Supporting Independent Thinking Through Mathematical Education

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Introduction

Mathematical education is recognized internationally as central to society. The teaching of mathematics begins at a young age, because basic mathematical concepts are at the heart of both personal and social development. There is no doubt that issues connected with mathematics education need to be at the center of attention of political leaders, educationalists the general community, and, of course, parents and teachers.

Teaching mathematics is important both for supporting the development of the child and for solving critical problems in a global society. Number sense, numerical literacy, spatial abilities and other fundamental skills and concepts of mathematics, are critical to social and personal growth and understanding. By means of it, mathematical knowledge gained with the help of teachers can favour the pupils with logical thinking and reasoning, which aids the conduct of dialogue and negotiation. In this way, mathematics supports ethical behavior, especially understanding human rights and obligations. The ability to organize and use data is valuable in almost all spheres of individual and social life. The search for solutions promotes creativity, flexibility, and adaptation to new situations, and succes in finding (multiple) solutions supports the development of self-esteem.

The quality of teaching and learning mathematics depends on many elements, affected and determined by each other. While many factors, such as social structures of inequity and diversity, are seemingly beyond the remit of the individual teacher, he or she remains a central element, responsible for what is going on during lessons in their classroom. Teachers must understand their role, both within the classroom, and as a part of larger social and political structures. They must blend their interactions with pupils and their understanding of mathematical content objectives with their own ethical and moral commitments in order to effect change in society.

Teacher-training in mathematics goes far beyond subject-specific and pedagogical content. It connects with many other realms: psychology (creation of concepts, emotions, motivations, interactions, ...), linguistics (communications, language in learning and teaching mathematics, symbol creation and its understanding, ...), socio-cultural theory (ethno-mathematics, equity and diversity, ...), history and epistemology (developments of mathematical concepts, historical obstacles in understanding mathematical concepts, ...), technology (application of technology in mathematics, using computers in teaching mathematics, ...), and so on.

Few people enter the field of teaching with a comprehension of the complexity that such work entails. The education of the teachers of the future, and the ongoing professional development of practicing teachers must help them to negotiate these complexities and to reconcile the potential conflicts between the realities of teaching and their own personal moral and professional commitments.

Open questions in mathematical education

Part 1

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Sense and representation in elementary mathematics

Two concepts are central to contemporary mathematics education theory and practice: the support of sense-making by pupils, and support for developing facility with representations. This presentation problematizes and recasts both of these concepts by framing learners as artists – creators and producers – within a curriculum that usually wants them to "consume and use" instead. A common assumption is that mathematics curriculum is content that represents and interprets. Applying work of Sontag that argued against representational art, we can generate new forms of learning activities where artists evoke parody, abstraction, decoration, and non-art in ways that make mathematics vibrant and relevant to several of our conference themes.

The Introduction of our book calls our attention to the ways that our efforts are tools for both the development of the child and for solving critical problems in a global society. Indeed, these two ways of thinking about our work are always interwoven, since individual children are always present and at the same time future members of our society. That is, our global society is nothing other than ourselves. Education in general, and mathematics education in particular, is central to the basic existence and aims of social life. I start today by reminding us that this introduction is, if anything, gentle in its call for principals, education officials, the general community, and, most importantly, parents and teachers, to consider how and why fundamental mathematical concepts are at the heart of both personal and social development. Toward this aim, I ask us to think about two concepts that undergird most of the intellectual work of teachers and curriculum workers in mathematics education, and to think critically about the ways in which they impact on our beliefs about what we should do and what might be changeable: the support of sense-making by pupils, and the overarching aim of facility with representations.

By 'sense-making', I am referring to the common assumption that our task as mathematics educators is to help young people to make sense of mathematics. We receive mathematics as a reasonable and logical world within which one should be made to feel comfortable and secure. We often agree that mathematics is the one place where we can be certain about what we know and whether or not we are correct. My comments today have to do with the power of these assumptions to enable specific kinds of educational experiences while also perhaps failing to allow pupils to fully appreciate the wonders and powers of mathematical modes of inquiry and understanding.

With 'representations', I ask us to consider also the power of our typical pedagogies, which tend to lead students from the concrete to the abstract, and also to move students away from specific instances of mathematics in the world toward general representations of these instances. We might think of the representations (of the ideas) as the actual material and content of the mathematics itself. I mean here simple things like numerals to represent numbers of things; drawings of shapes to represent ideal geometric relationships; fractions to represent parts of wholes, proportions, and ratios; equations to represent functional relationships, letters to represent variables that may take on different values, and so on. Other representations model mathematical concepts and relationships, such as base-ten blocks for arithmetic operations, drawings of rectangles or circles for fractions and ratios, or graphs which visually represent algebraic equations. In my experience, much of mathematics education aims to help students to develop artistic virtuosity with mathematical representations for communicating their ideas. However, if we take this artistic virtuosity seriously, then critics of artistic practice sometimes suggest that representation is not always the aim of art, and in fact, representation often violates art itself. What could young children, as mathematical artists, do, then, if they would not primarily be practicing forms of mathematical representation? I will return to this question, because it is connected most directly with our Introduction's call to consider the broad, social contexts even as we focus on the individual mathematicians in our kindergartens, primary school classrooms, and on the adolescent mathematicians with whom we work day-to-day.

Stop making sense

Brent Davis (2008) recently wrote, "In the desire to pull learners along a smooth path of concept development, we've planed off the bumpy parts that were once the precise locations of meaning and elaboration." We have, he says, "created obstacles in the effort to avoid them." Davis describes "huh" moments, when it is possible to enter authentic mathematical conversations. For example, we might ask someone to describe what we mean when we write '2/3 = 14/21'. Responses vary from pictures of objects to vectors on a number-line, but all share a conceptual quality of relative change so that increasing one thing leads to a proportional increase in another thing or group of things. However, when we ask the same person to describe what is happening in the expression '-1/1 = 1/-1', we usually get a kind of "huh", which communicates a moment where the mathematics has lost its sense, but which also potentially begins an important (mathematical) conversation. In my own work on what Davis calls the "huh" moments – when mathematics stops making sense to us, and we grope for models apparently not available (Appelbaum, 2008) – I, too, have noted the potential for the non-sense-making characteristics of mathematics to generate different kinds of teacher-student relationships, and most significantly, different kinds of relations with mathematics within associated critical mathematical action (Appelbaum, 2003). Mathematics curriculum materials too often hide the messiness of mathematics where sense dissolves into paradox and perplexity, but more importantly they construct a false fantasy of coherence and consistency. As most professional mathematicians understand, mathematics at its core is grounded in indefinable terms (set? point?), inconsistencies (Gödel's proof? Cantor's continuum hypothesis?) and incoherence (the limit paradox in calculus?). At a more basic level, multiplying fractions ends up making things smaller even though 'multiplying' conjures images of 'increasing' to many people; two cylinders made out of the same piece of paper (one rolled length-wise, on width-wise) have the same surface area but hold different volumes; we're taught to add multiple columns of numbers from right to left with re-grouping, when it is so much easier to think left to right starting with the bigger numbers. In some cases, it is impossible, speaking epistemologically, for mathematics as a discipline to 'make sense'; in others, it might be more valuable pedagogically to treat mathematics as if it does not make sense. To do so would celebrate the position of the pupil, for whom much of the mathematics is new and possibly confusing anyway.

Yet, so much of contemporary mathematics education practice is devoted to helping students make sense of mathematics! What if, instead, we stopped trying to make sense, and instead worked together with students to study the ways in which mathematics does and does not make sense? Instead of school experiences full of memorization and drill on techniques, we would imagine classroom scenarios full of conversation about the implications of one interpretation over another, or of explorations that compare and contrast models and metaphors for the wisdom they provide.

Elizabeth de Freitas (2008) describes our desire to make mathematics fit a false sense of certainty as ‘mathematical agency interfering with an abstract realm’. She encourages teachers to intentionally ‘trouble’ the authority of the discipline, in order to belie the ‘reasonableness’ of mathematics. In this way, we and our pupils can better understand how mathematics is sometimes used in social contexts like policy documents and arguments, business transactions, and philosophical debates, to obscure reason rather than to support it. Stephen Brown called this kind of pedagogy, “balance[ing] a commitment to truth as expressed within a body of knowledge or emerging knowledge, with an attitude of concern for how that knowledge sheds light in an idiosyncratic way on the emergence of a self” (Brown, 1973, p. 214)

So, you may wonder, what does this mean about curriculum materials and textbooks? “Obviously somebody somewhere with a lot of authority has actually sat down and written this Numeracy Strategy,” says one teacher with whom Tony Brown (2008) spoke. “it’s not like they don’t know what they are talking about.” Tony Brown blames the administrative performances that have shaped mathematics for masking what Brent Davis calls the huh moments, and what de Freitas describes as the self-denial that accompanies “rule and rhythm”. Teaching in this “senseless world of mathematical practice” need not abandon science and the rational. It merely shifts teaching away from method and technique toward what Nathalie Sinclair calls the “craft” of the practitioner, as she evokes the metaphor of teaching as midwifery from Plato’s *Theaetetus* (see also Appelbaum 2000). As midwives, teachers assist in the birth of knowledge; students experience not only the pain and unpredictability of the creative process, but also the responsibility for the life of this knowledge once it leaves ‘the womb’. One must care for and nurture one’s knowledge, whether it acts rationally or not. Can we be confident that the ways we have raised our knowledge will prepare it for when it is let loose upon the world? Will our knowledge be embodied with its own self-awareness and ethical stance?

A dubious theory

A demand that everything make sense, and that this sense be so simple that it is virtually instantaneous if at all possible, dominates the way we work with mathematics in school. We design a curriculum that introduces a tiny bit of new thought once per week or even less often, because we worry that a pupil will feel lost or confused, and not be able to move on to the next tiny new step that follows. I imagine instead a curriculum where children beg for new challenges, and where these children delight in the confusion that promises new worlds of thinking and acting, of children we do not just ‘get by’ in mathematics class, but who love mathematics as part of their sense of self and their engagement with their world. The French philosopher and social theorist Michel de Certeau (1984) blamed the social sciences for reducing people to passive receivers of knowledge. And indeed, educational research and practice has been dominated by the social sciences for the past century, so we have been living the successes and failures of these approaches to education and now need to look at them critically as we reassess our work in mathematics. de Certeau suggested that the social sciences cannot conceive of people as actors who invent new worlds and new forms of meaning, because they study the traditions, language, symbols, art and articles of exchange that make up a culture, but lack a formal means by which to examine the ways in which people re-appropriate them in everyday situations. This is a dangerous omission, he maintained, because it is in the activity of re-use that we would be able to understand the abundance of opportunities for ordinary people to subvert the rituals and representations that institutions seek to impose upon them. With no clear understanding of such activity, the social sciences are bound to create little more than a picture of people who are non-artists (meaning non-creators and non-producers), passive and heavily subjected to ‘receiving’ culture. Social sciences thus typically understand people as passive receivers or “consumers” rather than as makers or inventors of culture, ideas, and social possibilities. Indeed, I believe this is exactly the situation we find ourselves in as we seek ways to make mathematics meaningful for young people and for young people to take advantage of mathematical skills and ideas as they participate in their local and global communities.

This kind of misinterpretation is critical to our „consumer culture," in which people are assigned to market niches and sold products, concepts, modes of life, and predictable desires. In curriculum as in advertising, such social science persists, so that we see students as consumers of knowledge whose desires are shaped by the curriculum via the teacher, teachers as consumers of pedagogical training programs, and so on. de Certeau employs the word „user" for consumers; he expands the concept of „consumption" to encompass “procedures of consumption" and then builds on this notion to invent his idea of „tactics of consumption". School curriculum tries to sell students on the value of mathematical knowledge; we sometimes call this ‘motivation’. New curriculum materials are published and sold as part of a global economic system that demands new and improved products in a cycle of perpetual obsolescence and innovation.

What would it mean for youth who are learning “stuff that many adults already know” to be artists – creators and producers – when we seem to want them to “consumer and use” instead? The critical notion turns out to be how we make sense of the “art.” Susan Sontag (1966) wrote about what she named a “dubious theory” that art contains content, an approach that she claimed violates art itself. When we take art as containing content, we are led to assume that art represents and interprets stuff, and that these acts of representation and interpretation are the essence of art itself. Likewise in school curriculum, we often imagine the curriculum as content, and move quickly to the assumption that this curriculum represents and interprets. This makes art and curriculum into articles of use, for arrangement into a mental scheme of categories. What else could art or curriculum do? Well, Sontag suggests several things: To avoid interpretation, art may become parody. Or it may become abstract. Or it may become (‘merely’) decorative. Or it may become non-art.” (Sontag 1966: 10)

New worlds of mathematics education

Parody, abstraction, decoration, and/or non-art are three types of tactics for art and curriculum. I think, too, that they can be used to stop making sense of mathematics *for* young children, and instead, in the words of our introduction, they can help us ‘not only to pose questions, but also to look for solutions’. Common work of our book is focused around four main issues: Mathematics as a school subject; Teacher-training; Teachers’ work; and Learning Mathematics. I conclude with a brief outline for applying the de-Certeauian-Sontagian ‘tactics’ in each of these four realms. With my suggestions, I am encouraging each of us to consider how school mathematics could be experienced as something *other than* a representation of content, or something other than an abstract representation of ideas. This does not mean that I want us to abandon representations or the representation of ideas, but that our methods of teaching would not stress this as our primary purpose.

Mathematics as a school subject: Normally, we emphasize two kinds of experiences in school mathematics, and through these we create an implicit story about what mathematics ‘is’. We either develop ideas out of concrete experiences, or we model real-life events with mathematical language. An example of the first would be to work with numerals to represent numbers of objects, in order to stress for young children the differences between cardinality and ordinality, or to develop arithmetical algorithms for adding, subtracting, multiplying or dividing numbers.

We might work with base-ten blocks, number lines, collections of objects, drawings of objects, and so on. An example of the second might be to create a story problem out of a real-life situation, such as to ask how many tables we need for a party if each table can seat six people, and we expect fifteen people to attend our party; or, to ask, given eighty meters of fencing material, what shape we should use to have the most area for our enclosed playground. Now, suppose we wanted to transform our pedagogy so that the work in our classroom were one of parody, abstraction, decoration, or non-art, rather than representational art. Children might parody routine questions by acting out seemingly absurd situations where the reckoning leads to ludicrous results, or they might ask and answer questions that shed humorous or critical light on typical uses of the mathematics. For example, 4-year-olds who have counted the number of steps from their classroom to the door of the building, in ones, threes, and fives, might then count the number of drops of water to fill a bucket in ones, threes and fives, even though it seems to make no sense to do so … this would only be a parody, though, if the children themselves suggested it as a silly thing to do that they wish to do nevertheless. Similarly, ten-year-olds might design alternative arrangements of their classroom that make use of unusual shaped desks, such as asymmetric trapezoids, circles, etc. Mathematics might be abstract if children did more comparing and contrasting of questions, methods, and types of mathematical situations, rather than focusing on the particular questions or on practicing specific methods. For example, 8-year-olds might first organize a collections of mathematics problems first into three categories, and then the same problems into four new categories, rather than solving the problems themselves; the classification of the problems into types would constitute the mathematical work, rather than the solution of the mathematical problems. Mathematics as 'decoration' might be accomplished through a classroom project where students experiment with different representations of a mathematical idea for communicating with various audiences. After working with ratio and proportion, for example, a class of 11-year-olds might form small groups, one of which creates a puppet show for younger children, one of which composes a book of poems for older children, and another of which prepares a presentation for adults at their neighborhood senior citizens community center, all on the same subject of applying ratio and proportion to understand the ways that a recent election unfolded. In this sense of considering the appropriate way to describe ratio and proportion for a particular audience, the mathematics is more of a decorative from of rhetoric than a collection of skills or concepts; the important concepts have more to do with democratic participation in elections than with the mathematics per se. Mathematics as non-art uses artistic work that is not considered 'art' as its model – we could ask, when is creative mathematical work not mathematics? One answer is, when it is something else other than mathematics per se – for example when it is an argument for social action presented at a meeting; when it is an example used to demonstrate a philosophical point; when it is a recreational past-time; etc. In other words, mathematics as non-art would be mathematics not done for its own sake; mathematics as non-art would be mathematics for the purposes of philosophy, anthropology, literature, poetry, archaeology, history, science, religion, and so on. As long as the activity has purpose other than the mathematics itself.

Teacher's Work: So, mathematics as a school subject can and should take on the character of parody, abstraction, decoration or non-art. If this is to occur, there are important implications for the teacher's work. For one thing, the teacher would not be providing clear presentation or explanations of mathematical concepts or procedural skill. Instead, we can learn from current work at the University of Amsterdam on the types of teacher help that support mathematical level-raising (Dekker & Elshout-Mohr 2004, 2005; Pijls 2007; Pijls, Dekker & Van Hout-Wolters 2007). In their studies, they have found that teacher help directed at mathematical content –

explanations and demonstrations, is rarely more valuable than teacher help directed at collaborative learning and groups processes, and in fact sometimes teacher help focused only on the group processes leads to more significant conceptual level-raising. In other words, the nature of useful teacher work involves making it possible for pupils to participate as creators and consumers of mathematical art that is not representational, and which does not aim at simplifying the path to sense-making. Instead, teacher work essentially makes it possible for pupils to experience together the authentic practices of sense and non-sense through events such as parody, abstraction, decoration, and non-art. In the group processes that are supported by teacher-help, the mathematics is secondary to the group process in the teacher's mind. The teacher is helping the pupils to use mathematics in order to accomplish the group process, rather than using organization of the group in order to accomplish representation or sense-making of mathematics. This seems backward, given that our job is to teacher mathematics! It is almost counterintuitive! But, indeed, when we think this way, perhaps, there is a new "sense" to be made of mathematics teaching.

Teacher-training: What, then, are the implications for teacher training? I believe the key things to think about are the differences between preparation for representation and sense-making, which has been the primary direction of mathematics education for the last century, and preparation for the support of artistic practice. We have inherited a technology of teaching methods steeped in cognitive psychology which direct the teacher's attention to individual cognitive development. This has certainly been useful, and will continue to be useful to all of us in our work. However, I am suggesting today that we foreground another orientation to our work, which Eliot Eisner (1991) called criticism and connoisseurship. Ordinarily, the teacher training that I am most familiar with involves extensive practice in the application of methods, diagnosis and remediation. Eisner's ideas suggest instead that teachers-in-training spend more time immersed in experiences that are not directly focused on the representation of teaching and learning, or on making sense of what pupils can and cannot do, but instead on criticism and connoisseurship in the context of schools.

Connoisseurship is the art of appreciation. It can be displayed in any realm in which the character, import, or value of objects, situations, and performances are distributed and variable, including educational practice. The word connoisseurship comes from the Latin *cognoscere*, to know. It involves the ability to see, not merely to look. To do this we have to develop the ability to name and appreciate the different dimensions of situations and experiences, and the way they relate one to another. We have to be able to draw upon, and make use of, a wide array of information. We also have to be able to place our experiences and understandings in a wider context, and connect them with our values and commitments. Connoisseurship is something that needs to be worked at – but it is not a technical exercise. The bringing together of the different elements into a whole involves artistry.

It may sound like I am advocating an elitist notion here, but I do not mean this; indeed, I want us to think mainly about the depth of knowledge that all people have in their everyday lives as connoisseurs of those things they taste deeply, and to imagine how we could help young people to take those ways of learning and thinking and making meaning, and see that they are relevant in school (Gustavson & Appelbaum 2005; Appelbaum 2007). Now, what Eisner makes clear in his writing, is that educators need to be *more than* connoisseurs. They need to become critics. Our models for ourselves need to be those reviewers of films, albums, music videos, and video-games that we read and listen to for pleasure, and that help us to know which artistic works we will enjoy and find valuable, even those critics with whom we love to disagree. Criticism is the art of disclosure, of revealing more than the obvious; as John Dewey pointed out in his book *Art as Experience*, criticism has as its aim the re-education of perception. The task of the critic is to help us to see.

> Thus … connoisseurship provides criticism with its subject matter. Connoisseurship is private, but criticism is public. Connoisseurs simply need to appreciate what they encounter. Critics, however, must render these qualities vivid by the artful use of critical disclosure. (Eisner 1985: 92–93)

I see direct connections with our introduction, which describes teachers as crucial to the evolution of mathematics education:

> The quality of teaching and learning mathematics depends on many elements, affected and determined by each other. While many factors, such as social structures of inequity and diversity, are seemingly beyond the purview of the individual teacher, the teacher in the classroom remains a central element, responsible for what is going on during lessons in the immediate context. Teachers must understand their role, both within the classroom, and as a part of larger social and political structures. They must blend their interactions with pupils and their understanding of mathematical content objectives with their own ethical and moral commitments as a change agent in society.

So, in my own work in teacher education, I strive to work as a connoisseur and critic, in order to support the artistry of my students who wish to be teachers. And I welcome conversations with you over coffee, tea, a beer, wine, and so on, to share such stories. But back to the main theme of this presentation: what sort of mathematics learning is enabled by a teacher with extensive background in connoisseurship and criticism?

Learning Mathematics: Well, we could simply say, pupils of mathematics would be succeeding when they are demonstrating abilities to use mathematics in order to achieve a parody, to communicate an abstraction, as a decorative element in other contexts, or as non-mathematics across the curriculum. But more directly, I offer the following: Young people learning mathematics are artists whose tactics of parody, abstraction, decoration and non-art are forms of consumption that re-appropriate school mathematics as a tool of connoisseurship, and thus, of remaking their world anew in each act of mathematics they commit. Here is a very active and vibrant way to imagine mathematics learning: as artistry, as doing, as alive, and as transforming the world in every tiny moment. Mathematics in this “sense” is a collection of tactics for doing this. And learning mathematics is an apprenticeship in the artistry of social participation. Their mathematical actions, as art, are not aimed at a purpose that involves curricular illustration, but instead become the embodiment of critical pedagogy that engages both the mathematical artist and the artistic mathematician in critical citizenship (Springgay and Freedman 2007). I end, then, with a challenge to you: are you ready to allow the children in your life and work to become connoisseurs of mathematics? That is, to become more than knowers, to become critics of mathematics? Mathematics as criticism is an art of disclosure, of revealing more than what is obvious on the surface. Here is the magic recipe for achieving this: think more about coordinating activities where the children are active artists of mathematics than about how to represent or explain clearly a mathematical concept. I know, it goes against so much of our desires to make things easier for the child. In the end, though, if we stick to this plan, we will be lucky enough to spend time with current and future crafters of beautiful worlds, young people who use mathematics to shed insight on contemporary society, to ironically critique common sense practices, as tools for appreciating and interpreting culture and societal problems, as the medium of decoration and entertainment, and simply as so valuable as to be part of all things not usually named ‘mathematics’.

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Factors hampering independent mathematical thinking

Examples of adverse educational effects of the false stereotypical public image of mathematics are discussed. While the role of the context of a task given to children in primary education is indispensable, it is argued that the role of understanding the purpose of the task by the children is also essential. Some ways of developing independent mathematical thinking through critical thinking and of stimulating the development without overdemanding are outlined.

Introduction

The subject of the conference is not only the question of supporting independent *mathematical* thinking, which by itself is very important, but it also provides a much wider perspective: mathematics education as a means to support *independent thinking in general*.

I will concentrate on obstacles which hamper the implementation of the idea of independent mathematical thinking and on some ways of overcoming them.

False stereotypical public image of mathematics and its consequences

Some thirty years ago I read a feature article on the front page of a leading Polish weekly *Polityka*, written by a well-known author. The title “I defend the students’ right to do their own thinking” would fit the topic of our conference. However, I was astonished by his introductory remark: “Of course, this applies only to the humanities; in mathematics the student can only copy the teacher”.

My immediate reaction was that the opposite statement would be more fitting: *at the beginning of schooling, mathematics is the only subject in which children can check basic facts on their own*. Generally, students have to accept most of school knowledge on the grounds that the teacher has said so or that “the book says so”. This includes, say, the fact that the river flowing through Cracow is the same river as the one that flows through Warsaw; few students are able to verify this fact. Similarly, the information that the ruler of Poland accepted Christianity in 966 must come from external sources.

In contrast, the child, gently encouraged by the teacher, can use fingers to show that 5 + 3 is 8 and 7 + 5 is 12. Knowledge of such number facts does not have to depend on information given by the teacher or stated in a textbook. Similarly, children can find out that, say, 4 times 7 is 28 by repeated addition. Basing themselves on such experience and encouraged by the teacher, they can gradually memorize the multiplication table with the feeling that they have validated each of its items and can check them again if needed. Admittedly, when the subject matter becomes more advanced, such verification is only partially possible. Nevertheless, children's conviction that they can verify mathematical facts seems to be a crucial factor influencing their attitude towards the subject.

Misunderstanding of what is mathematics, illustrated by the above quotation from an essayist, is not uncommon. It is part of thinking of mathematics in terms of "musts", "dos" and "don'ts". Such an image (particularly conspicuous in case of school algebra) is often part of the hidden curriculum.

Teaching methods often do not distinguish between *laws* (such as, say, that of commutativity of multiplication) and *human conventions* concerning the system of notation. The fact that 4 times 7 equals 7 times 4 is true regardless of the symbols used to express it and will be true forever; the rule that in an expression such as $5 + 7 \cdot 4$ multiplication has priority over addition is not an objective law but a historically developed way of reading and interpreting such symbols, similar to the convention that the cross + denotes addition while the double bar = denotes equality.

Of course I do not mean that such subtleties (objective law versus notational convention) should be explained to children, but we should be aware of the potential obstacles involved. Let me illustrate the problem with some unexpected difficulties arising in apparently clear situations.

My first example concerns an excellent elementary teacher who felt (long ago) uneasy when she read a rule in a textbook: "If there is only addition and subtraction of numbers and there are no parentheses, the operations are performed from left to right" and compared it with the change of the order of operations: $24 + 37 + 6 = 24 + 6 + 37 = ...$, which appeared later in the same textbook (intended to show how commutativity may help someone with mental computations). She found the change of order incompatible with the previous rule. I then realized how serious was the problem. The source of her difficulty was the conception of the arithmetic symbol system as based on "must" and "must not" rules. The formulation "you *must* perform operations from left to right" is not what we mean. However, the formulation, say, "you *can* perform operations from left to right" is ambiguous. A theoretic explanation based on the concept of the value of an arithmetic expression would be too sophisticated for primary school students.

