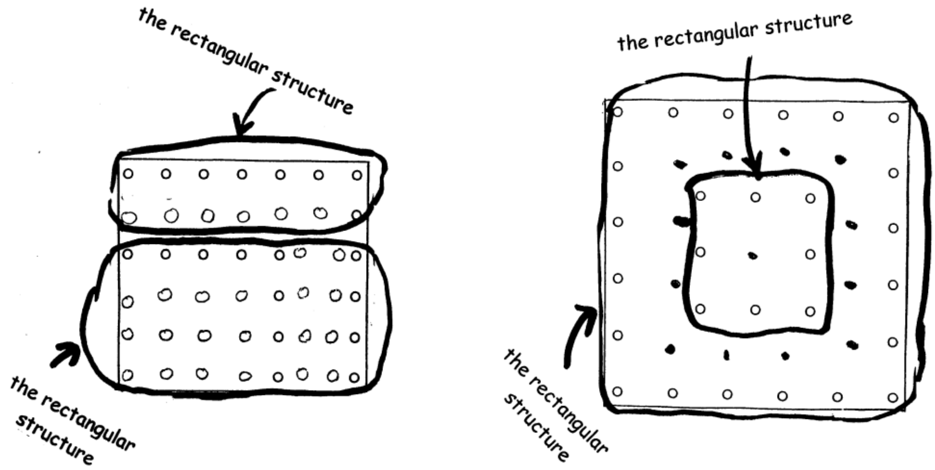


Kasia - 10;9 Kasia - 13;0

Figure 7

– **regulating the distances in order to obtain a rectangular structure**; some children paid no attention to the distances between rows. They were able to add dots to a slant tablecloth in such a way that the distances between rows were not the same but, as a result, they obtained a rectangular structure.

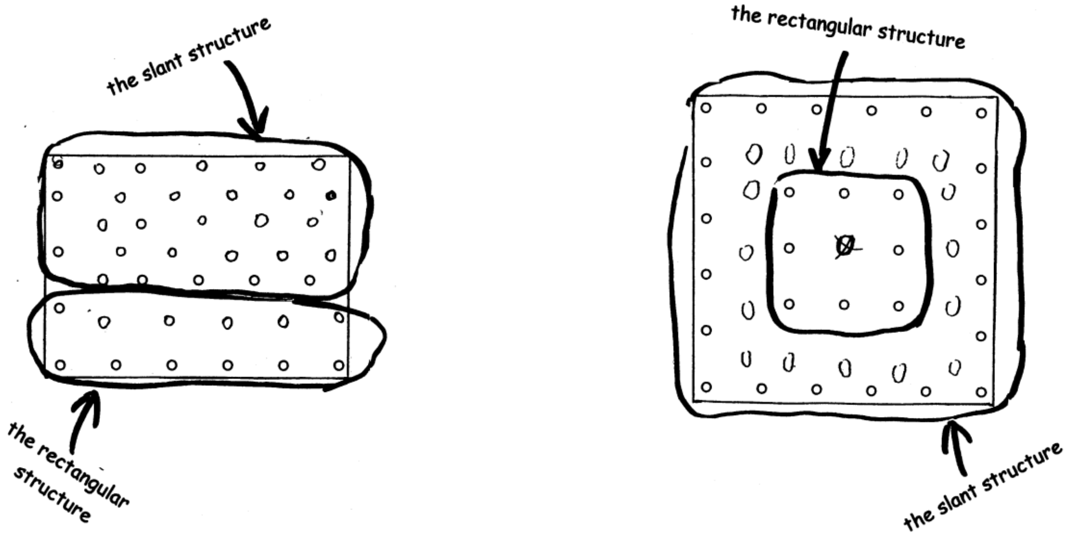
– **creating a partially rectangular structure** (fig. 8); quite often children concentrated on fragments of tablecloths without analysing the pattern as a whole. They added dots to a section of a tablecloth in a way which guaranteed preserving a rectangular structure of this fragment. Another section of the same tablecloth was treated by them as a separate part and they completed it to a rectangular structure. The whole tablecloth, however, was characterised by neither a rectangular structure nor a slant one.



Joasia - 12;0 Marcin - 11;0

Figure 8

– **creating tablecloths of a mixed structure** (fig. 9); there were also certain worksheets in which one fragment of a tablecloth was completed in accordance with the rectangular structure, while the remaining part displayed a slant structure.



Kamil - 10;3 Krzyś - 11;1

Figure 9

SUMMARY

The variety of methods employed by children to reconstruct tablecloths in a drawing seems to be a manifestation of the complexity and diversity of children's problems with structuring two-dimensional arrays. Perceiving the features of a slant tablecloth proved to be considerably more difficult than drawing a rectangular structure. Pupils' effort directed at changing slant tablecloths into rectangular ones reveals characteristics specific to children's thinking, which differs from the mode of thinking of adult people. The disproportion in the degree in which the drawings reflected these particular types of regular arrays is significant and important. It suggests that in the case of children of this age the structures of two-dimensional arrays are not formed yet but still undergo the process of being constructed. Only by thorough studying children's behaviour is it possible to succeed in describing the natural mode of proceeding which leads to structuring two-dimensional arrays.

REFERENCES

Battista, M. T., Clements, D. H., Arnoff, J., Battista, K., & Borrow, C. V. A.: 1998, Students' Spatial Structuring of 2D Arrays of Squares, Journal for Research in Mathematics Education, Vol 29(5), 503-532.

Freudenthal, H.: 1991, Revisiting Mathematics Education, Mathematics Education Library Kluwer Academic Publishers.

Outhred, L.: 1992, The array: convention or confusion, paper presented during ICME-7, Quebec.

Outhred, L. & Mitchelmore, M.: 2000, Young children's intuitive understanding of area measurement, Journal for Research in Mathematics Education, Vol 31(2), 144-167.

Outhred, L. & Mitchelmore, M.: 2002, Across the great divide: From counting to subdivision of rectangular area, In R. Nata (Ed.) Progress in Education, Vol 5. New York: Nova Science Press.

Rożek, B.: 1997, Struktury szeregowo-kolumnowe u dzieci w wieku od 6 do 8 lat, Dydaktyka Matematyki 19, 29-46.

Rożek, B., Urbańska, E.: 1998, Children's Understanding of the Row-Column Arrangement of Figures, Proceedings, CIEAEM 50, Neuchâtel.

Rożek, B.: 2002, Denkschema eines Reihen-Kolonnenfigurensystems bei Kindern im Alter von 6 bis 9 Jahren, Beiträge zum Mathematikunterricht: Vorträge auf der Didaktik der Mathematik in Klagenfurt, Verlag Franzbecker.

Rożek, B., 2003, Strukturbetrachtung in regelmäßiger Figurenanordnung bei den Kindern, Beiträge zum Mathematikunterricht: Vorträge, ed. Henn H.-H., Verlag Franzbecker, Hildesheim und Berlin, [37 Tagung für Didaktik der Mathematik], 549-552.

Rożek, B.: 2005, Strukturwiedergabe bei Zeilen-Spalten-Systemen in Zeichnungen von Grundschulkindern, Beiträge zum Mathematikunterricht, Vorträge auf der 39. Tagung für Didaktik der Mathematik, Franzbecker, Hildesheim und Berlin, s. 489-492.

Rożek, B.: 2005, Strukturyzacja dwuwymiarowego szyku prostokątnego przez dzieci w wieku od 10 do 13 lat, Roczniki Polskiego Towarzystwa Matematycznego. Seria V. Dydaktyka Matematyki 28, s. 221-274.

van Hiele, P. M.: 1986, Structure and Insight. A Theory of Mathematics Education, 1986, Academic Press, London.

IMAGINING A MYSTERIOUS SOLID: THE SYNERGY OF SEMIOTIC RESOURCES

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Recent studies have pointed out the significance of perceptuo-motor and embodied activities in mathematics learning. Assuming such a viewpoint, we use the theoretical notion of 'Action, Production and Communication Space' (Arzarello, 2008) and the practical tool of 'semiotic line' (described in this report) to analyse the cognitive and semiotic dynamics that occur during mathematical lessons. We discuss the case of a classroom activity introducing fourth grade children to 3D geometry. In group-work with the supervision of the teacher, children are asked to imagine a mysterious solid, i.e. a solid composed by the minimum number of equilateral triangles. We use the semiotic line to study the role and synergy of different semiotic resources used by the children and the teacher. In particular, gestures and gazes are analysed in order to identify their contribution to the cognitive activity (mainly imagining), as well as the communicative and the didactical ones.

INTRODUCTION AND THEORETICAL BACKGROUND

A complex interplay of languages and representations are involved in mathematical thinking and, consequently, in the processes of teaching and learning. It is well acknowledged that the learning subject tries to make sense of new inputs in terms of existing ones, widely drawing on cognitive mechanisms that link old experiences with new data. In particular, some studies framed in the "Embodied mind paradigm" have shown that different cognitive processes have a common ground in the human body, as well as in its location in space and time (Lakoff & Johnson, 1980; Lakoff & Núñez, 2000). Thus, conceptual knowledge is deeply embodied, that is integrated, in our sensor-motor system. The contribution of the cultural dimension has been emphasised by Sinclair and Schiralli (2003), claiming that sensor-motor experiences can be variously structured by those neurophysiological predispositions that human beings inherit genetically and are successively mediated by environmental factors that include symbolic and cultural systems. In short, individuals develop their cognitive (included learning) processes in two deeply intertwined frames, which refer to genetic and cultural evolution (Arzarello, Robutti & Bazzini, 2005; Arzarello, 2008). As a consequence, both bodily and cultural factors have to be taken into account to study mathematics teaching and learning.

Without disregarding such twofold dimension, in this report we focus on the contribution of embodied elements such as gestures and gazes to the process of imagination of the tetrahedron, as the solid made by the minimum number of (equal) equilateral triangles. As concerns gestures, there is experimental evidence that gestural and verbal modalities are strictly coordinated in communication and tend to converge synergically (McNeill, 1992; Goodwin, 2000). Furthermore, their strict interdependence in the production of thought and knowledge has been shown (McNeill, 1992; Goldin-Meadow, 2003): gestures are not limited to communication processes, but they are fundamental in the formation of thinking. Also in the educational context the need to merge the study of students' linguistic and mathematical activity in the classroom with an analysis of the related gestural component has emerged (Edwards, 2003; Radford et al., 2004; Sabena, 2008).

In this perspective, the construction of mathematical knowledge, at all school levels, should develop through activities that favour percepuo-motor learning and involve interactions with body experiences. Nemirovsky (2003) stated that

> subject itself…the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuo-motor activities, which become more or less active depending of the context while modulated by shifts of attention, awareness, and emotional states, understanding and thinking are perceptuo-motor activities; furthermore, these activities are bodily distributed across different areas of perception and motor action based on how we have learned and used the (p.108).

Thus, cognition lives in a complex landscape influenced by different components: the body, the physical world and the cultural environment. In the classroom, a realistic picture of what is going on contains all these elements. They have to act and interact in a synergic way, in order to create what Arzarello (2008) calls the ‘Action, Production and Communication Space’ (in short the APC-Space).

In the APC-space we can situate what Radford (2003) calls “semiotic means of objectification”, that are all those means that contribute to subjects’ knowledge formation, such as speech, tools, gestures, writing, and so forth. Arzarello (2006) considers all these different resources entering into teaching-learning processes. They constitute a sort of “semiotic bundle”, which includes signs of different kinds, in a deep integrated way: words (orally or in written form); extra-linguistic modes of expression (gestures, gazes, …); different types of inscriptions (drawings, sketches, graphs, ...); and so on (for some examples see Arzarello, 2006 and Arzarello at al., 2009). The semiotic bundle notion considers a very wide notion of sign, in Peirce’s sense (Peirce, 1931-1958): anything that can be interpreted by somebody in some respect can be considered as a sign. Differently from other semiotic approaches, it allows us to include gestures, gazes and more generally all the bodily means of expression, as semiotic resources in learning processes, and to look at their relationship with the traditionally studied semiotic systems (e.g. written mathematical symbolism):

> A semiotic bundle is a *system of signs* — with Peirce's comprehensive notion of sign — that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher (Arzarello et al., 2009, p. 100).

In teaching-learning contexts, the different semiotic resources are used with great flexibility: the same subject can exploit many of them simultaneously, and sometimes they are shared by the students and by the teacher. All such resources, with the actions and productions they support, are important for grasping mathematical ideas, because they help to bridge the gap between the worldly experience and the time-less and context-less sentences of mathematics. Previous research studies of our group confirmed such hypotheses in teaching experiments dealing with arithmetic (Arzarello et al., 2006; Bazzini, Sabena & Villa, 2009). In this paper we are concerned with the semiotic resources involved in a teaching experiment in geometry, and in particular in the process of imagining a mysterious solid on the base of given information.

In order to analyse the dynamics of the semiotic bundle in teaching-learning processes in the classroom, the research group in mathematics education at the University of Torino (composed by teachers and researchers) has pointed out a methodological tool called 'semiotic line'. The semiotic line is a table (produced using a worksheet) containing students' and teacher's utterances, gestures, and in general the semiotic resources used in the flow of time. It is based on video-recording of classroom activity, usually students' problem-solving in group-work, or discussions coordinated by the teacher. An example is shown below in Figure 8. In the horizontal dimension, it shows how things develop diachronically over time. In the vertical dimension, it shows how different semiotic contributions are related each other in a synchronic way: for instance, a gesture may be co-timed with an utterance by the same or another subject. As analysis tool, the semiotic line allows us to carry out analysis at a very refined level, taking into account, when necessary, sequences of just few seconds. On the other hand, it allows to get an overall look at certain semiotic features of the activity, e.g. at the gestures. Furthermore, the semiotic line is a flexible tool, modifiable according to the research need. In the analysis we are presenting in this paper, the component of the *gazes* of students is added. Gazes appear of great interest in a task strongly involving imagination, as in our case (see also, at a secondary school level and in another context, Ferrara, 2007).

In the following, we will use the semiotic line to analyse in detail some episodes of the teaching experiment concerning the mysterious solid. In particular, we will focus on the students' and teacher's gestures and gazes in a group activity, in order to highlight their cognitive, communicative and didactic contributions.

THE TEACHING EXPERIMENT

We carried out a teaching experiment framed in solid geometry. This subject is not frequently approached in primary education, notwithstanding its great educational value. In our perspective, 3D geometry is of great interest particularly because it is deeply connected with children's experience of the world around and, consequently, with their body actions and perceptions.

Traditionally, 3D geometry is approached through concrete models and 2D representations. Taking into account that geometrical reasoning is a process of close dialectic interaction between figural and conceptual aspects (Fischbein, 1993), in our experiment we foster children's imagination and investigation of solid figures before proposing 3D models, and we let children express their imagined solid in the way they prefer, including the use of gestures.

In the following, we analyse an experimental activity carried out in the fourth grade of primary school. First, the teacher asks the pupils to imagine a solid by using the minimum number of equal equilateral triangles. Then, the children are required to draw the solid and, finally, to built it by means of paper triangles. In the end, the teacher gives the name of the mysterious solid, the tetrahedron, and stimulates the children to discover its properties.

The activity lasted one hour and was videotaped. The pupils were divided in three groups (high, medium and low), according to their level, and each of them worked in a laboratory session with the support of the teacher.

ANALYSIS

We focus our analysis on the very first part of the activity, in which children are asked to imagine the solid without any tool or visual support. Here some passages from the high-level group (composed by seven children) are discussed.

A 'special pyramid'

The teacher submits the task to the pupils: *Imagine building a solid with the minimum number of equal equilateral triangles*.

One student immediately proposes the cube, but the idea is soon rejected, being it composed by squares and not triangles. Matteo (high level) silently forms a configuration with his hands (Figure 1).

Figure 1

1. Riccardo: The pyramid!
2. Teacher: Riccardo says the pyramid…
3. Matteo (interrupting): No because below, the base…
4. Valentina: They are not equilateral triangles.
5. Matteo: They can be that: just the base is a square! We could do a pyramid, but not a normal pyramid: a special pyramid. A pyramid with three

triangles (gesture similar to Figure 2). That is, putting one here, one here and one here (Figure 2).

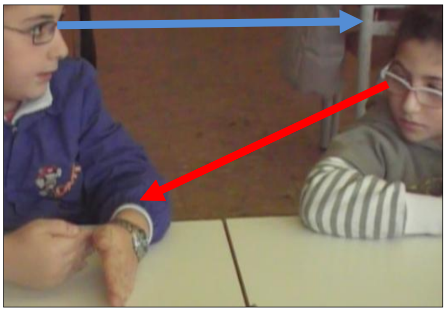


Figure 2

Riccardo introduces the idea of pyramid. All the students appear to associate to the term a prototypical idea of pyramid with squared base, as the very famous classical Egyptian pyramids, which children have studied in the history class. Matteo immediately reacts by pointing out that the pyramid (in the classical version) cannot be the mysterious solid, since it has a squared base. The pupil has already imagined a solid with triangular base: as we can notice in Figure 1, his hands and arms represent a triangular shape on the desk. Such gesture can be considered as a visual support to the child's imaginative process: let us observe how he is looking at his hands. On the contrary, in Figure 2 we can see how Matteo while speaking and gesturing is constantly looking at the teacher. We can interpret the different directions of the child's gaze as an index of the child's cognitive processes: in the first case (Figure 1), he is thinking and imagining, in the second case (Figure 2), he is describing what has imagined, with strong communicative intention (and probably in search for the teacher's approval). At the same time, we notice that Desiré, who stands close to Matteo, pays attention to Matteo's gestures and probably this contributes to her mental construction of the solid. To highlight the different direction of the children gazes during the activity, in the semiotic line we add coloured arrows to the pictures: yellow if the subject is looking at his own gestures, red if he is looking at another person's gesture, blue if he is looking at an interlocutor's eyes.

Matteo is still uncertain in his verbal description of the solid: he speaking of three triangles (line 5), neglecting to count the base. However, his gestures show that he is correctly imagining it: his hands in Figure 2 show at least two different features the solid, i.e. its triangular base (information already present in previous gestures, Figure 1) and its slanted sides.

To make explicit what is still compressed in the gestures, the teacher asks Matteo to clarify, with the aid of gestures:

6. Teacher: Let me show with your hands.

7. Matteo: One here (Figure 3a), one here (Figure 3b) and one here below (Figure 3c), so also at the base (tracing a triangle with his forefinger, Figure 3d) there is an equilateral triangle.

Figure 3 a-b-c-d

The teacher fosters the use of gestures. We can see how Matteo (line 7) is using a semiotic bundle made of speech and gestures. At first he is putting his hand to represent the three slanted sides of the pyramid, in three static configurations (Figure 3a-b-c). Then he concludes the representation by tracing with his index the triangular border of the base (Figure 4d). The accompanying utterance contains deictic references ('here') that are co-timed with the gestures.

Gesturing in a shared space

On the one hand, Matteo's account appears indicating that the child has built a good mental image of the solid. On the other hand, his description still contains some ambiguities that need to be clarified. For instance, it is not clear from his gestures if the solid is open or closed, and in his words he seems to forget one face (see both lines 5 and 7). Furthermore, even admitting that he is correctly imagining and is just having some problem in expressing himself, from a didactic point of view a further clarification is needed for the whole group of children.

As a didactic strategy, the teacher proposes to build the solid with the hands, starting from the base that she is forming with her fingers on the desk (Figure 4):

Figure 4

In this way, the teacher is using the same semiotic resources enacted by the children, and fostering some progress, i.e. she is making clear for all the children that the base is triangular. Furthermore, she is asking the children to actively contribute to her gesture, to build the mysterious solid by means of gestures.

Immediately Riccardo uses one hand to show how to complete the solid (Figure 5), then Matteo intervenes again and uses both his hands to simulate a face of the solid and tries to incline such face towards the interior of the solid (Figure 6), and finally all the children take part in the construction of the mysterious solid (Figure 7).

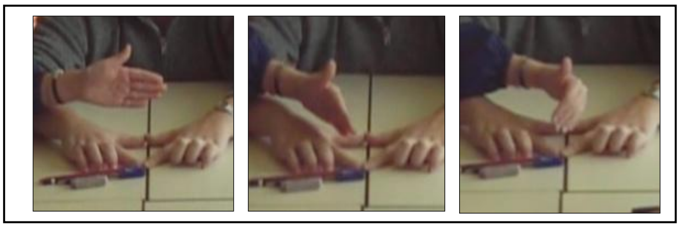


Figure 5

Figure 6

Figure 7

In other words, we can say that the teacher has promoted children participation in a sort of *'shared gesture space'*. The notion of 'gesture space' has been introduced by McNeill (1992) in gesture studies in psychology to indicate the space where a person is gesturing. In our case, the gesture space is physically shared among all the participants of the group, indicating an active involvement of all the children and fostering a suitable Space of action, production and communication (APC-space). Figure 8 shows a fragment of the semiotic line that has contributed to highlight the APC-space and the roles of words, gazes and gestures.

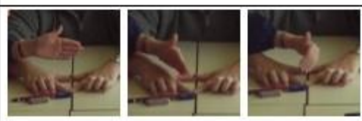
	0:03:39		0:03:49
31	32	33	34
	TEACHER	VALENTINA	RICCARDO
CHILDREN'S WORDS			One here, one here, and one here
TEACHER'S WORDS	Come to put the other faces. Let us suppose that this is the base, an equilateral triangle...Come to build the solid.		
CHILDREN'S GESTURES			
TEACHER'S GESTURES			

Figure 8

DISCUSSION

Our analysis of the teaching experiment by means of the semiotic line shows that the different semiotic resources activated by the children (ranging from natural language to gestures) are often in accordance, both as far as they concern time and meaning. This result confirms the assumption of their simultaneous activation during cognitive activities, which involve body and mind and their mutual interaction.

Several times we noticed that a pupil's gesture was reproduced by peers in the same modality or with additional details. The gesture of a pupil becomes the starting point for another pupil, and so on. This could confirm the hypotheses that gestures carry meaning and witness the cognitive progress of the students.

In the analysis shown in this report, we observed the intervention of the teacher who takes into account Matteo's gesture and reproduces it with some essential differences, i.e. reproducing the base of the solid. A similar phenomenon has been investigated by Arzarello and called *semiotic game* (Arzarello, 2006; Arzarello et al., 2009). A semiotic game happens in the teacher-students interaction when the teacher tunes with the students' semiotic resources and uses them to guide the evolution of mathematical meanings from personal to scientific concepts (Vygotsky, 1978). In many cases the teacher repeats a student's gesture, and correlates it with a new term or with the correct explication, by using natural language and mathematical symbolism. Semiotic games constitute therefore an important strategy in the process of appropriation of the culturally shared meaning of signs. Obviously, gestures must not be limited to imitation, but improve understanding and reasoning. In our example, the teacher (see Figure 4) has tuned with Matteo's gestures (Figure 3), but adding two important new contributions. The first one is given by the explicit shape of the base (triangular), fixed in static way on the desk, to remark all children that also the base is triangular (see the initial confusion with the prototypical pyramid with squared base). The second one is the invitation, soon accepted by all the children, to contribute to the gesture in a sort of *'shared gesture space'*. Such shared space, where gestures are performed, is interpreted as promoting the APC-space in the group, and thus a suitable environment for the solution of the task in social interaction and co-operation.

In our analysis we have also remarked the pupils' gazes, distinguishing when they are oriented towards one own gestures, someone else's gestures (by a peer or by the teacher), or to the interlocutor's eyes. We interpret the attention paid to one own gestures as an index of deep cognitive involvement of the subject, for instance when imagining the shape of the mysterious solid. On the contrary, the gazes at the interlocutor's eyes are interpreted as stressing more the communicative dimension of the interaction. Somehow in between, the attention paid to someone else's gesture is hypothesized supporting the process of understanding. In fact, looking at gestures completes listening to words: not only a process of communication, but a process of cognitive interaction and progress.

A final remark on the semiotic line. The semiotic line is an experimental tool, suitable for analyzing in detail the semiotic resources which play a role in the process of problem solving. Though its creation requires a huge amount of time, its use is flexible, according to the interests of the observer. For this reason we think it could be useful not only for researchers, but also for teachers and teacher education, in view of a detailed analysis of what happens in specific moments in the classroom. Through the semiotic line the teachers could get new insights into children cognitive activity and disclose hidden aspects of their didactic practice.

REFERENCES

Arzarello, F.: 2006, Semiosis as a multimodal process, *Revista Latino americana de Investigación en Matemática Educativa*, Vol. Especial, pp. 267-299.

Arzarello, F.: 2008, Mathematical landscapes and their inhabitants: perceptions, languages, theories, in: E. Emborg & M. Niss (Eds), *Proc. 10th Int. Congress of Mathematical Education* (ICME 10) (pp.158-181). Copenhagen, Denmark.

Arzarello, F., Robutti, O. & Bazzini, L.: 2005, Acting is learning: Focus on the construction of mathematical concepts, *Cambridge Journal of Education*, *35*(1), pp. 55-67.

Arzarello, F., Paola, D. Robutti, O. & Sabena, C.: 2009, Gestures as semiotic resources in the mathematics classroom, *Educational Studies in Mathematics, 70*(2), pp. 97-109.

Arzarello, F., Bazzini, L., Ferrara, F., Robutti, O., Sabena, C. & Villa, B.: 2006, Will Penelope choose another bridegroom? Looking for an answer through signs, in J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proc. 30th Conf. of the Int. Group for the Psychology of Mathematics Education* (vol. 2, pp 73-80). Prague, The Czech Republic: Charles University.

Bazzini, L., Sabena, C. & Villa, B.: 2009, Meaningful context in mathematical problem solving: a case study, *Proc. 3rd Int. Conf. on Science and Mathematics Education* (pp. 343-351). Penang, Malaysia: CoSMEd.

Edwards, L.D.: 2003. A natural history of mathematical gesture, *Paper presented at the American Educational Research Association Annual Conference*, Chicago.

Ferrara, F.: 2007, *Acting and interacting with tools to understand Calculus concepts*, PhD. Dissertation. Università di Torino.

Fischbein E.: 1993, The theory of figural concepts, Educational Studies in Mathematics, 24(2)*, pp.139-62.*

Goodwin, C.: 2000, Action and embodiment within situated human interaction, *Journal of Pragmatics*, 32, pp. 1489-1522.

Goldin-Meadow, S.: 2003, *Hearing gestures: How our hands help to think*, Cambridge, MA: Harvard University Press.

Lakoff, G. & Johnson, M.: 1980, *Metaphors we live by*, Chicago: University of Chicago Press.

Lakoff, G. & Núñez, R.: 2000, *Where mathematics comes from*, New York: Basic Books.

McNeill, D.: 1992, *Hand and mind: What gestures reveal about thought*, Chicago: University of Chicago Press.

Nemirovsky, R.: 2003, Three conjectures concerning the relationship between body activity and understanding mathematics, in: N. A. Pateman, B .J. Dougherty & J. T. Zilliox (Eds.), *Proc. 27th Int. Conf. of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 105-109). PME.

Peirce, C.S.: 1931-1958, *Collected Papers* (Vol. I-VIII), in C. Hartshorne, P. Weiss, A. Burks (Eds.), Cambridge, MA: Harvard University Press.

Radford, L.: 2003, Gestures, speech, and the sprouting of signs, *Mathematical Thinking and Learning*, *5*(1), pp. 37-70.

Radford, L., Demers, S., Guzmán, J., & Cerulli, M.: 2003, Calculators, graphs, gestures and the production of meaning, in N. A. Pateman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proc. 27th Int. Conf. of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 55-62). PME.

Sabena, C.: 2008, On the semiotics of gestures, in L. Radford, G. Schubring & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp 19-38). Rotterdam: Sense Publishers.

Sinclair, N. & Schiralli, M.: 2003, A constructive response to 'Where mathematics comes from', *Educational Studies in Mathematics*, *52*(1), pp. 79-91.

Vygotsky, L. S.: 1978, Mind in society. The development of higher psychological processes, Cambridge, MA: Harvard University Press.

VARIOUS INTUITIONS OF POINT SYMMETRY (FROM THE POLISH SCHOOL PERSPECTIVE)

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For acting in a geometrical world it is crucial to possess the ability of leading a reasoning which is based on the dynamic concepts' representations. In our approach we try to build the theoretical framework of creation of isometries. In this article we focus on the development of intuitional understanding of the point symmetry.

WHAT IS SPECIFIC IN EARLY GEOMETRICAL THINKING?

Theories which deal with the forming of geometrical concepts still need a lot of supplementation, mainly in their indications for the school practice. The most popular theories like van Hiele's (1986) levels of geometrical understanding or research results obtained by the Clements's team (Clements, Battista, 1992, Clements et al., 1999) create a quite broad theoretical framework. It is difficult to prepare any didactical proposal based on those backgrounds. From the other side, a traditional approach to the early geometry (at the intuitive stage) has focused on shape understanding only.

The approach we have chosen refers to the conception of the emergence of a geometrical world, as described by Vopěnka (1989) and Hejný (1993). In their opinion, a geometrical world cannot be perceived directly. It is hidden in the real world, and it is emerging from the surroundings through a special intellectual activity, which can be called 'the geometrical insight'. In this approach, the first geometrical recognition starts when a child focuses his/her attention on any geometrical phenomenon. It can be a shape, but also – any specific placement of two objects. The further stages of geometrical knowledge are based on perceiving connections among phenomena and relations among them.

On the basis of our previous research (Jagoda, 2004a, 2004b, Swoboda, 2006) we state that children between 5 and 10 can act in the 'world of regularities' by discovering regularities, making arrangements which are close to such geometrical relations like translations, rotations and a mirror symmetry. However, the results suggest that these phenomena are perceived in a static manner and are not connected with any movements. Children at this age do not make any reflection upon the way of manipulation, even while making the manipulation. They are focused on (and interested in) the final result of the manipulation.

Therefore, this is the opposite situation to one which supports the creation of arithmetical concepts. As it is stated in widely known Piaget's theories describing the process of creating of arithmetical concepts (Piaget, 1972, Aebli, 1982), the foundation of mathematical reasoning is the internalization of the actions, leading to its encapsulation into a mathematical concept. However, some theories question the action's priority for each type of mathematical cognition. Those convictions are mainly placed in relation to the geometrical cognition. It is believed that the development of geometrical concepts is different from that of the arithmetical ones (Gray et al., 1999, Tall 2001).

For acting in a geometrical world it is crucial to possess the ability of leading reasoning which is based on the dynamic concepts representations. A reflection upon the movement is especially essential in learning about isometries. For us it is important to build on a child's mind the connection among three elements: (a) the initial position of a figure, (b) its movement and (c) its final position. Although the overt description of symmetry as a transformation appeared rather late in mathematics (as it can be linked to the Erlangen Programme of F. Klein), the dynamic approach itself is crucial for geometry. A geometrical reasoning requires the mental transformation of objects. The history of mathematics shows the importance of the transition from a static to a dynamic interpretation of geometrical objects (Kvasz, 2000). This can be seen in Greek mathematics, in which the traces of general reasoning were based on dynamic object transformations.

In our approach we try to build a theoretical framework of the creation of geometrical concepts, by the didactical observation of ways of building the particular geometrical concepts. In this article we focus on the development of intuitional understanding of point symmetry.

THE FOUNDATION OF A SUBSTANTIAL LEARNING ENVIRONMENT FOR THE CREATION OF THE CONCEPT OF POINT SYMMETRY

The phenomenological analysis of the concept of point symmetry shows how complicated it is.

In the dynamic approach, the point symmetry can be treated as a particular case of rotation. This concept connects many others, like the general idea of a symmetrical figure, congruent figures or transformations. It creates a net of concepts which permeate each other. The proper understanding of one of them is the condition to understand the others.

When planning a didactical conception on teaching isometries at the level of basic school, we had some assumptions resulting from our previous observations (Swoboda, 2006, Jagoda, 2004b):

1. Mathematical concepts (including geometrical concepts) are multi-sided. Focusing only on one aspect of a concept leads to a limited understanding. From the other side, the process of integration of all aspects into one concept is very long and requires conscious didactical endeavors.

2. The preliminary understanding of a geometrical relation is static, as the arrangement between figures.
3. The mirror symmetry is the relation which is best recognized by children aged 4-10.
4. Translation is not treated as a relation, but as the repetition of the same object.
5. Rotation (of any angle) is intuitively understood locally, as the arrangement around any center which is brought into prominence for some reason.

We named our proposal ‘Tiles’, because the tools we used were tiles with one basic motif in two symmetrical versions: the left motif was the mirror reflection of the other one. Additionally, the motifs were placed on a square and on a rectangle. The designed tool was very simple, flexible and easy to adapt to the special conditions of a classroom. It also gave us the chance to realize the significant mathematical aims, contents and procedures at the certain educational level.

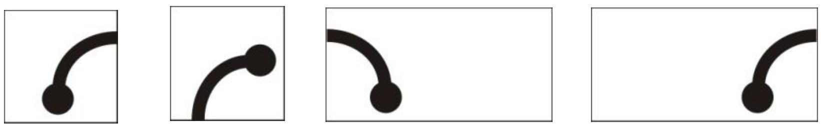


Figure 1: Variants of tiles

The creation of the didactical proposition went through several stages. In stage I, children received tiles and had to create ‘something interesting’. There was time for their free activity, but also time for recognizing various relations of one tile to the other one. At the second stage, we wanted to focus children’s attention on a one-dimensional pattern. The activities consisted of making patterns and correcting destroyed patterns. During the next stage children were familiarized with various relations – they coded and de-coded patterns. Additionally, they tried to verbally describe patterns. Stage IV was called the ‘guided patterns’. Children were preparing patterns according to a given title and a given music. The suggested topics were: (a) *a rapid river* – for using translation, (b) *a roundabout* – for using rotations, (c) *a mirror* – for using mirror symmetry. At stage V, children had the task to reconstruct a floor by using only a piece of the pattern.

The stages described above, lasted the whole school year. During that time children gathered experiences with various representations of geometrical relations like translation, rotation and mirror symmetry. They intuitively learned some of their properties. For example they learned (by their own experiences) that it is impossible to achieve mirror symmetry by any combination of rotations and translations.

For the researchers this was the period of gathering information about children's natural ways of work in the proposed environment. On the base of the gathered material we tried to recognize the possibilities of building such intuitions which are essential for geometrical concepts and reasoning.

Stages VI and VII were devoted for creating the intuition of point symmetry. The description of these stages is the main topic of this article.

NATURAL DIFFERENTIATION FOR CREATING THE CONCEPT OF ROTATION OF 180^0

1. Stage 'Dominoes'

At stage VI we wanted children to use some of the intuitions built during their previous activities. The higher class pupils (10-12 year old children) got the following task for solving:

How many different 'domino' blocks can be created by using two squared tiles?

Among other important discoveries which took place during the solving process, the students noticed one specific situation: there are some blocks, which after rotating them 'upside down' look the same. If they are created from one type of tiles (only 'right' or only 'left'), they look as follows:

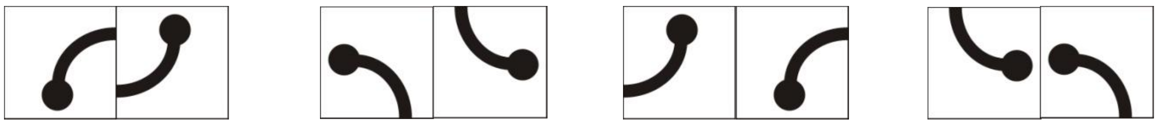


Figure 2: The 'rotated' dominoes blocks created by using 'left' tiles

In those dominoes both halves are in rotation of 180 degrees. When one takes the whole domino and makes a rotation of 180 degrees more, it gives the rotation about the full turn (360 degrees). If the domino blocks are built from a different type of tiles (left-right), after the rotation each block will have its twin, but in the set of right-left blocks.

During the activities students used the formulation 'rotation of 180 degrees' in a spontaneous way, although no features of those rotations were exposed.

2. Stage 'The floor which looks the same from both sides'

The next series of lessons were devoted to solving the following problem by the students:

Create a floor which looks the same while entering through doors located in the opposite walls of a rectangle room.

The main aims of this didactical experiment were:

- Observe, collect and describe the natural (spontaneous) students' approaches towards solving the problem, closely related to the concept of point symmetry.

- Comment on the natural approaches and make an interpretation of the current knowledge about the creation of geometrical concepts.

Students could use whichever tile they wanted (one type or simultaneously both left and right type). Everybody was working on a sheet of paper sized A4.

The group of students we describe below contained 10 persons (boys and girls). They were working during one lesson and their work was videotaped.

Various solving strategies

Every observed student started to work individually. During their work different strategies appeared. Sometimes initial ideas were transformed.

A. Using 'rotated' elements

Example 1. Two girls were working close to each other. One of them – Kasia – perceived the possibility of using the domino blocks, which after rotation about 180 degrees looked the same. When the whole floor had to look the same after rotation, its small pieces could have the same properties. She covered the whole sheet of paper by using only one type of domino block.

Her strategy went through several stages (as it was visible on the film). At the beginning she intended to use different types of 'rotated' blocks, but from a particular moment she started to standardize her pattern. She decided to make the whole floor by using only one type of block which looked the same after rotation of 180 degrees. It seemed she was working according to the conviction: when something worked locally, it should also work globally. She knew very well the inner relation in one domino block. Therefore, at the beginning she made the frame around the sides and then she filled in an interior part.

Figure 3: Kasia's work

The utterances quoted below show that she was thinking about repetition of one 'good' motif while she finished her work.

1. S: Yeees (*she corrects one tile*) – and now it should be good.
2. T: How do you recognize that it is good? How are you doing it?
3. S: Because **firstly I have put one block** consisting of two blocks (*tiles*) and then I checked if it looked the same after rotation (*she shows by hands how to turn a block upside down)* and then I built **one pattern** from two, … several, ….. two blocks by two blocks (*tiles*).

Example 2. A boy, sitting in the opposite room corner, has apparently implemented the same idea. He used a rotated motif. But he placed it in the alternate position on the sheet of paper. He has checked this placement by turning upside down the whole work.

1. T: Show me the beginning of your work.
2. S: *(he shows his work, containing two bands – one at the top, another one at the bottom. The bands contain two different rotated motifs)*
3. T: How did you check this work?
4. S: Firstly, I stowed like this, and after that I did such a turn (*he kept two opposite sides and did a rotation about 180 degrees).*
5. T: and?
6. S: it fits.

Figure 3: Turning paper upside down

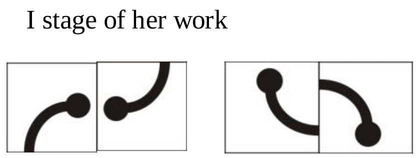
Here the approach is different. The boy knew that his floor should look the same after rotation. He proceeded systematically: he created one motif, stuck it on, then turned the paper upside down and stuck on an identical motif at the same place than the previous one. It was a secure strategy. From the pedagogical point of view, making rotation can be perceived as the first step toward interiorization and towards the mental model of this movement – the correspondent elements are placed in a very specific way. From the other side, no center of rotation (point symmetry) is distinguished.

B. Joining the idea of rotated motif with the mental picture of mirror symmetry

In the other students' works one may perceive the attempt of solving the problem going through experiences with the mirror symmetry. Sometimes this approach was provoked by the teacher, but very often students themselves were looking for such connections. It is difficult to state why they chose this direction. A stock of their experiences with making patters is strongly connected with the mirror symmetry. This concept is very close to the children's experiences; very often they use it to explain many geometrical phenomena. It is also the only geometrical relation which is exposed and elaborated during math lessons in the fourth and fifth grade.

Based on the gathered experiences students knew that the mirror symmetry enabled them to make an object which contained two identical 'halves'. Below we present variations of such an attempt.

Example 3. Marysia, sitting close to the girl described in the first example, was working only apparently similarly to her colleague's work. She started from making one motif at the table – she checked its appearance after rotation of 180 degrees. In this way she created few rotated motives and started to put them on the paper (fig 4).



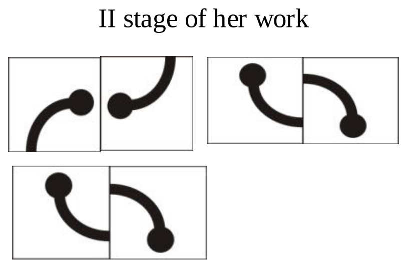


Figure 4: First two stages of Marysia’s work

After putting the third domino block, Marysia started to analyze the relations and to look for another domino which would fit into her pattern. The blocks were different and created a rich combination of relations with their neighbours. Marysia was working for a long time, thinking upon the next tiles. In this way she fulfilled the first half on her paper, by making some modifications on her work from the stage II.

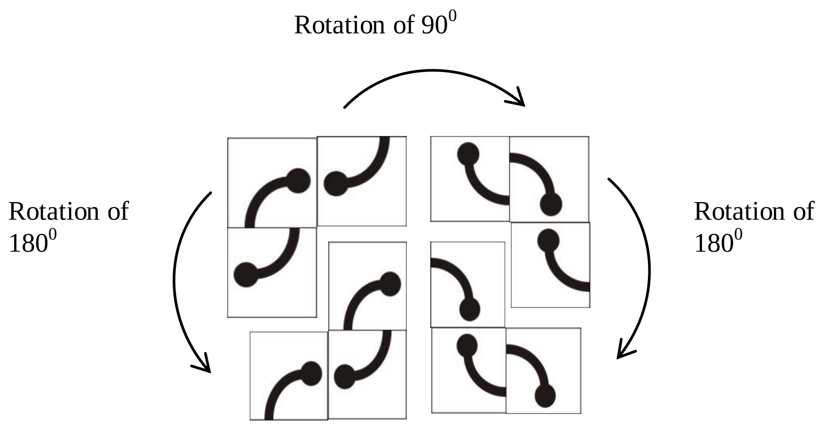
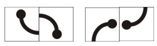


Figure 5: First half of Marysia’s

By looking carefully at this fragment we can see that she used a motif which contains three tiles. This motif was created by the specific configuration of tiles from domino blocks (). She repeated such a motif four times by using rotations. Such approach was time consuming and required a lot of mental effort from Marysia. Perhaps that was why she decided to use the idea of the mirror symmetry, after making the half of her work.

1. S: I have noticed something. When I have given the mirror here, it would be placed symmetrically.
2. T: and?…
3. S: Can I take ‘left’ tiles for putting them in this way?

Her neighbour, Kasia, already created the whole floor by using only one type of tiles. Marysia discovered that she can do it in another way, by using the mirror symmetry. She knew that such symmetry gave her the second half – the effect she needed. Till that time nobody from her colleagues stated that it is not enough to have only one type of the tiles. But she was sure that for realizing her idea of mirror symmetry she needed a second type of tiles.

As we see, Marysia modified her strategy. She tried to use *old* knowledge in a new situation. She was very consistent in the realization of her idea. It is seen in her utterances after finishing the whole work.

1. S: I made a pattern till the middle and since that I have done something like a mirror reflection of that first part.
2. T: Where was that middle?
3. S: *(shows by the gesture over the paper)* – here.

The student tried to combine the locally understood rotation of 180 degrees (domino blocks) with a more global understanding of mirror symmetry. During the work she did not make any manipulation on the paper, she created the pattern level by level. Undoubtedly, she was convinced that making the second part of the floor through mirror symmetry fulfilled all task's conditions. She put the line of symmetry in the middle of the paper, in parallel to the sides with the doors – this meant that the distance to both doors was the same. This resulted in complementary elements being placed in the same relation to the doors. In this way Marysia took into consideration many elements which determine a good solution.

Example 4. In spite of the general aim of the task (a floor which looks the same after turning), Bartek's work had an A, B, C, B, A structure, which was in connection to the mirror symmetry structure. During the talk with the teacher, the boy showed elements which were complementary to the rotation of 180 degrees.

1. T: Show me how did you arrange, and – how did you check your work.
2. S: *He shows his work and immediately turns it on 180 degrees. Then he shows one motif and says:* For example this one has a dot up and the arch from the right side (*turns the paper upside down)* – a dot up and the arch on the right.
3. T: Did you look only at this one element?
4. S: Not only at this one. This one is at down and I can make a circle from it (*with his finger he draws a shape of one of the motif).*An I have noticed something else (*he puts the paper in horizontal position and with his arm divides his work in two parts)*, when this [element] is here (*he shows one element below the arm)* then here is also in the same way *(he points at the complementary element from above the arm).* It works also in this way.
5. T: Bartek, is there a mirror reflection at your work? (*this question was provoked by the previous student's gesture).*
6. S: So, if we could put the mirror here …. (*he looks at his work carefully)*. No, there is no mirror reflection, but after rotation of 180 degrees it looks the same.

This example shows that permeating the ideas of mirror symmetry and point symmetry is difficult. Both those ideas existed independently and had a different epistemology. Bartek perceived the mirror symmetry statically. For him it was the form of the objects' arrangement and it was assessed visually. The point symmetry (at the intuitive level) was the new object for him and by this task it was in the centre of interest. The student not only searched the related objects (motifs), but discovered that such relation was independent from the paper position (*It works also in this way)*. During solving the task the student learned to recognize the mutual positions of the objects being in the relation of point symmetry, but his way of manipulation was unequivocal: firstly Bartek distinguished the basic elements, than he made a movement – rotation of 180 degrees, and after that he indicated an element which was in relation to the first one, and stressed that it looked the same. The sequence of actions is the same like in the construction of functional relation, transforming one object into another one. Additionally, the difference between intuitions of mirror symmetry and point symmetry were visible also in the last utterance (8): the mirror symmetry was assessed visually and was named *mirror reflection,* while talking about point symmetry he used the dynamic formulation; he said *after rotation of 180 degrees…*

C. Construction by using mirror symmetry

In a group of solutions one work occurred, in which a student decided to use two line symmetries in a conscious way. He was so fascinated of his discovery and did not want to disclose it before finishing his work.

Example 5. Episode 1

1. T: And you, Maciek, what kind of idea do you have?
2. S: it is funky… I'll show you, when I will have … an effect.

Episode 2

1. T: tra ta ta… (student, very proud, takes his work up and imitates fanfares)
2. T: And what – is this a good work?
3. S: I think "yes"
4. T: Tell us, what was the key to this work?
5. S: Simply, I tried … first I started from this central circle (he indicates this place by putting a tube with glue), and after that I did everything around it.
6. T: how, what was the rule?
7. S: simply, to have symmetrically – from this side and from this (he draws by the gestures two perpendicular lines over the work)
8. T: how did you check those symmetries? Could you show one more? You wanted top and down, right and left agreed?

Figure 6: Central circle of Maciek's work

9. S: Yes (he takes a stripe of paper and puts in the places where should be the line of symmetry – it was a system of two perpendicular lines with the crossing point at the same place which Maciek indicated by the tube of glue).
10. T: Did you look at two directions for the symmetry?
11. S:Yes.
12. T: and – in this way it looks the same after rotation of 180 degrees?
13. S: Yes.

The course of his work, registered on the video, confirms that this student started his work from the middle of the paper. In this way the central point (intuitively – the centre of rotation) was distinguished in a natural way. The boy made one symmetrical strip in the middle and after that he systematically extended his jigsaw puzzle – over and upper of the first line. He did not make any movements of the paper – probably he checked visually the effect of symmetry. During the talk with the teacher, the issue of the mutual array of the line of symmetry was underlined.

D. Ideas permeation

A loud discussion between the teacher and the students about different strategies caused that some students started to analyse their own work from the point of view of other solutions. Particularly interesting was the idea about two perpendicular lines of symmetry. Students critically checked the work, sometimes they tested the new approach.

Example 6. Marysia, described in example 3, says that her work is not correct.

1. T: No, Marysia, where did you find a mistake here?
2. S: here, when we look from this side, from the right there are such coming close tails *(she shows one motif in her work)* and then we turn it (*she rotates the whole work in 180 degrees, showing the motif in the correspondent place)* here are diverged tails.
3. T: Yees. Once again, please.
4. S: Because, from this side there is such smile (*she chooses another motif),* and when we rotated this (*again she turns the paper in 180 degrees)* there is also a smile but with dots inside.
5. T: But Marysia, here is mirror symmetry, for sure!
6. S: Yes, but…
7. T: Where is the place for a mirror?
8. S: here, but…
9. T: but?
10. S: Two symmetry lines are needed.

Figure7: 'Smiles' in Marysia's work

Marysia did not only find the mistake in her work, but also was able to diagnose the reason of the wrong solution. She concluded that two symmetry lines are needed, if she wanted to use the idea of mirror symmetry. Without doubt, this reflection was caused by Maciek's description, presented in example 5, because Marysia reported her remarks after Maciek's utterance.

It is worth stressing that during the talk with the teacher Marysia made her work more dynamic – she made various movements, rotations, localized corresponding motifs. This factor was not present in her previous work, leading by the idea of line symmetry.

Other students, fascinated by the idea of two mirror symmetries undertook the decision to make a new work, according to this new idea. It was, for example, Bartek's case (described in example 4).

Example 7. Bartek was very proud of his new mosaic. He showed it and simultaneously commented on its properties.

S: It is possible to look in this way (*he shows the paper in the vertical position),* turned upside down it looks the same. But it is possible to look in such a way (*he shows the paper in the horizontal position*) after rotation it also looks the same. And, in spite of this, it has symmetry lines here and here (*he shows with the hand two perpendicular lines).*

Figure 8: Two different Bartek's works

While analysing his first floor Bartek stated that it didn't have any mirror symmetry but fulfilled the task condition (it looked the same after rotation). Additionally, he stated that such regularity took a place independently of the paper position. Sometimes, during that work he did some paper movements.

When he was talking about his new work Bartek stated that it 'looks the same' in two opposite positions (bottom-top, right-left), but he also stressed the existence of two lines of symmetry. Clearly, he was aware that the second work was done in another way than the first one.

CONCLUSIONS

In our opinion, the task’s proper formulation directed students’ work towards building an intuition of point symmetry. It generated a large range of individual interpretations, independent searching and drawing out conclusions. A variety of approaches referred directly to own imagination, and the possibility to confront the ideas with colleagues caused, that those activities gave a chance for factual construction of individual understanding of the point symmetry.

During this lesson students in an active way treated the properties of the rotation of 180 degrees. They neither knew the name of this transformation, nor did they know that such concept existed in mathematics. In a spontaneous way they tried to generalize the properties of rotation understood locally (concerning rotated figures). Their trials were multidirectional:

- by covering the whole surface by elements which have locally a particular property,

- by the intuitive localization of places on the surface, which are in a mutual correspondence by rotation, and their local development by elements which have an expected property,

- by attempting to connect the properties of one particular element with the properties of the other global relations,

- by using the system of two perpendicular lines symmetry.

The preliminary stages of work were dominated by experiences gathered during the previous lessons. During them the students learned that there were elements which looked the same after rotation of 180 degrees. This piece of knowledge was inspired by action: the specific movement was the one that students did in a conscious way at the beginning of the whole action, they observed only the effect of the rotation.

The possibility to have an individual approach to the given problem, and after that the presentation of various solutions, turned out to be very beneficial as a method of work. The students utilized their previous geometrical experiences in various ways. The mutual ideas penetration influenced by the creation of students’ mathematical knowledge. It forced them to make the connections between other pieces of knowledge, to make the attempts for generalization, argumentation and verification of their own solutions.

One of the main problems which the students struggled with was the confrontation of two attitudes towards geometrical relations: the dynamic one (connected with rotation but understood locally) and the static one (connected with mirror symmetry, understood globally). This problem has a very complex structure.

In a spontaneous, natural way the students solved an important mathematical problem regarding the relation between line symmetry and point symmetry. At the end of the lesson they have drawn the conclusion that there is one ‘reliable’ strategy for obtaining the rotated area. It is enough that the area has two perpendicular lines of symmetry. But, they also created the rotated figures which after turning upside down looked the same and they did not have any line of symmetry! In addition to that, the existence of only one line of symmetry could be the non-expected result. Such an open situation provoked their treatment. Organizing the work around this topic could be a problematic task for the teacher. In mathematics as the science, the compound of two axis symmetries with a perpendicular axis is equal to point symmetry. What does it look like in a static understanding of geometrical relations? How to avoid false convictions, and at the same time not to squander real and valuable discoveries?

Another crucial problem can be connected with the existence of the rigid points in transformation – in this case – the middle of a rotation. Turning the figure ‘upside down’ is not equivalent with distinguishing the middle of such turning. It is possible, that our observations are in some way connected with the problem formulation, but we observed that the intuition of the point of symmetry was deeply hidden. We believe that it is necessary to turn children’s attention to this fact by a new problem formulation (in a continuation of our work), in which the middle point of rotation will be exhibited.

Observations presented in this paper cannot be taken as the base for generalizations. They turn attention on the possibility for creative activity of students while solving geometrical problems. They also indicate many problems waiting for elaboration. One of them is the problem of how multi-aspect ways for building the understanding of point symmetry can be included in the school teaching-learning process.

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REFERENCES

Aebli, H.: 1982, *Dydaktyka psychologiczna, Zastosowanie psychologii Piageta do dydaktyki*, PWN, wyd. drugie poprawione, Warszawa.

Clements, D.H., Battista, M.T.: 1992, Geometry and Spatial Reasoning, in: *Handbook of research on mathematics teaching*, ed: Grouws D.A., N.C.T.M.-Macmillan, pp. 420-464.

Clements, D.H., Swaminathan, S., Hannibal, M.A.Z., Sarama, J.: 1999, Young Children’s Concepts of Shape, *Journal for Research in Mathematics Education*, vol. 30, no. 2, pp. 192-212.

Hejný, M.: 1995, Development of Geometrical Concepts, *Proceedings of SEMT 95*, Prague, pp. 13-18.

Hejný, M.: 2001, Analiza dydaktyczna pojęć matematycznych – przykłady geometryczne, *Zeszyty Naukowe Uniwersytetu Rzeszowskiego*, Matematyka 1, Seria Matematyczno – Przyrodnicza. Rzeszów.

Hejný, M,. Jirotkova D.: 2004, Svět aritmetiky a svět geometrie, in: Milan Hejny, Jarmila Novotna, Nada Stehlikova (eds.), *Dvacet pět kapitol z didaktiky matematiky*, Univerzita Kralova w Praze – Pedagogicka fakulta, Praha.

Gray E., Pinto, M., Pitta, D., Tall, D.: 1999, Knowledge Construction and Diverging Thinking in Elementary and Advanced Mathematics, *Educational Studies in Mathematics*, **38**, pp.116.

Hiele van P.: 1986, *Structure and Insight. A Theory of Mathematics Education*, Academic Press Inc. London.

Jagoda, E.: 2004a, Perceiving symmetry as a specific placement of figures in the plane by children aged 10 – 12, www. ICME-10.dk

Jagoda, E.: 2004b, Dynamicznie czy statycznie, czyli położenie figur, *Dydaktyka Matematyki*, **27**, Kraków.

Kvasz, L.: 2000, The Changes of Language in the Development of Mathematics. *Philosophia Mathematica* (Series III), **8** (1), pp. 47 – 83.

Kuřina, F.: 1995, The First Geometrical Experience of a Child, *Proceedings of International Symposium of Elementary Math Teaching*, Prague, Charles University, August 28–September 1, pp. 42–45.

Littler, G., Kratochvilova, J.: 2003, Patterns & Conjecture, *Proceedings of SEMT '03*, Praga, pp.104-108.

Mammana, C., V. Villani. (eds): 1998, *Perspectives on the Teaching of Geometry for the 21st Century*, an ICMI Study, vol. 5, Kluwer Academic Publishers.

Marchini, C., Vighi, P.: 2004, The richness of patterns. Application to mathematics education in Italy, *Proceedings of 1st Elementary Mathematics Education Meeting*, 3-6 June, Braga, Portugal, Universidade do Minho.

Piaget, J.: 1972, *The Principles of Genetic Epistemology*, London: Routledge&Kegan Paul.

Rożek, B.: 2009, Shapes in games with the youngest - didactic episodes, in: Jan Guncaga, Ewa Swoboda (eds.), *Child and Mathematics*, Wydawnictwo Uczelniane Uniwersytetu Rzeszowskiego, pp. 77- 85.

Swoboda, E.: 2006, *Przestrzeń, regularności geometryczne i kształty w uczeniu się i nauczaniu dzieci*, Wydawnictwo Uczelniane Uniwersytetu Rzeszowskiego.

Tall, D.: 2001, What mathematics is needed by teachers of young children?, *Proceedings of SEMT' 01*, Charles University, Prague.

Vopěnka, P.: 1989, *Rozpravy s Geometrii*, Panorama, Praha.

PROPORTIONAL REASONING AND SIMILARITY[7]

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The main hypothesis of this work is the following: the approach to proportional reasoning would be better starting from geometrical concepts, in particular from similarity of figures. So, we prepared some activities with the aim to promote the comparison between numbers and the individuation of the fourth number after three numbers, working on similar figures. We analyze the results.

THEORETICAL FRAMEWORK

1. Understanding of proportional reasoning

Several mathematics educational researchers studied children's conceptions about ratio and proportion (Hart, 1984; Tourniaire, F.: 1986, Levain, 1992, Pesci, 2002), very important concepts in mathematics curricula.

According to Vergnaud (1988), "the notion of proportion is at the limit of the skill of the best pupils at the end of primary school" but it don't lead to conclude that "the teacher must avoid to introduce situations and observations about these notions". He must make it with prudence, without go too fast, and leaning on the most evident notions as that of operator. In particular, Vergnaud uses the locution "multiplicative conceptual field" to refer to "all situations that can be analyzed as simple or multiple proportion problems". He studied in deep these notions and he found in the concept of operator a possible useful instrument to work with young children.

Other important research studies dynamics of one fifth-grade student's construction of ratio and proportion schemes. The authors conclude that

> Our analysis demonstrated that ratio and proportion tasks were accessible to younger students … , and these tasks have the potential to encourage students to examine their knowledge of multiplication and division and to recognize the need for having non-integer numbers (Lo & Watanabe, 1997).

Moreover didactic research about problems concerning proportionality shows that on the one hand the success is precocious (8-10 years old children), but on the other hand the failure continues until 15 years old students. In particular, the 50% of pupils 8-9 years old are able to solve simple problems related the finding the fourth number after three numbers in proportion, when the context is

[7] Work done in the sphere of Local Research Unit into Mathematics Education, Parma University, Italy.

familiar and the numbers are small and integer (Tourniaire, 1986). This percentage increases notably in pupils 10-11 years old (Levain, 1992).

Some researches suggest to connect creation of the ratio concept with concept of similar figures. In geometry, proportional reasoning is necessary when we treat similarity between figures. M.van den Heuvel-Panhuizen states that

> ...special feature of the subject of ratio is that it is quite accessible in spite of this difficulty. The easy part about ratio is that it has strong informal roots based on visual perception. Long before its numerical approach and its formal notation children are already able to see ratio (M.van den Heuvel-Panhuizen, 1991, p.163).

The example of such attempt is a work done by Swoboda. She tried to find "How do children understand the concept of geometric similarity enlarging (or reduce in size) drawing in a given scale" (Swoboda, 1993). She observed that more than 90% of children were able to recognize the proper (or almost proper) relations between two similar figures. From the other side, studying behaviours of pupils 7-10 years old, Swoboda observes a big difference of performance between 'correct recognition of similar figures' and 'drawing, even if partial, of similar figures'. In the first case, she registers 64% of correct choice, in the second only 33% of correct drawing.

2. Problems in using semiotic description of concept "similarity"

In Italian schools when we work about similarity often improperly we say: "Similar figures are figure that have the same shape, but not same measures". This 'figural definition' falls when we try to apply it on rectangles, since using it we wrongly would conclude that all rectangles are similar! Sometimes teachers chose this locution with the aim to avoid proportions, but it could be dangerous. Other times they speak about 'enlarged or reduced figures', but these words can suggest bad ideas related to additive conceptual field. In particular, a young girl in primary school tells me: "To reduce or to enlarge means to increase or to decrease the lengths of the sides in figures". In effect, to solve proportionality problems some pupils incorrectly use additional procedures. The slow and difficult overcoming of additive structures is also documented in a publication of Hart (1981).

Semiotic representations have an important role in mathematics. According to 'theory of semiotic representations' (Duval, 2008), there are two kind of transformations of representations: 'treatment', passage from a representation to another inside a register, or 'conversion', passage from a representation to another in two different registers. The last is fundamental to realize a conceptual learning of mathematical concepts.

THE EXPERIMENT

1. Theoretical background for designing the research tool

Duval (2006) distinguishes four ways of visualising a figure in depending on the type of activity proposed by the teacher: as botanist, surveyor, builder or inventor. A botanist observes the shapes, distinguishes them by observing their boundaries and in this way he recognizes their names. A surveyor measures lengths, he reproduces shapes, and he uses geometrical properties to measure. Botanists and surveyors have 'iconic visualisation': they perceive the resemblance between a drawing and the shape of an object. A builder constructs figures by using instruments: he decomposes a shape to construct or to draw it. Lastly an inventor transforms the figures tracing 'reorganizing lines' and decomposing them in figural unities. Builders and inventors have 'non-iconic visualisation': it is based on the deconstruction of shapes. Duval affirms that in geometry learning 'iconic visualisation' obstructs a fundamental activity, the decomposition of a shape in figural units with an inferior number of dimensions.

Those theory was used by me while designed the research tool (worksheets), used for an experiment. According to Duval, the aim for series of four worksheets was to promote the passage from botanist behaviour to surveyor and especially to builder or to inventor behaviour. It needs the passage from 'iconic visualisation' to 'non iconic visualisation'. A basic motif used for each worksheet was "a flag". The use of resemblance among flags was not sufficient. Pupils needed decompose figures into one-dimensional parts and compare them. In this way, gradually pupils would construct their ideas about proportionality and similarity.

2. The research tool

During school year 2009 - 2010, I prepared some worksheets in collaboration with a team of teacher-researchers. Worksheets are mainly based on drawings of flags, already drawn or to complete[8]. All worksheets are attached in Appendix.

The aim of the worksheet 1 was to induce the pupils into observing the pictures and to face them with the problem of enlargement or reduction of a figure. Flags are drawn on different squared paper and they represent 'figures drawn in different scales' or 'similar figures'. Precisely, we use squared paper with sides that measure 0.5 cm (teacher-maestra and Piero), 1 cm (Luca) and 0.4 cm (Marco). We must make a clarification: following the commonsense, the teacher's flag would be bigger than the other, since it is drawn on a bigger flat, the blackboard. In fact, it doesn't constitute a problem for pupils. We asked if there are 'good copy' of teacher's flag, with the aim to analyse pupil's criteria of

[8] Many thanks to I. Aschieri for the realization of drawings and the editing.

equality or of similarity. Subsequently we ask to complete an enlarged figure (sailboat), with the aim to observe if completing the drawing children respect similarity or not.

In a second worksheet we changed the figures in the first part, maintaining questions. Now flags are drawn on one type of squared paper only. The aim of this worksheet was to turn attention into comparison of figures dimensions and recognition of similar shapes.

With worksheet 3, we wanted investigate if children understand and use the concept of geometric similarity or not, through the observation of figures drawn on a same sheet of squared paper. Only one of them, made from Anna, is obtained by enlarging of teacher drawing in a given scale (2:1), others are only partially enlarged or reduced, with mistakes or omissions.

The last worksheet 4 proposes to complete drawings. The second task was very difficult: it needed to evaluate the length of the pole and also how draw the missing part of the flag, respecting proportion. In this case the scale is 3:2, which is not integer ratio. In fact, is it possible to think only about simple ratios, regarding separately flat and pole: in the first the ratio between width and height is 2:1, while in the second the ratio between length of pole and height of flat is 3:1.

ANALYSIS OF THE PROTOCOLS AND RESULTS

The experimentation took place in an Italian Primary School[9], in a class of 21 students 9-10 years old and in two classes of 43 pupils 10-11 years old. The activity based on compilation of worksheets carried out in the ordinary time of lesson, in two different days. Gradually I presented the worksheets, pupils worked individually with the presence of their teacher and mine. In the first part of activity I was only observer, in the last I conducted the discussion. Subsequently I interviewed pupils one-by-one, posing questions about their individual work on worksheets.

Concerning worksheet 1, initially pupils notice that "children used different kind of squares", that "everybody drew the letter of the alphabet P like the teacher" and that flags are equal or smaller or bigger than teacher's flag. Problem of equal-different appears: "Pupil's drawings are equal to teacher's drawing, but with smaller or bigger squares" or "These figures can appear different, but if you carefully count we can notice that all poles have the same number of squares, flags also"; so, the number of squares used "is the same", but "measures are different", "dimensions also". A child write: "Children drew the same figure but with different squared paper". This aspect leads in the field of geometrical transformations and to face to problem of relativity of equality: the locution

[9] I wish to thank teachers E. Forti and M. T. Sabatino for their collaboration and helpfulness.