Second example. A preservice elementary teacher was to get credits for a math course. She got a series of arithmetic tasks together with the "responses" of a hypothetical child; she was to find out which computations were correct and to pinpoint errors, if any. One of the tasks was to judge the computation:

$8 \cdot 6 = (5 + 3) \cdot 6 = 5 \cdot 6 + 3 \cdot 6 = 30 + 18 = 48.$

The examinee has written that passage from $(5 + 3) \cdot 6$ to $5 \cdot 6 + 3 \cdot 6$ is incorrect, because the rule states that one first performs the operation in parentheses, that is, one must add 5 + 3.

Both examples vividly show what is lost in the "must-mustn't" image of mathematics. In either case the evoked rule ignores the *purpose* of a given step of computation (analogously, rules of the traffic code concern forbidden actions and not the destination of the journey). Such a misleading image of mathematics, resulting from years of inadequate rule-based school instruction, is very hard to change.

In Poland, in the 1970's, primary education was influenced by Zofia Cydzik, a leading educator and author of textbooks. She distinguished between a *way* and a *method* of performing an operation on numbers. A way depends on specific numbers and may be modified whereas a method is general and has to apply to all numbers. She argued that one should teach methods, not ways, because mastering a method guarantees success for all computations of a given type; a "way" may be easier for certain numbers but may fail in new cases. She also believed that teaching general methods fosters a true image of mathematics.

I mention this because her point of view, although erroneous, is not unusual and still affects education adversely. The scheme of presenting methods in Cydzik's textbook was to give a model example and then a series of tasks in the common format: "Imitate this example". One of the series devised for "going over ten" started with a *didactic template*

(*) $8 + 5 = 8 + 2 + 3 = 10 + 3 = 13$,

which was followed by analogous additions. The child was to copy the pattern using different numbers. The last item was $9 + 10 = 9 + 1 + 9 = 10 + 9 = 19$. Many parents-mathematicians found this irritating as a spectacular example of a nonsensical manner of adding 9 + 10.

Such teaching schemes resulted from, and contributed to a false stereotypical image of mathematics. In turn, this distorted view is one of main obstacles to students' independent mathematical thinking.

Didactic templates, albeit in more reasonable and less conspicuous forms, can be traced in many textbooks for primary and secondary schools; usually the ways of pushing arithmetic (or algebra, geometry, calculus, ...) are more subtle. Templates are convenient for teachers as workable schemes of what students should learn; their main disadvantage is that they fail to foster deeper understanding of the subject.

Before the 1960»s many educators insisted on showing children only one method (the best one) of performing a given operation. The main argument was that otherwise various ways would get mixed up. In the 1970»s an opposite tendency prevailed: students should be encouraged to find their own ways of computation.

However, subsequently in certain books and educational standards the idea has became completely distorted by declaring that at the "basic level" students have to get the result in any way while at the "extended level" they *have* to get it in two different ways. Thus, the previous preference for exploring a variety of ways and permitting children to perform the operation in their own way was replaced by a "must" condition, which shattered the educational sense of it.

Hassler Whitney (1973) gave a vivid description of the weaknesses of the usual way of "pushing mathematics". He also showed specific examples of helping children to learn in their own ways and to think for themselves. One of them is the addition of the type 8 + 5. Children put 20 beads on a piece of wire: ten red beads and then ten blue, say. — Show eight red beads by sliding them to one side! Put a cardboard spacer and slide five more! How many is this together? Children count; the point is that "thirteen" is *their* answer, not teacher's. Let us note that those five are two red and three blue beads. All thirteen form together a configuration 8, 2, 3 of concrete objects; yet, the decomposition 8 + 2 + 3 is not explicit and is not needed to get a correct answer. Gradually the abstract version 8 + 2 + 3 may be perceived by children as a result of reflecting on their actions. From a formal point of view, this is the same decomposition as in (*) above, but didactically there is an enormous chasm between them.

Since 1973 a great deal of fundamental research has been carried out, but the problem of how to encourage independent mathematical thinking in mass education *in an effective way* remains unsolved.

Context of a task, its purpose, and understanding

The significance of concrete objects in the early mathematics education has been repeatedly stressed in the literature. Moreover, they should be *objects to manipulate*; static pictures should not be regarded as a substitute of movable objects (Aebli, 1951). In certain situations, however, it may also be highly important what kind of objects are used.

A celebrated conception introduced by Piaget and Szemińska (1941) is that of *conservation of* (cardinal) *number*. In a standard task the child is first shown a two-part array where the two parts look identical; the equality of cardinal numbers is easy to judge on direct perceptual evidence. The next step is to introduce a mathematically irrelevant transformation which destroys the obviousness of the equality, e.g., elements of one set are spaced out. Finally the experimenter aims to discover whether the child is able to discount the change and maintain the equality (Donaldson, 1982).

Typically, conservation means the child after having seen a set of, say, 10 red counters in a one-to-one correspondence with a second set consisting of 10 blue counters, watch the blue counters be spread out and is then convinced that after the change there are still as many red counters as blue ones; some of them even react: "Of course, there are as many! Why do you ask?". In contrast, non-conserving children react by saying that there are more items in the set covering a larger area.

Piaget insisted on one-to-one correspondence and did not allow counting. However, even after having counted both sets non-conservers give much the same responses (Gruszczyk-Kolczyńska, 1994). If one considers how counting is grounded in the common practice of adults dealing with questions of equality, a demand of one-to-one correspondence from children would be excessive. The principle of parallelism (saying that the learning process of an individual should follow the order of historic development of human kind) does not apply here!

The conservation of number is attained at the age between 6 and 7 years on average. However, Alina Szemińska (1976–77) modified the standard Piaget test, replacing counters by suitable toys, e.g., by 10 houses without roofs and 10 roofs. Otherwise the method closely followed the original one. The strong semantic component made the new tests much easier. Szemińska used the term *pseudo-conservation* to describe the level of positive responses in such a test. It is quite remarkable that it was attained already by four-year-olds!

What is the main reason *why standard conservation tasks require a much higher level of mental development than those of pseudo-conservation?* A mathematician would find small, flat, round, identical counters as concrete as toys, so what makes the difference? Szemińska explained this by arguing that conservation is a consequence of the *identification of pairs*, which is easier when a correspondence suggests itself, e.g., houses-roofs or handles and baskets without handles.

Yet, another explanation seems more relevant: although the counters are concrete and manipulative, the question of whether there are as many red counters as blue ones does not have any real purpose for the children. It has no reference to their lives and thoughts and has to be understood literally. On the other hand, the purpose of the question whether there are as many houses as roofs is clear: each house must have a roof. Houses without roofs or roofs without houses are useless. Therefore the question whether there are enough roofs for houses makes sense and the child is not deceived by moving the roofs farther apart. The counters, although familiar to the child and concrete, are semantically neutral and give no hint how to answer the conservation question; the child has to decode the linguistic meaning of words "Are there as many ...".

This example illustrates that *children should perceive for themselves the purpose of a mathematical activity in order to appreciate and understand it.* Also many examples of post-Piagetian interpretations of children's behaviour reported by Donaldson (1978) confirm this.

The difficulty of many mathematical tasks depends on their contexts and on the language used. A modification of the context and/or the format of a problem, which appears mathematically irrelevant, may drastically change its difficulty for children. Conversely, a mathematical idea may have a multitude of different formulations, at various levels.

Developing critical thinking

Independent mathematical thinking should develop through critical thinking. Children should acquire the habit of checking their computations by themselves and of reflecting on their solutions of real-life problems. This postulate is in marked contrast to the attitude of those teachers who tend to make authoritarian decisions on what is right in students' work and what is wrong.

> While mathematics was proposed to be a subject structured purely by reason, the teaching of mathematics as a global concern developed rigourous structures far removed from any critical enterprise. Instead of being a discipline reflecting critical thinking, mathematics education became associated with domination, control, tests, and rigid forms of communication (Skovsmose and Nielsen, 1996, p. 1259).

The idea of fostering independent mathematical thinking in children is also incompatible with the long-standing fear (widespread in traditional teaching) of exposing a child to any error whatsoever. This fear has been based on the out-of-date empirical theory of reflection (for details, see Aebli, 1951) and also on stimulus-response models of learning. Clearly, if learning consists of a series of carefully devised similar tasks and the child is to imitate the correct way shown by the teacher, then any method appearing to involve possible confusion must be renounced.

This fear has also been influenced by the experience with children learning correct spelling, which is mostly based on visual memory and not on logic; therefore any case of a misspelled word written on the blackboard may have a lasting adverse effect. However, this kind of experience must not be automatically transferred to mathematics. It is not true that a look at $5 + 3 = 9$ may later cause the same mistake when the child adds 5 and 3. Moreover, it is advisable to present students with an opportunity to be exposed to certain intentional errors and to detect and overcome them with the teacher's assistance, when the situation is under control.

Students may be given a series of tasks of the form: Which of the following equalities $5 + 3 = 8$, $9 – 6 = 2$, ... are true? The child may be told to cross out each wrong equality, or to cross out the wrong result and to write down the correct one, or to write YES or NO by each equality. The words such as "true", "untrue", "false", "correct", "incorrect" should come into the children's vocabulary as clear natural words (understood in a context). I would not be afraid of the words "wrong" and "error" provided that they reflect *the child's* opinion (they are criticized by some educators for being politically incorrect, as the phrases "being wrong" and "making errors" are negatively connoted).

The child should have *the right to make errors*. Errors are inherent in learning. Making mistakes by a math learner should be regarded as natural as falls of a ski learner. Errors reveal incompleteness of knowledge; they do not occur randomly and may be rooted in misconceptions, in erroneous beliefs, in an incorrect underlying premise. *The child should not be punished for not understanding something*. However, the attitude of always saying "it's OK", regardless of whether the child's answer is correct or not, is definitely unacceptable. Incorrect results should not be readily tolerated; students should be informed about them or advised to find mistakes themselves, but this must be combined with help and attention so that they can learn from their mistakes. Errors may be springboards for exploration and discussion.

We should also distinguish between a definite error (as in, say, $8 + 5 = 12$) and unprecise or vague wording, which normally accompany learning.

Independent thinking should also be developed through critical approach to verbal problems. I believe that students should occasionally be exposed to problems which are intentionally distorted versions of standard well-formulated problems (the distortion should be clear and conspicuous). Such problems are intended to cause a deliberate conflict situation to make students aware of the necessity of checking the reasonableness of the text. There are several types of such problems: problems with *missing* data (incomplete information), problems with *surplus* (extraneous, redundant or not relevant) data, problems with *contradictory* or *impossible* data, absurd problems, "pseudoproportionality" problems, problems with unstated or irrelevant questions (Puchalska and Semadeni, 1987). Their common feature is that they require meaningful, critical interpretation of the text; children should not jump into computations before thinking a little.

Although non-standard problems are absent in traditional teaching, their idea sporadically appears in papers on education. Eighty years ago a leading Polish educator wrote:

> When we formulate problems, it is advisable that we occasionally pose them so as to make children realize that one has to find arithmetical relations between the data. If we present a problem of the form "Sophie is 5 years old, how old is her brother?", then the child has to note that the problem cannot be solved because no relation between the age of Sophie and the age of her brother is given. Posing problems with many irrelevant data [...] we habituate pupils to paying attention to the relations between the data (Jeleńska, 1926, p. 201).

A distinguished American educator has expressed a similar view.

> The objection to so-called "absurd" problems is based upon *a priori* grounds. [...] Part of real expertness in problem solving is the ability to differentiate between the reasonable and the absurd, the logical and the illogical. Instead of being "protected" from error, the child should many times be exposed to error and be encouraged to detect and to demonstrate what was wrong, and why (Brownell, 1942, p. 421, 440).

Markovits, Hershkowitz and Bruckheimer (1984) gave children problems of the type "The height of a 10 year old boy is 140 cm; what will be his height when he is 20 years old". Many students gave absurd answers based on proportionality. Bender (1985) reported similar attitude of children who got the problem: "A postage stamp for a standard letter from Aachen to Munich costs 60 pfennig. The distance from Aachen to Munich is 600 kilometers. The distance from Aachen to Frankfurt is 300 kilometers. How much is the postage for a standard letter from Aachen to Frankfurt?" Many children answered "30 pfennig". Mathematics was viewed as an activity with artificial rules and without any specific relation to out-of-school reality.

In an oft-quoted paper (IREM de Grenoble, 1979), children's responses to the following absurd problem were reported: "There are 26 sheep and 10 goats on a boat. How old is the captain? What do you think of the problem?". Almost 75% of children aged 7 to 9 and 20% of those aged 9 to 11 performed some arithmetical operations on the given numbers without expressing any doubt.

Non-standard problems of various types were included in some Polish textbooks for primary grades (children aged 7 to 10) as extra material, but most teachers skipped them (a likely reason was that dealing with such problems required independent thinking). When they were given, the students were initially surprised and reacted similarly to those in Grenoble. However, usually after the third problem, they grasped the idea, were very active, emotionally involved, and asked for further funny problems of this sort (Puchalska and Semadeni, 1987). The initial confusion may be explained as a result of an unexpected change of the social contract. Indeed, the children knew they had always been supposed to perform some suitable arithmetic operation(s) and it had been tacitly assumed that each word problem had a unique solution. Therefore it is highly important that students getting such problems should be properly introduced to the new convention and have enough time to adjust to it. Presenting them with one single non-standard problem is pointless.

A new dimension to the topic was given by Gruszczyk-Kolczyńska (1986, p.129) who reported on her remedial program for 61 students in grades 1–3 in Katowice. In her efforts to overcome the children's emotional block she used intentionally ill-formulated problems (having contradictory data, say) with success. During conversation with the child she would formulate such a problem and pretend that this was *her* error; her intention was to convince the child that the teacher *can* also make a mistake and to reduce the child's fear of error.

Stimulation without overdemanding

Through astonishment at certain unexpected facts or regularities children may gain an insight into sophisticated mathematical ideas (to paraphrase Aristotle's apt remark "Through astonishment men have begun to philosophise"). Children's surprise: *Why is it so? How is it possible?* opens a way to exploration of such regularities. Szemińska (1991) used the word "amazement" several times in her descriptions of children's efforts to deal with unexpected situations.

A remarkable way of stimulating children's interest has been used by Edyta Gruszczyk-Kolczyńska in her activities with preschoolers in Warsaw. First she tells each child to take ten pebbles (or, say, ten small sticks). When the children are ready, she closes her eyes and covers them with her hands. Then she tells the children to hold some of the ten pebbles hidden in one hand and the remaining pebbles hidden in the other hand. When the children say this has been done, she opens her eyes and comes near to each child successively. She asks the child to show the pebbles in one hand, looks at them, and says how many are hidden in the other hand. Children are amazed. She does not disclose her secret, but asks: "Who knows in what way I find out what is the number of hidden pebbles? If somebody has an idea, please come to me and whisper it in my ear".

Five-year-olds had only two ways of explaining the phenomenon to her: a) you know everything, b) you have magical powers.

In contrast, six-year-olds were able to gain insight into the problem. At the beginning only few of the children were able to find out how many pebbles had been hidden; they became the teacher's "assistants" and replaced her in telling the numbers of pebbles. They conspicuously used their fingers in counting on, but were unable to describe clearly how they got the right numbers. Those children who could not figure out the "secret" did not get any extra hints, but watching peers act was meaningful. After several repetitions, gradually, all the children in the group grasped the idea. It should be noted that no child used subtraction, although many educators had insisted that in such a problem students should use subtraction.

In this way children were given an excellent introduction into what is a mathematical problem. It is significant that such activities face them with one the of most important features of mathematics: computation can help you to find out about something without seeing it. This was also an example of stimulating children to think intensively without requiring anything over their heads.

The maxim: *stimulation without overdemanding* is strikingly illustrated by children's activities devised and described by (Swoboda, 2006).

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Various manipulation functions in solving geometrical tasks

In educational studies of mathematics, the role of manipulation is highlighted. The action is a base for learning early arithmetic. Manipulation in learning geometry is an argumentative topic, because of different theoretical bases for creation of geometrical concepts. Some theories underline a great importance of visual information in forming the first level of understanding geometry. Such approach is present in works of P. Vopěnka or M. Hejný. It result, from our former experiments, that children are able to act in their early years in the geometrical world. Assuming that visual information gives the first stimulus for creation of geometrical concept, we undertook the experiment to observe the role of manipulation in early geometry.

Geometrical concepts as a result of action interiorization?

Piaget's theory of interiorization has a great impact on methods of teaching mathematics in early childhood. It is assumed that child's mathematical concepts emerge by operations and interactions with the real world. An action on the object leads to creation of schemata. As the result, through the process of ***reflective abstraction***, actions can be replaced by symbols and words.

Piaget was in opinion that "it is useful to distinguish empirical abstraction, which draws its information from the objects themselves, from what we call 'reflective abstractions'. The latter proceeds from the subject's actions and operations" (Piaget, Garcia, 1989). This process is important also for geometrical concepts – in his opinion, the child is able to identify the properties of objects by the way in which different kind of actions affect them.

But actions undertaken by children up to 7 years old can lead only to a very limited number of examples that are important for mathematics.

> Experimentation shows that (…) only operations appropriate to class and relational groupings and to the numerical and spatiotemporal structures which resulted from them are used. (…) In the course of stage I (up to about 7–8 years) subjects are most concerned with their practical success or failure without consideration of means. (…) Although the children demonstrate by their behavior that they know how to act in the experimental situation, sometimes successfully, they never internalize their actions as operations, even as concrete operations. (Piaget, Inhelder, 1958).

Taking this point of view, perceptions (sights) have a very small impact on the development of mathematical – logical thinking, including geometrical thinking and the creation of geometrical concepts. The role of actions and manipulations is also problematic. It is not clear to what extend they support the creation of geometrical objects.

A complete acceptation of Piaget's ideas regarding geometry pose many questions. For example, some authors (Clements, at el, 1999, Clements Battista, 1992), criticize the fact, that in Piaget's theory the creation of geometrical concepts starts from topological features.

It seems logical to analyze a relation between Piaget's theory and other theories devoted to creation of geometrical concepts. Apart from other important issues, it is worth tracing the role of perception and the role of manipulation in learning geometry at an early educational stage.

Theories which underline the importance of perception in geometry

Some theories stress the fact that geometrical knowing and understanding is created in a specific way. In those theories, the priority is given to perception, although geometrical „seeing" is not identified with the literal meaning of that word. Geometrical world cannot be perceived directly. It is hidden in the real world, and it is emerging from the surroundings through the special intellectual activity which can be called the geometrical insight (Hejný, Vopěnka).

> The first, and the basic understanding of the real world is the understanding via senses. We look at the world of geometry, but not with our eyes; we learn the world of geometry, but not with ordinary senses. Geometrical seeing is possible only because of the sixth sense. This seeing is not less obvious than seeing the real world using the sense of sight. [...] Who looses the geometrical seeing, can not approach the geometrical world; he can only listen to us, talking about this world. He is as the blind, who finds himself in a gallery and listens, what the others talk about the pictures. (Vopěnka, 1989)

At the beginning, there is no geometrical world nor geometrical object in a child's mind. Only objects from the real world exist. But we focus our attention on those objects in various ways. Sometimes we perceive "something". Vopěnka (1989, p. 19) describes such a situation in the following way: *To see "this", means to focus attention on "this", to distinguish "this" from the whole rest. This, what can absorb the whole attention on itself, we call "phenomenon".* Perceiving "something" creates the <u>first understanding.</u> For example, a child can focus his or her attention on a shape of an object or on a specific position of one object in relation to another. *Phenomena* open the geometrical world to a child. In spite of the fact that our attention is attracted by these phenomena, this first understanding is passive: stimulus goes from the phenomenon. In this depiction, the role of perception is large – the perception of "something" is the first step to creation of the child's own geometrical world.

M. Hejný transforms P. Vopenka's philosophical depiction. He relies upon his own experimental studies and on conclusions derived from Piaget's, Vygotski's and van Hiele's work. In Hejný's theory, the development of understanding of geometrical world goes through various levels. On the first level, there is a possibility to perceive shapes and some relations, but these are (both – shapes and relations) attributes of real objects. A verbal isolation of these phenomena is also possible, by talking about them and calling them. Nevertheless, words such as: *triangle, pyramid,… , long, high,* or skills of making comparisons like: *longer, broader, ….* are still words and concepts related to the real, physical world.

In these depictions, the role of an action is lost. Results of psychological researches confirm that in understanding of shapes, the great importance lays upon the pictorial designate. But the next stage is needed. Acts of perception are important but are not a sufficient source of geometrical cognition. Szemińska (1991, p. 131) states that: *perception give us only static images; through these, we can catch only some states, whereas by actions we can understand what causes them. It also guides us to possibilities of creating dynamic images.*

On the other hand, widely known Piaget's results show that children (on the pre-operational level) have great difficulties in movements reproduction – they are not able to foresee a movement of an object in a space. The process of acquisition of such skills is lengthy and gradual. During manipulations, child's attention should be focused on *action*, not on the very *result of action.* It requires a different type of reflection than the one that accompanied his or her perception.

Experiment

In our experiment, as the basis, we took Vopěnka' and Hejný's theories about the opening of the geometrical world. First of all, we based on the assumption, that the first understanding takes place when a child turns its attention on any geometrical phenomenon. We were interested in situations where children can manipulate. Results of our previous experiments (Swoboda 2006; Swoboda, Synoś 2007) showed that making patterns (arranging them out of blocks, folding out of puzzles, drawing), can fulfil our expectations. Patterns are a friendly environment for children. They are close to their natural, spontaneous activities. Such work gives a chance to connect the process of concept forming with an individual child's actions, which are adapted to his/her own specific activity.

In order to test the possibilities of creating a "path" from perception to manipulation, we prepared an experiment, which took place in March–April 2008. Children from a nursery school, aged 5–6 , were the subject of the series of observations.

Children were tested individually. As a research tool we used "tiles" (two types), shown below (fig.1).

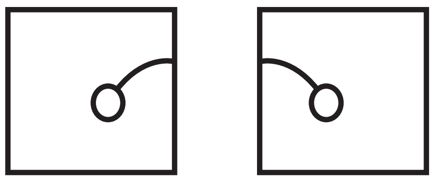


Figure 1. Research tool

Part I.:

A teacher makes a segment of the pattern (fig.2) :

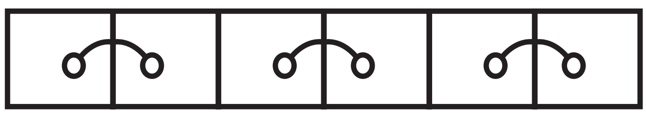


Figure 2. A segment of the pattern prepared by a teacher

On the table, there are also tiles arranged into two separate piles.

Teacher says: Look carefully at this pattern and try to continue it.

If a child doesn't undertake the task, the teacher will say: look how I do it. After that you will continue.

If a child undertakes a task. then after having finished making the pattern, he/she will take part in the next part of an investigation.

Part II.:

Teacher says: Now, please close your eyes, and I will change something in your pattern. After that ,you will say what has been changed. (Teacher exchanges one tile in the pattern, so that the regularity is distorted). Then, the teacher shows the pattern and asks a child: Is there something wrong here? Why? Regardless of the answer received from the child, the teacher says: and now try to correct the mistake I have just done.

Results of the experiment

The importance of visual information

A pattern prepared in our experiment represents an idea of a mirror symmetry (axis symmetry). Axis symmetry functions differently in visual (static) representation than in a dynamic creation of the object's symmetrical image. For a correct pattern continuation, it was necessary to use two different types of tiles, those with the same shape, but oriented differently.

While making the pattern (continuation of the pattern) children started their work by trying to compose one motif using two tiles. This motif was taken as "the whole", which was important for the next work phase. This method was related to the psychological aspect of perception. The holistic understanding was supported by the verbal explanation given by children; when asked: "what does this pattern look like" they answered: *cherries, headphones, if you drew one line at the bottom it would be a car, a tunnel ,a bridge, setting suns.* Just one object arises to multiple ideas. So, these children were working on the pre-conceptual level (according to M. Hejny's theory), their geometrical world was still strongly connected with the real world. The shape was an attribute of the real object.

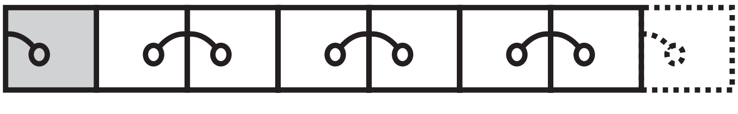
Some other children worked accordingly to the other strategy; they arranged tiles in a row and took them in turns – one tile from the first pile, one tile from the second pile.

On the basis of these observations, it is hard to say whether children were working accordingly to the idea of a mirror symmetry. Their action and manipulation was stimulated by visual information, which was sometimes supported by a rhythmical movement.

Sometimes children used to wait a long time before they started to manipulate. They were looking at the pattern prepared by a teacher and at the tiles. It took a long time – sometimes one minute or more – until they have analyzed the task, and after that they started their work with a clear, right idea. Therefore, we are in opinion that the main reason for children's work was the visual perception of the regularity.

Example 1. (6 year old boy)

1. T: Look carefully at this pattern and try to continue it.
2. B: *he catches quickly one tile from the left pile, puts it away, takes the second tile from the same pile, gives it back. He looks at the table (6 seconds)*
3. T: You can take it to your hands.
4. B: *He reaches the same pile, takes one tile, gives it back, takes another tile from the bottom of the same pile, gives it back again. 3 second pause. Now he reaches the right pile, takes one tile, leaves it, takes another one and finally decides to place it in the pattern – firstly, from the right side but very quickly changes his mind and puts it on the left side* (see the picture)



Now he works very fast. He continues his work on the left, taking tiles in turns from both piles (second motif). For the third motif, first he takes two different tiles, connects them in hands and only after that connects the whole motif with the pattern. In the meantime, the tiles from both piles mix up on the table and it is not easy to recognize two different sets. The boy stops his work and looks carefully at the table.

5. T: Did you finish?
6. No. *Now he takes one tile from the mass, tries to connect it with the last one in the pattern. On seeing that his choice is not correct, he changes the tile and continues his work, taking successive correct tiles form the table without doubt. In this way he builds a very long pattern, extending it on the right and left side. In the end, there are only three identical tiles on the table. The boy sits motionlessly (18 sec.), looks at the tiles.*
7. T: Do you still want to work?
8. B: No.

This boy spent a lot of time looking at the pattern and at the tiles. He preferred to make a visual analysis than a manipulation – supposedly, the visual information was more important for him. In addition – he knew how to use these information. Perception was the foundation for any his decision, manipulations only supported and verified the undertaken actions.

Children's word argumentations, derived from the second phase of the experiment, also support the visual level. Children experienced great difficulties in explaining why the regularity is destroyed. Their argumentation referred to the observed phenomenon. They tried to explain what *is wrong*, but not *what was changed*. Here are some statements:

Ola: Because there are … two… in the same direction… sides.
Kasia: Because they are the same.
Dominika: Because here it is like this (she shows the "old" tile) and here it is like this (she shows the "new" one).
Michał: Because a tunnel does not exist.

This type of argumentation was supported by indicative gestures. Children used gestures while pointing at the place where the regularity was broken. The importance of such gestures (of pointing at something) is discussed by psychologists: thanks to that gestures, these objects are highlighted. In our observations, children immediately identified the place with the wrong configuration. At this stage, a reflection was related to a figural aspect of the given task.

Such action (pointing) is strongly related to the visual level of understanding and children from our experiment worked at this level. Mental activity was not caused by the physical activity – it was evoked by the transformation of the visual information. A child followed one tile up to another, trying to control the global shape of the whole motif. At this level, all manipulations have the supplementary function. A child was able to detect errors visually, to find irregularities. A child was also able to check visually whether these correct relations exist.

Movement related to the geometrical phenomenon

Other types of manipulations were related to the action of "correcting mistakes" in patterns.

Two different tiles which were used in patterns constituted a symmetrical couple. Children's behavior undoubtedly showed that frequently they were not conscious whether it is and how it is possible to make a motif consisting of two tiles of the same type. The correction of regularities progressed in two different ways:

- A. A child rejected a "wrong tile" immediately and replaced it with the correct tile, taken from the proper pile.

- B. A child started to manipulate the "wrong tile" (despite of his previous experience gathered while he was making the pattern), trying at all costs to obtain the mirror position.

The table below contains the quantitative specification which shows the presence of these strategies in children's work.

Age	Replaced strategy (A)	Manipulative strategy (B)	Helpless
6 year old	10	6	
5 year old	7	7	1

Table 1. Pattern correction strategies

Although convinced that the tiles are in two different types, children sometimes undertook attempts of matching up two tiles of the same type. Such behavior can lead to a conclusion that maybe they were not completely aware of the nonsense of such actions.

Example 2. Martynka, 6 years old (second phase work: correction of mistakes in the pattern)

1. M: A *girl throws away two non-related tiles from the pattern immediately. She looks at the table but no free tiles are on. In this situation, she decides to manipulate these two that she has in her hands: rotates them, attaches one to another in various ways. She does not look at the teacher*.
2. T: So, probably it is impossible to do something with these two tiles.
3. M: *She nods in assent*
4. T: Do you have any idea how to correct this pattern?
5. M: *She does not say anything, still manipulates – one tile she keeps in one position, turns the other one.*

The fact that children undertook manipulations is the foundation for the statement that they tried to put some hypothesis about relations between these tiles. Probably they presumed that thanks to manipulations, they can come across such a position that will give a chance to built the whole motif. In that case, they assumed that an eligible movement leading to a specific, interesting arrangement exists. Such situation is seen it the next example:

Example 3. Karolek, 6 years old (second phase work: pattern mistakes correction)

1. K: *He starts to work immediately. Takes into his hands two identical tiles, which are laying one to another. He leaves one tile without any movement and he puts the second tile in many various positions.*
   After some time, he moves these two tiles closer to him and makes manipulations using only them.
2. T: Keep on trying…
3. K: *He makes various movements on a plane – mainly rotations. At some point, he looks at the reverse sites of the tiles (where no picture was printed): firstly, he looks at one tile, and after some time at the second one.*
4. T: Do you know what have I done? I've changed one tile from these piles (*she points at tiles on the table)*
5. K: *He does not react, he goes on manipulating the tiles for some time. Finally, he takes a casual tile from the table, checks if it fits, takes another one and after the third attempt, he finds the proper tile. Then he makes the whole motif and connects it with the pattern.*

Karolek starts correcting the pattern from manipulating two tiles of the same type. At the beginning he makes movements using only one element. It seems that he has a clear imagination of the results that he wants to obtain. Then, he supports his work by manipulating two tiles at the same time. He sees that the second tile should be "turned" in some way, but he does not know in which. Finally, he displayed a good intuition by looking at the reverse side of the tile. He checked if the picture did not pierce the paper. It is obvious that such experience led him to the following conviction: if he is to have only one-sided tiles, then in order to build a motif, he needs two different types of them. He finalizes his work: patiently, he chooses elements among these lying on the table, looks for those which create a couple with the previous ones.

The shape is accessible to a child in an intuitive manner, and the figure is recognized as the same if its shape is maintained (Williams, Shuard, 1970). Although accordingly to some researcher's opinion the axis symmetry is very difficult for the sight (Demidow, 1989), it makes invariant transformations in the manner which is non-conscious for us. Such an interpretation may not necessarily be contradictive to some Piaget's results (Piaget, Inhelder, 1968) regarding the development of mental images. Therefore, children's actions described in this paper are not yet the base for interiorization in Piaget's sense. They are perceived as a necessary step to obtain the next developmental level. This level is necessary for gathering experience needed for the level, in which the reflection upon the movement would be possible. As it is seen in the table 1, the consciousness of some relations changes accordingly to age. Children know that shapes on the tiles are connected to each other in some way, but they were not quite aware of the relation type. In spite of the fact that they participated in the first stage of experiment during which they gathered experience in building the pattern, they frequently changed the work strategy during the second stage. They tried to force the idea of connecting two one-type tiles, making mainly rotations. By putting the tiles "upside down", they were checking the effect visually. The children's reflection was focused on the result of actions (whether it is possible to fit two tiles) and not on the type of movement leading to success. It seems that there is a huge distance between the children's actions described here, and the understanding of visual dynamic imagines in Szeminska's (1991) sense. In our experiment, manipulations only supported the visual, static information. The rhythm, order and regularity were the factors that inspired children to act and the ones that we controlled. Such model of activity was in accordance with the initial Greek meaning of the word „symmetros", which means „harmonious", „with right proportions". Such feelings were verified visually by the children.

Summary

Observations and conclusions gathered by us should not lead to a conclusion that on the fist level of creation of geometrical concept, children should be deprived of a possibility of manipulations. It should be just the opposite! The perceived geometrical phenomenon should be investigated by means of a spontaneous manipulation. Therefore, the direction should be as follows:

$$\text{Phenomenon} \rightarrow \text{manipulation}$$

At this stage, manipulations are evoked by perception and are subordinated to perception. The manipulation itself is only a tool which enables to reach the aim. While solving the problem, child does not consider what kind of manipulation he/she makes. Thus, it is a didactical abuse to say that all tasks in which a child makes any manipulative activity, lead simply to interiorization of actions. A reflection upon the result of the experiment should not be identified with the experiment itself.

While making conclusions regarding the role of manipulation in creation of geometrical concepts, we are in opinion that, at the level of 5–6 year old children, manipulations can occur in (at least) two different types:

- Deictic motions (through pointing and showing, when a child has a clear idea what he wants to obtain). These are motions that indicate an awareness of relations and connections between particular elements. These are also motions that indicate an awareness of certain disturbances in relations. This awareness is built by visual inquiry of the geometrical phenomenon.

- Manipulation for searching for an effect (I know, what I want to obtain, but I do not know how to reach the objective). A child has a vague feeling that some kind of manipulations can establish an expected relation between objects, but has no idea what kind of movement is needed. Manipulations only support visual imagination.

Second type movements will probably have a great significance for creating concepts of geometrical transformations or dynamic visual imaginations of geometrical objects.

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Is it possible to teach our pupils to think independently?

To think independently means to think on one's own. People who think independently feel the ***need*** *to make sense of everything based on* ***personal observations and experiences*** *rather than on* ***information they were given without questioning it****. It does not mean that independent thinkers have to invent everything themselves, but it does mean that to think independently they have to trust their own ability to make judgments, even if it contradicts what others say and even if it results in making mistakes.*

Do we need independent thinkers, particularly in the process of teaching-learning mathematics? On the one hand, teaching creativity and independent-thinking are among the goals of teaching, and on the other hand, we have to teach pupils to recognize typical situations and react to them in the most suitable (usually algorithmic) ways. How is it possible to combine both these things? It is very tempting to concentrate just on the memorization of facts and practise algorithmic skills, than to work on creativity and independence in thinking – especially when we don't have much time. But independent thinkers usually bring to our work a new and very often interesting perspective. For example, when pupils have to draw on a chosen piece of paper as many points as possible almost all children did it in the usual way (picture 1), but one boy did it independently (picture 2) – and it was amazing!

Picture 1

Picture 2

So, what we can do to help our pupils develop the **desire** and **ability** to think on their own?

We should start with developing the **desire** to think independently.

First of all, we have to create an environment in which pupils will **feel confident enough to voice their opinion**. It is not easy because very often teachers tend to reward the answers **they** want to hear, and not pay much attention to pupils' thoughts or even discourage those pupils who have different ones.

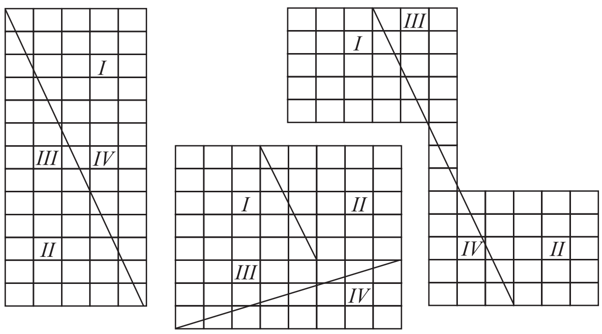
Once we have created a friendly atmosphere, we have to show **the need** to use thinking. For example, instead of just showing a set of tasks and their solutions, we can prepare a situation (cotextualized would be helpful) which introduces a chosen problem. Then, we have to ask questions which the **pupils will think** about it. We should also remember that general questions are in those situations more suitable then direct recall or knowledge questions. And now, we **have to be very patient** and let pupils **express and justify** their opinions. All the time we have to control ourselves to resist **the temptation** to tell pupils what they **should** think. In the end, we should show our pupils the **joy** that comes from being able to think independently and that we are proud of them – rewarding is the best way to motivate pupils and encourage this kind of thinking to happen again. What is more, we have to stop answering the question "why", with "because I said so". Could this kind of answer foster any kind of thinking? The result of this demand for unquestioning obedience is that the pupils stop asking questions, which eventually leads to not thinking.

To develop the **ability** to think independently among our pupils we have to use activity formats similar to the one described above and foster it in almost each situation we face (as often as we can).

How often do we create situations which foster independent thinking? It is awfully time-consuming, and demands a lot of preparation, attention and energy. In addition, the teacher should be able to perceive and appreciate the worth of independent thinking in the pupil and, of course, an independent-thinking teacher can only be an advantage. No wonder that we usually prefer just telling or showing pupils what they should know as opposed to trying to arrange situations in which pupils can find certain knowledge using their independent thinking. For example with small children, with whom we have to share the knowledge of different ways of adding numbers, instead of letting them to invent their own ways, differentiate and group them and lastly to draw conclusions, we just enlighten children with the final product. We can find the same format on every level of learning – for example with multiplying fractions, grouping quadrangles, proving Pythagoras Theorem or finding out the value of geometric series.

In preparing a situation which provokes independent thinking we can also use situations which confused us, for example the "proof" that 65 = 64 = 63 (picture 3).

65 = 64 = 63.



Picture 3

In those situations we can also use critical thinking, which is the ability and willingness to asses claims and make objective judgments on the basis of well-supported reasons, as a tool to think independently.

I would like to finish with the conclusion that although we can't teach our pupils to think independently, we can create situations which provoke independent thought and constantly support them with their struggle which is nicely described in a quotation from E.E. Cummings "A Poet's Advice to Students":

> To be nobody but yourself in a world which is doing its best, night and day, to make us like everybody else – means to fight the hardest battle which any human being can fight; and never stop fighting.

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Problems of formation learning environment

We discuss on major problems and ways of Ukrainian pedagogical innovation which based on democratic transformations in education. Our special study is formation learning community (environment) of successful learners. That modifying methodical frameworks focus on providing student's capacity to be engaged in cooperative learning as a capable individual that know how to initiate and create independent opinions, negotiate and build consensus for problem solving and risk-taking. We present a set of the modern methods and innovative leaning strategies in classrooms that increase the students' capacity to form learning environment as inquiry one. We focus on transforming classroom practices so that they provide a climate of trust, engage students in interesting activity and foster deep inquiry and genuine debate.

The realization of this approach is based on the idea that successful learning activities assume some features of inquiry; a lesson begins to resemble a project with meaningful classroom dialogue and inquiry [1]. It involves taking ideas and examining their implications, exposing them to polite skepticism, balancing them against opposing points of view, constructing supporting belief systems to substantiate them and taking a stand based on those structures. We have convinced that environment stimulates purposeful and productive activities, not traditionally „study work": students are engaged in the practical intellectual work of finding solutions to problems that originate from the real world. It improves the educational process that enables the students to acquire the mathematical knowledge more firmly, to form the practical abilities to use them.

We pay attention to the main peculiarities of learning environment and conditions of one's realization:

- **(i)** <u>goals of education:</u> they reflect the students' hopes based on dialogue (the productive exchange of ideas, attitudes such as tolerance, careful listening to others, assuming responsibility for one's own positions and so on);
- **(ii)** <u>a role of the pedagogue</u>: it reconstructs reality in a problematic form, with the students perceiving and analyzing this reality, the curriculum assumes that students' interests are to be taken into account;
- **(iii)** <u>subject:</u> a teacher (active) and a student (active), <u>object:</u> the entire surrounding world;
- **(iv)** <u>the knowledge is subject to doubt</u>: the doubt must stimulate dialogue, a critical approach and creative activity, education is a creative task;
- **(v)** drastic change of <u>reality</u> according to human needs;
- **(vi)** <u>existence of a problem</u> stimulates the search for its solution.

Three points advance students in their intellectual development [2]:

(i) they should be faced with choices, with materials that invite their comfortable and familiar ways of considering things, get more than one interpretation and the challenge to make and defend their own interpretations;

(ii) they should hear their classmates express points of view different from their own;

(iii) they should be encouraged to reflect, especially in writing, on the ways in which their thinking is changing.

We pay attention to the meta-cognitive processes, i.e. mastering of “thinking strategies”, “implementation rules” for cognitive activity. In this context it is essential to overcome stereotypes like “right” and “wrong” responses. Formation the learning environment is impossible without usable knowledge as well as knowledge concerning essence of inquiry and different ways of one. The students create more knowledge and solutions to practice problems, but also to be able to the systematic and efficient habits of idea creation based on the knowledge of the key discipline concepts.

The students know the valuable thing of collaborative/cooperative work and the improvement of their argumentation during conducting research and presenting results to others. In the learning environment the students are attracted to listening to different opinions of their classmates and creating an atmosphere that support free acceptance of the another ideas or argumentative rejection them, tolerant and interdependent social behavior.

In the formation of learning environment the teacher’s role is facilitating students, that realized by three objective: to design contexts that promote one’s inquire for thinking; to develop strategies for encouraging thinking skills; to develop strategies, especially portfolio, for authentic assessment of thinking as a mean of evaluating work. The main results of that approach is changing students’ role, that realized in self-searching activities and grouping forces (instead of remembering ready-made knowledge). Moreover it makes available shifting their attention to the individual peculiarities and uniqueness that provides possibilities of solving educational problems as individually as in group; encouraging their interests to following and training methods of inquiry (investigating, experimenting, interviewing, surveying, writing, and so on).

There are some work questions [2]: What is the main question posed by this piece? What answer does it offer? What reasons are offered in support of that answer? What evidence is offered in support of each reason? What reasons or facts are left out – things that might have supported different answer to the question? What “facts” are we expected to accept on faith? What nominal assumptions are made? What value assumptions are made?

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Suggestions regarding the support of independent thinking in mathematics

I. Changes and trends in arithmetics and early algebra

Part 2

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Scheme – oriented educational strategy in mathematic

Scheme is understood as a memory structure that incorporates clusters of information relevant for comprehension. It gets embedded in a person's mind by a repeated "stay" in a certain kind of environment (one's house, school, shopping centre). Scheme–oriented mathematical education is described and illustrated on a primary level. This paper surveys the experience with the implementation of this teaching method in teacher's training.

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1. Introduction

The aim of the contribution is to analyse and discuss one cognitive phenomenon which, to our opinion, might be used to improve contemporary education of mathematics (not only) in elementary schools.

Contemporary educational strategy of mathematics in the majority of our (Czech and Slovak) elementary classes is topic oriented. It means that each time, the whole period of mathematical lesson is focused on a particular topic: counting, sharing, measuring, etc. The alternative educational strategy presented in this paper is scheme oriented. It means that mathematical lesson is focused on solving problems, in which more mathematical schemes are addressed at the same time. A set of what we call mathematical environment is an educational tool for such an approach.

2. Scheme

When someone asks you about the number of windows or lamps in your flat or house, probably you will not be able to give an immediate answer. However, after a little while you will answer the question with absolute certainty. You will imagine yourself walking from one room to another and counting the objects that you were asked about. Both of the required pieces of information and many other data about your dwelling is embedded in your consciousness, as a part of the scheme of your flat. We use schemes to recognize not only our dwellings, but also our village, our relatives, interpersonal relationships at our workplace, etc.

Specialized literature gives various connotations of the term 'schemes'. The following quote by R.J. Gerrig provides a rather loose definition that serves our purposes. "Theorists have coined the term schemes to refer to the memory structure that incorporate clusters of information relevant to comprehension… A primary insight to scheme theories is that we do not simply have isolated facts in memory. Information is gathered together in meaningful functional units." (Gerrig, 1991, pp. 244–245).

A scheme-oriented education is based on creating two kinds of schemes:

semantic schemes rooted in everyday life experiences of a pupil and *structural* schemes which are 'pure mathematical' and have no direct linkage to pupil's life experience.

Structural schemes of early mathematics are created within different semantic schemes and after introducing a structural language of ciphers, they start to shake off this semantic supervision.

As an example, let us consider the concept of 'number 3' as one of the ten basic elements of the early mathematic structural scheme. This mathematical concept originates from the semantic scheme of rhymes, creating the ability to produce the rhythm, which synchronizes words and movements. A child's speech *one, two, three* is accompanied by handling obejcts. After the performance, its last word *three* must be repeated to point at the product of the process of counting: the set of three objects. Once the synchronization is created, a child is able to count objects. Both abilities – handling objects and synchronization – are nested in everyday life experiences. Word *three* in both its appearances starts to create a concept of 'three objects' as the first pre-structural concept of the concept 'number 3'. The first word 'three' brings the processual and the second 'three' the conceptual understanding.

The described pre-structural concept of 'number 3' is not completed yet. So far it is supported only by one semantical scheme and three others have to be added: number as an address, number as an operator of comparison and number as an operator of change (see below).

3. Semantic schemes breeding up the early arithmetic structural scheme

As mentioned above, there are four different semantic schemes in which a number appears: status S (number, magnitude), address A (in terms of place or time; the temporal address can be either linear of cyclical), the operator of change Ch and the operator of comparison Co. The symbol O will stand for the operator, when there is no need to specify its particular type.

In some cases there is no sharp boundary between these schemes. Take for example this situation: Ann (who stays at the second floor) has to go three floors up in order to see Betty (who stays at the fifth floor). Here number 3 can be regarded either as an operator of change (Ann moves) or as an operator of comparison (Betty stays 3 floors above Ann).

Numbers are the soil of an early arithmetic scheme. The core of it is an operation – addition and subtraction. Here the variety of semantic types comprise at least eight issues:

S + S = S	3 female and 5 male pupils, 8 pupils in total.
S – S = S	If 5 out of 8 pupils are boys, then the remaining 3 are girls.
S ± Co = S	E has 3 pets. F has 1 pet more/fewer than E. Thus F has 4/2 pets.
A ± Co = A	J. is 8 years old. R. is 1 year older/younger. R. is 9/7 years old.
S ± Ch = S	Eve had 5 €. Today she received/lost 2 €. Now she has 7/3 €.
A ± Ch = A	Cid used to live on the 5th floor. He moved 2 floors up/down. Now he lives on 7^{th} /3^{rd} floor.
Co ± Co = = Co	Eva read 5 pages more than Fay, who read 2 pages more/fewer than Guy. Eva read 7/3 pages more than Guy.
Ch ± Ch = = Ch	The number of bus-passengers increased by 7 persons at the first stop. At the second stop it increased/decreased by 5 persons. At these two stops the number increased by 12/2 bus-passengers.

Table 1

The key semantic model, mastery of which is the decisive step towards understanding Early Arithmetic scheme, can be written as ±O ± O = O. Our longterm experience substantiated by the experimental research of Ruppeldtová (2003), clearly indicate that the problems of using only operators are among the most demanding problems for pupils. We would like to know why operators are so demanding?

Commentary 1. The answer to the given question is rooted in different perceptions of statuses and addresses on one hand, and the operators on the other. The status and address are both enclosed data. Information such as "there are 5 chairs around the table" does not generate any further questions concerning numbers.

The operator is, by contrast, an example of an open data. The information "there are two chairs fewer" provokes the question such as: 'what was the original number of chairs?' and 'how many chairs are there now?' These two numbers are *virtually* present in the operator of change. The accuracy of the above thesis is confirmed by the behaviour of pupils who are assigned to such operator problems. When given such a problem, they keep asking for virtual data and for explanations as to how to deal with them. These pupils clearly did not have enough experience with numerical situations that feature exclusively the operator of change. That is why the current situation might be improved by incorporating operator tasks already in first-grade primary school curricula. In order to achieve this goal, we elaborated several environments. Three of them are presented in this paper.

4. 'Walk' environment

The teacher (and later one of the pupils) gives an order and another pupil(s) walks accordingly to it. Sample commands: 1. Three steps forward, go! 2. Two steps, then one step, forward, go! 3. Three steps forward, then two steps backwards, then one step forward, go! To keep steps of pupils equal, there is a set of about a dozen marks on the floor of the class. After this warm-up stage, the addition is introduced by the following scene: Two pupils, C and D, are standing side by side. Pupil C receives the following command: Three steps forward, then two steps forward, go! Pupil D receives the command: Five steps forward, go! Both pupils, C and D, eventually end up standing side by side again. The entire scene is accompanied by words and body movements, and can be classified as a walk representation of the addition $2 + 3 = 5$.

The problem originates by concealing one of the three numbers. The given situation, therefore, leads to three problems: $2 + 3 = ?$, $2 + ? = 5$, $? + 3 = 5$. The concealed number here has been replaced by a question mark. In the class scenario it is replaced by the word 'what?'; e.g. problem written here as $2 + ? = 5$ will be presented as

Problem 1. Pupils C and D are standing side by side. Pupil C goes 5 steps forward. Then a teacher says 'Pupil D two steps forward, then *what?* steps forward, go!'.

The class already knows that it is necessary to replace the word 'what?' by a suitable number. In the given case the number is 'three'.

'Walk' environment brings a natural possibility to introduce the pre-concept of negative numbers (which is impossible within the environments dealing only with a status). Negative numbers are represented by backward steps. The experiment proved that even firstgraders can easily solve the problem $2 – 3 = ?$ Pupil C receives the following command: Two steps forward, then three steps backward, go! Few pupils immediately and the whole class after a while found the solution as a command for pupil D: one step backward. In such a way a concept of negative number starts within the backward movement.

It is necessary to stress that on this stage, there is no numerical notation for negative numbers. Symbols like '-1' will be introduced later, not before the fourth grade. At that time, in each of our experimental classes, pupils used a sign minus as a natural description for both: addresses of places/years below the zero and operators of change in decreasing directions.

Commentary 2. The 'Walk' environment allows pupils to build their semantic schemes, from which four Early Arithmetic fundamental sub-schemes emerge: number ordering, addition, subtraction within natural numbers and pre-concept of negative number. The most important in this environment is a great support for understanding of addition and subtraction of operators, particularly the operator of change.

5. 'Footprint' environment

So far we dealed with short commands only. When a longer command with five or even more numbers appears, it will be difficult for a pupil to remember it. Thus, there is a need to find a way how to record a long command. Pupils start to create their own recording systems using fingers, dots, lines,… Finally one or more of pupils finds an arrow as a suitable tool for recording steps. A teacher now can take this pupils' discovery as a common language for describing commands and walk performances. It is important that no authority such as a teacher or a textbook actually brought this new language. Pupils found it themselves and therefore it is their own language. In such a way the Footprint environment is introduced[1] .

The arrow representation of the addition 2 + 3 = 5 is given by Figure 1.

$$\boxed{\rightarrow\rightarrow|\rightarrow\rightarrow\rightarrow} = \boxed{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow}$$

Figure 1

On Figures 2a, 2b, 2c there is an arrow representation of tasks 2 + 3 = ?, 2 + ? = 5, and ? + 3 = = 5 respectively.

$$\boxed{\rightarrow\rightarrow|\rightarrow\rightarrow\rightarrow} = \boxed{\qquad}, \quad \boxed{\rightarrow\rightarrow|\qquad\quad} = \boxed{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow}, \quad \boxed{\qquad|\rightarrow\rightarrow\rightarrow} = \boxed{\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow}$$

Figure 2a Figure 2b Figure 2c

There are two substantial differences between environments: Walk and Footprints. The first one is due to the fact that the Walk is ephemeral, while the Footprints is permanent. Words and steps will fade away, but Figure 1 will remain.

The second difference resides in the fact that the permanent language allows to create more demanding tasks than the language of word commands. This can be illustrated by one problem dedicated for fourgraders.

Problem 2. Fill in the three empty boxes with six arrows to fulfill both equations:

$$\boxed{\qquad\quad|\leftarrow\leftarrow\leftarrow} = \boxed{\rightarrow|\qquad\quad} = \boxed{\qquad\quad}. \tag{1}$$

Remark. Only arrows of the same direction are allowed in each box.

[1] J. Slezaková (2008) performed a number of experiments with the sole aim of finding appropriate graphemes for this model. In the end, arrow symbols were chosen as the most appropriate for children of 6 to 8 years of age.

Commentary 3. Having translated problem 2 into algebraic notation, it can be written as the system of three equations:

$x - 3 = y + 1 = z, |x| + |y| + |z| = 6;$

number x is positive if arrows in the first empty box in (1) are oriented right (→) and negative if these are oriented left (←). The same procedure is valid for letters x and z.