"same figure" means "similar figures". We notice also mistakes: "All figures are equal and congruent, but they are on different sheet of paper" ('equal' and 'congruent' used as synonyms, with meaning of 'same shape') or "All children used correct measures" (confusion between 'number of squares' and 'measure'). The conflict between same shape and different measures emerges: "Pupils drew the same figure, but using squares with different areas". The idea of 'good copy' is associated to 'congruence': "Piero made a good copy, since he has the same sheet of paper of the teacher". We obtained the 84% of answers that assign a 'good copy' to Piero, but the general comment is that all pupils were very good designers.

The completion of 'doublet flag' provided the following percentages:

%	correct	incorrect	other
9-10 y.o.	**38**	**38**	**24**
10-11 y.o.	**67**	**23**	**10**

Table 1: Percentage related to 2° question of worksheet 1

The results of worksheet 2 show the recognition of similarity between Luca flag and teacher flag. Some child speaks about "to make double": sometimes this is an activity suggested from the teacher in classroom. Only 13% find incorrect Marco drawing; in particular, a girl write that "Only Marco don't respects proportions", using also an appropriate language. Subsequently, at the end of the activity in classroom, I suggested some activities like this: "The teacher draws a pole long 18 'squares'. You must indicate the measures of related flag". Majority of pupils responded without incertitude, the play of double, triple or half was known. So, we decided to report the numbers in a list and to work with numbers and almost inevitably with operators:

Pole length	Flag width	Flag height
6	3	2
12	6	4
18	…	…
3	1,5	1
…	…	10

Table 2: Proportional measures in flag's drawings.

Children say that "Since 18 is triple of 6, the other remaining numbers will be 9 and 6" or "If flag height width is 10, pole length 30 and flag width 15". In this way this work was based on intuitive understanding of proportionality (simple or multiple proportionality), and using operators. In particular, pupils observed that, except in fourth row, in the columns there are multiples of 6, or 3, or 2.

Subsequently, we drew all flags on transparencies and using overhead projector we show this particularity, based on homothetic transformation (Mason, 2003). Pupils named this strategy “diagonal law”. The conclusion was: “if flags are ‘good-drawn’ diagonals are superimposed, otherwise not”. Using superimposition of drawings, it become evident that Marco flag don’t respect diagonal law.

Concerning worksheet 3, using common sense, 62% of pupils 9-10 years old gave this answer: “Everyone drew well, except Maria”; in 10-11 years this percentage begin 21%, showing a considerable decrease that would be interpreted as a passage from real life concepts to mathematical meanings of enlargement or reduction. One half of older pupils (50%) and only 30% of younger chooses Anna’s drawing as good enlargement. Some child chooses also Carlo flag since they don’t observe carefully the figure, in particular the incorrect length of the pole. We noticed some expected misconceptions: reduce as ‘subtract squares’, enlarge as ‘add squares’.

Concerning worksheet 4, quickly pupils understood the mistake in the length of Carlo’s pole (78%). The last task resulted very difficult; we registered very few correct answers in individual work, while the discussion in classroom gave satisfying results: the recourse to ‘diagonal strategy’, suggested from a pupil, allowed to solve the problem. A difficulty was to consider isolated parts of figures, but when teacher suggested to put numbers in an opportune table, using operators, children found without hesitation the length of pole (9 squared), while to right position of the ‘point’ of the flag created uncertainty. The passage from geometric to arithmetic representation and vice versa appeared very useful and important: conversion in Duval’s sense has been realized.

In particular, a protocol revealed a sliding in additive field: noticing that teacher’s flag is large as the part of pole without flag (four squares), a girl thought the make it also in Andrew flag; so, in this case she was lucky and she obtained a correct drawing, but using an incorrect reasoning.

CONCLUSIONS

The present work, based on few worksheets and their analysis, is only a first step and it don’t allows to make general conclusions about the passage from common meaning to geometrical meaning of ‘enlargement’ and ‘reduction’, but we can observe a fast improving in this theme for pupils of primary school. As other concepts, also similarity would be presented early, to prepare a ground for to following steps.

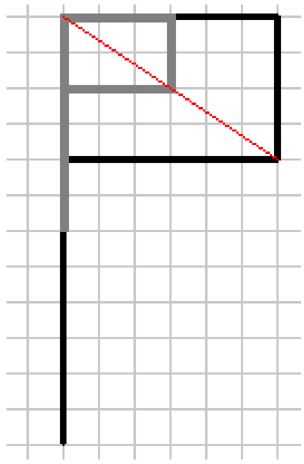


Figure 1: Diagonal lines highlight proportions in flags

After working about flags worksheets, we presented a sequence of problems as the following: "To make a phone call of 24 minutes, it needs 16 euros. How many euros to make a phone call of 15 minutes?". We noticed two different kind of strategies: the first was based on division (to have the unit price) and afterward on multiplication (to have the cost), the second based on a table as Table 2. A boy wrote: "I noticed some regularity in Table 2, so I made a similar table for this problem and I found the solution, 10". In effect, he created a table with multiples of 3 and 2 in two columns, he observed the relative position of numbers 24 and 16 (written in the same row), he looked for 15 in the table and in the same row he saw 10. So, to solve the problem he used table as in the previous activity, that could be a starting point to work with problems of proportionality.

REFERENCES

Duval, R.: 2005, Les conditions cognitives de l'apprentissage de la géométrie: développement de la visualisation, différenciation des raisonnements et coordination de leur fonctionnements, *Annales de Didactique des Mathématiques et de Sciences Cognitives*, **10**, pp. 5-55.

Duval, R.: 2008, Eight problems for a semiotic approach in Mathematics Education, in: *Semiotics in Mathematics Education*; *Epistemology, History, Classroom and Culture*, Radford, G. Schubring, F. Seeger Eds., Sense Publishers, pp. 39-61.

Duval, R.: 2006, Les conditions cognitives de l'apprentissage de la géométrie: développement de la visualisation, différenciation des raisonnements et coordination de leur fonctionnement, *Annales de Didactique et de Sciences Cognitives*, **10**, pp. 5-53.

Hart, K.M.: 1981, *Children understanding of Mathematics*, Murray, London.

van den Heuvel-Panhuizen, M.: 1991, Ratio in special education, in: *Realistic Mathematics Education in Primary School*, L. Streefland Ed., Freudenthal Institute, Utrech.

Levain, J.P.: 1992, La résolution de problèmes multiplicatifs à la fin du cycle primaire, *Educational Studies in Mathematics*, **23**, pp. 139-161.

Lo, J.-J. & Watanabe, T. : 1997, Developing ratio and proportion schemes: a story of fifth grader, Journal for Research in Mathematics Education, **28** (2), pp. 216-236.

Mason, J.: 2003, Structure of attention in the learning of mathematics, *Proceedings of International Symposium Elementary Math Teaching*, Novotná J. Ed., Prague, pp. 9- 17.

Pesci, A.: 2002, *Lo sviluppo del pensiero proporzionale nella discussione di classe,* Pitagora, Bologna.

Swoboda, E.: 1993, On the understanding of figure similarity by children 7-11 years old, *Proceedings of International Symposium Elementary Math Teaching*, Prague, pp.101- 109.

Swoboda, E.: 1995, The development of understanding of similarity of figures, *Proceedings of International Symposium Elementary Math Teaching*, Prague, pp.57-59.

Tourniaire, F.: 1986, Proportion in elementary school, *Educational Studies in Mathematics,* **17**, pp. 401-411.

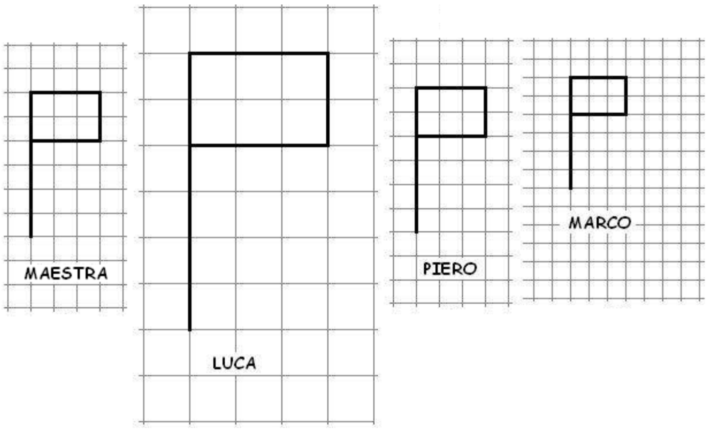
Vergnaud, G.: 1988, Multiplicative structures, in: *Numbers concepts and operations in the middle grades*, Hiebert J. and Behr M. (Eds.), Reston, VA: NCTM Hillsdale N.J, pp. 141-161.

Vergnaud, G.: 1991, La théorie des champs conceptuels, *Recherches en Didactiques des Mathématiques*, **10** (2), pp. 135-169.

APPENDIX

WORKSHEET 1

1) A teacher made a drawing on the blackboard and she asked to the pupils to copy it, respecting squares. Below there are the drawings made by the teacher and by three pupils: Luca, Piero and Marco.
   Observe the drawings and then write below your comment.

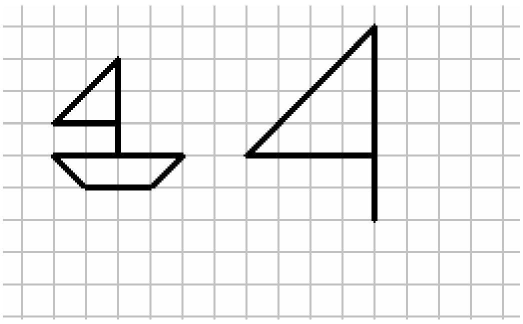


....

Who copied well? .... Why? ....

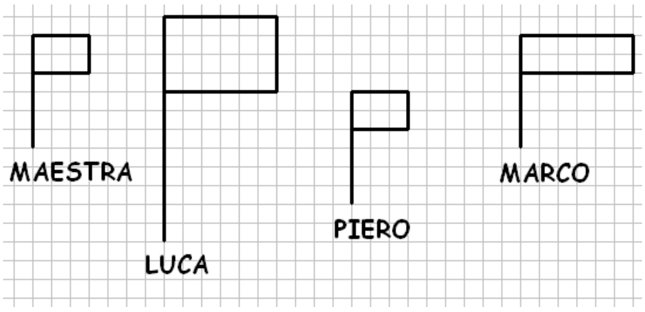
Write your comments about drawings of the pupils. ....

2) In the following drawing, complete the figure enlarged:



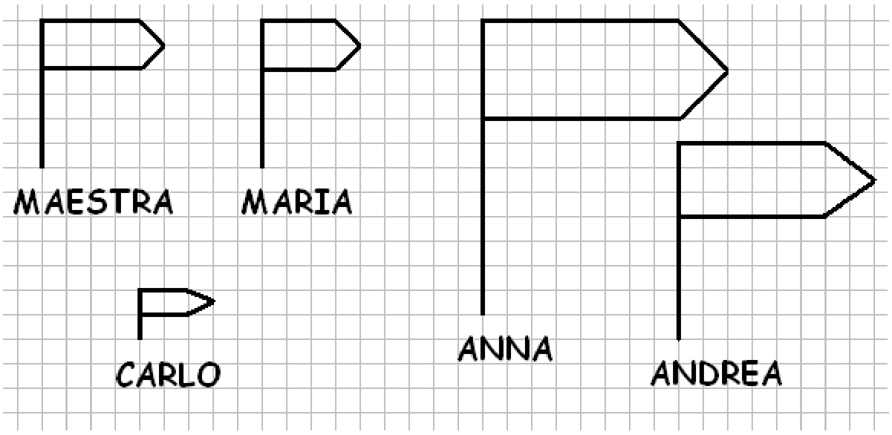
WORKSHEET 2

A teacher made a drawing on the blackboard and she asked to the pupils to copy it, but enlarged or reduced. Below there are the drawings made from the teacher and from three pupils of the classroom, Luca, Piero and Marco.



WORKSHEET 3

Later the teacher drew another flag on the blackboard and she asked to draw the same, but enlarging or reducing it.



Who better realized the task? ....

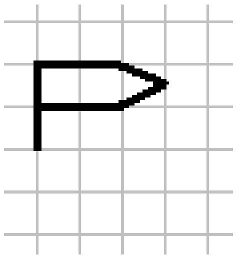
Why? ....

WORKSHEET 4

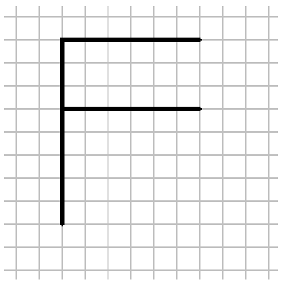
Drawing made from Anna is it exact? **....................**

Why? **...**

Carlo drawing not is exact. Could you modify it, obtaining a ‘good reduction’ of teacher’s drawing?



In the previous worksheet Andrea drawing is not finished and not exact. Could you modify it, obtaining a ‘good enlargement’ of teacher’s drawing?



THE PERIMETER AND THE AREA OF GEOMETRICAL FIGURES – HOW DO SCHOOL STUDENTS UNDERSTAND THESE CONCEPTS?

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The paper presents some results of a research focused on the following questions: (1) How do primary and secondary school students understand the perimeter and the area of geometrical figures? b) Do the students know that these two concepts are independent from each other? The research reveals an unexpected result: the higher level of education students are the less they are conscious of the aforementioned independency. The question is WHY?

INTRODUCTION

Measure and measurement constitute a very important element in the content matter of the school teaching and learning of mathematics. Regarding their practical applications and utility in everyday life they need to be well mastered by the student in the course of school education. But despite their close connection with daily reality and students' wide experience and relevant intuition in this area, those notions prove to be difficult for them. The topic of measure is in the school education distributed over all levels. The process of development of the concept of measure and measurement needs time and it should be carried in several stages, each leading to a better understanding of the idea of measure. Such stages (called phases) in connection with objectives were proposed by Konior (2002).

It should be remembered that though the properties of invariance with respect to isometries, and additivity, of area are not introduced as formal theorems, they are used from the very beginning of learning about measure and measurement. Gucewicz-Sawicka (1982) stressed that from the beginning, starting with the simplest exercises carried out by the students, the properties of measure should be highlighted: they are non-negative real numbers, for figures with disjoint interior the area of their sum equals the sum of their areas, and there exists a unit figure with area equal 1.

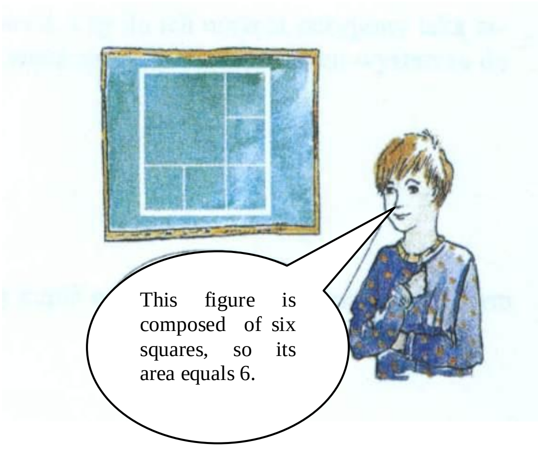
RESEARCH AND OUTCOMES

The main goal of the reported research was diagnosing the state of students of the two educational levels knowledge about the area and perimeter of a plane figure, specifically their mutual independence.

The research was carried out in January 2010. Participants were 20 students of the 6th grade of elementary school and 33 students of the 3rd grade of the junior secondary school. The students received sets of problems adapted to their level of education. Conclusions were based on the written answers to the problems.

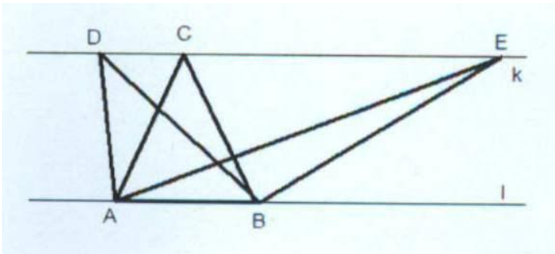
Elementary School Problems

Problem 1



Regard Waldek's drawing. Did Waldek correctly determine the area of the figure? Justify your answer.

Problem 2



Which of the triangles in this figure has the greatest area (lines k and l are parallel)?

Problem 3

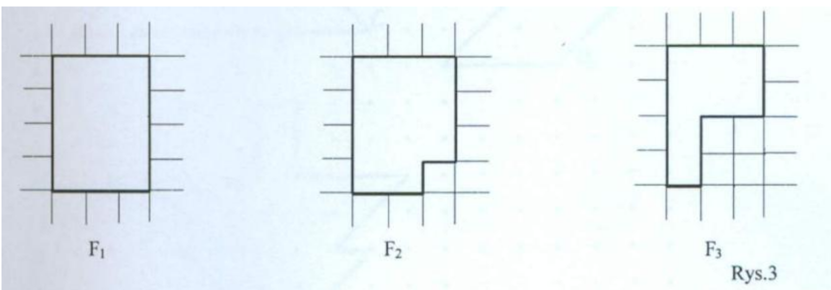
Say YES or NO to the following questions. Each time justify your answer.

a) Two window glasses have the same perimeter. To wash them, shall we use out the same amount of washing liquid? (We know that 1 ml of the liquid is need for 0,5 m^2 of glass.)

b) Can we know how much wire net should be purchased for a garden fence if the perimeter of the garden is known?

Problem 4

Estimate (without calculation) which of the figures below has the greatest area, and which has the smallest one. What can you say about perimeters of those figures?



RESULTS

Problem 1 – results

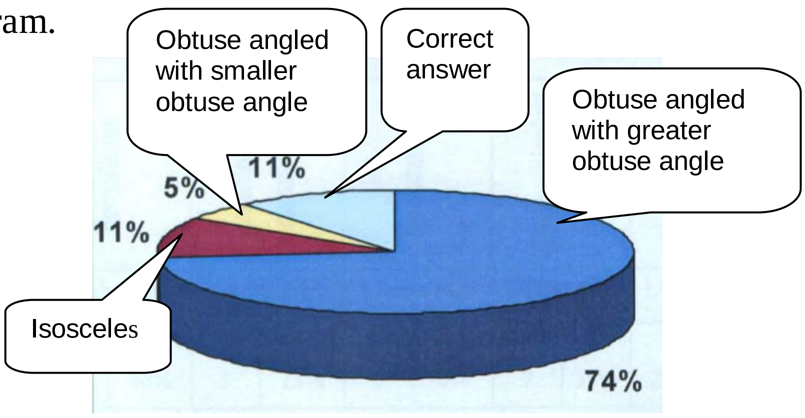
This problem was to diagnose the understanding by a 6th grade student of elementary school the concept of area of a plane figure and the process of measuring it. The expected correct answer was that Waldek is wrong, based on the presumption that all squares the figure is built of have the same area. The intention was to examine the intuitive understanding of area and its connection with the process of measuring, and, as a consequence, with its outcome i.e. the number called area.

A majority of the students answered correctly (saying that Waldek was wrong) – 17 persons; only two answered incorrectly, and one did not give any answer. A disquieting phenomenon is that among those answering correctly only 6 persons correctly justified their answer. Others did not justify it or wrote, for example: “The figure is composed of 10 squares” (the student probably counted also the square blackboard in the picture). Other justifying expressions: “There are no given lengths of the sides of the large square or the inside squares”, “Area is calculated side times side”, “The number of squares does not influence the figures’ area”. The last answer may suggest that for this student the area of a figure is absolutely not associated with the process of measuring. Another student said that Waldek correctly determined the area because this information is sufficient for defining the figure.

It seems that most students correctly understand the area of a plane figure as the number of square units that it can be filled with. But some of them connect the area of a figure only and exclusively with a known formula for the area of the given figure. Then, lacking the data needed for calculating the area using the formula they think that any other way of determining it is wrong.

Problem 2 – results

There were 20 answers in the investigated group, with one not taken into account in the analysis because it was a humorous reaction of the student only. It proved that the situation of the problem was interpreted properly by two persons in the group only. The quantitative distribution of answers is presented in the following diagram.



The task did not require of the student an explanation of the answer but it seems reasonable to say that in the students' presumption the greater the angle of a triangle the greater is its area. It is probable that this was the thinking of 14 students of the investigated group.

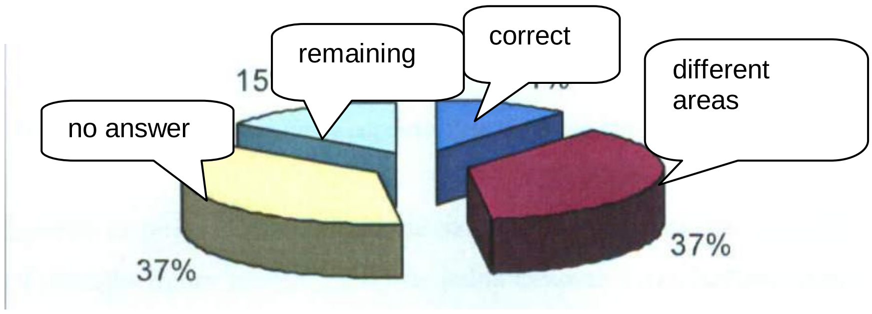
Problem 3 – results

This problem was solved by all students. 9 persons gave correct responses, while 11 wrong. Interesting are their justifications, for example: "As they have the same perimeter they are the same" (3 students), "They have the same perimeter then they would rather have the same area" (1 student), "Area and perimeter of one and the same figure have to be the same" (2 students). In those statements linking the two investigated measures is visible. There were also students who attempted to use the information in brackets: they multiplied the numbers included there. It seems that they simply took that information as an obligating one trying to exploit it some way, but did not know how to. Other statements give evidence of mature thinking, e.g. "The perimeter does not influence the area" (1 student), or "Glasses may have different shapes/areas" (2 persons).

Item b) shows that students do distinguish the meaning of area and perimeter in some real life situations. The correct answer was given by 16 students, and almost all correctly justified their answer. One student wrote: "One has to calculate the area; the perimeter is not sufficient". Such assertion evidenced the insufficient comprehension of the question.

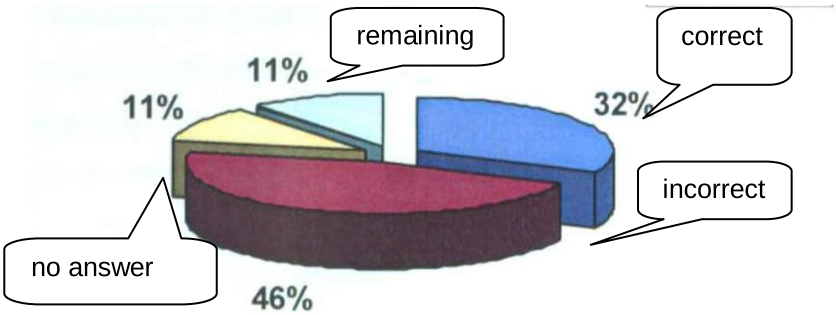
Problem 4 – results

Two persons only answered correctly and one did not answer this problem at all. The diagram below presents the distribution of answers to the question, which of the figures has the greatest, and which the smallest, area.



The correct answer was that figure F1 has the greatest and figure F3 the smallest area. So answered two persons only, and, interestingly, those were persons who also answered correctly the question on perimeter. Seven persons only answered that the areas of those figures would be different, which was of course true but not precise; it caused me to count them in a separate category "different areas". Those students may have read the problem carelessly, which could be the reason of the number of correct answers to this question being so small.

The following diagram presents the distribution of answers to the question on perimeters.



The correct answer was that all the three figures have the same perimeter. So answered 6 students. All incorrect answers were same: F1 has the greatest perimeter, and F3 the smallest one. It may have resulted from confusing the perimeter and the area occupied by the figure, then the area. So said 9 students. Two students answered the question on area in such a manner that it is impossible to decide if the answer refers to perimeter or area. They probably confused both notions.

This problem has shown that elementary school students probably do not understand the independence of area and perimeter of a plane figure. The most frequent error was confusing area and perimeter of the given figures. To the question on perimeter they usually answered so as if they were thinking about areas, pointing the figure with smallest area as the one with smallest perimeter. Another cause of this error might be resorting to a false assumption: if the area of a figure decreases then the perimeter also must decrease. Probably then, for the majority of elementary school students the concepts of area and perimeter of a plane figure depend on each other.

Junior secondary school problems

Problem 1

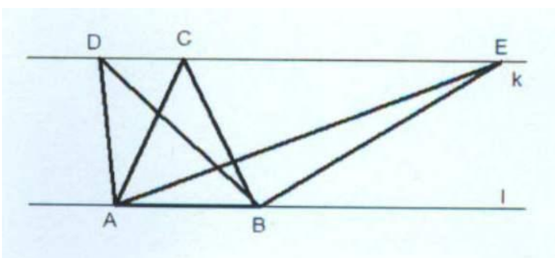
Janek said that the area of the figure he had drawn is 7. His sister, when looked at the drawing, decided that the area is 28. Is it possible? Answer and justify.

Problem 2

Calculate the area of an isosceles triangle with two sides of length 20 cm, the third one of 5 cm, and the height of 5 cm.

Problem 3

Kasia said: “If lines k and l are parallel then the three triangles have equal areas” (see figure below). Was she right? Justify your answer.



Problem 4

Which of the following statements is true? Evaluate putting TRUE or FALSE. For each of the questions draw a picture that would justify your answer.

a) Figures of equal areas have equal perimeters. ....
b) Figures of equal perimeters have equal areas. ....
c) Figures of equal areas may assume different shapes. ....

Those problems were solved by 33 junior secondary students.

Problem 1 – results

Five persons failed to answer the question included in this problem, which may indicate a lack of adequate knowledge or its uncertainty. Analysis of the solutions showed that 14 persons gave a good answer, but none gave a satisfactory justification. The students wrote, e.g., “Janek and his sister regarded the figure at different distances” (3 students), “An area may have different lengths” (1 student), “There are similar figures of ratio 1:4” (2 students), or – most frequent – “Someone was wrong” (7 persons), supported by a possible explanation: “Janek may have calculated the area of a triangular pyramid. He found the area of one face but forgot to multiply is by 4. Only his sister corrected his error”, “Because Janek may have looked at the length of the side only, which was 7, and his sister multiplied one side times Janek’s side, so it came out the whole area”, “Janek calculated it wrong. The sides had to be 2 x greater in order that sister’s result were in accord”, or even “Sister only looked at the picture, she did not know the measurements.”

The quoted statements imply that the situation the students met here was very untypical for them and a correct answer did not mean that the student was thinking as the teacher would expect.

The incorrect answer was here the answer NOT. So answered 14 persons. Two of them did not propose any justification, others wrote e.g. “A figure cannot have but one area”, “It has to be calculated, without calculation it cannot be decided” or “After extending the sides the area will change”. No student put down a fully correct answer, i.e. Janek and his sister gave the area of the same figure, but in different units. Most frequently the students considered the statement in the problem as being wrong saying that someone must have made a mistake. There were also worrying answers that “an area cannot have different lengths”, probably meaning different values. There were persons, though, who asserted that a geometric figure cannot have but one area. They did not draw therefore the conclusion that if so then the area of this figure is the same, but given in different units.

Problem 2 – results

The students were put in an untypical situation: the area of a “non-existing” isosceles triangle was to be found. The impossibility of the existence of such a triangle the students were to establish based on the information of the measurements of its sides and height given in the content of the problem. The purpose of the problem was to investigate if students thinking is purely algorithmic or they verify some way the data and calculation results. A closer analysis showed that only 2 persons gave answers implying that the area of such a triangle cannot be calculated. In fact, 23 persons calculated the area applying the formula, while 15 among them even did not draw a picture for the problem.

As comes out of the analysis of solutions of this problem, a majority of the students of the studied junior secondary group applied here an algorithm, right away using the formula for a triangle. Even a picture, which, when thought over, would instigate doubts on the existence of the triangle, did not suggest to the students any other course of reasoning.

Problem 3 – results

Five persons did not answer this problem. One attempted marking angles and their measures on the picture, but did it incorrectly; another one said “don’t know”. 14 students answered rightly but 5 of them did not justify the answer. A fully correct argument appeared in the papers of 4 students.

Students saying that Kasia does not say the truth justified their decision as follows: “They have different sizes/lengths of sides (5 persons); “They are of different size”, “They are different” (4 persons); “It does not depend on the parallel lines k and l” (2 persons); “We do not know because we do not have the dimensions of those triangles” (1 student).

Only 4 students correctly answered the question in this problem and correctly justified their judgment. So we did not receive any better results as compared with answers by the elementary school students. Indeed, many persons presented wrong intuitions that suggested to them that the triangles “are of different size”.

Problem 4 – results

Not all answers to this question were complete, but no student returned the paper with a blank below this problem. Most students evaluated the given sentences and justified the answer by a corresponding picture, according to the assignment. Part of them only proposed the value without any justification. Others omitted one of the items.

Item a). The correct answer was given by up to 31 persons, among them 6 did not give any justification. Correct justifications included the picture of a square and a rectangle of equal areas but different perimeters. There were 2 incorrect answers to this point, but students who so answered did not give any justification, which suggests that the answers were haphazard.

Item b). Correct answers were given by 27 persons, 10 among them provided a fully satisfactory examples of figures that evidenced the answer, e.g. pictures of a square and a rectangle of equal perimeters and different areas (9 students) or other figures. There were 5 students that answered incorrectly. Only one attempted to illustrate his answer but failed.

Item c.) The correct answer was returned by 28 persons, among them 13 quite correctly substantiated it with a picture and calculations. In 6 papers justification was lacking. Three students returned incorrect answer to this point.

Summing up results obtained by problem 4 as a whole we can claim that a majority of persons in the investigated group answered correctly at least two included questions. Hardship occurred when it came to substantiation of own judgements.

The lack of a justification (or any answer) in 27 cases out of total 99 required solutions may evidence the absence of any students' imagination concerning this topic; their intuition may link unbreakably the two investigated concepts or they had never met this kind of dilemma.

GLOBAL SUMMING-UP

The first result provides an image of haw students understand the concepts of area and perimeter of a plane figure. I will refer to the distribution of answers to problems 1, 2 for elementary school, and 1, 2, 3 for junior secondary school. All answers were qualified as GOOD or POOR. The Category "good" means adequate understanding by the student of the concepts of perimeter and area of a plain figure (of course, as far as it was observable in the study). Konior says:

> It appears that a student who grasps with apprehension the concept of area should in the act of getting to understand somehow connect three elements: a plane figure, the process of measurement and the result of this process – the number assigned to the figure. A mechanic and detached manipulating with formulas exposes only the third element that functions without a wider context and, in a sense, isolated from the two others (In Rabijewska, 1999, p. 80).

So the understanding of area (similarly length and volume) is "good" if the student is aware that the number he/she gets after applying a known formula for the area of the given figure is the same number as one resulting from measuring out the figure with square units. Above that, the student must know that a number so assigned to the figure is unique. "Good" understanding of a concept becomes visible also through flexible use of formulas: only where they are really needed; sometimes it is more practical to use a long being formed intuition of the concept than mechanically apply a formula. An evidence of the sufficient understanding of the concept of area is also mastery in using units and understanding that the area of a figure may assume different values if different units were used.

Category “poor” is opposite to the former one. A characteristic feature of students’ answers assigned to this category was the mechanic use of formulas for areas of figures. These students operate with formulas detached from the process of measuring the given figure. Eventually, even if a formula should be used in order, for example, to find the value of some involved parameter, the student does not know to do it because he/she was only taught to mechanically substitute data in a formula and calculate the result. Above that, the student does not understand the meaning of a unit, does not accept the possibility of a given figure possessing two values of area expressed in different units.

The distribution of categories in relation to the two investigated educational levels is the following:

Elementary school	Junior secondary school
Problem 1: GOOD – 42%, POOR – 58%	Problem 1: GOOD – 0%, POOR – 100%
Problem 2: GOOD – 11%, POOR – 89%	Problem 2: GOOD – 28%, POOR – 72%

Indeed, even accounting for the small size of the investigated group and small number of diagnostic questions, one cannot but say that these results are worrying. When moving up the consecutive educational levels students less and less understand intuitively the concepts of area and perimeter of a figure, and the older the students the more frequently they apply algorithmic thinking, evidenced by a mechanic use of formulas for the area. One is then encouraged to claim that burdening students with a large number of formulas for areas, volumes, and perimeters of figures brings unwanted didactic effects. The limited range investigation has shown that it is the youngest students that have got best intuitions of the discussed concepts.

In the second global summing up of the results of the investigation of understanding the independence of area and perimeter, students’ answers were qualified in two categories: YES and NO.

The category “yes” includes answers showing evidently that the student well understands the independence of area and perimeter of a plane figure. First of all, the student does not confuse those concepts. He does not call “perimeter” the area occupied by a figure, nor “area” its border line. A good understanding of the independence of those concepts also includes knowledge of the fact that if perimeters (areas) of given figures are equal then their areas (perimeters) do not need to be equal. Similarly, answers assigned to this category are those, which clearly show the student’s understanding that growth of the perimeter of a figure does not necessarily cause the growth of its area. A student who well understands the independence of area and perimeter can provide adequate and accurate examples and counterexamples that evidence his/her statements.

The second category includes those students’ answers, which show that the student probably confuses the concepts of area and perimeter and is led by false intuitions concerning dependence of those concepts. For a student whose answers were accounted in this category it is evident that if perimeters (areas) of the given figures are equal so are their areas (perimeters), too. He would give “examples” supporting his/her judgment. Of course, this is true, but only for squares or circles; it cannot be applied for the set of all figures. Yet students are confident that when they show an example of figure, for which this property is fulfilled, this property will be fulfilled for the family of all figures. Additionally, a student who misunderstands the independence of area and perimeter of figures, being rather certain they are dependent of each other, would thoughtlessly confirm the statement that with the growth of the perimeter the area also grows.

The distribution of the two categories is the following:

Elementary school	Junior secondary school
Problem 3: YES – 25%, NO – 75%,	Problem 5: YES – 21%, NO – 79%
Problem 4: YES – 25%, NO – 75%.	

CONCLUSION

The results above show a big dispatch between the sizes of groups of answers belonging to the identified categories: with their advancement to the higher level of education the students more and more rarely notice the independence of area and perimeter of a figure. One would ask if the guilt is theirs. An analysis of some mathematics textbooks, in particular those for the junior secondary level, shows the deficiency of problems directed to an investigation and exploration concerning the discussed relationship. Often problems are missing in which the two concepts would be involved together.

REFERENCES

Biernacki, M.: 1954, O równoważności wielokątów, *Matematyka* **29**/1954, pp. 4-4.

Batko, M.: 2010, Miara i mierzenie w nauczaniu i uczeniu się matematyki (unpublished diploma dissertation), Pedagogical University, Mathematics Institute, Kraków.

Ciosek, M.: 2005, *Proces rozwiązywania zadania na różnych poziomach wiedzy i doświadczenia matematycznego*, Wydawnictwo Naukowe Akademii Pedagogicznej, Kraków.

Gucewicz-Sawicka, I.: 1982, *Podstawowe zagadnienia dydaktyki matematyki*, PWN, Warszawa.

Howe, R.: 1999, Introduction to Liping Ma: Knowing and Teaching Elementary Mathematics – Teachers’ Understanding of Fundamental Mathematics in China and the United States, Lawrence Erlbaum Associates (in Polish in *Dydaktyka Matematyki*, 21)

Kąkol, H., Powązka, Z., Treliński, G.: 1995, *Przewodnik metodyczny 7,8. O nauczaniu i uczeniu się matematyki w klasie siódmej i ósmej szkoły podstawowej*, Wydawnictwo Kleks, Bielsko-Biała.

Konior, J.: 2002, *Materiały do studiowania dydaktyki matematyki*, J. Żabowski (Ed.), T. IV, Płock.

Krygowska, Z.: 1974, Miara Jordana i pole figury w ujęciu elementarnym. Part. I, in: B. Nowecki, *Wybrane zagadnienia nauczania matematyki w szkole średniej, Wykłady telewizyjne część II,* WSziP, Warszawa.

Moszner, Z.: 1961, *O mierzeniu w matematyce*, PSWSZ, Warszawa.

Rabijewska, B.: 1999, *Materiały do zajęć z dydaktyki matematyki cz.2*, Wydawnictwo Uniwersytetu Wrocławskiego, Wrocław.

Trelińska, U., Treliński, G.: 1996, *Kształtowanie pojęć geometrycznych na etapie przeddefinicyjnym*, Wydawnictwo MAT&MET, Kielce.

Turnau, S.: 1990, *Wykłady o nauczaniu matematyki*, PWN, Warszawa.

Turnau, S.: 1994, Pola na kolorowo, *Nauczyciele i matematyka* 10/1994, pp. 37.

Turnau, S., Ciosek, M., Legutko, M., Urbańska, E.: 2004, *Poradnik metodyczny. Matematyka dla Ciebie. Klasa 4,* Wydawnictwo Nowa Era, Warszawa.

Turnau, S., Ciosek, M., Legutko, M., Urbańska, E.: 2006, *Poradnik metodyczny. Matematyka dla Ciebie. Klasa 5*, Wydawnictwo Nowa Era, Warszawa.

Zaremba, D.: 1993, *Sztuka nauczania matematyki w szkole podstawowej*, GWO, Gdańsk.

Zięba, A.: 2003, Tworzenie intuicji pojęcia pola, *Matematyka* 4/2003, pp. 233- 233.

Differentiation in building mathematical knowledge

III. Knowledge related to other areas

DEFINING AND PROVING WITH TEACHERS: FROM PRESCHOOL TO SECONDARY SCHOOL

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This paper describes our work with K teachers' on definitions of triangles, with elementary school teachers on definitions of parity, and with secondary school teachers on proofs, validating or refuting elementary number theory statements. We illustrate and examine the Pair Dialogue approach that we use in professional development programs when working with teachers on their knowledge needed for teaching. Here we focus on issues related to teachers' mathematical knowledge.

There is a wide agreement that teachers may play a significant role in learners' mathematical development. Consequently, various attempts have been made to design, implement and evaluate professional programs that influence the nature and quality of teachers' knowledge and practice (Ball et al., 2008; Borko et al., 1992; Cooney, 1994; Ebby, 2000; Hiebert et al., 2003). We have devoted considerable attempts during the last two decades to promote teachers' mathematical knowledge needed for teaching. These attempts are accompanied with explicit discussions of the interplay between knowledge, reflective-practice and related affect issues. We work with individual teachers, small groups and large courses of prospective teachers and teachers (e.g., Tirosh & Tsamir, 2004; Tsamir & Tirosh, 2005).

In our interactions with teachers, we use the Pair-Dialogue (PD) approach, which we have developed and implemented in various teacher education and professional development programs for preschool, elementary and for secondary school teachers. In this paper we briefly describe and illustrate the PD approach when working with preschool teachers on the definition of triangles, with elementary school teachers on the definition of even numbers and with secondary school teachers on proofs by validating or refuting Elementary Number Theory (ENT) statements.

WHAT IS THE PAIR-DIALOGUE TEACHING APPROACH?

The Pair-Dialogue (PD) approach is a specific form of team-teaching. Team teaching approaches are forms of instruction in which at least two instructors work purposely, regularly and cooperatively to help a student or a group of students learn (Buckley, 2000). The frequent model is that of a team of experts with different expertise sharing a responsibility for an interdisciplinary course (e.g., Gosetti-Murrayjohn & Schneider, 2009; Sandholtz, 2000; Shibley, 2006).

Three, quite unique aspects of our PD approach are the *Who*, the *How* and the *What*. Regarding the *Who* aspect: The PD approach is based on the engagement of two teachers who share expertise (mathematics educators with common fields of interest), status and affiliation (professors at a mathematics education department at the same university). That is, our approach is largely based on our intertwined professional identity and on our shared seniority and less on complementary contributions.

Regarding the *How* aspect: The PD approach is based on our solo pair-performances, and on our interactions with the audience. That is, the PD approach is based on (a) semi-staged thought-provoking dialogue-episodes conducted by the two of us; and (b) segments of "inviting the audience" (e.g., teachers, prospective teachers) to express their views on different ideas that are presented and to "help us out" in resolving the dilemmas that we raise. The dialogues are semi-structured, allowing for both pre-prepared and in-action adaptations to different populations of teachers, within specific settings (e.g., an academic course, field guidance). The interactions with the audience have several appearances: participants' work individually on worksheets that they occasionally hand in to us; small group communications and whole class discussions. All in all, the PD approach is based on three major didactical components: (a) continuing, *formative evaluation* of the participants' knowledge, (b) *teaching-learning interactions*, addressing issues that are known to be challenging, i.e., error or dilemma-eliciting, and (c) discussions of teachers' *reflective practices*.

Regarding the *What* aspect: The PD approach based professional development programs for mathematics teaching typically include three main content elements: (a) mathematics (<u>Subject Matter Knowledge</u>: SMK c.f. Shulman, 1986), (b) knowledge about learners' mathematical reasoning, common errors, and possible sources of these errors; and knowledge about designing meaningful and enjoyable mathematics engagements for learners (<u>Pedagogical Content Knowledge</u>: PCK c.f. Shulman, 1986), and (c) practice experiences, accompanied by reflective discussions on occurrences in the teachers' mathematics classrooms. While the pivotal topics are identical in all our programs, the general time frame and the time allotted to each part differ.

In this paper, some aspects of our PD approach are illustrated by describing and analysing some segments of our work with preschool, elementary school and secondary school teachers on mathematical issues. Here we discuss SMK related themes. The first two sections deal with discussions of mathematical definitions, while the third addresses mathematical proofs.

WHY DEFINITIONS AND PROOFS?

Definitions and proofs are two central constructs that play a crucial role in mathematics. Yet, studies have shown that learners often face difficulties when working with these mathematical entities, that intuitive obstacles are a main cause for these hurdles and that it is not a trivial matter for teachers to familiarize students with definitions and proof constructions (e.g., Alibert & Thomas, 1991; Fischbein & Kedem, 1982; Tall & Mejia-Ramos, 2006; Tall, 1999; Vinner, 1991). A layman may expect that this issue to be present only from secondary school since it is a commonly held belief that mathematical proofs and definitions are first addressed within Euclidean geometry in high school. This, however, is not the case. Many mathematics educators have recommended to start developing solid mathematical foundation, including reference to definitions and proofs, as early as possible. For example, according to the Principles and Standards for School Mathematics, mathematical definitions, reasoning and proofs may be and should be nurtured from a young age on. "…Instructional programs from prekindergarten through grade 12 should enable all students to recognize reasoning and proof as fundamental aspects of mathematics" (NCTM, 2000, p. 122). Thus, classrooms performances should provide opportunities, even for very young children, to address definitions and proofs in a natural, systematic and coherent manner. Students should be encouraged to raise questions and assumptions, to suggest solutions and to provide acceptable justifications to explain their ideas and to consult definitions and proofs (e.g., Fischbein, 1993).

One may wonder what types of explanations, definitions and proofs are expected at different developmental stage. Evidently, the types of reasoning and justifications suitable for young children may differ from those appropriate for older children. At early ages we may focus on informal explanations, based on students' real-world experiences, rather than (or much more than) on formal explanations that consist of rigour, symbolic representations. Koren (2004) differentiated between Mathematically Based (MB) explanations that employ only mathematical notions and rules, and Practically Based (PB) explanations that may also use daily references (also, Tsamir, Sheffer & Tirosh, 2000). Mathematics education researchers illustrated how young children offer MB explanations in classroom discussions (e.g., Ball & Bass, 2000); and showed that many elementary school students understand, use and even prefer such explanations (e.g., Levenson, Tirosh, & Tsamir, 2006).

Clearly, a major aim for mathematics educators is to promote learners' ability to produce and communicate MB and even formal explanations. That is, to promote students' ability to justify and explain his/her mathematical solutions by familiarizing them with related mathematical terminology and with relevant sets of mathematical rules. What is meant by "terminology" and by "set of rules"? A mathematical "term" is a concept that is given an identity by means of a definition that draws clear borders between examples and non-examples of that concept. For instance, when being asked "is this a …?" (e.g., is this a triangle"?), the answer 'yes' or 'no' is justified by addressing a definition.

A mathematical "set of rules" is formed by theorems that determine what "can be done with" certain concepts or "what are possible relationships between" concepts. Theorems are statements (rules) that have been proven to be valid. A proof can either validate or refute a statement. For instance, when being asked "is this [mathematical statement] correct?" (e.g., is the sum of three consecutive numbers divisible by three"?), the answer 'yes' or 'no' should be justified either by a validating or by a refuting poof. All in all, mathematical concepts as a terminology-base and mathematical theorems as a rule-base entities are two pivotal constructs of the mathematical realm; and consequently, concepts, definitions, theorems, and proofs play a central role in doing mathematics and in discussing mathematical issues. In the following sections we describe and analyze episodes taken from three different teacher professional development courses, one with preschool teachers, one with elementary school teachers and one with secondary school teachers. We focus on some SMK fragments.

WORKING WITH PRESCHOOL TEACHERS ON DEFINITIONS OF TRIANGLES

There is a growing awareness among mathematics educators of the importance of early childhood mathematics education (e.g., Tsamir & Tirosh, 2009). However, there is consistent evidence that many preschool teachers have limited knowledge of mathematics and of young children's mathematical reasoning and that geometry is a major hurdle (e.g., Clements, 2003). There are calls for initiating professional development programs for early childhood teachers that focus on the mathematics knowledge needed for teaching geometry (e.g., Clements & Sarama, 2007). Yet, there is still only little research addressing types of instruction that have a potential to enhance preschool teachers' geometric knowledge (e.g., Clements, Sarama, & DiBiase, 2004).

We have devoted, in the last decade, extensive efforts to working in low-income areas in Israel, in an attempt to meet the challenge of making geometry friendlier to the preschool teachers of these young children. In one of these professional development courses 17 preschool teachers participated in six four-hour sessions. The participants stated that they were suffering from geometry anxiety, and that in their preschools geometry was commonly neglected.

In this paper we focus on the first session, in which we addressed the topic: triangles. The teachers were initially asked to answer a questionnaire (formative evaluation). Their responses served in designing the following parts. We first briefly described the data that served us in formulating the questionnaire.

Designing the task: Is this a triangle?

The tasks that we formulated for the triangle-sessions and the related Pair Dialogues were based on reported, research findings, on our past studies on students and teachers' geometrical knowledge, and on the accumulated data that we collected from the specific group of 17 teachers.

An overview of the data collected led us to consider two dimensions when discussing examples and non-examples of triangles: the mathematical dimension and the psychological dimension (see Figure 1). The mathematical dimension is based on mathematical definitions, and therefore consists of two well defined, disjoint sets of figures: examples and non-examples. The psychological dimension consists of two sets of figures: intuitive and unintuitive, a distinction based on studies on children and adults' conceptions and misconceptions when addressing each figure (e.g., Tsamir, Tirosh, & Levenson, 2008). Intuitive triangles are easily identified as such (e.g., the triangles that have one side parallel to the "down edge" of the paper, see Figure 1, Cell 1), while unintuitive triangles are commonly misjudged as non triangles (e.g., "upside down" triangles, "thin" triangles, see Figure 1, Cell 2). In the same spirit, intuitive non examples of triangles are easily identifies as "not being triangles" (e.g., circles or squares, that learners are familiar with their image and with their name, e.g., Figure 1, Cell 3). Unintuitive non-examples of triangles are figures that are not triangles, but there is a tendency to regard them as triangles (e.g., a seemingly triangular shape with one bent side, see Figure 1, Cell 4).

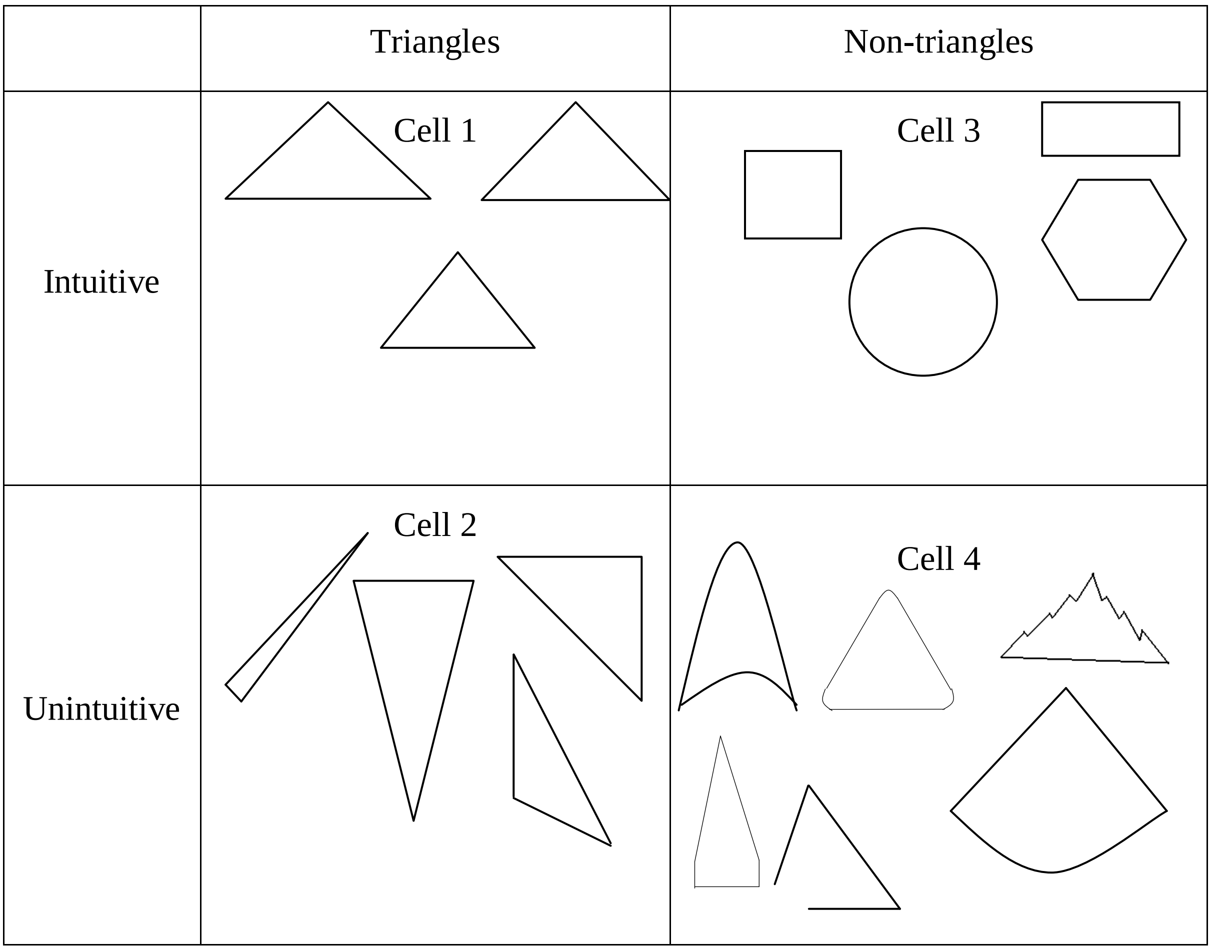


Figure 1: Examples of intuitive and unintuitive triangles and non-triangles

A worksheet that included intuitive and unintuitive, examples and non-examples of triangles was handed out to the preschool teachers, and they were asked to determine if each figure: (1) is a triangle? (2) Why? (See parts of the figures in Figure 2. The questionnaire included additional items that are not reported here (e.g., how confident are they in their answer).

This worksheet was designed to assess teachers' responses to identification-of-triangle tasks, as well as their tendency to use critical attributes of a triangle in their justifications (van Hiele & van Hiele, 1958).

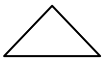
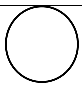
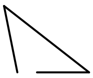
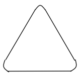
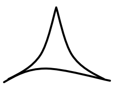
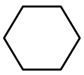
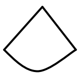
The Figure	**Triangle?** ***Why?***	**Comments...**
1 	Yes 17 *It has three sides* 15 *No Explanation* 2	
2 	No 17 *It's a circle* 16 *It has no sides* 1	
3 	No 14 *It's missing a part* 14 Yes 2 *It has 3 sides* 1 *No explanation* 1 Almost 1 *It's triangular with 3 sides* 1	*I am not sure* *One side is a bit too short* *One side is a bit broken*
4 	Yes 17 *It has three sides* 13 *It's the shape of a triangle* 2 *No Explanation* 2	*Like the triangular road-sign* 5 1
5 	Yes 9 *It has 3 sides* 3 *It has 3 bent sides* 3 *No explanation* 3 No 7 *No explanation* 7 Sort of 1 *It's triangular with 3 sides* 1	*I'm not sure. It may still be a triangle* 2 *The sides should be more stretched*
6 	No 17 *It's a hexagon* 9 *It has 6 sides* 8	
7 	Yes 17 *It's like a pizza triangle* 15 *It has three sides* 1 *No Explanation* 1	

Figure 2: Preschool teachers' responses to "Is this a triangle?" (Partial worksheet)

Evaluating the teachers' images of triangles

Figure 2 indicates that all preschool teachers correctly identified the intuitive triangle, and the two intuitive non-triangles (the circle and the hexagon).

However, all 17 teachers incorrectly identified the "pizza-triangle" (Shape 7), and the "road-sign triangle" (Shape 4), as triangles. There was also a tendency to incorrectly view the "arcs-triangle" (Shape 5) as a triangle, and some hesitations regarding the "open-triangle" (Shape 3). In the latter two cases teachers further described entities as "sort of" triangles, and as "almost" triangles, expressions might suggest that they were not aware of the sharp mathematical distinction between examples and non-examples of triangles. After studying the data on the general, "most popular" errors and the "right and wrong ideas" for each participant, we conducted the following PD.

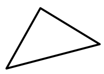
Pair-Dialogue: How can I know whether this is a triangle?

The aims of this session were (a) to challenge the preschool teachers' images of 'a triangle' and to develop triangle-images that are consistent with the related mathematical definition (the notions *concept images* and *concept definitions* are taken from Tall & Vinner, 1981); and (b) to increase the teachers' awareness of the need to consult the definition when making decisions about the nature of the figures (whether it is an example or a non-example of a triangle). Several Pair-Dialogues were employed for this purpose. Here we present the first part of the dialogue "*How can I know whether this is a triangle?"*

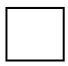
P: I feel a bit confused about triangles… I mean… the identification of triangles, can you help me?

D: Sure.

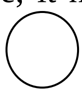
P: Please draw a triangle.

D: [draws] 

P: This seems to be easy… I… kind of know that it is a triangle; I see it's a figure that has three sides. OK. It has to have THREE SIDES.

D: Right. So, this [draws a square]  is not a triangle.

P: Sure. It's a square.

D: Yes. It's a square, and therefore, it has FOUR, and NOT THREE sides. And this [draws a circle]  is also NOT a triangle.

P: Sure.... It's a circle…

D: It has NO SIDES.

P: Eh… I believe I get it… a figure with THREE SIDES… right? Like this… [Draws a "road-sign" shape] 

D: No… No… No… This is not a triangle.

P: Why? It has three sides.

D: But the corners, the vertices are round…

P: So what? We said nothing about vertices… Do we need to?

D: Yes. There should be three vertices…. Sharp corners…

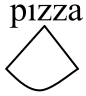
P: OK. OK. OK... If I get you right… you mean that a triangle is a figure with three sides and pointy vertices, right?

D: Yes.

P: OK. So the traffic-sign triangle is ALMOST a triangle.

D: No. No. In geometry there is no "ALMOST". It is either YES… I mean a triangle... an example, or NO.

P: [mumbles quietly as if to herself] either yes or no... [turns to D] I can surely draw a good example now… [draws] like this pizza triangle - It's even called [in Hebrew] a pizza TRIANGLE…



D: No… No… No… This is not a triangle. Not in geometry.

P: WHY? It has three sides and three vertices... and EVERYBODY calls it a pizza TRIANGLE…

D: But one side is not really a side… not geometrically… it is NOT STRAIGHT…

P: Still… It's a side… I don't get it. Every time you add conditions… I'll never know what a triangle is…

D: You need to address the definition… I mean ALL the critical attributes…

P: ALL? What do you mean by ALL?? How do I know that I addressed ALL attributes? And suddenly you added another term… What is this CRITICAL thing that you mentioned? [Turns to the class] Can someone else help me check my ideas? Do you agree with Dina? [Writes on the side of the blackboard, under the title: Dilemmas and assumptions:

1. How do we determine that a figure is a triangle?

2. What are critical attributes?]

This dialogue challenged the justification: "*it* has three sides" that most preschool teachers provided to justify their correct assertion that shape 1(Figure 2) is a triangle. The participants erroneously regarded this explanation as sufficient or as a definition, and many used the term 'side' in a daily manner, employing the concept image of a wall or a fence that is not necessarily straight.

This episode illustrates one possible way of working with the preschool teachers on incorrect or incomplete responses. In this PD one teacher educator (P) acted as a "model learner", presenting students' opinions, dilemmas, questions; the other (D) acted as a knowledgeable guide. A main gain is that the preschool teachers were confronted, in a gentle manner, with their incorrect responses.

This opening served as a springboard to a thorough discussion of the common errors.

At this stage, Gal, one of the preschool teachers said:

Gal: I agree with you [P], the pizza triangle is DEFINITELY a triangle. It's called so!

Here we see a member of the "audience" cutting in our solo part of the pair-dialogue. Gal felt confident to interrupt us and to declare that "the pizza triangle is DEFINITELY a triangle". Her confidence in her erroneous solution is evident by her bursting into the dialogue, the terminology that she used (definitely), and her tone when voicing this word. The episode continued by one of the teacher educators (D). She opened the discussion to the entire class, asking all participants to vote (triangle / not triangle) for each figure.

D: Wait a minute, [smiles at P] I see that we have some disagreements here. [Turns to the class] Let's do what my friend asked us to do... let's have another look at each of the figures and vote... Let's think about each of the figures [draws on the board the figures and the outline of Figure 3], you can vote for each figure only once – 'Yes' it is a triangle, 'No' it isn't, or 'I have not decide yet'.

P: Why can't they vote twice, if they feel like... that it... I mean, if someone thinks that a certain figure in a way IS a triangle, but in another way it IS NOT?

D: that's an important question [to the class]. What would you say?

[Giggles and voices]: No. No it can't be. If it's a triangle then it's not a NOT triangle.

Galit: But it can be SIMILAR to a triangle.

D: If it's ONLY SIMILAR, please vote NO. We'll discuss it further later. OK. OK. So... let's vote.

During this invitation (to vote), the other teacher educator (P) raised a substantial question: Can a figure simultaneously be a triangle and a non-triangle? And in general terms, can "something" simultaneously be an example and a non-example of a mathematical concept? This encouraged Galit to use the problematic notion of "SIMILAR TO". At this stage D guided the participants to vote "no" when it's "only similar", but a profound discussion of this issue followed in a session that is not presented here.

Discussing the teachers' images of triangles

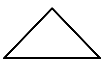
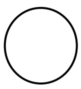
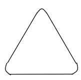
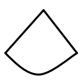
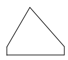
The Figure	**It's a triangle**	**It's not a triangle**	**Don't know / almost**
1 	17		
2 		17	
3 	9	2	6
4 	13	1	3

Figure 3: The preschool teachers' vote on "*Is this a triangle?*"

Figure 3 shows that after this preliminary PD, before a more profound discussion, eight and four preschool teachers, respectively, changed their minds (in the correct direction) regarding the "rounded-edges" shape, and the "pizza shape". Two stated that the rounded-edges shape is NOT a triangle, and six confessed "I don't really know". One of the latter said that "it's almost a triangle, so by Dina's guidance I should vote that it is not, but I don't feel good about it.

In brief: The teachers' SMK at the end of the course

In the final assessment the preservice teachers were asked to address a rich collection of figures, to state, for each figure, whether it is a triangle, a quadrilateral, a pentagon or none of the above, and to justify their judgments. The 17 teachers correctly identified all the triangles, and only one of them wrote that to her the "pizza-triangle" feels like a triangle although she knows it is not. They also provided mathematical, correct, although not always full definitions to justify their answers. However, when addressing the pentagon, six of them incorrectly claimed that the figure  is not a pentagon, because "it seems like a triangle" (2 teachers), "it does not look like a pentagon" (4 teachers); and when addressing the quadrilaterals, nine participants argued that the square is not a quadrilateral "because it is a square" or "because it is called 'square'" (7 teachers). The findings indicate that the preschool teachers' concept images of polygons at the end of the course were: (a) more consistent with definitions, than before the course; (b) still not completely and not always consistent with the mathematical definitions; and (c) vulnerable when a figure could be labelled by more than one term (e.g., a square that is also a quadrilateral).

Working with Elementary School Teachers on Even-Definitions

It is commonly recommended to introduce the concept even number in early elementary school (e.g., NCTM, 2000). Research have indicated that parity is not always a straight forward concept for learners. For example, it was found that students tend to claim that "a number is even if the last / units digit is even", and to reject connections between 'evenness' and 'divisibility by two' (e.g., Zazkis, 1998). Students may believe that a number can be both even and odd (e.g., Ball & Bass, 2000). Also, students tend to claim that zero is neither even nor odd, because "zero is not a number", "zero is 'nothing'" (e.g., Levenson, Tsamir, & Tirosh, 2007; Tsamir, Sheffer, & Tirosh, 2000).

The following segments are taken from a lesson that was given to 25 elementary school teachers who participated in a mathematics professional development three-year, four hours per week, course.

Pair-Dialogue: Can you give an interesting example of an even number?

The aim of this session was to challenge elementary school teachers' personal definition of parity and to increase their awareness of the need to include in it not only a minimal, sufficient collection of critical attributes, but also a statement about the "reference-set" (domain) to which these attributes apply. For example, when defining a quadrilateral in Euclidean geometry, the definition "it has four sides" is vague and unsatisfactory (the 'reference-set', is unclear). "A polygon with four sides" is a good definition, but "a figure with four sides" is incorrect, because the reference-set does not determine that the figure is two dimensional, and thus the definition lacks a critical attribute.

Several PDs were employed for this purpose. We present here the opening part of our *"Can you give an interesting example of an even number"* PD that addressed common errors related to the notions of parity and zero. The dialogue started by making reference to the statement "an even number is a number that is two times something" that participants offered as definitions of even number by the end of the previous lesson. This 'definition' gave rise to various bizarre ' examples' rooted in the vagueness of the term "something".

P: Last time… by the end of the lesson someone suggested that 'an even number is a number that is two times something'... right?

D: Right.

P: We know that when we have a definition we can produce examples

D: and non-examples...

P: Right...So let's do just that. Let's play a game of 'producing interesting even numbers'.

D: Great! Like 2, 4, 6, 8, and so on?? [Writes the numbers on the blackboard while mentioning them]

P: Yes… and NO…

D: What do you mean?

P: These are not interesting. Let's go crazy and think about INTERESTING examples, like, 76543210, or 135792 and… [Adds the numbers on the blackboard]

D: What about 2π and $\sqrt{2}$? It's TWO TIMES π, and TWO TIMES $\sqrt{2}$ [Writes them under the previous examples]

P: These are SPECIAL ones… What do you think about $\frac{2}{4}$? [Adds $\frac{2}{4}$ to the list]

D: I'd rather have decimals, like 0.4 or 44.44… [Adds then to the list]

This dialogue makes explicit (*two times* π and *two times* $\sqrt{2}$) and implicit references to the problematic 'definition' "'an even number is a number that is two times *something*" that was suggested in the last lesson. While we were presenting our correct (2, 4, 6, 8, and 76543210, 135792) and incorrect ideas, we simultaneously made a list of all our suggestions on the board, so that we would be able to keep track and address each number in the following discussions.

This episode illustrates another modus operandi of the PD approach. In this segment both teacher educators offered correct and erroneous examples. This PD performance differs dramatically from the one illustrated before, in the preschool teachers class. There, we had a "clever, always-right" character (D) and a "puzzled-erring" character (P). Here, we have two confident performers (D and P), both throwing into the air a mixture of correct and incorrect ideas. One reason for our altering roles in the different PD segments is not to have irrelevant hints, such as "D is always right", that may take away the mathematical essence when examining ideas to be *right-or-wrong.*

During our PD show, until this very moment, the teachers were listening with great care and amusement. At certain points, several faces expressed surprise (after all, these were *strange* examples), yet, no critique or question marks were voiced. The following point was a turning point.

P: WOW, It's so much fun!! I never thought about this kind of examples before…and what about ZERO???

Anat [bursts out]: No way! Don't write zero! Definitely NOT ZERO. [P and D look at each other. The class is silent]

Yael [quietly]: Zero is NOTHING. It's NOT A NUMBER, so it can't be an EVEN NUMBER.

Anat: It's not even and not odd...

D [to P]: What does she mean? What do you say? What NOW?

P: Let's see... [Turns to the class] You all were quite silent during our performance, and we can't know what you have in mind... so... let's first see what are your opinions regarding the different examples. We'll ask you to answer a brief worksheet, quietly... and individually. Afterwards we'll all discuss each of our interesting examples of even numbers.

D: But, what about ZERO?

P: This will be the first example that we'll address in our discussion.