Within this environment even such a difficult rule

minus out of minus makes plus (2)

can be presented as a walking performace by means of the command 'turn about' abbreviated by TA. For example the expression 3 – (2 – 4) can be produced as shown in arrow languge:

→ → → | TA | → → | ← ← ← ← | TA

Figure 3

Our experience with this interpretation of the rule is very positive. Many of our pre-service teachers, future elementary teachers declare this performance to be ‚the proof of the rule'.

Commentary 4. While solving various Walk & Footprints problems, a pupil gets familiar with this double-environment and develops his/her mathematical understandig in different areas: ordering, addition, and subtraction of whole numbers; later on also solving system of equations, pre-concept of the absolute value of a number and even several ideas from probability and statistics.

6. A 'bus' environment

The bus route is marked by several (shall we say five) stops in the classroom, which we shall label A, B, C, D, and E. The stops are at particular places within the classroom, e.g. the teacher's desk, a washbasin, map, whiteboard, wardrobe, the piano, … A cardboard box stands for the bus and plastic bottles stand for the passengers. The bus departs from the initial stop A and ends up in the terminus E and anyone can get off and get on each stop. The decision-making is done by the pupils who act as conductors at individual stops. All the pupils see how the passengers are getting on and off, but only the driver can see the inside of the bus (the box); the driver is the pupil who is carrying the box. When the bus has reached the terminus, the teacher asks the pupils how many passengers they think there are in the bus. Each pupil writes his/her tip into a table and then checks it by looking inside the box.

The pupils first try to remember the number of passengers, later they start to keep written records. After the fifth or sixth round of the game, the teacher asks whether anybody remembers how many people got off at stop B. The teacher asks such questions during every subsequent performance, which forces the pupils to invent a more resourceful way of recording the entire process.

Story. In one experimental class, where each stop had a distinct colour, the teacher, after a tenth round asked: "What happened at the green stop? Did the number of passengers increase or decrease? By how many?" Only several pupils understood those difficult questions. One pupil immediately gave a correct answer. Then he explained to his classmates the secret of his solution.

Prior to the performance, the boy drew up 5 oval shapes in five different colours which matched the colours of the bus stops. Each oval stood for a bus standing at the appropriate stop. Then, using arrows, he recorded the performance. Two arrows directing the green oval represented two passengers entering the bus at the green stop and four arrows directing out of the oval represented four passengers getting out of the bus. Having used this record, the boy immediately saw that at the green stop the number of passengers decreased by two.

After several performances, some pupil came up with a method of table- recording. On request of the teacher, the discoverer showed the record to the class and the teacher started writing his/her own performance records on the blackboard. The discovery was not made in all experimental classrooms, in some of these the teacher had to clue the pupils and gave them the table record. The pupils eventually used the record found in the upper half of table 2. The table shows us that 2 passengers got off at stop B, while 3 got on; on the stop C one passenger got off and 4 got on. After a month, the table was complemented with a row entitled "go" which stands for the number of passengers on the bus between individual stops. E.g. Table 2 indicates that the highest number of people – 7 – went from stop C to stop D.

	A	B	C	D	E
out		2	1	4	5
in	3	3	4	2	
go	3	4	7	5	

Table 2

The teacher gives the pupils tasks that reside in blotting out some of the numbers. It can be illustrated by a rather demanding task assigned to 4th-graders.

<u>Problem 3</u>. Design a performance table if you know that the same number of people got on at each of the A, B, C, D stops. 5 people got off at stop D and the maximum of 5 people and the minimum of one person got off on the subsequent stops.

<u>Commentary 5</u>.The Bus environment was tested by seven different teachers in 4 first-grade and 3 second-grade classrooms. The environment was given a very warm welcome, especially by pupils. This influenced even those teachers who did not trust the environment at first. The environment uses the experience of pupils with bus route.

After illustrating three semantical environments, we turn our attention back to the starting concept of our research, to the scheme.

7. Theory of generic models – a tool for understanding mathematical scheme

Remark. From now on, under the term 'scheme' we mean 'mathematical scheme'.

In chapter 2 we gave Gerrig's definition of the concept scheme. Then, we gave several illustrations of mathematical schemes. However, the concept still remains in a theoretical level. So far, we have no clear idea how to use this concept in our educational praxis. Namely we do not know

a) How to evaluate the quality of particular mathematical scheme
in a given pupil's mind? and (3)
b) How to help this pupil to overcome possible developmental obstacles?

The goal of this chapter is to find such a tool. To do this, we will use our theory of generic models, which we briefly (i.e. without illustrations) describe[2] here.

[2]The theory designed by the author's father, Vít Hejný was first published in 1977 in the Slovak language. Its first English presentation is found in the article Hejný (1988), and its current version is in the paper Hejný, Littler (2006).

Our model of the process of gaining knowledge is based on stages. It starts with motivation and its cores are two mental lifts: the first (generalisation) leads from a concrete knowledge to a generic knowledge and the second (abstraction) from a generic to an abstract knowledge. The permanent part of the knowledge gaining process is crystallisation – inserting new knowledge into the already existing mathematical structure.

The whole process can be depicted in the following scheme consisting of two consequent levels:

motivation → isolated models → *generalisation* → generic model(s)

generic model(s) → *abstraction* → abstract knowledge → crystallisation

As we see, the generic model, the pivot between experiences and abstract knowledge, plays a decisive role.

<u>Motivation</u>. We see motivation as the tension which occurs in a person's mind as a result of the discrepancy between the existing and desired states of knowledge. The discrepancy comes from the difference between 'I do not know' and 'I need to know', or 'I cannot do that' and 'I want to be able to do that'. Sometimes this discrepancy comes from other needs too.

<u>Isolated models</u>. First experiences of a new piece of knowledge come into mind gradually and have a long-term perspective. For instance, the concepts of fraction, negative number, straight line, congruency or limit develop over many years at a preparatory level. For more complex knowledge, the stage of isolated models can be divided into four sub-stages:

1. The first concrete experience – the first isolated model appears and this is a *source* of new knowledge.
2. A gradual 'collecting' of more isolated models, which at this stage are separate.
3. Some models begin to refer to each other and create a *group*. The feeling develops that these models are 'the same, in a sense.
4. Finding out the reason for the 'sameness', or even better, the correspondence between any two models. These models create a *community.*

The above sub-stages can be useful for us when we investigate how a new idea gradually develops in a pupil's mind. It often happens that a new sub-stage, not presented here, appears and that one of those presented does not appear at all.

The stage of isolated models ends with the creation of the community of isolated models. In the future, other isolated models will come to a pupil's mind, but they will not influence the birth of the generic model. They will only differentiate more detail in it.

<u>Generalisation and generic model(s)</u>. In the scheme of the process of gaining knowledge, the generic model is placed over the isolated models indicating its greater universality. The generic model is created from the community of its isolated models and has two basic relationships to this community:

1. it denotes both the core of this *community* and the core of *relationships* between individual models and
2. it is an example or representative of all its isolated models.

The first relationship denotes the construction of the generic model; the second denotes the way the model works.

<u>Abstraction and abstract knowledge</u>. The generic model remains an object representative and does not allow for a higher level of structuring acquired knowledge. Therefore, the next step of knowledge development must be abstraction, i.e. disconnection from an object characteristic of a generic model. This shift is accompanied by a change of language and an object representative is exchanged for a symbolic representative. The symbolic representative brings about higher abstract understanding of the knowledge or knowledge area in question than the previous

object representative does. This process is intellectually demanding and requires a lot of time and effort from a pupil. The abstract knowledge is only rarely a consequence of AHA-effect, i.e. a sudden sight of truth. A majority of abstraction processes run in small stages. Creating abstract knowledge is based on the assumption that the symbolic representative is autonomously constructed or at least interiorized by an individual. If the symbolic representative is implemented in a pupil's mind from the outside in a ready-made form it usually only stays on a memory level as 'the knowledge without understanding'.

Crystallisation. After its entrance into the cognitive structure, a new piece of knowledge begins to look for relationships with the existing knowledge. When it discovers disharmonies, the need arises to remove them by adapting the new knowledge to the previous knowledge and, at the same time, to change the previous knowledge to match the new knowledge.

The above description of crystallisation is imprecise in two aspects: first, it suggests the image that crystallisation only begins when the abstract knowledge has been constructed. Second, it supposes that the only thing that is added to the cognitive structure and takes part in the process of crystallisation is the abstract piece of knowledge. Neither is true. Each new mental step, which plays a role in creating the new abstract knowledge, immediately becomes a part of the whole cognitive structure and plays a role in crystallisation. None of the pieces of knowledge which a pupil constructs has a final form and each is being polished, changed and broadened all the time. This permanent development of knowledge is a typical sign of the quality of non-mechanical knowledge.

8. Cognitive mechanism of the birth and the rise of mathematical scheme

Now we are prepared to answer questions (3). In brief we can say that

a) the quality of particular mathematical scheme in a given pupil's mind can be evaluated accordingly to the set of its generic models and a web by which these models are connected;
b) the most frequent developmental obstacles originate from the lack of generic models and their connections; thus the way of overcoming these obstacles is to build these missing models and connections.

In more details we describe the birth of the scheme and its internal organization.

Firstly, we clarify the birth of scheme. Isolated models and clusters of these models provide a breeding ground for a scheme. A scheme only appears with the origination of the first generic model. A child may discover that the total of 2 footballs and 3 footballs equals 5 footballs, the total sum of 2 and 3 dolls is 5 dolls, but s/he has not yet developed a scheme for adding small numbers. This scheme is only developed once the child has discovered that these calculations can be done by counting on fingers, which thus become a generic model for adding small numbers.

Secondly, we underline that scheme is a dynamic organisation of heterogeneous elements. The word *organisation* emphasises the fact that it is not just a set of elements, but also a set of bonds between these elements. The adjective *dynamic* refers to both short- and long-term mutability of the set of elements and of the entire organisation. Schemes may be either more stable or more flexible. Some flexible schemes originate by the amalgamation of smaller schemes. E.g. the scheme of the term "rational number" was created by amalgamating the schemes of the terms "fraction" and "negative number". The dynamism of a scheme is shaped by an internal conflict following the introduction of a new isolated model: a 1st-grader discovers that one half is a number, or a 4th-grader realises that a quadrilateral can be non-convex, or an 8th grader finds out that there can be a triangle with indefinitely large circumference and indefinitely small area.

9. Conclusions

As stated in the Introduction, one of the goals of this paper is to prepare our future activity while working with teachers. The key role in this work will be played by schemes, isolated and generic models. Here, these concepts are illustrated mostly in the area of arithmetic. However, in this work we will deal with geometry as well. As a base for the geometrical activity we will use ideas described in Swoboda (2006), Jirotková (2007), and Hejný, Jirotková (2006).

The concept of the scheme is elaborated in several theories. For example in a famous study of Gray, Tall (1994), in which the concept of procept is introduced. We read:

"The ambiguity of notation allows the successful thinker the flexibility in thought to move between the process to carry out a mathematical task and the concept to be mentally manipulated as a part of the *wider mental scheme*. Symbolism that inherently represents the amalgam of process/concept ambiguity we call a 'procept'" … (p. 116).

The concept of the scheme is also incorporated in the APOS theory. It presupposes "… that mathematical knowledge consists in an individual's tendency to deal with perceived mathematical problem situations by constructing mental *actions*, *processes*, and *objects* and organizing them in *schemes* to make sense of the situations and solve the problems. … Finally, a *scheme* for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemes which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept" (Dubinsky, McDonald, 1999).

Interpretations of the scheme in procept theory and in APOS theory are similar to our interpretation. The comparison of these theories can be found in Hejný (in print)

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Perceptions of numbers of 5 to 6 year old children

In this article we deal with observation of children's perceptions of number. We investigate process in which to numerical information the conceptions of numbers are assigned. The numerical information is word three and the child's drawing is used to mediate the numerical conceptions of children. The experiment was realized with the children in the kindergarten. We analysed drawn children's conceptions of number 3 and created concept map from those drawings.

Theoretical basis of the surveyed problem

We concentrate on the observation of children's perceptions of number in our research. When investigating the exteriorization of the processes of children's thinking about numbers, we will distinguish two processes. The process, in which to different conceptions of numbers (C) the numerical information (I) is assigned: ; and the process, in which to numerical information the conceptions of numbers are assigned: . The numerical information is such information (a word, move, gesture …) that contains at least one number. If we want to take a deep look into the primary conceptions of numbers of children, the information created in the first phases of mental process and closely connected to his/her private world of numbers are very important. We analyze the process in our experiment, whereby the child's drawing is used to mediate the numerical conceptions of children.

According to Piaget (1970), the drawing is a form of semiotic function, which has its place between the symbolic game and figurative conception.

R. Davido (2001) claims about the child's drawing that a child probably does not already know or does not already want to express himself/herself verbally; however, his/her drawings can reveal much about him/her and his/her real and imaginary world. The quality of not only motoric but also of cognitive and emotional growth of a child is reflected exactly in the child's drawing. It is the drawing that helps to organize the world into the unity of shapes.

That is the reason why we have chosen drawing as an appropriate entrance gateway into the inner world of children.

The publication written by Hejný, Stehlíková (1999), in which the authors analyze the process of the emergence of the world of numbers from the world of things, provided the theoretical basis for the realization of our research. The introduction of the world of numbers Hejný realizes in accordance with Popper's idea of three worlds. He classifies the world of numbers as a part of the second and the third world that Penrose (1994) entitled as the mental world and the world of culture. Hejný understands the world of numbers as the structure with the essence in person's semantic conception of a number. The conception consists of three components: the number, its attachment to the world of things, and the knowledge of an individual, in which the number and its attachment are situated. The world of things (the first world according to Popper) interprets

Hejný as the set of all the human's conceptions of things, events, situations and relations existing within the perceived world. The world of numbers originates within the world of things in the course of the intellectual growth of children.

The process of the emergence of the world of numbers from the world of things has two components:

- Verbal: the emergence concerns the words -number words. The child acquires them as the sounds whose meaning he/she understands only vaguely. Words three, four, five are intuitively grouped together such as words white, blue, green
- Semantic: the emergence concerns the meaning of the numerals. This component is crucial for the construction of the world of numbers. The process of construction is divided into four developmental stages:
  1. The stage of opening of the world of numbers – the child begins to distinguish between the singular and the plural, i.e. between one and lots of.
  2. The stage of separated conceptions – the child already has the conception of what three balls means or what three fingers means, but perceives them in isolation. He/She already does not know that these conceptions represent the same amount of things.
  3. The stage of universal conceptions – the child knows that individual conceptions of numbers may stand one for another. Fingers or counters of an abacus are becoming universal models for the child. The achievement of this stage means the construction of the world of numbers that is associated with the world of things.
  4. The stage of abstract conceptions – the child is already able to manipulate with the conception of "three", "four", etc. meaningfully. He/She does not need to frame this conception within the world of things. The world of numbers gained independence.

There are various classifications of conceptions of natural numbers into classes and subclasses. In our paper, we will sort the children's conceptions into the following groups:

1. the natural number as a cardinal number, i.e. for counting (the number of dots on the face of a cube, the number of fingers on the hand, three pears, five cars, …)
2. the natural number as an ordinal number, i.e. for sequencing (the first in the finish, the third floor, …)
3. the natural number for identifying (the class 3.A; the bus No. 7; its 12 a' clock; …)

Within the experiment, we were working with children attending kindergarten. Therefore, we mention some skills and knowledge about numbers that children are supposed to know at the age of 5 and 6 (according to the educational activities realized in kindergarten):

– to recite the numbers from 1 to 6 in correct order,
– to create a group of objects with a stated number of elements (bring one scarf, pass me two papers, put aside two dolls, take one plate, …)
– to determine the number of objects in a given set (How many cars are there?)
  - by estimation (dots on the die, fingers on the hand),
  - by counting the one by one,
– to express the number of objects of a set with the use of fingers, dot symbols, respectively to assign the number symbol to a given group.

The experiment

The aims of the experiment

For the realization of the experiment we chose number 3, because it is the number from the numerical scope 1–6 and it is freely distributed in fairy tales, advertisements and other areas of ordinary life.

The following aims were stated:

1. to obtain drawn children's conceptions of numbers created according to the provided numerical information in the process (I $\rightarrow$ C),
2. to classify created number conceptions into stated groups and to analyze in which phase of formation of semantic component of the world of numbers do these conceptions appear,
3. to compare the number conceptions of children before and after the intervention of the experimentalist,
4. to create a concept map from children's drawings,
5. with the help of a video recording, to evaluate the behaviour and activity of children in the experimental activities.

The course of the experiment

The experiment was realized in January 2008 with the group of 20 children in the age of 5 to 6 years in the kindergarten in Nábrežie Mládeže Street in Nitra between 8 a.m. and 10 a.m. The lesson procedure was recorded on video. The experiment was divided into four phases. The first three phases covered the interaction between the experimenter and the children, the fourth one between the children and their teacher:

1. The first phase passed without our intervention. The children were asked to draw what they imagine when we say number 3. The process (I $\rightarrow$ C) was monitored, in which the numerical information was the word 3.
2. On the basis of the reactions of children we told them at the beginning of the second phase several examples from real life where they might come across the number (talk between experimenter and children):

   E: Has anybody seen something with number three in advertisements?
   Ch: Markiza (TV)
   E: Is Markiza with three? Where is number three there?
   Ch: Because you just press three on the remote control and that is Markiza.
   E: I live on the third floor. Is there anybody else living on the third floor?
   Ch: I live on the tenth. And I on the third.
   E: When you use an elevator, what do you press?
   Ch: Three.

   A chocolate bar 3Bit is shown to children.

   E: And what is this?
   Ch: Chocolate bar 3Bit.
   E: Do you know it? The three again!
   E: Adam was on the football tournament. Do you know what their rank in the order was?
   Ch: What does it mean "the order"?
   E: That somebody is the first, second, third.

We asked them to draw their imagination of number 3 again. We approached to children individually and talked with them about their picture and tried to help the weaker ones. Children presented their drawings. We observed how their conception of the number changed after our intervention.

3. In the third phase of the experiment, all the pictures were fixed to the board. Children were asked to arrange the pictures into groups according to some similarity and to explain their choice. Thus the concept map of children concerning their images of number three were created. At the end of the third phase the children were asked to look for the number three in their domestic environment.

4. The fourth phase passed without the presence of experimenter. The next day the teacher made a record about the reactions of children on this topic.

The evaluation of the experiment

The whole activity took 60 minutes. All children participated actively in the activity; they were concentrated for 35 minutes. After this time had passed, 10 children were able to concentrate on the next activity. Children that up to now are not able to concentrate for longer time on one activity kept running away from our activity. Two of them are diagnosed as ADHD.

The evaluation of the first and the second phase of the experiment

The pictures of children were arranged into the groups according to their conceptions of numbers. Children figured number 3 as the quantity or as an identifier. Six children produced rich pictures, however, without any connection to number 3; one child figured only the shape of number 3 that did not express any conception of the number.

After the dialogue was held with children about number 3 and its various forms around us, the children drew pictures related to number 3 again. In comparison with the first illustration, in the new portrayals number 3 appeared represented as an identifier more times, and for the first time as the order.

The illustration of number 3	The first phase Number of children	The second phase Number of children
Quantity	12	13
Order	0	2
Identifier	1	4
Shape of the number	1	1
No model	6	3

Table 1

Comparison of pictures from the first and the second phase

– three children among those six children who did not assign any concept to number 3 in the first phase could not react correctly even in the second phase. Two children produced models of quantity and one child depicted number 3 as the identifier. These models did not reproduce the conceptions of number 3 introduced by us. Children that did not create any model seemed in general to be very weak in other educational activities, too.
– Five children expanded their primal conception of number of new models as follows:

Number of children	The first phase → The second phase
2	quantity → identifier
1	identifier → quantity
2	quantity → order

Table 2

The enhancement of children's conceptions of numbers was influenced by our preceding dialogue.

Two children created interesting conceptions, which we have labelled “3 in 3”, i.e. in one conception of number 3 another conception of this number appears – three tables with three drawers, three cubes with three dots, three numbers 3 …

The first phase	The second phase
	

Figure 1. Figure of one child

The number conceptions of one child from the first and the second phase of the experiment are depicted in the Figure 1. There are many conceptions of number 3 in both pictures. The enrichment of number conceptions occurred in the second phase – the child did not depict number 3 only as the quantity, but also as the order – the third on the podium. The models “3 in 3” can be found in both phases. Following the richness of drawings, we can conclude that the child is aware of the fact that the same amount of different objects represents the same number, i.e. the child has reached the stage of *universal conceptions*. Similarly various pictures were produced by another four children. Other twelve children were able to assign to numerical information *I* only one or two conceptions of number 3. Thus it can be concluded that they so far do not realize that the same amount of different things represents the same number, they perceive them in isolation – they have reached only the stage of *separated conceptions.*

The evaluation of the third phase of the experiment

Each child presented his/her picture and fixed it to the board. After that, the children were asked to group those pictures that they think have something in common. Thus the groups of three trees, three flowers … were created. It can be seen that children grouped only the pictures depicting the same amount of the same things. They also created a group of two pictures depicting train, which, however, did not represent any conception of number 3. It can be concluded that children’s conceptions are at this age strongly fixed to the world of things.

We have produced the following concept map of number 3 together with children (Figure 2).

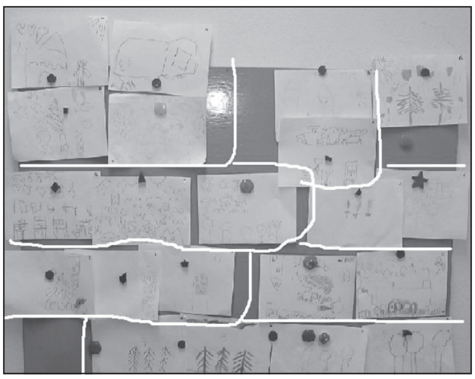

Figure 2. Concept map

The evaluation of the fourth phase of the experiment

The next day, all children, except for one that did not take part in the experiment, reacted to the appeal of the class mistress. Children created miscellaneous number conceptions, in which they presented number 3 as the quantity. These new concepts were independent from the previous shared activity and they manifested deep interconnection with the inner world of children. Children expressed the amount of number 3 with those objects from their lives, to which they have the emotional attitude.

Conclusion

Educational activities aimed at improvement of mathematical conceptions in nursery schools embody mainly in the process, ergo in assigning the numerical information to different number conceptions. In the experiment, the task of the children was to assign to given numerical information the different number conceptions. We were monitoring the process children do not commonly come across. The overall activity of children, as well as the richness and diversity of their drawings confirm that the preschool age children are playful and spontaneous in expressing their conceptions and feelings. The drawing proved to be a very appropriate form of expression, because even a withdrawn girl refusing to communicate with us verbally expressed herself in this way. Even the children unable to connect their conceptions with the world of numbers were drawing; moreover, they were able to react appropriately the next day, too. This confirms that our creative activity had particular influence on children of this age, even if they seemed to be passive at that time. The created concept map points to the ability of children to group the same number conceptions. Within the process of shared sorting of pictures, the children expressed themselves on the level of separated conceptions of numbers within the semantic component of the process of the emergence of the world of numbers from the world of things. Particular pictures are, however, the evidence of the fact that some children have already reached the stage of universal conceptions. Presented experimental activity was enhancing and interesting for the children, as they were able to react appropriately upon this topic and to look for another “threes” in their surrounding spontaneously even later on. We hope to pursue the improvement of number conceptions and the construction of the world of numbers in our further research on the chosen sample.

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Primary school pupils' miscoceptions in number

A group of four universities from the UK, Czech Republic, Israel and Italy collaborated to find common misconceptions in number across the four countries. The paper cites some of the misconceptions founding all the countries and looks in detail at a specific task which the authors developed together with the results gained from the task over the whole primary range of pupils and beyond.

Introduction

Over the past two years researchers in four universities in the United Kingdom, the Czech Republic, Israel and Italy have been looking into the misconceptions which primary school pupils have in mathematics. What are the major misconceptions? How do they arise and what can we do to eliminate them? Are specific misconceptions only found in one country or are some of them found across the four countries involved in the project?

These were some of the questions we asked ourselves at the first meeting. We also were of the same opinion that mistakes/misconceptions should not been seen by the teacher and the pupil to be avoided at all costs and that some form of punishment should be given for making mistakes. We firmly believe that mistakes can be used as a great educational tool to bring about understanding of the underlying concepts behind a particular piece of mathematics.

We clarified in our own minds the difference between mistakes and misconceptions. A mistake often occurs because a pupil is trying to work too quickly and thus miscopies, makes a simple mistake in a calculation or makes an error remembering one of the many facts which s/he has had to commit to memory such as 7 x 8 = 54! Where as a misconception, as the name suggests means that the pupil does not understand the concepts on which the particular mathematical topic is based. For instance a pupil who has been told that when you multiply by 10 you 'add a zero to the right hand side of the number you are multiplying', when asked the cost of 10 pens which cost Ł1.35 each gives the answer Ł1.350. In other words the pupil does not know why s/he should add a zero in the first place probably because he got the knowledge second-hand either from the teacher or a peer. The important point being that the 'rule' gave the correct answers for the level of mathematics the pupil was doing at the time but was never told that the rule only applied to natural numbers.

Background

Many books and papers have been written on misconceptions in primary mathematics but few if any have looked at the underlying reasons for these, for instance Gelman and Gallistel, (1978), Hughes, (1986). Several books have been written on specific problems which arise from children's misunderstanding of concepts such as the equality sign and zero, (Haylock, Cockburn, 2008 and Lakoff, Núńez, 2000 on matters relating to zero and Jones, 2006, relating to equality). Pupils use the equality sign in several ways, first and foremost as the completion of a process such as 3 + 4 = 7 to which they use the words '3 add four is 7'. They seldom see it as an equivalence as in 3 + 4 = 1 + 6 or as in the case, when as far as the pupils are concerned the question is set the wrong way round, 6 = 2 + ?, I am sure we have all seen the pupil who strings equality signs together which do not make sense but still the pupil gets the right answer, for instance $32 + 56 = (30 + 2) + (50 + 6) = (30 + 50) = 80 = (2 + 6) = 8 = 80 + 8 = 88$.

Similarly in our research we have found many instances of pupils not understanding that zero has a quantity. They only have the idea that it is a place holder. Later in the paper we will give examples of students crossing out the digit 1 rather than zero when they say they are crossing out the 'smallest' digit. This was found not only with primary school pupils but also with pupils up to Grade 10.

Obviously the problems with zero are also linked to problems with place value. Our research supports the view expressed by Ashlock (2002) that pupils are taught the value of the digits 1 to 9 and the concept of the value of the places in the denary system, that is if you ask a young child to bring 4 sweets from then dish then this can be done, and if you point to a particular place they can probably tell you 'those are tens' but these two concepts are rarely put together so that the pupil can recognise that the value of the 2's in 203 and 472 are very different. A feature of modern life has contributed to this. Rarely do you hear a pupil say the number 125 as one hundred and twenty five, it is more usually read as one, two, five! Hence the pupils are not recognising the value of the digit in its place setting.

When dealing with decimals in the upper primary school we found exactly the same misconceptions which had been found in the long-term UK research by the APU (1975–80,1985) For instance when the number of digits after the decimal point in one number is different from the number in the second then this causes problems eg. To the sum 5.07–1.3 three types of misconception arose giving the answers: 4.4, 4.04, 4.94. In the first answer the pupil ignored the zero in the first number and treated the seven hundredths as seven tenths. In the second the three was put under the 7, the decimal points under each other and the units under each other, is this a case of zero meaning 'nothing'? The third answer was derived from putting the 1.3 under the 0.07 ignoring the decimal point in the second number and then subtracting. These results all came from one class.

Another area which high-lighted many misconceptions were tasks dealing with the number line which was unmarked apart from the end points. Values near the endpoints were generally approximately true but in over 95% of the pupils work there was no idea of proportionality when putting numbers on the line. For instance when Grade 2 pupils were asked to put 0, 5 and 8 on a number line showing the position of 1 and 10 most of the answers given had all the numbers between 1 and 5 on the line. Many older pupils showed decimals of the form 0.25 as being below zero.

Experiment

We obviously could extend the misconceptions we found considerably but the authors had specific tasks to test across the four countries. At one of the first meetings of the team, various misconceptions which the members had met were discussed and some ten tasks devised which it was hoped would show whether the misconceptions were found in just one of the countries or across all of them. The experimental design was the same for all the tasks in all the countries. We involved teachers we knew and with whom we had worked before, presenting them with the tasks which were carefully aimed at specific age-groups or syllabus development and asked them to administer the tasks on our behalf. Before they gave the pupils the tasks several in-service sessions were given at which the tasks were discussed, what we hoped they would show and all the tasks were attempted by the teachers so that they were clear what was required.

In this paper we want to concentrate on one task for which we devised four levels. The basic task which was set was:

Given an n-digit number, (n being dependent on the age/ability of the pupil), strike out a digit so that what is left is the largest possible (n –1)-digit number. The digits in the original number must not be reordered. Starting again with the original number, strike out a digit to make the smallest possible (n – 1) digit number. Thus 'n' in the case of 594 would be 3 with the largest 2-digit (i.e. n–1) number result being 94 and the smallest 54.

The numbers we gave the four levels were determined by the number syllabuses in the four countries involved. If pupils were working on two digit numbers we gave them three digits in the task, for them to cross out one as so reduce the problem to a two digit number which should have been within the pupils' competence. This methodology was continued through the levels.

Level I. (Knowledge 0 to 20) 213, 120.

Level II. (Knowledge 0 to 100) 2109, 892.

Level III. (Knowledge >1000) 23015, 15023.

Level IV. 352091, 432502.

We developed this task because our experience in schools suggested that some pupils had 'rules/strategies' in their memories which they applied to determine largest and smallest numbers. The ones we had met were:

- Strategy 1: Cross out the 'smallest' digit to get the largest number;
- Strategy 2: Cross out the 'largest' digit to get the smallest number;
- Strategy 3: Cross out the right hand digit to get the largest number;
- Strategy 4: Cross out the left-hand digit to get the smallest number;
- Strategy 5: Cross out the zero to get the largest number.

Table 1. Strategies met in school prior to research

Hence our objectives when designing the task were to:

(i) give us insight into the pupils' knowledge and understanding of place value. So, for example, in the case of 213 do children consider *place value* (as we would hope!) or opt for crossing out the 'largest' digit (i.e. 3) when endeavouring to create the smallest 2-digit number?
(ii) see if there were common misconceptions across the four countries of the project. If so, what could we learn, for example, about the different teaching methods used?
(iii) analyse the misconceptions to determine their origin. This included looking for patterns in the children's responses to see if, for instance, they responded correctly to all the tasks they were given excepting those involving zero(s).
(iv) how early in their school life did these strategies occur?
(v) provide ideas for re-education/education to eradicate the misconception(s).

To give an example of the range of possible answers we could expect if all the strategies listed above were used by a class, we have tabulated the possible results for level 1 task, using the numbers 213 and 120

Number	**Correct solution**	**Smallest/largest digit – strategies 1 and 2**		**Right/left digit – strategies 3 and 4**		**Zero, largest smallest – strategy 5**	
213							
Smallest	**13**	21		13			
Largest	**23**		23		21		
120							
Smallest	**10**	10		20		12	
Largest	**20**		12		12		12

Table 2. Possible answers to level 1 task using strategies above

Note: for the number 120, if the pupil uses the right/left digit strategy then the 'smaller' number determined is bigger than the determined 'larger' number. This actually happened in several cases and would suggest that these pupils had a strategy in their minds which they were convinced would give them the correct answer, so they never checked to see whether the results they gave were sensible or not.

Results

Analysing three classes of 58, 6 to 7 year-old pupils who worked with the numbers 213 and 120 the following facts emerged:

For **213**

Largest number

Cross out the smallest digit (correct answer)	27 pupils
Cross out the right-hand digit (units)	10 pupils
Cross out the left-hand digit	12 pupils

Smallest number

Cross out the left-hand digit (correct answer)	24 pupils
Cross out the largest digit	7 pupils

Less than half the 58 pupils got the two parts of the task correct. There were 9 pupils who used the twin strategies 'cross out the left-hand/right-hand digits for the smallest/largest numbers respectively'. No pupil who used crossing out the smallest digit to get the biggest number used cross out the biggest digit to get the smallest number.

In one of the three groups the pupils must have had some instruction of how to find the smallest number for 213 since the whole group gave the same incorrect solution, cross out the '3' and then reversed the remaining digits '21' to get the number 12. Their solutions for the other tasks were not significantly different from the other two groups.

For **120**

Largest number

Cross out the left-hand digit (correct answer)	42 pupils
Cross out the smallest (zero, right-hand) digit	12 pupils

Smallest number

Cross out the biggest digit (correct answer)	33 pupils
Cross out the right-hand (zero) digit	19 pupils

Eleven pupils used the twin strategies 'cross out the left and right-hand to get the smallest and largest numbers respectively'. Zero caused many pupils difficulties since many are not sure of its function. They probably have been told that 'zero is a place holder' or that 3 – 3 'is nothing' which is then written as '0'. Another phenomenon connected with zero arose with older pupils.

Looking across the three groups different phenomena occurred in each group. In one group the most common misconception for finding smallest number for 213 was crossing out the '3'– the largest digit. Another group had different misconceptions for finding the largest number in 213, in fact contradictory misconceptions since the same number of pupils crossed out the right digit as crossed out the left one. In this class more pupils crossed out the right digit zero than got the correct answer. A number of pupils re-arranged the digits if their strategy did not seem to give the expected result, even though they were told at the start of the tasks not to alter the order of the digits.

Thus we did not have to look very far to see how early these inappropriate strategies were used. Responses to the level 2 task, showed that all the strategies used in level 1 appeared again at this level with crossing out of the largest and smallest digits being the dominant misconception of these pupils whose ages ranged from 8 to 10 years. A new phenomenon appeared with these pupils. Some pupils who said they were crossing out the smallest digit to get the largest number in 2109 and 9120 crossed out the '1' and not the zero getting 209 and 920 respectively. This gives the incorrect answer for the first number and a correct answer for the wrong reason in the second. We considered that this was evidence that these pupils did not see zero as a digit having a value but purely as a place holder in the denary place value structure.

At level 3 exactly the same misconceptions occurred with those classes working with five digit numbers as occurred earlier. That the misconceptions are perpetuated is worrying since it means that these pupils have not been given tasks which will help to identify these misconceptions or possibly the teachers have marked an answer correct which was obtained by incorrect reasoning.

We gave the level 4 task to secondary as well as primary school pupils and even with pupils as old 14 to 15 years the same misconceptions as were found in Grade 1 were apparent. Many grade 6 pupils used the twin strategies 'cross out the right hand digit to get the biggest number and the left-hand digit to get the smallest number' for 352091, getting 35209 and 52091 for the biggest and smallest numbers respectively! Even at this age there were some pupils who crossed out the '1' rather than the zero.

Conclusions

The important results of our analysis were:

- the strategies listed earlier in the chapter were prevalent in Grade 1 and were found in every grade up to 10;
- most pupils were inconsistent in the strategies they used to solve the problems both across tasks and within tasks. This would suggest that these pupils considered each task individually. Not many pupils used both of the twin methodologies – smallest/largest digit or right/left-hand digit to solve one task;
- pupils did not check to see if their answers were sensible;
- many pupils did not connect the cardinal value of the digit with the place value where it was situated;
- very few pupils took a number and crossed out the digits in turn to determine smallest and largest.

Teachers need to give their pupils tasks which will high-light misconceptions and these particular misconceptions are particularly hard to eradicate because they sometimes give the right answer for the wrong reason and in many cases have been used throughout their school career.

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Focusing on children's early ideas of fractions

This paper describes children's understanding of quantities represented by fractions in quotient, part-whole and operator situations. The studies involve two samples of first-grade children, aged 6 and 7 years from Braga, Portugal. These children were not taught about fractions before. Two questions were addressed: (1) How do children understand the equivalence of fractions in quotient, part-whole and operator situations? (2) How do they master the ordering of fractions in these situations? Quantitative analysis showed that the situations in which the concept of fractions is used affected children's understanding of the quantities represented by fractions; their performance in quotient situations was better than their performance in the other situations.

This paper aims to describe part of a research project focused on the effects of situations on children's understanding of the concept of fraction.

Independent thinking is only possible with understanding. To improve children's understanding of a mathematical concept one needs to know how the concept develops. According to the Vergnaud's (1997) theory, to study and understand how mathematical concepts develop in children's minds through their experience in school and outside school, one must consider a concept as depending on three sets: a set of situations that make the concept useful and meaningful; a set of operational invariants used to deal with these situations; and a set of representations (symbolic, linguistic, graphical, etc.) used to represent invariants, situations and procedures. Following this theory, this paper describes studies on children's informal knowledge of quantities represented by fractions, focused on the effects of situations on children's understanding of the concept of fraction.

Different classifications of situations that might offer a fruitful analysis of the concept of fractions are distinguished in the literature. (Kieren, 1988, 1993) distinguished four types of situations – measure (which includes part-whole), quotient, ratio and operator – referred by the author as 'subconstructs' of rational number, considering a construct a collection of various elements of knowing; Behr, Lesh, Post and Silver (1983) distinguished part-whole, decimal, ratio, quotient, operator, and measure as subconstructs of rational number concept; Marshall (1993) distinguished five situations – part-whole, quotient, measures, operator, and ratio – based on the notion of 'schema' characterized as a network of knowledge about an event. More recently, Nunes, Bryant, Pretzlik, Evans, Wade and Bell (2004), based on the meaning of numbers in each situation, distinguished four situations – part-whole, quotient, operator and intensive quantities. In spite of the diversity, part-whole, quotient and operator situations are common to these classifications. These were the three situations selected to be included in the studies reported here.

In part-whole situations, the denominator designates the number of parts into which a whole has been cut and the numerator designates the number of parts taken. So, 2/4 in a part-whole situation means that a whole – for example – a chocolate was divided into four equal parts, and two were taken. In quotient situations, the denominator designates the number of recipients and the numerator designates the number of items being shared. In a quotient situation, 2/4 means that 2 items – for example, two chocolates – were shared among four people. Furthermore, it should be noted that in quotient situations a fraction can have two meanings: it represents the division and also the amount that each recipient receives, regardless of how the chocolates were cut. For example, the fraction 2/4 can represent two chocolates shared among four children and also can represent the part that each child receives, even if each of the chocolates was only cut in half each (Mack, 2001; Nunes, Bryant, Pretzlik, Evans, Wade & Bell, 2004). In operator situations, the denominator indicates the number of equal groups into which a set was divided and the numerator is the number of groups taken (Nunes et al., 2004). In an operator situation, if a boy is given 2/4 of 12 marbles, means that the 12 marbles are organized into 4 groups (of 3 marbles each) and the boy receives 6 marbles – that is 2 groups of the 4 into which the 12 marbles were organized. Thus number meanings differ across situations. Do these differences affect children's understanding of fractions when building on their informal knowledge?

Applying Vergnaud's (1997) theory to the understanding of fractions, one also needs to consider a set of operational invariants that can be used in these situations. Extending Piaget's analysis of natural numbers to fractions, one has to ask how children come to understand the logic of classes and the system of the asymmetrical relations that define fractions. How do children come to understand that there are classes of equivalent fractions – 1/3, 2/6, 3/9, etc – and that these classes can be ordered – 1/3 > 1/4 > 1/5 etc? (Nunes et al., 2004). It is relevant to know under what condition children understand these relations between numerator, denominator and the quantity. The invariants analysed here are equivalence and ordering of the magnitude of fractions, more specifically, the inverse relation between the quotient and the magnitude.

Thus these studies considers a set of situations (quotient, part-whole, operator), a set of operational invariants (equivalence, ordering of fractional quantities), using linguistic combined with pictorial representation. In this paper we investigate whether the situation in which the concept of fractions is used influences children's performance in problem solving tasks. The studies were carried out with first-grade children who had not been taught about fractions in school. Two specific questions were investigated: (1) How do children understand the equivalence of fractions in part-whole, quotient and operator situations? (2) How do they master the ordering of fractions in these situations?

Previous research (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 2005) on children's understanding of division on sharing situations has shown that children aged 6 and 7 understand that, the larger the number of recipients, the smaller the part that each one receives, being able to order the values of the quotient. However, this studies were carried out with divisions were the dividend was larger than the divisor. It is necessary to see whether the children will still understand the inverse relation between the divisor and the quotient when the result of the division would be a fraction. The equivalent insight using part-whole situations – the larger the number of parts into which a whole was cut, the smaller the size of the parts (Behr, Wachsmuth, Post & Lesh, 1984) – has not been documented in children of these age. Regarding equivalence in quotient situations, Empson (1999) found some evidence for children's use of ratios with concrete materials when children aged 6 and 7 years solved equivalence problems. In part-whole situations, Piaget, Inhelder and Szeminska (1960) found that children of this age level understand equivalence between the sum of all the parts and the whole and some of the slightly older children could understand the equivalence between parts, 1/2 and 2/4, if 2/4 was obtained by subdividing 1/2. Concerning operator situations, previous research on children's informal knowledge (Empson, 1999) shows that children aged 6 and 7 found it difficult to understand the operator concept.

Although some research has dealt with part-whole, quotient and operator situations with young children, these were not conceived to establish systematic and controlled comparisons between the situations. There have been no comparisons between the three situations in research on children's understanding of fractions. Research about the impact of each of these situations on the learning of fractions is difficult to find. We still do not know much about the effects of each of these situations on children's understanding of fractions. This paper provides such evidence.

Methods

Participants

In a first study, Portuguese first-grade children (N = 80), aged 6 and 7 years, from the city of Braga, in Portugal, were assigned randomly to work in part-whole situations or quotient situations with the restriction that the same number of children in each level was assigned to each condition in each of the schools. In a second study, another group of Portuguese first-grade children (N = 40), aged 6 and 7 years, from the same two schools were working in operator situations with the same restriction for each level in each of the schools. The children had not been taught about fractions in school, although the words 'metade' (half) and 'um-quarto' (a quarter) may have been familiar in other social settings.

The tasks

An example of each type of task presented to the children is given below (Table 1). The instructions were presented orally; the children worked on booklets which contained drawings that illustrated the situations described. The children were seen individually by an experimenter, a native Portuguese speaker.

Problem	Situation	Example
Equivalence	Part-whole	Bill and Ann each have a bar of chocolate of the same size; Bill breaks his bar in 2 equal parts and eats 1 of them; Ann breaks hers into 4 equal parts and eats 2 of them. Does Bill eat more, the same, or less than Ann? Why do you think so?
	Quotient	Group A, formed by 2 children have to share 1 bar of chocolate fairly; group B, comprising of 4 children have to share 2 chocolates fairly. Do the children in group A eat the same, more, or less than the children in group B? Why do you think so?
	Operator	Anna and Martha each have a bag with 4 marbles. Anna splits hers into 2 equal groups and puts 1 group in her red bag. Martha splits hers into 4 equal bags and decides to put 2 groups in her blue bag. Does the red bag have more marbles than the blue bag? Does the blue bag have more marbles than the red one, or do they have the same number of marbles? Why do you think so?
Ordering	Part-whole	Bill and Ann each have a bar of chocolate the same size; Bill breaks his bar into 2 equal parts and eats 1 of them; Ann breaks hers into 3 equal parts and eats 1 of them. Who eats more, Bill or Ann? Why do you think so?

Problem	Situation	Example
Ordering	Quotient	Group A, formed by 2 children has to share 1 bar of chocolate fairly; group B which consists of 3 children has to share 1 chocolate fairly. Who eats more, the children of group A, or the children of group B? Why do you think so?
	Operator	Eve and Ruth each have a bag with 6 lollypops. Eve splits hers into 2 equal groups and puts 1 group in her red bag to eat later. Ruth splits hers into 3 equal bags and decides to put 1 group in her blue bag. Does the red bag have more lollypops than the blue bag? Does the blue bag have more lollypops than the red one, or do they have the same number of lollypops? Why do you think so?

Table 1. Types of problem presented to the children in each type of situation

Design

In both studies, the six equivalence items and the six ordering items were presented in a block in random ordered at the beginning of the session. The numerical values were controlled for across situations.

Results

Study 1

Descriptive statistics for the performances on the tasks for quotient and part-whole situation are presented in Table 2.

	Problem Situation			
	Quotient (N = 40; mean age 6.9 years)		Part-whole (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	2.1 (1.5)	2.95 (1.54)	0.6 (0.7)	0.6 (0.5)
Ordering	3.3 (2.1)	4.25 (1.3)	1.45 (1.4)	1.2 (0.83)

Table 2. Mean (out of 6) and standard deviation (in brackets) of children's correct responses by task and situation

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7-year-olds) and problem solving situation (quotient *vs* part-whole) as between-participants factor, and tasks (Equivalence, Ordering) as within-participants factor.

There was a significant tasks effect, ($F(1,76) = 18.54$, $p<.001$), indicating that children's performance on ordering tasks was better than in equivalence tasks. There was a significant main effect of the problem situation, ($F(1,76) = 146.26$, $p < .001$), and a significant main effect of age, ($F(1,76) = 4.84$, $p < .05$); there was a significant interaction of age by problem solving situation, ($F(1,76) = 7.56$, $p <.05$). The older children performed better than the younger ones in quotient situations; in part-whole situations there was no age effect. There were no other significant effects.

An analysis of children's arguments was carried out and took into account all the productions, including drawings and verbalizations. Table 3 shows the frequency of children's arguments and the rate of correct responses for problems in quotient and part-whole situations. Children presented more valid arguments based on the inverse relation between the number of recipients and the size of the shares, when solving problems in quotient situations. In part-whole situations, the valid arguments were based on the inverse relation between the number of parts into which the whole was cut and the number of parts eaten/taken. In part-whole situations the most frequent arguments used when were based on the number of parts eaten/taken, ignoring their sizes and the number of parts into which the whole was cut.

	Type of situation							
	Quotient (N = 240)				Part-whole (N = 240)			
	Equivalence		Ordering		Equivalence		Ordering	
Type of argument	Freq.	Prop.	Freq.	Prop.	Freq.	Prop.	Freq.	Prop.
Invalid	17	0	17	.01	10	.01	6	.02
Perceptual comparisons	46	.03	50	.09	–	–	–	–
Valid	88	.27	94	.38	14	.03	15	.06
Only to the dividend (numerator)	76	.09	64	.14	172	.18	177	.13
Only to the divisor (denominator)	13	.03	15	.01	44	.05	42	.01

Table 3. Frequency of arguments type and proportion of correct responses when solving the tasks in quotient and parte-whole situations

Study 2

	Problem Situation			
	Part-whole (N = 40; mean age 6.9 years)		Operator (N = 40; mean age 6.9 years)	
Tasks	6 years	7 years	6 years	7 years
Equivalence	0.6 (0.7)	0.6 (0.5)	2 (1.9)	2.6 (1.9)
Ordering	1.45 (1.4)	1.2 (0.83)	2.6 (2.2)	2.7 (2.1)

Table 4. Mean (out of 6) and standard deviation (in brackets) of children's correct responses by task and situation

Descriptive statistics for the performances on the tasks for part-whole and operator situation are presented in Table 4.

A three-way mixed-model ANOVA was conducted to analyse the effects of age (6- and 7--year-olds) and problem solving situation (operator *vs* part-whole) as between-participants factor, and tasks (Equivalence, Ordering) as within-participants factor. There was a significant mains effect of tasks ($F(1,76) = 15.23$, $p < .001$), indicating that the children's performance in ordering was better than in equivalence problems. There was a significant main effect of the situations, ($F(1,76) = 22$, $p < .001$), indicating that the children's performance was better in operator than in part-whole situations. There was no significant age effect and no significant interactions.

	Operator Situation (N = 240)			
	Equivalence		Ordering	
Type of argument	Freq.	Prop.	Freq.	Prop.
Invalid	2	0	1	0
Based on number of units	229	.36	231	.42
Valid	4	0	4	0
Only to the dividend (numerator)	5	.02	4	.02

Table 5. Frequency of arguments type and proportion of correct responses when solving the tasks in operator situations

Children's success on solving problems in operator situations relies on the comparison of the number of units, ignoring the existence of groups and judging the number of units by counting. The inverse relation between the divisor and the quotient is lost in operator situations.

Discussion and conclusion

The situations in which fractions are used have an effect on children's understanding of fractions. Children's ability to solve problems of equivalence and ordering of quantities represented by fractions is better in quotient than in other situations. The levels of success in children's performance in quotient situations, supports the idea that children have some informal knowledge about equivalence and ordering of quantities represented by fractions. These results extend those obtained by Kornilaki and Nunes (2005), who showed that children aged 6 and 7 years succeeded on ordering problems, in sharing situations, where the dividend was larger than the divisor. These results showed that the children still be able to use the same inverse reasoning when dealing with quantities represented by fractions. The findings of these studies also extended those of Empson (1999) who showed that 6-7-year-olds children could solve equivalence and ordering problems in quotient situations, after being taught about equal sharing strategies. The children of these studies were not taught about any strategies. These findings suggest that children possess an informal knowledge of fractions that can be successfully explored using quotient situations. If it is so, why should we keep introducing the concept of fraction to children using part-whole situations? Maybe we should explore more about which is the best situation to introduce children to fractions in the classroom. What sequence of situations should be explored in the classroom to offer a better support to children's independent thinking about fractions?

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Connections – as a fundamental element to constructing mathematical knowledge (exemplify of one pre-algebra task)

Polish primary school do not talk about algebra a lot. Teaching mathematics on this level of education is oriented on arithmetic. In my paper, I would like to present a part of results from my research carried out in a primary school among fourth grade students. This research concerns discovering the regularity, which leads to algebraic reflection.

Introduction

During the last few years, much time has been spent on discussion about learning algebra in a primary school. We say about an "early algebra" and algebraic thinking. (Mutschler, 2005). It is very hard to separate an algebraic thinking from an arithmetic one. One of the ways of developing algebraic thinking is a "superstructure" of arithmetic thinking. It is made by means of generalizing arithmetic contemplations through un-changing constants. Tasks that concern discovering of arithmetic-geometric dependences, which it is necessary to generalize and write by symbols, serve this purpose. It is very hard for the primary school's student to move to symbolic notation. First of all, the student says the general relation and next s/he tries to notice it by symbols. In order to understand algebraic language, the student has to start from understanding its basic component – the letter (Turnau 1990).

Discovering and perceiving regularity by students is an important problem present in the world trends in teaching mathematics. In many countries, teaching mathematics is closely connected with the rhythm and the regularity. We can find references to the description of researches concerning discovering and generalization of noticed rules (Zazkis, Liljedahl, 2002; Littler, Benson, 2005).

Searching for a regularity is an extremely effective method while solving mathematical problems; it is a strategy of solving tasks. As E. Swoboda (2006) says:

> … perceiving a regularity is a desirable skill. Activities, during which the child has to perceive the regularity and act accordingly to the rule are stimulating his or her mental development. These activities are the basis of mathematical thinking for each level of mathematical competence.

The child learns/develops its mathematical knowledge through building its own cognitive structures, a web of interrelationships, mental "maps" (Hejný, 2004, Skemp, 1979). Accumulated experience enables to create a so-called data set, used by a child to build up its mathematical knowledge. This inner structure of mathematical knowledge (internal mathematical structure – IMS, Hejny 2004) is – as prof Hejny says:

> ... dynamic web of connections with many elements of knowledge, such as concepts, facts, relations, examples, strategy of solutions, algorithm, procedures, hypothesis, ..., creating nodes of this web. All this cause the existence of IMS. IMS is a web by itself, connecting these all elements. At the same time IMS is the way of organizing all these elements which create knowledge.

The essential factors which help a child to develop its mathematical knowledge are interactions with its environment, particularly during the teaching-learning process (e.g. during mathematics classes). It is present during a teacher-student interaction and a student-student interaction as well. The best way to activate these interactions is a group work in cooperating teams.

The appearance of a reflection is very important. Reflection on our experience is a perfect starting point for understanding the world (constructivism). Everyone creates his own 'rules' and mental models, which we try to apply in order to understand and use our experience of mastering the knowledge of our environment. The reflection appears when we have to manifest our ideas. While expressing our thoughts, we look for an appropriate form of words (Wygotski, 1989). Reflection does not appear automatically among 7–11 children. Therefore, a conversation during a cooperation with students is an opportunity to recognize their mental processes while solving their tasks

The aim of research

I have been dealing with a perception of regularities and appliance of discovered rules by students for some time. Presented investigation is a part of a series of research concerning the perception of regularities by students on different levels (Pytlak, 2006, 2007). The results of Polish students from PISA test and one of PISA's task (called "Apple trees" – Białecki, Blumsztajn, Cyngot., 2003) were an inspiration to take on this subject.