It may seem surprising that zero was the first even number example that incited a sparkle of erroneous objection. But, it was expected in light of the publications on learners' grasp of zero and on the parity of zero. This was the reason for using the zero-example in our dialogue. At this stage, we moved from presenting thought-eliciting ideas, to evaluating the participants' knowledge

Evaluating the teachers' images of even-numbers

We asked the teachers to individually answer the worksheet *Is this an even number*? (see Figure 4). All the participating teachers correctly identified the parity of (positive) natural numbers (items A, B, J and L in Figure 4).

#	**The Number**	**Even? Yes / No** (n=25)		**(Common) Explanation**	
A	4312678	Yes	25	*The unit digit is even (8)* *It's divisible by 2*	*23* *2*
B	268430	Yes	25	*The unit digit is zero* *It's divisible by 2* *It's two times 134215*	*21* *2* *2*

C	18.9	No 25	*Because of the nine* 18 *It's not divisible by 2* 2 *No explanation* 5
D	$2\sqrt{2}$	Yes 21 No Answer 4	*It's two times* 12 *It's divisible by 2* 9 *It's strange. I don't know.* 3
E	444.44	Yes 25	*It's divisible by 2* 10 *It's two times 222.22* 9 *Because of the last digit* 5 *It's all fours* 1
F	$\frac{2}{4}$	Yes 23 No Answer 2	*It's two times* $\frac{1}{4}$ 11 *It's divisible by 2* 6 *2 and 4, two even numbers* 6
G	0.6	Yes 25	*It's two times 0.3* 11 *Because of the 6* 7 *It's divisible by 2* 7
H	2π	Yes 22 No Answer 3	*It's two times* π 14 *It's divisible by 2* 8 *I'm not sure* 3
I	$3\sqrt{5}$	No 25	*Because of the 3* 8 *Because of the 3 and the 5* 7 *It's not divisible by 2* 7 *No explanation* 3
J	34761242	Yes 25	*The unit digit is even (2)* 13 *It's divisible by 2* 12
K	Zero	No 23 Yes 2	*Zero is not a number* 9 *Zero is neither even nor odd* 9 *Zero is nothing* 3 *Two times WHAT?* 2
L	579989889	No 25	*The unit digit is odd* 15 *Because of the nine* 5 *It's not divisible by 2* 5
M	$\frac{2}{3}$	Yes 24 No Answer 1	*It's two times* $\frac{1}{3}$ 19 *It's divisible by 2* 4

Figure 4: Teachers' responses to "*Is this an even number?*" (Partial worksheet)

Had we stopped here (addressing only positive numbers), we would have concluded that the notions 'even' and 'odd' are well understood in our class. However, the other data shows that all the participants erred at least once (e.g., in item E, in Figure 4), overextending the 'reference-set' (the domain) of the notion 'even number' from whole numbers to any number set. Almost all (and several times all) the participants erroneously applied the notion of 'even number' to decimals (items C, E, G) to fractions (items F, M) and to irrational numbers (items D, H, I). Even more striking is the observation that participants either granted the attribute of parity to all kinds of numbers, or (a few) did not answer at all. None explicitly rejected the use of the term "even" for numbers that are not whole numbers.

The zero was another interesting, erroneous case. While the prevalent error was that teachers *did* grant the attribute of 'even' / 'odd' to numbers that could not have it, the zero elicited a reverse error. The teachers tended to prevent the zero of its parity attribute. Moreover, 19 teachers claimed that 268430 (item B) is an even number, because "the unit digit is zero", but 17 of them stated that zero (item K) is NOT even.

In brief: The teachers' SMK at the end of the course

In the final assessment the teachers were asked to determine if each number in a list of various numbers is even / odd / another answer. All participants correctly explained that the attribute of parity is applicable only to whole numbers, and identified other sets of numbers as irrelevant for this attribute. They also judged zero to be even, and correctly provided MB explanation. However, when having to address zero in another part of the questionnaire, three teachers wrote that "zero is not a number", and two mentioned that "zero is nothing". For them, the intuitive image of zero was still in contrast with the formal definition.

Working with High School Teachers on ENT Proofs

Proofs are often addressed in high school mathematics. Studies have shown that students often face various types of difficulties when having "to prove". Various researchers reported that students are not always aware of the necessity for a general, covering proof when proving the validity of a universal statement for an infinite number of cases (e.g., Bell, 1976) and that they tend to encounter difficulties in constructing a complete proof based on deductive reasoning (e.g., Healy & Hoyles, 1998; 2000). When refuting a statement, students tend to relate to a counter example as a bizarre instance rather than as sufficient to refute a universal statement (e.g., Balacheff, 1991).

Several studies have focused on teachers' content knowledge of proofs (e.g., Knuth, 2002; Dreyfus, 2000), but only a few examined teachers' related knowledge with reference to "prove" tasks (i.e., produce a proof) vs. "evaluate a proof" tasks (i.e., right or wrong?) (e.g., Barkai et al., 2004). Here we briefly address the latter two issues with reference to ENT statements.

Designing the sessions: Proofs - Validating and refuting ENT statements

The tasks that we formulated for the *Validating and Refuting* sessions were based on relevant publications, on our studies on students and teachers conceptions of proofs and on the data that we collected from the 23 secondary school teachers that participated in our program. The participants were first asked to answer a questionnaire consisting of six ENT statements (see Table 1: Validity is determined based on a combination of predicate and quantifier).

Predicate / Quantifier	**Always true**	**Sometimes true**	**Never true**
Universal	**S1**: The sum of **any** 5 consecutive numbers is divisible by 5. *True*	**S2**: The sum of **any** 3 consecutive numbers is divisible by 6. *False*	**S3**: The sum of **any** 4 consecutive numbers is divisible by 4. *False*
Existential	**S4**: **There exist** 5 consecutive numbers so that their sum is divisible by 5. *True*	**S5**: **There exist** 3 consecutive numbers so that their sum is divisible by 6. *True*	**S6**: **There exist** 4 consecutive numbers so that their sum is divisible by 4. *False*

Table 1: Classification of statements

The teachers were asked to determine, for each of the six statements, if they are 'true' or 'false', and to prove it in various ways (see also Tirosh & Vinner, 2004; Barkai, et al., 2004). All knew which statement is true and which is false, and provided correct proofs to validate or to refute the statements (frequently using only algebraic representations for proving the universal, true statements). These findings are consistent with our findings in an extensive study that we carried out with the support of the Israeli Science Foundation (ISF, 900/06) with fifty secondary school teachers (e.g., Tsamir, et al., 2008). Here we focus on a PD that presented teachers with two attempts to prove the same statement, asking them to state their opinions regarding the correctness of each suggestion.

Pair-Dialogue: Let's prove in different ways

The aim of this session was to challenge secondary school teachers' tendency to accept algebraic attempts to prove universal statements and to reject numeric ones. We provided two student-proofs for validating the statement *the sum of any 5 consecutive numbers is divisible by 5*: A numeric, valid, cover-proof and an algebraic representation of an attempt to prove (no reference is made to the domain for x).

D: It might be interesting to find several proofs for a statement. For instance to prove that *the sum of* ***any*** *5 consecutive numbers is divisible by 5...*

P: I like this idea...

D: I'd like to show you a nice numeric proof that a student once gave… The sum 1+2+3+4+5 is 15, right? So, it's divisible by five. To advance to the following five-consecutive-numbers you need to add one to each of the original numbers. So you have 2+3+4+5+6

P: That's 20 and it's divisible by 5.

D: The great idea is NOT to look at the 20. But at the process, when advancing from one 5-consecutive-numbers to the next 5-consecutive-numbers you add one to each number so all in all you add FIVE to the sum, so the new sum is again divisible by 5, and so on. [P has a puzzled expression]. Let's call it *The Numeric, 'Adding Five' Proof* [writes on the board]:

The Numeric, 'Adding Five' Proof

The sum of the first 5-consecutive numbers is:
1+2+3+4+5 is 15 and it's divisible by 5
The sum of the next 5-consecutive-numbers is:
2+3+4+5+6 =
(1+1)+(2+1)+(3+1)+(4+1)+(5+1)=
(1+2+3+4+5) +(1+1+1+1+1)
(Divisible by 5)+ 5
Divisible by 5

P: I n t e r e s t i n g… I'd rather have an algebraic proof; it gives a stronger sense of generality… This is also a solution that was once given by a student, look: the first number is presented as 5x, then 5x+1… and so… the sum is [writes on the blackboard]:

The Algebraic, 5x+n Proof:

5x+(5x+1)+(5x+2)+(5x+3)+(5x+4)=
(5x+5x+5x+5x+5x) +(0+1+2+3+4)=
25x +10
Divisible by 5 + 10
Divisible by 5

D: To me, NOT using algebra and still addressing the generality is stronger...

P: Perhaps we should consult our friends here [turns to the class] what would you say? Is the Numeric proof correct? Is the algebraic proof correct? Would you discuss both in class? Which one do you prefer?

At this stage, the teachers were asked to write and submit their opinions regarding each of the suggested proofs. We report on the significant findings.

Evaluating the teachers' knowledge and images of ENT proofs

The teachers analyzed the statement and the proof according to its mode of argumentation and its modes of representation (Stylianides, 2007). They expressed unease with the numeric representation, and all but three stated that "it doesn't seem right". In respond to the question: Could a numeric representation be a correct proof? Ten teachers wrote 'yes', and eight of them added "but" ("not really", "not in high school", "I wouldn't use such a proof and / or I wouldn't like my students to use it"). Seven teachers wrote 'no', explaining that "it isn't general", occasionally adding comments like "we can't know what about REALLY LARGE numbers...". Six teachers claimed that they cannot state whether it is correct, because "it's strange", "I never use such methods".

When referring to the algebraic suggestion, all 23 participants stated that "algebra is the right way for proving that such statements are valid". Seven teachers praised the given algebraic proof: "it's good" / "interesting", because "it brings forward the divisibility by five, right from the first expression"; and three of them added "it is definitely better than the other [numeric] one". Five teachers wrote "they are not sure", "I never used such a sequence", and the other 11 teachers referred to the presented "proof" as "partial", not covering *all* cases, yet 5 of them added that it's general and thus better than the numeric one.

By the end of the course all participants accept numeric representation which cover all cases, and rejected algebraic representations that failed to provide the needed cover. They were also very careful about the examination of the domain of algebraic representations.

A CONCISE SUMMARY

Mathematics teacher researchers constantly search for promising, sensitive ways of enhancing teachers' mathematical knowledge needed for teaching. In this paper we briefly describe the application of the PD teaching approach that we have developed to preschool teacher, to elementary school teachers and to secondary school teachers. It seems worthwhile to study the short-term and the long-term implications of using this approach. There is still a long way to go with developing, implementing and assessing the impact of the PD teaching approach with individuals, with small groups and with whole classes of prospective and practicing mathematics teachers who are engaged in teaching mathematics.

REFERENCES

Alibert, D., & Thomas, M.: 1991, Research on mathematical proof, in: D. O. Tall (Ed.), *Advanced mathematical thinking*. Kluwer: Dordrecht, The Netherlands, pp. 215-230.

Balacheff, N.: 1991, The benefits and limits of social interaction: the case of mathematical proof, in: A. J. Bishop, S. Mellim-Olsen, & J. van Dormolen (Eds.), *Mathematical knowledge: Its growth through teaching*, Dordrecht, The Netherlands: Kluwer Academic Publishers, pp. 175-194.

Ball, D., & Bass, H.: 2000: Making believe: The collective construction of public mathematical knowledge in the elementary classroom, in: D. Phillips (Ed.), *Yearbook of the national society for the study of education, constructivism in education*, Chicago, IL: University of Chicago Press.

Ball, D., Thames, M., & Phelps, G.: 2008, Content knowledge for teaching, *Journal of Teacher Education*, ***59***, pp. 389-40.

Barkai, R., Tsamir, P., & Tirosh, D.: 2004, Is it a mathematical proof or not? Elementary school teachers' responses, in: M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME Conference,* vol. 1, Bergen, p. 283.

Bell, A. W.: 1976, A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, ***7***, pp. 23-40.

Borko, H., Eisenhart, M. Brown, C.A., Underhill, R.G., Jones, D. & Agard, P. C.: 1992, Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education,* **23**, pp. 194–222.

Buckley, F. J.: 2000: *Team Teaching: What, Why, and How*? Thousand Oaks, CA: Sage.

Clements, D. H.: 2003, Teaching and learning geometry, in: J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards of school mathematics,* Reston, VA: NCTM. pp. 151-178.

Clements, D. H., & Sarama, J.: 2007, Early childhood mathematics learning, in: F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning,* Charlotte, NC: Information Age Publishing, pp. 461-556.

Clements, D. H., & Sarama, J., & DiBiase, A. (Eds.): 2004, *Engaging young children in mathematics*. Mahwah, NJ: Lawrence.

Cooney, T.J.: 1994, Teacher education as an exercise in adaptation, in: D. Aichele & A. Coxford (Eds.), *Professional development for teachers of mathematics*, Reston, VA: National Council of Teachers of Mathematics, pp. 9–22.

Dreyfus, T.: 2000, Some views on proofs by teachers and mathematicians, in: A. Gagatsis (Ed.), *Proceedings of the 2nd Mediterranean conference on Mathematics Education*, Vol. I, Nikosia, Cyprus: The University of Cyprus, pp. 11-25.

Ebby, C. B.: 2000, Learning to teach mathematics differently: The interaction between coursework and fieldwork for preservice teachers, *Journal of Mathematics Teacher Education*, **3**, pp. 69–97.

Fischbein, E.: 1993, The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity, in: R. Biehler, R. Scholz, R. Straber, & B.Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline*, Dordrecht, the Netherlands: Kluwer, pp. 231–245.

Fischbein, E., & Kedem, I.: 1982, Proof and certitude in the development of mathematical thinking, in: A. Vermandel (Ed.), *Proceedings of the 6th PME Conference,* Antwerp, Belgium, pp. 128-131.

Gosetti-Murrayjohn, A., & Schneider, F.: 2009, Dialogues in performance: A team-taught course on the afterlife in the classical and Italian traditions, *College Teaching, **57***, pp. 37-43.

Healy, L., & Hoyles, C.: 1998, *Justifying and Proving in School Mathematics*, University of London, Institute of Education: Technical Report.

Healy, L. & Hoyles, C.: 2000, A study of proof conception in algebra, *Journal for Research in Mathematics Education*, **31**, pp. 396-428.

Hiebert, J., H., Morris, A., & Glass, B.: 2003, Learning to learn to teach: An "experiment" model for teaching and teacher preparation in mathematics, *Journal of Mathematics Teacher Education,* ***6*,** pp. 201–222.

Knuth, E. J.: 2002, Secondary school mathematics teachers' conceptions of proof, *Journal for Research in Mathematics Education*, ***33***(5), pp. 379-405.

Koren, M.: 2004, Acquiring the concept of signed numbers: Incorporating practically-based and mathematically-based explanations, *Aleh* (in Hebrew), ***32***, pp. 18-24.

Levenson, E., Tirosh, D., & Tsamir, P.: 2006, Mathematically and practically-based explanations: Individual preferences and sociomathematical norms, *International Journal of Science and Mathematics Education, **4***, pp. 319–344.

Levenson, E., Tsamir, P., & Tirosh, D.: 2007, Neither even nor odd: Sixth grade students' dilemmas regarding the parity of zero, *Journal of Mathematical Behavior, **26***, pp. 83–95.

National Council of Teachers of Mathematics: 2000, *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Sandholtz, J. H.: 2000, Interdisciplinary team teaching as a form of professional development. *Teacher Education Quarterly, **27***, pp. 39-54.

Shibley, I. A.: 2006, Interdisciplinary team teaching: Negotiating pedagogical differences, *College Teaching, **54***, pp. 271-274.

Shulman, L.: 1986, Those who understand: Knowledge growth in teaching, *Educational Researcher, **15***(2), pp. 4-14.

Styliandes, A.: 2007, Proof and proving in school mathematics, *Journal for Research in Mathematics Education*, **38**, pp. 289-321.

Tall, D. O.: 1999, The Cognitive Development of Proof: Is Mathematical Proof For All or For Some?, in: Z. Usiskin (Ed.), *Developments in School Mathematics Education Around the World*, vol, 4, Reston, Virginia: NCTM, pp. 117-136.

Tall, D. O., & Mejia-Ramos, J P.: 2006, The long-term cognitive development of different types of reasoning and proof, presented at the *Conference on Explanation and Proof in Mathematics: Philosophical and Educational Perspectives*, Essen, Germany.

Tall, D. O, & Vinner, S.: 1981, Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics, **12***, pp. 151-169.

Tirosh, C., & Vinner, S.: 2004, Prospective teachers' knowledge of existence theorems, in: M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th PME Conference,* Vol. 1, Bergen University College: PME. p. 360.

Tirosh, D., & Tsamir, P.: 2004, What can mathematics education gain from the conceptual change approach and what can the conceptual change approach gain from its application to mathematics education? *Learning and Instruction*, ***14***, pp. 535-540.

Tsamir, P., Sheffer, R. & Tirosh, D.: 2000, Intuitions and undefined operations: The case of division by zero, *Focus on Learning Problems in Mathematics*, ***22***, pp. 1-16.

Tsamir, P., & Tirosh, D.: 2005, In-service elementary mathematics teachers' views of errors in the classroom, *Focus on Learning Problems in Mathematics*, ***27***(3), pp. 30-42.

Tsamir, P., & Tirosh, D.: 2009, Affect, Subject Matter Knowledge and Pedagogical Content Knowledge: The Case of a Kindergarten Teacher, in: J. Maaß, & W. Schlöglman (Eds.), *Beliefs and attitudes in mathematics education: New research results*, Rotterdam, the Netherlands: Sense Publishers, pp. 19-32.

Tsamir, P., Tirosh, D., Dreyfus, T., Barkai, R., & Tabach, M.: 2008, Inservice teachers' judgment of proofs in ENT. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the 32nd PME Conference*, Vol. 4, Morélia, Mexico, pp. 345-352

Tsamir, P., Tirosh, D., & Levenson, E.: 2008, Intuitive nonexamples: The case of triangles, *Educational Studies in Mathematics*, ***69***, pp. 81-95.

van Hiele, P. M. & van Hiele, D.: 1958, A method of initiation into geometry, in H. Freudenthal (Ed.), *Report on methods of initiation into geometry*, Groningen, pp. 67-80.

Vinner, S.: 1991, The role of definitions in the teaching and learning of mathematics, in: D. O. Tall (Ed.), *Advanced mathematical thinking*, Kluwer: Dordrecht, The Netherlands, pp. 65-79.

Zazkis, R.: 1998, Odds and ends of odds and evens: An inquiry into students; understanding of even and odd numbers, *Educational Studies in mathematics*, ***36***, pp. 73-89.

ANALYSING THE EFFECTS OF SITUATIONS ON FRACTIONS LEARNING ENVIRONMENTS – THE CASE OF QUOTIENT SITUATIONS

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This paper describes a study on the effects of quotient interpretation on students' understanding of the concept of fraction. An intervention program and pre- and post-tests were conducted with students between 11 and 12 years of age (N=84) from Braga, Portugal. A quantitative analysis showed that students improved their ideas about the equivalence and ordering of fractions after working with fractions presented in quotient situations; their performance on solving problems of naming, ordering and equivalence of fractions presented in part-whole and operator situations improved as well. Implications for rational numbers learning environments are discussed.

FRAMEWORK

At elementary school levels students are supposed to develop their number sense. This includes the acquisition of rational numbers. In agreement with several authors (see Behr, Wachsmuth, Post & Lesh, 1984; Kerslake, 1986; Kieren, 1993), the concept of fraction is one of the most complex concepts that children learn during the elementary grades. Knowing the concept of fraction demands the understanding of the logical aspects of fractions (ordering and equivalence of fractions) and the ability to use distinct modes of representation, in different interpretations of this concept (Behr, Wachsmuth, Post & Lesh, 1984; Nunes, Bryant, Pretzlik, Wade, Evans & Bell, 2004; Mamede & Nunes, 2008).

Several authors have distinguished interpretations that might offer a fruitful analysis of the concept of fraction. Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Kieren (1993) considers measure, quotient, ratio and operator as mathematical subconstructs of rational number; Mack (2001) proposed a different classification of interpretations using the term 'partitioning' to cover both part-whole and quotient interpretation. More recently, Nunes, Bryant, Pretzlik, Wade, Evans and Bell (2004) presented a classification based on the notion of situation, distinguishing quotient, part-whole, operator and intensive quantities situations, according to the number meanings that occur in each situation.

In spite of the differences, part-whole, quotient and operator situations are among the situations identified by all of them. However, there is not much research producing no unambiguous evidence about whether students behave differently in different situations or not. Literature provides information about students' difficulties and misunderstanding with fractions (see Behr, Wachsmuth, Post & Lesh, 1984; Hart, 1981; Kerslake, 1986; Mamede & Nunes, 2008). However, little research has been produced on the effects of situations on students' understanding of fractions. This paper focuses on the use of fractions in three situations - part-whole, quotient and operator situations - and provides such evidence.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, 2/4 in part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken (Nunes *et al.*, 2004). In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In this situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes *et al.*, 2004). Finally, in an operator situation, the denominator designates the number of equal groups into which a set was divided and the numerator designates the number of groups taken. In operator situations, the connection between the numbers that describe the situation and the fraction is created by operating on these numbers. For example, if Bill has 12 sweets and eats 2/4 of them, the numbers 2 and 4 are not perceived directly in the situation; this means that one has to divide the set of sweets into 4 and take 2 groups (Nunes *et al.*, 2004). Thus number meanings differ across situations. Do these differences affect students' understanding of fractions?

This paper focuses on the effects of quotient, part-whole and operator situations on students' understanding of fractions. Portuguese programme and curricular guidance for mathematics at the elementary grades includes these situations. By the end of sixth grade (11 and 12-year-olds), Portuguese students are supposed to be fully acquainted with the labelling, ordering and equivalence of fractions in different situations. Do part-whole, quotient and operator situations affect students' understanding of fractions?

Little research has been done concerning the effects of situations in which fractions are used on students understanding of the concept of fraction. This study reaches a deeper understanding of the effects of quotient situation on students' ideas of fraction. The investigation was motivated by the results of a survey on students' performance on fractions tests. This survey involved 158 students, between 11 and 12 years of age, from two schools in Braga, Portugal. The tests included items of labelling, equivalence and ordering of fractions in three situations – part-whole, operator and quotient. The results revealed that students' performance is affected by the type of situation in which the concept of fraction is used. Although part-whole and operator were the most frequently explored situations in the classroom, students' performance was surprisingly better in solving problems involving equivalence and ordering of fractions in quotient situations, even without being familiarized with this type of situation. These results led us to the following question: does a deeper exploration of the quotient situation affect students' understanding of fractions and its representation on other situations that they are more familiar with? Are there relevant differences on students' performance with fractions after a deeper exploration of the quotient situation? If it is so, fraction learning environments are affected by the type of situation in which fractions are used.

METHODS

Participants

Four classes of Portuguese sixth-grade students (N=84), aged 11 and 12 years, from a school of the city of Braga, in Portugal, participated in this study. Two of the classes (n=46) were the experimental group and the other two formed the control group (n=38). All the participants gave informed consent and permission for the study, obtained from their teachers. The participant school was attended by students from a wide range of socio-economic backgrounds.

The teachers of the participants of this study informed the researchers about the type of situations that students were familiar with. These situations included predominantly part-whole and operator situations; quotient situations were described by them as little explored in the classroom.

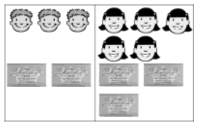
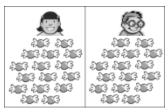
Design

This study was developed under a quantitative methodology following a quasi-experimental design, with non-equivalent control group. In order to identify eventual modifications in students' performance, a pre-test was given at first hand, followed by a post-test after intervention. The individual tests comprised tasks related to ordering and equivalence of fractions, and labelling of fractions (with pictorial and verbal support). These types of tasks were presented in quotient, part-whole and operator situations. Tasks involving only the formal symbolic representation of fractions, without any explicit situation, were presented as well and are referred here as algebraic representation. The tasks of the tests were inspired on the studies of Kerslake (1986), Streefland (1991) and Nunes *et al.* (2004). The fractions involved in the tasks were all smaller than one.

Pre- and post-tests

A quantitative analysis of the results of the pre- and post-tests was developed in order to identify the effects of the teaching intervention on students' performance. Pre- and post-tests were identical; the number and order of questions in the pre-test were maintained in the post- one, only a few fractions involved in the problems were changed.

The tests comprised 30 tasks: 7 presented in quotient situations (QT) (2 of ordering of fractions, 2 of equivalence of fractions, 3 of representation); 7 presented in part-whole situations (PW) (2 of ordering of fractions, 2 of equivalence of fractions, 3 of representation); and 9 without any explicit situations, using only algebraic representation (AL). Table 1 shows an example of a task presented in each type of situation and also an example involving the three situations. The fractions were the same across situations, according to the type of task. Thus, for instance, an ordering task involving 2/3 and 3/5 had a correspondent task presented also in part-whole and operator situations.

Situation	Problem	Example
QT	Ordering	Three boys are going to share fairly 2 chocolate bars. Five girls are going to share fairly 3 chocolate bars. Tick the right statement:  ☐ Each boy eats more than each girl; ☐ Each girl eats more than each boy; ☐ Each boy and each girl eat the same amount of chocolate Write the number that represents the amount of chocolate eaten by each child.
PW	Naming (verbal support)	Bill ordered a pizza and divided it into 4 equal parts. He decided to eat 3 of them. What part of pizza did Bill eat? ☐ $\frac{4}{3}$ ☐ $\frac{3}{4}$ ☐ $\frac{1}{4}$ ☐ $\frac{1}{3}$ ☐ 3 ☐ Other: __
OP	Equivalence	Rita and Lewis have 16 caramels each. Rita ate $\frac{3}{4}$ of the caramels. Lewis ate $\frac{6}{8}$ of the caramels. Tick the right statement.  ☐ Rita ate more caramels than Lewis; ☐ Lewis ate more caramels than Rita; ☐ Rita and Lewis ate the same amount of caramels.

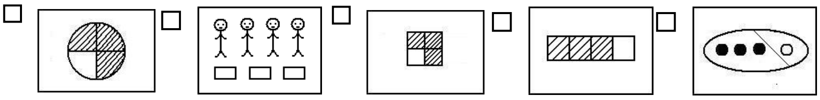
QT, PW, OP	Naming (pictorial support)	Which figure(s) best represents the fraction $\frac{3}{4}$? 

Table 1: Examples of tasks presented involving different situations.

An example of a task without any explicit situations, using only algebraic representation, is listed on Table 2.

Situation	Problem	Example
AL	Equivalence	Tick the right statement: ☐ $\frac{6}{8}$ is two times $\frac{3}{4}$; ☐ $\frac{3}{4}$ and $\frac{6}{8}$ are equivalent fractions; ☐ $\frac{3}{4}$ is smaller than $\frac{6}{8}$; ☐ $\frac{6}{8}$ is found by multiplying $\frac{3}{4}$ by 2; ☐ $\frac{3}{4}$ is two times $\frac{6}{8}$.

Table 2: Example of a task using only algebraic representation.

The intervention

The intervention comprised two 90 minutes lessons taught by the mathematics teachers of each class. These lessons were planned in cooperation of the researcher, and all the tasks presented to the students were selected in agreement with teachers of the participant classes.

The experimental group was firstly introduced to the quotient situation as this is considered the situation that better matches the students' informal ideas of fractions (see Mamede & Nunes, 2008; Nunes *et al.*, 2004; Streefland, 1991). The teacher explained the meaning of the numerator and denominator in this type of situation. Then students were given worksheets containing tasks of identification of fractions, and problems of ordering and equivalence of fractions presented in quotient situation. After this, students were challenged to work in part-whole situation, which was the most explored situation in the classroom, and finally the operator situation, solving problems of ordering and equivalence of fractions in these situations.

The intervention carried on with the control group included only part-whole and operator situations, both already known by the students. In this case, students solved problems contained in the textbook and no contribution of the researcher took place to plan the sessions. Both experimental and control groups worked with fractions involving only algebraic manipulation.

The tasks during the intervention

The tasks used in the intervention were based on the work of Kerslake (1986), Streefland (1991) and Nunes *et al.* (2004). These tasks involved ordering and equivalence of fractions, and naming fractions. Seven of the tasks were presented to students in quotient situation (QT) (5 of ordering of fractions, 5 of equivalence of fractions, 2 of representation); 5 in part-whole situation (PW) (2 of ordering of fractions, 2 of equivalence of fractions, 1 of representation); 7 in operator situation (2 of ordering of fractions, 2 of equivalence of fractions, 3 of representation); and 6 without any explicit situations, using only algebraic representation (AL). Table 3 shows an example of a task presented to the experimental group, in each type of situation.

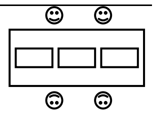
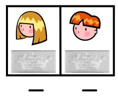
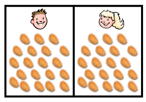
Situation	Problem	Example
QT	Naming	Four friends are sharing 3 bars of chocolate fairly. Can each child get a whole bar of chocolate? _____ Can each one get at least a half bar of chocolate? __ What fraction of the chocolate does each one get?__ 
PW	Equiv.	Mary and John have each a bar of chocolate of the same size. Mary broke hers into 3 equal parts and ate 1 of them. John broke his into 6 equal parts and ate 2 of them.  Write the number that represents the part of chocolate that each child ate. What can you conclude? __________
OP	Order.	Michael and Sarah have 20 hazelnuts each. Michael ate $\frac{2}{4}$ of his hazelnuts; Sarah ate $\frac{4}{5}$ of hers. Who ate more hazelnuts, Michael or Sarah? Why? _______________________ 

Table 3: Some examples of tasks presented to the experimental group.

Situation	Problem	Example
AL	Equiv.	Fill in the missing numbers: $\frac{7}{9}=\frac{....}{27}$ $\frac{5}{11}=\frac{....}{....}$ $\frac{7}{....}=\frac{28}{36}$

Table 4: A task presented to the experimental group using algebraic representation.

An example of a task without any explicit situations, using only algebraic representation, is listed on Table 4.

RESULTS

Descriptive statistics of students' performance on the tasks for each working situation, in pre- and post-tests are presented in Table 5, reporting the proportions of correct responses and standard deviations by task and group. As the problems of ordering and equivalence relate to quantities represented by fractions, they demand the understanding of basic logical aspects. Thus, these problems will be referred to here as logic of fractions problems.

	Experimental Group (n=46)		Control Group (n=38)	
	Pre-test	Post-test	Pre-test	Post-test
Logic of fractions	.31 (.20)	.55 (.20)	.37 (.17)	.40 (.20)
Naming of fractions	.58 (.15)	.72 (.14)	.61 (.13)	.60 (.15)

Table 5: Proportions of correct responses (standard deviation) by task and test.

An ANOVA was conducted to analyse the effect of the intervention and the type of problem (logic of fractions, naming) on students' performance (experimental group (EG), control group (CG)). Table 6 shows the Adjusted Means and (Standard Errors) for problems of logic and naming of fractions by group.

	Logic of fractions			Naming of fractions		
	ME (SE)	t(83)	p-value	ME (SE)	t(83)	p-value
Experimental	.57 (.02)	8.6	<.001	.73 (.02)	7.08	<.001
Control	.38 (.02)			.59 (.02)		

Table 6: Adjusted Mean and (Standard Error) of proportion of correct responses.

The results of ANOVA suggest that students' performance improved after the intervention in both types of problems (logic and naming problems). However, students performance on problems of logic improved better (ME=.57; SE=.02) than on problems of naming fractions (ME=.73; SE=.02), when compared to the control group. In order to identify changes on students' understanding of fractions with the intervention sessions, experimental and the control groups performances were compared when solving problems presented in quotient, part-whole, operator situations and algebraic representation. Descriptive statistics of the performance of experimental group on the tasks for each working situation are presented in Table 7, reporting mean and (standard deviation) for the proportion of correct responses of experimental group in pre- and post-tests, in problems of logic and naming of fractions, by type of situation (n=46).

The intervention sessions allowed students to improve their performance in problems of fractions across the situations. This improvement was stronger when quotient situations were involved, as the students were not very familiarized with this situation before the intervention.

	QT		PW		OP		AL	
	Pre-t	Post-t	Pre-t	Post-t	Pre-t	Post-t	Pre-t	Post-t
Logic of fractions	.39 (.24)	.71 (.23)	.38 (.29)	.61 (.32)	.32 (.31)	.64 (.35)	.16 (.16)	.23 (.18)
Naming of fractions	.19 (.23)	.45 (.32)	.72 (.34)	.81 (.27)	.30 (.31)	.55 (.27)	.85 (.12)	.91 (.11)

Table 7: Mean and (Standard deviation) of proportions of correct responses by task, test and situation.

Analyses of Co-variance were carried out to assess the effective improvement of students in each type of situation and task, with the intervention. Table 8 summarizes the results of the ANCOVA, giving the adjusted means and (standard errors) for the proportion of correct responses solving the problems of logic and naming of fractions in each situation (Quotient (QT), Part-whole (PW), Operator (OP), algebraic (AL)).

		Logic of fractions			Naming of fractions		
		ME (SE)	t(83)	p-value	ME (SE)	t(83)	p-value
QT	Experimental	.72 (.04)	3.52	<.05	.45 (.04)	2.47	<.05
	Control	.42 (.04)			.18 (.04)		
PW	Experimental	.62 (.04)	6.85	<.001	.83 (.03)	7.82	<.001
	Control	.44 (.04)			.71 (.03)		
OP	Experimental	.65 (.04)	5.31	<.001	.56 (.04)	3.77	<.001
	Control	.38 (.05)			.34 (.04)		
AL	Experimental	.30 (.03)	5.23	n.s.	.90 (.01)	4.04	n.s.
	Control	.26 (.03)			.87 (.02)		

Table 8: Adjusted Means and (standard errors) for the proportion of correct responses solving the problems in each situation.

The results of ANOVA suggest that students' performance improved after the intervention in both types of problems (logic and naming problems). Students performance improved strongly with the intervention on problems of logic of fractions presented in quotient situations (ME=.72; SE=.04) and on naming problems (ME=.45; SE=.04), when compared to the control group.

The intervention focused in quotient situations seemed to help students to improve their understanding of the equivalence and ordering of fractions, not only in this type of situations (which was expected), but also in part-whole and operator situations. For some students this short intervention created the opportunity to learn fractions representation in quotient situation. A surprisingly result was the one achieved by the students on problems of naming fractions in part-whole situations; they are able to succeed with fraction labels even without understanding the logical issues of fractions. This can give the teacher the wrong idea about students' knowledge of rational numbers.

In spite of the good effects on students' performance on problems presented in particular situations, the intervention of this study did not have a relevant impact when algebraic representation was involved.

DISCUSSION AND CONCLUSIONS

The results suggest that the intervention contributed to an improvement of the student's performance on solving problems of equivalence, ordering and representation of fractions in all three selected situations. Most important, the improvement in the performance seems to express a development and acquisition of a wider concept of fraction, since a better performance was also acquired in part-whole and operator situations. The results also suggest that the improvement in the performance was particularly notorious in quotient situation. This is an unexpected result as these students were exposed to an introduction of fractions based essentially on part-whole and operator situations. This improvement after a brief contact with quotient situation suggests that this type of situation seems to be easily understood by the students. This idea converges with the one presented by Nunes *et al.* (2004), who conducted a study with 62 students, aged 7 to 10 years, analysing their strategies for problems in quotient situation. Notwithstanding the fact that students only explored fractions in part-whole situations during the period of formal instruction, the results suggest that 11 of the 12 groups of participants used the notion of division to justify the equivalence of fractions. The results of Nunes *et al.* (2004) are partially comparable to those obtained here as these authors limited their study to the resolution of tasks of equivalence in quotient situation, excluding problems of ordering and also part-whole and operator situations.

The results of this study are in partial agreement with those of Kerslake (1986) who, under the Strategies and Errors in Secondary Mathematics Project (SEMS), carried out an intervention with 59 students of 13-14 year-olds, during six sessions of 40 minutes each. Pre-, post- and delayed post-tests were conducted. The intervention was mainly focused on: the interpretation of fraction as a quotient; the equivalence of fractions; and the recognition of fractions as numbers. Concerning the interpretation of fractions as a quotient, Kerslake (1986) results indicate that students improved their performance from pre- to post-test (from 30% to 65%), but reduced it again in the delayed post-test (to 50%), indicating that their ideas of fractions in quotient situations were not consistent. For Kerslake (1986) these results are due to a short period of the intervention and also to an excessive use of the part-whole situation. Regarding this last point, the author argues that the almost exclusive use of the part-whole situations can led students to develop wrong ideas about rational numbers and also inhibits the development of other interpretations of fractions. Similarly, the results presented in this paper suggest that the predominance of parte-whole and operator situations is not enough for the appropriate development of students' concept of fraction.

However, further research is needed in order to achieve a better knowledge of the role of situations in the understanding of fractions. For this purpose, it is of the utmost importance to understand students' mental processes in each situation as it should have an impact on rational numbers learning environments.

REFERENCES

Behr, M., Lesh, R., Post, T. & Silver, E.: 1983, Rational-Number Concepts, in R. Lesh and M. Landau (Eds.), Acquisition of Mathematics Concepts and Processes, Academic Press, New York, pp. 92-127.

Behr, M., Wachsmuth, I., Post, T. & Lesh, R.: 1984, Order and Equivalence of Rational Numbers: A Clinical Teaching Experiment, Journal for Research in Mathematics Education, 15 (5), pp. 323-341.

Hart, K.: 1981, Fractions, in K. Hart (Ed.), Children's Understanding of Mathematics: 11-16, John Murray Publishers, London, pp. 66-81.

Kerslake, D.: 1986, Fractions: Children's Strategies and Errors – A Report of the Strategies and Errors in Secondary Mathematics Project, NFER-NELSON, Berkshire.

Kieren, T.: 1993, Fractional numbers: from quotient fields to recursive understanding, in T. P. Carpenter & E. Fennema & T. Romberg (Eds.), Rational Numbers: An Integration of Research, Erlbaum, Hillsdale, NJ, pp. 49-84.

Mack, N.: 2001, Building on informal knowledge through instruction in a complex content domain: partitioning, units, and understanding multiplication of fractions, Journal for Research in Mathematics Education, 32, pp.267-295.

Mamede, E. & Nunes, T.: 2008, Building on Children's Informal Knowledge in the Teaching of Fractions, in O. Figueras, J. Cortina, S. Alatorre, T. Rojano & A. Sepúlveda (Eds.), Proc. 32th Conf. of the Int. Group for Psychology of Mathematics Education and PME-North America XXX, Vol. 3, pp.345-352. PME, Morélia, México.

Nunes, T., Bryant, P., Pretzlik, U., Evans, D., Wade. J. & Bell, D.: 2004, Vergnaud's definition of concepts as a framework for research and teaching, Annual Meeting for the Association pour la Recherche sur le Développement des Compétences, Paper presented in Paris: 28-31, January.

Streefland, L.: 1991, Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research, Kluwer Academic Publishers, Norwell, MA.

ACQUIRING A SENSE OF MOTION: TOWARD THE CONCEPT OF FUNCTION AT PRIMARY SCHOOL

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The research discusses a long-term teaching experiment in which some children have to study graphs representing motion data using technological devices. The analysis of two episodes, related to Benny's behaviour, will show the relevance of perceptuo-motor activities in understanding processes, and traces of the hypothesized multimodal nature of mathematical learning.

INTRODUCTION AND THEORETICAL BACKGROUND

This paper presents a long-term study that has involved some children from the 2nd to the 5th grade. The children participated in experimental activities in which they used two technological devices for graphing motion. The first device is a motion sensor, called CBR (*Calculator Based Ranger*); the second device is a computer software, which allows representing motion in two components (*Motion Visualizer*). The main goal of the research is to analyse the *processes* through which the children make sense of the graphs they work with. But the experiment has also a primary didactical goal: to design a path toward the concept of function, founded on the *visualization* of graphs of functions as results of movements. The subject of functions is relevant for the curriculum as the basis of many advanced topics. On the other hand, there is a mutual relationship between the mathematics used to describe change (Calculus) and kinematics in physics. It is frequently difficult for learners to realise this relationship, which has been evident in the history of function. An epistemological assumption lies at the core of the research then: building the concept of function on *cognitive roots* (according to Tall, 2000) related to motion may be didactically effective, for learners and for teachers. The effectiveness stays in that the historical origins of the concept are echoed (Edwards, 1979). So many years ago, Newton was already convinced that: "Lines [curves] are described, and thereby generated, not by the apposition of parts but by the continued motion of points. […] These geneses really take place in the nature of things, and are daily seen in the motion of bodies." (from *De quadratura curvarum*, quoted in Struik, 1986, p. 303).

Along with epistemological reasons, there are cognitive reasons: understanding what a function is, is a cognitive task. Both goals are based on a chief idea coming from the theory of *embodied cognition* and from its later streams (Lakoff & Núñez, 2000; Wilson, 2002). Basically, the idea is rooted in considerable evidence showing that the body (and the sensory-motor experiences) plays a central role in thinking processes. Developing on this view and discussing on mathematics learning, Nemirovsky (2003) argues:

> It is a very different experience to watch a movie displaying a geometrical object than it is to touch and walk around a plastic model of the same object. Clearly both experiences can be useful, but even if one would argue that they both reflect the same mathematical principle, they are not mere repetitions. One difference is that the use of appropriate materials and devices facilitates the inclusion of touch, proprioception (perception of our own bodies), and kinesthesia (self-initiated body motion) in mathematics learning (p. 103).

The relevance of perceptuo-motor experiences concerning mathematics learning is not new, of course. It comes from an ancient tradition on the use of manipulatives, based on the ideas of famous educators, such as Maria Montessori, Georges Cuisenaire, Caleb Gattegno and Zoltan Dienes. Since that time many steps further have been made, up to arrive to conjecture that: "We think of, say, a quadratic function, by enacting "little thrusts" of what writing its equation, drawing its shape, uttering its name, or whatever else the use of a quadratic function in a particular context might entail. The actions one engages in mathematical work, such as writing down an equation, are as perceptuo-motor acts as the ones of kicking a ball or eating a sandwich; elements of, say, an equation-writing act and other perceptuo-motor activities relevant to the context at hand are not merely accompanying the thought, but are the thought itself as well as the experience of what the thought is about." (Nemirovsky, 2003, p. 109).

Recent neuroscientific results, deepening the above embodied perspective and using evidence on the firing of *mirror neurons* (Gallese, 2007), say something more about conceptual knowledge in general. In the last years, these results started to affect research in Mathematics Education, giving fresh insights into the role of perceptual, sensory and motor experiences in relation to learning. They have shown that the sensory-motor system of the brain has an inherent *multimodal* character being active during thinking processes: many modalities are linked together, infused each with properties of others (Gallese & Lakoff, 2005). Gallese & Lakoff (2005) sustain that "sensory modalities like vision, touch, hearing, and so on are actually integrated with each other *and* with motor control and planning. This suggests that there are no pure "association areas" whose only job is to link supposedly separate brain areas (or "modules") for distinct sensory modalities." (p. 459). Some experiments revealed that the same brain areas are active when one performs actions relative to diverse modalities. Such multimodality of the brain is exploited by language; it seems to be at the foundation of social cognition (Gallese, 2007). Based on these considerations, the hypothesis that *mathematics learning happens in a multimodal manner* emerges. In the classroom, among the numerous modalities at play it is possible to observe that children have often recourse to gestures, and even gazes seem to be relevant. This hypothesis is crucial in all the phases of the research study: on the one hand, it influences our choice of filming the children at school and their interactions with both other subjects and tools; on the other hand, it justifies our interest in the role of gestures and other modalities in children's cognitive processes. In Arzarello (2006), first traces of this interest appear with particular attention drawn to gestures, from a semiotic-cultural viewpoint.

Embracing this whole theoretical stance, we believe that modelling motion is a fertile ground where making experience with the variables space and time and with their *co-variation*. Understanding the concept of function is highly linked to that of co-variation (Slavit, 1997). Briefly speaking, we see modelling motion as an approach to the mathematics of change, which allows developing sense of the graph (Robutti, 2003) and knowledge of functional relations. The actions performed with one's own body and the direct interactions with the representations furnished by the tools may favour a way of learning that is multimodal. Through such an approach, children may acquire what we call a *sense of motion*, as a metaphor recalling the sense of graph, the number sense (Sowder, 1992) and the symbol sense (Arcavi, 1994), present in the literature. We could thus say: the sense of motion as a first step to gain sense of function. This also explains the choice of working at primary school, with children. Cognitive roots are possibly established at early stages of development and understanding. Later, they can be refined along with meanings closely associated to the formal face of mathematics, when Calculus comes to be the core of mathematical instruction. Again, modelling motion activities at secondary school on the study of the mathematics of change have been experimented, and their value explored, using both the CBR and the Motion Visualizer (Ferrara, 2009; Ferrara *et al.*, in print).

METHODOLOGICAL AND EXPERIMENTAL SETTINGS

According to what we said, the *mathematics laboratory* methodology, proposed for the curriculum in Italy, fits our teaching experiment (Anichini *et al.*, 2004). The mathematics laboratory is not meant as the traditional room, different from the classroom, with computers to be programmed. Nowadays, it is synonymous of a structured set of activities aimed at the construction of meanings for mathematical objects. We can imagine it as a Renaissance workshop where the apprentices learned by doing, seeing, and communicating with each other and with the experts. In the laboratory, the construction of meanings is strictly tied, on the one hand, to the tools used in the activities, and on the other, to the interactions among people that work together. In addition,

> the *laboratory* echoes the idea of labour, effort, zeal, the *lesson* echoes a treatment by the expert, an imparted teaching. The *laboratory* evokes a bodily and mind involvement; the *lesson* echoes an exclusively intellectual participation. The handmade labour carried out in the laboratory takes a long time" (Paola, 2007, p. 13, emphasis in the original, English translation by the authors).

Therefore, the image of the mathematics laboratory wraps all those situations in which the traditional lesson is modified by the introduction of specific artefacts and modelling activities. This is just the case of our activities, in which the children model motion through graphing.

In our methodological choices, we considered as a fundamental competence for the children the capacity of *argumenting*. They have always been used to explain "why", as a part of both their classroom culture and the teacher's practice. In the activities, the children were always asked to explain their reasoning, and in the structure of each worksheet, it is always possible to find a page at the end we entitled "*Space of reasoning*". Especially, the activities in the long period showed us that "the *written argument* allows children to walk a meta-cognitive path toward the awareness of their understanding. Since the first years of primary school, the written argument is a cross means of communication that fosters and organises the ways of comprehending and solving problems" (Ferrara & Savioli, 2009, p. 134, emphasis in the original, English translation by the authors).

In the international panorama, the Trends in International Mathematics and Science Study (TIMSS), which provides reliable and timely data on the mathematics and science achievement of U.S. 4th- and 8th-grade students compared to that of students in other countries, considers as fundamental, among the reasoning competences (one of the three cognitive domains where competences are assessed), the capacity of making conjectures, arguing and justifying results (for details, Mullis *et al*., 2009).

Math in motion. The experiment we are telling is called "Math in Motion", from the series of designed activities carried out with children. The name is also in line with the philosophy of our research, as we explained it above. In the activities, first a one-dimensional motion sensor (CBR), then software to gather motion in two-dimensions (Motion Visualizer DV) were used. They gave in real time graphical representations describing movements (always projected on the wall through additional means). The children interacted with the tools both for interpreting graphs related to given movements, and for the inverse task: checking movements associated to given graphs. So, there was a double passage: from motion to model and from model to motion. The presence of technological tools is not at all secondary. In fact, the tools are attractive for children that feel induced to the discovery of relationships between the phenomenon and the representations, almost magically arising from the devices. In addition, the children can see the effect of the movements they perform (moving their body, or an object) on the shape of the graphical representations. In a way, their actions are transparent in the corresponding graphs. In terms of the multimodal manner in which learning may occur, these characteristics of the technology we chose have a crucial function. The real time affection of an action on the curve, displayed on the wall, returns in a *perceptual-sensuos-imaginary* *readiness* to look for a sense of the graph. We argue that *this sense is not separated, but embodied, in the shape of the curve as it originates by a specific movement* (say, with a certain trajectory, pace, speed, acceleration). This is why we call it sense of motion or, better, mathematical sense of motion, and we believe it can be made with a high percentage of awareness. Finally, the use of technology provided mathematics with a new connotation: it showed that mathematics is not only made of algorithms, but it can also explain everyday phenomena. In the literature, it has been showed up that all the experiences in which students can interact with tools to create phenomena help them to understand the mathematics connected to those phenomena (Nemirovsky *et al.*, 1998; Nemirovsky & Monk, 2000).

Students and curriculum. Our activities involved a primary school classroom for four years, from 2006 to 2009. The school is a little school of a peripheral zone of Turin (Chieri), in the countryside. At the beginning of the experiment, the children were attending the second grade. At that time, the classroom group was made of 15 children (7 females and 8 males), and it modified during the successive year (3^{rd} grade), when a child changed school and a new one (that had to repeat the year) came. In the last year (5^{th} grade), again a student joined the group growing up to 16 children. A handicapped child took part in the whole experiment: it was our choice not to exclude him from the activities. The children were taking regular mathematics lessons two days per week, for a total amount of 8 weekly hours. Each school year, the activities were carried out in the second period (after February), for a mean of ten weekly meetings. We both took part in all the research phases: from the design to the planning of the project, to its implementation at school. During the activities, children were filmed using two video cameras, one mobile and the other fixed. The videos provided us the stuff to be analysed together with the collected written materials. The project adhered to National Indications for the primary school curriculum, although its non-standard nature. For example, regarding the targets for the development of competences reachable at the end of primary school, we read:

- “to use suitable data representations and to make use of them to get information in various situations”;
- “to learn to produce (even not formal) reasonings and to validate them, by means of laboratory activities, peer discussions and manipulation of the models created with mates” (this point is consistent with the methodology we discussed above).

Activities and technologies. We now discuss the organisation of the activities and the functioning of the technological devices used in the experiment. We left out no typology of work: there were individual and pair activities, small group works and classroom discussions. Since the first grade, the children were accompanied, in their mathematical way, by some fantasy characters, as the Wizard of Numbers, and by playful situations. To support them in discovering the links between the phenomenon and its representations, we thus had recourse to some stratagems. For examples, using the CBR we often spoke of animals or cartoon characters to distinguish velocities of motion. These velocities could be compared with those of the children's movements. In the case of the Motion Visualizer, we took a metaphor speaking of Movilandia, the Land of Motion, and Cartesiolandia, the Land of Descartes. These lands were created covering a wall with a big paper. The computer screen (where the software displayed graphs) was projected on one side of the paper; on the other side, movements were performed of coloured objects, like an orange glove or an orange ball on a stick (in fact, whereas the sensor works through sound, the software functions thanks to light). A paper on the wall was also used with the sensor. The blackboards were always available. The great difference between the CBR and the Motion Visualizer is that the former gathers motion in one direction, and the latter in two components (in respect of a plane), both showing spatial or temporal functions by choice. The CBR then provides a single graph, and the Motion Visualizer affords two graphs at once with reference to motion directions. Our choice was to work with position-time graphs in both cases. The technological settings seem to be rather complex but the potentialities the tools offer in visual terms are a lot. We will enter in details of how the devices run whether this will be necessary for the sake of the analysis.

DISCUSSION: THE CASE OF BENNY

In what follows, we focus on two specific episodes of the experiment relative to the behaviour of a child, Benny. For the sake of clarity, we take this child as an example to show the ways children understand and communicate about the graphs they work with. The multimodal productions we will see support our hypothesis that learning happens in a multimodal manner. The initial episode comes from a little piece of a classroom discussion on the shape of the position-time graph representing the first movement in front of the CBR. The second episode takes into account a written protocol of an activity involving the Motion Visualizer, and regarding the passage from a given graph to the corresponding movement.

The CBR and the first experience. The very first movement in front of the motion sensor arose from the request that a volunteer performed a free movement along a red band put on the floor. The CBR works in real time detecting for 15 seconds, each tenth of a second, the position of one that moves in its action cone between 0.5 to 6 meters (this is why the band was there). Benny came to move, and the rest of the group, seating on the floor, watched on the wall the creation of the position-time graph given by a calculator linked to the sensor. He walked back and forth covering the red band five times, ergo stopping at the band's end farthest from the CBR position. Benny's walk resulted in a graph with the shape of the "mountains", to use the word children adopted after a first surprise. Right away, the discussion aims at understanding the graph: "*How can we explain this drawing?*". Benny answers as follows (in Figure 1, the band is marked in red; the arrows indicate the gestures):

Benny: "While I arrived to the end [pointing to the band; Fig. 1 left], the thing [the CBR] arrived upward [tracing an increasing line in front of his torso: Fig. 1 centre]"

Figure 1: Benny's gestures

Francesca [researcher]: *"Can you also show me this there* [on the wall]*?"*

Benny: *"When* [gazing at and pointing to the band while walking to the wall], *assume this* [pointing to the right end of *t*-axis] *is the start. I go* [running *t*-axis to the left end], *I arrived here* [pointing to the left end] *and this piece came* [tracing the first ascent: Fig. 1 right]*"*

Benny's *gestures* serve to highlight first relations between the shape of the graph and his movement, which would be difficult to express only in words. At the beginning, Benny gesticulates in that space he can use for communicating: the extended bodily space he can reach in the surrounds of his body with his arms, what would be called by neuro-physiologists peripersonal space (Rizzolatti *et al.*, 1997). In the effort of imagining why the shape of the graph is the way it is, gestures and words are not well coordinated at once. The first gestures do not confuse motion and its representation: pointing conveys positions on the trajectory, while the trace in the bodily space mimes a piece of the graph. However, in words Benny still makes con*fusion* between the trajectory ("the end" of the band), the tool ("the thing") and the graph ("upward"). It is not only *fusion*, i.e. "merging qualities of symbols with qualities of the signified events or situations" (Nemirovsky *et al.*, 1998, p. 141). In this confusion, the CBR combines with the qualities of the graph and the qualities of the movement. In fact, it is what allows having the graph. So, when Benny says: "the thing arrived upward", he does not distinguish the role of the CBR and the representation of motion; he merges the two. But, as Benny repeats his reasoning in front of the wall something changes. His bodily space actually blends with the space of the representation (the paper on which the graph is projected). At this point, Benny introduces a *metaphor* ("assume") taking *the point of view of the tool*. He treats the horizontal axis of the representation (the axis of time, or *t*-axis) as if it were the motion trajectory, the red band. The metaphor supports him to recollect the first forth part of his movement along the band, by running the *t*-axis from right to left, exactly as the real movement occurred from the point of view of the children ("this is the start", "I go", "I arrived here"). Imagining the occurrence of this first fraction of the movement in the space of the representation (the same space where the graph appears) allows Benny to clearly explain the shape of the graph ("this piece came"). The previous confusion is overcome also in words, where distinct subjects are used to refer to the subject of motion ("I") and the graph ("this piece"). Gestures and words are well coordinated. In this way, Benny makes apparent the link between a moment of motion and the corresponding part of the mountains. The fact of being the one who moved is not at all a secondary aspect; it is what helps him, indeed. In such brief excerpt, we see the start of an evolution in understanding the relations between the physical experience (motion) and its abstract representation (the graph). The evolution happens through three phases: recollection of moments of motion, imagination of qualities of motion in relation to qualities of the graph, and interpretation of its shape.

The Motion Visualizer and the first worksheet. The activity we consider is the first individual worksheet on the Motion Visualizer. It followed the real time experience in which children discussed on the two graphs of $x(t)$ and $z(t)$ associated to an orange glove fixed in a generic (free) position (x, z) on the vertical plane xz (3^{rd} grade). The children knew x and z as Mister x and Mister z, since we told them a story before starting. The present worksheet was structured in three pages: in the first page, a specific (upper-left) position of the glove in Movilandia and the corresponding position-time graphs were given. In the second page, children were required to observe the new (lower-right) position of the glove in Movilandia and to represent what they would have seen in Cartesiolandia. In both cases, the software gives two straight lines as representations of $x(t)$ and $z(t)$. But, the lines are different for their position concerning a reference line that can be taken as the *centre* (in the representation). So, according to where, out of four possibilities (upper-left, upper-right, lower-left, lower-right), the glove is kept in Movilandia, in Cartesiolandia the two straight lines of $x(t)$ and $z(t)$ can be *above* and/or *below* the centre. The software uses as a reference for the representation the position of the glove with respect to the centre of Movilandia. Being on the right or on the left of the centre in Movilandia gives information on the x-position, while being above or below the centre refers to the z-position. But in Cartesiolandia, both x-position and z-position are graphically represented on the vertical axis. This is one of the most difficult issues children encountered in this activity. The final page of the worksheet contains the space of reasoning. In general, all the children solved the worksheet correctly, even if not all of them expressed their reasoning in a good shape. The correct solution is given by a straight line above the centre for $x(t)$, and a straight line below the centre for $z(t)$ (lower-right position of the glove in Movilandia). Benny proposed the following argument (his words in italics; the recalled sketches in Fig. 2):

> *"At page 2, I completed like that because in the "space" of M.x* [Mister x]*, placed this way* [sketch 1, Fig. 2]*, pretend that you turn it this way* [sketch 2, Fig. 2]*, pretend that you place the glove where it was before, that is placed this way* [sketch 3, Fig. 2]*, then I turn it again this way* [sketch 4, Fig. 2]*. Hence the line has to be placed this way* [sketch 5, Fig. 2] *and, in the other little scheme, only I don't turn it, but I move it* [the glove] *away and it* [the graph] *comes this way* [sketch 6, Fig. 2]*".*

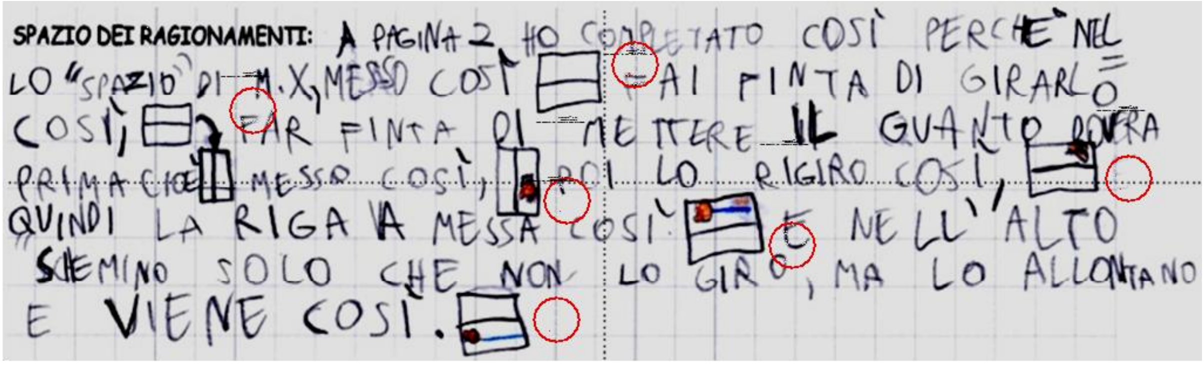

Figure 2: Benny's argument and his sketches

Benny's argument is an example of a multimodal effort of explaining the reasoning followed to solve the task. Benny uses a short explanation, in which many sketches intertwine with words. He has to write thoughts; he cannot gesticulate. Nonetheless, he needs some form of representing things he is not able to express in words ("this way" many times). As a result, he draws a lot. Some of the sketches are like *gestures crystallized* on paper (sketches 2, 3 and 4). It is like he imagines making the gestures while writing. The space of reasoning is at the moment the *imaginative* space where gestures become sketches. Benny's peripersonal space embodies the piece of paper. This makes very clever the way Benny distinguishes the variables x and z. He sees they have similar representations but they do not behave equally. Having understood that position is always displayed in Cartesiolandia on a vertical axis, Benny finds a tactic to explain why the graph of $x(t)$ is given by a straight line that he drew above the centre (there is no problem for $z(t)$). Again, he uses a *metaphor*: he thinks of the x-position in Cartesiolandia as if it were the same x-position in Movilandia just rotated clockwise of 90 degrees ("turn it this way"). We speak of a metaphor because he uses the word "pretend". He repeats the rotation twice, referring first to the passage from Cartesiolandia to Movilandia ("pretend that you turn it this way"), and then to the inverse passage ("pretend that you place the glove where it was before", "then I turn it again this way"). A remarkable feature of Benny's argument is also the fact that, in its shortness, it is essential and clear. Each sketch just contains necessary elements, nothing more. Sketches 1 and 2 show only the centre, the crucial factor to introduce the rotation image. Sketches 3 and 4 introduce the glove, key to link the two worlds. Sketches 5 and 6 close the argument ("hence") revealing the positions of the lines.

CONCLUSIVE REMARKS

The discussion of the two episodes highlights the ways a child, Benny, makes sense of position-time graphs related to movements. The cases are different from each other for many reasons: used technological devices, experiential knowledge (larger in the second case), complexity of the involved mathematics, type of activities, and times. Though, they disclose that understanding processes develop through many resources, more than oral and written words. Additional forms of representations are used to integrate what words are not sufficient to explain. Benny needs to express images in the effort of communicating his explanations: gestures and sketches then appear in his oral and written productions. We believe this is a striking feature of a multimodal way of learning. We also think that the kind of activities children have been involved in, and the use of technological tools affected their ways of learning, fostering its having a multimodal character. Based on the neuroscientific results on the relevance of perceptuo-sensory-motor (we would add imaginary) experiences with respect to thinking processes, we claim that it is possible through suitable activities to activate senses for readiness to construct meanings. Regarding the experiment we presented, suitable activities were prompted by the technology that allowed grounding the sense of the function on a sense of motion. Having motion at the foundation of the concept of function also gives a means to recover the historical path that brought to the concept itself. Finally, it may be meaningful to design experiments where children, students in general, discover and recover relations between the mathematics of change and kinematics.

REFERENCES

Anichini, G., Arzarello, F., Ciarrapico, L. & Robutti, O. (A cura di): 2004, *Matematica 2003. La matematica per il cittadino. Attività didattiche e prove di verifica per un nuovo curricolo di Matematica (Ciclo secondario)*, Matteoni stampatore, Lucca.

Mullis, I.V.S., Martin, M.O., Ruddock, G.J., O'Sullivan, C.Y. & *Preuschoff, C.* (Eds.): 2009, *TIMSS 2011 Assessment Frameworks. Chestnut Hill*, TIMSS & PIRLS International Study Center, Boston College, MA.

Arcavi, A.: 1994, Symbol sense: informal sense-making in formal mathematics, *For the Learning of Mathematics*, **14**(3), pp. 24-35.

Arzarello, F.: 2006, Semiosis as a multimodal process, *Revista Latino Americana de Investigacion en Matematica Educativa*, vol. Especial, pp. 267-299.

Edwards, C.H.: 1979, *The historical development of the Calculus*, Springer-Verlag, New York.

Ferrara, F.: 2009, Sonar e grafici di movimento per risvegliare le radici del concetto di funzione, *La Matematica e la sua Didattica*, **23**(2), pp. 216-223.

Ferrara, F. & Savioli, K.: 2009, Il guanto magico, Mr. x e Mr. z, nel mondo di Cartesiolandia e di Movilandia. Tra movimento e grafici nella scuola primaria, in O. Robutti & M. Mosca (A cura di), *Atti del III Convegno Nazionale di Didattica della Fisica e della Matematica DI.FI.MA. 2007. Curriculum e successo formativo in matematica e fisica: proposte, esperienze, problemi*, Grafiche Ambert. Verolengo, Torino, pp. 132-146.

Ferrara, F., Laiolo, P., Paola, D. & Savioli, K.: in print, Movimento, visualizzazione e costruzione di significato nella scuola secondaria, *L'insegnamento della Matematica e delle Scienze Integrate*.

Gallese, V.: 2007, Embodied simulation: from mirror neuron systems to interpersonal relations, *Novartis Foundation Symposium*, **278**, pp. 3-12.

Gallese, V. & Lakoff, G.: 2005, The brain's concepts: The role of the sensory-motor system in conceptual knowledge, *Cognitive Neuropsychology*, **22**(3/4), pp. 455-479.

Lakoff, G. & Núñez, R.: 2000, *Where Mathematics comes from: How the embodied mind brings Mathematics into being*, Basic Books, New York.

Monk, S. & Nemirovsky, R.: 2000, "If you look at it the other way…": An exploration into the nature of symbolizing, in P. Cobb, E. Yachel & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms. Perspectives on Discourse, Tools, and Instructional Design*, Lawrence Erlbaum Associates, Mahwah, NJ, London, pp. 177-221.

Nemirovsky, R.: 2003, Three conjectures concerning the relationship between body activity and understanding Mathematics, in N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1). University of Hawai'i, Honolulu, HI, pp. 105-109.

Nemirovsky, R., Tierney, C. & Wright, T.: 1998, Body motion and graphing, *Cognition and Instruction*, **16**(2), pp. 119-172.

Paola, D.: 2007, Dal laboratorio alla lezione: descrizione di un esempio, *Innovazione Educativa, Supplemento per l'Emilia Romagna*, **8**, pp. 13-20. TECNODID, Napoli.

Rizzolatti, G., Fadiga, L., Fogassi, L. & Gallese, V.: 1997, The space around us, *Science*, **277**, pp. 190-191.

Robutti, O.: 2003, Il senso del grafico con la mediazione delle tecnologie: metafore attivate e significati costruiti, *La matematica e la sua didattica*, **2**, pp. 173-195.

Slavit, D.: 1997, An alternate route to reification of function, *Educational Studies in Mathematics*, **33**, pp. 259-281.

Sowder, J.T.: 1992, Estimation and number sense, in D.A. Grows (Ed.), *Handbook of research on mathematics teaching and learning*, Macmillan Publishing Company, New York, pp. 371-389.

Struik, D.J. (Ed.): 1986, *A sourcebook in Mathematics, 1200-1800*, Princeton University Press, Princeton, NJ.

Tall, D.: 2000, Biological Brain, Mathematical Mind & Computational Computers (how the computer can support mathematical thinking and learning), in Y. Wei-Chi, C. Sung-Chi, C. Jen-Chung (Eds.), *Proc. 5th Asian Technology Conference in Mathematics*. ATCM Inc, Blackwood VA, Chiang Mai, Thailand, pp. 3-20.

Wilson, M.: 2002, Six views of embodied cognition, *Psychonomic Bulletin & Review,* **9**(4), pp. 625-636.

NUMBER BANKNOTES IN CHILDREN'S ACTIVITY

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Children of pre-school age have a certain command of natural numbers, which reveals itself in various situations. In particular, they use numbers when they think about money and shopping. We should make use of this in teaching children mathematics.

The infinite sequence of natural numbers is the quintessence of mathematics and the dominating structure of the system of natural numbers is the decimal positional notation, which is indispensable in all situations, from gaining a merely linguistic grasp of numerals to mastering operations on numbers in practice (Freudenthal, 1987).

Contemporary studies revealed the fact that "children at every stage are intellectually much more sophisticated than Piaget believed" (Zimbardo, 1999). Notions correlated with the number appear in their utterances quite early. The development of these notions is stimulated by circumstantial learning, consisting in skilfully taking advantage of every opportunity to inform a child about something he or she notices but is not well acquainted with or does not fully understand or is unable to explain by themselves (Szuman, 1985). The circumstances particularly beneficial to the development of the notion of the number are the situations connected with using money.

Banknotes and coins have different denominations, which are denoted by numbers. These are selected numbers (1, 2, 5) and their products by powers of ten. This enables comparing banknotes with respect to their values on various levels of abstraction.

Episode 1

Jaś (3 years and 1 month old) has a sister, Hania, who is a few years older. Hania is going away for her holidays.

Jaś asks – How much money are you giving to Hania for the road, mummy?

Mum – And how much does Hania want?

J – One hundred and fifty zloty.

Jaś takes out of his mother's purse 5 grosz coin and gives it to Hania.

J – Here's one hundred and fifty zloty.

Jaś recognises just the digit 5 and associates it with numbers he which he has heard of and which contain the sound "five".

Episode 2

Jaś's mum (3:5) is wearing a new necklace of beads.

Jaś – Wow, what beautiful beads! How much does it cost?

Mum – These are Czech beads.

J – Hundred thousand. These are Czech. Mum bought them in a shop. In a supermarket.

Among the numbers Jaś has heard or he distinguishes the "great" ones. Something which is "beautiful" must have cost a lot, this is why he used the number hundred thousand.

Episode 3

Ula (4 years and 6 months old) examines the banknotes her parents brought home on their payday (Previously she used to name them by the pictures visible on banknotes: Waryński, Dąbrowski, Kościuszko, Copernicus). She can tell banknotes apart and knows which ones have greater value.

Ula – How much is Chopin?

Mum – Five thousand.

U – And this one (Copernicus)?

M – One thousand.

U – And if there is 3 here and three zeros, will it be three thousand? And who is drawn on the three thousand? Is there such money?

Before now, Ula was able to rank banknotes with respect to their value distinguishing between them by pictures printed on them. Now she has become interested in numbers written on banknotes with the aid of digits. (At that time in Poland the following banknotes nominations were used: 10, 20, 50, 100, 200, 500, 1000, 2000, 5000 zloty**.**) She finds out that the symbol "5000" means five thousand and "1000" is one thousand. She advances a hypothesis that there exists a number written as "3000" and it should be "called" three thousand. Similarly, she would like to associate this number with some picture on a banknote but, since she has never seen such a thing, she has some doubts about the existence of such a banknote (but not about the existence of the number). Banknotes (numbers) with three zeros and some other number in front of them are called thousands. She knows that there is as many thousands as the first number indicates.

Episode 4	Episode 5
Jaś's (4:9) grandma brought a certain amount of money from the bank. Jaś follows his mum and grandma around the house making himself tiresome by asking. J – Lend me some money, grandma, I'm gonna give it back. Mum, lend me some money. I will give it back. Finally, mum gives him a new 5 zloty note and a 2 zloty coin. [...] Jaś is having a close look at the money his mother has given him and then goes to his grandma and asks in the tone of request. J – Will you lend me this one hundred zloty from the bank? I'll stop being a nuisance, do lend me this money! If you say you will I'll stop being a nuisance. Will you lend me a hundred zloty? Will you? M – Money is not a toy. [...]	Jaś (4:10) is writing numbers on a sheet of paper. Jaś – Mum, how much is it if there is a two first and then a one? Mum – Twenty one. J – And if there is a five and a one and once more a five and a one, how much is it then? M – Five thousand one hundred and fifty one. J – And three eights and a one? M – Eight thousand eight hundred and eighty one. J – No, but I've made a mistake, because I wanted to say four eights. M – Then it's eight thousand eight hundred and eighty eight.

Fascinated with large numbers in their digit notation, Jaś searches for some information on this subject. This is the reason why he repeatedly "borrows" money, scrutinises banknotes and coins and gives them back.

Episode 6
Ula (4:11) dreams about an aquarium with fish. Mum – On Friday we will go and buy an aquarium and some fish. Have you got lots of money? Ula – Yes, 500 from grandpa. M – And if they cost 1000? U – Then you will give me another 500. M – And if they cost 2000? U – Then daddy will give us another 500 and we'll still have to find 500 somewhere.

From her experience Ula knows that a 2000-zloty note may be substituted by two 1000-zloty banknotes and a 1000-zloty banknote may be replaced with two 500-zloty notes (doubling numbers and money values is an operation initially preferred by Ula - and other children, as well). In the above described situation Ula uses imaginary banknotes in just the same way symbols of written numbers would be used.

Episode 7

Jaś (4:11) is drawing columns of digits.

Jaś – Are coats and shoes expensive?

Mum – Yes.

J – And how much do they cost?

M – Shoes cost two hundred to one thousand zloty and coats one thousand to four thousand.

J – Five hundred and five hundred is one thousand?

M – Yes.

J – Give me half a thousand, five hundred!

Jaś asks about expensive things to hear large numbers, he also looks for relations between them: “five hundred” and “five hundred” is one thousand, so one “five hundred” is a half of a thousand. He has not yet begun to read numbers emerging from the digits written by him.

Episode 8

Jaś (4:11) – What number is it two and two zeros?

Mum – Two hundred.

J – And a three and two zeros?

Mum – Three hundred.

J – And a five and two zeros?

Mum – Five hundred.

J – And a six and two zeros?

Mum – Six hundred.

200

300
500
6000

Episode 9

Ula (4:11) is writing some numbers. All of a sudden, she asks - How much is it a two and a zero?

Mum – Twenty.

U – And a two and two zeros?

M – Two hundred.

U – And a two and three zeros?

M – Two thousand.

U – And a two and four zeros?

M – Twenty thousand.

U – And a two and ten zeros?

M – Twenty billion.

U – And a two and eleven zeros.

M – I don't know now; there are some more names, but later on, when there are lots of zeros, they are not used any more.

U – And when you write, for example, 2 and lots and lots and lots of zeros, is it also a number?

M – Yes.

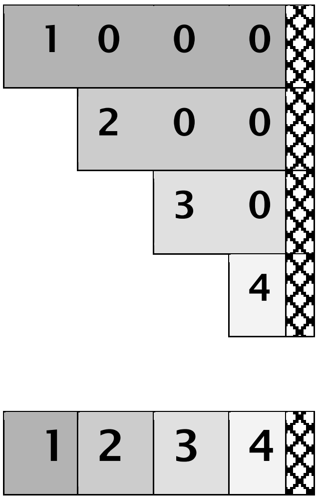
U – Grandma, I'll write you such a number that you won't be able to read. (With a gesture she writes 2 and six zeros). Grandma interrupts her – I can't read it even now.

U – Huh, but it's only six zeros!

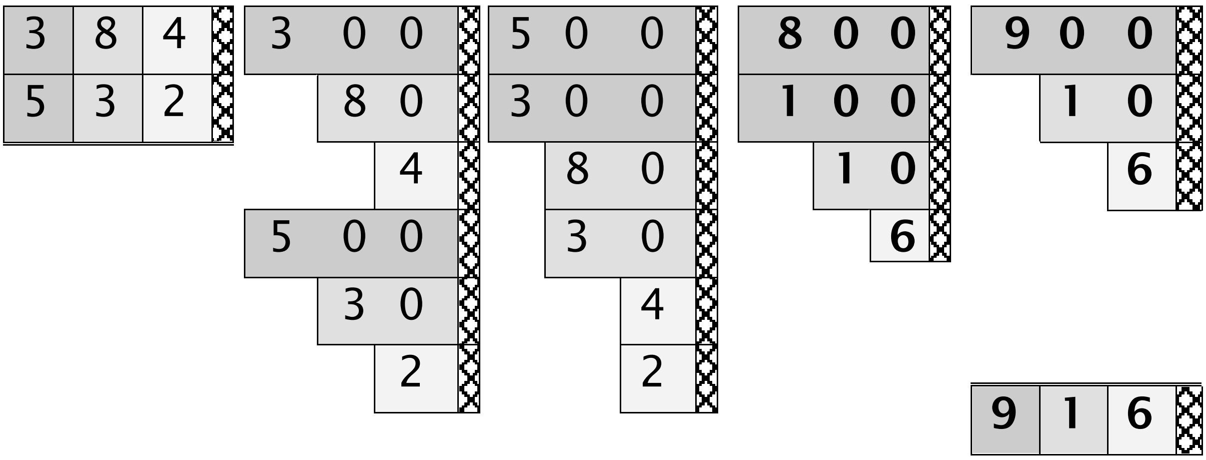
20
200
2000
20000000000
200000000000
200000000000...

"Whole" denominations of banknotes provoke children to produce numbers-inscriptions of digits by themselves. Children do it methodically. Jaś verifies the existence of the number consisting of two and two zeros; next, he changes the first digit leaving two zeros following it; then he adds the third zero and realises that all his propositions of symbols denote some numbers. Ula demands reading out the numbers 2, 20, 200, 2000, 20000. Afterwards, she predicts that the symbol made up of numerous zeros preceded by two denotes some number, as well.

Consciously and consistently, the children put forward hypotheses and modify them in accordance with the information obtained. In this way they seek rules of writing numbers with the use of digits, including numbers they have never seen written. After reading and reconstructing in their imagination numerical symbols, there comes the time for creating them independently by generalising the previously noticed regularities.



The process of assimilating numbers together with their symbolical notation in the decimal system by a child was significantly influenced by the fact that a child as well as its environment use money. This is what suggested the idea of creating a teaching aid – “number banknotes”. The “banknotes” are of denominations expressed by numbers 1, 2, 3, 4, 5, 6, 7, 9 and their products by 10, 100, 1000, ... and differ by their length, which depends on the denomination. Every amount between 1 and 999 may be “paid out” with the banknotes 1, 2,..., 10, 20, ..., 100, 200, ..., 900; they may be added and subtracted, as well.



For example, in order to add 384 and 532 it suffices to “pay out” both amounts in “number banknotes” (300, 80, 4 and 500, 30, 2), arrange them according to their values (500, 300, 80, 30, 4, 2), substitute one banknote for the banknotes of the same order: first 800, 100, 10, 6, and, finally, 900, 10, 6. During the process of performing this operation “banknotes” should be placed one under another and in the end they should be arranged in one pile, with the greatest denomination at the bottom. In this way, the correctly written result 916 is obtained.

For a child, money is a well-known manageable context in which large numbers are immersed. It is easy to read them from the banknotes, since their name depends solely on the first digit and the lengths of the “tail” made up of zeros only. Comparing, adding and subtracting is not difficult, because these activities are preceded by the experience in using money in spontaneous play as well as in reality.

REFERENCES

Freudenthal, H.: 1987, Co znaczą struktury naukowe i struktura nauki w rozwoju poznawczym i w nauczaniu, *Dydaktyka Matematyki* **7**, PWN, Warszawa, pp. 27 - 42.

Piaget, J.: 1966, *Studia z psychologii dziecka*, PWN, Warszawa.

Piaget, J., Szemińska, A.: 1969, Genezis czisła u rebionka, in: Piaget, J.: *Izbrannyje psichołogiczeskije trudy*, Proswieszczenije, pp. 233-565.

Smoczyńska, M.: 1998, Krakowskie dane językowe dzieci w systemie CHILDES, in: Smoczyńska, M. (red.), *Studia z psychologii rozwojowej i psycholingwistyki*, Universitas, Kraków, pp. 283-296.

Szuman, S.: 1985, *Podstawy rozwoju i wychowania w ontogenezie*, WSiP, Warszawa.

Urbańska, A.: 1996, O aktywności matematycznej dziecka przedszkolnego - na przykładzie kształtowania pojęcia liczby, in: *Problemy Studiów Nauczycielskich* **6**, Wydawnictwo Naukowe WSP, Kraków.

Zimbardo, P.G.: 1999, *Psychologia i życie*, PWN, Warszawa.

Jaś's utterances come from the so called *Szuman Speech Registers* (*Szumanowskie dzienniki mowy*), which are stored in a typewritten form in the Jagiellonian University Institute Department of Developmental Psychology. In the last few years Magdalena Smoczyńska prepared the Registers in the electronic form, according to the international format CHAT within the framework of CHILDES.

I would like to express my gratitude to Ms Magdalena Smoczyńska for giving me permission to use them in this form.

Different approaches to organizing the learning process

Part 3

YOUNG CHILDREN'S ORGANIZATION AND UNDERSTANDING OF DATA IN EVERYDAY MATHEMATICS SITUATIONS

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Democritus University of Thrace, Greece

This study examined how young children organize and interpret data that come from real-life situations presented graphically. Fifty Kindergarten and fifty Year 1 children were asked to construct and interpret picture graphs and symbolic graphs with two and four bars. Their performance in constructing the graphs was not affected neither by the age nor by the type of graphs. Differences in interpreting the data in the graphs were observed with the oldest children showing more complex reasoning. Similar reasoning across age groups was found in the different types of graphs. Educational implications are discussed in accordance to the need of using graphs as a tool to help children solve the range of mathematics problem they could encounter.

INTRODUCTION

In children's everyday lives and before they start school, they have experiences of comparing situations and talking about them. For example, a group of children can decide among two or three alternative games which one to play: they collect information and follow the option taken by the majority of the children. Later on at school, they learn to record these experiences and present them graphically using a range of pictures and diagrams. They make predictions, make up questions, read information from graphs and summarize results; in other words, they process statistical data.

Conscious of the need to provide materials which enable students to recognize, improve, help and develop their abilities in solving math problems, school mathematics curricula in different countries promote the use of diagrams (e.g., National Council of Teachers of Mathematics in the US, 2000; DES in England and Wales, 1989; National Curriculum published by the Greek Ministry of Education in Greece, 2001). The collection and representation of information is an established part of work in primary schools, where block graphs, pictograms and diagrams are commonly on display. In the last years, mathematics education focuses on the importance of learning tasks relating to real-world situations, mainly coming from children's everyday life. The challenge is to produce a range of realistic activities and questions which will allow the early development of all the statistical work considered appropriate. In other words, the purpose is to 'propose and justify conclusions and predictions that are based on data' (NCTM 2000, p. 176).

In fact, handling data in reading a diagram is described as one of the most effective strategy to improve efficacy in solving mathematics problems. For example, Hembree (1992) suggested that using diagrams was the most efficient among strategies that had been suggested as helpful for problem solving. This idea that statistics is taught and learned more effectively in the context of projects that explore issues and is used as a helpful heuristics rather than as a set of skills and processes is common in school mathematics curricula. Furthermore, Robinson and Kiewra (1995) found that students learned more and could apply that learning more if they had been presented with text that included graphic organizers compared with students given text outlines. The reasons why diagrammatic representations are so effective are firstly discussed by Larkin and Simon (1987): it allows a large number of perceptual inferences, minimizes labels, and obviates the need to search for problem-solving inferences.

Several studies (e.g., Mevarech, & Kramarsky, 1997, Meira, 1998, Fennell & Rowan, 2001) investigated the importance of symbolic representations in mathematics conceptualization as pedagogical issue. For example, Selva, Da Rocha Falcao and Nunes (2005) proposed a didactic sequence to six- and seven-year-old children for solving additive-structure problems, based on graphical representation of quantities derived from three-dimensional histograms built with Lego-blocks. Their results suggested that the graphics used were a representational tool and provided great support for dealing with quantities and their relations.

Although graphics are considered powerful heuristics in problem solving situations, graph comprehension and critical sense in graphing remains an issue (see Friel, Curcio, & Bright, 2001, for a detailed discussion of critical factors that influence graph comprehension). The pedagogical approach behind children's use and understanding of graphics is presented in Nisbet, Jones, Thornton, Langrall and Mooney's study (2003), who examined how Grades 1 to 5 children –who already had taken instruction in data use- represented and organized categorical (school transport) and numerical (pet fish) data. Children had to represent the data given by drawing graphs which then compared with alternative graphs provided by the researchers. Their results showed that younger children were more idiosyncratic and incomplete in their thinking, whereas children beyond Grade 1 were good at making connections between different aspects of the data, at least for the categorical data.

Despite the official inclusion of teaching graphic skills as a topic in the mathematics curriculum, children's preference for graphing activities has been restricted. This issue is very complex because is associated with several factors, among those we can highlight students' perceptions as well as the role of the teacher. For example, Uesaka, Manalo and Ichikawa (2007) reported that many secondary school students from Japan and New Zealand do not share the positive view of diagram use that teachers and researchers tend to hold. As they suggest, this lack of spontaneous use could be due to students not knowing how to construct diagrams and not appreciating the potential benefits of using diagrams. From the teachers' point of view, their role is very important as they have to construct their teaching context in terms of meaningfulness and purposefulness for the participants (Ainley, 2000). It is great scope for the teacher to exploit the children's own direct experience in the questions posed and the data collected as well as the tasks to have some purpose from the child's perspective. In their study with student teachers, Monteiro and Ainley (2004) revealed that this is very hard to be gained. An excellent example, however, of working with third graders in a project about the most popular main dishes among children is reported by Manchester (2002). The project involved children in figuring out how to design the study and offered them the opportunities for learning how to organize information logically and how to present it accurately in graphs. Manchester's worth of the study comes from posing a question that is meaningful to the children and provides emphasis to the importance of children's working on an area of their interest (Ainley, 2000).

The main aim of the present study was to examine how young children beyond school age organize and interpret data that come from everyday situations and are presented in particular kinds of graphs. The research questions were: a) Is children's way of organizing and interpreting data different according to age? b) Is children's performance affected by the type of graphs? c) What are children's strategies when interpreting data?

METHOD

Participants

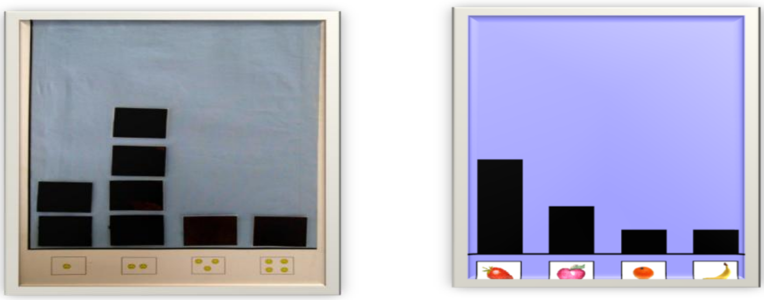
The study was conducted among 50 kindergarten students (mean age 5 years 4 months) and 50 Year 1 students (mean age 6 years 6 months) who were randomly selected from urban and rural schools in Northern Greece covering a wide range of social backgrounds. Participants had not taken any typical instruction on data exploration at school.

Design

Participants were interviewed individually by the researcher in a quiet area within their school and their answers were audio-taped. An introductory trial was presented in order to confirm children's understanding of the tasks demands. All children were asked to carry out five tasks, two related to picture graphs (with two and four bars), two related to symbolic graphs (with two and four bars) and one related to bar graph (with four bars). The particular types of graphs were chosen in the study following the suggested way of teaching graphs at primary school.

For each task an L-shape framework was given, a story was presented and follow-up questions were asked. The stories presented referred to where a group of children live (house or flat), how they go to school (feet, car, bike, bus), what pets they have (dog, cat), how many kids they are in their families (one, two, three, four) and what fruit they like most (strawberry, apple, orange, banana). Children were asked to construct picture graphs using pictures that represented the data, and symbolic graphs using black cards, for the first two tasks and for the third and fourth tasks, accordingly. A bar graph was also presented to the children who only had to interpret it. Finally, children were probed to interpret the data in all graphs with simple questions like ‘Can you tell something about the kids that have a dog and those that have a cat?’. The procedure lasted for roughly 20 minutes. Figure 1 presents examples of the graphs children dealt with for each task.

Picture graph – two bars   Picture graph – four bars   Symbolic graph – two bars



Symbolic graph – four bars   Bar graph – four bars

Figure 1: Examples of the tasks

RESULTS

Children’s overall performance

Regarding the first four tasks, results showed that kindergarten children and Year 1 children had similar high performance (82% and 83%, accordingly) when they constructed the graphs ($\chi^2_{(4)}$=1.835, p=.766). Figure 2 presents the percentage of children’s correct responses according to age.

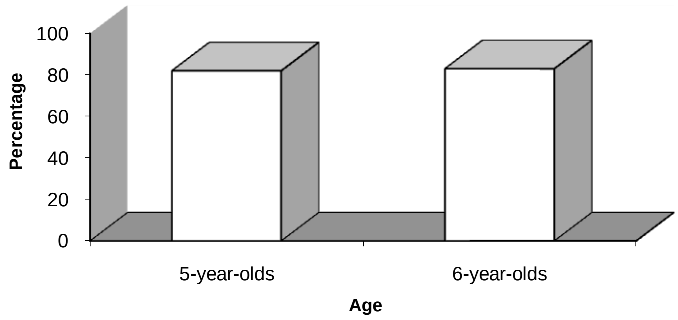


Figure 2: Percentage of correct responses by age

Types of graphs

In order to examine whether children's performance is the same across different types of graphs when organizing data, a comparison was made between their performance in picture graphs and in symbolic graphs. Significant differences were not found for all children (t=.191, df=49, p=.850) neither for each age group separately (t=.054, df=24, p=.983 and t=.327, df=24, p=.746 for 5- and 6-year-olds), indicating that the more complex type of graph –such as the symbolic graph– did not pose great difficulties to young children. This is illustrated in Figure 3.

A comparison between children's performance in picture graphs and symbolic graphs depending on whether those involved two or four bars was also conducted in order to test whether the number of bars affected children's organization of data. Not significant differences were found for all children (t=-1.632, df=49, p=.109). When this comparison was made separately for each age group, however, it was found that only the 6 years old children performed significantly better on 4-bars graphs compared to 2-bars graphs (t=-2.823, df=24, p<.01), while the number of bars in the graphs did not affect the 5 years old children's performance (t=.811, df=24, p=.425). These differences are presented separately for each age group in Figure 4.

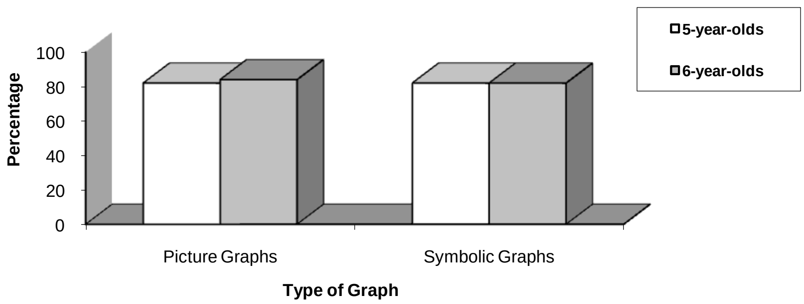


Figure 3: Percentage of correct responses by type of graph and age

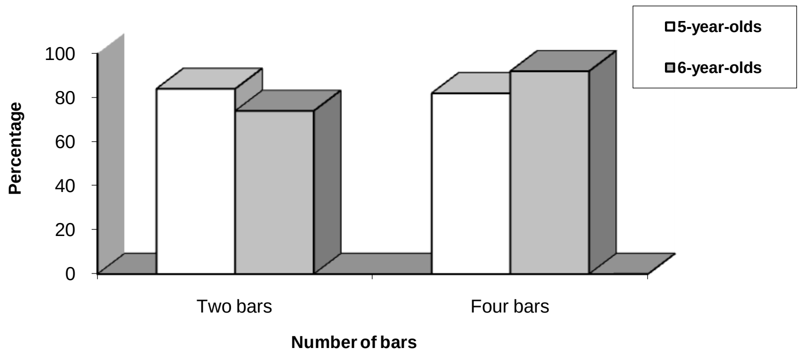


Figure 4: Percentage of correct responses by number of bars in graphs and age

Children's interpretation of data

All children were asked to interpret the graphs they constructed independently of whether or not they constructed them correctly. If children are able to talk about the information presented in the graphs, their reasoning would be reflected in the justification they give when interpreting them. Children's justifications were classified into the following nine distinct categories:

1. Other (idiosyncratic responses) - No interpretation
2. Falsification of the facts in the story - No interpretation
3. Repeating part or the whole story – No interpretation
4. Considering one term in the comparison – mainly using 'the more'
5. Considering two terms in the comparison – using 'the more' and 'the less'
6. Telling the story and considering one term in the comparison
7. Telling the story and considering two terms in the comparison
8. Considering three terms in the comparison when the task allowed
9. Telling the story and considering three terms in the comparison when the task allowed.

Significant age differences were observed between children's use of justifications in all types of graphs ($\chi^2_{(6)}$ = 23,000, p<.01) showing that older children interpreted the graphs in a different way than the younger ones (see Figure 5). More specifically, both age groups showed great preference for Justification 4 (concentrating on 'the more'). For example, Konstantina (5 years, 7 months) said: 'strawberries are liked more' when interpreting the bar graph with four bars. Whereas 58% of the youngest children mainly preferred this justification, the six-year-olds interpreted the graphs using a variety of justifications (only 33% used Justification 4). A high percentage of idiosyncratic responses were given by the 5- and 6- years old children (almost 25% and 17% accordingly), as well as responses that led to no interpretation. It is also interesting that none of the kindergarten children gave Justifications 7 and 9 which involved telling the story as well as considering the terms in the comparison. About 10% of the Year 1 children did prefer these justifications, showing that the 6-year-olds could manage to read the information presented in the graphs and make connections between the data being measured and frequency (e.g. type of transport and number of children) compared to their counterparts. Associations between types of justifications and age were also found when children's responses were analyzed separately for each type of graphs, showing the 6 years old children's greater readiness in interpreting data.

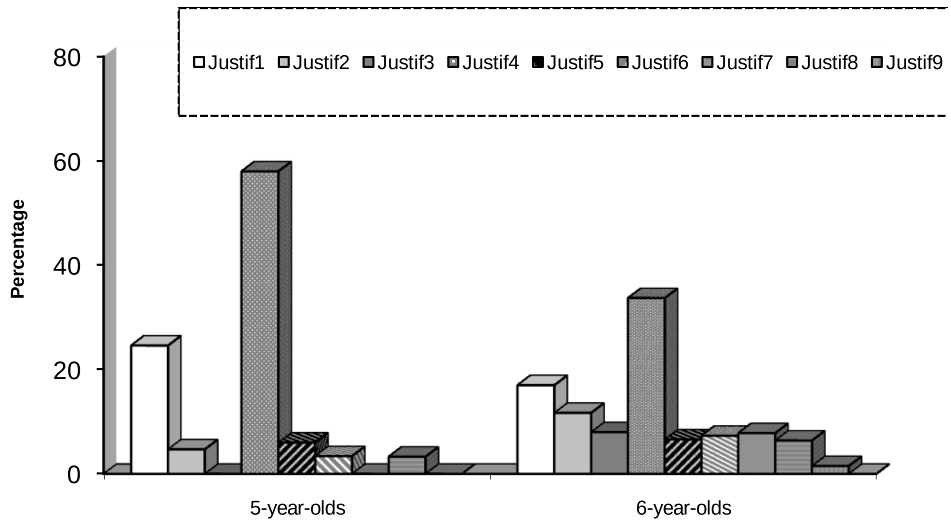


Figure 5: Percentage of responses in each type of justification by age

CONCLUSIONS

The present study was exploratory in viewing very young children's – such as kindergarteners' – first attempts in organizing and interpreting data presented graphically. The results suggest that children from early years dispose an important ability related to organizing data and talking about them when presented with simple forms of graphs, such as picture graphs and symbolic graphs. The high percentage of correct performance shown by the 5- and 6-years old children might result from the fact that familiar everyday situations demanding the use of graphs were presented. This finding allows to say that, although graphs have recently become an important part of primary school mathematics curriculum in Greece and other countries, children's introduction to graphing and data could start earlier on even from the kindergarten.

Consistent with previous findings (Nisbet et al., 2003) the analysis of children's responses revealed that children's readiness when interpreting the data in the graphs comes with age. More specifically, the six-year olds showed a more complex way of reasoning about data compared to the 5-year-olds: whereas the older ones were able to make connections between different aspects of data (such as type of transport and frequency) and compare frequencies, the youngest children exhibited more incomplete responses mainly concentrating on one aspect of the data (e.g., the choice with the greatest frequency). Overall, children have some ideas about reading graphs before they are taught and these ideas should be considered when one designs instruction about graph reading.

The findings also suggest that activities like those presented in the study promote mathematical language, starting with the articulation of a question to be investigated and concluding with the implications drawn from the data and the graph. For example, the representation of the data made relations such as ‘more than’, ‘less than’, ‘best’, ‘least’ transparent and it was found that the majority of children used them to a great extent. Furthermore, the majority of children – at least the 6-year-olds - gave reasonable responses in the stories presented to them. More research is needed, however, in order to look deeply to the predictions that children are able to make from the beginning as well as how the interpretation of the data matches their predictions. It would be interesting to give children the opportunity to explore questions that involve ‘going beyond the data’ (Pereira-Mendoza, 1995, p.6).

The real-life situations used in this study illustrate the presence of some spontaneous examples as learning opportunities for the very young children, both mathematically and pedagogically. One important limitation, however, of the present study is that the construction of the graphs as well as the interpretation of the data did not come as a part of a project arising from children’s interests and needs as it happened in the Lunchroom project (Manchester, 2002). It would also be interesting to examine in the further work whether a change in the interpretation styles of children occurs when other types of graphs are constructed, such as pie graphs or histograms.

The analysis though of the nature and conditions of these learning opportunities can shed light on ways which very young children’s craft knowledge about organizing and understanding data in graphs develops. It would therefore be useful to promote young children’s skills in using graphs in order these to be used as a tool for conceptualizing mathematics problems (Larkin, & Simon, 1987, Selva et al., 2005, Ainley, 2000).

REFERENCES

Ainley, J.: 2000, Constructing purposeful mathematical activity in primary classrooms, in: C. Tikly & A. Wolf (Eds.), *The maths we need now: Demands, deficits and remedies*, London: Institute of Education, Bedford Way Papers, University of London, pp. 138-153.

Department for Education and Skills: 1989, *Mathematics in the National Curriculum*. London: HMSO.

Fennell, F. & Rowan, T.: 2001. Representation: An important process for teaching and learning mathematics, *Teaching Children Mathematics*, **5**, pp. 288-292.

Friel, S.N., Curcio, F.R. & Bright, G.W.: 2001, Making sense of graphs: Critical factors influencing comprehension and instructional implications, *Journal for Research in Mathematics Education,* **32(2),** pp. 124-158.

Greek Ministry of Education & Pedagogical Institute: 2001, *National Curriculum of Greece*.

Hembree, R.: 1992, Experiments and relational studies in problem-solving – a metaanalysis, *Journal for Research in Mathematics Education,* **23**, pp. 242-273.

Larkin, J.H. & Simon, H.A.: 1987, Why a diagram is sometimes worth ten-thousand words, *Cognitive Science,* **11**, pp. 65-99.

Manchester, P.: 2002, The lunchroom project: A long-term investigative study, *Teaching Children Mathematics,* **9(1)**, pp. 43-47.

Meira, L.: 1998, Making sense of instructional devices: The emergence of transparency in the mathematical activity, *Journal for Research in Mathematics Education,* **29(2)**, pp. 121-142.

Mevarech, Z. R. & Kramarsky, B.: 1997, From verbal descriptions to graphic representations: Stability and change in students' alternative conceptions, *Educational Studies in Mathematics,* **32**, pp. 229-263.

Monteiro, C. Ainley, J.: 2004, Developing critical sense in graphing, *Proceedings of the 3rd Conference of the European Society for Research in Mathematics Education* (CERME3), Thematic Working Group 5: Stochastic Thinking. Bellaria, Italy.

National Council of Teachers of Mathematics: 2000, *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council.

Nisbet, S., Jones, G., Thornton, C., Langrall, C & Mooney, E.: 2003, Children's representation and organization of data, *Mathematics Education Research Journal,* **15(1)**, pp. 42-58.

Pereira-Mendoza, L.: 1995, Graphing in the primary school: Algorithm versus comprehension, *Teaching Statistics,* **17**, pp. 2-6.

Robinson, D.H. & Kiewra, K.A.: 1995, Visual argument: Graphic organizers are superior to outlines in improving learning from text, *Journal of Educational Psychology,* **87(3)**, pp. 455-467.

Selva, A.C.V., Da Rocha Falcao, J.T. & Nunes, T.: 2005, Solving additive problems at pre-elementary school level with the support of graphical representation, in H.L. Chick & J.L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 161-168). Melbournes: PME.

Uesaka, Y., Manalo, E. & Ichikawa, S.: 2007, What kinds of perceptions and daily learning behaviors promote students' use of diagrams in mathematics problem solving?, *Learning and Instruction,* **17**, pp. 322-335.

GRAPHIC CALCULATOR AS A TOOL FOR PROVOKING STUDENTS' CREATIVE MATHEMATICAL ACTIVITY

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The article presents a part of a research, whose goal was to study and describe the ways of applying the graphic calculator by 14-year-old students when solving a specific kind of tasks. This article attempts to answer a question: What mathematical activities does the graphic calculator provoke?, with an emphasis on the manifestations of the students' creative activities. A unique calculator program enabling recording as well as replaying the work with this particular tool has been applied during the research.

INTRODUCTION

To achieve student's full mathematical development it is essential to elicit and shape their mathematical activity, meaning both gaining basic knowledge and achieving the skills that are indispensable for solving mathematical problems as well as to release behavior typical of mathematical thinking. None of these can be performed by itself in the course of developing basic knowledge and skills. The latter is often the privilege of the talented and mathematically oriented students. Thanks to this kind of activity students acquire certain behavior that is characteristic of a mathematician's work, such as formulating hypotheses, discovering new information or proving. From a teacher's point of view, it is necessary to organize and control students' work in such a way that it would be possible, as Maciej Klakla states in one of his works, not to "miss diamonds in a school sieve", referring to the students who reveal a gift for the mathematical creative activity at the stage of the gymnasium level.

What is the mathematical creative activity? It is, like every other mathematical activity, a mental effort aiming at shaping notions and mathematical way of reasoning, stimulated by situations leading to formulating and solving theoretical and practical problems, manifesting in various mental activities (Nowak, 1989). General student's activities include:

- adopting and assimilating mathematical information,
- practicing basic elementary mathematical skills,
- solving typical problems with the use of basic mathematical techniques and methods,
- using mathematical language in its various forms,
- clearing up and memorizing knowledge,

- specifically creative activity (Krygowska, 1977).

The latter is described by Maciej Klakla in the following way: creative mathematical activity in terms of its shaping and development can be observed in the following forms:

1. formulating hypotheses and their verification (particularly formulating inequality hypotheses on the basis of empirical data);
2. method transfer (transferring the method of reasoning or solving the problem to other issues which are similar, analogical, more general, resulting from increasing the dimension, exceptional or limit cases;
3. creative collecting, processing and using mathematical information;
4. discipline and critical thinking;
5. problem generating in the course of method transfer;
6. problem continuation;
7. stating problems in open situations (Klakla, 2002).

Many mathematical didactics experts pay attention to the fact how difficult the performance of creative mathematical activity in the traditional math teaching is, we still devote more time to average students than to the skilled ones, we still fail to prepare young people to creative activities, we fail to develop creative attitudes (Klakla, 2002). It is certain that one of the reasons for this situation is a shortage of didactic tools leading to provoking mathematical activity, another one maybe disability to effectively apply available didactic aids during math classes. Zofia Krygowska underlines the necessity to seek the means of provoking mathematical activity, including the creative activity at any educational level (Krygowska, 1977).

On the gymnasium educational level (post-primary school), since the students of this particular age group are the subject of this article, it is necessary to develop the elements of creative mathematical activity. It is indispensable to encourage students to cognitive activities developing (Klakla, 2002).

GOALS, ORGANIZATION AND RESEARCH METHODOLOGY

The appearance of the graphic calculator on math classes in Polish schools[1] has caused the necessity to study the influence of this tool on the didactic process and teaching mathematics.

[1] For a few years Polish teachers have been using graphic calculators, which however, have been used rarely and unsystematically. It has been caused by the lack of integration of the tool with mathematics, which involves the insufficiency of the curriculum, course books and methodology for using the tool. So far only one such syllabus has been developed for gymnasium: "Mathematics with a graphic calculator and a computer in a gymnasium".

The study that has been performed so far has stated that the application of the graphic calculator during maths lessons can bring numerous advantages, including:

- it can improve the understanding of mathematical notions, particularly the notion of functions (Dunham, 2000; Waits, 1997);
- it can lead to the improvement of skills at solving mathematical problems (Dunham, Dick, 1994; Dunham, 2000; Kutzler, 2000; Waits, 1997);
- it can contribute to students' observations of certain regularities and mathematical hypothesis formulation through enabling them to perform many mathematical trials quickly and thoroughly (Kąkol, 1987);
- it can improve and perfect mathematical thinking, inculcate the ability to conclude sensibly in the students (Dunham, Dick 1994; Waits 1997);
- by incorporating it into programming it can trigger the development of activities involving coding and creating algorithms (Herma, 2004).

The research, whose parts I am going to present, aimed at studying and describing the ways of the graphic calculator TI-83 Plus. The research, whose parts I am going to present, aimed at studying and describing the ways of using the graphic calculator by the students for solving a specific type of problems. I have been looking especially for the answers to the following questions:

1. What do the students use a graphic calculator for when they solve a problem?
2. What are the students' strategies [2] in the course of solving tasks?
3. What mathematical activities does the graphic calculator provoke?

In the article I am going to concentrate on the answer to the third question, with an emphasis on the manifestations of creative mathematical activity of the students under research. The study was based on the students of grade one (14 years old) of a gymnasium in Bielsko-Biała, where mathematics was taught on the basis of the course book and syllabus called "Mathematics in gymnasium with a graphic calculator and a computer" („Matematyka w gimnazjum z kalkulatorem graficznym i komputerem") (Kąkol, Wołodźko, 2002). The study was performed on the subject of "Functions". Four students solved problems during weekly 45- minute, extra-schedule sessions, for four months. At every session the students were given the content of the task and a graphic calculator TI 83 Plus. The course of the session was recorded by the author with a cassette recorder. Every meeting finished with a conversation with a student about the method and the purpose of the student's use of the calculator.

[2] By strategy I understand the way of a student's performance with the use of a graphic calculator leading to the solution of the problem.

The main research tool was a unique at that time[3] calculator program[4] enabling the work performed on the calculator to be recorded. The program made it possible to record the work of each student participating in the individual meetings. It is essential that the recording program enables succeeding screen views to be shown in a form of an accelerated movie, as well as it enables to scan the list of the buttons pressed by the student in the course of their work. The program allowed me to have an insight into each student's thinking process and learn the exact sequence of their work stages on the task, not narrowing the analysis to merely the final record of the solution on the paper, or, as it happens most often, limiting it to the result itself.

The tasks given to the students come from a course book and a collection of practice exercises called "Mathematics with a graphic calculator and a computer" („Matematyka w gimnazjum z kalkulatorem graficznym i komputerem"), from a part devoted to Functions (Kąkol, Wołodźko, 2002). The curriculum of teaching mathematics with the use of the graphic calculator and the computer in grade one of the gymnasium assumed the performance of the following issues: the Cartesian coordinate system, geometrical figures in the coordinate system, the notion of function, the graph of a function, simple proportionality, the properties of the function f(x) = ax, linear function and its properties, linear equations with one unknown and inequalities with one unknown. They were adapted to their skills connected with the use of a graphic calculator, as well as to their current mathematical knowledge. A few tasks "slightly outdistanced" their knowledge and skills in mathematics, however, they fitted in the "range of their nearest future development", which was mentioned by Piaget. The students solved 13 tasks during individual sessions.

The group consisted of : Dorota (D), Monika (M), Janek (J) and Szymon (S). The survey they had filled in at the beginning of the study reveals that all of them had a positive attitude towards the graphic calculator and they were not afraid of using it. Those students acquainted with the tool for the first time at their math classes when they were in grade one of the gymnasium. For the initial six months of their work with a graphic calculator Dorota and Monika used it for ready-made programs and for making auxiliary calculations, while Janek and Szymon wrote their programs in addition.

For a better recognition of the level of the students' knowledge, their final primary school results in Math were known, as well as those ones at the end of the first semester of grade one in a gymnasium. The aforementioned grades are given in table 1.

[3] At present, the Casio company offers a program which records the course of the work of a student using the emulator of a graphic calculator

[4] prof. John Berry from the Plymouth University is the author of the program.

	Student			
Math grade	Dorota	Janek (J)	Monika	Szymon (S)
At the end of the primary school	very good	excellent	good	very good
At the end of the first semester of grade one in gymnasium	good	excellent	very good	very good

Table 1: The students' under research Math grades

The results of a test checking the level of knowledge and skills before the notion of functions was introduced are presented in table 2.

	Student			
	Dorota	Janek (J)	Monika	Szymon (S)
result (number of tasks)	14	21	16	18
correct solution	7	4	8	4
partly correct solution	2	0	1	2
incorrect solution	2	0	0	1
no solution	14	21	16	18

Table 2: The results of the quantity analysis of the test task solutions

The data presented in both tables 1 and 2 show that all students achieved good or very good results. Moreover, it can be assumed that Janek is the best one in the group, while Dorota is the weakest one.

MATHEMATICAL ACTIVITIES OF THE STUDENTS IN THE COURSE OF THEIR TASK SOLVING WITH THE USE OF A GRAPHIC CALCULATOR

The analysis of the research results revealed the following mathematical activities of the students, which had been provoked by the use of the calculator:

1. empirical conclusions,
2. symbolic language usage,
3. generalization,
4. hypothesis formulation and verification,
5. formulation of new problems,
6. deduction.

The activities being the manifestations of the students' creative mathematical activity deserve special attention. Among them, generalization, hypothesis formulation and verification and formulation of new problems should be mentioned in particular.

For each of these activities I am going to present the content of the task and a copy of the student's notes accompanied by a short commentary to the solution. Because of a limited number of pages of the article, I am not including the students' solutions.

Generalization

The generalization of statements and mathematical notions is a process so characteristic of the mathematical activity that it is worth looking for such didactic situations where we evoke that process in a student's mind. The activity was observed in 6 tasks, it had been provoked with all the four people.

Task

Give the formula for a linear function which takes the following values:

1. positive for x belonging to the interval $(-\infty, -4)$,
2. negative for x belonging to the interval $(2, +\infty)$.

Commentary

The student very quickly found formulas adequate for each part of the task with the help of a graphic calculator. He did not stop after finding one example but he continued looking for another one. Consistent application of the chosen method, together with the rational selection of the values of coefficients enabled him to quickly find consequent correct graphs. A thorough observation and comparison of the coefficient values of the function formulas, fulfilling the conditions of the task, contributed to the formulation of a general rule, which indeed might have educed any of the previously found cases by means of specification.

The description of Szymon's work[5]

1) he introduces the expression x + 4 into the editor of function rules
2) he draws the graph of the function

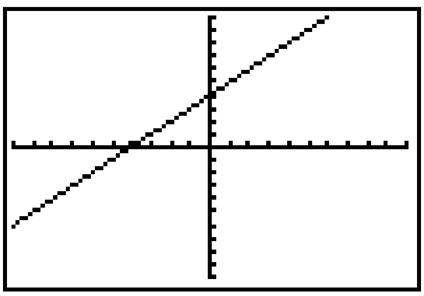


Figure 1

3) he draws the graphs of the functions:

[5] Because of limited number of pages of this article I only put this one detailed description of the pupil's work over the task together with windows of screen of calculator.

$y = -x + 4$, $y = x + 4$, $y = x - 4$, $y = -x - 4$,

4) He draws the graph of the functions: $y = x + 2$, $y = -x + 2$,

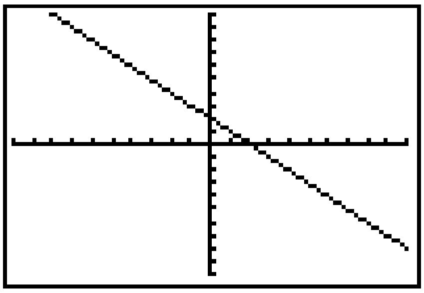


Figure 2 (for $y = x + 2$)

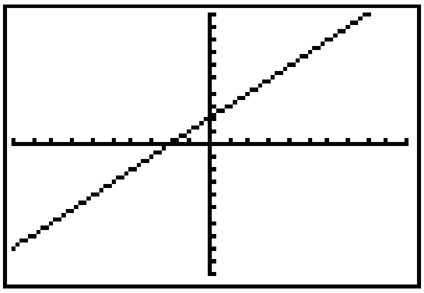


Figure 3 (for $y = -x + 2$)

5) The student starts looking for other formulas of functions fulfilling the conditions of the task.

6) He draws the graphs of the functions:

$y = -2x - 4$, $y = -2x – 8$, $y = 2x + 8$, $y = -2x – 8$,

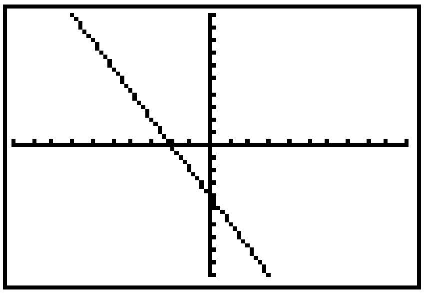


Fig. 4 (for $y = -2x – 4$)

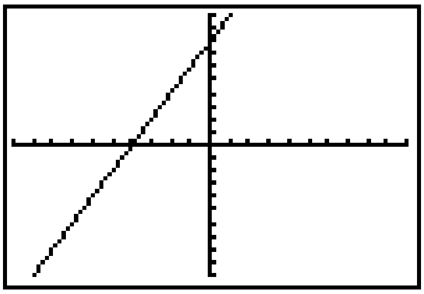


Fig. 5 (for $y = 2x + 8$)

7) He draws the graphs of the functions:

$y = -8x + 2$, $y = - 2x + 2$, $y = -2x + 3$, $y = -2x + 4$, $y = -4x + 8$, $y = -8x + 16$.

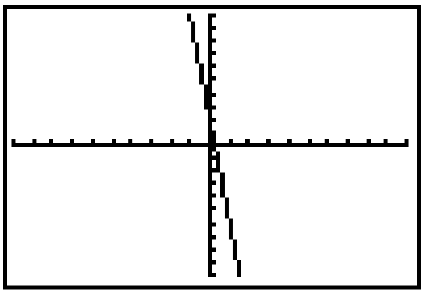


Fig. 6 (for $y = -8x + 2$)

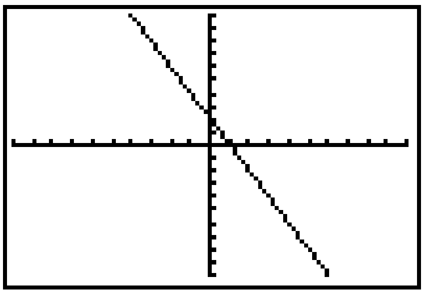


Fig. 7 (for $y = -2x + 2$)

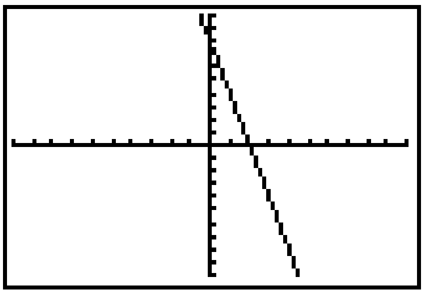


Fig. 8 (for $y = -4x + 8$)

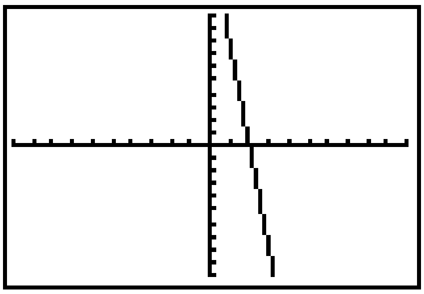
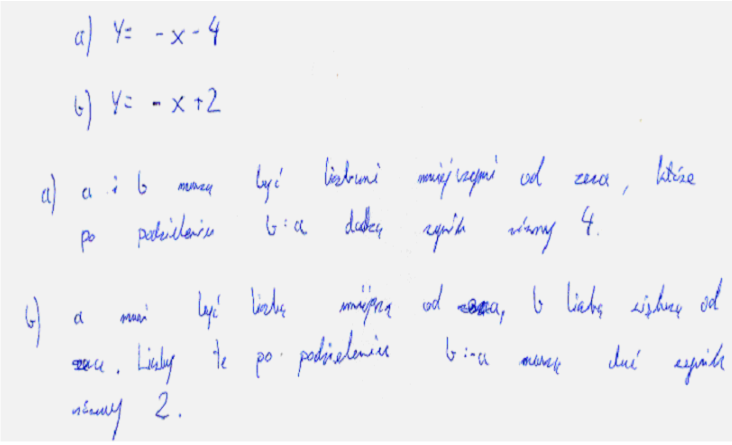


Fig.9 (for $y = -8x + 16$)

A copy of Szymon's notes



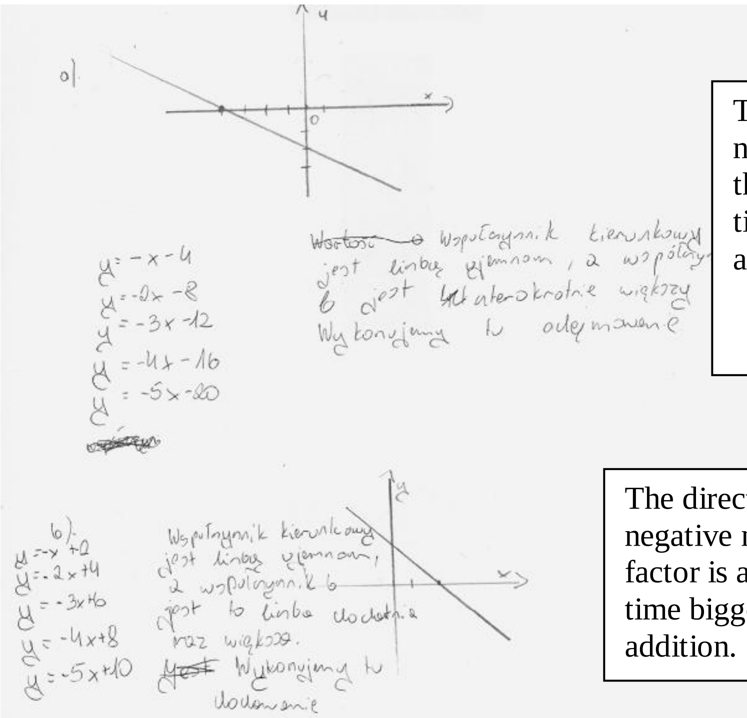
a and b must be such a number less than zero, which after dividing b:a will result 4

a must be a number less than zero, b – a number bigger than zero. The numbers must equal 2 after dividing b: -a

Commentary

Dorota made the graphs of many functions with the use of the graphic calculator, and then she concluded if the fulfilled the conditions given in the task. The possibility of making many graphs in a very short time enabled the student not only to find one formula of the function fulfilling the conditions of the task, but it also enabled her to make an attempt to formulate a general rule describing the families of functions fulfilling the conditions given in the task. She formulated that rule orally, however not very precisely.

A copy of Dorota's notes



The linear factor is a negative number and the b factor is four times bigger. We make a subtraction here.

The directional factor is a negative number and the b factor is a positive number one time bigger. We make an addition.

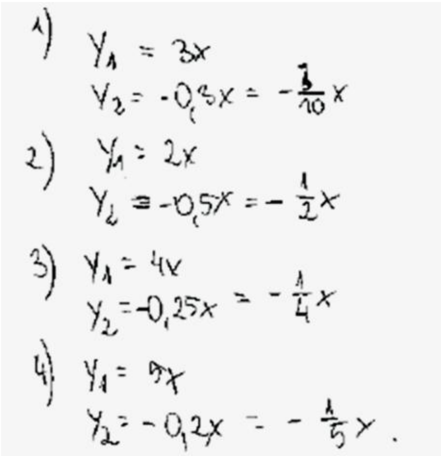
Task

For which values of a the graphs of the function f(x) = ax will be perpendicular to each other? Can they be parallel?

Commentary

The student made 31 trials. She found three examples of the pairs of functions whose graphs were mutually perpendicular. She also found a rule making it possible to give the formulas of other pairs of functions fulfilling the conditions of the task, which is supported by the correct examples of function formulas given by the student. The conversation with the student led to a conclusion that she was unable to formulate in written the general condition she had found.

A copy of Monika’s notes


1) $y_1 = 3x$
$y_2 = -0{,}3x = -\frac{1}{10}x$
2) $y_1 = 2x$
$y_2 = -0{,}5x = -\frac{1}{2}x$
3) $y_1 = 4x$
$y_2 = -0{,}25x = -\frac{1}{4}x$
4) $y_1 = 5x$
$y_2 = -0{,}2x = -\frac{1}{5}x$.

Hypothesis formulation and verification

Hypothesis formulation and verification is a specifically creative activity. With the use of a graphic calculator that activity is revealed as the result of collecting many empirical data (very often in the course of many trials) as well as detailed observations and deduction. This refers both to the formulation of a hypothesis and to its verification. The activity was observed in few situations, however, each of the students from the experimental group faced this activity.

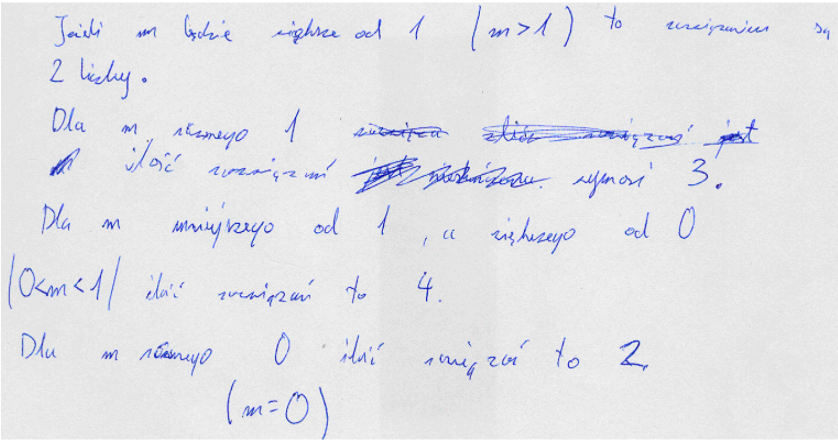
Task

> Check (with the use of a calculator) the number of the roots of the following equation: $|x^2 - 4x + 3| = m$ depending on the parameter m.

Commentary

The student took advantage of the potential of the calculator in order to solve the problem. He drew the graphs for both functions and on the basis of thorough observations he formulated a hypothesis about the number of results of the equation for particular values of the parameter M. The student assumed (it was stated during a conversation) that for M = 0 and M = 1 there are infinitely many solutions. Hypothesis verification was the result of many manipulations: enlarging and diminishing the graphs, establishing new windows for drawing graphs, following the points of the coordinates, calculating the common points of both graphs, following the points of the coordinates in the table. All these activities contributed to refuting the hypothesis which had been stated before, and to giving the correct answer.

A copy of Szymon's notes



Jeżeli m będzie większe od 1 (m>1) to rozwiązaniem są 2 liczby.

Dla m równego 1 ~~rozwiązanie~~ ~~ilość rozwiązań~~ ~~jest~~ ilość rozwiązań ~~prawdziwych~~ wynosi 3.

Dla m mniejszego od 1, a większego od 0 |0<m<1| ilość rozwiązań to 4.

Dla m równego 0 ilość rozwiązań to 2 (m=0)

If m is bigger than 1 (m>1), two numbers are the solution.
For m equal1 the number of solutions equals 3.
For m less than 1 and bigger than 0 (0<m<1), the number of solutions equals 4.
For m equal 0, the number of solutions equals 2.

Formulating new problems

Not often does it happen that a student asks an interesting and not commonplace question during a lesson, all by themselves, or they formulate a certain mathematical problem. Solving a task usually means the end of a student's thinking process. It does not become the source of new problems. Until recently, in Polish schools the teacher's work was mainly focused on transmitting the ready-made knowledge and training students for certain skills. Today, when problem-oriented teaching with big steps enters schools, and when more and more often we have didactic tools encouraging new methods of teaching at our command, it is easier to arrange such specifically creative situations as formulating new problems. The activity was observed in few situations. However, each student in the experimental group faced it personally.

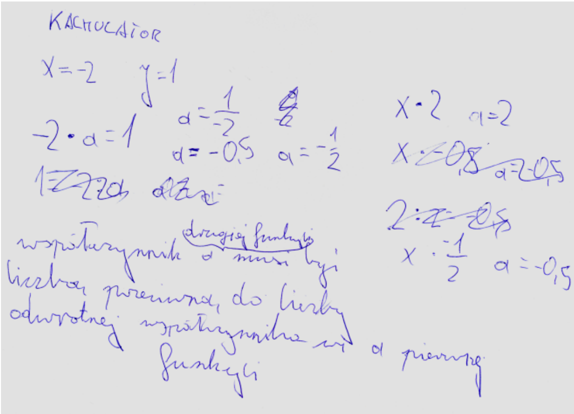
Task

For what values of a the graphs of the function f(x) = ax are perpendicular to each other? Can they ever be parallel?

Commentary

Janek and Szymon at the end of their work on the task formulated a new problem: will the discovered rule work for the functions with their graphs cutting at a different point than the origin of the coordinates? Investigating the reason for such students' activity reveals many essential factors, such as awareness of conscious work with a graphic calculator, possibility of performing quickly and correctly such complicated and time-consuming activities as drawing graphs, which encourages stating questions, interest in the tool, which allows to trigger a student's thoughts, stimulating explorations of mathematical situations, and at the same time , deepening the knowledge of the tool, as well as the inborn curiosity of the students themselves.

A copy of Janek's notes



Coefficient a of the second function must be a number opposite to the inverse number of the coefficient a of the first function.

RESULTS AND CONCLUSIONS

The analysis of calculator recordings showed the variety of ideas among the students during their work on each of the tasks, particularly with the students achieving very good learning results (Janek, Szymon). Very often the process of solving the tasks was performed in different ways, with the use of different abilities of the calculator and different types of mathematical knowledge. The students in the experiment very thoroughly and courageously explored the situations emerging from the task, and when the activities they had undertaken did not lead to the solution, they quit the polarization of thought and started drawing another graph or analyzed another situation. The graphic calculator contributed to triggering an active (this refers to all the students), or even a creative attitude (this mainly refers to Janek and Szymon) towards the task to solve. With the girls (Dorota and Monika) the calculator triggered a creative attitude towards the tasks less frequently. However, it needs to be said, that because of the limited character of the research, the conclusions cannot be treated too universally. They only refer to a particular, small group of students, to a certain type of problems and to certain research conditions. The results presented in this article can constitute a starting point for further studies.

It is necessary to continue searching for new working methods and techniques provoking the release of mathematical activity of all students. The operative character of mathematics makes it possible to apply new technologies, like computers and calculators, facilitating certain operations. Working with the use of these tools enables a student to work independently, to look for solutions to problems through carrying out their own ideas, to discover and check their own mathematical objects. The results of my study show that the application of a graphic calculator can provoke a student's creative attitude towards the tasks they solve. The tool triggers visualization, gives an opportunity to experiment within the mathematical environment. Those exertions play a crucial heuristic role in creative thinking and reinforce the students' mathematical activity.

Being aware of the fact, that it is impossible to discuss active teaching in the situation, when knowledge is elicited from pure perception, it is worth mentioning the benefits of working with a computer and a calculator, which show that these tools may provoke and develop mathematical activities.

REFERENCES

Duda, J.: 2009, Twórczość matematyczna uczniów uzdolnionych a kalkulator graficzny (fragment badań), *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Didactica Mathematicae,* **32**, pp. 43-85.

Dunham, P., Dick, T.: 1994, Research on Graphing Calculators, Mathematics Teacher, *www.tenet.edu/tcks/math/resources/graphcal.html*

Dunham, P.: 2000, Hand-held Calculators in Mathematics Education: A Research Perspective, in: *Hand-Held Technology in Mathematics and Science Education: A Collection of Papers,* The Ohio state University.

Herma, A.: 2004, Wpływ kalkulatora graficznego na rozwijanie wybranych aktywności matematycznych (fragment badań wstępnych), *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki,* **26**, pp. 81-94.

Juskowiak, E.: 2005, *Sposoby wykorzystywania kalkulatora graficznego w procesie nauczania i uczenia się matematyki,* praca doktorska, UAM, Poznań.

Kąkol, H.: 1987, Rozumowanie matematyczne a komputer, *Matematyka*, **3**.

Kąkol, H.(red): 2002, *Program nauczania matematyki w gimnazjum z wykorzystaniem kalkulatorów graficznych i komputera,* Wydawnictwo Dla Szkoły, Wilkowice.

Kąkol, H, Wołodźko, S.: 2002, *Matematyka w gimnazjum z kalkulatorem graficznym i komputerem", klasa 1, Podręcznik,* Wydawnictwo Dla Szkoły, Wilkowice.

Klakla, M.: 1982, Z badań nad rozpoznaniem uzdolnień matematycznych, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki,* **2**, pp. 33-81.

Klakla, M.: 2002, Kształcenie aktywności matematycznej o charakterze twórczym na poziomie szkoły średniej w: *Materiały do studiowania dydaktyki matematyki, tom III, prace Macieja Klakli*, Wydawnictwo Naukowe NOVUM, Płock, pp.261-273.

Krygowska, Z.: 1977, *Zarys Dydaktyki Matematyki*, części 1, 2, 3, WSiP, Warszawa.

Krygowska, Z.: 1986, Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich, *roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyka,* **6**, pp. 25-41.

Kutzler, B.: 2000 , The algebraic calculator as a pedagogical tool for teaching mathematics, *in: Hand-Held Technology in Mathematics and Science Education: A Collection of Papers,* The Ohio State University.

Laborde, C.: 2000, *Why technology is indispensable today in the teaching and learning of mathematics?,* http://emptweb.mps.ohio-state.edu.dwme/t3/post/icme/papers.asp

Nowak, W.: 1989, *Konwersatorium z Dydaktyki Matematyki*, PWN, Warszawa.

Polya, G.: 1964, *Jak to rozwiązać?*, PWN, Warszawa.

Waits B.: 2000, *Rola kalkulatorów w nauczaniu matematyki,* Magazyn Matematyków i Przyrodników „Edukacja z TI", **1**.

MATHEMATICS IN KINDERGARTEN: GROWN-UP THINGS

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Teaching mathematics in kindergarten gives the opportunity to interpret the adult world through children's eyes, rejecting the idea of a child as a small adult. In this paper we intend to suggest opportunities and ways to carry out this view of scientific education aiming to integrate imagination with a scientific approach and to solve adults' problems from a child's point of view.