The aim of my research was to get answers for following questions:

- Will 9–10-years old students be able to perceive mathematical regularity and, if yes – in which way do they "think" about regularities and what is their thinking processes while solving tasks in which they have to discover and use noticing rules?
- Will they be able to cooperate while solving the task?
- To what degree this common work will have an effect on the way of solving the task and discovering the regularity as well as using them in the task?

Presented research was carried out in February 2008 among students from a fourth grade of primary school. The research contained four following meetings, during which students were solving following tasks. All meetings were recorded by a video camera. After the research, the report was made. Students worked in pairs. Researcher talked with every group of students while solving tasks by them.

Twelve students from fourth grade of primary school took part in this research (9–10 years old children). Students had work sheets, matches (black sticks), ball pens and a calculator. Before students started their work, they had been informed that they can solve this task in any way they would recognize as suitable; their work would not be graded; teacher would be videotaping their work and they can write everything on the work sheet which they recognize as important. The research material consists of work sheets filled by students, as well as of a film recording of their work and a stenographic record from it.

The research tool consisted of four sheets and each of them consisted of two tasks.. Tasks were as following: students make a match pattern consisting of geometrical figures – once there are triangles and another time there are squares with a side length of one match. In the first two sheets the figures were arranged separately, in the second two – connected in one row. The question was: How many matches do you need to construct 1, 2, 3, 4, 5, 6, 7 of such figures? Results should be written in the table. In the task two, there was a question about a number of matches which are needed to construct 10, 25 and 161 of such figures (Littler 2006).

Patterns, which were the subject of next tasks, were as following:

1. Separate triangles
2. Separate squares
3. Connected squares
4. Connected triangles.

Choice of tasks and the order of sheets were not random. The problem was to check if students will benefit from their earlier experience while solving new tasks; will once elaborated solving strategy be applicable during the next task.

This task and the way of its presentation (four following sessions) were something new for students. Up to this time, during math lessons they did not solve tasks concerning the perception of the appearing rules and generalization of noticed regularities. It was a new challenge for them.

The results

The first two sheets students solved very quickly. They did not need to construct the pattern consisting of proper figures, they already started to fill in the table and next, they answered the question two. They were perfectly able to give the rule according to which the pattern was constructed. Some of students arranged only one figure (one triangle in case of the first sheet and one square in case of the second sheet). It was rather a kind of marking what kind of figure was applied in the task than supporting the task solving process.

Difficulties appeared during the work with the third sheet. The first difficulty concerned the expression: "connected in one row". Next obstacle appeared while shifting from the table to the question two. Students did not have enough sticks in order to continue arranging the pattern. Besides, in the table they gave consecutive values and in the question two a "jump" appeared. At the beginning, the problem for students was to fill this gap. In order to give the answer, they started to analyze the previous solution of the task and the way of constructing the pattern.

Solution strategies were as following:

<u>The first sheet</u> – filled automatically; the discovered rule is: add three to a previous value (relate to the table) or multiply the number of triangles by three. All answers and formulated rules were correct, students were able to make a generalization, they did not use any symbolic notation.

<u>The second sheet</u> – students noticed an analogy to the previous task (from the first sheet). Some of them applied the rule "multiply by three"... Solving this task lasted less time than in the case of the first sheet.

<u>The third sheet</u> was a challenge for students. At the beginning, they were trying to transfer a solving method from previous sheets. But seeing that it is ineffective, they looked for another solution. They started to analyze contents of the task. Next, they arranged, using matches, a fragment of a pattern – for two, three squares. They discovered the rule: the first square made of four elements, each following of three elements. Therefore, in order to give the number of needed matches, one should add three to the previous number. After filling the table, two ways of actions appeared: continuation of "adding numbers three" to ten squares or searching for "components" in the table, using previously obtained data. Initially, students were convinced of correctness of their method. Only a conversation with the teacher, as well as verification of the applied method for data from the table (i.e. will it be as such for the number seven) caused the change in the way of thinking and discovering the proper dependence.

<u>The fourth sheet</u> was also a challenge for the students. Here however, they used their own experience gained while working on the third sheet, so the solving process of the task progressed quite efficiently.

I would like to look closer at the solution of the task from the third sheet, which was made by two girls: Sylwia and Nicola.

Sylwia and Nicola's work

Nicola is a girl who copes with school mathematics very well and operates with knowledge on the abstraction level (that is available on her level). Sylwia is a much worse student who needed a visual representation to solve the task. While solving tasks from two previous sheets, the girls did not work together. Sylwia worked with Paulina. Before they started to solve the task, Sylwia arranged a few triangles, and next she used it to fill the table together with Paulina.

As a justification for their activity concerning the table, they gave the rule: every time I add three. Moved to the second question, girls changed the rule: multiply the number of triangles by 3. As a general rule they gave: it is needed to multiply the number of triangles by 3 because each triangle has 3 sides. In the case of the second sheet, both girls worked in a similar way. In the table they added 4, and in the second question they multiplied by 4. As a general rule they gave: multiply the number of squares by 4. Two first sheets Nicola filled individually. She did not arrange any figures, at once she moved to the action of filling in proper values in the table. From the very beginning she used correct rules, she was able to generalize them for any element as well. Both in Sylwia and Paulina's solution, and in Nicola's work as well, there was no symbolic notation but only verbal expression of the general rule.

Only during the work with the third sheet, Nicola and Sylwia joined together in one team. Girls cooperated very harmoniously. Nicola allowed Sylwia to solve the task first, and when Sylwia had some trouble with them, Nicola took the initiative and afterwards she told her friend to repeat the whole reasoning. Nicola took the role of a teacher. The teacher was only an observer.

At the beginning Sylwia did not understand what does "in one row" mean. Nicola explained that to her by arranging the pattern with the sticks on the desk and commentating the manner of its development:

1. S: [*reads aloud contents of task*] … what does it mean "in one row"?
2. N: Like this [*she shows by hand a row on the desk*] (…) Look, construct the square [*Sylwia is constructing a square*]. And now you are building the second like this [*Nikola is adding three matches to a square constructed by Sylwia*]. You see, you have three here. And you keep on constructing like that.
3. S: Right
4. N: So in the first one there will be four, and next you will add three
5. S: Aha, I know it now. (…) [*she is starting to fill in the table*] well for one square there will be four, (…) now two times three, because there are three matches here [*she takes away one match from the first square*]
6. N: [*she is adding a match*] now look. There are two squares. We do not add three and three, here there was only four [*she shows the first square*]…
7. S: Seven
8. N: Yes, seven. For this one you add three so that is
9. S: Ten.

Sylwia did not understand the way in which the pattern emerges. Nicola showed the manner of building next squares to her friend. For Nicola only two elements were sufficient to "see" the whole and to understand the general rule of building the pattern. For Sylwia it was too little. After two elements, she did not see the whole structure. The expression "here you have three, and you go on arranging like that" [1] or "next you add three" [3] could have been associated with two previous tasks.

It was adding the same numbers all the time, and it had a reference to the general rule: multiply the number of figures by the number of matches which you have to add. Hence Sylwia applied a solution: for two squares it will be two times three matches – because I am adding three. Only one more explanation by Nicola concerning the manner of arranging and paying attention to the fact that for the first square we use four matches and for each next square only three matches, caused that Sylwia understood the matter of the task and filed the table correctly.

In order to answer the question about 10 squares, the girls "extended" the table, adding by three to the previous number until they received 10 squares. This strategy was very clear and understandable for the girls, and it directly rose from previous established strategy of solving the task. At this stage girls did not discover any dependences between numbers of squares and number of matches which occurred in the pattern.

After having answered the question about 10 squares, Nicola decided to check if the received result is correct. Maybe in this way she wanted to anticipate the teacher's question, as the teacher, after every solution of the task from the previous two sheets stated questions such as: "why will it be that result?", "how do you know that it will be like this?". And maybe she intuitively used, postulated by G. Poly (, "a glance backward", and by this she showed a great mathematical maturity. Verification of correctness of the result happened as follows:

10. N: … Ten times four, or full squares, it will be forty. And now not all were with four matches, nine were with three, weren't they?
11. S: Yes
12. N: So, then we subtract these nine matches, that is thirty one, a kind of a verification.
13. S: Right.

While checking if the result is correct, Nicola referred to the previous task and to the manner of forming the pattern in that task. Therefore, she connected two different, separate experience that led her to create a different thinking model from the executed procedure. Her reasoning looked as following: I have to built ten squares; if these would be separated squares, for each one I need to use four matches, that is 4 x 10 = forty matches.

But my squares have to be connected so only the first will be made of four matches and for each next one I need only three ones, that is one match less than it was. I will build nine of these "incomplete" squares, that is for nine times, I will have one match less. This way of understanding turned out to be very helpful in further work with the task and benefited with discovering some interesting dependences. Here, girls used different connections: they used the separation rule. Focusing attention on the property of the operations, in order to "use arithmetic in a correct way" caused that the girls forgot about the structure of the pattern and occurring dependences.

While answering the question about 25 squares, girls used a following strategy (a well known for us: additive function property): 25 that is 2x10 and 5. For 10 squares we need 31 matches, what we know from the task 2a). So 20 squares are 2x31 that is 62 matches. Moreover, I add 5 squares that is 5 x 3 matches, so in total I have 77 matches. While making a verification, girls notice a mistake:

14. N: There are 25 squares, times 4 [*Sylwia counts on calculator*] that is 100. But yet, we must subtract 24 (…) [*she reads the result from calculator*] 76. Why did we make a mistake?

Girls want to find the source of their mistake. For this purpose, they counted the task again, using the calculator. But they repeated that reasoning which led them to a mistake:

15. N: So, well, it will be like this [*she counts on calculator*]: 10 x 4 = 40, subtract 9 equals 31 [*she checks the result on the sheet*]. We made it right. Now 31 x 2 = 62 (…) And now we still have to add these 5 squares that is 5 x 3 so that is 15.
16. Ex: And why do you add two times 31?
17. N: So that were 20 squares.
18. Ex: Well, these 31 squares will form one row composed of 10 squares, won't it?
19. N: Yes
20. Ex: And for this you add the second row of 10 squares, don't you? To the first row of squares we add the second one in order to create one long row composed of 20 squares. (…) And as you have already arranged the row of 10 squares and beside it the second one of 10 squares, and if you moved them closer now …
21. N: So, we have to take one match away… Aha [*she corrects the mistake*] that is right, 76. Ok. Well that is 161. So let's do it like this: [*she writes on the sheet*] 161 x 4
22. S: [*she counts on the calculator*]
23. N: Ok., and now … [*she takes the calculator and makes calculations 644–160*] …484

Now as a justification Nicola used the following reasoning: from 484 matches I subtract 4 – because this much is needed for the first square. Now I divide 480 by 3 – because the rest of squares have three matches. I get 480: 3–160. These are 160 incomplete squares plus that first one, which gives us 161.

The strategy, which up to this time was only the way of justification of the result, became the rule according to which students were solving the task. It turned out that this rule is reliable – it allowed to find the mistake during the previous task. Besides, it is clear and comprehensible. It is very easy to apply, it is applicable for all examples:

24. Ex: And if you had to arrange a thousand squares?
25. N: It is easy. Look. We write one thousand, don't we? [*she writes 1000*] As we did it here and we multiply by 4. (…) And from these four thousand what do we need to do?
26. S: Subtract
27. N: Subtract this one match from every square except the first one
28. S: Well. That means we have to subtract three
29. N: Why three? (…) Look, you have 1000 squares, you multiply them by each of this kind [*she points out the arranged square with four matches*] that is by four, as if each one had four sides, right? And you have four … no, it can be done differently
30. S: Four thousand.
31. N: One thousand multiplied by three, that will be three thousand.
32. S: Yes
33. N: And now what do we do? You add one square, that one made of four. This way seems easier to me.

During the conversation Nicola noticed another dependence occurring between the number of squares and the number of matches. She succeeded in noticing this thanks to using each time the procedure of "verification" for the correctness of the result. This is a different dependence, which does not reflect the way of the pattern creation. Although, for us this is evident, for a child it is difficult to notice. In order to discover it, the student has to make a "division" of the first square between 1 + 3. S/he must see the square not as a whole, unitary and static figure, but as a dynamic creation. In this case the pattern is built "from the end" that is of incomplete squares (I take three matches, add next three ones, later next three, etc), and after having built the whole, it "closes" the first square, adding the lacking side.

The general rule for that task which was given by the girls goes: multiply the number of squares by three and add one.

Summary:

In the carried out research, students coped with the new task very well. They were able to perceive the dependences occurring in the task, they used the noticing rules correctly. While solving the task, students were using the set of information which they succeeded to gather earlier. They used information about a square and information about the shape of the pattern. These data allowed them to make a general model – "generic model" (Hejný, Kratochvílová 2004) concerning the development of next puzzle elements.. They were able to generalize the dependence discovered by them and to give the dependence's oral record. So they were able to wander off from the concrete facts and start their abstract thinking.

The part of research which was presented here, shows, as far as ways of thinking are concerned, how different children are. One can also see that the verbalization of their own thoughts is very important during the process of solving the task. So is the verification of their own progress of reasoning – the verification of the already obtained results. G. Polya called this "a glance backwards". While checking the correctness of the task execution, students discovered a new approach to the task. Therefore, through words' verification they were able to change their view on the task. The verbalization caused the change in their activity. Thanks to it, they were able to generalize the perceiving rule.

The most important thing for attaining the success while solving tasks of the whole series was the ability to create both: connections among experience assembled on previous levels, and connections with other pieces of mathematical knowledge. The first task gave the chance to make the generalization, to search for the answer to the question about the number of matches needed for building separate figures. That experience was helpful in the second task – in general, students transferred the strategy from the task about squares to the task about triangles without any problems. The girls from the team described above, used the same procedure.

The strategy of counting the amount of elements for the pattern consisting of 10 connected squares concerned the transfer of the way of work from the task about separate squares. But on the level of checking this task, a new discover developed, which was not a verbalization of the applied procedure.

The next important moment which caused new discoveries was the shift from a thinking procedure of counting matches in the pattern in the third task, to applying the rule of divisibility of multiplication with regard to adding. While counting the amount of matches for the pattern built of 25 squares, girls automatically moved to the strategy of counting matches for 20 squares (understood as 2 x 10) and 5 squares. That strategy turn out to be ineffective, but the fact that it was applied by students is worth emphasizing. The necessity of changing the correctness of obtained results forced girls to verify that strategy and to made them search for new connections. This time, the experience from the first and the second task was used constructively.

The girls presented different levels of knowledge. Regardless of that fact, both of them were able to find themselves in an "algebra reality". Both showed that they are able to think algebraically despite of the fact, that they became familiar with arithmetic on different levels. Simultaneously, the arithmetic and extensive focusing on calculations and proper usage of arithmetic rules hindered the correct approach to the solution, shaded or covered the matter of the task. So maybe it is worth developing algebraic thinking regardless of the arithmetic one?

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Preparing of pupils for notion of limits

In this article we present our results concerning qualitative and quantitative research carried out at the St Andrew secondary school in Ruzomberok in October 2007. The research was aimed at the relationships between input and output factors in the teaching process. In the article we describe the results of input test.

MESC: H30, E30

Introduction

The following two notions are included in the senior year curriculum at secondary schools: *limit of a sequence* and *sum of an infinite series*. We realized an experimental teaching devoted to understanding of these notions by students. Our experimental group consisted of students of the St Andrew secondary school in Ruzomberok.

The goal of the research was also to analyze the students' mistakes and to find their roots. The problems we have solved with students are usually not contained in typical mathematical textbooks. Similar research has been described in Hejný, Michalcová, 2001 and Sierpinska, 1987. In this article we analyze input test.

Input test

We assumed that the students had preliminary knowledge from some other areas of school mathematics. Based on our previous qualitative research, we have considered the following input factors: L – logic, AV – algebraic terms, PN – understanding of infinity, N – inequalities. Accordingly, the students wrote the following input test:

Variant A

1. (factor L) Negate the next propositions:

a) There exists a state in which every law is at least two times revised. (3 points)

b) For each natural number x there exists a natural number y such that $x + y = 5$. (3 points)

c) All numbers are even. (2 points)

2. (factor AV) Simplify the following expressions (specify conditions):

a) $\frac{u^2 - 4}{u + 2}$ (3 points)

b) $\frac{u^3 - 8}{u - 2}$ (3 points)

c) $\left(\frac{u^2-4}{u+2}+\frac{u^3-8}{u-2}\right):(u+1)$ (4 points)

d) $\left(\frac{u^2 v^{\frac{1}{3}}}{u^{\frac{3}{2}} v^3}\right)^{\frac{4}{3}}$ (3 points)

3. (factor PN) Answer the following questions:
   a) Is the set of natural numbers greater than one milliard finite or infinite? (1 point)
   b) Which number is the greatest? (1 point)
   c) Let n be a natural number. Is the expression $2n$ increasing or decreasing in n? If it is increasing, find the greatest number, which we can get. If it is decreasing, find the smallest number, which we can get. (2 points)
   d) Let n be a natural number. Is $\frac{1}{n}$ increasing in n or it is decreasing? If it is increasing, find the greatest number, which we can get. If it is decreasing, find the smallest number, which we can get. (2 points)
   e) Given a line in a plane, how many parallel lines do there exist? (1 point)
   f) In the plain, let A be a point and let p be a line which does not contain the point A. How many lines containing the point A and parallel to the line p do exist in the plane? (1 point)
   g) How many points does contain a line segment, which is 10 centimeter long? (1 point)
4. (Factor N) In the set of real numbers, solve the next systems of inequalities:
   a) $4x+6<2x+5<5x+8$, (5 points)
   b) $2x+7<4x-5<x+6$. (5 points)

Variant B consisted of similar examples. The points obtained by students in corresponding example was the value of the factor. Because we need values from interval $\langle 0,1\rangle$, the values were normalized for statistical processing.

The topic of the factors L, AV is a part of the thematic unit *Introduction to study of mathematics* in the curriculum for first year of secondary schools in Slovakia (see Curriculum, 1997). The topic of the factor N is a part of the thematic unit *Functions, equations and inequalities* also for first year of secondary schools. The factor PN was devoted to the intuitive understanding of infinity.

Frequent mistakes of students

In example 1a) we find next mistakes:
- Wrong negation of first quantifier: *"There does not exist any state, in which every law is at least two times revised." (Zuzana)*. The students negated only the first quantifier, but they haven't the negation of complex proposition.
- Wrong negation of second quantifier: *"In each state is every law at most one times revised." (Júlia)*

In example 1b) some students negated correctly the quantifier, but forgot negate the propositional form $x + y = 5$. Another group use the wrong terms $\forall x \notin$ N or $\exists y \notin$ N. Sometimes the students handled formulas only formally, not understanding the content.

In example 1c) we can find compositions of mistakes from 1a) and 1b). One group of students wrote *"There does not exist any number, which is even" (Zuzana)*. The second group of students wrote: *"There exists a number which is odd" (Mária)* or *"There exists a number which is even" (Andrea)*.

In example 2 a lot of students had problems with term $u^3 - 8$. They tried to simplify this term similarly as $u^2 - 4$:

Petra: $\dfrac{u^3-8}{u-2} = \dfrac{(u-2)(u+2)(u+2)}{u-2} = (u+2$

Barbora used the wrong equation $(u + 3)^2 = u^2 + 3u + 9$. She exchange it with equation $(u + 3)^2 = u^2 + 6u + 9$.

Many students did not solve example 2d. Some students had problems with powers and with simplifying of terms. Katarina wrote $u^3 = 3u$, Lucia L. $\left(\dfrac{u^2 v^{\frac{1}{3}}}{u^{\frac{3}{2}} v^3}\right)^{\frac{4}{3}} = \dfrac{(u^2)^{\frac{4}{3}} \left(v^{\frac{1}{3}}\right)^{\frac{4}{3}}}{u^{\frac{3}{2}} v^3}$

and Lucia S. $\dfrac{u^{\frac{4}{9}}}{u^{\frac{3}{4}}} = u^{\frac{4}{9}:\frac{3}{4}}$.

In example 3 the typical mistake was that the line segment has two points and there is no line in a plane, which contains the point A and it is parallel to the line p. Some students wrote that the greatest number is infinity ∞.

In example 4 some students used a right algorithm for solution, but they did not find a correct result. For example, Katarína wrote $x < \dfrac{3}{2}$ and $x > -\dfrac{5}{3}$ and the result was $x \in \left(\dfrac{3}{2}; \infty\right)$ because she drew incorrect picture for this result:

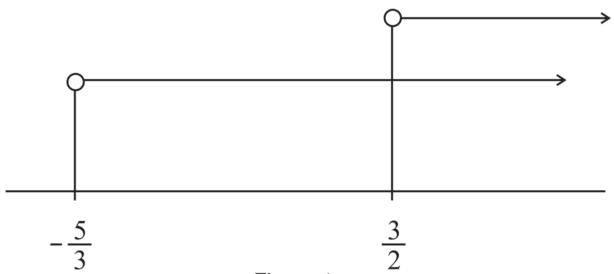

Figure 1

Some students did not multiply correctly the both sides of inequality with a negative number:

Ján: $-3x < 5$ Jana: $-3x < 3$

$x < -\dfrac{5}{3}$ $x > 1$

Marek had problems with equivalent simplifying: $2x + 5 < 5x + 8$

$$7x > -3$$

We can resume this students' mistakes in the next table:

Factor L	Factor AV	Factor PN	Factor N
Wrong negation of first or second quantifier Forgetting of negation of the propositional form Using wrong terms $\forall x \notin$ N or $\exists y \notin$ N	Using wrong equations of the type $a^3 - b^3$ problems with powers and with simplifying of terms with powers	Infinity as a number line segment has two points	Bad solutions of logical composition of the propositional forms (inequalities) Bad multiplication of inequalities with negative numbers Bad equivalent simplifying

Table 1

Some results of quantitative research

The quantitative research was oriented to the relationships between input factors. We analyze now the next hypotheses:

Ha: The factors AV and L influence to factor N.

Hb: The factors AV and L do not influence to factor N.

The correlation indexes are shown in next table:

	AV	L	N
AV	1	0,53	0,34
L		1	0,41
N			1

Table 2. Correlation indexes

In the experimental group we had 54 students and the critical value of correlation index is $r_{54}(0,05) \approx 0,279$. Factors AV and L correlate with factor N, because the correlation indexes are more than the critical value ($0,34 > 0,279$ and $0,41 > 0,279$).

The influence of the factors can be better seen from the implicative graph prepared by software CHIC:

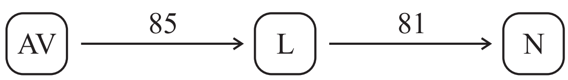

Figure 2. Implicative graph

The values in the figures are implicative indexes, which have values between 0 and 100 percent. These indexes show how strong is the implicative relationship between factors.

Correlative analysis and implicative graph supports hypothesis *Ha*, so we can reject the hypothesis *Hb*. Interestingly, there is strong influence of factor AV to the factor N through factor L.

Conclusions

The results of next qualitative research show that the students have problems with solving examples dealing with the limits of sequences and the sum of infinite series and the problems are conditioned by the lack of knowledge of previous parts of school mathematics (solving of inequalities, simplifying of the algebraic terms – part of mathematics, which is taught in the firs year of secondary school in Slovakia). This show also the pupils' solutions of the input test.

The results of quantitative research show that factors AV and N have influence to the factor N. That means the successful solution of inequalities depends from the ability of pupils simplify the algebraic terms and their logical knowledge. Other researches (see Gunčaga, 2004; Tkačik, 2004; Vancsó, 2003) follows the results of this research. Another problems for propedeutics of calculus teaching in the first year of secondary school is possible to find in Wachnicki, Powązka, 2002 and Zhouf, Sykora, 2002.

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Suggestions regarding the support of independent thinking in mathematics

II. Changes and trends in geometry

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How to reveal geometrical independent thinking in the lower primary years[1]

In School Year 2005/2006, we extended, in Italy, a part of a research initiated by Ewa Swoboda in Poland. Our research used simple tools but our results indicate that certain geometric aspects are present in the mind of pupils before a formal teaching of geometry.

Foreword

The search of pupils' pre-conceptions can be a first step in a teaching activity supporting independent thinking. Teacher cannot consider children as empty vessels but s/he must take into account as pupils personally organize their knowledge and their limited ability of verbal expression. The researcher aiming to study younger pupils' knowledge has a double commitment: to investigate a specific topic and to assess whether the used tool is suitable for that task.

Theoretical framework

The problem of suitable tools for investigating geometrical pre-conceptions is faced in many papers e.g. Vighi (2003), Marchini & Rinaldi (2007). Following these contributions we adopted an activity like a drawing activity.

The mathematical subject of this paper can be considered in between spatial and geometrical activities. We interpret the term 'space' following the concept epistemological analysis of Speranza (1997).

For educational aspects we assume Van Hiele's theory (Van Hiele, 1986) stating that in the educational processes, student undergoes several stages. On the first one – *visual level* – concepts develop on the basis of experiences and observations from the reality. Visual level is a main step in spatial knowledge. On the visual level, students recognize the figures as entire ones and are able to represent them as a mind visions. Notice that Van Hiele states that "[…] the levels are situated not in the subject matter, but in the thinking of man" and Arcavi (2003) suggests that visualization can be considered as a method of 'seeing the unseen'. Moreover Viholainen (2006) tells us that "*Visual thinking* is probably the most usual type of informal thinking in mathematics."

In the literature, the term *spatial ability* is identified as interpreting figural information and the visual processing of abstract information (Bishop, 1983), as spatial cognition, spatial intelligence, spatial reasoning and spatial sense. It is also defined as the mental process used to perceive, store, recall, create, edit, and communicate spatial images (Linn & Petersen,

[1]Work done in the sphere of Local Research Unit into Mathematics Education, Parma University, Italy

1985), while Gardner (1983) identifies spatial intelligence as one of the seven distinct types of intelligence.

This assumption compels to distinguish among the uses of terms that sometimes are employed in the literature as synonyms: *visual*, *spatial*, and *perceptive*. We adopt, here, the following distinctions: the concrete objects are placed in an actual environment and 'moving' them requires a *spatial ability*; the objects interact with the thinking man by perception therefore there is a *perceptive activity* (which can be involuntary) on the ground of all knowledge processes, but knowledge is distinct from understanding and from thinking processes. The concrete features of the objects give raise to visual level of thinking, by the means of perception, first, and personal reflection or intuition, afterwards.