INTRODUCTION

Teaching mathematics in kindergarten must take into account Italian and American reports on the education system, in particular the revolutionary document known as the Charter of Children's Rights. Such documents deal with the child's identity in a different way from the stereotypes in the adults' collective imagination.

Mathematics education has to take into consideration changes in the child's social and cultural status, as well as the main aspects of scientific education in kindergarten. The distinction between adults and children is no longer valid. In the past, the child had no particular social status nor rights. He was not yet a well-defined being. Now however, he is considered a *person* (Andreoli, 2000). The child is no longer a sort of miniature or immature man, but is considered a person with rights and duties.

Therefore the aim of kindergarten is to contribute to children's education in a harmonious and complete way giving value to their particular abilities and learning capacities while respecting the differences and identities of individual children. (NCTM, 2000, NRC, 1989, INC, 2007). This is the only way for the child to have a true social position and to become a person with equal rights in relation to others. By contrast, adults no longer have the right to deal with children as they please but must defend children's rights. It is therefore very important to exclude any kind of formal and premature learning in kindergarten and to identify the adult as a person who encourages personal relationships between peers and with adults in the several different fields of experience, thus creating the conditions for thinking, doing and acting (NCTM, 2000, INC, 2007).

An adult capable of this kind of teaching is not like the conventional parent whose love is often conditional and focused on producing an obedient child who resembles the parent (Andreoli, 2000). Determining how a kindergarten can be a place where families cooperate in their child's education is a problem which mathematics teaching can be instrumental in solving (NCTM, 2000, INC, 2007).

Education has to be evaluated according to its ability to make children self-reliant, creative and independent rather than mere imitators of their parents (Steffe, Cobb, 1988).

THE CHILD IN THE ADULT WORLD: CULTURAL REFERENCES

There is a world of adults where the child is put and is "forced" to deal with adults. In this world, the child reasons just as adults do but with behaviour that is very emotional and typical of childhood (Piaget, Inhelder, 1967). A ten-year-old child can do complex mathematical operations but be capable of crying and other forms of childish behaviour, thus demonstrating the inability to be separated from their mother. A child can also use their linguistic competence to test both the limits imposed by adults as well as their own power over adults while, at the same time, trying to please them (Louis, 2005).

Because children live in an unreal world of magic and fantasy, it is important to teach scientific methods and knowledge from the very beginning of their scientific education (Cattanei, 1995). While we do not believe children should blindly copy adults, we are, nevertheless, convinced that they should be taught to understand the world of adults without losing the childlike qualities of spontaneity and wonder (Cattanei, 1995, Steffe, Cobb, 1988, Louis, 2005). For example, it seems to be important for children to grasp the meaning of acts typical of the adult world that, in their opinion, can take on magical power such as buying, building and working.

Demands from children often receive an answer that they are unable to understand: "It's too expensive!", "Oh, if only Third-World children could have all this good food". The meaning, source and use of money, as well as food, starting from raw materials that are processed before they are ready to eat, are things and events that can introduce the child into the world of adults. This can be done in a playful and child-like way typical of that age even though kindergarten can put it in a scientific contest through the teaching of mathematics (Clements, 2004).

The learning objective in kindergarten is to enter the world of adults by following the "who, what, where, how, why" method in order to make a concept clear and to explain the meaning of a process (Ginsburg, Pappas, Seo, 2001) This objective can be realized by resorting to well-defined mathematical concepts, such as the ability to invent and plan, make similarities and relationships, as well as to analyze the different forms of natural language that are the starting point of every activity of formalization. It seems to us that we have followed guidelines, related to everyday activities, knowledge of personal history, time rhythms and cycles, space orientation and exploration of nature. It also seems to us very relevant to point out the importance of gathering, arranging, counting and measuring by resorting to more or less methodical ways of comparing and arranging, in relation to different properties, quantities and events through the invention and use of objects or sequences or symbols to record and having recourse to some simple measuring instruments and, finally, by making quantification, numeration, comparisons. (Clements et all, 1999, Geary, 1994, Ginsburg, Seo, 2004, Clements, 2004, Copple, 2004)

The teaching experience gained in kindergarten has shown us the way to take in order to lay the foundations for a personality (from three years onwards) that firmly refuses to imitate adults and does not became "self-centred and narcissistic" (Andreoli, 2000).

The experience of rediscovering an interest in "street games" and in "playing in the street", the discovery of the importance, use and source of money, and the invention and implementation of the "barter market" (Ancona, Montone, Pertichino, 2004, 2005) encourage not only a way of performing an action, being in contact with nature, things, materials, the social environment and culture, but also the ability to plan and invent, make projects and shapes copied from real-life or create entirely new ones (Clements, 2004). In kindergarten should contradict those adults who argue that a child must grow up quickly. In that regard, it is worth noting what D. Hawkins says: "Before initiating the child into the discovery method (made up of rules, of a series of actions, etc.) it is profitable to set aside quite a long time for the *hands-on activities* that involve a sort of exploratory game, the easy availability of materials to handle, to try out, and to use without having to follow instructions, give any kind of explanation, and make a determined object, etc." (Macchietti, 1995). *Hands-on activities* will then give room for more structured learning. That is how games take shape and play a crucial role in the life of the child and in kindergarten.

The National Guidelines point out how "games must be exploited" in all their forms and expressions (especially the games of make-believe, imagination and identification so as to develop the ability to elaborate and turn experience into symbols): the play therapy element in the teaching activity provides for a learning experience in their personality" (NCTM, 2000).

However, children's games are often used by adults as a functional exercise rather than as on opportunity for children to express their sense of freedom and joy of living. Adults often tend to turn play-time into a competition, to encourage a competitive spirit. In that way, children miss the opportunity to invent games and infringe rules in order to make new ones which are more appropriate to their needs (Ginsburg, Inoue, Seo, 1999). If children are left free to play, they are encouraged to develop their own rules of play which are not necessarily less rigorous then those imposed by adults (Andreoli, 2000).

Our proposal to go back to the old habit of playing in the street has allowed children and their parents to discover the unstructured world of play that their grand-parents enjoyed. The children find this way of playing very appealing and they adopt it to their playtime activities in playgrounds and gyms. We were motivated to invent games having more or less structured and shared rules to meet the goals expressed in the Italian kindergarten guidelines concerning space, order and measurement.

METHODOLOGY

The didactic methodology uses the inquiry approach, a model based on assumptions of knowledge, learning and teaching derived from criticisms of the traditional method of transmission. Through the inquiry approach, it is possible to:

- encourage students to explore;
- help students to verbalise their mathematical ideas;
- bring students to understand that many mathematical questions have more than one answer;
- make students aware that they are capable of learning mathematics; and,
- teach students, through experience, the importance of logical reasoning .

In other words, we try to enable students to develop the mathematical capabilities necessary to pose and solve mathematical problems, to reason and communicate mathematical concepts and to appreciate the validity and the potential of mathematical applications (Borasi, Siegel, 1994). This has been recommended in numerous important American and Italian studies on reforming the teaching of mathematics (NCTM, 2000, NRC, 1989, INC, 2007).

Several researchers who have studied the learning of mathematics have found that students must actively demonstrate a personal understanding of mathematical concepts and techniques. Only in this way can they reach a level of significant understanding (Ginsburg, 1983, Steffe, von Glaserfeld et all, 1983, Baroody, Ginsburg, 1990). This position is reflected in constructivism. The influence of constructivism on mathematics teaching can be seen in requests for teaching environments that encourage students to actively participate in developing their knowledge rather than receiving it from teachers or books (Steffe, 2004). In these classes, the roles are reversed. Instead of passively listening, the students assume responsibility for their learning. The teachers, on the other hand, speak considerably less and listen a great deal more to the students' reasoning in order to help them understand what they have deduced (Confrey, 1991).

In other words, to be good students, children today must be researchers ("inquirers"). Therefore, only doubt and uncertainty can motivate the search for new knowledge (Skagestad, 1991). Our experience was based on the inquiry approach model which allowed us to alternate problem posing with problem solving. It showed children solving problems which arise and for which no one has the answer rather than solving problems prepared by the teacher. For example, when they have to assign a price to one of their products, they decide on the basis of their different personal daily experiences.

This model led us to use the problem posing method in which the children's answers, their questions and the data they used are analyzed. In other words, with this methodology the children can make observations, ask questions and formulate proposals. Moreover they can compare an external investigation with an internal one. It is also possible to compare and contrast exact and approximate investigations, using the strategy of "and what if…" to generate new hypotheses.

Similarly, it was useful to analyze the clinical conversation not with a view to verifying the correctness of the answers but rather to gain an understanding of the social and cultural motivations behind them. It was an extremely important method for forming, informing and maintaining the teacher's "intermediary inventive mind" (James, 1958). For example, when the children talk about how money is used in their family, the different approaches to buying and saving become evident and are manifested in the different prices they assign to their products (Ginsburg, Pappas, Seo, 2001).

EXPERIENCES

In our experience of planning and developing learning experiences, we decided to focus on two areas, the use of money and bartering, rather than on mathematical concepts. We wanted to emphasize the different roles played by parents and children in both situations. In our classroom encounters, we encouraged children to look for and discover the meaning of situations and objects to enable them to form their own view of the grown-up world, answering questions concerning what is an automated teller machine, what is a salary, which criteria are applied to set the price of an object, what sort of exchange value can be given to an object, etc (Montone, Pertichino, 2003).

Our first objective was to encourage an exploration and understanding of the reality connected whit the meaning and the uses of money and its symbols (the different denominations of coins and banknotes). We created a situation using banknotes and coins which then simulated a clinical conversation based on questions like "What is money?", "How do you get it?", "What is this for?" and "What is the bank?". From this it was possible to observe how the interaction with the environment and the world of adults leads to the internalization of mathematical language, discriminatory, classing and sorting operations leading to simple generalizations and abstract concepts. Clearly, there were different interpretations of the use of money according to the child's personal experiences with money in the family context (Ginsburg, Pappas, Seo, 2001).

Some questions were asked to try to find out what children know about the meaning, origin, value, and use of money as well as the meaning of terms such as "bank" and "automated teller machine". Here are some excerpts from the conversation:

Q. What has fallen from the wallet?

A (Rossella) Small coins and money; yes, banknotes and pieces of metal, one has got a head and the other one has a cross and the number five, the banknotes have all the same picture.

Q. What is money for?

A. (Silvana) For buying, for paying.

(Pasquale) You can buy things and then eat them.

(Rossella) If I have to buy a candy or stickers I use coins, but if I go to the "Everything at 1.00 euro" store I give a 1.00 euro coin and other pieces which are bigger.

Q. How do you get money?

A. (Vanessa) By working, because when you work you get money.

(Silvana) Dad gets a salary because he is paid by the person he works for.

(Pasquale) What! … They work hard and then they aren't paid?

Q. Have you ever played with money?

A. (Sonia) You can set up a play shop … a greengrocer's. When I went camping and I decided that things were old-fashioned I set up a sort of market stall near toilets and people came to buy them and give me money.

Q. Do you know what a bank is?

A. (Rossella) A bank is a place where you keep your money, because if you have got a lot of money, then you keep accounts and you can see how much money the bank gives you, you give money to the bank and it increases. And then you go withdraw money. The bank makes calculations very well so you can get much more money and then it sends you a statement to tell you how much money you have got or you have to be paid.

Q. If I need money when the bank is closed, for example at night, what can I do?

A. (Gabriella) You can take money out when the bank is closed, because mum has got a card and she can insert it and then the money comes out.

Q. Who knows the name of the special card?

A. (Vanessa) Yes… credit card.

Our second objective was to familiarize the children with situations involving money through simulation activities. We created a board game with dice and cards that required the children to manage their salaries and expenditures. Because the children did not know numbers, they created face symbols for the dice and gave values to cards that represented the money. These two phases, clinical conversation and familiarization, favoured the development of both divergent and convergent knowledge processes, which are important characteristics of the learning process. The objective was to encourage the children to learn mathematics in a social context of necessity. To facilitate this, we organized a real trading situation where the children were each given a sum of money. They then bought materials wholesale and made articles or created services which they sold in their stores. They used different sizes of coloured plastic discs as money and it could be withdrawn from teller machines thanks to a code connected to their names. The children demonstrated the ability to price their articles adequately to cover labour costs and other expenses and make a profit. In the end, those with the highest profits were declared the winners. This experience showed that children can learn mathematical concepts through everyday activities.

They learned to establish connections and functional relationships, formulate shared rules, develop strategies and apply real life experience to their commercial activity.

CONCLUSION

On the basis of this experience, we can draw some conclusions. First of all the so-called “child-centred approach” is no longer needed because children are often put at the centre of the world and all the members of the family live for them, and so there is the risk that the needs of the adult (reassurance, search for the meaning of life) prevail over the needs of the child (Andreoli, 2000). Moreover, since the child seems to be able “to astonish both the social environment and himself” with a set of more and more sparkling results, there is the risk that parents feel well disposed towards him, hence a “lack of childhood” (Miller, 2007). Scientific and mathematical education can, on the contrary, play a crucial role in the proposal of a childhood education, i.e. the right to be free to play and take part in educational activities in the direction of independence.

REFERECES

Ancona, R.L., Montone, A., Pertichino, M.: 2005, *Street games for the acquisition of spatial orientation in Primary School*, proc. SEMT Praga, pp. 66-70.

Ancona, R.L., Montone, A., Pertichino, M.: 2004, Contare e misurare nella scuola dell’infanzia, in *Atti del Convegno Nazionale Incontri con la matematica: La Didattica della Matematica: una scienza per la scuola*, Ed. Pitagora, Bologna, pp.75-79

Ancona, R.L., Montone, A., Pertichino, M.: 2004, Tu dai una cosa a me, io do una cosa a te. Ovvero l'arte del baratto si impara in sezione, in *Atti del Convegno Nazionale Incontri con la matematica: La Didattica della Matematica: una scienza per la scuola*, Ed. Pitagora, Bologna, pp. 103-105.

Andreoli, V.: 2000, *Dalla parte dei bambini*, Biblioteca Universale Rizzoli.

Baroody, A.J., & Ginsburg, H.P.: 1990, Children's learning: a cognitive view, in: R.B. Davis, C.A. Maher and N. Noddings (Eds.), Constuctivist view on the teaching and learning of mathematics, Reston, VA: NCTM, pp. 51-64.

Borasi, R., Siegel, M.: 1994, Un primo passo verso la caratterizzazione di un "inquiry approach"per la didattica della matematica, *L'insegnamento della matematica e delle scienze integrate*, vol. 17A- vol. 17B- N.5, pp. 468-493.

Cattanei, G.: 1995, *Scientificità e conoscenza scientifica nei bambini*, Scuola Materna. Editrice La Scuola, **18**, pp. 10-16.

Clements, D.H., & Conference Working Group: 2004. Part one: Major themes and recommendations, in: eds Clements, D.H., Sarama, J., & DiBiase, A.-M., *Engaging young children in mathematics: Findings of the 2000 National Conference on Standards for Preschool and Kindergarten Mathematics Education*, Mahwah, NJ. Lawrence Erlbaum, pp. 7-73.

Clements, D.H., Swaminathan, S., Hannibal, M.-A., & Sarama, J.: 1999, *Young children's concepts of shape*, Journal for Research in Mathematics Education, **30**, pp. 192-212.

Confrey, J.: 1991, Learning to listen: a student's understanding of power of ten, in: E. von Glasersfeld (Ed), Radical constructivism in mathematics education, Dordrecht, the Netherlands, Kluwer, pp. 111-138.

Copple, C. E.: 2004, Math curriculum in the early childhood context, in: D.H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Findings of the 2000 National Conference on Standards for Preschool and Kindergarten Mathematics Education*, Mahwah, NJ: Lawrence Erlbaum, pp. 83-90.

Geary, D.C.: 1994, *Children's mathematical development: Research and practical applications*. DC: American Psychological Association, Washington.

Ginsburg, H.P., & Seo, K.-H.: 2004, What is developmentally appropriate in early childhood mathematics education? in: D.H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Findings of the 2000 National Conference on Standards for Preschool and Kindergarten Mathematics Education*, Mahwah, NJ: Lawrence Erlbaum, pp. 91-105.

Ginsburg, H.P., Inoue, N., & Seo, K.-H.: 1999, Young children doing mathematics: Observations of everyday activities, in: J.V. Copley (Ed.), *Mathematics in the early years*, Reston, VA: National Council of Teachers of Mathematics, pp. 88-100.

Ginsburg, H.P., Pappas, S., & Seo, K.-H.: 2001, Every mathematical knowledge: Asking young children what is developmentally appropriate, in: S.L. Golbeck (Ed.), *Psychological perspective on early childhood education: Reframing dilemmas in research and practice*, Mahwah, NJ: Lawrence Erlbaum, pp. 181- 219.

Ginsburg, H.P.: 1983, The development of mathematical thinking, New York, Academy Press.

Indicazioni Nazionali per il Curricolo: 2007, *Scuola dell'Infanzia, Primaria e Secondaria di Primo Grado*, Ministero della Pubblica Istruzione, Ed Erikson.

James, W.: 1958, *Talks to teachers on psychology: and to students on some of life's ideals*, W.W. Norton & Company, New York.

Louis, J.M.: 2005, *Riconoscere e aiutare i bambini precoci*, Edizioni Magi.

Macchietti, S.S.: 1995, *La scuola materna fonte di conoscenze scientifiche e di pensiero,* Scuola Materna. Editrice La Scuola. 18, pp. 38-53.

Miller, A.: 2007, *Il dramma del bambino dotato e la ricerca del vero sé*, Universale Bollati Boringhieri.

Montone, A., Pertichino, M.: 2003, *La matematica nella vita quotidiana*, Bambini anno XIX, edizioni junior, 1, pp. 47-51.

Montone, A., Pertichino, M., Fiore, C., Milone, M., Roselli, G.: 2000, "le vie del denaro"…Ovvero: l'uso del denaro nella scuola dell'infanzia, in Atti del *Convegno Nazionale Incontri con la Matematica: Didattica della Matematica nel III millennio,* Pitagora Editrice Bologna, pp. 203-205.

National Council of Teachers of Mathematics: 2000, *Principles and standards for school mathematics*, Reston, VA: Author.

National Research Council (NRC): 1989, *Everybody counts: A report to the nation on the future of mathematics education*, Washington, DC: National Academy Press.

Piaget, J., Inhelder, B.: 1967, *The child's conception of space*, W.W. Norton, New York.

Skagestad, L.P.: 1991, The road of inquiry, New York, Columbia University Press.

Steffe, L.P., & Cobb, P.: 1988, *Construction of arithmetical meaning and strategies*, Springer-Verlag, New York.

Steffe, L.P., von Glasersfeld E., Richards J., & Cobb P.: 1983, Children's counting types: Philosophy, theory and applications, New York, Praeger Scientific.

Steffe, L.P.: 2004, PSSM from constructivist perspective, in: D.H. Clements, J. Sarama, & A.-M. DiBiase (Eds.), *Engaging young children in mathematics: Findings of the 2000 National Conference on Standards for Preschool and Kindergarten Mathematics Education*, Mahwah, NJ: Lawrence Erlbaum.

USING SIMPLE ARITHMETIC CALCULATORS AS A DIAGNOSTIC TOOL ON PLACE-VALUE

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In this study 3rd graders are coping with tasks relevant to place value using simple arithmetic (broken) calculators. Main aim of the study is to signify the potential usage of calculator by the teacher as a diagnostic tool that could reveal misconceptions of the students or limited understanding of the concept of place value.

INTRODUCTION

The usage of calculators in the classroom setting is not something new. A lot of research done on that field has showed that appropriate calculator use: i) improves students' computational skills (Hembree & Dessart, 1992); ii) helps students especially in early grades to develop number sense and mental computation strategies (Groves & Stacey, 1998) stimulating thus their problem solving thinking (Campbell & Stewart, 1993); iii) enables students to make conjectures and generalizations in relevance to numbers and operations (Charles, 1999). All this potential usage of calculators in classrooms is summarized into the well known "Calculator-Aware Number" (CAN) curriculum created in the UK by a project team in collaboration with teachers (Ruthven, 2009). The major aim of this project was to study the effect on the learning of numbers of introducing young learners to the calculator as a resource tool. In this paper we try to adopt the usage of simple arithmetic calculator as a diagnostic tool for the teacher to uncover students' misconceptions related to place-value in early grades of primary school. In the next section we will present some elements concerning our theoretical context. Then the description of the study follows and after that we close with the results of the study and some concluding remarks.

THE THEORETICAL CONTEXT OF THE STUDY

There is no single, universally accepted view of the best use of calculators in classroom. We already mentioned some kinds of calculator usage and obviously we could add more (for example, using calculator '*as a way of developing understanding of the structure of the computational algorithms that are most often taught and learned by rote with little conceptual understanding*' (Schwartz, 1999) or using calculators to check answers by making inverse operations for subtraction and division (Chick & Baker, 2005)). And it is obvious (as Stigler & Hiebert (1999) suggest) that calculators (among others) could be seen as valuable allies in the teachers' struggle to help students understand mathematics. However, a very essential component of teachers' pedagogical content knowledge is their knowledge about students' misconceptions. This means that teachers try to teach in a way that helps students avoid misconceptions and at the same time they have different approaches to deal with these misconceptions that in any case arise. Place-value domain is a fruitful one as far as the arising of misconceptions is concerned. There is a plethora of evidence about the fact that students face great difficulties to understand correctly place-value especially in early grades (Ross, 1990). These difficulties include the treatment of 10-groups as units (Fuson, 1988), the fact that only few students see the validity of doing and undoing groupings (partitioning) to solve multidigit number problems (Bednarz & Janvier, 1988), the case of seeing multidigit numbers as unitary collections (Sinclair & Scheuer, 1993) and the case of zero as 'place holder' (Wheeler & Feghali, 1983).

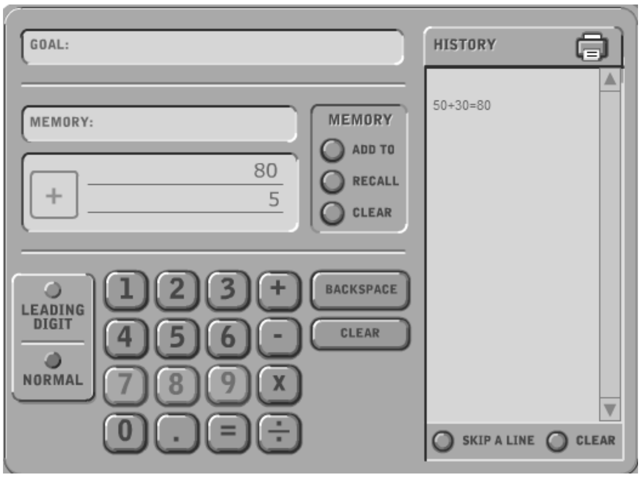

Figure 1: Web-based broken calculator

DESCRIPTION OF THE STUDY

For the purpose of the study the usage of 'broken' calculator activities was considered as the most appropriate approach. Judah Schwartz initially published the Broken Calculator software in 1989 under the title "What Do You Do with a Broken Calculator". (A web-based Flash version can be found in http://seeingmath.concord.org/broken_calculator/) (see Figure 1). Broken calculator activities are essentially puzzles in which the students must figure out how to perform certain tasks on a calculator that has one or more non-working keys. Such activities '*help students focus on and analyze the structure and elements of arithmetic and gain skills along with understanding, rather than have the calculator replace their thinking*' (Goldenberg, 2000). According to Schwartz (1999) the partial availability of the keys forces students to devise alternative ways of doing what the missing tools would ordinarily do. This serves as a way for accessing the students' mathematical understanding.

The participants were 19 students attending the third grade in a primary school in Greece. The tasks were posed to them after they completed the unit of the 3-digit numbers which means that they had spent time on being familiar with these numbers as also on making operations and solving problems that use these numbers. More specifically and according to the official curriculum after finishing this unit the students must be able: to read and write 3-digit numbers, to move from oral to written form of numbers and vice versa, to distinguish units, tenths and hundredths and their place value, to apply easily numbers in everyday situations and finally to write numbers in their extended form.

The tasks are presented below in Figure 2.

Task-1

Press the keys to get the number 432. Add or subtract any number so as 3 is substituted by 0.

Task-2

In your calculator the only keys working are 0, 1, and the keys of the four operations. How we get the number 5000?

Task-3

In your calculator the keys 7, 8 and 9 are broken. How we get the number 78? Do the same for 777.

Task-4

Keys 9 and + are broken. Explain how to get 90. Do the same for 109. Do the same for 99.

Task-5

In your calculator the key 8 is broken. Explain how you can make the subtraction 800-130.

Task-6

The only keys working in your calculator are 1, 0, +, -. Try to get the number 345.

Task-7

The only keys working in your calculator are 2, 5, +, -. Try to get the number 205.

Figure 2: The Broken Calculator Tasks

Students were asked firstly to draw on a piece of paper the keys they would press before using the original keys of the calculator in order to have evidence of the way they are thinking about place value. Thinking in terms of pressing calculator’s keys is important since -as we shall see below- this would serve for detecting certain misconceptions related to place value. Their worksheets were collected and the students' answers were coded according to two criteria: a) whether a specific misconception was apparent and b) whether a correct answer could be considered as an indication of a poor or deeper knowledge of place value.

RESULTS - DISCUSSION

The data collected from the students' worksheets as also from their efforts to solve the tasks in the original environment of the broken calculators allowed us to distinguish three types of answers. In the first one there were answers showing certain misconceptions relevant to place value. In the second one the answers showed limited understanding of place value whereas in the third one there were answers indicative of a deeper knowledge of place value.

Type 1. Misconceptions relevant to place value

1. Zero is not place holder

The use of zero as a placeholder is a source of difficulties for young students. It appears to create confusion to students (Cockburn and Parslow-Williams, 2008). Part of this difficulty perhaps has been caused by considering zero as nothing and this difficulty concerns equally students (Horne and Livy, 2006) and teachers (Wheeler and Feghali, 1983). Task 2 could reveal the handling of numbers that include zero as 'following a pattern'. Actually many students use digits mechanically and with some degree of automaticity. For example, in order to make the sequence 100, 200, 300,… one has just to follow the pattern: *increase the first digit by one and then just place two zeros*. In the specific task 5000 was the wanted number. So, according to the above mentioned pattern the students had just to reach to number 5 and then to place three zeros. This could be done easily working on paper. So, the lack of knowledge concerning the arithmetic relationships that result to 5000 would be unnoticed. But it was different with the calculator environment. In the latter case the students had to reveal the way they think so as to achieve the three zeros and subsequently the specific number. Here are some examples of the students' work (From now on each couple of brackets [ ] means one key pressing):

[1][+][1][+][1][+][1][+][1]**[=] 5 [0][0][0]**[=] (obviously pressing the zero key would give 0 instead of 5000)

Something similar we found in Task 5. The students had somehow to get the number 800 since the [8] key was not available. Here is the way some students preferred to work:

[4][+][4][=] 8 **[0][0]**[=] (obviously 0 instead of 800)

In both cases the students got the wanted hundredths (5 and 8) and then just like in the above mentioned pattern they pressed three or two consecutive times respectively the zero key. This meant that for them: i) zero was not related with units, tenths and hundredths and ii) it was not required a specific multiplication to obtain 5000 or 800.

2. Unitary concept of multidigit numbers

In this case the multidigit numbers are seen as unitary collections. It is known that the value represented by a digit is the product of the 'face value' (Sinclair and Scheuer, 1993) of the digit and the value associated with the place of the digit in relation to the point of reference (i.e., the ones places). However, for students, very often tenth digits represent single units and not multiple of 10 ones (Hiebert and Wearne, 1992). So, the question is whether children understand that in case of 25 objects the number 2 represents twenty objects rather than 2 (representing thus a two digit number as two sets of units). We will present three different examples of using the number's digits as single units.

In Task-1 students had to get the number 3 (tenths) substituted by 0. This is the procedure followed by some of the students:

[4][3][2]**[-][3]**[=] (429 instead of 402)

Number 3 is considered as representing units and this is why the students decided to subtract 3 instead of 30.

In Task-6 students had to get the number 345. The written steps of some students were:

[1][+][1][+][1][=] 3 [1][+][1][+][1][+][1][=] 4 [1][+][1][+][1][+][1][+][1][=] 5

All the digits of the 3-digit number were considered independently as being just sets of units without any arithmetical relationship among them.

In Task-7 the students had to get the number 205. The problem was that the zero was not available. So some students worked as:

[2][-][2][=] 0.

Then according to them it would be easy to get 205 by having [2][0][5]

In another case the rationale was to get firstly the 200 and then by adding 5 to get 205:

[2][-][2][=] 0.

Then again the same process was repeated (so as to get the second zero) and finally the idea was to take 200 by putting the digits [2][0][0] one after the other.

All the cases show clearly that in the students' mind digits are not carrying a certain value that is associated with their placement in the number.

3. Reversing values of tens and ones

In this case the digits are reversed when writing numbers. Students regard numbers such as 52 and 25 identical. Profoundly, understanding them as identical means that students do not take place value of the digits into consideration. In Task-3 students were asked to get the number 78 given that the keys 7, 8, and 9 were broken. According to this type of misconception the asked number 78 is considered as identical to 87. This is why some students followed the process:

[5][0][+][3][0][=] 80 [+][5][+][2**][=] 87** (instead of 78)

4. Counting sequence errors

Counting sequence errors are occurring when students use counting approaches to work out answers for questions involving multidigit numbers. The most common mistakes of this kind are: a) mistakes when changing decade. The problem is on naming the next or previous decade in a counting sequence (e.g., 60, 71, 72, … or 63, 62, 61, 50, 49, 48, …), b) omitting numbers (e.g., 31, 41, 61, 71, …).

In Task-4 two of the wanted numbers were 99 and 109. Some of our students worked as follows:

[8][0][-][1][=] **99** (instead of 79)
[2][0][0][-][1][=] **109** (instead of 199)

The incorrect results 99 and 109 were the results that the students wrote in their worksheets and thought that would be displayed on calculator's screen. In both cases there was a problem in finding the decade that was before 80 and 200 (70 and 190 respectively).

5. Mistaken number representations (that reflect the base-10 numeration system)

To be successful in understanding place value, it is necessary for the students to posses certain mental skills such as the correct recalling of the numbers' names. Good number sense is important for the students to grasp the meaning of mathematical questions and for working out potential answers. There exists an association between the written symbols and the number words which are alternative representations of numerical quantities. So, it is a very challenging task to help students to reach the stage of learning numbers and their names. The way students respond to specific tasks can show a limited possession of the above mentioned association. In Task-3 the students were asked to get the number 777. However, for some of the students getting 777 was the same as getting 707 or 770:

[6][0][0][+][1][0][3][+][4][=] **707** (instead of 777)
[1][0][0][+] [1][0][0][+] [1][0][0][+] [1][0][0][+] [1][0][0][+] [1][0][0][+] [1][0][0][+][4][+][3][=] **707** (instead of 777)
[6][0][0][+][1][1][0][+][6][0][=] **770** (instead of 777)

Obviously, there is a lack of place value understanding. These students were not able to use features of the base-ten numeration system so as to represent accurately certain quantities by written numerical symbols.

Type 2. Limited understanding of place value.

Some examples of students' work are presented here showing limited understanding of place value. The way the students worked leads to the correct result. However, the path they followed indicates that they did not understand place value in depth.

Figure 3: Unitary concept of multidigit numbers

In Task-6 the students wanted to get the number 345. One student thought to get separately the numbers 300 and 45. In order to get 45 he added units.

[1][+][1][+][1][+][1][+]…[+][1][=] **45** (Fig. 3)

This kind of working is referred as *Unitary concept of mutlidigit numbers*. The students represent a multidigit number as a collection of ones only. The specific student did not see the 45 as 4 tens and 5 ones, but as 45 ones. This view of the numbers could be considered as one of the first steps towards understanding place value.

The next two examples of our students' working are in the same spirit. The difference is that in these examples the students worked with collection of units rather than single units.

In Task-7 the students were asked to get 205 given that the only operating keys were [2], [5], [+], and [-].

[5][+][5][+][5][+][5][+]… [+][5][+][5][+][5][+][5][=] **205**

In Task-2 and in order to get 5000, some students followed this path:

[100][+][100][+][100][+]… [100][+][100][+][100][=] **5000**

Even though the rationale behind this path was the same as in the unitary concept of multidigit numbers, the students now used fives or hundreds instead of ones.

Type 3. Answers showing understanding of place value

In this section we present some of the students' responses that show a refinement as far as the understanding of the place value is concerned.

In Task-1 all of the correct answers were based on subtracting 40 from 432. It is interesting to mention here that no one worked with addition (e.g., adding 70) even though the [+] key was available.

In Task-2 the students managed to get 5000 using a lot of different ways. We could mention two of them:

[1][0][0][0][+] [1][0][0][0][+] [1][0][0][0][+] [1][0][0][0][+] [1][0][0][0][=] **5000**
[1][+][1][+][1][+][1][+][1][=] **5** [*][1][0][0][0][=] **5000**

In Task-3 it was impressive how many different approaches were applied by the students. Some of them made only one addition (45+33, 64+14, 66+12). Others used two consecutive additions (60+16+2) or more than two (50+20+4+4, 60+10+5+3). The landscape was similar for the second part of the Task concerning the number 777. However, it is worthwhile mentioning here the fact that in some cases the paths in both parts followed by the same student were parallel as can be seen in the two examples below:

Student A

For 78: [3][5][+][3][5][+][4][+][4][=]
For 777: [3][5][0][+][3][5][0][+][3][5][+][3][5][+][4][+][3]

Student B

For 78: [5][0][+][2][0][+][5][+][3]
For 777: [5][0][0][+][2][0][0][+][5][0][+][2][0][+][5][+][2]

It is clear that the first one was based on the half of 70 and 700 (the core idea is 3.5+3.5=7). The second one was based on the fact that 5+2=7.

In Task-7 the handling of the available combinations of the allowed keys demanded higher level of skills so as to get 205. However, again there were many different answers revealing a broad understanding of place value:

[2][5][5][-][2][5][-][2][5][=]
[2][2][5][-][5][-][5][-][5][-][5][=]
[2][2][5][-][2][5][+][5][=]
[5][2][+][5][2][+][5][2][+][5][2][-][5][+][2][=]

CONCLUSIONS

Knowing the common mathematical errors and misconceptions of young students can provide teachers with an insight into students' thinking. This allows teachers to adapt the focus of the teaching and learning process in the classroom. Calculators should be seen as tools to assist teachers in detecting such errors and misconceptions. Carefully designed tasks especially for the calculator environment can be used to diagnose the extent to which students have grasped the mathematical meaning of the topic (place value in our case) and consequently where further teaching is needed. In the light of the information that the teacher is able to extract from students' responses about the nature of students' misconceptions, the teaching process is revised and retested. Our approach based on calculator's usage allowed us to detect some of the commonest misconceptions concerning place value: Zero is not place holder; Unitary concept of multidigit numbers; Reversing values of tens and ones; Counting sequence errors; and Mistaken number representations. At the same time we had the chance to record students with limited understanding of place value or students who showed a performance that ensures an exploitation of the acquired mathematical knowledge on place value. So, it seems that the usage of calculators can be broadened. The major view for calculators considers them as tools for speeding up arithmetic computations or for ascertaining the accuracy of those computations. There also exist studies proposing calculators for exploring patterns and discovering more about mathematical concepts and even more for investigating mathematical relationships. However, the purpose of this paper is to add and demonstrate another role for the calculator, that of the diagnostic tool in the domain of place value. We hope that as our teachers have started progressively to adopt the perspective of the usage of technology in their classrooms, they would be better able to incorporate calculators in their daily practice. This would give them the chance to assess the usefulness of technology as a tool for modifying their lesson so as to respond to the students' needs and level of knowledge.

REFERENCES

Bednarz, N., & Janvier, B.: 1988, A constructivist approach to numeration in primary school: Results of a three year intervention with the same group of children, *Educational Studies in Mathematics,* **19**(3), pp. 299-331

Charles, R.: 1999, Calculators at the elementary school level? Yes, it just makes Sense! *Mathematics education dialogues,* **2**(3), p. 11, N.C.T.M.

Chick, H. & Baker, M.: 2005, Investigating teachers' responses to student misconceptions, in H. Chick & J. Vincent (Eds.), *Proc. 29th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol.2), Melbourne:PME, pp. 249-256.

Cockburn, A. & Parslow-Williams, P.: 2008, Zero: understanding an apparently paradoxical number, in A. Cockburn & G. Littler (Eds.), *Mathematical Misconceptions*, London: Sage Publication, pp. 7-22.

Fuson, K. C.: 1988, *Children's counting and concept of number,* New York: Springer-Verlag.

Goldenberg, P.: 2000, *Thinking (and talking) about technology in math classrooms,* Education Development Center, Inc.

Groves, S., & Stacey, K.: 1998, Calculators in primary mathematics: Exploring number before teaching algorithms, in L. J.Morrow & M. J. Kenney (Eds.) *The teaching and learning of algorithms in school mathematics.* 1998 Yearbook of the N.C.T.M., pp. 120 -129.

Hembree, R., & Dessart, D. J.: 1992, Research on calculators in mathematics education, in J. T. Fey (Ed.) *Calculators in mathematics education;* 1992 Yearbook of the N.C.T.M., pp. 22-31

Hiebert, J. & Wearne, D.: 1992, Links Between Teaching and Learning Place Value With Understanding in First Grade, *Journal for Research in Mathematics Education,* **23** (2), pp. 98-122.

Horne, M. & Livy, S.: 2006, Young children developing place value understandings. In Novotná, J., et al. (Eds.). *Proc. 30th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 313-320). Prague: PME

Ross, S. H.: 1990, Children's acquisition of place-value numeration concepts: The roles of cognitive development and instruction, *Focus on Learning Problems in Mathematics,* **12**(1), pp .1-17

Ruthven**,** K**.:** 2009, Towards a calculator-aware mathematics curriculum, *Mediterranean Journal for Research in Mathematics Education,* **8**(1), pp. 111-124

Schwartz, J.: 1999, Can technology help us make the mathematics curriculum intellectually stimulating and socially responsible? *International Journal of Computers for Mathematical Learning,* **4**, pp. 99-119.

Sinclair, A., & Scheuer, N.: 1993, Understanding the written number system: 6 year olds in Argentina and Switzerland, *Educational Studies in Mathematics,* **24**(2), pp. 199-221.

Stigler, J., & Hiebert, J.: 1999, *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press

Wheeler, M., & Feghali, I.: 1983, Much ado about nothing: preservice elementary schools teachers' concept of zero, *Journal for Research in Mathematics Education 14*(3), pp. 147-155.

THE IMPACT OF FORMATIVE ASSESSMENT ON PUPIL'S ACADEMIC ACHIEVEMENT AT THE ELEMENTARY SCHOOL

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In this article we deal with the question of the evaluation of the pupil's learning results. We observed the impact of formative assessment on pupil's learning outcomes which they achieved in our research. We present the results of an experiment in which we stepped into the teaching process by formative assessment of teaching materials prepared by us. Experimental and comparison group consisted of 14-15 years old students of secondary school.

THEORETICAL BASIS OF THE SURVEYED PROBLEM

Our article is dedicated to formative assessment and its impact on the achieved learning outcomes of students in mathematics. We consider it an important factor affecting the level of achievement in the learning process. It is important for the humanization of evaluation in the learning process also. The qualitative characteristics are observed quite often in the teaching of mathematics by the phenomena analysis of the tasks in assessment. This way is possible to identify individual problems of the pupil and remove them more rationally on the individual basis.

G. Littler (2002) identifies the three main methods of evaluation:

a) Summative – is the final evaluation, which takes place at the end of the semester, school year, after completion of the course etc. This evaluation is used to compare students with each other according to the different perspectives. It is called a testing relative to standard.

b) Formative – is the informal as well as formal evaluation, which should be adopted by every teacher in all to help the teacher to determine how much the particular student learned and to plan the further work with the individual or group in a way to achieve their goals. It is called testing in relation to requirements.

c) Diagnostic – the teacher chooses few specific issues with intention to determine the cause of misunderstanding or failure to properly solve the task or to reach the correct result.

The evaluation in mathematics as well as in other subjects is specific. According to G. Littler (2002) we evaluate the following in mathematics:

- facts - what we want students to know,

- skills - what we want students to be able to manage,
- terms - what we want students to understand,
- usage - what we want students to be able to use in every situation,
- behavior - what social skills we want students to gain,
- attitude - what attitude we want students to gain.

The formative assessment is not usually included in the formal grading of student's work. Its purpose is feedback, detection and diagnosis of the mistakes, issues and their causes in the learning activities of the pupil. The aim is to eliminate these deficiencies and streamline learning activities of the pupil. We can consider the use of the formative assessment to correct teaching methods of teachers and learning methods of students as an appropriate instrument that can be applied in the teaching of mathematics as well. The assessment is formative when the determined information is really used to eliminate deficiencies in student's work (Formative assessment, 2005). Students will be informed quite soon where in the reading and solving of mathematical problem solving tasks are mistakes being made. Moreover, students are not affected by stress as they know their performance won't be graded and included in the final formal grade from mathematics.

OECD CERI Project Formative Assessment Improving Learning in Secondary Classrooms (2005) analyzes the formative assessment approach. It was implemented in eight countries from 2002 to 2004 and has produced results that can be used in the education reform. The project has provided the analysis and evaluation of the formative assessment and teaching strategies. The formative assessment is characterized as a frequent and early interactive evaluation of the student's progress and understanding. Teachers are then able to adjust teaching methods to determine student's learning needs better. In this way the summative information is used formatively at each level of the system (see Figure 1).

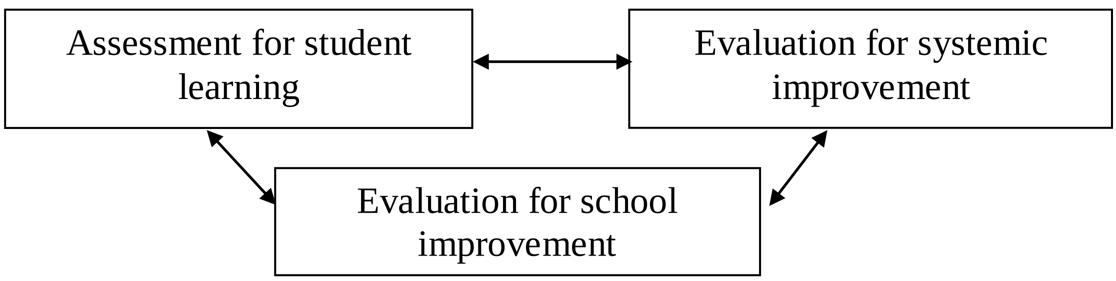


Figure 1: Coordinating assessment and evaluation (Formative assessment, 2005)

Information gathered at each level can be used to identify strengths and weaknesses and to shape strategies for improvement.

Formative assessment is one of the most effective strategies to support student's high performance. There are six key components of the formative assessment.

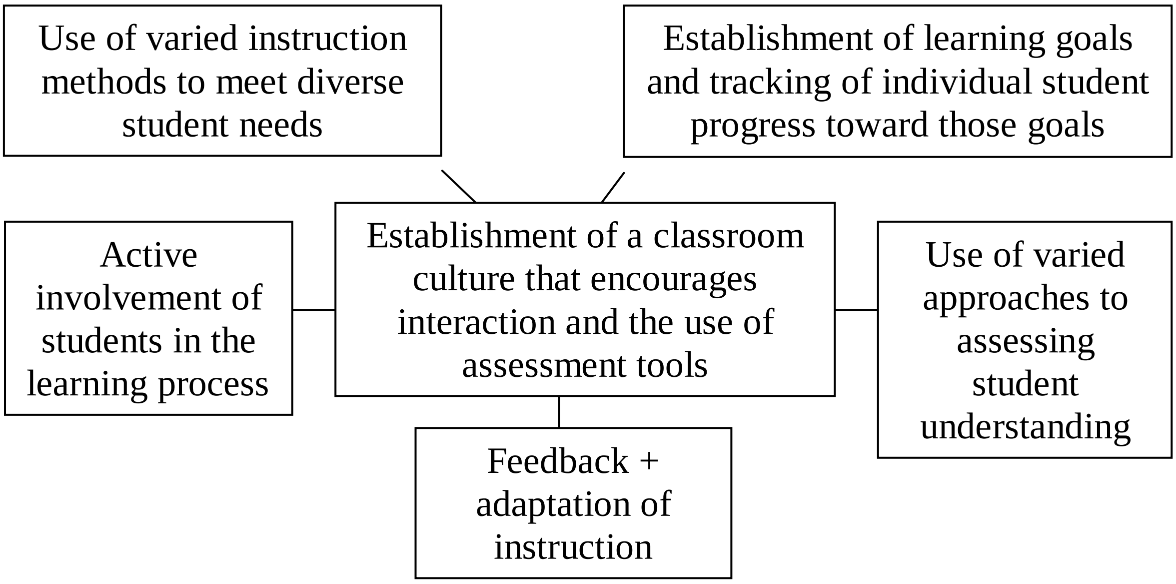


Figure 2: The six key elements of formative assessment (Formative assessment, 2005)

During the time when project was carried out in schools teachers were using formative assessment as a tool for teaching and learning. The change of attitude was focused to create and maintain the regulation of formative assessment. Teachers accentuated on each of these elements in order to create a dynamic work environment in the classroom and move students closer to achievement of educational goals. OECD developed principles for the support of formative assessment where it is stated that importance of summative and formative assessment approaches is equal. Countries and their politics should pay attention to investment to the education and the promotion of formative assessments to encourage teachers to include this approach of evaluation in their practice. To achieve this, it is necessary to support the research and innovation in teaching and to support the development and implementation of projects in this field also. It is also important to encourage the active participation of students and their parents in the process of formative assessment because of the fact that it is an interactive process between teachers and pupils. By increasing emphasis on formative assessment the amount of the formal grades as a result of summative assessment is decreased – it requires parents to understand that it will have a positive impact on the future life of their child. OECD CERI in this project highlights the importance of formative assessment in the light of effective teaching and assessment in the lifetime learning. Each country developed different issues during the project. This led to the conclusion that it is necessary to support further research in the field of widespread use of formative assessment to gain the experience in the forthcoming decades.

THE EXPERIMENT

The aims of the experiment

The main objective of our research was to compare learning outcomes of students of mathematics where the learning process was supported by using formative assessment and the learning outcomes of students without such support of the teaching.

To achieve this we set the following tasks:

1. to enter the teaching process through formative assessment,
2. to develop didactic materials for this formative assessment,
3. to identify and evaluate its impact on the realized level of knowledge of students of mathematics.

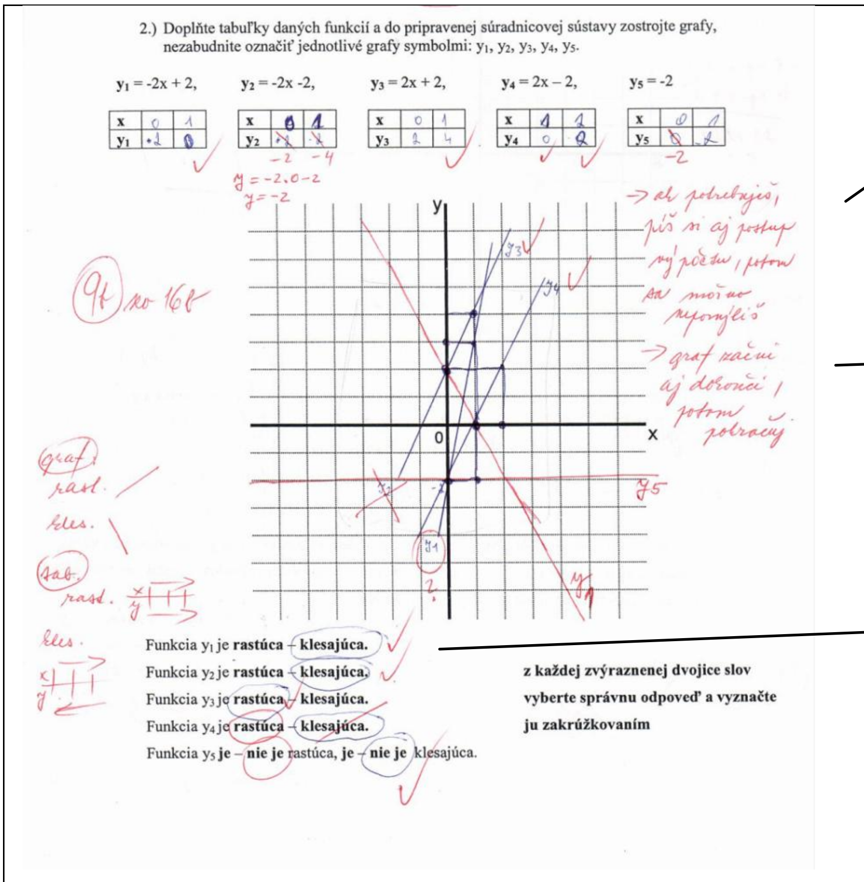
The course of the experiment

The research sample consisted of 80 students of 9th year class, who were divided into two groups: experimental and comparative. The experiment ran from March to June 2008 in two parallel classes in two schools where the learning process went as follows: in one of them as common (control group) and in the second one (experimental group) we used the didactic materials related to formative assessment. Frequency of use of these didactic materials was 3-5 lessons depending on the nature of the curriculum. This was the way how curriculum of two thematic units was assessed, however there was also used a summative assessment in the teacher's common way in each of them. Didactical materials have been solved by students of experimental group during our presence and they were also assessed by us. Pupils had the assessment recorded in writing directly in the tests and this written assessment has been completed by individual oral assessment on the forthcoming lesson after the exam. They had the opportunity to ask what they weren't sure about however, this option was rarely used. The reason why they rarely had any questions might be clarity in the assessment work or even disinterest of students, eventually some personal barriers. Before the start of cooperation pupils were informed about our goal and that their achievements won't be assessed by formal grade. We have determined the initial knowledge and skills of surveyed subjects by pre-test and outgoing by post-test to discover the impact of experimental exposure. We have adopted the test results from Testing 9-2008 in mathematics as a result of pre-test (nationwide testing of the 9th year class of elementary school students carried out by National Institute for Education of the SR) which were sufficiently credible to compare the equivalence of groups due to professional development and the way how it had been carried out. We processed the results of pre-test via the statistical data processing methods. We concluded that students of experimental and control groups will form equivalent groups which mean their ability to achieve certain learning outcomes in mathematics will be at a comparable level.

Tests used in the experiment

We created 12 didactic materials constructed as tests, papers and worksheets with content adjusted to actual curriculum in the classroom. They included text-based tasks, questions with making answers and multiple choice answers questions. There were also solved tasks whose solutions contained some mistakes made on purpose. Pupils were supposed to find and correct them. One of these worksheets pupils were supposed to evaluate in addition. However, some pupils did not evaluate them: that is why we decided not to assess this part of their work. Worksheets also contained some nets of solids which pupils were supposed to cut out and put them up to individual solids. Some tasks were focused on reading mathematical text with understanding and also on increasing mathematical literacy. According to the difficulty of tasks, their number varied from 2 to 7 in one test. Time for their solving was from 10 to 30 minutes. Types of these tasks were not different from the tasks which are commonly used in summative assessment. Difference was in the way they were assessed. We assessed these orally without the final grading.

Tests no. 1, 2, 3 were focused on linear functions, namely: direct and indirect variation, formula and graph of functions, domain and co-domain of function, plotting points in Cartesian coordinate system, text-based tasks used in practice. Tests no. 4 to 10 were focused on the geometric solid objects and their nets, volumes and surfaces of objects, conversion units of length, geometric text-based tasks. Tests no. 11 and 12 had amended form and content, they were processed written tests which pupils had to check, correct errors in the solutions and evaluate by grade. As a sample of pupil's solution and evaluation please see the task of test no.3. Handwriting notes in the sample are notes of the teacher.



Teacher ´commentary:

- *if you need to, please write each step of you solution for yourself, it might prevent your mistakes*
- *you should start and finish the graph, continue afterwards*

Function is increasing - decreasing

Figure 3: The sample of test no.3

Examples of the other verbal assessments:

...try to work regularly in order to obtain comprehensive knowledge,

...try to improve your graphic expression,

...try to read carefully to avoid unnecessary numerical errors,

...you would know how to solve this task if you read the text of the task more carefully,

...you won't be able to calculate the volume of the object if you do not know the appropriate formula,

..if you have weaknesses in this curricular area, please come to see me we can try to eliminate them,

...your solution notes are not clear, you have got lost in them. Would you prefer to record it this way....? (followed by the teacher's demonstration),

...although you solved the task, you did not answer the question in the assignment. Because of your inconsistency you lost some points.

We have observed the positive impact of the verbal assessment on pupils.

At the end of the experiment we used the *post-test.* It was a classic exam at the end of the classification period. In this case it was the last quarter of the school year. We agreed on the choice of 5 assignments in this exam as well as the way of how to asses and award points with both teachers that cooperated with us. The grade wasn't important for us; teachers evaluated exams by formal grades for themselves. We have focused on the overall success of this exam and analysis of solutions for each task.

The evaluation of the experiment

Obtained data have been processed by *statistical data processing methods* used in educational research, namely: descriptive statistics, Levene's test for homogeneity of variances, Median test. The final set of hypotheses evaluation confirmed that the experimental and control groups were equivalent in pre-test and there was a statistically significant difference in favor of the experimental group in post-test. It was confirmed that the *use of formative assessment has a positive impact on the achieved learning outcomes of students in teaching mathematics.* Below are the descriptive statistic tables containing results of experimental and control groups in the pre-test and post-test, where there was chosen 95% interval of confidence.

Effect	Descriptive Statistics							
	Level of Factor	Level of Factor	N	Pretest Mean	Pretest Std. Dev.	Pretest Std. Err.	Pretest -95%	Pretest 95%
Total			73	55,27397	23,97757	2,806362	49,67959	60,86835
Group	con		38	51,05263	20,70246	3,358382	44,2479	57,85736
Group	exp		35	59,85714	26,63715	4,5025	50,70696	69,00732
School	ZŠ1		36	67,08333	16,22938	2,704897	61,5921	72,57457
School	ZŠ2		37	43,78378	24,87204	4,088939	35,49103	52,07654
Group*School	con	ZŠ1	18	62,5	13,95897	3,290162	55,55837	69,44163
Group*School	con	ZŠ2	20	40,75	20,60116	4,60656	31,10836	50,39164
Group*School	exp	ZŠ1	18	71,66667	17,40521	4,102446	63,01126	80,32207
Group*School	exp	ZŠ2	17	47,35294	29,37461	7,124389	32,24991	62,45597

Table 1: Descriptive statistics – pre-test

Effect	Descriptive Statistics							
	Level of Factor	Level of Factor	N	Posttest Mean	Posttest Std. Dev.	Posttest Std. Err.	Posttest -95%	Posttest 95%
Total			73	53,60274	28,1695	3,29699	47,03031	60,17517
Group	con		38	47,13158	24,80319	4,023609	38,97897	55,28419
Group	exp		35	60,62857	30,22084	5,108254	50,24735	71,00979
School	ZŠ1		36	55,77778	25,56349	4,260582	47,12834	64,42722
School	ZŠ2		37	51,48649	30,69801	5,046722	41,25126	61,72171
Group*School	con	ZŠ1	18	46,16667	19,09804	4,501452	36,66943	55,6639
Group*School	con	ZŠ2	20	48	29,49576	6,595453	34,19556	61,80444
Group*School	exp	ZŠ1	18	65,38889	28,01919	6,604187	51,45527	79,32251
Group*School	exp	ZŠ2	17	55,58824	32,4674	7,874502	38,89504	72,28143

Table 2: Descriptive statistics – post-test

During the experiment there were some factors that significantly complicated the realization of our didactic materials with students (disruptions to lesson caused by various external influences and associated disruption of continuity of action, personal failure of individuals to attend the lesson, etc). In selecting the research sample, we chose 9th year class students. One of the reasons was Testing 9-2008 as we wanted to use its results as results of pre-test to compare the equivalence of experimental and control groups. Another factor was the influence to the group in which it was difficult to motivate pupils to work in order to consider the results of our survey valid for pupils where the motivation to work was less demanding and students are willing to cooperate.

CONCLUSION

Based on the results of educational research and theoretical background of the survey we have come to results which lead us to formulate the following recommendations for teachers of mathematics:

- to include more tasks which are developing mathematical literacy to the curriculum,

- to give the greater attention to reading with comprehension in teaching of mathematics,
- to lead students to minimize the amount of numerical errors in solving of mathematical tasks,
- to use a calculator appropriately in the teaching of mathematics so that the students could practice the estimation of the result
- to lead students to the regular systematic preparation for the lessons in mathematics,
- to assess often and continuously the work of pupils in teaching of mathematics,
- to alternate used assessment methods of pupil's work in teaching of mathematics,
- confirmatory assessment to be used more frequently as we consider it important for the humanization of evaluation,
- positive assessment to be used more frequently, find a positive element even in the poor work and admire student.

The formative assessment helps the pupil to assess his or her work properly. The main problem in its use is its time requirement. It is important to have didactic materials which teachers can adapt on their own according the abilities of students. The information obtained from these materials can greatly help students in eliminating their weaknesses in the work. We consider increased use of the formative assessment as a significant element of humanization in learning activities evaluation.

REFERENCES

Čeretková, S., & Šedivý, O.: 2005, Aktuálne problémy teórie vyučovania matematiky, FPV UKF, Nitra.

Kolář, Z., Šikulová, R.: 2005, Hodnocení žáků, Grada, Praha.

Velikanič, J.: 1973, Skúšanie, hodnotenie a klasifikácia žiakov, SPN, Bratislava.

Education today, The OECD Perspective [Available on: http://browse.oecdbookshop.org/oecd/pdfs/browseit/9609021E. PDF, 2009-03-25]

Formative assessment, Improving learning in secondary classrooms, OECD [Available on: http://browse.oecdbookshop.org/oecd/pdfs/browseit/9605021E. PDF, 2008-10-21].

Littler, G.: 2002, Comenius course EMTISM (Empowering Mathematics Teachers for the Improvement of School Mathematics), Praha, CD [Available on: ww.tnantik.sk/Hypertext/Texty/ratio_studiorum.htm. 2008-03-10].

Matematická gramotnosť, PISA SK: 2003, [Available on: http://www.statpedu.sk/ Projekty/PISA/ PISApubl/ 04_matematicka_gramotnost.pdf, 2007-06-20]

Národná správa PISA: 2006, [Available on: http://www2.statpedu.sk/buxus/docs/projekty/PISA/pisa2006nsprava.pdf, 2008-05-10].

Správa o priebehu a výsledkoch testovania žiakov 9. ročníka ZŠ MONITOR 9-2007, [Available on: http://www.statpedu.sk/buxus/docs/cert_merania/2007/sprava_monitor9_2007.pdf, 2007-12-16.]

Teachers' training

Part 4

TEACHERS SUPPORTING MATHEMATICAL DEVELOPMENT

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INTRODUCTION

In a recent TED-talk[1], Dan Meyer (2010), a high school mathematics teacher in the United States, explains why he thinks mathematics teachers are not doing a good job. In his talk he mentions five symptoms in students that indicate that a teacher is doing math reasoning wrong in a classroom: lack of student initiative; lack of perseverance; lack of retention; aversion to word problems; and eagerness for formulas. Many teachers and many former students will recognize these symptoms. Meyer proposes that we: redefine what a word problem is; redefine what math is; redefine what problem solving is; and redefine what doing math is.

He recognizes that textbooks are not necessarily helpful. All too often, a problem in a textbook is just a series of directed questions that students need to answer, which buys you out of reasoning and problem solving. As a result, the students only have to use a set of tricks, algorithms, or formulas, and when observing students at work, one often hears them wonder what rule to use. This kind of problems takes the mathematical thinking away from children, or as one of the people who reacted to this talk wrote: “(we) feed them (students) information and knowledge before they are hungry for it.”

In his talk, Dan Meyer suggests to use real world problems, and to make sure that the students buy into the problem. His problems are not a set of connected short questions, but he asks the shortest question possible, even trying to have the students define and build the problem. Implicitly he suggests that the classroom norms and the role of the teacher will have to change too, but in this talk he does not touch on that.

Neither his message nor his solution is new. For instance, the Freudenthal Institute has been working on the design of realistic mathematics and problem solving for years, while others have focused more on the classroom community, or on the norms in the classroom (for instance Cobb, Wood & Yackel, 1991; McClain & Cobb, 2001).

[1] TED is a small nonprofit devoted to Ideas Worth Spreading. It started out (in 1984) as a conference bringing together people from three worlds: **Technology, Entertainment, Design.** Since then its scope has become ever broader. Along with two annual conferences -- the TED Conference in Long Beach and Palm Springs each spring, and the TEDGlobal conference in Oxford UK each summer -- TED includes the award-winning TEDTalks video site, the Open Translation Project and Open TV Project, the inspiring TEDx program and the annual TED Prize.

PROBLEM-ORIENTED INTERACTIVE MATHEMATICS TEACHING

Realistic mathematics is based on the notion that students can solve realistic, meaningful problems using common sense, and during and after solving the problem slowly (progressively) develop mathematical knowledge. We believe that problem-oriented, interactive education offers specific affordances in this area. Freudenthal (1973) argued that students should engage in ‘mathematics as a human activity’ instead of being taught mathematics as a ‘ready-made product.’ According to Freudenthal, students should be given the opportunity to reinvent mathematics with the help of teachers and well-chosen tasks. We expect that problem-oriented, interactive mathematics education will be more motivating and interesting for students than a traditional curriculum would be, and will therefore contribute positively to developing an interest in mathematics.

We offer students larger investigations to explore in small groups, and they are expected to be able to explain, justify, and defend their thinking in their small group work and during whole class discourse. This kind of problem-solving is not directed at the solution in itself. In contrast to what many students and teachers seem to think, mathematics education is directed towards the development of mathematical understanding and mathematical knowledge, not at finding solutions to problems. As Steffe (1991, p. 187) puts it, “the purpose for engaging children in goal-directed activity that includes problem solving is not simply the solution of specific problems. The primary reason is to encourage the interiorization and reorganization of the involved schemes as a result of the activity.”

When a solution is found, the development of mathematical understanding continues. The solution serves as a jumping board to another level of mathematical thinking and development, from horizontal to vertical mathematization.

> And, although there will always be a natural differentiation in the various levels of mathematizing in a group of pupils, the (vertical) development from lower level to higher level solution strategies is not the intention of horizontal mathematization. When, however, teaching focuses on this development of strategies and concepts in a certain area of the mathematical system itself, it is called vertical mathematization (Treffers & Beishuizen, 1999, p. 32).

Regardless of the number of problem solved, vertical mathematization requires a reorganization of the mathematical thinking.

Solving mathematical problems can be seen as a part of horizontal mathematization: the transformation of a realistic problem into the world of mathematics. In horizontal mathematization, students will for instance schematize, visualize, reformulate problems, use mathematical tools, and identify and use similar mathematical aspects in different problems. “Horizontal mathematisation leads from the world of life into the world of symbols”

(Freudenthal, 1991, p. 41). For him – quoting Adri Treffers – moving in the world of symbols is vertical mathematization. "(There) symbols are shaped, reshaped, manipulated, mechanically, comprehendingly, reflectingly (ibid, pp. 41-42)."

NATURAL DIFFERENTIATION

In my own experience, problem-oriented interactive mathematics teaching allows students to work together on the same problem and hence on the same mathematical content, even though each student works at his or her own level. This will definitely occur when the problem is designed to allow students to choose their own level of difficulty depending on the situation, supporting them to develop a deeper mathematical understanding. They can solve the problem at their level of abstraction, but later during the classroom discourse they develop and expand their mathematical theory. Each child will refine its mathematical theory in a personal way and will be left with personal questions and possible answers. In my opinion problem-oriented interactive mathematics teaching can be described as a substantial learning environment (SLE). Wittmann et al. (2004, 365-366) described four criteria for a substantial learning environment that would allow for natural differentiation:

1. It represents central objectives, contents and principles of teaching mathematics at a certain level.
2. It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities.
3. It is flexible and can be adapted to the special conditions of a classroom.
4. It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research.

Allowing and supporting natural differentiation is not easy. When allowing for differences amongst children, many teachers are not sure what to tell the children, how to confer with the different groups, or how to support each child. Teachers have to support students pursuing their own questions and taking mathematical and problem solving decisions on their own; furthermore, with problem solving activities in class, the role of the teacher as initiator of mathematical development becomes crucial. However, many teachers feel they do not have a handle on what the students are doing. They need to be able to solve the problems themselves, and to anticipate on children's' strategies. Therefore, they themselves need to understand the mathematics students are using and constructing.

REFLECTING ON A CLASSROOM EXPERIMENT

During a session of an in-service course, Arianne, a second grade teacher, discusses with other participants what happened in her classroom.

She had given the students a problem she had re-designed herself. The first classroom experiment had started a few weeks before. At that time one of her colleagues had designed an investigation for his grade 1 children about the younger sister of a famous Dutch skater. This girl saw her sister train and wanted to know for how long she was training. She could only count to three. Her mother counted how often she did that: 23 times. The younger sister now wondered what she had counted.

The investigation seemed very challenging for children that age. 23 x 3 is not a common problem in grade 1. At least that was what the other participants thought when they saw the investigation. However, when they focused on the children's work, they were amazed. The children were capable of more mathematical thinking than they had imagined. They were also surprised to see that the large numbers did not really matter.

The discussion that followed focused on the numbers used. What is the role of the context, or more specifically, what numbers – so was the question the mathematics educator posed – could support the children? The connection to the place value system was made and some teachers wanted to try a new investigation using the number 5. Arianne, one of the teachers, created an investigation where an artist used 5 pencils a week and the question was how many pencils she would need for a period of 168 weeks. She used this in a classroom experiment in grade 1 and 2.

In analyzing the children's work and their thinking, the teachers noticed that they did not use any visualization or representation to support their thinking.

Dave: "No one made groups, did they?"

Jane: "Indeed, they just wrote numbers, they did not use any models."

Arianne: "When I asked them to help me understand what they did, they could not really do that."

Ariane: "Maybe it has to do with… Maybe that is because of the context. There was no place … uh to group, to realize you could think in grouping. Did I consciously do it? No, of course not."

Jane: "No, in reflection we start to realize what is needed."

Arianna: "We can understand why they did not use any visualization. The children had no reason to draw. The context was very verbal."

Jane: "Yes, you had given them a verbal context."

Dave: "And they reacted verbally."

John: "I cannot say our expectations are met. I had not expected they would add all these fives, but that they would first make a drawing and then skip count with fives."

Arianne: "I have to tell you I worked and reworked the context so often. I never planned that children would not draw. Now I see the link between the context and the thinking of the children. The context

> supports the children. A context not only challenges children, it also supports them."

Reflecting on the investigation, Arianne realized the context had not supported the children in developing a visualization. She also realized that this kind of challenging investigation is more fun than she ever thought. The children enjoyed the work and she slowly changed her role. This time she started to ask the children for explanations and justification of their thinking. So, looking back, she realized that while her context and teaching had been challenging for the children, it had not supported them enough. Teachers have to develop a repertoire that walks a fine line between challenging and supporting. On the one hand they want to develop challenging contexts that allow children to investigate and construct mathematics; on the other hand they want to give some support too. Some teachers are trying to be less comforting while they walk around and talk with students as they are working on the investigation, but at the same time they are tempted to transfer their old role to the context. They would like the context to be so clear cut that the children will always know what to do, and would thus once again take the mathematics away from them. This also occurred in the follow-up discussion about Arianne's investigation. Some teachers preferred giving the children a drawing of 16 pots with 5 pencils, while others suggested using a drawing where children could only see an idea of grouping without doing all the work for them. This debate between 'leading children to the answer' versus 'encouraging them to reflect on what they are doing and grow from understanding what it is that they are doing' or the debate between 'capitalizing on students' inventions' and 'planning instructional interventions' was not resolved. Subsequent classroom experiments should give the teachers more information to fine-tune their theories. These experiments would also have to support the teachers in seeing how these two positions are not automatically opposites, but can be merged; additionally, the experiment would also support teachers to develop a repertoire that allows using the students' inventions in instructional interventions.

TEACHERS' ROLES

Problem-oriented interactive mathematics teaching asks for different roles and norms in the classroom compared to more traditional teaching (McClain & Cobb 2001). The way students and teachers interact has to be redesigned and the way mathematics learning is seen has to be re-conceptualized. Students have to become used to the kind of mathematics class that involves working on fewer but more substantial problems, using common sense (Freudenthal, 1991), taking decisions on their own, and pursuing their own questions. The teachers' role changes as well. Teachers have to support students with pursuing their own questions and with taking mathematical and problem solving decisions on their own; for instance, they have to give students just enough information about the problem situation to allow them to start to construct one or more solutions.

Classroom culture is related to teachers' beliefs about teaching and learning. In most countries the view on teaching as 'direct transmission' and the view on teaching as 'facilitating students to construct knowledge' co-exist (OECD, 2009). Teachers who support a 'talk and chalk' approach to teaching often believe that teachers have to explain the theory, demonstrate correct solutions, and have students apply this new knowledge.

Teaching is more than facilitating. I suspect that the notion of facilitating could be misleading teachers. Of course, teachers need to support children in developing mathematical knowledge. They design learning environments where a student can actively construct knowledge. However, their role is more subtle as well, as they have to navigate the borders between focusing on understanding and meaningfulness, and on students' motivation and participation, between mathematical structures and students' development, between the community of students and the individual student, between supporting children's development and keeping the responsibility for learning with the children.

Teachers constantly have to keep mathematical explorations and conversations meaningful. For instance, we know that students start using partly-understood ideas, strategies and models. It is through this use that they start to understand them better. So questioning these partly-understood ideas will help the students to create a better understanding, and at the same time it might also create an unwillingness to participate. This delicate work is more complicated than the word 'facilitating' suggests.

The TALIS-report (OECD, 2009) concludes that it would be wrong to simply introduce constructivism. Teachers need to be convinced that they can be successful in communicating deep content and in involving students in cognitively demanding activities, thereby following constructivist principles, while maintaining a positive disciplinary climate and providing student-oriented support.

TEACHER PRACTICES

Creating And Organizing Substantial Learning Environments

Problem-oriented interactive mathematics starts with a real investigation that somehow captures students' interest and that will also allow them to investigate and discuss a mathematical big idea, strategy, or model. In several projects in the Netherlands (Dolk, Garssen, te Selle, 2010) teachers have been working on designing, using, and researching this kind of investigation. In designing an investigation, the teachers started to think about the big ideas, strategies or models that students might develop or use with a certain investigation. They needed to take into account how the students' learning would take place. If needed, they would redesign the investigation – by changing the context, or changing the numbers – to make sure. These teachers mentioned that an investigation needs to be challenging enough. Children should not be able to solve it immediately; there is a need to ‘kick the idea around.’ However, a context also has to contain potentially realized suggestions (Fosnot & Dolk, 2001); for instance, the numbers or the context suggest a certain strategy or the development of a model.

The teachers also developed a repertoire to make sure the students understand the investigation. They started to have students paraphrase or (re)build the problem to establish that they own it, and they looked for signs that students felt comfortable enough to use their common sense (Freudenthal, 1991). If they did not see signs to support these points, they reformulated the presentation of the situation. It became important for them to suggest that it was a real problem for someone in the story, as that would help the children to connect to the investigation.

Stimulating, Organizing, And Facilitating Mathematical Dialogue Among Students

When an investigation allows solutions at different levels of abstraction, (groups of) children can start at their own level. To support this, groups need to be made up carefully. We suggest creating groups of children that have an optimal mismatch, in the sense that they are heterogeneous enough (different in mathematical development and understanding) to inspire each other, but homogeneous enough to be able to understand each other and to work together in solving a problem. In other words, children who are too close in development might have nothing to say to each other; when the group consists of children who are far apart in development, there is a risk that one child will tell the others what to do to solve the problem. The latter situation also does not allow each student to work at a level within his grasp in collaboration with the group members.

When different groups of children solve the problem in different ways, there is the possibility to compare those solutions in a mathematics congress. To make this happen, we often ask children to construct a ‘poster’ that shows their solution, their thinking, and justifications of their thinking. Such a poster serves as a first reflection by the students on their solution process. A second reflection will happen when the students have to explain and defend their poster during the math congress. Teachers need to enable the children who created the poster to explain what they did and thought, and to make sure all students understand what the creators of the poster did and that they could all put that in their own words.

A mathematical conversation in the class where students are explaining, questioning, defending, and justifying a solution will help all children to understand the different solutions better, but it does not in itself allow them to reflect on and abstract the mathematical ideas, strategies, and models used. To allow for this vertical mathematizing, the teacher needs to recognize the mathematical moments in the conversation, and the mathematical development that might arise from this discussion.

Expectations about learning are important. Children need to be aware that they are expected to understand rather than have an (correct) answer. Establishing this norm is not enough. As Yackel and Cobb (1996) argue, it is just as important to establish the norms in the community about what constitutes mathematics and what makes a mathematical argument. These conversations are not easy; but in reality almost every argument used in class also defines what mathematics is in this class.

Charlotte, a teacher in grade 7, for instance, started a mini-lesson with a simple problem. She had announced that some of her problems would be so easy that they could be seen as insulting, while other problems could prove to be too difficult. She started with 12 x 3. It was not her intention, but this problem became the starting point for a complicated classroom discussion about what mathematical arguments stand up in class. Irene said that 12 x 3 is 36 because 2 x 3 is 6 and 1 x 3 is 3. “Oh”, Charlotte reacted, “Help us, here you say it is 36, but to me 6 and 3 is 9. Why is it 36?” Irene continued to follow a set of rules and even became agitated and angry because Charlotte refused to understand her argument. Irene could only repeat what a teacher had taught her a few years earlier. Other children – halfheartedly – joined the conversation. On the one hand, they understood what Irene said and they agreed. On the other hand, they also agreed with Charlotte. During that conversation many of them became impatient: “Let’s continue; what is the next problem? We need to continue.” However, every time Charlotte expressed her concern that the math did not make sense, they returned to the conversation. Slowly, more and more students tried to find words for what we call place value. The discussion became a discussion among the children. And all of a sudden, even Irene started to use two arguments. Unwilling to lose her safeguard – the procedure learned a long time ago and used for so many years – she continued to defend that rule and at the same time she argued that 1 x 3 could also be seen as 1 x 30. When in the end, most children used a place value based argument, Irene stated “And still my rule works.”

This classroom episode reveals how Charlotte and Irene differed in their opinion of what constitutes mathematics. To Irene, mathematics was practicing a set of rules and procedures to find an answer. For Charlotte, mathematics was about understanding and being able to explain what you are doing. Charlotte probably agrees with Keith Devlin (2010) for whom: “Mathematics is a way of thinking about problems and issues in the world. Get the thinking right and the skills come largely for free.” She started a dialogue about what is an acceptable mathematical argument in this class and how to discuss what we understand and what we do not understand. To do this, Charlotte listened carefully and intensely to the students’ arguments. She questioned these, gave counter-arguments or played devil's advocate to have the students confront the tension between following a not completely understood procedure and understanding. She forwarded questions and arguments to other children. She allowed students to share ideas, to convince themselves and others of their arguments. She made sure that the dialogue flowed mainly between the children. In all this she was supporting students' mathematical development.

Supporting And Fostering Mathematical Development

Sometimes teachers hinder mathematical development. With the best intentions they can take the mathematics and mathematical thinking away from children. Often, this is done out of compassion. The list of these moments is long: they try not to give the children problems that – in their opinion – are too hard for them; they confirm mathematical ideas before questioning these ideas; they acknowledge correct or incorrect answers. Although compassion is a good attribute for teachers, it is not helping children. The side effect of compassion is the lack of initiative and lack of perseverance in students that Ted Meyers (2010) remarked on. If a teacher is always there to support you, there is no reason to get started, nor to continue if you do not know how to.

A focus on teaching by the teacher can also be an obstacle for development. When focusing on teaching, teachers often listen for that moment that one student 'gets it'. The teachers' questions often focus students' thinking in such a way that the teacher is doing the mathematical work. One student reacting with understanding is seen as a sign that they can continue. A focus on teaching also causes the teacher to be responsible for the pacing; in Dutch primary schools this pacing is directed by a pacing calendar suggested by the text books. Changing the focus on the mathematical development by the children produces another pacing regime. In discussing a mathematical idea, children will only start discussing these ideas when they are not too easy and not too hard to understand and they continue the discussion as long as there is no shared understanding. Therefore, focusing on children's understanding allows them to decide the pacing.

What Action Do Teachers Use

At the risk of presenting a cookbook of routines related to problem-oriented mathematics teaching, I will conclude by listing some of the actions that I have observed teachers using and that were successful in the situation where I observed them.

Teachers listen to the learners and create moments for further discussions. They

- listen and question students to monitor their development,
- support students to explain their thinking to allow other students to help them build a deeper mathematical understanding,

- stimulate students to ask clarifying questions to support sense making by the students,
- ask students to pair talk at important math moments, to have all students talk about important mathematical ideas and to emphasize the importance of students' listening and reacting to each other,
- pose important questions to have students discuss possible confusions,
- ask open ended questions that probe student thinking, help them express their ideas more clearly, and develop a better understanding,
- ask authentic questions. For instance, teachers did not ask for the answer, as they knew that already. Teachers asked how students solved the problem and how they could convince themselves and others that their solution worked,
- paraphrase, at crucial moments, what students say to highlight the important mathematical ideas. At other moments they ask students to paraphrase, to be sure the students listen to each other and understand the mathematical ideas being discussed and to allow for an in-depth conversation when one or more students are not sure of [....],
- provide construction space for students to solve problems themselves, to discuss the ideas among themselves, allowing them to make sense of the problem and find a solution in their own way and in their own pace,
- give the students thinking time,
- carefully choose contexts, and numbers in the context to support mathematical development,
- choose a context that is appropriate and accessible to all students regardless of their development,
- keep the students grounded in the context or bring them back to the context to create meaning,
- bring the students back to the context to resolve questions and confusion,
- avoid that students generalize before they really understand the mathematical solution they have been working on,
- decide which solution or poster will be discussed by the community and why (based on the math development of the students)

Of course, this list is not complete. Teachers' actions are more varied than this summing up suggests. Most actions are related to the following questions that are at the basis of teachers actions:

- how to create and organize substantial learning environments,
- how to communicate mathematically with students,
- how to support and foster mathematical development,
- how to confer with students, to listen and pose important questions,

- how to stimulate, organize, and facilitate mathematical dialogue amongst students,
- how to develop a community of learners.

REFERENCES

Cobb, P., Wood, T., Yackel, E.: 1991, A constructivist approach to second grade mathematics. In E. Glaserfield (Ed.), *Radical constructivism in mathematics education*, Dordrecht: The Netherlands: Kluwer Academic Publishers. pp. 157-176.

Devlin, K.: 2010, WANTED: An Apollo Program for Math. *Stanford [Magazine – a Publication of the Stanford Alumni Association], May/June,* **32**.

Dolk, M., Garssen, F., te Selle, A.: 2010, Teachers' repertoire to promote the development of mathematical knowledge through problem solving. Paper to be presented at the EARCOME conference, Tokyo.

Fosnot, C. T., Dolk, M.: 2001, Young Mathematicians at Work (Vol I): Constructing Number Sense, Addition, and Subtraction. Portsmouth NH: Heinemann.

Freudenthal, H.: 1973, *Mathematics as an educational task.* Dordrecht: Reidel.

Freudenthal, H.: 1991, *Revisiting Mathematics Education*. The China Lectures. Dordrecht Boston: Kluwer Academic Publishers.

McClain, K., Cobb, P.: 2001, An Analysis of Development of Sociomathematical Norms in One First-Grade Classroom, *Journal of Research in Mathematics Education*, **32**(3), pp. 236-266.

Meyer, D.: 2010, *Math class needs a makeover*. Retrieved 20100712, from http://www.ted.com/talks/dan_meyer_math_curriculum_makeover.html.

Steffe, L. P.: 1991, The constructivist teaching experiment: Illustrations and implications. In E. Glaserfield (Ed.), *Radical constructivism in mathematics education.* Dordrecht, The Netherlands: Kluwer Academic Publishers.

Treffers, A., Beishuizen, M.: 1999, Realistic mathematics education in the Netherlands. In I. Thompson (Ed.), *Issues in teaching numeracy in primary school* (pp. 27-38). Buckingham, UK: Open University Press.

Wittmann, E. Ch. et al.: 2004, *Das Zahlenbuch. Mathematik im 1. Schuljahr. Lehrerband*. Leipzig: Klett Grundschulverlag.

Yackel, E., Cobb, P.: 1996, Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, **27**, pp. 458-477.

PROFESSIONAL DEVELOPMENT FOR TEACHERS OF MATHEMATICS: ENGAGING WITH RESEARCH AND STUDENT LEARNING

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Drawing on data from a national project in England, researching effective professional development for teachers of mathematics to students of all ages (NCETM 2009), the paper considers the relationship between professional development, educational research and students' learning. It considers three case studies of professional development for elementary and early childhood practitioners and explores how they made use of research findings and considered student learning in the ways in which they worked with teachers. The teachers' responses to these ways of working are presented through a mixture of evidence taken from classroom observation and interview and the possible long term implications of the professional development are assessed.

This paper draws on data from the 'Researching Effective Continuing Professional Development in Mathematics Education' (RECME) Project in the UK, funded by the National Centre for Excellence in the Teaching of Mathematics (NCETM, 2009), to explore the processes involved in effective professional development for teachers of mathematics. The project studied thirty continuing professional development initiatives for teachers of mathematics of learners ranging from three year olds to adults. In this paper we present case studies of three initiatives to illustrate how different approaches to professional development made use of research findings in their work with teachers and how they supported teachers in engaging with their students' learning . We also consider the teachers' responses to these different approaches.

Professional development initiatives often introduce teachers to new ways of working. However there is considerable evidence in the literature (Eraut, 2001) to suggest that teachers are resistant to change unless they are convinced that the change that is being proposed has credibility and will lead to improvements in their students' learning. In the paper we suggest possible approaches drawn from the case studies that sought to convince the teachers that change was worthwhile through using research and through paying attention to children's learning.

There has been some suggestion that teaching should be a research informed profession (for example (Hargreaves, 1996)) but Joubert and Sutherland's literature review (2008) found that very little research has considered how research is used in the professional development of teachers. Zeuli (1994) suggests that teachers tend to use research findings only when they match their own personal experiences of practice and we consider this assertion in relation to our findings. We consider the ways in which research was presented to the teachers in the course of their professional development as well as how they engaged with it and what they perceived to be its impact on their practice. We also look at the ways in which these cases of professional development initiatives supported the teachers in examining their students' learning. We suggest that if the teachers see that their students are learning better then they are more likely to continue with the professional action that has led to that improvement (Guskey, 2002).

The project adopted a theoretical framework based on the premise that the experiences and contexts of the teachers would have a major influence on their learning and professional development. This led us to pay particular attention to the teachers' interpretations of what their involvement meant to them as well as to the situation, opportunities and contexts in which they engaged with the professional development itself.

Data for the project included observations of lessons with two teachers from each initiative; interviews with these teachers about their professional development; observations of sessions; interviews with the leaders of the professional development initiatives; and responses to an online questionnaire sent out to participating teachers. Analysis of the data was undertaken using a grounded approach. Teachers were invited to co-construct reports about themselves and their initiatives. The findings have been reported in the final report (NCETM, 2009) and a number of other papers (for example De Geest et al., 2008).

The research identified three different organisational structures for professional development within its sample: courses, network groups and within-school initiatives. Networks were the most informal and were usually developed by teachers for groups of teachers drawn from several schools. They arose in response to pressing issues within professional practice and the teachers involved were often very committed to them. Courses were generally more formal than the other types of professional development and were typically offered by providers such as universities to teachers from a number of different schools. Within-school initiatives involved all members of staff involved in teaching mathematics and frequently addressed an issue which was of concern to the school as a whole. In some cases advice was provided by an external expert or consultant and the initiative was formally set up, in others approaches were more fluid. In this paper we present three cases: one network, one course and one within-school initiative, all of which involved teachers and other professionals working with children under the age of twelve years. We present the case studies in turn before discussing the role of research in professional development.

A NETWORK FOR PROFESSIONALS WORKING IN EARLY CHILDHOOD EDUCATION.

This initiative is set in the context of early childhood education in which a group of teachers and other practitioners established a network group. The impetus for the network came from two researchers, who ran courses and conferences for teachers. The researchers (2006) based their courses and conferences on their own research work on the notion of *children's mathematical graphics*.

Understanding the concept of *children's mathematical graphics* is key to the understanding of this professional development initiative. Although *children's mathematical graphics* have been compared to 'emergent writing' they are not the same. However, in both emergent writing and *children's mathematical graphics* children make and attach meanings to the graphical marks and symbols they use. In the past children's early mathematical understanding and development has been overlooked in early childhood education, to the detriment of their mathematical education. Carruthers and Worthington (2006) have drawn attention to the ways in which children attach mathematical meanings to some of the marks they make in the course of their play and develop their own symbols and representations for quantities and calculations. When these *mathematical graphics* are supported by sensitive adults who understand and value them, situations develop which allow children to explore and communicate their personal mathematical thinking, helping them to understand the standard abstract symbolism of mathematics. The network group gave the teachers and other professionals the opportunity to focus on their students' mathematics in this way.

The network group was led by Sarah, a teacher who went on a course run by one of the two researchers, who said:

> This was an inspirational course and came at exactly the right time for me and my school where I am maths coordinator. I initiated the Group in March 07 under the umbrella of the two researchers' organisation.

At the course, the researchers suggested that forming groups at a grass-roots level would help to encourage and support teachers and other professionals in working with children's own mathematics and Sarah went ahead with this venture. She said that the course had helped her to deal with a problem with the transition between pre-school and the first years of formal schooling.

Sarah, described how her work with the group led her to pursue her interest in *children's mathematical graphics*:

> It has made me research an area of the curriculum about which I am strangely passionate, reflect on my own understanding and practice, collect

> and collate evidence and share this with fellow maths enthusiasts within my school and the group.

Sarah and the other members of the group saw its function as supporting them in developing effective strategies to support children's mathematical development and helping them to develop an understanding of the importance of *children's mathematical graphics* and the ways in which children make sense of 'written' mathematics. The researchers hoped that, in their practice, the participants would move away from imposing mathematics on children and work towards supporting children in developing their own mathematical understandings and representations in meaningful contexts. In this way they hoped that the teachers would be able to support children in adopting conventional symbols, such as the numerals, more effectively. The relationship between the researchers and members of the network group was one in which support and encouragement provided when asked for but in which the researchers took no direct role in the meetings or their organisation.

The content of each session was decided co-operatively by the whole group which meant that it was relevant to each participant. During the meeting observed by the RECME researcher, the participants contributed examples of children's spontaneous mathematical problem solving from their own practice. These examples were shared with the group and the scenarios from which they had arisen were discussed. The topic had been chosen at the previous meeting in response to concerns related to government advice on practice. Sarah described how the work of the group was leading them on to further work:

> At the group meetings we share examples of our children's mathematical learning supported by photographs, quotes, samples of work and so on. We are currently working towards a shared file of examples of children's problem solving as a resource for all members of the group. Sharing our experiences, children's work, information from research, other professional training and ideas, adds to our collective knowledge of teaching mathematics.

This sharing of children's work formed the substance of the observed meeting and included a wide variety of examples which had been carefully analysed by the professional presenting it. In many cases, these examples involved accounts of what the children had done, examples of their productions in terms of marks made or artefacts created and photographs of the children in action. The group discussed in detail the mathematical aspects of each example and talked about how they could support the mathematical thinking that it represented.

Sarah described how her involvement was:

> … enabling me to continue to keep abreast of current thinking, be reflective and share my ideas and experiences … in a safe, supportive, non-threatening environment.

Her involvement in the group and attendance at various conferences in the area run by the researchers had developed her understanding and enthusiasm and she was in the process of becoming a researcher in her own classroom. As a result of her involvement Sarah was now committed to practice focused on children's mathematics following on from her extended study of children's mathematical graphics and problem solving. This involved a way of teaching that was completely different from a worksheet- and textbook-based approach that used to exist in her school.

It was evident from examples of the children's work on notice displays and in their books in Sarah's classroom that these new approaches were being adopted and the children were able to articulate their mathematical understandings clearly in relation to these examples. As Sarah said:

> The children in our classes have a positive attitude to sharing and representing their mathematical thinking. They are developing confidence in their *mathematical* graphics which are valued, they are developing fluency and a willingness to talk about their thinking.

The displayed work demonstrated Sarah's detailed observations and analysis of the children's mathematical thinking and understanding on a day-to-day basis. This emphasis on the display and annotation of children's mathematical work was also evident in the classroom of the other teacher participant in the school and illustrated the importance of mathematics for these children in these classes.

We suggest that participant ownership of this initiative helped to sustain involvement and that the members supported one another in sustaining their passion and enthusiasm. Overall, the initiative supported the participants in their professional change by giving them a space for the detailed and joint consideration of children's mathematical thinking. It supported them in following up research sources that would support their analysis of the *children's mathematical graphics* and enabled them to encourage children to take charge of their own mathematical activity. It also offered them a supportive and encouraging arena in which their professional concerns and difficulties could be discussed.

The focus in the group was very strongly on the research related to *children's mathematical graphics* and the participants became involved in contributing informally to this research through gathering evidence from their own practice. In a sense these teachers became researchers in their own classrooms although they did not publish or disseminate their findings except to the researchers whose work had been the inspiration for forming their group.

A COURSE FOR TEACHERS IN ELEMENTARY SCHOOLS

This course was part of programme involving primary school teachers in 10-day courses, which provided eight meetings out of school and two days within school during which they were free to act on some of the things that they had learnt on the course, while other teachers took responsibility for their class. In most cases, two teachers attended from each school. Each participant had the opportunity to submit assignments for accreditation at Masters level as part of the course.

The initiative was a response by a local authority to perceived problems with the quality of mathematics teaching in its schools and formed part of a drive to raise standards of mathematics teaching at all levels within the county. The initiative aimed to build capacity across the county by building confidence and expertise in teaching mathematics amongst elementary school teachers; to develop their mathematics subject knowledge, mathematics pedagogy and understanding of solving mathematical problems and thinking mathematically; and to give them opportunities to work with colleagues to develop their practice. Its design was based on an earlier model of 20-day courses for teachers[2] which were common in England in the 1980s, which are generally perceived as successful.

The course was based on the premise that mathematics in primary schools is different from other subjects and requires a different kind of support. As the course leader, Robert, said: ‘Maths is different from other subject areas in primary school; the teachers need to experience doing mathematics themselves in order to improve their subject knowledge.’ Through the use of reflective journals to relate their own and their students’ responses to mathematical tasks and through the reading of related literature, it was hoped that the teachers would develop their understanding of mathematics, of ways of teaching it and students’ responses to the subject.

The teachers were expected to examine the difficulties and misconceptions that they and their students and peers might have in response to a particular problem and to read research literature that related to these observations. The content of the course combined mathematical subject knowledge and pedagogy with collaborative working with colleagues on changing practice, as well as working on mathematical tasks and engaging in mathematical thinking.

One of the tasks involved ordering groups of fractions from smallest to largest, including examples that people might have difficulty with and which might reveal common misconceptions. The participants were asked to convince one another of the validity of their mathematical reasoning and addressed some common misconceptions about fractions. They were also expected to adapt the fraction activity for their own students before the next meeting and to collect evidence of the children’s mathematical learning in response to the task to share with the rest of the group. In doing so they paid attention to their student’s mathematical learning.

[2] During the late 1980s in response to the new demands of the National Curriculum, the Department for Education and Skills (DfES) initiated a programme of courses to support primary teachers in National Curriculum mathematics and science. For an evaluation of these courses see (Harland & Kinder, 1992).

Teachers were asked to read, and think about, research and they were asked to write reflective journals about all aspects of the course including their reading. We suggest that this helped teachers to articulate clearly their approaches to teaching mathematics. Their comments suggested that they had given their practices, and the changes they had made in them, considerable thought. One of the teachers commented on her engagement with the theory underpinning her practice:

> I really like looking at the theory behind why we do certain things and the misconceptions people have.

A WITHIN SCHOOL INITIATIVE

This initiative was set up by a head teacher of a primary school, and involved an external commercial provider. The head teacher wanted to improve attainment in mathematics and she asked a consultant to lead the initiative. Setting up the initiative had involved collaboration between the consultant and the school to establish the needs of the school. The head teacher, mathematics subject leader and the consultant decided to focus on developing the mathematical subject knowledge of the teachers and teaching assistants and encouraging children's engagement with mathematics and their mathematical creativity. There was also an emphasis on developing speaking and listening activities and using the notion of 'assessment for learning' in mathematics lessons.

The rationale for the course was grounded in research findings such as those related to 'assessment for learning' (Black, Wiliam et al. 1998) but most of the teachers and teaching assistants involved were not expected to engage with reading research articles or papers although the mathematics subject leader did so. The course offered the teachers resources to use in the classroom, as well as suggestions for ways of working with children on their mathematics. It was hoped that the involvement of all members of staff would encourage teachers and teaching assistants to talk about issues related to the changes that were being suggested.

The planned programme ran throughout the academic year and included a number of twilight sessions (running for an hour and a half after school) and whole and part day meetings. The external consultant and mathematics subject leader mentored other teachers, delivered the course and supported teachers in their classrooms. They developed documentation such as a calculation policy.

Kerry was the subject leader for mathematics in the school and played a key role in the initiative. She saw her role as leading the development of mathematics teaching throughout the school. She was aware that if she was setting herself up as in some way an expert, she needed to be following her own advice in her own practice: 'Because if I am making judgements of people I have to be able to do what I am suggesting well.' She felt strongly that her own involvement in the project and her support of the development of her own colleagues had led to her own professional development too. She suggested that not only had she become more confident to try out new approaches in her own classroom, but she had also become more confident in leading professional development for her colleagues:

> I am more confident with helping colleagues and able to support them and make judgements about their teaching and be more helpful for them. I am able to advise so that when we look at results and see there is a group of children here who need more input I can make suggestions like 'you could try…'. Having the outside input has helped me to develop.

Kerry commented that her involvement with leading the initiative had changed her practice so that she had become less reliant on textbooks. She said that she had begun to use open-ended tasks, which sometimes meant that there was no record of the activity in their exercise books. She said that she sometimes used post-it notes to record what the children had done:

> I jot down the children who have or haven't got it and I offer them much more practical work.

This approach to assessing students learning is based on research of which the teachers were aware although they were unlikely to have read the original research (Black, Wiliam et al., 1998).

For Angela, an experienced teacher who had been working at the school for over 15 years, the ways of working that were being suggested were very different from the ways in which she had previously taught mathematics and involved much more practical work and games which led to a reduced emphasis on writing things down. As she said:

> I use more activities, far more resources and practical activities. Now if people were working practically and they didn't have written evidence of their learning, that wouldn't worry me. I don't feel that a child has to write something down to know it. … There has been a lot of support for the change and the children have become more involved and they enjoy it so much more. For me that is a very positive way of showing that it is the right way of going about it. It is the reaction of the children that has made me realise that this way of teaching is better.

Angela also wrote observations of children's mathematics on post-it notes as part of her strategy for assessing their achievements in mathematics. She made these during the course of the lesson and encouraged teaching assistants to do the same.

Both observed teachers reported using more practical mathematical activities and this was seen in both their lessons. Developing the use of formative assessment was another aim and the teachers reported making using on post-it notes as a strategy for assessing pupils, so that they were observing the children's mathematical behaviour rather than necessarily expecting the children to record answers in their books. This showed that they were using research findings in their practice even though they did not talk about reading or engaging with the research literature. The focus on assessing the children's work through making detailed observations of their mathematical activity showed that the professional development encouraged the teachers to pay attention to children's mathematical learning and that they were doing so as a result.

The extended programme of workshops for teachers and teaching assistants over a long period of time may also have helped with the changes, as their importance for the school and its development was repeatedly stressed throughout the academic year at each meeting. During this academic year most of the school's time available for professional development was devoted to this initiative which probably also served to raise the importance of the initiative for all those involved.

CONCLUSIONS

The three cases show different ways of paying attention to student learning that may be of significance to other professional development providers. In the first case the teachers collected examples of children's work and talked to the children about it, using the children's responses to annotate presentations of the work which were valued by inclusion in the children's portfolios or on displays in the classroom. In the second case the teachers focused on analysing children's responses to tasks that they offered them and sharing those responses with colleagues. In the third they collected evidence of children's mathematical learning through classroom observations of the children's engagement in mathematical activities. As such these cases offer models for ways of working with teachers that pay attention to student learning.

All three case studies based the ways in which they worked with teachers on research and used research evidence to support the changes in teaching practice they advocated. However they did so in different ways and it seems that the teachers in them engaged with the research to a variable extent. In all three cases the teachers acted in their classrooms in ways that revealed an awareness of research findings so the teachers' practice could be characterised as being research informed but in some cases we would question whether they were convinced of the rationale for the changes they had made.

In the case of the course, the teachers were encouraged to collect examples of children's work and to make connections between what they observed in their classrooms and the research findings relating to them about which they were reading. We would suggest that for them this remained something that they did in connection with the course and had not become embedded as part of their professional identity. In the whole school initiative the teachers expressed a willingness to try things out and to see whether they worked in their classrooms with their students but this was taken at the level of considering advice from outside and they did not articulate a rationale for ways of working with learners that they justified with reference to research findings.

In the case of the network the teachers had become engaged themselves on research in their own classrooms. They sought evidence of children's mathematical development and made clear annotated records of the children's mathematical activity. In their conversations they referred frequently to the research publications they had read and this suggests that they had become practitioners whose practice was informed by their knowledge of research that was relevant to it. They spoke with conviction about the new ways of working they had adopted and in which they had become engaged with children's learning and the gathering of evidence of children's learning.

In this paper we have not provided evidence to 'measure' whether one of these ways of working might have been more successful in sustaining changes in professional practice than the others but we have a strong feeling that in the case of the network the teachers were more likely to persevere because it seemed to us that they really believed in what they were now doing. We suggest that this may have been due to the depth to which they had engaged with the research findings that they were using and their engagement with action research in their own classrooms that was coupled with a focus on students' mathematical learning.

REFERENCES

Black, P., Wiliam, D., Harrison, C., Lee, C., and Marshall, B.: 1998, Working inside the Black Box: Assessment for Learning in the Classroom, London: King's College London.

Carruthers, E., and Worthington. M.: 2006, Children's Mathematics: Making Marks, Making Meaning, 2nd ed., London: Sage.

De Geest, E., Joubert M., Sutherland R., Back J., and Hirst C.: 2008, Researching Effective Continuing Professional Development in Mathematics Education (RECME), paper presented in International Congress on Mathematics Education, Monterrey, Mexico 2008.

DfES (Ed.): 2007, The Early Years Foundation Stage: Practical Guidance for the Early Years Foundation Stage, DfES Publications.

Eraut, M.: 2001, Do Continuing Professional Development Models Promote One-Dimensional Learning? Medical Education, 35, pp. 8-11.

Guskey, T. R.: 2002, Professional Development and Teacher Change. Teachers and Teaching: Theory and Practice, 8, pp. 381-91.

Hargreaves, D. H.: 1996, Teaching as a Research-Based Profession: Possibilities and Prospects, in Teacher Training Agency Annual Lecture, London: Teacher Training Agency.

National Centre for Excellence in the Teaching of Mathematics: 2009, Final Report: Researching Effective CPD in Mathematics Education (RECME).

Zeuli, J.: 1994, How Do Teachers Understand Research When They Read It? Teaching and Teacher Education, 10, pp. 39-55.

MATHEMATICS AND MATHEMATICS PEDAGOGY KNOWLEDGE OF FUTURE TEACHERS IN POLAND THE RESULTS OF THE TEDS-M 2008 STUDY

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Using the results of the TEDS-M 2008 study, we show the relative performance of Polish future teachers in comparison with future teachers in other countries with similar characteristics of teacher training. We also discuss the substantial differences that have been found in the knowledge of mathematics and mathematics pedagogy of students of different study programs in Poland.

INTRODUCTION

The Teacher Education and Development Study in Mathematics 2008 (TEDS-M, 2008) is an international study undertaken by the International Association for the Evaluation of Educational Achievement (IEA). It covered teacher education systems of 16 countries: Botswana, Chile, Georgia, Germany, Malaysia, Norway, Oman, the Philippines, Poland, the Russian Federation, Singapore, Spain, Switzerland, Taiwan, Thailand and the United States (see Tatto, 2008, for the general framework of the study).

The main instruments used in the TEDS-M study were cognitive tests designed to measure the knowledge and skills of future teachers trained, respectively, to teach in primary and lower-secondary schools. Both tests were scaled to measure knowledge and skills in two domains: Mathematical Content Knowledge (MCK) and Mathematical Pedagogical Content Knowledge (MPCK). The results of the four tests were scaled separately and are reported on the standardized scales with the international average of 500 and standard deviation of 100.

For the purpose of reporting the results, the diverse program types existing in participating countries were grouped according to the grade span and the degree of specialization. Accordingly, in case of future teachers of primary schools four program groupings were distinguished: 1) generalists prepared to teach mathematics and other school subjects in the first grades of primary school; 2) specialist prepared to teach mathematics in the first grades of primary school; 3) generalists for the remaining grades of primary school; 4) specialists for the remaining grades of primary school. For the secondary level, two program groupings were distinguished: 5) future teachers of lower secondary schools and 6) future teachers qualified to teach in upper secondary schools. Out of the six program groupings, only four exist in Poland, and only these will be presented in the article. The average performance in program types with similar characteristics existing in other countries will be shown.

TEACHER TRAINING IN POLAND

The school mathematics is taught in Poland mostly by specialists that graduated in mathematics. However, in the first three grades of primary school there is no distinction between school subjects. With exception of foreign language teaching, there is only one teacher for all subject areas, timetable of educational activities is flexible and the assessment is descriptive. Teachers for this stage of school education have qualifications to teach so called “integrated teaching” gained as part of studies in the field of pedagogy. The stage of integrated teaching was devised to provide the smooth transition from the pre-primary and the primary education. Regular school subjects, including mathematics, start only in the fourth grade.

Therefore, while the graduates of pedagogy that specialize in integrated teaching are qualified to teach only in grades 1-3, the graduates in mathematics gain qualifications to teach mathematics both in primary school (grades 4-6) and lower secondary schools (grades 7-9). Those who complete the master degree can also teach in upper secondary schools. This contrasts with the tradition in many countries, where there are separate routes of training for primary and secondary schools. Moreover, despite a unified structure of primary school, there is a marked transition point in the school education. This is also seen in the core curriculum, which distinguishes grades 1-3 and grades 4-6 as separate stages of education.

An important peculiarity of the Polish system of teacher education is that it is organized as a separate specialization in the major field of study, rather than a separate field of study (for detailed discussion of the system see Wiłkomirska, 2005). This means that students of mathematics need to complete the predefined list of courses that cover pedagogy, general pedagogy, the mathematics pedagogy and complete the practicum while the core of their curriculum is common with other specializations in the mathematics field of study (such as theoretical mathematics or applied mathematics in finance). An advantage of this framework is that the mathematical content knowledge is relatively advanced. Its disadvantage is an overly academic orientation of the studies and relatively limited opportunities to acquire the knowledge and skills useful in teacher practice. It can be noted that this weakness extends beyond the teacher education (see Fulton et al., 2007). The curriculum of the pedagogy studies has limited focus on subject-matter content that will be taught by future teachers. Instead, it offers comprehensive education in different subfields of pedagogy included in the standards of study for the pedagogy degree. In particular, there are no required courses in mathematics in most of the study programs.

Another complication of the Polish system is the structure of degree levels. Traditionally, it took five years of studies to gain a higher education degree (*magister*). However, in the 90s higher education institutions started to shift towards two-degree structure, which became mandatory from the academic year 2006/2007 as a part of adaptation of Polish higher education to the Bologna process. Therefore, as of 2008, when the TEDS-M study was conducted, three types of programs operated: three-years long first-cycle (undergraduate) programs, two years long second-degree (graduate programs), which can be accessed by candidates with the first-degree diploma, and, finally, the so called, long-cycle programs, which usually take five-years to complete.

It is also important to note the distinction between full-time and part-time programs. In part-time programs the courses are shorter and it is assumed that student will work intensively at home, most-often having full-time jobs. In public universities, only full-time day studies need to be provided education which is free of charge. Because the tuition fees are an important source of funding for public universities and part-time programs are more attractive for private higher education institutions, these programs started to flourish and about half of the students in Poland study part-time.

All this contextual information is important for understanding the learning outcomes of students. It is obvious that graduates of pedagogy have much more limited mathematics content knowledge than graduates of mathematics, who will teach in upper grades of primary school or secondary schools. One could also expect that the performance of part-time students will be worse than full-time students. The results of the TEDS-M 2008 study give an opportunity to test these hypotheses empirically in a large-scale setting.

FUTURE TEACHERS IN PRIMARY SCHOOL

Generalists: Future Teachers For Early Grades of Primary School

Only in five out of sixteen countries participating in the TEDS-M study there exist program types that prepare generalists for early grades (up to the grade four) of primary school (program grouping 1). Both in terms of mathematical content knowledge (MCK) and mathematical pedagogical content knowledge (MPCK), Polish future teachers were outperformed by future teachers in Russia, Switzerland and Germany. The average performance was the lowest in Georgia.

	Sample size	Mean performance	s.e.
Russia	2266	535	9.89
Switzerland	121	512	6.43
Germany	935	501	2.86
Poland	1799	456	2.28
Georgia	506	345	3.85

Table 1: MCK. Primary Future Teachers. Lower Primary Generalist (Grade 4 Maximum) (in Poland, Students of Pedagogy).

	Sample size	Mean performance	s.e.
Switzerland	121	519	56.3
Russia	2266	512	8.09
Germany	935	491	4.75
Poland	1799	452	1.87
Georgia	506	345	4.93

Table 2: MPCK. Primary Future Teachers. Lower Primary Generalist (Grade 4 Maximum) (in Poland, Students of Pedagogy).

Primary Future Teachers For Upper Grades: Mathematics Specialists

Within the program grouping of future teachers that specialize in teaching mathematics in upper grades of primary school, the performance of students from Poland and Singapore was the best. However, the performance of future teachers on the MPCK scale was significantly better in Singapore than in Poland.

It is not surprise that both in Germany and Poland, where there are separate program types prepare generalists for the early grades of primary schools, the performance of specialists was better than generalists. In Germany, the difference was relatively small: 55 points on the MCK and 61 on the MPCK scale. In Poland it was very substantial. It amounted to 158 points on the MCK scale and 123 points on the MPCK scale, which is equivalent, respectively, to 1.58 and 1.23 of international standard deviation.

	Sample size	Mean performance	s.e.
Poland	300	614	4.79
Singapore	117	600	7.76
Germany	97	555	7.48
Thailand	660	528	2.31
U.S.	191	520	6.57
Malaysia	576	488	1.82

Table 3: MCK. Primary Future Teachers. Mathematics Specialists (in Poland, Students of Mathematics)

	Sample size	Mean performance	s.e.
Singapore	117	604	7.04
Poland	300	575	4.04
Germany	97	552	6.82
U.S.	191	544	5.89
Thailand	660	506	2.26
Malaysia	576	503	3.09

Table 4: MPCK. Primary Future Teachers. Mathematics Specialists (in Poland, Students of Mathematics).

Lower Secondary Future Teachers – Students of Mathematics

Students classified in this program grouping, defined as those program types that can teach up to grade 10, solved more difficult test. The students from Taiwan obtained the best results. They outperformed the second best country Singapore by 123 points on the MCK scale and 100 points on the MPCK scale. The results of Polish students (here represented by first cycle programs) were above the international average and were similar to those obtained by students from Singapore and Switzerland, although the performance of students from these countries was slightly better on the MPCK scale (with difference statistically significant in case of Switzerland). Performance of all other countries was below the international average.

	Sample size	Mean performance	s.e.
Taiwan	365	667	3.86
Singapore	142	544	3.65
Switzerland	141	531	3.75
Poland	158	529	4.25
U.S.	121	468	3.72
Norway	148	461	4.54
Philippines	733	442	4.6
Botswana	34	436	7.31
Chile	741	354	2.53

Table 5: MCK. Lower secondary future teachers. Lower secondary, no higher than Grade 10 (gymnasium in Poland). (in Poland, Students of Mathematics)

	Sample size	Mean performance	s.e.
Taiwan	365	649	5.25
Switzerland	141	549	5.88
Singapore	142	539	6.06
Poland	158	520	4.5
Norway	148	480	6.24
U.S.	121	471	3.87
Philippines	733	450	4.67
Botswana	34	436	8.51
Chile	741	394	3.77

Table 6: MPCK. Lower secondary future teachers. Lower secondary, no higher than Grade 10 (gymnasium in Poland) (in Poland, Students of Mathematics).

Lower And Upper Secondary Future Teachers: Mathematics Specialists

Although the target population in the TEDS-M 2008 study were the future teachers of primary and lower secondary schools, several program types in participating countries qualify to teach above the lower-secondary level (here defined as above grade 10). In this program grouping, students from Russia and Singapore obtained the best results of both mathematical content knowledge and mathematical pedagogical content knowledge. The results of Polish students (here represented by long-cycle programs) were clearly above the international average and were similar to the results of students from the United States.

	Sample size	Mean performance	s.e.
Russia	2 139	594	12.78
Singapore	251	587	3.84
U.S.	354	553	5.07
Poland	139	549	4.4
Norway	43	503	9.75
Malaysia	388	493	2.43
Thailand	652	479	1.56
Oman	268	472	2.44
Botswana	19	449	7.52
Georgia	78	424	8.91

Table 7: MCK. Students of Mathematics. Lower secondary future teachers. Lower and Upper Secondary; above Grade 10 (in Poland, Students of Mathematics)

	Sample size	Mean performance	s.e.
Russia	2 139	566	10.15
Singapore	251	562	6.05
U.S.	354	542	5.81
Poland	139	528	6.17
Norway	43	495	17.75
Thailand	652	476	2.49
Oman	268	474	3.79
Malaysia	388	472	3.32
Georgia	78	443	9.63
Botswana	19	409	15.64

Table 8: MPCK. Lower secondary future teachers. Lower and Upper Secondary; above Grade 10 (in Poland, Students of Mathematics).

RESULTS OF POLISH FUTURE TEACHERS BY PROGRAM TYPES

Pedagogy

There are statistically significant differences between the average performance of students of different program types in Poland. In pedagogy, the performance of students in long-cycle programs was the best. However, this program type had the highest variation of performance and the highest difference between the average performance of full-time and part-time students. What is surprising is that the performance of the students of the second-cycle programs is, on average, lower, than those of their colleagues from the studies of the first-cycle. The performance on the MPCK scale was more variable both across and within programs.

	Full-time programs		Part-time programs	
	Mean	SD	Mean	SD
first cycle	469 (3.4)	63.6 (3.2)	444 (2.1)	60.6 (2.5)
second cycle	457 (5.9)	59.7 (10.1)	430 (4.7)	68.2 (4.7)
long cycle	482 (7.5)	71.2 (4.6)	437 (7.7)	76.4 (7.8)

Table 9: Average performance on the mathematical content knowledge of pedagogy students by program type.

	Full-time programs		Part-time programs	
	Mean	SD	Mean	SD
first cycle	473 (2.8)	75.3 (3.2)	435 (3.1)	90.4 (3.7)
second cycle	437 (7.8)	74.7 (7.2)	423 (4.8)	87.4 (5.1)
long cycle	488 (6.3)	77.8 (4.3)	417 (7.6)	105.4 (5.6)

Table 10: Average performance on the mathematical pedagogical content knowledge of pedagogy students by program type.

Mathematics

Similarly to the pedagogy programs, the performance of long-cycle programs was, on average, the highest. The mean result of students of these programs on the MCK scale (test for future teachers in primary school) was 633 points and was 46 points higher than those of first-cycle programs' students. The difference was not so marked on the MPCK scale, where it was only 24 points. However, long-cycle programs are the most diverse with standard deviation as high as 99 points in MCK test and 79 in MPCK. Contrary to expectations, the difference in average results of first and second-cycle is not statistically significant.

Because of the fact that the sample was split into two groups, that is in every participating program half of the students solved the test at the primary level and the other half solved the test at the lower-secondary level, the standard error on the performance estimate of the full-time and part-time students was too high to show the relative performance as it was done in case of pedagogy students.

	Mathematical Content Knowledge		Mathematical Pedagogical Content Knowledge	
	Mean	SD	Mean	SD
first cycle	587 (5.6)	73.34 (5.3)	560 (5.6)	71.16 (4.2)
second cycle	583 (8.7)	79.64 (7.6)	550 (6.9)	81.59 (8.1)
long cycle	633 (7.4)	98.67 (6.8)	584 (6.2)	78.85 (4.7)

Table 11: Students of Mathematics. Primary test.

DISCUSSION

Polish students of mathematics performed markedly well in relation to students in other countries. In contrast, the results of the students of pedagogy were among the worst. This may have important consequences for the quality of education, as the early grades of primary education form the foundations of mathematical knowledge and skills and build a basic understanding of the nature of mathematics in pupils.

The negative impact on the quality of mathematics education in Poland may be exerted by the very diverse knowledge and skills of future teachers. While many of them are among the best-performing, even compared to a leading countries of the world, there are also students with surprisingly low level of competencies. The system of higher education does not provides the clear qualification system, which would signal the school principals responsible in Poland for hiring teachers, the actual levels of knowledge and skills of the candidates for the teaching post. As a consequence, this may increase the variability of the quality of teaching.

A more detailed analysis of the data shows that the Polish students perform best when presented with items that require factual knowledge of mathematics. On the other hand, they had problems with test items which required solving non-typical problems or items, where modelling, selection of an appropriate mathematical model to the situation was required. Surprisingly, this conclusion mirrors the results of the OECD PISA study, where similar problems were found in the performance of 15 years' olds (IFiS, 2007). This suggests that the weakness in solving non-routine tasks and using more advanced skills of mathematical reasoning and modelling extends beyond school education.

On the basis of relative performance of Polish mathematics students compared to future teachers from other countries one may also conclude that the knowledge and skills in mathematics pedagogy was often worse than in mathematics content knowledge, which may reflect the fact that mathematics pedagogy seems to be underemphasized in the study programs of mathematics teacher education.

In the area of mathematics pedagogy, future teachers performed relatively well in diagnosis of typical students errors. When faced with examples of non-typical reasoning of pupils their performance was relatively worse. This may mean that their teacher education is overly oriented towards teaching a group of students rather than towards the work with individual students. Students also had problems in defining mathematical concepts and working with the curriculum. As a consequence, they may face difficulty in using the new Polish core curriculum, which has been gradually introduced in schools starting from September 2009 and which is defined in terms of learning outcomes to be achieved rather than content to be taught.

REFERENCES

IFiS .: 2007, *Program Międzynarodowej Oceny Umiejętności Uczniów – PISA 2006. Wyniki badania w Polsce [The Programme for International Student Assessment – PISA 2006. The results in Poland),* Warszawa: Instytut Filozofii i Socjologii PAN (http://www.ifispan.waw.pl/ifis/badania/program_pisa/)

Fulton, O. P. Santiago, Ch. Edquist, E. El-Khawas, E. Hackl.: 2007, *Poland. OECD Reviews of Tertiary Education,* Paris: Organisation for Economic Cooperation and Development.

Wiłkomirska, A.: 2005, *Ocena kształcenia nauczycieli w Polsce [The Evaluation of Teacher Education in Poland],* Warszawa: Instytut Spraw Publicznych.

Tatto, M.T., J. Schwille, S. Senk, S. L. Ingvarson, , R. Peck, G, Rowley.: 2008, Teacher Education and Development Study in Mathematics (TEDS-M): Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics. Conceptual Framework. East Lansing, MI: Teacher Education and Development International Study Center, College of Education, Michigan State University (https://teds.educ.msu.edu/20080206_TEDS-M_conceptual_framework.pdf).

DIDACTIC MATERIAL AS A MEDIATOR BETWEEN PHYSICAL MANIPULATION AND THOUGHT PROCESSES IN LEARNING MATHEMATICS

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The use of didactic material in mathematics classes has an important role in the formation of mathematical thinking. This article focuses on some problems related to the use of didactic material in teaching and learning mathematics from the aspect of associating physical manipulation and thought processes. The article presents the results of an empirical study that aimed to determine whether the views on the issue of didactic material in teaching and learning mathematics depend on the status of respondents. The answers of teachers and students to a set of questions explored the influence of teachers' practical experience on their attitudes towards the role of didactic material in mathematics classes.

THE ROLE OF DIDACTIC MATERAL IN TEACHING AND LEARNING MATHEMATICS

There is a common view among teachers and parents that children learn mathematics more easily by manipulating some concrete material while learning. Research on this matter varies. For example, during the 1960s and 1970s, the Dienes blocks were widely used in the Netherlands, but the criticism of their use – they were seen as helpful for the representation of abstract number structure but not very suitable for the representation of more complex number operations (Beishuizen, 1999) – led to the increasing use of the bead frame and bead strings (Anghileri, 2001). Among other authors who researched the role of the structured apparatus and unstructured material in the process of teaching and learning mathematics were Fennema (1972) and Fridman (1978), who showed a positive role of counting strategies based on the use of concrete material at primary level but not at secondary school level. Suydam and Higgins (1977) also found manipulating concrete material useful throughout elementary school. On the other hand, Labinowicz (1985) observed young children using the Dienes blocks and came to the conclusion that they had problems establishing relations between these blocks and the place value system of integer numbers. Moreover, while Fuson and Briars (1990) found a very positive role of the blocks in learning how to add and subtract integer numbers, Thompson (1992) and Resnick and Omanson (1987) concluded that they had very little influence on children's understanding of arithmetic algorithms in primary school.

These often contradictory findings suggest that the use of concrete material in the learning process does not automatically ensure successful learning. The input data that children acquire via physical activity and their manipulation of concrete objects should result in a certain deduction, i.e. in a mental activity required to understand an abstract mathematical concept. Didactic material has the function of a mediator between the teaching aims that drive the teaching process and the result of this teaching process – mathematically educated children.

We should also investigate whether children are aware of the didactic value of teaching material; whether they use it in a way that is expected from them by their teachers; and whether this material really leads to the desired mathematical aims. Teachers see a certain mathematical structure in the didactic material which is meant to encourage the desired mental activity but this does not guarantee that the same structure is perceived by children and that they use the didactic material in a way that would develop their mathematical thinking.

Children can perceive a type of didactic material in a desired way – mathematically. In such a case, didactic material functions as a representation of an abstract mathematical concept. However, didactic material can also be perceived non-mathematically, i.e. children may only see it as a physical object and may not see the mathematical relations in the background (Gravemeijer, 1991, in Streefland, 1991).

Various psychological theories emphasize the significance of inducing mental activity as a fundamental characteristic of didactic material. According to them, the key question is: ‘Is physical activity isomorphic to the intended mental activity?’ (Gravemeijer, 1991, in Streefland, 1991, p. 57). Using didactic material can lead to a mental activity that is not necessarily isomorphic to physical activity. The problem of division between physical and mental activity can be observed in the manipulation of various didactic resources.

Example: Number line.

> If children want to use the number line to calculate the sum of 5 + 3, they will start at 5, count ‘one, two, three’ and move towards the right of the line. This procedure of counting differs from the one that we perform when counting three steps further in our thoughts. When doing so, we say to ourselves ‘six, seven, eight’ and the last number represents the sum of five and three.

In the described case, didactic material functions as a technical aid that enables children to solve a certain problem in an easy manner. But, as was shown, this aid does not necessarily encourage the type of thinking that is required when working on the mental level.

THE INFLUENCE OF DIDACTIC MATERIAL ON THE TEACHING PROCESS

As was previously established, the success of using a certain type of didactic material in the classroom is largely dependent on the way teachers use it and on the way they perceive its use in teaching mathematical concepts. This can be based on their own experience, the experience of their colleagues, as well as on various examples from textbooks and teacher's manuals.

Related to this is the issue of how teachers are supposed to respond to a new type of didactic material. Should they adapt their teaching to the new material or vice versa – should they adapt the material to their teaching style and their existing classroom routines? Mathematics teachers cannot easily adopt a newly developed didactic material to their teaching practice. They usually adapt it to the goals that they pursue: they fit the use of didactic material into their existing approaches to teaching mathematics.

Adapting the teaching to new didactic material is reasonable only when the teacher is the carrier or the trigger of such an adaptation: not because of the material itself but because of their new perspectives on the process of teaching (Gellert, 2004).

The two responses to new learning material – adapting it and adapting to it – can be examined further through the example of the geoboard.

1st option: the teacher adapts the new didactic material to existing circumstances: the geoboard is used to achieve teaching aims specified in the curriculum that were already developed in the past but by using some other didactic material. The teaching aim 'The student forms a triangle' can be achieved by drawing the shape, cutting it out of paper or even using new didactic material – the geoboard. In this way, we do not influence the development of a new, additional teaching aim but strengthen the existing one by introducing new didactic material.

2nd option: the teacher adapts to the new didactic material: in such a case, the teacher uses additional possibilities that the new didactic material provides and uses the material for problem solving. Children can explore how many different triangles it is possible to form on the 3 by 3 geoboard. The introduction of the concept polygon is thus enriched by an activity which would not be possible without the use of the new didactic material. Such an adaptation of teaching to new didactic material is only possible if teachers are open to new ideas and new, problem-oriented approaches to teaching mathematics.

To sum up, the adaptation to new didactic material only makes sense under certain conditions: the teacher should be open to new ways of doing things with the clear purpose of improving learning and teaching mathematics.

EMPIRICAL RESEARCH

Problem Definition and Methodology

We were interested in comparing the attitudes of teachers and students towards the role of didactic material for teaching mathematics. By comparing their answers to a set of questions, we also explored the influence of teachers' practical experience on their attitudes towards the role of didactic material used in the mathematics classroom.

The empirical study was designed on the descriptive and causal non-experimental method of pedagogical research.

The aim of the study was to answer the following research questions:

1. What are the attitudes of students and teachers towards:
   a) using didactic material for learning abstract mathematical concepts?
   b) using didactic material for problem solving in mathematics?
2. Does didactic material have an impact on teachers' lesson planning?
3. Do the attitudes of teachers and students towards different statements about didactic material significantly differ?

Description of the Sample

A purposive sample was used in the study. 76 teachers and 94 students (20 third-year students and 74 fourth-year students of Primary Teacher Education) completed the questionnaire; 5 of them were male (2 teachers and 3 students), the rest female. All the teachers in the sample worked in primary schools and had on average 19.7 years of work experience (standard deviation is 8.9 years).

50 of the investigated teachers had university degrees and 26 had higher education degrees.

Data Processing

Data acquisition was carried out from March to May 2008.The data from the questionnaires was processed using methods of descriptive and inferential statistics. Teachers' and students' attitudes were evaluated on a five-stage scale where the grade 5 means that they fully agree, 4 – they agree, 3 – cannot decide, 2 – disagree, 1 – fully disagree. The statistical procedures employed were: frequency distribution (f, f %), the basic descriptive statistic of numerical variables (mean, standard deviation), χ^2 – test of hypothesis independence.

Results and Interpretation

The above sample of respondents was used to determine the attitudes of teachers and students towards didactic material, especially towards new didactic material and towards problem solving in mathematics. The results and analyses are given below, separately for each research question.

First we were interested in the respondents' attitudes towards the role of didactic material in the formation of mathematical concepts (research question 1a). The respondents were asked to give their opinions about the following statements:

		P- value	Mean teach.	σ_t	Mean stud.	σ_s
1	Using didactic material is essential for learning mathematical concepts.	0.24	4.75	0.59	4.59	0.68
2	Children better understand a mathematical concept if they can manipulate the didactic material useful for that concept.	0.46	4.87	0.34	4.84	0.45
3	The use of didactic material does not influence children's understanding of mathematics.	0.13	1.59	1.15	1.49	0.88
4	Too frequent use of didactic material prevents the development of a mathematical concept at an abstract level	0.03	2.46	1.12	2.71	0.92
5	Didactic material could move children's attention away from the mathematical concept to be learned (children are more conscious about the material itself than about manipulating it).	0.00	2.62	1.17	2.94	1.08

Table 1: Statements for checking the research question - What are the attitudes of students and teachers towards using didactic material for learning abstract mathematical concepts?

The results in Table 1 show that both teachers and students find the use of didactic material in teaching mathematics important; they believe that its use has an impact on learning mathematical concepts, however it is also shown that their views on the relations *frequency of use – formation of abstract concepts* and *didactic material – focusing children's attention* depend on the status of respondents. Most of the teachers (46.05%) do not agree with the statement that didactic material can distract children's attention from the mathematical concept dealt with in the classroom. On the other hand, 39.36% of students remain undecided, which may be related to their lack of teaching experience.

With regards to the role of didactic material for learning abstract mathematical concepts, we were also interested in the opinions of survey respondents on the usefulness of specific didactic material for teaching different mathematical topics.

		P- value	Mean teach.	σ_t	Mean stud.	σ_s
1	Link cubes are useful for dealing with numbers up to 100.	0.00	4.14	1.04	3.28	1.31
2	The Dienes blocks best represent the numbers up to 1000.	0.00	4.07	0.93	3.21	1.26
3	The geoboard is useful for teaching plane geometry.	0.22	4.38	0.78	4.24	0.79
4	Abacus is useful for dealing with numbers up to 1 000 000.	0.00	4.05	1.09	3.06	1.50
5	Children must know the place-value system in order to be able to manipulate the Dienes blocks.	0.07	4.13	0.91	3.84	0.92
6	The use of the 100-square for calculating up to 100 is helpful for less able children.	0.00	3.84	1.19	3.19	1.07
7	Using didactic material for introducing written algorithms is essential.	0.34	4.49	1.10	3.36	1.01
8	The best didactic material for representing written algorithms is the Dienes blocks and abacus.	0.19	3.79	0.88	3.42	0.94

Table 2: How is specific didactic material useful for teaching different mathematical topics?

Table 2 shows that the attitudes towards statements 1, 2, 4 and 6 depend on the status of respondents. The majority of teachers agree with these statements. Students' answers lean towards agreeing with the statements, yet there are many of them who do not agree completely, do not agree or cannot decide on an answer. It is obvious that practical experience contributes to the use of didactic material in teaching mathematics.

It is also interesting to examine the reasons for deviations between teachers' and students' attitudes about statement 6: in contrast with students, teachers find the 100-square more useful when working with weaker children. One of the reasons for this could be that the majority of textbooks that teachers use for teaching mathematics encourage the use of the 100-square, while students are taught in the course of their studies at university that this didactic material does not necessarily support cognitive processes that are required in arithmetic to 100.

		P- value	Mean teach.	σ_t	Mean stud.	σ_s
1	Didactic material enables problem solving situations in mathematics.	0.17	3.87	1.01	3.74	0.97

Table 3: A statement for checking the research question - What are the attitudes of students and teachers towards using didactic material for problem solving in mathematics?

Secondly, we were interested in the respondents' attitudes towards the influence of didactic material on problem solving (research question 1b).

We can see that the distribution of answers does not depend on the status of respondents. Based on the mean values, we can conclude that both teachers and students are aware of the significance of didactic material for problem solving in mathematics. In relation to this, we were especially interested to find out which didactic material teachers and students would know how to use in problem solving. The responses are presented in Table 4.

	Teachers		Students		
Didactic material	Number of responses	Share of responses	Number of responses	Share of responses	P - value
Geoboard	35	46%	61	65%	0.01
Link cubes	59	78%	75	80%	0.73
100-square	47	62%	62	66%	0.58
Pocket calculator	34	45%	62	66%	0.00

Table 4: Which didactic material should be used in problem solving?

We can see that the status of respondents has a substantial impact on the use of two types of didactic material – the geoboard and the pocket calculator. The higher share of students who are able to use them in problem solving can be attributed to the fact that the course on didactics of mathematics places emphasis on the use of modern didactic material mostly from the point of view of solving mathematical problems. Teachers, especially those who have been teaching for a longer period of time, are not really familiar with the use of the geoboard in teaching and still see the calculator as an arithmetic aid and not as a type of cognitive didactic material which can be used to solve problems (Hodnik, Čadež, 2000). These results confirm our findings that teachers tend to adapt didactic material to the ends they pursue and consequently they do not completely make use of the potential that new didactic material has for teaching and learning mathematics.

In the continuation, we focused on the attitudes of teachers towards new didactic material (research question 2):

		Mean teach.	σ_t
1	New didactic material does not have any influence on my teaching mathematics (I use it when it fits my teaching style).	3.22	1.34

Table 5: A statement for checking the research question – Does didactic material have an impact on teachers' lesson planning?

The chart (Figure 1) shows that most teachers (columns 4 and 5; 56.58%) agree with the statement that didactic material has almost no impact on their teaching style. This finding coincides with the study of Gellert (2004) which also finds that teachers adapt didactic material to their teaching and consequently do not fully utilize its potential for a different didactic approach or teaching methods, the problem solving approach being only one of them.