In our task we ask the pupils to perform a (free) drawing, with the aim of investigating their personal thinking, by the means of their spatial abilities by working with direct manipulation of specific tools. Manipulation in central in the first three Van Hiele's (de Lange, 1987):

1. A pupil reaches the first level of thinking as soon as s/he can manipulate the known characteristics of pattern that is familiar to him.
2. As soon as he learns to manipulate the interrelatedness of the characteristics he will have reached the second level.
3. He will reach the third level of thinking when he starts manipulating the intrinsic characteristics of relations.

Student's action is inspired by each time differently recognized and understood characteristics that relate to patterns, properties, relationships. The tools, the simple paper 'tiles' of Figure 2, have features helping student in manipulating them in creating patterns, by making use of the evident geometric interrelations among them. Their protocols give evidence of different understanding of space, different structures and geometrical intuitions.

Remember that Van Hiele distinguishes between rigid or feeble structures. In his opinion feeble structures are worth noticing, they fill out the majority of our everyday life. They come from a non-verbal, intuitional way of thinking, but mathematical thinking is not superior to the intuitional one. Feeble structures may be a beginning of knowledge on a higher level of thinking where we may have something to do with, ex. rigid structures or still a feeble one (Swoboda, 2005).

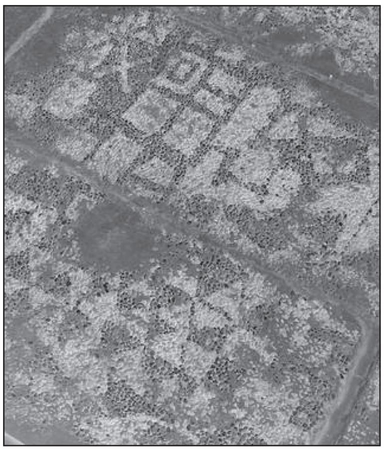


Figure 1. A particolar of Gordion mosaic

This intuitional way of thinking can be considered visual rather than geometrical (Panoura *et al.*, 2007; Pittalis *et al*., 2007). Moreover feeble structures can be used to study and to analyse an activity based on the creation of a floor. Feeble structures are very important for educational research in order to detect child's thought. They can reveal the process of early geometric knowledge appropriation. They are expressions of spatial intuitions that cannot be expressed by word, but only by graphical language, they are the first steps in geometric understanding (Bishop, A., 1980, 1983). Feeble structures are characterized by presence of connections, rotations, parallel translations, symmetries, applied only locally in the drawing. Feeble structures can be compared with additive patterns (Trilling, 2001), the most ancient ones.

In figure 1 a particular of one of the earliest known mosaics of artistic significance, (8th century B.C.).

Rigid structures reveal the presence of a mental project using both geometric shapes or isometries or the sake of regularity. From the point of view of ornaments, this kind of structures corresponds to unitary, repeating and hypotactic patterns, in the classification proposed by Trilling (2001). Pupil could pass through feeble structures to rigid ones by awareness of the 'regularity' and of isometric transformations, using sight. Swoboda (2006) shows that there is a relation between mastery of rigid structures and school success.

We gave to pupils four types of tiles as a sort of alphabet for a language, which can be considered the first step of a future expression by words. Therefore, following Vygotsky (1992), we helped the coming into existence of geometrical concepts. Moreover activities like as the one we proposed, i.e. to pave with these tiles an A4 paper sheet, with the aim of constructing a 'floor', arrange a *milieu* (Perrin-Glorian & Hersant, 2003), from which pupils have the occasion of improving their knowledge, since

> In each case, the situation organises a «material» milieu that allows experiments for pupils; and the milieu gives feedbacks. The material milieu is made of "material things" to act with (when we say "things" we mean that for the students they do not necessarily represent mathematical objects) (Bloch, 2006).

By the means of this '*milieu*' we can explore, geometrical aspects and children's ability and potentiality relating with visual and geometrical thinking, since activities such as these described here give the opportunity to analyse different intuitions that can be treated as a basis for developing not very simple geometrical notions (Swoboda 2005), by appropriate teaching.

For other aspects of the theoretical framework of this research we refer to (Swoboda 2005) and the references therein and to Marchini & Vighi (2007) and Marchini *et al*. (2008).

The research

Our research uses the setting of (Swoboda 2005) with some modifications. The tiles in figure 2 are presented in (Swoboda 2005), inspired by Kuřina (1995):

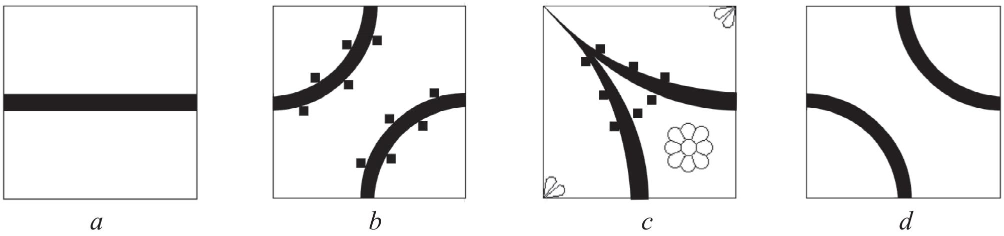


Figure 2. The four types of tiles

The first phase of the research consists of manipulation activity. At the beginning we ask the class to give a name to each kind of tile for distinguishing them. The aim is to familiarise children with the characteristics of each drawing in the spirit of first de Lange's remark.

The task is: "Create from these tiles as beautiful floor as possible" Swoboda (2005). The 'floor' consists of an A4 blank sheet (21 cm × 29.7 cm); it must be paved gluing tiles on. Sides of the square tiles measure 2.5 cm.

Child works, with a stick of glue, an A4 blank sheet, four containers, one for each kind of tiles, prepared in advance. Scissors are not allowed. The works is performed singly in the classroom environment, one child after the other.

The second phase is centred on the use of colour: we give a photocopy of the protocol to the author-pupil for colouring it. Afterwards we ask for a title. The introduction of colour and of title for protocols is the main distinction in the methodology of the research respect to Swoboda (2005). In this way we get each protocol in black and white and in colour.

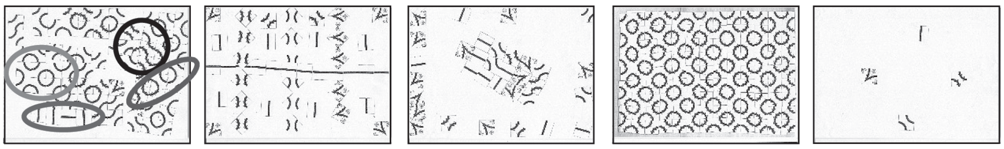
The whole experimental activity took one school year (2005–2006). In this research were involved 212 pupils (97 – 5 year old – Kindergarten, 68 – 6 year old – grade 1, and 47 – 7 year old, grade 2). The experiment involved Kindergartens and Primary Schools of Viadana in the province of Mantova, Northern Italy[2]. Viadana is a small town with agriculture and artisan industry, where there are many immigrants from other Italian regions and from abroad.

The experiment results

Our request can be presented very early and it should be motivating and well accepted also by kindergarten 5 year old pupils. We can state that pupils reserved a great attention to the task: the average number of tiles used for a protocol is 69.47 (Kindergarten 53.53, First graders 80.62 and Second graders 86.51) with more than 25 minutes of work.

We can discuss the task „Create from these tiles as beautiful floor as possible": this requirement could seem too ambiguous, but it allows the children express themselves in a good way. In other words, the children are completely free to choose which, how, where, how many tiles, and how many times, in order to obtain the most beautiful flooring that they can, so that individual artistic taste and choice of design are what determines the choice of tile. Looking for their intuitions, we are interested in some geometrical order; other more precise statement of the task would be more difficult to be interpreted by these young pupils.

We obtained protocols showing feeble and rigid structures.



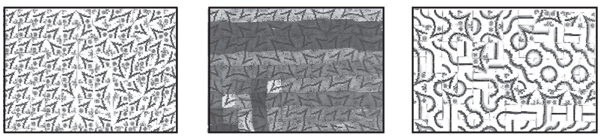
a *b* *c* *d* *e*

Figure 3. Some black and white protocols

Protocols 3.*a* and 3.*c* were produced in Kindergarten; protocol 3.*d* was made by a first grader, the authors of remaining protocols are of second grade pupils. We superimpose some oval on the reproduction of original protocol in figure 3.*a*, in order to draw reader's attention to feeble structures. The other protocols show the presence of rigid structures.

The colour gives other opportunities for revealing different interpretation of geometrical aspects.

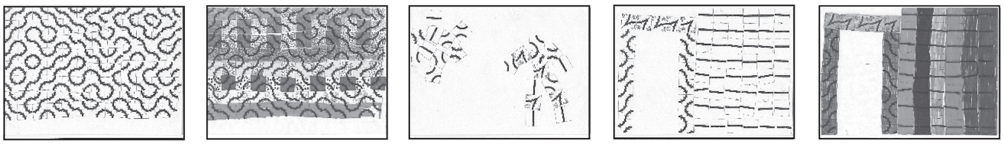
Figure 4. Some coloured protocols



a *b* *c*

[2]We would like thank the School Heads and the teachers of Bedoli, Carrobbio, Cogozzo, and San Pietro Kindergartens, and Classes 1A, 1C, 1E, 2A and 2C of Primary School, for permitting and collaborating with our work.

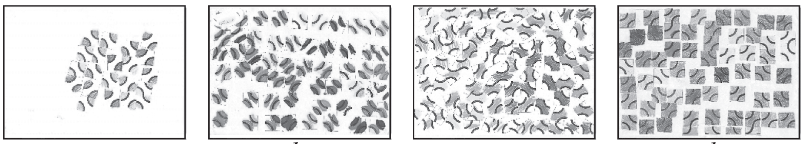
The colour can be also a kind of language, therefore we asked children to colour the protocol as this might reveal the criteria the learners base their design on. The colour and the title given to the protocol, take place of semantics for the black and white drawing. The colour sometimes afforded new information about pupil's aims; it changes a lot the possible interpretation of protocols: in some cases the colour respects the rigid (feeble) structure of the black and white protocol (figures 4.*a*, 4.*c*), but we have also the case that colour transform a feeble black and white structure, figure 5.*a*, in a more rigid one: figure 5.*b*. Figures 5.*a* and 5.*b* show the superposition of a 'vertical-horizontal' structure given by colouring, to a 'diagonal' one present in the black and white protocol.



a *b* *c* *d* *e*

Figure 5. Other protocols

The tiles were employed for figurative/decorative aims, as each tile was the sign of a pencil. An example is in figure 5.*c*, *'The boy looks at the sun'*. In other occasion the colouring and title are revealing of the child's aims (figures 5.*d* and 5.*e*: 'A tree and the rain'). Colour can reveal where the child attention is focused on. Protocols 4.*a* and 4.*b*, using tiles of the same type, display pupils' local or a global attention.



a *b* *c* *d*

Figure 6. Colour and space in some protocols

Drawings in figure 6 use tiles of the same type, but children 'found' space in different parts of the, tile highlighting them with colour. It is evident the different approaches witnessed by the four drawings of figure 6. In 6.*a* the space of the tile is in the 'concave side' of the lines, making a non-connected space. The author of 6.*b* is interested on lines in themselves and the colour points out them. The space between the 'convex parts' of the lines on the tile, attracts the pupil of figure 6.*c*. For the author of 6.*d*, the space is the square tile independently by the drawing on it.

The tile measures do not fit the sides of the A4 sheet. The pupils, therefore, have to face problems regarding in filling the space as actual tiles layer does with an actual floor. This feature will allow investigating pupils' conception of space (*independent* / *non-independent*; for an examination of this issue using different tools, see Marchini, 2004). Our attention will be focused on the way the child fills the sheet, if s/he choose a sort of 'frame' drawing (figure 3.*c*), or if s/he will glue tiles on an imaginary sheet larger than the actual one (figure 3.*d*).

The geometric tile structure could limit degrees of freedom for children's expression. Furthermore the mind activity required to construct and to colour the drawing is, in our opinion, a suitable, right task about a spatial – geometric activity necessary to prepare the next more formal treatment of geometry.

Tiles (and colour) can be used to express the need of continuity, realizing locally or globally, connections, see figure 4.*c*.

It is possible that the number of tiles the child glues on the sheet is determined by pupil's attention span, by her/his manual coordination, commitment to the task, and by interest in producing her/his own design; also it can be related related with age and teacher's practice. The average number of tiles (in Table 1) can be used as a rough indicator of all these aspects. It shows clear differences among ages and between genders. From Table 1 we can conclude that our tools are sufficient for investigating different way of thinking.

	tile *a*	tile *b*	tile *c*	tile *d*	no. of tiles
Sample Males	14.12	16.18	15.77	**21.91**	8 294
Sample Females	13.23	10.12	**36.21**	12.06	6 446
Kindergarten Males	17.15	17.27	18.17	**29.82**	5 851
Kindergarten Females	13.50	12.66	**46.25**	11.61	3 697
1st grade Males	12.55	14.36	14.69	**37.50**	3 332
1st grade Females	16.46	5.27	**35.88**	16.35	1 923
2nd grade Males	**23.83**	21.48	23.21	18.69	2 529
2nd grade Females	8.78	9.33	**61.22**	6.06	1 537

Table 1. Rates of tiles use

Taking in consideration the presence of isometries in the pupils' protocols we get the following data:

Years	Local relation			Global relation				Children
	translation	axis symmetry	rot	translation	axis symmetry	rot	line continuity	
4	43	0	0	0	0	0	0	100
5	30	12	8	4	1	1	24	100
6	28	18	9	5	4	1	34	100
7–8	34	26	13	13	6	2	43	100

Table 2. Presence of isometries in pupils' protocol (in percentage)

Table 2 reveals that the rate of application of the more complex isometries grows with the pupils' age, even if the children of the sample did not have a formal teaching of geometry. We can justify these data observing that the enhancement of manipulation ability obtained by school teaching likely has good effects also for improving geometrical and spatial intuitions. Marchini *et al.* (2008) treats in deep these and other interesting geometric features of these 212 protocols.

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Activities in the preparation for and establishment of the concept of lenght and perimeter

In this paper an experiment is described, which focuses on the contribution of the various activities and devices to the preparation for and establishment of the concept of perimeter. The formation of the concept of rectangles and squares and their interrelationship are also analysed. The aim is to demonstrate to what extent and how the various methods such as whole class teaching, group and pair work help learners to gain knowledge, to develop their thinking and creativity in this area.

Introduction

Problems in teaching geometry in Hungary

- Activity is quite often is neglected in teaching measurements and the emphasis is laid on primarily on the introduction of measurements and the relationships between them, and their transformations.
- The majority of children learn the characteristics of geometric shapes, the definitions without having any concrete experience based on activities. Thus one of the most important aspects of the evolvement of cognition, the principle of continuity and versatility is impaired.
- A large amount of geometrical concepts can also be found in everyday life, i.e. children have acquired some of them by the time they go to school. However this sort of prior knowledge is not always taken into account in education.
- Sometimes the concepts children are taught are not linked and they do not think in terms of patterns, which makes it difficult for them to recall the concepts.

The historical and theoretical background of activities

Learning through activities is a not totally new teaching method. Hints at it can be found even in antique times.

E.g. the saying of a scholar from antique times goes like this:

"What I hear I forget, what I see I remember, but what I do I know."

In the history of Hungarian math's teaching several mathematicians underlined the importance of activities.

Farkas Bolyai, professor in Marosvásárhely, the father and also the teacher of János Bolyai, formulated several educational principles at the beginning of the 19th century, which is still valid even these days. In these principles he pointed out the primacy and importance of activities.

> (the teacher) should always start with things that can be seen and touched and not with general definitions (the first utterances (of children) are not based on grammar. Neither should the teacher torture the children prematurely with longwinded reasoning. It is more than enough to prepare the mind without damaging it, so that the child could find the activities both engaging and pleasurable.

> Geometrical forms and reading should be the first things to start with… we should leave the sheet (the plane), … and later on land-measuring not only on board but in the field as well. (Bolyai, The Beginning of Arithmetic, 1830)[1]

In the fifties of the 20th century in maths teaching a worldwide reform movement started and the aim of the new approaches and theories was to modernise maths teaching. Some of the innovators intended to introduce the standardising approach of the Bourbaki group into teaching whereas others relied on psychological principles, mainly the ideas of Piaget. There were also followers of the heuristics of György Pólya and the 'playful maths' of Zoltán Dienes.

Over the past one hundred years innovations in maths were also initiated by psychological research, which gave insight into the way children think and these results had to be built into learning theories.

Bruner's representation theory also based on activities: According to his theory in order that learners could understand the teaching material, they should 'process' it intuitively before.

Experience is emphasized in constructivism, however in its theory it is the interpreting human being of cognition that plays an essential part. According to constructivist theory of learning children not only store knowledge but also construct it.

Several researchers have dealt with the formation and development of children's geometrical knowledge. (Van Hiele 1959, Piscalo 1968, Piaget 1966, Freudenthal, 1983)

Two of the Van Hiele levels can be found in the first three classes of lower primary. In the first two classes children consider not only rectangles and squares as separate but they separate rectangles as well. Children sometimes see rectangles, whose side is one cm, just as a strip. Some children do not consider the 'upright' rectangle as a rectangle.

The second level is reached by some of the children by the end of the third class of the primary, whereas the majority reaches that level in class four or even later. They are able to realize that every square is a rectangle, but e.g. when making a rectangle of 20 cm perimeter, most children came up with a rectangle whose side was four and six centimetres, since rectangles with this proportion of sides were the most common for them in real life.

The role of activities in assisting linguistic problems and in the development of the use of mathematical terms

Learners quite often have problems expressing themselves in terms of mathematics and activities help them overcome these language barriers and to express their ideas and thoughts.

> Mathematical concepts normally are not established spontaneously, but rather in a conscious, deliberate way. The words used in mathematics come into being through the co-operation with the way of thinking of adults and conscious development. Sometimes even complex thinking and linguistic processes are required. The precondition of the formation of mathematical concepts is the proper level of everyday concepts, since in the process of establishing mathematical concepts we several times rely not only on direct presentations and objects, but everyday language as well. (Szendrei p. 401)

Knowledge gained through meaningful and appropriate activities, which can be fitted into the learners' pattern of thinking, will be more lasting on the one hand, and will prepare the reception of new knowledge much better on the other.

If we manage to link the knowledge under discussion to the children's sphere of activity, we can arouse their interest in mathematics.

> Appropriate situations ought to be found at the various levels of teaching. An appropriate situation can be described as natural, what is interesting for the persons we offer them, and what

[1]Translation by Eva Tohol

is simple enough to facilitate the analysis of mathematical ideas, and opens the way to further topics and it also makes a better approach to reality possible. (André Revuz, p.106)

Research question

How long does it take to prepare and to teach the concept of length and perimeter through measurements in our environment, drawing broken lines creating triangles and squares and determining their length?

Hypothesis

We assume that plenty of experience in measurement and practical activities, games and references to application in everyday life will contribute to a better understanding of the concept of length and perimeter.

Group work and pair work will improve children's ability to co-operate and to help each other in the activities and gaining knowledge, which eventually will result in better understanding.

Methodology of the research

The experiment, which took place in a class of István Bocskai Primary School, started in class 1 including class 3. The school is at a housing estate, and the equipment and teaching methods are very much the same as anywhere in the region. The pupils can be described as a mixed ability group and their attitude to learning and mathematics is varied.

The experiment started in the second term of the first class, and ten or twelve afternoon sessions were held in the afternoon day care lessons by the author. The sessions were taped (audio or video) and photographs were also taken. Occasionally worksheets were handed out in order to make children's work easier.

The preparation of the concept of perimeter in class 1

In class one child had already been involved in tasks in which they stuck a ribbon round a small paper box or jewel-box by way of decoration. In this case the aim was to prepare for perimeter as a quantity of length. The groups of children were asked 'to order' strips according to measurement. If one group ordered less than necessary, the strip was not long enough to decorate the box, thus the box did not look really nice. And if another group ordered more than necessary, the ribbon hang down, therefore their box was not nice either. Accurate measuring was essential and so was the adding the data measured.

Determining the length of open and closed lines

In class 2 we continued preparing the concept of the perimeter by drawing broken lines, which were called tourist routes. The tasks were carried out on grids on which the sides of the squares were exactly one cm, in order that determining the length of the line drawn could be easier for the children. Most children counted the squares along the route instead of the segments fitting onto the line. Some of them came up with the number of inner squares on the curve of the line, others counted the outer ones. And some of them also changed directions while they were counting.

The length of the routes was determined in various ways by the learners: by counting the fitting squares and measuring by thread and ruler.

On the worksheets the steps were available in writing and according to them broken lines had to be drawn and their length had to be measured. When they provided these lengths not everybody counted the grids, but they added the various steps. The lines were both closed and rectangular as well.

Children will need plenty of experience to be able to recognize rectangles in any position among various polygons. This was an easy matter for them when they were asked to select rectangles out of polygons cut from paper, because they could turn them into a position when they were able to decide without hesitation. In case of the polygons drawn on the blackboard it was a lot more difficult for them to decide. It happened e.g. that even an 'upright' rectangle was not considered as a rectangle by them. At the beginning they did not recognize the rectangles in a rotated position. We facilitated the recognition by cutting polygon on the board from paper, and when in this way they were able to rotate it, they were able to solve the problem.

Turning triangles and quadrangles into rectangles by means of cutting into pieces and completion also aimed at the same purpose i.e. to improve their concept of rectangle. They came up with several solutions to the transformation of polygons.

We paid so much attention to rectangles, because without the clear-cut concept of rectangle it would not have been possible to determine its perimeter.

Excerpts from a case study:

Teacher: How can you describe rectangles?

Pisti: They have not height, and even if I turn them upward, they are rectangles (What he meant by saying 'They have no height', they are plane figures. 'Turning them upward' refers to the relationship of upright and horizontal position.)

Ági: They have four sides and four angles.

Tamás: A proper rectangular prism, if we turn it in a way that its larger part is here, and here…. it is also a rectangle. (Tamás mixes up the words rectangular prism and rectangle only verbally; he is also talking about the 'upright' and 'horizontal' position of the rectangles, actually pointing at what he was talking about.)

A case study on turning polygons into rectangles:

Stella: Number one is a triangle.

Dávid: It has three points and three sides. (Point is and everyday word for vertex)

Tamás: Number six is not a rectangle. (Some of the children don't agree with him, but actually he is right, because it is a rhombus.)

Tamás: I'd like to say that if we drew the same to number one, than we would have number two. (Number one is a right-angled triangle, and number two is an equilateral triangle.)

Csaba: Number three is neither a square nor a rectangle, because two of its lines are slanting. It has four vertexes but it is not a rectangle. (It was a square on a vertex)

Stella: Number six is not a rectangle because its side is a slanting line, not straight. (Although what she is saying is correct, but it demonstrates that she can recognize a rectangle if its sides are parallel with the edges of the blackboard.)

The above – mentioned three plane figures are turned into rectangle; some children do it by completion others by cutting. In the meantime the others comment on and if necessary correct what their partner is doing.

Formation of rectangles of a given perimeter

We intended to approach the preparation of the concept of perimeter from another aspect, therefore children made squares and rectangles from strings, or rickrack of a given length (3 m, 4 m, 6 m, 8 m) first in the classroom, then, using even longer strings in the schoolyard. In these activities they discovered that various kinds of rectangles could be made, but only one square.

The task proved to be rather difficult, because they did not manage to pull the opposites sides of equal length. The reason for this might have been the small size of the classroom, because in the schoolyard they had a better view of the plane figure they formed. Thus they managed to do the task there a lot better. They still had problems with determining the length of the sides and the string, because not everybody was able to measure properly. Some of them read only the last number on the tape measure, and they had no idea what it meant. If they had used a ruler, this error would have been unnoticed.

Creating polygons from straws

In term two of class two the preparation for the concept of perimeter was introduced by a test in order to find out how much children remembered what they had previously learned. The correction of the test results was analysed by interviews first in groups then in pairs.

In case of some children, the concept of perimeter was not firmly established. Therefore they had the task of making polygons (triangles, squares, rectangles etc) from straws. In this situation they can see only the borderline of the figures; we assumed that the concept of perimeter and area would be more clear-cut. There was a gradual process from free creation towards tasks with several conditions.

Children measured the length of the pieces, which they cut, and they put the data measured into tables, and after stringing the straws they measured again the whole length. Then they compared this length with the sum of the lengths in the tables. The eventual differences may be due to the fact that the cutting or fitting of the pieces was not quite right.

Making triangles, squares and rectangles of a given perimeter was more stimulating for the learners.

First they made a triangle of 18 cm perimeter, which they expected to be an equilateral triangle with side of 6 cm. In case of a triangle of 16 cm perimeter Máté immediately said: *"We cannot make such triangle as 16 cannot be divided by three."*

By means of some illustrations and drawings on the board they realized that not every triangle is equilateral. Several pairs created isosceles triangles, but none made triangles whose sides are different.

Tamás and David broke up 16 cm in several ways such as, 10 cm, 3 cm, 3 cm. When stringing the straws they realized that no matter what combination they try to tie them up to get a triangle, they did not manage. Finally they turned to the teacher who reassured them there was nothing wrong with their measurement or stringing, it was impossible to make it. And Tamás immediately solved the problem: *"Of course, the two small sides together should be longer than the big one."*

As to the rectangles they have realized that the opposite sides are of equal length, and moreover this is not enough to make a rectangle, the side should also be properly "adjusted." What this meant for them was that they observed the position of the neighbouring sides and they were aware of the position of the angles.

Defining perimeter

In the first term of class three children were aware of the concept of perimeter; they only had to formulate it. Emphasis was laid on mainly on activities.

> Teacher: How could we measure the perimeter of the classroom?
> Dávid: We could measure there, pointing at one side of the classroom, and also pointed at the opposite side, because it is the same over there, then he pointed at the adjacent side and said that: we could also measure it because the opposite side is the same. (David is very creative.)
> Teacher: Who can come up with other solutions?
> Ági: We could stand forming a fathom.

The children stood by the walls at one arm's length from each other and they realized that 19 children were not enough to encircle, thus the perimeter must be about 22 or 23 fathoms.

Summary of the results of the experience

- During these sessions even shy learners could get into the limelight, especially when drawing routes, using straws and measurements in groups.
- Almost all of them were able to use the ruler properly, which can be due to the fact that in the pair interviews the more experienced pupil helped his or her partner to eliminate the mistakes.
- When measuring with thread, they realized that the line can be straightened, i.e. perimeter is a quantity of length.