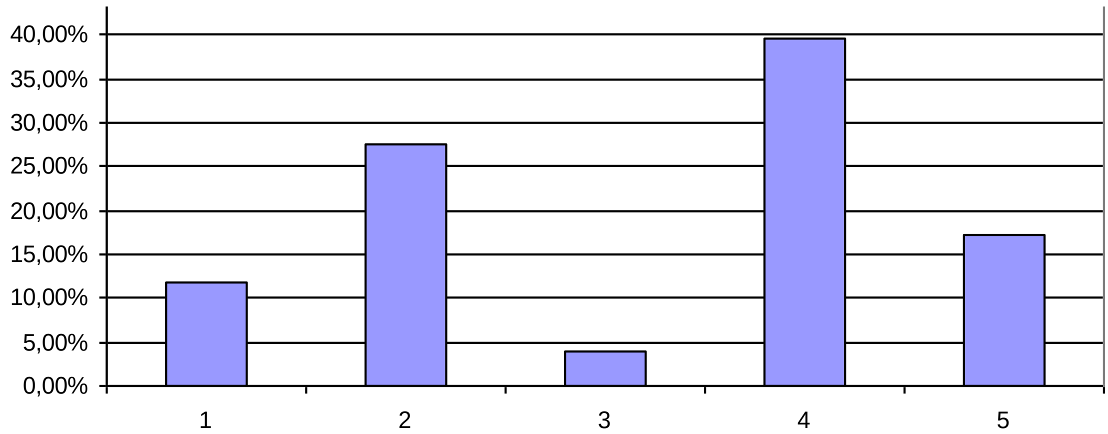

Figure 1: Didactic material does not have any influence on my teaching mathematics (I use it when it fits my teaching style). (Statement 1 in Table 5)

For each of the questions above, we included some commentary on the influence of each respondent's status on the choice of their answer. This helped us answer our last research question: Do the attitudes of teachers and students towards different statements about didactic material significantly differ? The following is a summary of our findings.

Both students and teachers are aware of the importance of using didactic material for learning mathematical concepts. On average they do not see any negative effects that the use of didactic material could have on learning mathematics but they differ in their attitudes towards the role of some specific types of didactic material:

– Teachers are more aware of the role of link cubes, the Dienes blocks, the abacus and the 100-square for the development of number concepts.

– The share of students who advocate the use the geoboard is much higher than the percentage of teachers who actually use it.

Both students and teachers are aware of the role of didactic material for problem solving.

In some cases the observed attitudes depend on the status of respondents. Due to their lack of teaching experience, many students were undecided about certain statements, for example about the statement ‘Too frequent use of didactic material prevents the development of a mathematical concept at an abstract level’ or about the statement ‘Didactic material could move children’s attention away from the mathematical concept to be learned.’

DISCUSSION

By comparing the attitudes of two different groups of respondents, teachers and students (future teachers), we hoped to gain an insight into their awareness of the importance of didactic material in the process of teaching and learning mathematics. At the same time, we also wanted to identify some differences between them. One of our findings is that the experience of practicing teachers significantly influences some of their attitudes, even though a lot could be done to raise their awareness about the role of didactic material and its use in the classroom while they are still at university as students.

Despite years of experience in teaching mathematics, teachers seem surprisingly uncritical towards some of the statements. For example, they find the 100-square very useful for less able children, which means they are not aware of or do not consider its potential negative impact on children’s learning. In our opinion, the 100-square does not necessarily trigger the ‘right’ kind of mental activity. Children can use it because it works, but they are unaware of why it works. It is enough for them to know that it will lead them to the right result. In other words, they take the 100-square to be a type of primitive calculator, which leads to the right result but not to the desired mental activity. Moreover, we can observe that the majority of teachers (59.21%) do not see any negative consequences in overusing didactic material, even though the instruction of mathematics based on the use of didactic material does not aim to remain at this level but aims to surpass it. Didactic material should only serve as a mediator facilitating the transition from concrete to abstract thinking. The survey also revealed a very high (excessively high, as we see it) share of teachers who failed to consider the possibility that didactic material could move children’s attention away from the mathematical concept to be learned (53.94%). In addition, many children tackle didactic material in a non-mathematical way, i.e. they do not see it as a representation of a mathematical concept or relation (for example, Dienes blocks or abacus).

We realize that in the field of arithmetic there remain a number of open questions related to teaching various algorithms. The educational system in Slovenia emphasizes written arithmetic and separate treatment of individual decimal units in the derivation of an algorithm, which is very different from the holistic approach (Anghileri, 2001) where the role of didactic material, for example the role of Dienes blocks, is substantially smaller. The emphasis on written arithmetic is also problematic for the use of pocket calculators, either as didactic material or as a problem solving means.

Future studies should focus on the way teachers use didactic material in problem solving, especially considering the different interpretations of problem solving among teachers and the relatively rare inclusion of such tasks in textbooks.

The role of didactic material in teaching mathematics is at least three-fold: it can help students with learning difficulties; it can promote and enable problem solving situations; and, most importantly, it can contribute towards the concretization of abstract mathematical concepts. All these functions call for further research, yet we already believe that their most important characteristic is the relation between physical handling and mental processes. In other words, what is important is the progress in mathematical knowledge achieved by using didactic material. Furthermore, we should not forget that it is children who know best what material helps them with their learning and in what way. We should strive to overcome the belief that teachers are the only ones who know what material best fits different groups of children – children should be actively involved in this choice and should have more control over their learning.

CONCLUSION

The study has shown that both teachers and students need to be systematically educated in the use of didactic material. We can conclude that in most cases teachers do not explore the potential that different types of didactic material offer, and rather employ textbooks and their teaching methods as key guidelines for dealing with didactic material. We are aware that it is not possible to achieve a direct and linear link between curriculum materials and teaching, or between curriculum materials and the learning of teachers. Researchers have found that close analyses of teachers' beliefs and the knowledge of mathematics can explain how they structure their lessons (Thompson, 1984). Studies on how teachers establish relations with the teaching resources that they use mostly focus on the ways teachers draw on resources and assume that doing so also involves interpreting the meaning and intent of these resources (Doyle 1993, Lemke 1990, Snyder, Bolin and Zumwalt 1992). These relationships are complex and often oversimplified. According to Clandinin and Connely (1992), the curriculum, often referred to as enacted curriculum, is not what is written in textbooks or policy guidelines but what actually takes place in the classroom. Teachers' ideas about mathematics and how it is learned as well as their views about teaching contribute significantly to their use of didactic material (Collopy 2003).

We believe that didactic material has an important role in problem solving when it is used in a careful and considered way, with an appropriate synthesis of physical manipulation and mental activity, which should lead, as often as possible, towards generalizations or derivation of mathematical laws. The results have also shown that students have an extensive knowledge in the area of didactic material that can be used in teaching mathematics. Especially pleasing is the fact that they are familiar with modern didactic material (for example, the geoboard and the calculator) and its role in problem solving. Nevertheless, it will be their work in the classroom that really shows the value of our findings.

REFERENCES

Anghileri, J.: 2001, Contrasting Approaches that Challenge Tradition, in: J. Anghileri (ed.), *Principles and Practices in Arithmetic Teaching*, Open University Press, Buckingham.

Beishuizen, M.: 1999, The Empty Number Line as a New Model, in: I. Thompson (ed.), *Issues in Teaching Numeracy in Primary Schools*, Open University Press, Buckingham.

Clandinin, D. J., Connelly, F. M.: 1992, Teacher as curriculum maker, in: P. W. Jackson (Ed.), *Handbook of research on curriculum*, Macmillan, New York, pp. 363-40.

Collopy, R.: 2003, Curriculum materials as a professional developmental tool: How a mathematics textbook affected two teachers' learning, *Elementary School Journal*, **103**, pp. 287-311.

Doyle, W.: 1993, Constructing curriculum in the classroom, in: F. K. Oser, A. Dick, J. Patry (Eds.), *Effective and responsible teaching*, Jossey-Bass, San Francisco, pp. 66-79

Fennema, E. H.: 1972, Models and Mathematics, *Arithmetic Teacher*, **18**, pp. 635-640.

Friedman, M.: 1978, The Manipulative Materials Strategy: The Latest Pied Piper?, *Journal for Research in Mathematics Education*, **9**, pp. 78-80.

Fuson, K. C., Briars, D. J.: 1990, Using a Base-Ten Blocks Learning/Teaching Approach for First and Second Grade Place Value and Multidigit Addition and Subtraction, *Journal for Research in Mathematics Education*, **21**, pp. 180-206.

Gellert, U.: 2004, Didactic material confronted with the concept of mathematical literacy, *Educational Studies in mathematics*, **55**, pp. 163 – 179.

Gravemeijer, K. P. E.: 1991, An instruction – theoretical reflection on the use of manipulatives, in: L. Streefland (ed.), *Realistic Mathematics Education in Primary School*, On the occasion of the opening of the Freudenthal Institute.

Hodnik Čadež, T.: 2004, Children's interpretation of arithmetic operations within a theory of representational mappings, in: A. Lazaridou (ed.) *Education*, Institute for Education and Research, Athens, pp. 305 – 314.

Hodnik, T.: 2000, Calculator Modulated Arithmetic in Elementary School, in: V. Kokolj-Voljč, B. Kutzler, M. Lokar, J. Palčič (Eds.), *Exam Questions and Basic Skills in Technology – Supported Mathematics Teaching*, Lehrmittel GmbH&CoKG, Hagenberg, pp. 85 – 89.

Labinowicz, E.: 1985, *Learning from Children: New Beginnings for Teaching Numerical Thinking*, Addison-Wesley Publishing Co, California.

Lemke, J. L.: 1990, *Talking science: Language, learning and values*, NJ: Ablex, Norwood.

Markovac, J.: 1990, *Metodika početne nastave matematike,* Školska knjiga, Zagreb.

Resnick, L., Omanson, S.: 1987, Learning to Understand Arithmetic, in: R. Glaser (ed.), *Advances in Instructional Psychology*, (Vol. 3), N.Y.: Lawrence Erlbaum Associates, Hillsdale, pp. 41-95.

Snyder, J., Bolin, F., Zumwalt, K.: 1992, Curriculum implementation, in: P. W. Jackson (Ed.), *Handbook of research on curriculum*, Macmillan, New York, pp. 402-435.

Suydam, M. M., Higgins, J. L.: 1977, *Activity-Based Learning in Elementary School Mathematics: Recommendations from the Research*, Columbus, Ohio: ERIC/SMEE.

Thompson, A.: 1984, The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice, *Educational Studies in Mathematics,* **15**, pp. 105-127.

Thompson, P. W.: 1992, Notations, Conventions, and Constraints: Contributions to Effective Uses of Concrete Materials in Elementary Schools, *Journal for Research in Mathematics Education*, **25**, pp. 297-303.

ANALYZING MATHEMATICS STUDENTS' LESSON PLANS: FOCUSING ON CREATIVE MATHEMATICAL ACTIVITIES

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This paper presents an analysis of mathematics students' lesson plans. The students were participating in a series of workshops focused on creative mathematical activities. The aim of the lessons was to develop some kinds of such activities for the pupils. The analysis of these lesson plans was made in order to examine the students' – future mathematics teachers' – ability to plan and organize the work of their students in such a way that they can have the opportunity to undertake different kinds of creative mathematical activities. The results of our analysis have shown that most students can design a lesson which fulfils the initial aim and their work revealed some aware actions to the direction of creativity.

INTRODUCTION

Contemporary mathematics education follows in the direction of mathematical activities. In many professions creativity, ingeniousness and a creative attitude to the problems are required even from a young person who just entered the field. At the same time, that creative side of education is almost absent at school. Mathematics teaching very often has an imitative and reproductive character. It is focused on elementary activities and skills and the students learn schematic behaviors. This is because the teachers are not sufficiently prepared to promote the creativity in mathematics among their students. They don't have sufficient knowledge, skills, experience and didactical tools to develop creative mathematical activities among their students (Klakla, 2008; Maj, 2006).

School mathematics, apart from ready-made knowledge (a set of definitions, theorems and procedures) is mostly a domain of a specific human intellectual activity whose product is ready-made knowledge and the tool used is specific mathematical thinking. Thus, that view of mathematics should be formed among the students (Hejny & Kratochvilova, 2005; Klakla, 2002). Mathematics knowledge is not only the main goal of mathematics education, but should also be the tool which enables the student to engage in mathematical activity. As a result of the work on mathematics lessons a student should learn to work like a mathematician – that is s/he should be able to put hypotheses, notice some regularities and relations, argue, justify, etc.

In this connection there is a need for paying special attention to developing creative mathematical activities and elaborating some methods of instruction for these activities. This view appeared and still appears in the literature of mathematics education (Burton & Stacey, 2005; Klakla, 1982, 2002; Krygowska, 1985, 1986; Mason, 2005; Mnich, 1980; Polya, 1975, 1993; Ponte, 2001).

THEORETICAL FRAMEWORK

The mathematical activity of a student is

> a work of mind oriented to the formation of concepts and to mathematical reasoning, stimulated by the situations which lead to formulating and solving theoretical and practical problems (Nowak, 1989).

It is worth underlining that the mathematical activity is a work of mind that should be stimulated. Therefore, it is not a work of a student which appears in a natural way.

A conception of forming creative mathematical activities was worked out by Klakla (2002). He distinguishes particular kinds of creative mathematical activities, which are present in an essential way in activities of mathematicians. These are:

a) hypotheses' formulation and verification;

b) transfer of a method (of reasoning or solutions of the problem onto similar, analogous, general, received through elevation of dimension, special or border case issues);

c) creative receiving, processing and using mathematical information;

d) discipline and criticism of thinking;

e) problems' generation in the process of the method transfer;

f) problems' prolonging;

g) placing the problems in open situations.

The essential element of learning mathematics (especially during mathematics investigation) is social interactions. Participation in a community of learners requires an active teacher – somebody who can lead the discussion, establish the rules of cooperation and motivate the learners (Cobb, McClain, 2006). Various authors describe the role of the teacher in such community. It is undoubtedly very important role in developing the social mathematical norms in the classroom. It is promoting the group of students as a community of inquiry, in which they feel more comfortable to share their ideas and to justify their opinions (Yackel, Cobb 1996; Lampert 1990).

According to Nęcka (2005) "a creative teacher will educate creative pupils, a not much creative teacher will rather discourage pupils from unconventional thinking" (p.201).

Da Ponte (2001) makes a detailed description and presents various roles of a teacher in leading mathematics investigation of the students. They are related to the professional knowledge of teachers, including their mathematical knowledge, and their didactical knowledge. The author mentions the following roles of a teacher: (1) challenge pupils, (2) support pupils, (3) evaluate pupils' progress, (4) think mathematically, (5) supply and recall information, (6) promote pupils' reflection.

The first three roles are related to the actions of the teacher:

> These roles are connected to the logic of the development of any activity. The teacher challenges the pupils with situations and questions in order to involve them in investigative work. The teacher supports them, asking questions, making comments, or providing suggestions. The teacher also tries to evaluate the progresses already done and possible difficulties, collecting information, and, based on that, decides to continue, to modify some aspects of the work, or to move to another phase of the activity (Ponte, 2001, p. 17).

The didactical work of the teacher is strictly connected with the mathematical knowledge of a teacher:

> All the didactic work carried out by the teacher requires an understanding of the task and its mathematical connections. The most specific aspect of the activity of the mathematics teacher, as a teacher of a discipline, is supporting the development of mathematical thought, before, during, and after the lesson (Ponte, 2001, p. 17).

Moreover, we may claim that the role of the mathematics teacher consists of various aspects besides the mathematical ones.

METHODOLOGY

In this paper we present the analysis of 39 mathematics students' (future mathematics teachers) lesson plans. The students belonged to three groups: group [1]: 9 students in the second year of their master's course, group [2]: 17 students in the first year of their master's course and group [3]: 13 students in the first year of their master's course. They participated in a series of workshops focused on creative mathematical activities. These workshops contained multistage tasks, in line with Klakla's (2002) conception. The workshops were organized as part of the course 'Didactic of Mathematics' and their duration was as follows: group [1]: 24 hours, group [2]: 24 hours and group [3]: 15 hours.

Our purpose was to assist the development of the future teachers' skills in organizing situations that – under certain circumstances – can lead to creative mathematical activities which are favourable to be undertaken by their pupils.

After the end of the workshops the students had the task to prepare a two-hour mathematics lesson with the main aim to develop some creative mathematical activities among pupils. The lessons' scenarios had to be related to mathematics specialization classes at high school. They were supposed to contain a number of detailed questions directed to the pupils and the description of creative mathematical activities being developed at that particular time. In their previous experience concerning the preparation of the lessons and the determination of the lessons' aims, the students were used to focus on the mathematical content enclosed in the curriculum. Now they had to concentrate on mathematical activities which should form and develop around a theme of a lesson.

The analysis of the lessons' plans was aimed to show us whether the future teachers can plan and organize a work of their pupils in such a way that they can have the opportunity to undertake different kinds of creative mathematical activities. However, it was of less importance the class and the mathematical content of the lesson.

That analysis was focused on the three following topics:

a) mathematical problems and tasks which were used in planned lesson,

b) the role that the teacher would play during the planned lesson,

c) creative mathematical activities.

Particularly, regarding the mathematical problems we interested in:

a) The kind of tasks that the students chose for the planned lesson (open problems, closed problems, problematical situations, which can or cannot be prolonged).

b) If the mathematical problems are somehow connected to each other (except the topic of the lesson), if they constitute a sequence or are separated from each other (if they are prolonged, if the pupils can have the occasion to use transfer of the method, if they give the opportunity to undertake the set of mathematical activities).

c) If the structure of the tasks or their order leads the pupils from the concreteness to abstraction (process of mathematization).

Regarding the roles of the teacher that s/he would play during the planned lesson, we wanted to learn about:

a) The kind of questions the 'teacher' asks (open, closed, suggesting).

b) If s/he asks for an explanation or a justification.

c) How s/he organizes the work in the classroom (forms of work).

d) If s/he plays a 'challenging' and 'supporting' role or imposes to the pupils his/her own way of thinking; if s/he is a leader or only the organizer of the learning process.

Regarding creative mathematical activities, we focused on:

a) The kind of these activities – if any – the author of a scenario could develop among pupils.

b) If s/he stimulates the pupils in the direction of undertaking such activities.
c) If the planned way of 'conducting' the lesson supports independent and creative thinking of the pupils.

RESULTS

After the analysis of the lesson plans we distinguished three categories: the scenarios which comply with the aim of developing creative mathematical activities, the scenarios which included some elements or fragments that comply with that aim, and scenarios of typical lessons which do not fulfil that aim. Table 1 shows the results according to the categories and the three students' groups.

	Category (1)	**Category (2)**	**Category (3)**
[1] group of students	7	1	1
[2] group of students	10	3	4
[3] group of students	4	1	8
Total number of the students	**21**	**5**	**13**

Table 1: The quantitative results of the analysis

We can notice that two thirds of the students could design a lesson whose aim was to develop the creative mathematical activities or a lesson with fragments fulfilling that aim (in total 26 from 39 students). The other students could not fulfil that task and most of them came from group [3], who participated in the smallest number of workshops during the didactical course.

The topics of the planned lessons were various: there were lessons related to geometry, calculus or theory of probability. The forms of classroom work were also different: collective work (15 lesson plans), collective and individual work (17 lesson plans) and group work (7 lesson plans). Especially we were interested in the last mentioned form because that form was mainly used during the workshops. However, collective work was not preferable among the students.

We will present now a couple of examples of the lessons plans representing all three categories.

Category (1) – scenarios which comply with the aim of developing creative mathematical activities:

<u>Example 1</u>

The theme of the lesson (in the third class of high school) was "Summarizing known facts about prisms". The author of the scenario underlined that the main aim was to "develop creative mathematical activities among the pupils through tasks which require work of mind in the direction of forming mathematical concepts and stimulating the formulation and verification of hypotheses". This expression was based on the definition of the mathematical activity by Nowak (1989). Except that main aim the student mentioned some detailed ones:

a) using basic (mentioned before) knowledge and skills in solving nonstandard problems, using mathematical language which helps in creative and critical thinking;
b) noticing and using analogies;
c) mathematizing;
d) defining and interpreting new definitions;
e) generalizing which can lead to discovering a theorem, and proving that theorem.

The lesson was devoted to the Pythagorean Theorem in three dimensions. The scenario included a two-hour lesson around this topic.

The pupils' work was organized as collective work with the teacher. The initial situation was the following problem: "Find the three-dimensional version of the Pythagorean Theorem". The task was an open problem and the way of introduction to the class did not impose the method of solving it. 'The teacher' directed some supportive questions, like: "What is the three-dimensional equivalent of a triangle in the plane?" (an answer: "a tetrahedron"), "What will be the equivalent of the right triangle then?" (an answer: "a right tetrahedron"), "So which tetrahedrons according to you can be right?", "What will be analogical theorem in the three dimensions? Try to formulate the hypotheses". It was assumed that the pupils can consider the particular cases of the three-dimensional equivalent of the Pythagorean Theorem, however the choice of the cases depended on them, e.g.: a right tetrahedron cut from a 1-cm edged cube and instead of the areas they could consider the length of the edges. That particular case led the pupils to discover another theorem. After that more questions were asked: "Can we find another example of a right tetrahedron?", "Can we investigate that problem in a different way?", provoking the pupils to consider a general right tetrahedron and then to change the formulated hypothesis (with the areas of the faces). The work on this problem led the pupils to formulate the conclusion that "a right angle in three dimensions is an angle which is one eighths of the space". That observation resulted in considering another case: a right tetrahedron cut form a regular tetrahedron.

The lesson was planned in such way in order to give to the pupils the opportunity to work with a series of problems whose solution methods could be transferred from one into the other. These considerations were always directed from the particular to the general case. The author of the scenario was also asking the following questions: "How can we check that it is really so?", "What conclusion can we reach?", etc. These questions were aimed to provoke the pupils to verify their hypotheses.

Example 2

The theme of the planned lesson (in the third class of high school) was "The relations between a function and its derivative". In the assumptions the author wrote that the lesson could be conducted when the pupils:

> will acquire the knowledge of the derivative of a function in a point and will know the definition of the derivative of a function, as well as its geometrical interpretation.

The lesson had the aim:

a) to acquaint with the properties of the derivative of a function and its relation with the function;

b) to develop the skills of using the derivative of a function to investigate its monotonicity and extrema;

c) to develop the intuitive understanding of the derivative of a function in a point as 'the speed of change' (increasing, decreasing) or as the tangent's angle.

The main aim according to the student was to develop discipline of thinking and critical thinking.

The work was organized in groups of four and by using the program 'Graph' for drawing the graphs of the functions. The student prepared the following worksheet for her pupils:

> What you should do:
>
> Draw the graph of the first from the given below functions and the graph of its derivative. Watch them and try to find some relations between them. Discuss it with your colleagues in your group. Then investigate if similar relations are in the other given functions. You can also think up your own functions. Can you explain why such relations are (remember what is the derivative of a function and its geometrical interpretation)?
>
> The functions to consider:
>
> 1. $f(x) = x(x - 4), f'(x)= \ldots$,
> 2. $f(x) = -x(x - 4), f'(x)= \ldots$,
> 3. $f(x) = x^3, f'(x)= \ldots$,
> 4. $f(x) = -x^5 - 2x^3 + 10x$, $f'(x)= \ldots$,
> 5. $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)(x - 5), f'(x)= \ldots$,
> 6. $f(x) = sin(x), f'(x)= \ldots$

The task which was the starting point of the pupils' work, was a problematic situation (Bonafé, 2002) whose aim was to construct new knowledge by discovering theorems not known by the pupils before. Being based on that problematic situation, formulating general instructions: "try to find some relations between the graph of the function and the ‘behaviour’ of its derivative”, the author of the scenario created the conditions for the pupils to discover unknown relations between the concepts which they already knew. In case of difficulties, she prepared a list of supportive questions: “what is happening with the function when the derivative is positive?”, “what when it is negative?”, “where does it ‘meet’ the x axis?”.

The aim of group work was to discover the following hypotheses:

a) a function increases when its derivative is negative,

b) a function decreases when its derivative is positive,

c) a function has ‘a hill’ or ‘a hole’ when its derivative equals to zero in that point and the sign of it is changing from positive to negative (‘a hill’) or from negative to positive (‘a hole’).

The author of the scenario let the pupils use everyday language which according to her can facilitate the work and make the pupils more easily imagine those situations.

The next step was the discussion about putting hypotheses and verifying them using ‘brainstorming’. During that discussion the teacher can also support the pupils by questions, e.g. “what it is the graph of the derivative of the function?”, “what is the relation of the coefficient of an angle of the tangent in particular points?”. Those questions could help the pupils when they had some difficulties and at the same time they could form their mathematical language. Only then the author of the scenario planned the formulation of these relations as theorems and their formal proving. That was expected to be the effect of the work of all groups.

The student chose a difficult topic for the lesson. But instead of ‘giving’ ready knowledge to the pupils she decided to let them work in groups with a problematic situation. Thus, she created the occasion for pupils to construct new knowledge and work by social interactions. She selected the functions in the task in such a way to give them the opportunity of searching, discovering, putting hypotheses. Then asking for justifying she would ‘force’ them to verify their own hypotheses and because of the common editing of the proofs she would develop pupils’ discipline of thinking and creative thinking.

Category (2) - scenarios which included some elements or fragments that comply with the aim of developing creative mathematical activities:

<u>Example 3</u>

The theme of the lesson (in the first class of high school) was: “Mathematical induction”. As the first aim the student wrote: “acquainting the pupils with mathematical induction” and as the second: “developing creative mathematical activities, like: transfer of a method, putting hypotheses, critical thinking”. The lesson was planned as a collective work with some elements of individual work. The starting point was the analysis of drawings prepared by the author of the lesson (Figure 1).

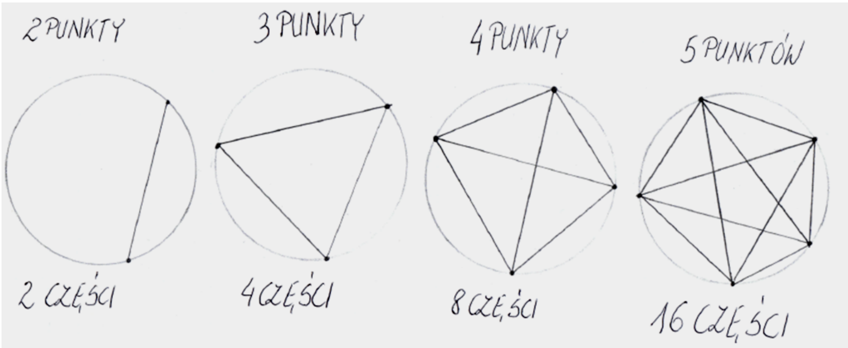

Figure 1: The initial task (‘punkty’ – ‘points’, ‘części’ – ‘parts’)

We can notice that the initial mathematical situation chosen by the student was interesting. But she could not use the potential of that situation. Even the description of the drawings was implying the direction of searching. Another intervention was the instructions given to the pupils: “Look at the drawings. On them we have 2 points, 3 points, 4 points and 5 points. The chords from these are also drawn. These chords divide the circles into 2, 4, 8 and 16 parts. How many parts we would get if we had n points?”. Then the student assumed that the pupils would answer: “for n points the circle would be divided into 2^{n-1} parts”. We cannot know how they would get that answer, and what is more, that answer is not correct. The pupils seem to not have any chance to discover, experiment, search different relations (not only that one). In order to check the given hypothesis the student planned to place 6 in the formula which would show the pupils that the answer is not correct. Then she would tell the pupils the theorem of mathematical induction: “in mathematics there is a reliable method of checking if the general conclusion made on the base of some cases is true. This method is called mathematical induction”. After giving the theorem to the pupils, she planned the second task (figure 2):

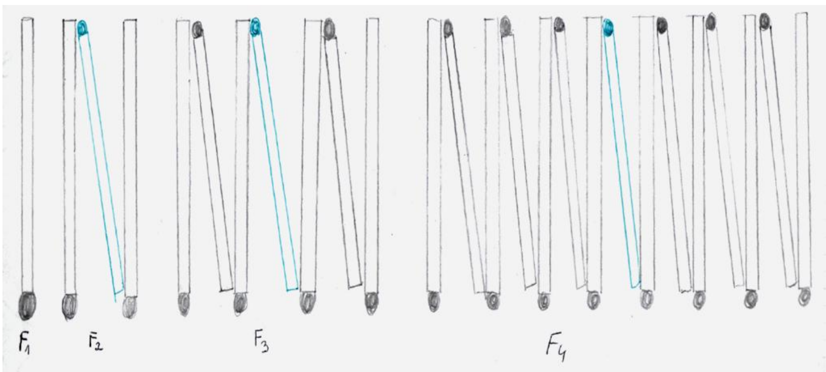

Figure 2: The second task

This is another not fully used open situation. The instruction to the task is again imposing, not supporting independent thinking: "The drawing presents some figures from the matches. Every one, except the first, was built from the two copies of the previous one connected with an additional match. How many matches we need to build the fifth figure? From how many matches we can build F_n?" The pupils doing some calculation should get the formula 2^n-1. This time the student assumed that the pupils will calculate the matches in every drawing and they will try to present the results as a formula which fits to every situation. Then they would generalize for the n-th figure. The proof would be done using the mathematical induction while the initiative was taken by 'the teacher'. All the next examples were typical, since their aim was to prove properties among natural numbers.

The student chose some interesting mathematical problems but they were not enough to develop creative mathematical activities. The significant intervention of 'the teacher' would probably result in the pupils putting only two hypotheses but even during that process the analysis of the drawings limited that opportunity. Also the method of verifying the hypotheses was imposed to the pupils without any explanation of it. We can notice that even having some 'good tasks' and a 'good idea of the lesson' is not enough; the crucial element is the way of working with the pupils.

Category (3) - scenarios of typical lessons which do not fulfill the aim of developing creative mathematical activities:

The characteristic of those lessons was that they presented typical mathematics lessons in which the teacher played the role of the leader – authority who is asking the questions and leading the pupils to the desired aim. The tasks used during the lessons were closed, related to the concrete theme and the questions were also directed to a concrete answer. Even if the form of work was different – e.g. group work – and the role of a teacher-leader changed, this was not a guarantee that the scenario of the lesson is prepared in such a way to let the pupils work creatively.

Example 4

The theme of the lesson (in the second class of high school) was: "The area of quadrangle, the equation of a line and a circle – solving the tasks". The work of the pupils was organized in groups of five, by the "tasks' tables" method (every group gets a different task(s) and then they share the results which should assist the formulation of another relation). The tasks prepared for the groups were very similar. An example of one of them is the following:

> In the parallelogram ABCD the following vertex are given: A=(-3,-2), B=(5,-l), C=(7,4). The point of intersection of the parallelogram's diagonals is the center of a circle with diameter AC. Calculate the coordinates of the vertex D, the area of the

parallelogram ABCD, the equation of the circle and the equation of the diameter AC. Present an interpretation of the task.

The presented task is a closed task, typical for using the elementary mathematical skills. It does not challenge the pupils (in the sense of making them being interested, provoke some mathematical investigation, searching). After solving it, a presentation of the solutions and common discussion was planned. It was not mentioned what issues connected with the tasks would that discussion concern. The role of the teacher was determined as a coordinator, but there was not any description how it would look like. From the scenario we can only notice the organization of the working groups; however it is not an organization in the direction of creativity. The planned lesson was directed into mathematical content and the skills of working in teams. However, it did not fulfil the aim of developing creative mathematical activities.

CONCLUSIONS

We are aware that analysing only the scenarios of lessons is not enough in order to predict if the lessons would run according to the plans. The school reality would verify both the scenarios and the skills of the students conducting the lessons. However, in the education of future teachers we can only require that we should make them able to plan a lesson according to the aims which they would like to achieve. Most students could design the lesson by developing creative mathematical thinking. Their scenarios seemed to be a conscious action in the direction of creativity (because of the frequent comments almost in every step by ‘the teacher’). The students treated the lesson as a multifaceted process by having in mind the mathematical content of the lesson, the activities which they wanted to develop, the attitude of ‘the teacher’ and the choice of the appropriate didactical tools. The fact that most students who ‘failed’ in that task (8 from 13) were in group [3] which had the smallest number of the workshops during their didactical course, is also significant.

REFERENCES

Bonafé, F. et al.: 2002, Les narrations de recherche de l`école primaire au lycée, I.R.E.M. de Montpellier.

Cobb, P., McClain, K.: 2006, Guiding inquiry-based math learning, in: R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences,* New York, NY: Cambridge University Press, pp. 171–185.

Hejný, M., Kratochvilova, J.: 2005, From experience through generic model to abstract knowledge, Proc. of CERME4, Sant Feliu de Guixols, Spain, http://cerme4.crm.es.

Klakla, M.: 2008, Universal Basic Mathematical Education and Education through Mathematics, *Acta Didactica*, **3**(2), University in Nitra, pp. 9-30.

Klakla, M.: 2002, Kształcenie aktywności matematycznej o charakterze twórczym na poziomie szkoły średniej, in: J. Żabowski (Ed.), Materiały do studiowania dydaktyki matematyki, Tom III, Prace dr Macieja Klakli, Płock.

Klakla, M.: 1982, Z badań nad rozpoznawaniem uzdolnień matematycznych, Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki, 2.

Krygowska, A. Z.: 1985, Matematyka współczesna i nauczanie w świetle dyskusji na zachodzie Europy (1958), in: G. Treliński, H. Siwek (Eds.), *Modernizacja kształcenia matematycznego i jej wpływ na rozwój dydaktyki matematyki*, Kraków: Wydawnictwo naukowe WSP.

Krygowska, A. Z.: 1986, Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich, Roczniki Polskiego Towarzystwa Matematycznego, Seria V, *Dydaktyka Matematyki*, **6**.

Lampert, M.: 1990, When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching, *American Educational Research Journal,* **27**(1), pp. 29–63.

Maj, B.: 2006, Teachers' Beliefs and Competencies of Creative Mathematical Activities, *Proceedings of PME 30*, Volume 1, Praga, pp. 292.

Mason, J., Burton, L., & Stacey, K.: 2005, *Matematyczne myślenie*, Warszawa: WSiP.

Mnich, W.: 1980, *Aktywności matematyczne jako kryterium doboru zadań w nauczaniu matematyki* (Unpublished doctoral dissertation), Akademia Pedagogiczna, Kraków.

Nęcka, E.: 2005, *Psychologia twórczości*, Gdańsk: Gdańskie Wydawnictwo Psychologiczne.

Nowak, W.: 1989, *Konwersatorium z dydaktyki matematyki*, Warszawa: PWN.

Polya, G.: 1975, *Odkrycie matematyczne*, Warszawa: WNT.

Polya, G.: 1993, *Jak to rozwiązać?*, Warszawa: PWN.

Ponte, J. P.: 2001, Investigating in mathematics and in learning to teach mathematics, in: F. L. Lin & T. J. Cooney (Eds.), *Making sense of mathematics teacher education,* Dordrecht: Kluwer.

Yackel, E., Cobb, P.: 1996, Sociomathematical norms, argumentation, and autonomy in mathematics, *Journal for Research in Mathematics Education*, **27**(4), pp. 458–477.

PRE-SERVICE TEACHERS' FIRST-TIME CREATIONS OF OPEN-ENDED PROBLEMS

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Open-ended problems are said to enhance students' creativity by offering them multiple solutions and solution paths. Thus, teachers are expected to have some experience in such problems. This paper describes a part of an instructional series for pre-service teachers, who were – among other activities – asked to create their own open-ended problem based on a given phrase. Few working groups completed the task. The problems were analysed from a mathematical and linguistic point of view and the results of the analysis show that despite their lack of experience these few pre-service teachers created interesting problems by including everyday and complex data; however, the complexity and subjectivity were eventually interpreted negatively by their colleagues.

INTRODUCTION – THEORETICAL REMARKS

The importance of problem solving is stressed by all mathematics educators. In numerous documents, from curriculum guidelines (NCTM, 2000), to influential and well-known monographs (Shoenfeld, 1985), problem solving is presented as a substantial element of students' mathematical thinking. The main characteristics of a mathematical problem is that is has "no obvious solution or path to the solution" (Southwell, 2004, p. 3) and that "it involves engagement on the part of the solver" (ibid.); this means that what constitutes a problem for one student might not be a problem for another one. Following Foong's (2002) categorization we see that problems can be either closed (one solution) or open-ended (multiple solutions). Open-ended problems are said to enhance students' creativity by offering multiple solution paths and serving multiple goals (Silver, 1997). Moreover, in order for teachers to be able to use open-ended problems in their classrooms they should have some experience in solving such problems by themselves.

Having this in mind I designed a series of instruction units for pre-service teachers. The aim of these units was to familiarize students with the problem solving approach not only by solving but also by posing and evaluating problems made by their colleagues. The aim of the present paper is to examine students' initial ideas and interpretations of what constitutes an open-ended problem. For that purpose, students' own constructions and evaluations of open-ended problems will be presented and discussed.

CONTEXT AND METHODOLOGY OF THE STUDY

The study is based on three instruction units that took place in the first three weeks of March 2010 during a course entitled "Didactics of Mathematics I" which is offered to students at the third year of their studies (in a total of four years). The participants of the study were 189 students whose only previous experience in university Mathematics was a course entitled "Basic mathematical concepts" which includes basic elements of number theory, number systems, introduction to functions, percentages, elementary Euclidean geometry and basic statistics. The aim of the "Didactics of Mathematics I" course is to provide an outlook on the current approaches in Mathematics Education, together with engaging students in problem solving, modelling activities and generally group work. This course is followed by the "Didactics of Mathematics II" course, which is more practice-oriented and the students participate in lesson planning and evaluating.

The instruction units under consideration were the third to fifth in a sequence of 13 three-hour units that comprise the course. In the previous two units the students were introduced to the current approaches in Mathematics education (e.g. constructivism, socio-cultural approaches, interactionism, realistic Mathematics), usually in comparison with the 'traditional' approaches (e.g. transmission of knowledge). Besides that, the students were engaged in solving problems, usually in groups of three to four. The different solutions and approaches were discussed and the instructor – the author of the paper – tried to initiate the students in a mathematical culture that involved the following norms (cf. Yackel and Cobb, 1996):

- a solution is accepted if it can be mathematically justified; the same is the case with the rejection of a solution which has to be done only on a mathematical justification basis;
- mathematical justification is based on logical connections between well-known and commonly accepted mathematical facts;
- commonly accepted mathematical facts are a cultural product and not a part of an out-of-the-world sphere of thoughts;
- there may be more than one solution strategies and solutions to a problem;
- real-life problems usually require more than simple numerical skills; particularly, they may require decision-making, hypothesis formulation by 'filling-in' the missing information, etc.;
- each participant is expected to (be able to) evaluate the others' but also his/her own solution, based on the discussion that takes place.

The reason I decided to focus on the above was that I had clear indications of a common attitude on behalf of the students on the existence of only one 'correct' solution and on the authenticity of the instructor, which is seen as unquestionable and reliable. And this was one of the reasons that I also chose to focus on the initial productions of students, once they were faced with open-ended problems. What I was expecting to obtain was an image of how students see, handle/solve and evaluate open-ended problem when they do not have any prior experience. In order to do so, I organized the three teaching units according to the following scheme:

1. Introduction to the categorization of problems according to the number of their solutions: the students were introduced to Foong's (2002) categorization and they were given examples of all cases.
2. Solving all types of problems: the students were asked to work in groups of three to four in order to solve one problem from each category mentioned before; this process was followed by discussion on each other's solutions and possible implementation in the classroom.
3. Creating two problems based on the phrase: "Eight olive trees can give approximately 72 kg of olive oil". This activity was taken from the Greek Mathematics textbook for the 4th grade of Primary School (9-10 year old children). The instructor suggested that one of the problems should be an open-ended one.
4. Solving, evaluating and categorizing the open-ended problems produced by the students: there were finally only ten open-ended problems produced; each group was given a working sheet containing these problems and adequate space for solving, evaluating and categorizing them.
5. Rephrasing the above problems: the students' evaluation of their colleagues' problems was rather poor; that is the reason why I decided to give them the opportunity to 'correct' the problems according to their own standards and norms.
6. Creating a single open-ended problem based on the phrase: "The students of the two last classes of a Primary School decided to put two types of flowers in the two flower-beds in their school's garden. The length of each flower-bed is 30 m".

The above activities were realized during three 3-hour sessions as follows: activities 1, 2 and 3 in the first session; activity 4 in the second session; activity 5 in the third session and activity 6 was given to the students as a homework after the third session. The data for the present study come from students' written work, but there will also be few comments from the discussion that took place during all activities. Due to space limitations I will only focus on activity 3, which was expected to be the most creative one.

The analysis of students' open-ended problems consists of three parts. In the first part I focus on the mathematical aspects of the problems, i.e. their content and their solution process. Foong (2002) suggests that open-ended problems can be categorized into converted textbook problems with open-ended situations for conceptual understanding and applied problems with real-life context. The converted problems can be further categorized into missing data problems, problem posing and explanation of concepts/rules or errors. The solution process categorization was done by interpretation of the problem's text, focusing on the assumptions that are needed from the reader-solver in order to proceed to the solution process and the mathematical processes (e.g. operations) involved in the solution process.

The second part of the analysis also focuses on the problems' texts, but from a critical linguistic point of view. According to Morgan (1998) the basic functions of a text are the ideational (or experiential), the interpersonal and the textual. Each one can be identified by the following questions:

> What does this mathematical text suggest mathematics is about? (p. 78) What is the role of human beings in mathematics? (p. 80)
>
> Who are the author and the reader of this mathematical text? What is their relationship to each other and to the knowledge constructed in the text? (p. 78)
>
> Textual function: What is the mathematical text attempting to do? (p. 78)

The analysis on the ideational function can be done by identifying "the types of processes and the types of participants that are active in them" (Morgan, 1998, p. 80). Six main types of processes are distinguished, namely material, mental, relational, behavioural, existential and verbal. The process of nominalization which treats mathematical entities as objects at their own right is also important. Moreover, a focus on the participants in the text (e.g. whether it is a human being, a mathematical object, etc.) may assist us in examining the author's view on mathematics. The interpersonal function is usually revealed by the use – or the absence – of personal pronouns and the author's consistency in their use. The use of imperatives is also of importance. Finally, the textual function can be examined by internal features of the text. However, the small size of the problems created and their predetermined function (in the sense that they were supposed to be open-ended problems) did not allow for an examination of their textual function.

The third part contains students' evaluations of their colleagues' problems. This part has mostly a demonstrative and complementary function to the other two. In other words, I will use this part in order to demonstrate students' common assumptions on the characteristics that should be considered while evaluating a problem.

Before we move to the results of the study, we have to consider that during the study students could not 'escape' from the situation of a university course which – among other commitments – includes assessment by their instructor. In other words, the students did not have any other option but participate in all these activity, probably having in mind that their work is somehow evaluated. In order to minimize that stress, when I received questions like "Are you going to assess us according to our solutions?" (referring to activity 4) I answered "No, the important thing for me is just to see your work". Indeed, from the instructor's point of view, I was mostly interested in providing the students a fertile ground for cooperation in the context of open-ended mathematical problems.

RESULTS

After students solved a number of problems, closed and open-ended, their last task in the first session that we are looking into was to design two problems based on the phrase: "Eight olive trees can give approximately 72 kg of olive oil". It was apparently the first time that the students were faced with such a task and they were frustrated. Among their first questions were:

> What level of difficulty shall our problems have?
>
> Will that phrase be in the problem or should it be its answer/solution?

At that point I stressed that there are no constrains concerning the difficulty level of the problems that they will design; that is because I wanted to see the genuine (and maybe spontaneous) results of their creativity. After the necessary answers were provided, I stressed the fact that one of the problems should be an open-ended one. More questions arose from a small number of students:

> How shall we make an open-ended problem?
>
> Shall we make a similar one to the ones that we did before?
>
> Does it have to have many solutions?

In order to assist them, I showed on the data projector the slide of the presentation which was related to Foong's (2002) categorization. The students had around 15 minutes to complete the task, but many groups worked for longer. Finally, only ten out of 63 groups created an open-ended problem. The problems created were the following. During translating in English I have tried to not alter the students' grammatical forms and mistakes.

> P1. We have 8 olive trees and each one gives a different amount of olive oil. If altogether they give 72 kg of olive oil, how much oil could each one give?
>
> P2. How many olive trees are needed for the farmer in order to produce around 72 kg of olive oil?
>
> P3. A farmer has two fields with olive trees. In each field there are 6 olive trees. Some olive trees were destroyed from the recent storms. How many olive trees could be in each field so that 72 kg of olive oil is produced, knowing that 8 olive trees are needed for the production of 72 kg of olive oil?

P4. I possess 8 olive trees which can give 72 kg of olive oil altogether. Choose the type of vessel that we can put it and why. How many will be needed and why would you choose these vessels?

P5. If 1 olive tree gives around 9 kg of olive oil (or litres) how many olive trees are needed (for us) to fill a barrel?

P6. Mrs. Yiorgena produces around 72 kg olive oil from 8 olive trees, while uncle-Mitsos produced around 160 kg olive oil from more productive olive trees. How many were approximately uncle-Mitsos's olive trees?

P7. Mr. Yiannis produces 72 kg of olive oil from 8 olive trees. How many olive trees does Mrs. Maria need in order to produce little more kg of olive oil?

P8. Basili's father has a field with 90 olive trees. Eight olive trees give around 72 kg of olive oil. But this winter in Agrinio it snowed after many years and some olive trees were destroyed. How much oil did the remaining ones give?

P9. Eight olive trees which are in our garden produce 72 kg of olive oil. How many kg of olive oil could we produce in a 2-stremma[3] field?

P10. Eight olive trees give around 72 kg of olive oil, but can also give around 1000 olives. If you were the farmer, would you rather sell the olives or make them into oil and then sell them?

According to Foong's (2002) categorization all the above problems fall into the category "Converted textbook problems with open-ended situations for conceptual understanding – Missing data". Thus, this categorization was insufficient for the purpose of this study. A next thing that I examined was the type of extra information that was added explicitly in order to create the problem. That information varied from mathematical concepts (P1: different amount; P6: 160 kg; P9: 2-stremma, P10: 1000 olives) to quasi-mathematical (P3: some; P6: more productive; P7: little more; P8: some) or everyday-life concepts (P3: storms; P4: vessel; P5: barrel; P8: winter-snowed). Quasi-mathematical concepts do not have the clarity of formal concepts but they appeared in four out of ten problems. The next step was to identify the assumptions requested by the potential solver, together with the mathematical operations involved.

a) P1, P2, P3: require the investigation of possible combinations that add up to 72 (P1, P2) or 8 (P3). P2 does not fully adhere to the instruction to use the phrase given.

b) P4, P5: require an assumption by the reader, which will eventually lead to a division. The assumption is related to the capacity of an appropriate vessel (P4) or a barrel (P5) and can be based on real-life objective data. In P5 the given phrase is not used as it is, but as a starting point – hidden from the final reader.

[3] The stremma is a Greek unit of land area, equal to 1000 square metres.

c) P6, P7, P8: require an assumption by the reader, which may lead directly to the solution (P7), to a division (P6) or to more operations (P8). The assumptions are related to the reader's subjective interpretations of the expressions "more productive" (P6), "little more" (P7) and "some" (P8).

d) P9, P10: require some assumptions by the reader, concerning the distribution of olive trees in a field of known area (P9) or concerning the cost and the possible profit from selling olives and olive oil. These assumptions can be based on real-life data, but are characterized by a high degree of complexity.

It is interesting to note that none of the above problems requires explicitly the writing of all – or at least more than one – possible solutions. All problems but one (P10) include the question of "how much" or "how many". The last problem (P10) raised much discussion among students, who stressed the realistic and complex factors which influence the solution process in a significant way. Some of the factors mentioned were:

> The process of making olive oil requires special equipment; since we don't know if the farmer has such equipment it is hard to decide what would be best for him.

> It would be hard for someone to estimate the number of olives produced by an olive tree.

> Not all olive trees can be used both for olives and for producing olive oil.

The second part of the analysis aimed to provide some clues concerning how pre-service teachers see mathematics, themselves while doing mathematics (ideational function) and the relation with their potential problem solvers (interpersonal function). The processes involved in the problems could be all characterized as material, the main one being the production of olive oil by the olive trees. Only P10 contains a somewhat relational process, since the farmer has to compare two material processes (selling the olives or making them into oil and then sell it). Participant in most problems is a farmer, who is in two problems accompanied by another farmer (in an antagonist position). The production of olive oil is either attributed to the olive tree (P1, P3, P4, P5, P8, P9 (partly), P10 (partly)) or to the farmer (P2, P6, P7, P9 (partly), P10 (partly)). Concerning the interpersonal function, the use of first person pronouns was present, but not extended (P1, P4, P5, P9); in other cases the reader/solver is an observer of the situation (P2, P3, P6, P7, P8) and in P4 and P10 the reader is addressed directly by a question. In P4 there is a conflict between the various pronouns used.

Moving to the third part of our analysis we will meet students' evaluation of their colleagues. The students were asked to mark every problem from 1 (poor) to 5 (excellent), by writing down each problem's advantages and disadvantages and by reformulating the problems which were poorly evaluated by them (less than 4 in the scale given). The following table shows the average and the standard deviation of each problem.

Problem	Average	SD
P1	3,95	0,99
P2	2,33	0,93
P3	4,38	0,99
P4	3,37	1,05
P5	2,76	1,14
P6	4,26	0,93
P7	3,51	1,00
P8	3,67	1,08
P9	2,98	1,06
P10	2,63	1,25

Table 1: Students' evaluation of problems.

Beyond this numerical evaluation it is interesting to see the students' remarks. I have decided to focus on the remarks concerning the disadvantages of the problems, as expressed by the students. The answers provided come from three groups who have filled in all answers requested; I have avoided summarizing quantitatively all students' answers, since there were many cases of non-answered working sheets.

P1. Infinite solutions / Some solutions might not be realistic / Infinite solutions if you consider decimal numbers

P2. Missing data / Ambiguous / Missing data

P3. Ambiguous / Ambiguous / None

P4. Ambiguous / Ambiguous / Subjective solutions may lead the class to confusion

P5. Missing data / Missing and ambiguous data / Change of initial data – missing data

P6. Infinite solutions / None / None

P7. Infinite solutions / Ambiguous / No operations needed

P8. Missing data / Missing and ambiguous data – wrong structure / Missing data – too many possible combinations

P9. Ambiguous data / Missing data / Missing data

P10. Missing data / Wrong structure – non-connected data / Missing data – syntactical and notional mistake

A main remark may concern the students' insistence on considering missing data as a disadvantage despite the fact that they were introduced to the open-ended problems as sometimes having missing data.

DISCUSSION

It was interesting to note that during the discussion that took place after the students completed their work in activity 2, I refrained from giving them the 'final' solution (whenever there was only one); but at the end of the lesson, or even in the next lesson, the students insisted on me giving them the solution to the problem we discussed. Concerning the problems created, it is obvious that the students tried to make their problems as much realistic as possible, sometimes by adding contextual information which was either affecting the problem situation (weather conditions) or not (the farmers' names, which were used by the students resembled actual names that are used in Greek villages). However, the small number of teams that completed the task indicates the students' reluctance (or inability) to engage in such a challenging and new for them activity.

The human agent was present in most problems (9 out of 10) in the form of a farmer, a second farmer, the author or the author with the reader(s). Olive oil production was transferred from the olive tree (impersonal) to a human being. It seems that the students attempted to make their problems look more every-day, by putting a person in the control of the situation.

The use of the inclusive 'we' was extended and this could be attributed to the students' attempts to create problems that would somehow resemble textbook problems. The students' lack of experience in problem posing was not apparent from a linguistic point of view, since there was only one case (P4) which involved switching between different personal pronouns.

From the solution process point of view the potential solver of the problem would have to make some hypotheses in order to proceed to the solution. Most of these hypotheses were clearly subjective, which is sometimes the case with open-ended problems. In few cases the situations described resembled real-life situations with all the complexity and decision-making which is involved. However, in these problems the students would have to search for additional resources to obtain the necessary data. It is interesting that the most characteristic of these problems (P10) was at the same time the most controversial, thus receiving a very low evaluation.

Concerning students' remarks on the open-ended problems, it seems that their lack of experience in dealing with open situations guided their participation throughout the various activities, including evaluation. During all these units I could see their frustration, but at the same time I could see their attempts to live up to my expectations. The results show that although some students were able to create open-ended problems there is still a long way to initiating pre-service teachers in a culture where mathematics is not just a school subject, but the means to deal with and overcome everyday complex problems.

REFERENCES

Foong, P.Y.: 2002, Using short open-ended mathematics questions to promote thinking and understanding, in A.. Rogerson (Ed.), *Proceedings of the International Conference: The Humanistic Renaissance in Mathematics Education* (pp.135-140), Palermo, Italy.

Morgan, C.: 1998, *Writing mathematically: The discourse of investigation*, Routledge, London.

National Council of Teachers of Mathematics: 2000, *Principles and Standards for School Mathematics,* NCTM, Reston, VA.

Schoenfeld, A. H.: 1985, *Mathematical Problem Solving,* Academic Press, INC, USA.

Silver, E. A.: 1997, Fostering creativity through instruction rich in mathematical problem solving and problem posing. *Zentralblatt für Didaktik der Matematik*, **3**, pp. 75-80.

Southwell, R.: 2004, Investigating problem solving. Paper presented at the *10th International Congress on Mathematical Education 2004, Topic Study Group 18.*

Yackel, E., Cobb, P.: 1996, Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education,* **27**, pp. 458-477.

PRE-SERVICE MATHEMATICS TEACHERS' STRATEGIES IN SOLVING A REAL-LIFE PROBLEM

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The paper presents the investigation of pre-service mathematics teachers' behaviour and solution strategies for a real-life problem based on the context of a post office situation. Our students' initial reactions included frustration and discomfort, probably because of their lack of experience. Finally, no student presented a complete solution, namely the mathematical model used by the ticketing machine in the post office. However, some students managed to reach a partial solution. The basic action undertaken by the vast majority of students was the visual representation of the situation described in the problem.

INTRODUCTION

The importance of engaging students in meaningful mathematical activities is stressed by most mathematics educators and curricula worldwide. NCTM (2000) stresses the fact that students should be able to use Mathematics in their everyday life. Boaler (1993) considers vital the role of the context of mathematical problems:

> Contexts have the power to form a barrier or bridge to understanding and it is this realisation which prompts consideration of the range and complexity of influences upon a student's transfer of mathematics. (p. 370)

In other words, if our aim is a successful transfer of school mathematics in real-life situations, the context of the problems that we pose to the students should be carefully chosen, in order to reflect the reality's complexity and subjectivity. All these considerations are in the core of the Realistic Mathematics Education (RME) approach, according to which mathematics is a human activity which should be connected to reality (Freudenthal, 1978). That connection can be made possible by the use of word problems, which:

> ... can provide practice with real life problem situations, motivate students to understand the importance of mathematics concepts, and help students to develop their creative, critical and problem solving abilities" (Chapman, 2006, p. 212)

The above considerations have been reflected in the latest Polish curricula regarding classes of primary and secondary school. The following excerpt is taken from the latest national curriculum (Podstawa programowa z komentarzami, 2008):

> The most important skills acquired by a student in primary school are (among others):

- mathematical thinking – the skill of using the basic mathematical tools in everyday life and conducting elementary reasoning;
- scientific thinking – the skill of formulating conclusions on the base of empirical observations connected with nature and society (p.15).

Among the aims of mathematical education in the primary school are the following:

- Mathematical modelling: A student selects a proper mathematical model for a simple situation, uses known formulas and relations, and transforms the text of the task into arithmetical calculations and simple equations.
- Reasoning and creating strategies: A student conducts simple reasoning consisting of a little number of steps, sets the order of actions (and calculations) which lead to the solution of the problem and can reach a conclusion from information given in different forms (p.29).

Similar remarks can be found in the latest curriculum for 'Gymnasium' secondary school (13-16 years old):

The most important skills acquired by a student in gymnasium are (among others):

- mathematical thinking – the skill of using basic mathematical tools in everyday life and formulating judgments founded on mathematical reasoning;
- scientific thinking – the skill of using scientific knowledge to identify and solve problems, and formulating conclusions on the base of empirical observations connected with nature and society (p.19).

Concerning the aims of mathematical education at the gymnasium level we read:

- Mathematical modelling: A student selects a proper mathematical model to a simple situation, builds mathematical model of given situation.
- Using and creating strategies: A student uses a strategy which results from the task, creates a strategy for problem solving.
- Reasoning and argumentation: A student leads simple reasoning and provides the necessary argument (p.35).

From the above it is obvious that students' ability to deal and model 'practical' (real-life) problems is of central importance for the curriculum developers. One may expect that this should in turn affect mathematics teachers' education.[4] But these teachers come from Mathematical Departments, which means that most of their courses consist of 'pure' (advanced) Mathematics. Problem solving, problem posing and didactical engineering are given relatively less time, thus providing the future teachers with relatively few experiences about the methods and techniques that they are expected to teach. Such a course on Didactics of

[4] According to the Polish educational system graduates of the Mathematics departments (Bsc.) are entitled to teach from the fourth class of Primary School (10-11 years old) to the third class of 'Gymnasium' (16 years old). After obtaining their masters' degree the students are also entitled to teach in the 'Lyceum' (16-19 years old).

Mathematics provided the ground for our study. This course, taught at the Institute of Mathematics of Rzeszow University consists of a series of lectures accompanied by 'practice' lessons; these lessons include discussions on topics like curriculum and handbook analysis, didactical engineering and how these are informed by various theoretical approaches. It also includes solving problems. Our study took place during one of these 'practice' lessons and our aim was to investigate the students' behaviour when they would be faced with a real-life problem based on a familiar context (post office). Particularly, we were interested in examining:

- the students' initial reactions and emotions when they would face the problem.
- the students' solution strategies and particularly if they would follow a conventional (formal) approach, an informal one or if they would try to construct a mathematical model through a process of mathematization.

CONTEXT AND METHODOLOGY OF THE STUDY

Our study took place in December 2009 during a single two-hour session of "Didactics of Mathematics" course at the Institute of Mathematics of Rzeszow University. The authors of the paper played the role of the instructors and organised the session into small parts consisting of solving mathematical problems of various types and contexts (including non-context ones). Thirty-three students were present in the session, all of who were at the 3rd year of their mathematical studies with a specialisation in teaching. At that time, none of the students had any experience in problem solving through their studies, although they were aware of the theoretical assumptions underlying this approach.

The students were initially given a series of non-context ('zero-order' context according to De Lange, 1999) problems in order to trigger their interest. Most students solved these problems quite easily. Then they were given the following problems which may fall under the category of realistic problems:

1. A paper company is going to produce a number of concert tickets. The ticket dimensions should be 6 cm to 8 cm. The tickets will be cut from a paper with dimensions 30 cm to 21 cm. What is the maximum number of tickets that can be cut from a single piece of paper?
2. When I entered the post office I got the ticket shown in the image (see Figure 1). The machine also indicated that there are 22 customers waiting and I saw that the customer with the number 398 was the last being served. There were four cashiers operating at that time. The ticket showed that the estimated time of my waiting would be 13 minutes. Write the formula which is used by the machine to estimate the waiting time given the number of waiting customers and operating cashiers.

Figure 1: Post office ticket

For the purpose of this paper we focus only on the second task. Our data consists mainly of the students' work done on paper, but we also considered their verbal interactions. Students' work was analysed according to the mathematical actions and operations performed; no predetermined categories were formed and the data led us to the categorisation.

RESULTS

Once they were given the task, the students started asking questions like:

> What shall we do in this task?
>
> What shall we calculate in this task?

After a while two students – sitting in the same desk but working separately – raised their hands. Their work is shown in Figure 2 and Figure 3.

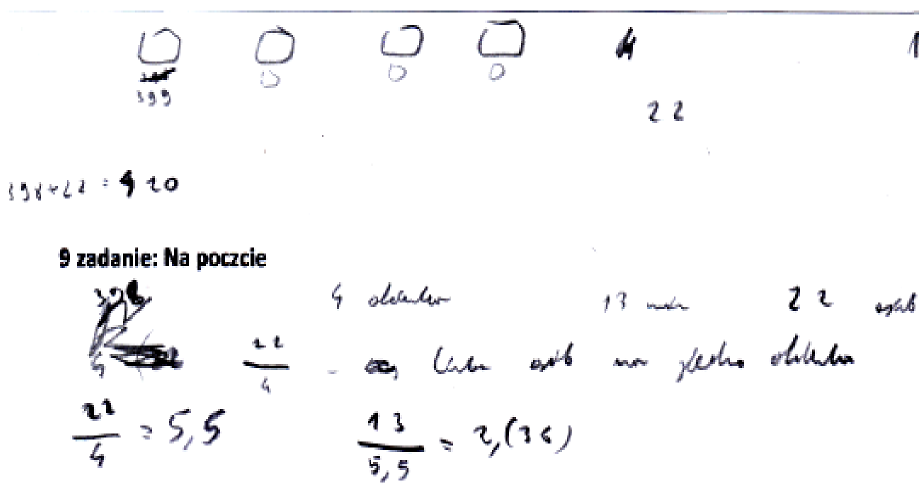

Figure 2: Initial work done by the first student

Figure 3: Initial work done by the second student

As it is shown in both figures none of these students created the general formula required. The first one calculated the average waiting time per client by initially using a visual representation of the cashiers' desks; it is noteworthy that he replaced number 398 with number 399 in the 'first' cashier. The second student explained how 13 can be obtained by using the numbers 22 and 4; his initial attempt was to divide 13 by 22, but it was deleted. The final result successfully represents the relationship between these numbers, but it cannot be characterised as a formula, because of the lack of variables.

The instructors decided to provide more time to the students. So, after few minutes two more students came up with solutions similar to the one shown in Figure 2. It was apparent that students had a hard time in solving the task and more questions were brought into the discussion:

> What if the clerk will be slower/faster than the others?
>
> What if the estimation of the machine is not correct?
>
> If the 398th customer is at one desk, what is the situation in the other three desks? Are there any customers there?

The instructors responded to these questions by implying some assumptions concerning the functionality of the ticketing machine. The assumptions that follow were not explicitly stated, but were rather expressed indirectly:

a. The ticketing machine is regulated by a person on the basis of some assumptions (probably based on prior observations and calculations of the average time needed per customer).

b. The ticketing machine can be re-regulated, but not on a daily basis.

c. Each customer needs the same time at the desk.

d. Each clerk works on a constant rate.

e. The ticketing machine coupon provides only an estimation of the expected waiting time.

f. The time spent at each post office desk is not the same with the time spent in another office's desk.

For example, the instructors gave the following responses to the students' questions:

> Do you think that it really matters if a clerk is slower or faster?
>
> What is according to you the meaning of the expression "estimated time"? If the estimated time is said to be 13, what if you finally wait for 14 or 12 minutes? Was the ticketing machine correct?
>
> Do you think that the machine was regulated after observing the time spent by only few customers?

The discussion described can be said to refer to the contextual constraints of the task. At the same time there were comments like:

We don't know how to solve this. It's too hard!

We'd prefer to have a task with integrals than this one!

All these realistic elements hinder us!

Although these comments refer to the task's context, at the same time they reveal a certain attitude of the students towards real-life problems (or to be more precise, the particular problem). Finally, there were also questions related to the solution process and particular decisions that should be made:

How to divide 22 people into 4 desks?

What if one customer will take more than 2.36 min?

In order to better categorise students' solution strategies we firstly distinguished four mathematical actions or operations that were performed by most students:

A1. Visual representations of the desks

A2. Use of proportions

A3. Mere calculations

A4. Writing down all customers' numbers

The first of these actions was usually combined with the other ones. This is clearly indicated in Table 1 which shows the various solution strategies followed by our students.

Category	Number of students	Solution strategy
S0	4	No solution (one student even wrote: "I don't understand")
S1	4	Merely writing down the data
S2	5	Writing down the data accompanied by a visual representation of the cashiers' desks [4 students: (A1, A4); 1 student: A1]
S3	1	Unidentified process (a female student who attempted to do an estimation of the waiting time) [A3]
S4	9	Combining data by performing various calculations without success; all but one accompanied by a visual representation of the cashiers' desks [3 students: (A1, A3, A4); 4 students: (A1, A3); 1 student: (A1, A2, A4); 1 student: A3]

S5	1	Merely writing down an incorrect formula with one variable representing the number of customers without any clue about the data used to reach it
S6	2	Writing down an incorrect formula with two variables representing the number of customers and the number of the operating cashiers [A2; A3]
S7	2	Calculating the average waiting time per customer accompanied by a visual representation of the cashiers' desks [(A1, A3); (A1, A2, A3)]
S8	5	Writing the formula $t = t_1 \cdot \frac{c}{d}$ using various representations for c, d and putting in the place of t_1 the average waiting time per customer: 2.3636… or 2.3 or 2.5 [A3; (A1, A3); (A1, A4); (A1, A3); (A2, A3)]

Table 1: The solution strategies of the students.

Then a student from the S8 category presented his solution to the class. This solution was good enough for the particular data given in the task ($t = 2.36 \cdot \frac{k}{o}$, k: number of the customers, o: number of the 'working desks'), but could not be generalised. So, the instructors asked the student to generalise, which he did. The formula that he came up with was the following: $t = 2.36 \cdot \frac{nr_{OT} - nr_{OB}}{o}$, nr_{OT}: number that we get from the machine, nr_{OB}: the number of the last served customer.

An interesting and unusual solution of a student belonging in the S8 category is shown in Figure 4:

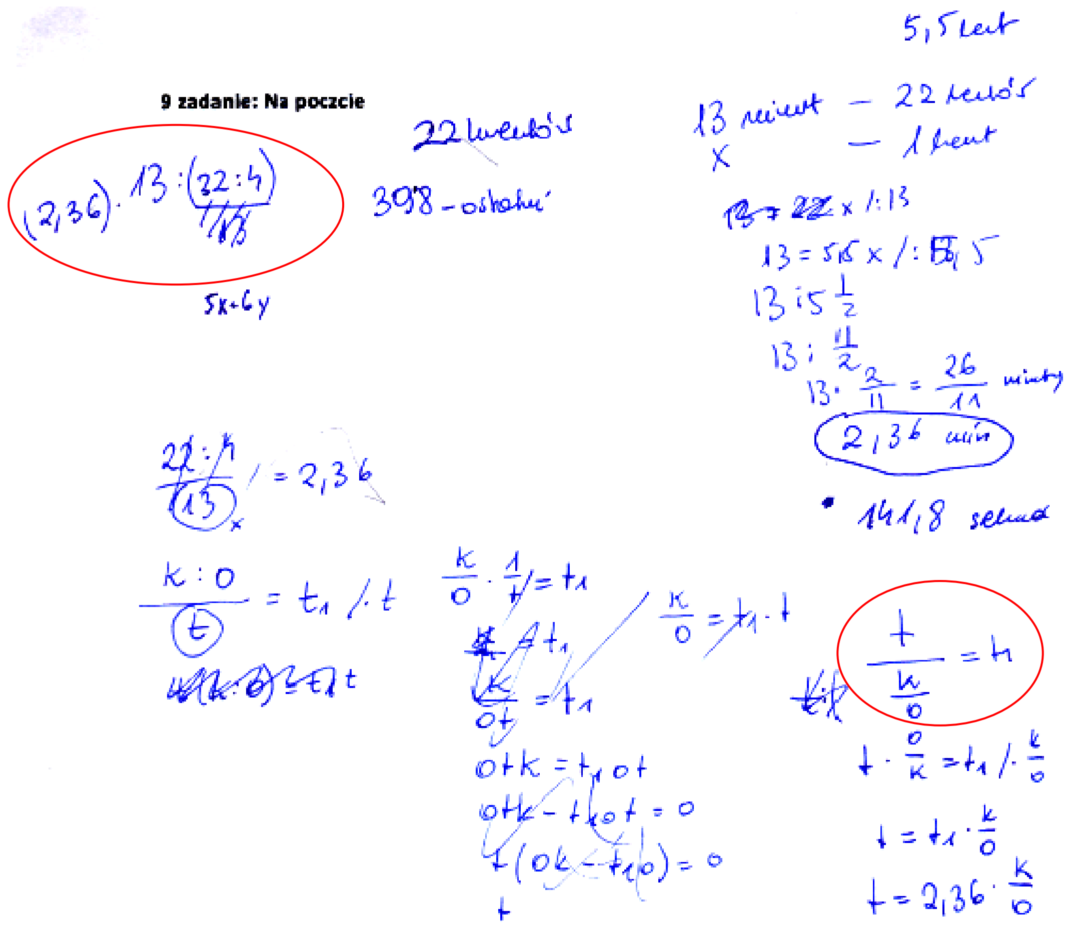

Figure 4: An unusual S8 solution.

The student used a proportion to obtain the average time (see the top right part) and then she put variables into the places of numbers (see the top left part). After many trials (which were deleted later), she transformed the formula to the 'accepted' form at the bottom right part by performing operations on the equations.

CONCLUSIONS

According to Krygowska (1980) the conception of mathematics created by a student passes through the prism of the tasks solved by him. In our case, the pre-service mathematics teachers demonstrated their existing conceptions of mathematics as a domain whose main characteristic is the reproduction of successful algorithms, in order to solve standard/typical tasks. So, once they were faced with a non-standard real-life problem, they felt unsafe or even threatened; that is the reason why they asked for the 'safety' of their known 'territories':

> We'd prefer to have a task with integrals than this one!

The challenge of the real-life problem was big, since they would have to overcome their existing conceptions and be involved in some real creative work. Creating a model takes more than the reproduction of known formulas and procedures. This fact made most –if not all – students feel uncomfortable and it is quite obvious that their lack of experience in such problems was the main reason for that feeling. However, they tried to come up with a solution, but most of the times they ended up with merely writing down the problem's data or trying to combine it in order to reach the 13 minutes of estimated waiting time.

Thus, from a mathematical point of view the basic conclusion is that none of the students managed to create the mathematical model that was necessary to solve the problem (i.e. a formula that could be used to calculate the estimated waiting time for a new customer knowing the actual numbers of desks working and of previous customers waiting). It seems that the realistic constrains of the problem were hindering factors, since students continuously asked for clarifications concerning e.g. the function of the ticketing machine. The concept of estimated time – thus the process of estimation – proved hard for the students to work with. However, one of the most pervasive actions performed by the students was the use of visual representations. The vast majority of students who solved the problem made a drawing of the four desks and the waiting customers, even if sometimes they did not use it. It seems that the realistic context of the task led them to deploy some informal strategies, at least initially. A characteristic example is shown on Figure 5, where the student put also herself in the drawing:

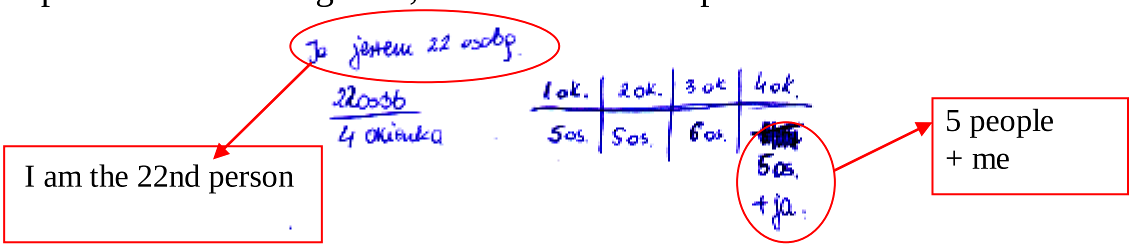

Figure 5: The drawing made by a female student in S8 category

From the above we may conclude that although the students tried to 'put themselves into the situation' they did not succeed and this is probably due to their lack of experience. This fact stresses the need for including more real-life problems in pre-service teachers' education, not only in order to make them capable of implementing the modelling approach, but also to improve their own mathematical literacy. Because mathematics is not only about effectively performing operations; it is also about being able to select and then implement the adequate operations to solve a problem, whether it is an abstract or a realistic one.

REFERENCES

Boaler, J.: 1993, Encouraging the transfer of 'school' mathematics to the 'Real world' through the integration of process and content, context and culture. Educational Studies in Mathematics, 25, pp. 341-373.

Chapman, O.: 2006, Classroom practices for context of mathematics word problems. Educational Studies in Mathematics, 62, pp. 211-230.

De Lange, J.: 1999, Framework for Classroom Assessment in Mathematics, National Center for Improving Student Learning and Achievement in Mathematics and Science, Freudenthal Institute, Madison, Wisconsin.

Freudenthal, H.: 1978, Weeding and Sowing. Preface of a Science of Mathematics Education, Reidel Publishing Company, Dordrecht.

Krygowska, Z.: 1980, Zarys dydaktyki matematyki, Part 3, WSiP, Warszawa.

National Council of Teachers of Mathematics: 2000, Principles and standards for school mathematics, Reston, VA: National Council of Teachers of Mathematics.

Podstawa programowa z komentarzami: 2008, Tom 6, Edukacja matematyczna i techniczna, w szkole podstawowej, gimnazjum i liceum: matematyka, zajęcia techniczne, zajęcia komputerowe, informatyka.

MATHEMATICS AND LANGUAGE INTEGRATED LEARNING – IDENTIFYING TEACHER COMPETENCES

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Teaching a content subject through a foreign language is an educational trend of growing importance, which proves to increase motivation and improve learners' attitude to the content subject. This report presents an ongoing research in the field of teacher training for mathematics and foreign language integrated teaching, aiming at shifting the focus towards mathematics and the specifics of mathematics teacher training.

INTRODUCTION

Content and Language Integrated Learning (CLIL) refers to situations where subjects, or parts of subjects, are taught through a foreign language with dual-focussed aims, namely the learning of content, and the simultaneous acquistion of a foreign language (Marsh, Langé, 1999). It is a trend of growing importance and extension in European education; among its other benefits, the research confirms the contribution of CLIL in complementing individual learning strategies, diversification of methods and forms of classroom practice, increasing learner motivation, and improving learner attitudes to both the language and the content subject (Hoffmannová, Novotná 2002; Coyle, 2006; Vollmer, 2008).

Positive influence of CLIL on learners' motivation and development being proved, and its implementation growing, a need for specific teacher training for CLIL is becoming more and more eminent. In 2006–2009, an extensive Socrates–Comenius 2.1 project was carried out, aiming at proposing a universal model for teacher education based on classroom observation and relevant research: CLIL across contexts: A scaffolding framework for teacher education (CaC).

Despite the fact that one of the key notions in CLIL is the balance between content subject and language, the vast majority of authors and project partners were specialists in language teaching and relevant research, which might weaken the relevance of the framework in terms of the content subject.

We are convinced that Mathematics, where not only the two languages, but also the symbolic and iconic 'language of mathematics' interact, has a specific position among other subjects, and requires a re-definition or specification of the proposed CaC framework.

This article presents an ongoing research which aims at verifying the legitimacy of the CaC framework, focusing on the specifics of mathematics education within the CLIL context, and at shifting the perspective of the framework towards mathematics. The overall target of the research is to propose a teacher-training framework adapted specifically for mathematics teacher-trainees and in-service teachers who adopt the CLIL challenge.

CLIL ACROSS CONTEXTS

According to its website, the CaC project aims at proposing a model for teacher education based on classroom observation and relevant research in selected areas of bilingual education and learning in general. It presents eight areas for the development of CLIL teacher competence, and claims that for each of the areas a careful integration of content and language is taken for granted.

1. In a student centred approach the first step consists in identifying *learner needs.*
2. Then the *planning* phase starts.
3. Aware of learner differences and of the special challenges that partial language skills cause, teachers will choose *multimodal approaches to learning.*
4. Planned and emergent forms of *interaction* are particularly important to stimulate cognitive and linguistic skills.
5. A focus on the specific aspects of *subject literacies* allows students to acquire the types of discourse required for an adequate appropriation of content.
6. Constant *evaluation or assessment* for learning gives support to all learners and encourages self-reflection.
7. Teachers' own *reflection* is crucial and in CLIL contexts it is significantly enhanced through the *cooperation* between subject and language specialists.
8. A last area, which could also be the first, encompasses the omnipresent but complex issues of *context and culture* that underlie all learning and teaching situations.

It formulates the key features divided into four categories of Knowledge, Values, Skills and Activities, aiming at offering a complex teacher-training framework culminating in the teacher's portfolio of training activities.

METHODOLOGY

Taking into account the results of the 'bottom-top' CaC project, this research opts for reverse approach, starting from the general notions, confronting the formulations with CLIL practitioners, and confirming them in a series of case studies, focusing from the very beginning specifically on mathematics as the content subject.

In the first phase, the framework is decomposed and analysed in detail to establish the areas of relevance for mathematics teachers, teacher trainees and teacher trainers.

Second, the initial hypotheses of focus and relevance are verified through a teacher-trainees and teacher reflection, and further confirmed in a series of scaled questionnaires piloted in diverse contexts among teachers, teacher-trainees and teacher trainers involved to a certain extent in CLIL approach; the results obtained from mathematics teachers, trainers and trainees are compared to responses from language and other subject teachers, trainers and trainees.

In the latter phase of the project, a video study will be carried out, focusing on the individual features of the remodelled framework for CLIL.

This report presents the first phase of the project, which is the analysis of CaC framework and the first verifications in a questionnaire survey.

ANALYZING THE CAC

We started the detailed qualitative analysis with the study of the working materials and studies in the two sections where the mathematics teacher-trainers were involved, that is, Multimodality and Subject literacy; subsequently, we followed with the materials presented in the remaining sections, focusing on the ratio of content/ language focus.

Despite the fact that according to the final report of the CaC project, "balance between content and language is taken for granted", both of the first two sections analysed dealt rather with the linguistic dimension of the activities, defining and working with language levels and skills. In spite of the original aim at Subject Literacy, this section seemingly took into account rather the general notion of literacy as "flexible and sustainable mastery of a repertoire of practices with texts of traditional and new communications technologies via spoken language, print and multimedia" (Luke, Freebody and Land, 2000), than a more subject-focused conception, as in e.g. the OECD Pisa survey (2006) definition of mathematical literacy as "the capacity of an individual to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen" – although the final text of CaC featured both of these definitions. Also, the majority of the rest of the activities offered focused above all on language.

This could support a hypothesis that the main concern is underlining the language dimension of subject literacy, and the main need for the CLIL teacher is to understand and promote this level.

In a teacher-training seminar for CLIL, at Charles University in Prague, 33 teacher-trainees, 3 CLIL practitioners and 2 teacher trainers were assigned to analyse the final framework and choose features that were most relevant for CLIL practice, in their opinion. Materials were studied individually; each of the participants wrote down a list of their choice of features and then reflected on them in a joint discussion.

Based on this discussion and the summaries of the individual work presented by the participants, a series of requirements for a CLIL teacher was summarized in 34 entries. The participants (both the teachers and the trainees) strongly argued that the knowledge, skills and activities sufficiently reflected the values and thus considered most of the items under the heading of 'values' neglectable, in the sense that they are implicitly present in the other features; further, they offered a significantly lower number of 'knowledge' items, arguing that knowledge needs to be demonstrated through skills and specific activities. The resulting 34 entries are thus distributed in the following manner: 7 items referring to 'knowledge', 14 referring to 'skills' and 13 referring to specific 'activities'. Further, this preliminary study proved the initial language-centric hypothesis to be wrong, for the items chosen by the participants were evenly distributed among diverse aspects. There were no significant differences between the output by future mathematics teachers and future language teachers, neither were there significant differences in the concepts underlined by CLIL practitioners.

The preliminary analysis leads us to the following hypotheses:

1. The teachers and CLIL practitioners will agree upon the requirements for a CLIL teacher collated during the first phase, that is, will consider all the items as relevant.
2. Being offered a more complex set of options, the teachers and CLIL practitioners will also focus on the practical implications rather than on the value systems.
3. There will be no significant difference between the results obtained from mathematics teachers and language teachers.
4. The individual features can be demonstrated not only in a CLIL-practitioner self-reflection, but also in their lessons and relevant materials.

These should be confirmed on two levels: first, running a questionnaire survey among teachers interested in CLIL and CLIL practitioners (hypotheses 1, 2 and 3), and second, confirming the execution of the individual items in video case studies, accompanied by interviews with the CLIL practitioners (hypotheses 3 and 4).

THE QUESTIONNAIRE SURVEY

A questionnaire was constructed along the items synthetised in the previous phase. The Likert scale was used for each item for the respondents to express the level of their identification with the statements; the succession of the items was randomized.

The questionnaire (in its Italian translation) was administered during a CLIL Teacher Training seminar (Metodologia CLIL e insegnamento delle discipline scientifiche) in Cascina, Italy. There were 50 participants, out of which 21 teachers of mathematics, 6 teachers of physics or other sciences and 23 teachers of foreign languages. Out of these teachers, 9 mathematics or science teachers and 4 language teachers are CLIL practitioners, with the average of 2 years of practice. The questionnaire (in its Czech translation) was also presented to 8 Czech CLIL practitioners in the field of mathematics. In the case of dual specialization of some of the teachers, they were asked to choose the dominant specialization based on how they perceive themselves and their answers were worked with accordingly. CLIL practitioners' data were considered first as part of the mathematics/language group and also considered as a separate group.

All the respondents underwent at least an introductory CLIL training and were familiar with the concepts and terminology used in the questionnaire. The questionnaires had been printed out, the respondents had sufficient time to fill them in individually.

The respondents were asked to express the level of their agreement with the following statements: "In my opinion, a CLIL (specifically: Mathematics through a foreign language) teacher needs to:

1. be able to design and use activities which integrate language and content with cultural awareness
2. be able to create an open and safe environment in the classroom
3. reflect on their relationship with the learners
4. be aware of and respect the cultural background of the learners
5. participate in international programs
6. be able to foster multiculturalism
7. understand the role of code-switching in bilingual context
8. use scaffolding
9. analyze and adapt the cognitive demands of the CLIL materials used
10. be able to analyze and reflect on their lesson plans
11. use cooperative teaching / learning methods

12. cooperate with another teacher to compensate for their language/content deficiencies
13. design the CLIL activities
14. design the CLIL lesson plans
15. design the didactical material for CLIL
16. know both didactics of language and of the content subject
17. know the specific CLIL didactics
18. be aware of the dual objectives of every CLIL lesson
19. be able to analyze linguistic demands of the materials used
20. know and teach coping strategies to deal with language barrier
21. have a clear evaluation system
22. discuss the objectives of the evaluation in the classroom
23. be able to integrate language and content on the level of planning, teaching and evaluating
24. be aware of the different levels of language in CLIL
25. foster learner autonomy
26. foster critical thinking in students
27. be aware of different models of classroom interaction
28. make recordings of their lessons and reflect on them
29. perceive an error as a learning opportunity rather than a failure
30. constantly work on improving their language skills
31. share experience with other CLIL practitioners
32. learn about CLIL research
33. present a variety of different representations
34. facilitate the ‘translation’ between individual modes of representation
35. use many non-verbal modes of representation

This survey will be followed up by further phases of the research, namely, administering this questionnaire also among Czech teachers to avoid possible influence of the national specifics (March 2010), and later continuing with verifying the further hypotheses (questionnaire survey to confirm hypothesis 2: February – April 2010, video study: May – October 2010).

DATA AND INTERPRETATION

As was expected, the teachers generally expressed a large ratio of agreement with the suggested items (average of 4.4 on a standard five-Likert scale), and overall balance among practitioners and non practitioners (4.3 and 4.4, respectively) was observed, as well as among mathematics and language teachers (4.4 and 4.3, respectively).

However, some individual items showed minor discrepancies. As for the differences between *CLIL practitioners* and *non-practitioners*, the main disaccord was observed in items 13, 14, 15, 30, 31 and 34 (see above). Practitioners attributed significantly larger importance to designing activity/materials/lesson plans (attributing 4.9 accord as opposed to non-practitioners' 4.3), and they attributed more importance to cooperation (4.8 compared to 4.1).

Mathematics teachers, as opposed to language teachers, attributed less importance to the cultural dimension (mathematics teachers: 3.6, language teachers 4.2), international programmes (3.6/4.1 resp.) and teaching coping strategies (3.7/4.4, resp.); on the other hand, they were more eager to analyse cognitive demands (4.3/3.7 resp.).

CONCLUSIONS

Within European education, the trend of Content and Language Integrated Learning is more and more promoted. Most research and training focuses on the language aspects and benefits of the CLIL approach. We try to shift this focus, and raise questions on what the specifics of CLIL teaching and training for mathematics are.

In the first phase of the research, we focused on teacher competences for CLIL, as viewed by teacher trainees and teachers themselves, parting from the general formulations of C*LIL across Contexts* framework.

Focusing on mathematics, the emphasis is on specific activities and skills. There is no significant difference between mathematics teachers' and language teachers' opinions on the relevance of key teacher competences for integrated learning of mathematics and foreign language – however, the sample studied so far had mostly rather theoretical background in CLIL and the results need to be verified.

We may sum up the implications for teacher-training for CLIL: despite the fact that the selection of requirements on a CLIL teacher contained a wide range of skills and competences that are dealt with during mathematics-teacher training, for effective CLIL training it is necessary to underline the importance of variety of modes of representation within mathematics teaching, and complement the training with both language-didacticts features (such as scaffolding) and specific

CLIL methodology, including focus on analyzing and designing CLIL-specific didactic materials.

REFERENCES

CLIL across Contexts: 2010, available online at <http://clil.uni.lu>. Retrieved 8. 2. 2010.

Coyle, D.: 2006, *Monograph 6 Developing CLIL: Towards a Theory of Practice.* Barcelona, Spain: *APAC* (Associació de Professors d'Anglès de Catalunya).

Hofmannová, M. and Novotná, J.: 2002, Implementation of Content and Language Integrated Learning (CLIL) in Mathematics Classroom. In Cockburn, A., Nardi, E. (Eds.) *PME 26 Proceedings, Vol. 1*. UEA.

Luke, A., Freebody, P., & Land, R.: 2000, *Literate Futures: Review of literacy education.* Brisbane, Qld: Education Queensland.

Marsh, D., and Langé, G.: 1999, *Implementing Content and Language Integrated Learning*. Jyväskylä, Finland: Continuing Education Centre, University of Jyväskylä.

OECD: 2006, *Assessing scientific, reading and mathematical literacy: a framework for PISA 2006*, OECD Publishing, Paris, pp. 71-73.

Raymond, E.: 2000, Cognitive Characteristics. *Learners with Mild Disabilities*. Needham Heights, MA: Allyn & Bacon, A Pearson Education Company, pp. 169-201.

Vollmer, H.: 2008, Constructing Tasks for Content and Language Integrated Assessment in J. Eckerth, and S. Siekmann, (Eds.), *Research on task-based language learning and teaching. Theoretical, methodological and pedagogical perspectives* , Lang, New York, pp. 227-290.

DIVERSITY OF FUNCTION SYMBOL INTERPRETATION AND ITS INFLUENCE ON THE UNDERSTANDING OF A PROBLEM BY PRE-SERVICE MATHEMATICS TEACHERS

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This paper shows different interpretations and ways of understanding of the notation of function and their influence on the understanding of a problem by pre-service mathematics teachers. Four kinds of sources of the difficulties are identified. All of them should be taken into account within the framework of mathematics teachers training as well as within the general mathematics teaching and learning process.

SUBJECT MATTER KNOWLEDGE FOR TEACHING

The quality of content-related vocational preparation by mathematics teachers and its influence on quality of teaching is undoubtedly one of the most important areas of research in the field of didactics of mathematics. The literature on the extent of teachers' knowledge is very extensive, and researchers exploring the subject have defined the phrase *subject matter knowledge* (SMK) in different ways. However, Shulman's work (1986) has had a significant impact on research defining teachers' knowledge and general vocational preparation. Shulman identified the following three categories of content knowledge structure: subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curriculum knowledge (CK). Different terms and definitions have since been used to analyse teachers' knowledge but most of them are rooted in Shulman's (1986) categories.

Accumulated research findings in past decades have led to the understanding that teachers' SMK for teaching is essential for effective teaching (e.g. Ball, 1991; Cooney & Wiegel, 2003). Moreover, Ma (1999) found that SMK structure should consist of profound understanding of fundamental mathematics (PUFM). Teachers' understanding of fundamental mathematics should be deep, wide, thorough and complete, and should include the awareness of the structure and the roots of mathematical concepts.

THEORETICAL FRAMEWORK

Taking into consideration the results of significant research in the field – Dyrszlag (1978); Even (1990); Freudenthal (1983); Semadeni (2002); Konior (1993); Krygowska (1986); Klakla (2003); Sierpinska (1992); Skemp (1971) – a general theoretical background was introduced (Sajka, 2006) that distinguishes the six general elements of teachers' SMK of a chosen mathematical concept, which structure the concept of function as follows:

SMK*f*-1: the essence of function,

SMK*f*-2: different representations and languages related to function,

SMK*f*-3: basic set of function's meanings,

SMK*f*-4: analysing function's meanings,

SMK*f*-5: the strength of function,

SMK*f*-6: mathematical culture.

The research analysis is carried out in the light of that theoretical background.

METHODOLOGY

The participants of the present study were prospective teachers. The research was carried out among 157 graduate students of mathematics in Pedagogical University of Krakow. These students had completed a three-year course preparing them to teach in primary and junior gymnasium schools, and had attended training courses for pre-service mathematics teachers preparing to teach in high schools.

The following problem served as the theoretical research tool:

PROBLEM

(a) Draw the graph of the function *h* knowing that $h(x) = |x|$ for $x \in [-1, 2)$ and that for any $x \in R$ the requirement $h(x) = h(x-3)$ is fulfilled.

(b) What properties does the function *h* have? Justify your answer.

The level of the problem does not exceeded the range of the mathematics curriculum for high school students in Poland, however, it is nonstandard in comparison with tasks which students usually solve and meet in the course of mathematics lessons.

RESULTS AND DISCUSSION

The written replies to the problem by the participants have been analyzed in the light of the six elements of the SMK of functions (SMK*f*) introduced above. The analysis presented in this paper is mainly focused on the diverse range of teachers' interpretations of function symbols, with the principal aim of highlighting the various difficulties provoked by the problem. Therefore, hypothetical sources of the interpretation of function symbols are identified in this paper. The study of the participants' answers has revealed many different (positive and negative) symptoms of the SMK-*f* elements. Due to limited space, only a selection is considered in the present paper.

HOW ARE THE SYMBOLS $h(x), h(x-3)$ INTERPRETED?

The following **five main kinds of interpretation** of the symbols $h(x), h(x-3)$ provided by pre-service teachers' answers to the problem can be identified:

I) value of the function h for arguments x and $x - 3$,

II) two different functions: $y_1 = h(x), y_2 = h(x-3)$,

III) algebraic expressions describing the formulae of two different functions,

IV) the names (labels) of two different functions,

V) others – idiosyncratic or no particular interpretation.

Each of the interpretations led to different conclusions regarding the understanding of the following requirements and their conjunction given in the problem:

(*) $h(x) = |x| \quad for\, x \in [-1,2)$ and

(**) for any $x \in R$ the requirement $h(x) = h(x-3)$ is fulfilled.

The various ways of understanding the conjunction resulted in different strategies for drawing a graph of the function h, which eventually led to different answers.

THE PROPER INTERPRETATION AND SOLUTION OF THE TASK

It should be noted that among all interpretations, only (I) is correct, and is thus the only one that can lead to the correct answer. This interpretation was reached by 83 of the 157 study participants (about 53%), while the correct graph (see Figure 1) was obtained by 68 teachers, of which only 63 gave the fully correct answer identifying the function as periodic (which constitutes only about 40% of the study participants).

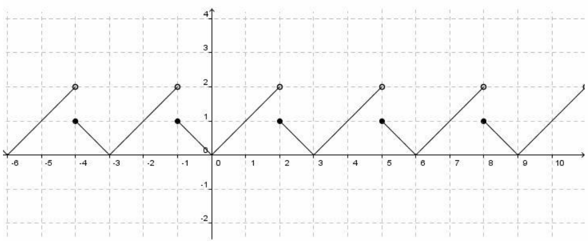

Figure 1: Correct graph of the function.

Pursuant to the correct answer to the problem we can formulate a strongly positive diagnosis of the SMK of functions in its authors, revealing the following elements of SMK-f :

▪ (SMKf-1) Understanding of the concept of a function and its domain.

▪ (SMKf-2) Understanding of the notation of function and ability to use and interpret it flexibly; ability to apply the functions' language; ability to use formulae of functions; ability to draw the graph of a function correctly.

▪ (SMKf-3) Knowledge of the graph of the function $y = |x|$.

▪ (SMK*f*-4) Knowledge of the definition of periodic function and its understanding, including understanding of the requirement $h(x) = h(x-3)$ fulfilled for any real numbers *x* as equivalent to the function *h* being periodic; ability to sketch the graph of a function given by requirements; demonstration of a preferred approach to analyzing functions; ability to use the proper method in a given context (in a global sense and not only interval-wise or point-wise).

▪ (SMK*f*-6) Ability to read the mathematics with understanding; knowledge of logic and set theory and ability to use it (understanding of the general quantifier and conjunction of the defining requirements); ability to work on nonstandard problems, aptitude for logical and flexible thinking, ability to self-control and self-observe mental activity.

The correct answer to the problem gives us a rich, important and positive insight of its authors' SMK-*f*.

OTHER ANSWERS FOR THE INTERPRETATION (I)

The proper interpretation of the symbols $h(x), h(x-3)$ is not sufficient to give the correct answer to the problem. It is an interpretation of the requirement (**), i.e. equality of the values of the function $h(x) = h(x-3)$ fulfilled for any real number *x*, that turned out to be crucial for solving the problem. Diagram 1 shows the diversity of interpretations of the requirement, and conclusions drawn on this basis, in interpretation (I). Numbers in brackets refer to how many answers followed the respective interpretation or conclusion.

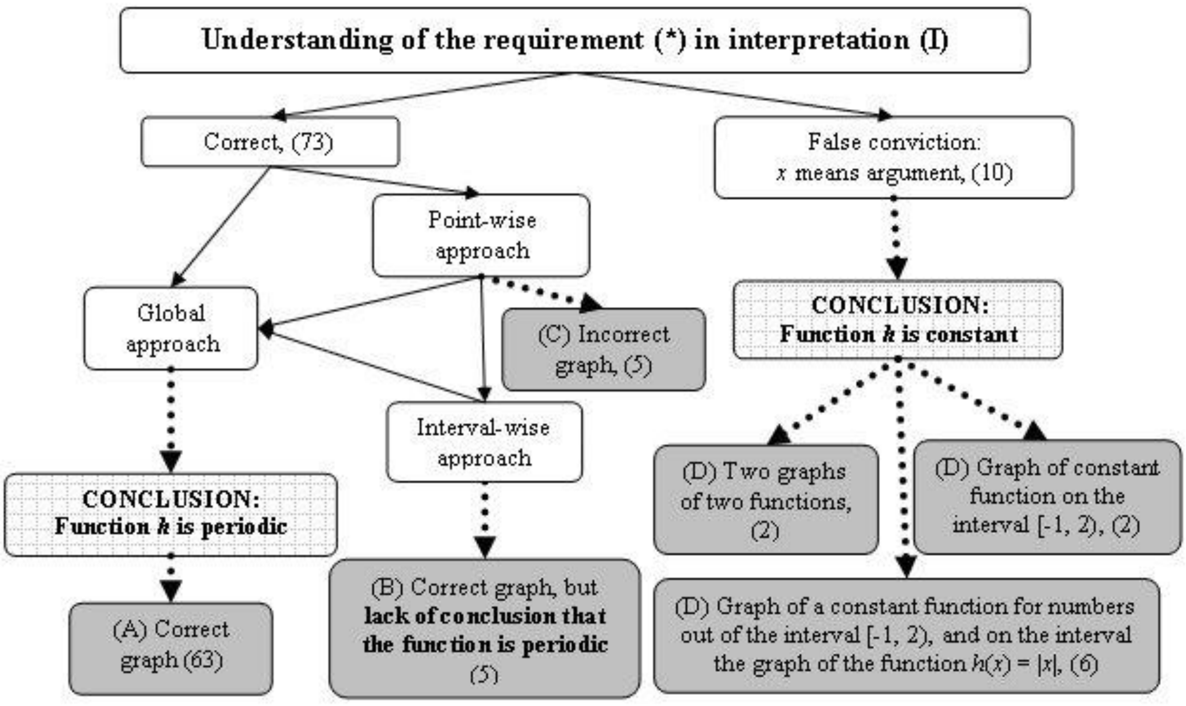

Diagram 1: Different conclusions and answers to the problem in interpretation (I)

Sources and analysis of answers B&C in interpretation (I)

It was somewhat surprising that five teachers, despite having drawn the correct graph as a result of finding the particular values of the function, did not identify the function *h* as periodic. When drawing the graph they paid attention to other properties of the function (e.g. zero points, local monotonicity) as was demonstrated by the correct graph construction and descriptions of the properties. In spite of obtaining such a suggestive graph, the concept of a periodic function was largely absent in the answers. This demonstrates an important lack of knowledge of periodic functions and their definition, and thus an inability to identify the shapes of graphs of periodic functions (SMK*f*-3,4). In this case, the reason for this lack of knowledge is almost certainly the **very limited context**, probably restricted to trigonometric functions, in which periodic functions had occurred in the course of their mathematics teaching. While this difficulty would be normal for high school students, it has a negative undertone in the context of examining the knowledge of mathematics teachers who had been studying mathematics for four years at university. Knowledge gaps in the area of SMK*f*-2 can be identified as an inability to interpret the requirement (**) as being equivalent to the requirement that function *h* is periodic. It should be emphasized that, despite the lack of knowledge connected with periodic functions, that this category of answers reveals nothing but positive symptoms of the elements of SMK of functions.

Meanwhile, five people from this group, while drawing the function graph, used only the point-wise approach – i.e. as a result of finding particular values of the function, they made some mistakes either in counting or in connecting the points of the graph. In this case we observe difficulties described in the above diagnosis, but moreover the lack of ability to take an interval-wise and global approach to functions (SMK*f*-4), which is an important negative symptom of teachers' SMK-*f* because of its likelihood of causing serious mistakes. Further, the point-wise procedure of drawing the graph yielded a graph of some continuous function for two participants who experienced an epistemological obstacle concerning the understanding of the notion of function. That epistemological obstacle is rooted in the need for the continuity of function and is well known from the history of the notion of function. Functions have appeared in history as graphs which can be drawn using a free, continuous motion of the hand. This obstacle is also well known and often experienced by students in school practice (e.g. Vinner, 1983).

REASONS FOR UNDERSTANDING REQUIREMENT (**) AS DEFINING A CONSTANT FUNCTION

Requirement (**) was interpreted by 10 study participants as defining a constant function. Two kinds of reasons for this interpretation can be distinguished.

The first reason is rooted in identifying x as an argument of a function. Dialogues with chosen study participants revealed that the phrase: "for any $x \in R$ " was in this context interpreted by some people as "for any real argument of a function" and then, without thorough analysis of the requirement, they added: "values of the function are equal".

It is worth mentioning that during the individual observation of chosen teachers undertaking other problems, the same misinterpretation was obtained in similar contexts, as described in the case study by Sajka (2007). For example, instead of searching for a function fulfilling the requirement $f(x - 1) = f(x - 3)$ for any real number x, they were looking for the function fulfilling another requirement: $f(x) = f(x + 1) = f(x + 3)$. It was x and not $x + 1$ or $x + 3$ that was the 'true' argument of the function f. This misinterpretation was unrecognized and moreover, it was not realized by the participants and demonstrates a particular type of *false conviction* for the notation of function (the term *false conviction* was described by Pawlik (2004) in the context of geometrical transformations).

It would be worthwhile to include the interpretation of **"identifying *x* with an argument of a function" as an additional *unconscious scheme of thought*** constituting another epistemological obstacle in understanding of the notion of function (see Sierpinska, 1992).

The second, less probable reason for the misinterpretation can be rooted in mistakes made whilst carrying out the procedure of finding particular values of the function. As a result of the procedure, an incorrect generalization could be made and the conclusion could be drawn that all the values of the function must be equal.

This interpretation of the requirement $h(x) = h(x - 3)$ excluded the possibility of sensible consideration of the requirement (*) that should have led to the conjunction and occurred in only 2 participants. These participants sketched two graphs of two different functions. One was $h(x) = |x|$ for $x \in [-1,2)$ and the second was a constant function for any real numbers. The following comment was accompanied by the graphs: "I am not sure which of these answers is correct – the answer 1) or 2)". Having experienced such a doubt the participants may not have shown self-control nor thought out the meaning of the requirement (**), and neither did they notice the mistake in their interpretation (SMK*f*-6).

Other teachers gave the answer D (see Diagram 1), not accepting the contradiction aimed at creating a graph of the function. Among them, six people took into account the requirement (*) and the fact that the domain of the function is R, and subsequently joined these requirements to create a graph of another function, given by the formula: $y = \begin{cases} |x| & for\, x \in [-1,2) \\ const.\, for\, x \in R \setminus [-1,2) \end{cases}$, which is given, for example, in Figure 2. This answer, selected here as an example, demonstrates the significant negative symptoms of the SMK*f*-6.

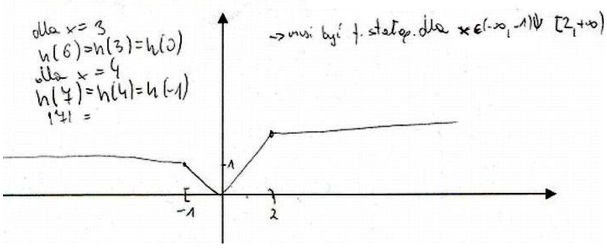

Figure 2. Requirement (**) means that *h* must be a constant function out of the interval [-1, 2).

Two people concluded that the function *h* must be constant on the interval [-1, 2). That answer stems from the combination of a conclusion about the constant function and a partial implementation of requirement (*). The interval [-1, 2) is taken from (*) as the domain of the function *h*, and the information that $h(x) = |x|$ is skipped. This kind of choice suggests that the interpretation of function *h* being constant dominated the decision-making of these participants.

INTERPRETATION (II): REASONING SOURCES AND TYPES OF ANSWERS TO THE PROBLEM

Sources of interpretation (II)

Interpretation (II) consisted of understanding the symbols $h(x)$, $h(x-3)$ as the labels of two different functions. One of them is the function "$h(x)$" given by the formula: $y_1 = |x|$, and the second is "$h(x-3)$": $y_2 = |x\text{-}3|$.

The first source of interpretation (II) is rooted mainly in **ambiguity of the notation of function**. Without reference to the problem we cannot say that this interpretation is incorrect. Gray and Tall (1994) point out that the notation of function, for example, $f(x) = 2x+3$ tells us two things at the same time – how to calculate the value of the function for particular arguments and how the whole concept of function for any given argument is encapsulated. The notation of function is ambiguous in yet another way. Sierpinska (1992) emphasizes that flexibility in understanding is necessary because, for example, $f(x)$ represents both the name of a function and the value of the function *f* for argument *x*. Interpretation depends on the context, which can be confusing. Despite the fact that many efforts have been made to specify the symbols, e.g. assuming that the name of function we mark with single letter *f*, and its value for some argument *x* with the symbol $f(x)$, we still meet the duality problem in teaching. In some didactic situations teachers intentionally make the opposite decision of the mentioned assumptions. For example in Poland teachers prefer to talk about "the function sin*x*" than about "the function sin". Their explanation is based on their teaching practice. They claim that it protects students against losing *x* in other contexts. Therefore interpretation still depends on the context and aims, leading to further confusion for pre-service teachers.

Moreover, high school students meet the symbols of the type $h(x-3)$ mainly in the context of geometrical transformations of the graphs of functions. Usually, in the context of the function for which the graph was created after translation by the vector [3, 0]. We come then to the second source of that interpretation: the very **restricted contexts in which the symbols occur in teaching**.

The interpretation (II) is incorrect in the context of the problem. Further, this kind of interpretation reveals a lack of understanding of the mathematical text (SMK*f* -6).

Different kinds of answers in interpretation (II)

Diagram 2 shows the diversity of interpretations of the requirement (**) and the respective conclusions drawn on this basis in interpretation (II). It is impossible to give the exact numbers of these answers in some categories of (II)B because some responses do not contain any description of the properties of the function *h*, so we cannot find out what the graphs of the function *h* were showing as valid in the opinion of their authors.

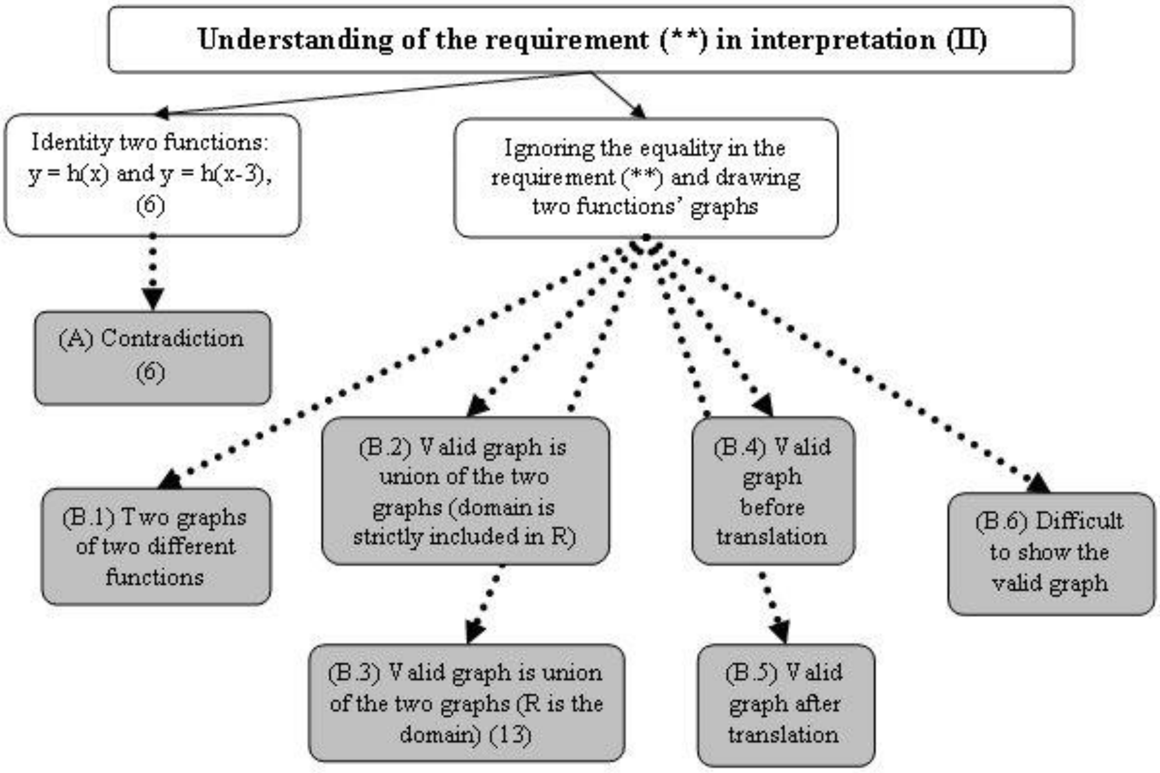

Diagram 2: Different conclusions and answers to the problem in interpretation (II)

Re (II) A. Identification of two functions

Six participants from this group interpreted $h(x)$, $h(x-3)$ as two functions, understanding that requirement (**) was the identity of them. The reasoning is very natural: if $h(x) = |x|$ means the first function and $h(x-3) = |x-3|$ is the second, and if the requirement $h(x) = h(x-3)$ has to be fulfilled, then from interpretation (II) it is deduced that the two functions must be identical.

Pre-service teachers in this case drew the conclusion that this is a contradiction. Some specified: "For $x \in [-1,2)$ the equality $h(x) = h(x - 3)$ is not fulfilled because $|x| \neq |x-3|$ this is the truth only for $x = 1½$. I do not understand this problem because I do not know whether I should draw $h(x) = |x|$ or $h(x - 3)$." Other participants paid attention to different domains of these functions because "for the function $h(x) = |x|$ it is the interval [-1, 2), and for the function $h(x - 3)$ it is the set of real numbers."

Re (II) B. Ignoring the equality in requirement (**) and drawing the graphs of functions $y_1 = h(x)$ and $y_2 = h(x - 3)$

In many answers for the problem we observed that the participants sketched two graphs of two different functions: $y_1 = |x|$, $y_2 = |x-3|$. The graph of the function $y_2 = h(x - 3)$ was constructed by translating (correctly or incorrectly) the graph of the given function $y_1 = h(x)$. Having drawn the graphs, which reveal positive symptoms of pre-service teachers' basic repertoire of function examples (SMK*f*-3), they then gave different answers. This interpretation of requirement (*) reveals the algorithmic approach to solving problems without profound connection with conceptual knowledge (lack of SMK*f*-4, SMK*f*-1). Ignoring the sign of equality in (**), they showed significant deficiencies connected with SMK*f*-6, i.e. an inability to understand mathematical text, lack of self-control, lack of checking an obtained answer with the formulation of a problem, and many others.

Due to space reasons the paper refers only to the groups B.2 and B.3, which are distinguished in the Diagram 3. They resulted in sketching the graph of function h as union of the two graphs $y_1 = h(x)$ and $y_2 = h(x - 3)$ defined either on R or on an interval strictly included in R.

Participants giving answer B.2 (see Figure 3) described in a detailed way the properties of the function h defined on the interval [-1, 5), which reveals that information on the function domain as well as the general quantifier was ignored. The problem formulation was not understood properly. Study participants giving this type of answer most probably reacted mechanically to the symbol $h(x - 3)$ by carrying out some translation and gave the answer without thorough analysis of the problem formulation (lack of SMK*f*-6). What is important is the fact that some of them identified the obtained function as periodic. This disclosed knowledge of the symbolic requirement of the definition of a periodic function but probably without its full understanding. Some participants identified the function as periodic but at the same time interpreted the symbol $h(x - 3)$ as a formula of the function, from which the graph was obtained as a result of translation. This interpretation was dominant and blocked the proper interpretation in the context of the problem of understanding of the symbol.

Other participants (B.3) took into account in their reasoning that the domain of the function *h* is R. An example of that kind of answer is shown in Figure 4.

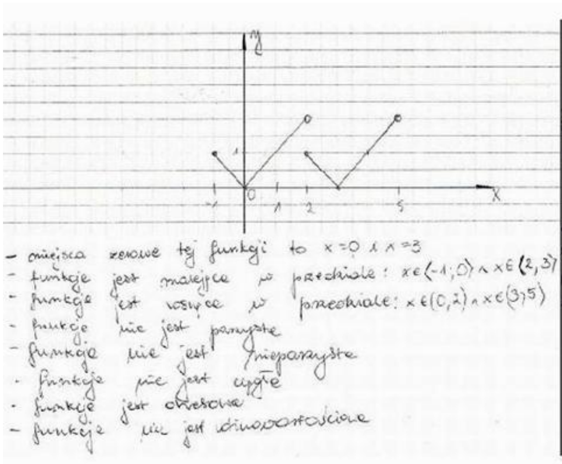

Figure 3: Graph of the function *h* defined on the interval [-1, 5).

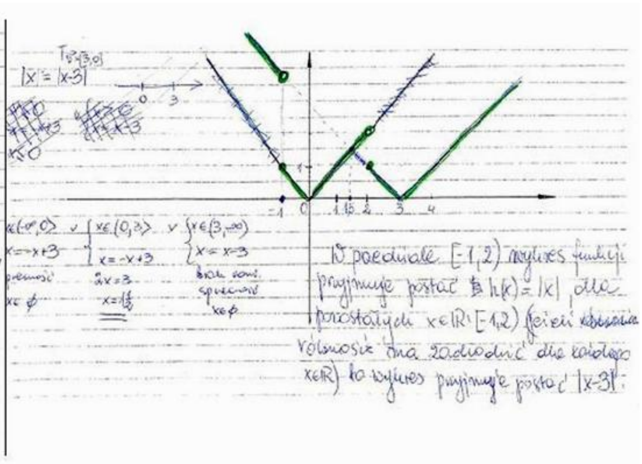

Figure 4: Graph of the function *h* as union of two graphs (R is the domain).

Interpretation of the symbols $h(x)$, $h(x - 3)$ as different functions could be strengthened in this case in another way. Participants could treat the symbol $h(x)$ alternately with the symbol y (these symbols are very often treated in this way in school practice). As a result they could obtain the formula of another function: $y = |x|$ for $x \in [-1, 2)$, and $y = | x - 3 |$ for any $x \in R$ (in supposition – for the rest of arguments). Such a function can be described by the formula:

$$y = \begin{cases} |x| & for\ x \in [-1,2) \\ |x-3| & for\ x \in R \setminus [-1,2) \end{cases}.$$

INTERPRETATION (III): REASONING SOURCES AND TYPES OF ANSWERS TO THE PROBLEM

Sources of interpretation (III)

Interpretation (III) is that the symbol $h(x)$ is understood only as an algebraic expression used in the formula of a function. This interpretation was revealed by 12 pre-service teachers.

The reasoning almost certainly was as follows: if $h(x) = |x|$, then $h(x - 3) = |x - 3|$, and since the requirement $h(x) = h(x - 3)$ has to be fulfilled, then the equation $|x| = |x–3|$ with unknown x has to be solved, graphically or algebraically.

The source of that interpretation is mainly **ambiguity of the notation of function,** intensified additionally by **linguistic inaccuracy** or even negligence and **mental leaps** present during mathematics classes or lectures. Mathematics teachers and university mathematicians often say for example: "let us take the function |*x*|” or “the function is given by the formula |*x*|”, which certainty does not mislead any mathematicians, but it can provoke terminology chaos and lead to such a kind of troubles in the interpretation of functions’ symbolism.

The consecutive source of that interpretation is also and again **limited didactical context and a restricted set of examples** in which those kind of symbols occurred in teaching.

Sierpinska (1992) identified “thinking in terms of equations and unknowns to be extracted from them” (pp. 37-38) as the fourth epistemological obstacle in understanding the notion of function, classifying it as an “unconscious scheme of thought” (p. 37). Therefore, not only pre-service teachers struggle with the obstacle, it is natural in the process of shaping the notion of function. Research on high school students also confirms the existence of the obstacle in students’ reasoning (e.g. Sajka, 2003). However, it should certainly not have been of any problem for proper interpretation by teachers.

Different kinds of answers in interpretation (III)

Study participants revealing interpretation (III) tried to solve the equation to answer the problem. They obtained the solution $x = 1½$, which was commented on in three ways (see Diagram 3).

For example, five people gave the answer that the point (1½, 1½) is the valid graph of function *h* this kind of answer. They graphically interpreted the following system of equations: $\begin{cases} y = |x| \\ y = |x-3| \end{cases}$. Figure 5 presents this kind of answer, showing additional inability to sketch the graph of function $y = |x – 3|$.

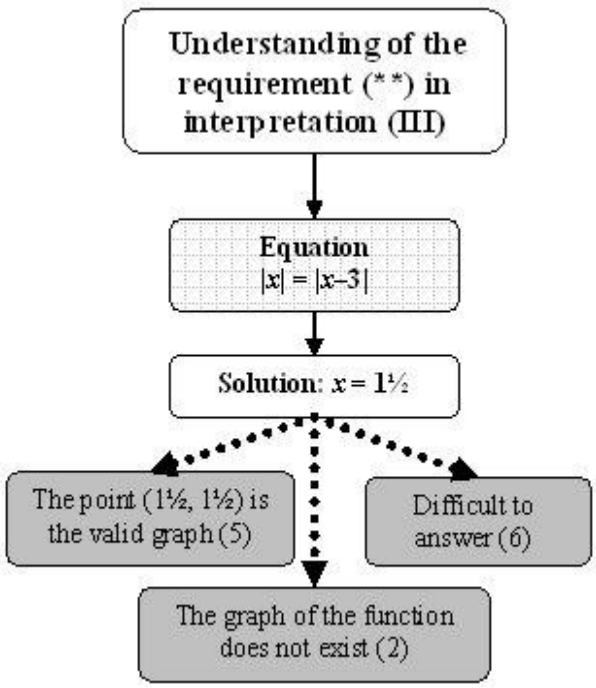

Diagram 3 (on the left). Different conclusions and answers to the problem in interpretation (III).

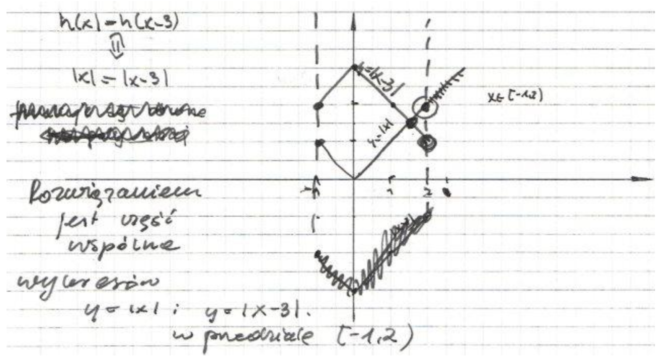

Figure 5: An answer interpreting requirement $h(x) = h(x – 3)$ as the equation solved geometrically.

One should pay attention to the fact that the teacher, having obtained one common point of the graphs, answered that the point was the graph of the function. Although the answer to the problem is incorrect, accepting the point as a graph of a function proves that arbitrariness of functions is also accepted. It is worth mentioning that this answer reveals positive symptoms of understanding of the notion of function and its graph (SMK*f*-1,2) among many negative symptoms of other elements of SMK*f*.

INTERPRETATION (IV): REASONING SOURCES AND TYPES OF ANSWERS TO THE PROBLEM

Sources of interpretation (IV)

In interpretation (IV), the symbols $h(x)$ and $h(x-3)$ are perceived only as names or labels of functions and do carry any content.

The main sources of that kind of understanding are similar to those distinguished for interpretations (II) and (III). Moreover, interpretation (IV) reveals an incorrect understanding of the symbols $h(x)$ and $h(x-3)$ as a whole, as they are understood only as labels or names of different functions. This interpretation could be also caused by idiosyncratic understanding of the symbols.

This result reveals a lack of understanding of the notation of functions and a lack of understanding of the concept of a variable, its role and relation with the symbols (SMK*f*-1,2). It is worth mentioning that the same interpretation was also observed in a high school student examining her understanding of the notion of function. Symbols $f(x)$, $f(a)$, $f(b)$ were perceived by her as labels of three different functions, because "they look different" (see Sajka, 2003).

Different kinds of answers in interpretation (IV)

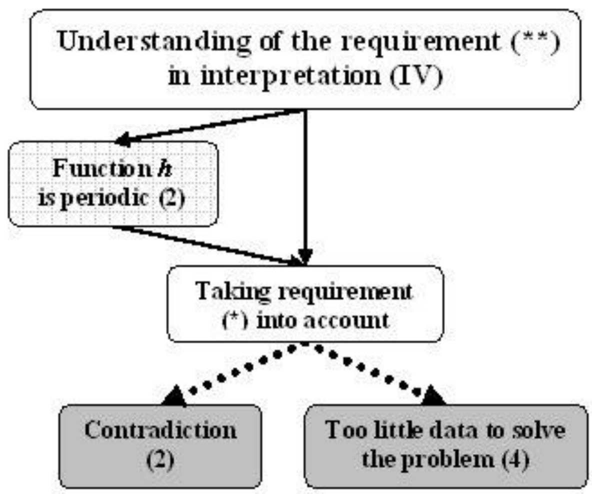

Diagram 4: Different conclusions and answers to the problem in interpretation (IV)

Four people answered that the function is given only for $x \in [-1, 2)$, because for x outside of the segment there was not enough information to find it. The requirement (**) for the participants probably did not carry any content. They claimed that the problem is badly formulated, they were not able to interpret it at all and answered that there was too little data to solve the problem. For these people symbols $h(x)$ and $h(x-3)$ were sensible only in the context of formulas of functions. Not having any algebraic expression, they could not deal with the problem.

This interpretation reveals a lack of SMK of functions and is very surprising when its authors are people who have been studying mathematics for four years.

Two people identified the requirement (**) as defining a periodic function, but when they tried to consider the conjecture in connection with the requirement (*) they came to a contradiction. Justifying the answer, one person wrote: “requirement $h(x) = h(x\text{-}3)$ cannot be fulfilled, because the function $h(x) = |x|$ is not a periodic function”.

The requirements (*) and (**) were considered separately and mechanically without thorough understanding. The answer was accompanied by the graph of functions $h(x) = |x|$ for $x \in [-1, 2)$. Probably the participants only mnemonically acquired the definition of a periodic function and recognized it in the requirement (**), then the symbols $h(x)$, $h(x - 3)$ were probably interpreted as the only name of a function – the same name: function h, alternately with the symbol y. The symbols were not understood as values of the function for respective arguments. In this case we observe lack of flexibility in interpretation of the function notation, which implies lack of its understanding (SMK*f*-2). Moreover it shows lack of SMK*f*-6, because of the inability to understand mathematical text, a lack of self-control, lack of understanding of the conjunction of requirements, and others. However, the answer that the function $h(x) = |x|$ is not periodic does reveal some positive symptoms of SMK*f*-3,4.

AN EXAMPLE ANSWER IN INTERPRETATION (V)

One example of idiosyncratic interpretations is provided below (see fig. 7), revealing a thorough lack of understanding of the notation of function. Symbols $h(x)$, $h(x - 3)$ were associated with zero points of a function. The person explained it as follows: “I understand $h(x) = h(x - 3)$ in the way that the zero points of the function are $x = 0$ and $x = 3$”. Further she wrote: “Only for the interval [-1, 2) I do know what the function looks like. Outside this interval I know from the requirement $h(x) = h(x - 3)$ only that its value is 0 for $x = 3$”. Claiming that outside the interval [-1, 2) there was not enough data to draw the graph of the function, she at the same time identified only the interval as the domain of function h.

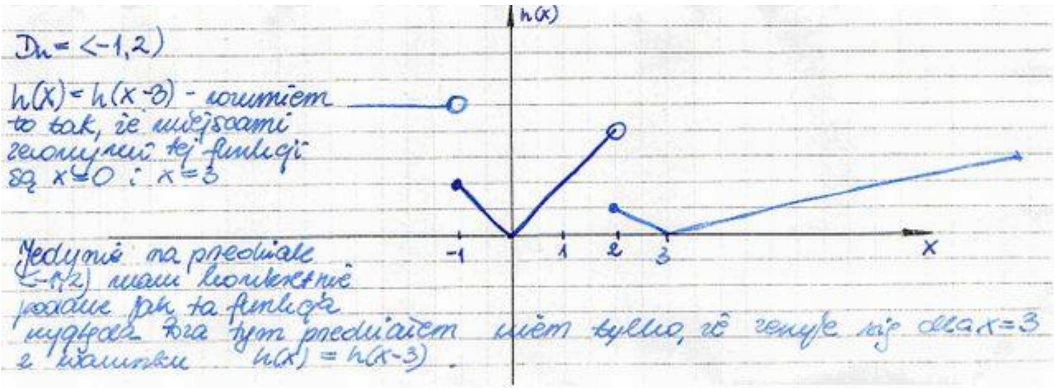

Figure 6: Idiosyncratic interpretation of the requirement (**)

It is worth mentioning that in fourteen answers the requirement (**) was not interpreted at all. Thirteen people only provided the graph of function *h* for $x \in [-1, 2)$ and one person did not draw the graph of function *h* even for $x \in [-1, 2)$.

FINAL CONCLUSIONS

Research carried out on a set of pre-service teachers revealed an unexpected diversity of ways of interpreting the notation of function, as well as of interpreting the requirements given by the problem. The participants' answers to the problem disclosed important information about their SMK of functions. The answers revealed many difficulties regarding their understanding of the notion of function and its symbols. Some of these difficulties are of basic nature and caused misunderstanding of functional symbolism by people who are legitimately able to teach mathematics, and in particular to teach functions.

From these results we can draw several basic and general conclusions:

1. Only 40% of pre-service teachers taking part in the study were able to solve the problem correctly (the percentage of correct answers was definitely worse for extramural students). The other answers to the problem revealed a thorough lack of understanding and interpretation of the notation of function as well as many other deficiencies concerning SMK of functions. The state undoubtedly needs to undertake remedial action that can be organized within the framework of the classes connected with didactics of mathematics.
2. The following four basic sources of improper interpretation of the function's symbols can be distinguished:
    a. The intrinsic ambiguities of the mathematical notation in connection with the lack of flexibility in its interpretation;
    b. The restricted contexts in which some symbols occur in teaching, and a limited choice of mathematical tasks at schools;
    c. Pre-service teachers' *false convictions* or *unconscious schemes of thoughts* (e.g. identifying *x* only as an argument of a function);
    d. Idiosyncratic interpretation of the symbols by the participants.
3. All the reasons for interpretations of the notation of function distinguished above are crucial, and their identification should help in mathematics teaching.
4. Universities preparing pre-service mathematics teachers should pay attention to the analysis of the intrinsic ambiguities of mathematical notation. Moreover, pre-service teachers should be stimulated to undertake nonstandard problems which require both flexibility in

interpreting the mathematical symbols and conceptual thinking. Pre-service teachers have to manage with untypical problems.

5. Similar difficulties and reasoning appear in the understanding the notion of function by high school students.
6. In junior gymnasium and high schools, attention should be paid to showing the diversity and ambiguity of the notation of function. Mathematically gifted students should be encouraged to undertake nonstandard problems of functions. Moreover, there is a need to both select examples for teaching very carefully and with wide range, and to accustom students to conceptual and critical thinking. Interpretations II, III, IV, V described here lack the interpretation of symbol $h(x-3)$ and are probably the result of solving typical and undifferentiated tasks algorithmically and without connections with conceptual thinking.
7. There is a need to be careful regarding the precision of statements in mathematical classes at schools and universities. Using mental shortcuts or metonymical figures of speech (*pars pro toto*) for example: "let us take the function x^2" or "the function is given by the formula $|x|$" can provoke terminological chaos which may result in difficulties with understanding the concepts.
8. The presented problem can serve as multifunctional and effective diagnostic and didactical tool that can be used for high school students as well as for students of mathematics studies.

REFERENCES

Ball, D. L.: 1991, Research on teaching mathematics: Making subject-matter knowledge part of the equation, in: J. Brophy (Ed.), *Advances in Research on Teaching,* vol. II, JAI Press, London, pp. 1-49.

Even, R.: 1990, Subject matter knowledge for teaching and the case of functions, *Educational Studies in Mathematics*, **21**(6), pp. 521-554.

Cooney, T. J., Wiegel, H. G.: 2003, Examining the mathematics in mathematics teacher education, in: A. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick and F.K.S. Leung (Eds.), *Second International Handbook of Mathematics Education*, Kluwer, Dordrecht, The Netherlands, pp. 795-828.

Dyrszlag, Z.: 1978, *O poziomach i kontroli rozumienia pojęć matematycznych w procesie dydaktycznym*, *Zeszyty Naukowe WSP w Opolu, seria B: Studia i Monografie nr 65*, Opole.

Freudenthal, H.: 1983, *Didactical Phenomenology of Mathematical Structures*, D. Reidel Publishing Company, Dordrecht.

Gray, E., Tall, D.: 1994, Duality, Ambiguity, and Flexibility: A "Proceptual" View of Simple Arithmetic, *Journal for Research in Mathematics Education*, **25**(2), pp. 116-140.

Klakla, M.: 2003, Dyscyplina i krytycyzm myślenia jako specyficzny rodzaj aktywności matematycznej, *Studia Matematyczne Akademii Świętokrzyskiej*, **10**, pp. 89-106.

Konior, J.: 1993, Samoobserwacja aktywności myślowej studentów na zajęciach z przedmiotów kierunkowych jako element przygotowania do zawodu nauczyciela matematyki (raport na podstawie doświadczeń z pracy ze studentami kolegium nauczycielskiego), *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki*, **15**, pp. 37-55.

Krygowska, Z.: 1986, Elementy aktywności matematycznej, które powinny odgrywać znaczącą rolę w matematyce dla wszystkich, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki*, **6**, pp. 25-41.

Ma, L.:1999, *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States*, Lawrence Erlbaum Associates, Inc.

Pawlik, B.: 2004, *Fałszywe przekonania dotyczące przekształceń geometrycznych na płaszczyźnie w rozumowaniach studentów matematyki*, Unpublished doctoral thesis, Akademia Pedagogiczna, Kraków.

Sajka, M.: 2003, A secondary school student's understanding of the concept of function – a case study, *Educational Studies in Mathematics*, **53**, pp. 229-254.

Sajka, M.: 2006, Refleksje na temat określania wiedzy przedmiotowej nauczycieli matematyki, *Annales Academiae Pedagogicae Cracoviensis, Studia ad Didacticam Mathematicae Pertinentia I*, **36**, pp. 297-319.

Sajka, M.: 2007, On diagnostic possibilities provided by a certain nonstandard problem, in: E. A. Avgerinos, A. Gagatsis (Eds.), *Current Trends in Mathematics Education, 5th MEDCONF2007*, Rhodes, pp. 645-654.

Semadeni, Z.: 2002, Trojaka natura matematyki: idee głębokie, formy powierzchniowe, modele formalne, *Roczniki Polskiego Towarzystwa Matematycznego, Seria V, Dydaktyka Matematyki*, **24**, 41-92.

Skemp, R.: 1971, *The Psychology of Learning Mathematics*, Penguin Books, Harmondsworth, England.

Sierpińska, A.: 1992, On understanding the notion of function, in: E. Dubinsky, G. Harel (Eds.), *The Concept of Function. Aspects of Epistemology and Pedagogy, MAA Notes*, vol. 25, pp. 25-58.

Shulman, L. S.:1986, Those who understand: Knowledge growth in teaching, *Educational Researcher*, **15**(2), pp. 4-14.

Vinner, S.: 1983, Concept definition, concept image and the notion of function, *International Journal of Mathematical Education in Science and Technology*, **14**(3), 293-305.

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