According to the above it can be said that the hypothesis was realistic and this also can be seen in the results.

Conclusion

According to the experiments there need to be a clearer focus on the following:

- The afternoon activities are absolutely suitable for carrying out measurements while the children are outdoors as if they were just walking or playing games in the playground.
- During the experiment we also realized that learners' knowledge gained in everyday life could be used education.

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Developing problem solving strategies via estimating the area of irregular shapes

In this paper we are interested in the work of 6th graders (11–12 years old) when they face non-standard tasks with the area of irregular shapes. These shapes add to the students an extra level of difficulty when they include curved lines in their boundaries. We record the strategies the students decided to apply in order to overcome these difficulties. We further analyze their work for whether there is evidence the strategies are primarily due to the computer environment or the use of paper and pencil.

Introduction

It has been acknowledged by the relevant research literature that when students are dealing for an adequate period with problem solving tasks of the same conceptual backdrop, they develop problem solving techniques. They also make connections among mathematical ideas using specific problem solving strategies (Rickard, 1996). In our case the conceptual backdrop is the area of irregular shapes, and the techniques are for the estimation of area (Mamona & Papadopoulos, 2006). For example, the pupils use the cut-and-paste technique in order to compare areas of irregular shapes, which indicates an understanding of the preservation of area through transformation (Baturo & Nason, 1996). They use a measurement unit of their choice to deal with the remaining parts of such shapes, since it is not possible to cover them with whole units or with simple fractions of them (Clements & Stephan, 2004; Reynolds & Wheatley, 1996). They use the grid a) as a device for measuring the total number of the whole measurement units; b) for combining sub-units; or c) for measuring lengths in order to apply known formulas. This problem-solving landscape has recently been influenced to a great extent by the introduction of new technologies. For example, they affect the way students explore properties of mathematical entities (functions, shapes) in addition to the way calculations are made. Students can use the computer as a device for exploring various solution paths and decision making; they can try out ideas and strategies and simultaneously receive feedback on those ideas and strategies (Clements 2000). Our research project focused on the area of irregular shapes. We can distinguish two types of such shapes. In the first type the shapes' boundaries are constituted completely by segments (i.e., irregular polygons). In the second type we have shapes that include exclusively curved lines on their boundaries (i.e., the outline of a lake), or a combination of curved lines and segments. What we present here is a part of our project that examines the way 6th-graders cope with non-standard tasks concerning the estimation of area of irregular shapes of the second type. The critical element of our choice was the presence of the curved lines in their boundaries. As noted, this causes a series of difficulties and provokes specific techniques for overcoming these difficulties. In this spirit, we present the strategies the students applied, and comment on how these strategies reflect certain aspects of visualization and problem solving.

Description of the study

The project involved 6th-graders in an urban area in Greece. There were 44 students. Eighteen of them worked in the computer environment and 26 in the traditional environment of paper-and-pencil. They had been taught through their regular mathematics classes the concept of area and the formulae for the calculation of the area of known shapes (triangles, squares, rectangles, circles and trapeziums). However in the official textbook there is only a tiny reference to irregular shapes. So the students during the last year participated in an earlier stage of our research project aiming to explore and enhance their comprehension of the concept of area. Emphasis had been on problem solving techniques for the estimation of the area of irregular shapes that do not have curves in their boundaries. This experience gave them an accumulated understanding of the usage of various tools for calculating the area of irregular shapes. We could mention here the usage of a grid in a geoboard, the division of a shape into sub-shapes, the usage of sub-units, and the cut and paste technique. These techniques were not presented directly to the students. They "invented" them through suitably designed tasks promoting implicitly specific techniques.

The map of the province of Serres is under scale 1:1500000. Estimate as close as possible what actually is the area of the province.

The task we present here demands a more refined application of these techniques since its outline is constituted completely by curved lines. In this sense it could be regarded as non-standard since there are no analogous tasks in the official textbooks and it can not be solved by merely retrieving known procedures–formulas or relying to the experience gained through their working with irregular shapes substituted by segments.

The task is presented with a grid, which constitutes a main tool in activities concerning the estimation of area. Such a grid facilitates the handling of the irregular shape since it allows the students to safely intervene and move parts of the shape. Moreover, it offers a guide for an arithmetical result (using the dots of the provided array as a device for measurement), especially when typical measurement units are not available. The students worked individually in each environment without interventions from the researcher. For the computer environment we used the Microsoft Paint, the known painting program of Microsoft Windows (that enables cut and paste processes, rotation, colouring, drawing segments); at the same time we recorded with capturing software (Camtasia Studio) in a movie format anything happening on the computer screen. The students spend enough time to be familiar with the software through activities that were irrelevant to mathematics. Our data in the paper and pencil environment were the students' worksheets. They had as many copies of the task as they wanted. After the session, each student was interviewed with direct questions about their motivations during the task. It was not so profound to see how the students worked by merely looking at their worksheet (sometimes they worked with more than one worksheets). So, it was necessary to ask them to describe their solving process, to explain why they abandoned a specific way of working and turned towards another one, etc.

Overcoming difficulties

We recorded 30 organized attempts to solve the tasks, i.e., 30 pupils (11 in the computer environment and 19 in paper and pencil). By organized we mean that we did detect in each case certain steps that revealed a concrete strategy for overcoming the difficulty of the curved lines, regardless of whether or not the task was successfully accomplished. The main idea behind the strategies the students decided to apply was in each of the 30 cases the preservation of area, and was expressed through the cut and paste method. The students did not show any difficulty in accepting the idea of the preservation of area; on the contrary, they realized that this method was not sufficient enough to obtain the solution. Thus the students combined this method with another one or modified it slightly, as we will present below.

Units and cut-and-paste strategy (Str. 1)

The students who applied this strategy (8 out of 30) initially counted the whole square units. Then, by cut-and-paste, they transferred some partial square units to another place, combining the partial square units in pairs, thus transforming the initial shape (Fig. 1). It is worth pointing out that this strategy was applied only in the computer environment; it can be explained because the image is powerful, and the students have the possibility to visualize in the computer screen the result of their action. They can colour, cut, move, rotate pieces, undo a previous action, etc. It was difficult for the students in the paper-and-pencil environment to apply this strategy since they had to keep in mind the whole process step by step, viz., which piece has been removed and to where. On the other hand in the computer environment the students could not avoid empty space among the pieces, or overlapping between them, which resulted in a loss of a part of the asked area.

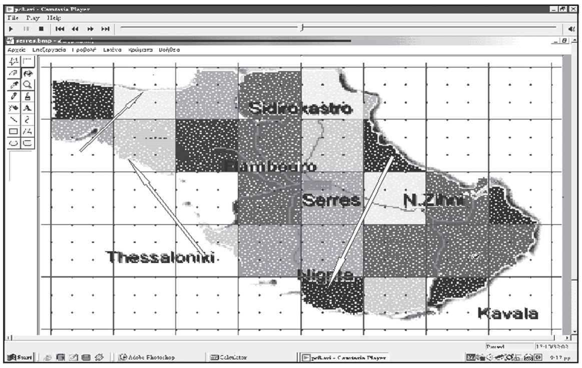

Figure 1. Counting for units and cut-and-paste for sub-units

Units and sub-units and cut-and-paste (Str. 2)

This strategy is an advancement of the previous one. The students (3 out of 30) initially counted the whole square units. After that, they used the provided array of dots to divide the partial square units into subunits. They counted the new smaller square units and then they applied cut and paste for the remaining parts of the shape (Fig. 2). This strategy (also applied only in the computer environment) allowed the students to reach the final area with a closer approximation, applying cut-and paste only for the small regions that were left over.

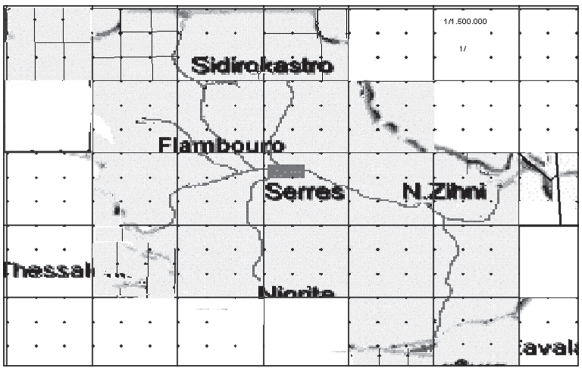

Figure 2. Counting of units and sub-units and cut-and-paste for the remaining part

Substituting curved lines with segments (Str. 3)

Figure 3. Substitution of the curved lines by segments

Some students (5 out of 30) in the paper-and-pencil environment tried to overcome the difficulty stemming from the presence of the curved lines by substituting these lines with segments (Fig. 3). So the partial square units became known shapes, and it was now easier for them to calculate their area using known formulas. However, this substitution sometimes caused an additional difficulty. The new shape was an irregular polygon and so the students had to apply extra strategies (i.e., dividing the shape to sub-shapes) in order to find its area.

Whole units and partial units in pairs (Str. 4)

This strategy is analogous to the first one presented (units and cut-and-paste), most prevalent in the computer environment. Students employing this strategy (11 out of 30, all of them in the paper-and-pencil environment) initially counted the whole square units. Then they assumed that the partial square units combined in pairs constituted a whole unit. So they counted these partial units and then divided their number by two in order to have an approximation in terms of whole square units.

Whole units and partial units per groups (Str. 5)

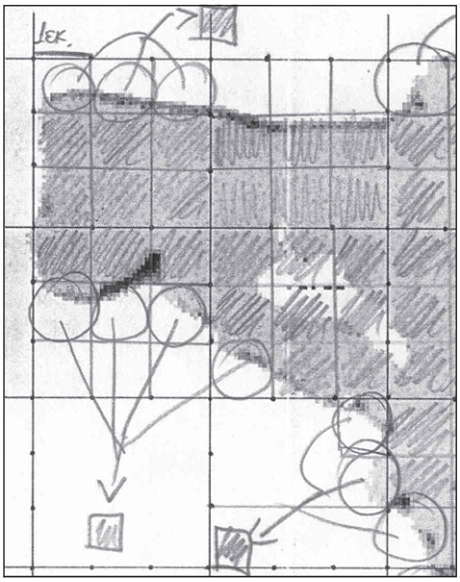

Figure 4. Grouping sub-units

A more refined version of the previous strategy (only 1 out of 30, in the paper-and-pencil environment) was to proceed to the creation of a more detailed grid. The student counted the new whole square units and then, instead of combining partial units in pairs, she preferred to combine groups of partial units, based on her appreciation regarding how many of them could create a whole unit (Fig. 4).

Whole units and mutual disablement (Str. 6)

One of our students in the paper-and-pencil environment based his decision about the contribution of the partial square units on a comparison. He compared the part of the area of the square unit that was occupied by the shape with the part that was left over. If the first one was smaller than the second then he ignored the partial unit (represented by X in Figure 5). If, however, the first one was bigger than the second, then he counted it as a whole unit (represented by tick in Figure 5).

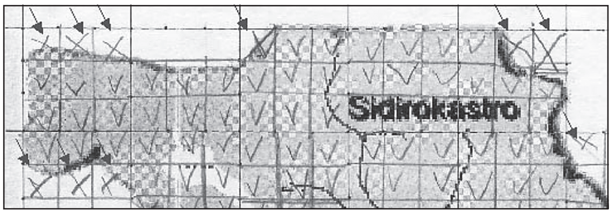

Figure 5. Mutual disablement of partial units

The case of complement (Str. 7)

Figure 6. Working on the complement of the shape

Finally, one student out of thirty (paper-and-pencil) worked somehow paradoxically. He tried to find the area of the shape estimating its “complement”. He calculated the area of the whole grid and then he tried to estimate the area of the region that was external of the initial shape in order to reach by subtraction the final result (Figure 6).

	Str. 1	Str. 2	Str. 3	Str. 4	Str. 5	Str. 6	Str. 7	Total
Computer environment	**8**	**3**	0	0	0	0	0	11
Paper and Pencil environment	0	0	**5**	**11**	**1**	**1**	**1**	19
TOTAL	8	3	5	11	1	1	1	30

Table 1. Distribution of the students strategies

Summarizing the students’ attempts and strategies as summarized by Table 1 above, it becomes clear that we have an explicit distinction between two groups of strategies. The main difference between them is that in the first group (str. 1, str. 2) we have strategies that relied on cut-and-paste actions. It could be explained on the basis that in the computer environment the students had the chance to visualize their actions. They had a sense of control over what is transferred and to where it is transferred; and furthermore, they could undo a wrong action through the available tools, allowing actions such as cut, move, rotate, undo, colour, etc. (Papadopoulos, 2004). Obviously these strategies could also be applied in the paper and pencil environment. However, in that case a hard mental effort was required for the student to keep track of each transference and how it contributed to the total area.

Conclusions

The above mentioned strategies share a common characteristic. All of them have their origin in visualization. Visualization constitutes an important process in Euclidean geometry since geometry learning is developed to a great extent through visual mental strategies. Visualization directs the students towards a solution procedure through a series of steps, viz., a) decomposition into units; b) creation of auxiliary constructions c) transformations; and d) recomposition (Hershkowitz, Arcavi & Bruckheimer, 2001). Reaching the solution through 'such problem-solving strategies could be regarded especially important for the primary school level, since in that age students rely heavily (compared to adults) on the credibility of what they see or what their imagination produces. Our students seemed to easily accept the idea of preservation of area that was behind the task. So the most important thing for them was to effectively apply problem-solving strategies to overcome the difficulties introduced by the curved lines. The fact that a grid existed in the task limited the students' freedom since it imposed a specific starting point. Moreover the available techniques they had at their 'tool-bag' (according to their past experience) could be applied only when the outline of a shape was constituted by segments. However, the students showed a broad range of different approaches: a) They counted the whole square units and applied the cut-and-paste method for the partial units. b) They counted the whole square units; and then created subunits for the remaining part of the shape, counting the subunits and applying cut-and-paste for the leftover partial units. In that case we had a more accurate approximation of the area. c) They substituted the curved lines with segments in order to create shapes familiar to them and estimated their area using formulas. d) They counted the whole square units and then combined the partial ones in pairs, estimating that two partial units formed an approximate whole square. We also had three other strategies applied (one student per strategy). e) One of them counted the whole units and combined the partial ones in groups based on her estimation of how many partial units formed a complete one. f) Another student counted the whole units and then for the partial ones made a comparison between the part off the unit that was occupied by the shape and the part that was left over. And finally, g) one student avoided directly estimating the area of the irregular shape; he calculated the area of the whole grid (a rectangle) and then tried to estimate the area of the shape's complement, that is, the area of the region that is exterior to the shape. After that, he estimated the asked area of the initial shape by subtraction.

Working with the last four strategies, the region is not transformed, as occurs when one applies the first two strategies. So, as far as the visualization is concerned, in the first case the solver respects the initial image, whereas in the latter case the estimation of the area of the shape depends on its transformation. From a problem-solving point of view there is a difference between a tool that directly analyzes a system (strategies 4, 5, 6) and the transformation of a system to another one, as is necessary for solving the problem through cut-and-paste actions. Furthermore, it seems that even such young pupils can enter such a task environment in order to apply, adapt or broaden an existing strategy.

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Geometrical transformations and the concept of cyclic ordering

In this paper we describe a research on the connection between geometrical transformations and orientation. We discuss the particulars of the thinking process and typical difficulties connected to the field of geometrical transformations. We pay special attention to the problem of cyclic order. We investigate pupils' competence in primary school, especially in Grade 2 (age 7–8).

Introduction

This study is part of a research on spatial orientation competence in primary school.

Spatial orientation describes the visualization of a spatial arrangement in which the observer is part of the situation (Maier, 1999).

On the basis of mathematical and historical analysis we may divide the relevant mathematics curriculum regarding the topic of spatial orientation into 6 subtopics (Kónya, 2006).

1. Using words to describe spatial relations
2. Describing routes (using simple maps)
3. Ordering cyclically
4. The coordinate system
5. Geometrical transformations
6. The front-, side-, and top-view of an object

We will discuss the particulars of the thinking process and typical difficulties connected to the 3rd and 5th subtopic.

Theoretical background

We use Guilford's interpretation of spatial ability, especially spatial orientation. Spatial ability has two main components: visualization and spatial orientation. Spatial orientation has five components: factor S3 of Thurstone, spatial relations, spatial perception, mental rotation and kinesthetic imagery (Maier, 1999).

Mental rotation and kinesthetic imagery are important preconditions of the development of spatial ability. (Aman and Roberts, 1993)

We studied the results of mathematical, historical and didactical theories connected to the spatial orientation and particularly to the geometrical transformations. Our analysis is based mainly on the work of Hilbert (1956), Kerékjártó (1937) and Freudenthal (1983).

Hilbert introduced the circulation sense of a triangle with the help of the concept of the left side of an oriented line. The circulation sense is the basis of the orientation on a plane, furthermore of the well-known property of transformations: preserving or inverting of orientation.

Kerékjártó introduced the concept of orientation in another way. The starting concept in his work was the cyclic order. If a, b, c are three half-lines with a common start point O on the plane, he says that the cyclic orientation of a, b, c is a function, which orders to them one of their permutations (Kerékjártó, 1937, p. 116). It is easy to see that permutations *(abc)*, *(bca)*, *(cab)* and permutations *(cba)*, *(bac)*, *(acb)* determine the same cyclic orientations. One of the two cyclic orientations corresponds to one of the two directions of rotation around point *O*. Kerékjártó highlighted the link between cyclic permutation and orientation.

Freudenthal compared the cyclic order with the linear one on the level of mental objects (Freudenthal, 1983, p. 414). He established that cyclic orientation is not deducible from linear orientation directly, so it is worth teaching it separated. He pointed out that cyclic orders are probably early mental objects and arranging cyclically is an early mental activity in the individual learning process then linear order and arranging linearly. He referred to such kind of activities as sitting around a table, standing or dancing in a circle, counting out, etc. Freudenthal called one's attention to two phenomenologically important sources of orientation: to the reflection and to the angle (Freudenthal, 1983, p. 424–425).

Research questions and methodology

Our research questions are the following:

1. Do pupils in grade 2 have the competence to construct the reflected image or the rotated image of an arrangement?
2. Can they identify the transformed image of a certain arrangement?
3. Which are the activities we can enlarge pupils' knowledge with or correct their recognized faults of thinking?

We assumed that in this age it is worth dealing with these questions through specific activities.

Our investigation consists of the following phases:

We planned a pilot study (in spring 2005) with pupils of grades 1–4. Our goal was to gauge the problems of elementary school-pupils in different ages in order to adjust the actual knowledge level for the full experiment. We chose three elementary schools in Debrecen, in Hungary. The first was the practicing school of the teacher training college. The pupils had very good abilities; they had been accepted to the school after a selection. We can say that average pupils attend the second school, and in the third school there are pupils whose abilities are average or below average, and whose social backgrounds are not optimal. We chose, in all, three classes from each grade. The classes were without any specification, their learning based on the normal curriculum of their school. With the selection of the classes participating in our experiment we tried to represent the real situation in the grades 1–4 in Hungary.

Grades	Grade 1	Grade 2	Grade 3	Grade 4	All
Number of participants	63	78	73	62	276

Table 1. The number of participants in the pilot study

We prepared the following paper-pencil tasks: (The first was used only for Grades 1–3, the second only for Grades 1–2.)

Problem of rotation

You can see the same disk in different situations. Colour the white squares!

(■: green: ■: yellow, ■: red, ■: blue, Figure 1–3.)

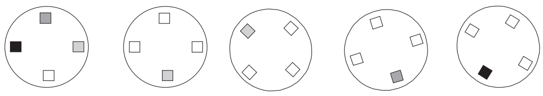


Figure 1. Rotation task for Grade 3

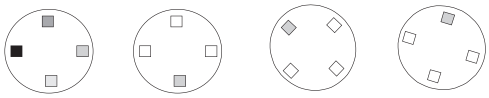
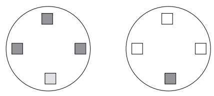


Figure 2. Rotation task for Grade 2

Figure 3. Rotation task for Grade 1

Problem of reflection



Colour the reflected images of the disks if the black lines means the position of the mirror! (Figure 4.)

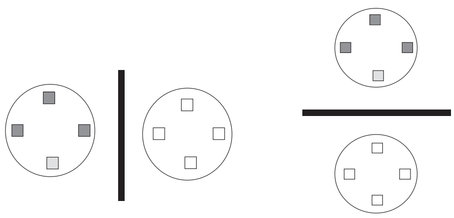


Figure 4. Reflection task

After this pilot study, in the next school year (in autumn 2005) we carried out a classroom experiment with pupils of Grade 2 (7–8 years old children). We chose the same class (27 pupils) from the practicing school of the teacher training college which participated as Grade 1 in our pilot study. Our aim was to try our ideas to develop pupil's ability in the field of spatial orientation, particularly of geometrical transformations. Grade 2 seemed a good choice because pupils are already familiar with school life, reading and writing. The results of the pilot study in grades 1, 3 and 4 were useful because of identification problems which remained and knowledge that was getting in every day life in this age. We prepared 3 problems on the topic of geometrical transformations in 3 different lessons. One problem required 10–15 minutes from the lesson. We planned the lessons together with the classroom-teacher, and discussed the problems after the lessons, but we did not teach.

We finished the classroom experiment with a post-test (in January 2006) and prepared a delayed-test for "our" second graders two months after. The post-test was solved not by the experimental class, but by other Grade 2 class from the same school (control class) too. We were interested in the development of "our" pupils comparing their results with other pupils' results. We wanted to know also about the spontaneous development of pupils who didn't pay special attention to the topic of orientation.

Results

Pilot study

Problem of rotation:

To solve the problem pupils have to imagine the process of rotation, have to do a mental rotation. Diagram 1 shows an overview of solutions.

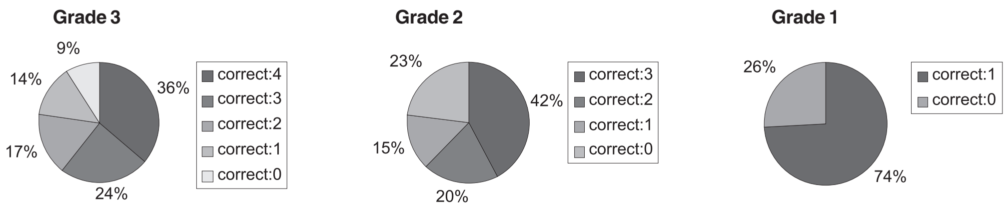


Diagram 1. Solutions of the problem of rotation in the pilot study

The direction of the rotation is indifferent; the cyclic order of colours will be the same in both directions. We wanted to know which graders are able to construct correct cyclic order. We can assume that pupils who colour all disks correctly, or made only one mistake have the competence to construct the rotated image of a discrete arrangement.

The number of these pupils is relatively low in any grade. In Grade 340% of pupils couldn't solve the problem. They didn't understand the task or couldn't construct the cyclic order.

Problem of reflection:

We allowed pupils to use mirror to colour the disks, but this tool didn't give support for everybody. Some of them weren't able to use it. Diagram 2 shows the solving strategies of first and second graders.

76% of pupils in both grades use the same strategies by colouring of disks independently from the position of the mirror.

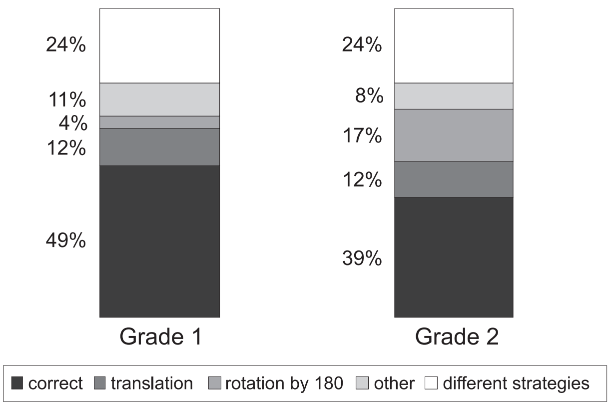


Diagram 2. Solving strategies in problem of reflection in the pilot study

"Translation" means that order of colours neither vertically nor horizontally changes (Y: yellow, R: red, B: blue, G: green, Figure 5).

"Rotation by 180°" means that order of colours changes both vertically and horizontally (Figure 6).

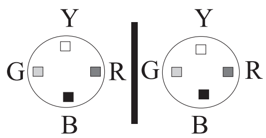


Figure 5. The "translation" strategy

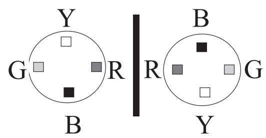


Figure 6. The "rotation by 180°" strategy

We can compare the results of the problem of rotation and reflection in Grade 1 and 2. (Diagram 3)

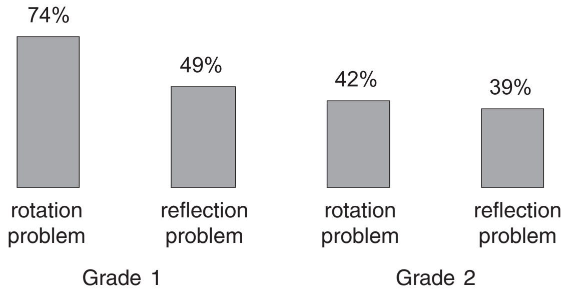


Diagram 3. Correct solutions percentages in the two problems of pilot study

Constructing of a reflected image is more difficult than constructing of a rotated image notwithstanding the use of the mirror.

Teaching experiment

Lesson 1:

We gave pupils a coloured hexagon from cardboard (Figure 7) and a mirror. We asked them to colour 3 rotated (Figure 8) and 3 reflected hexagons (Figure 9) on the paper adequately.

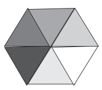


Figure 7

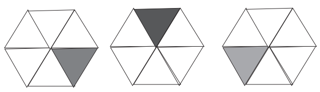


Figure 8

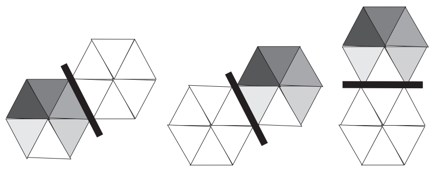


Figure 9

The work was very successful, almost all pupils coloured the hexagons correctly. (25 pupils, 5 hexagonal per person, only 5 hexagons from 125 was wrong)

Lesson 2:

Pupils worked with the same hexagon, but now we drew six coloured hexagons on a paper and they had to mark which the rotated image of the original one is.

From 27 solutions 14 were correct in all the six case.

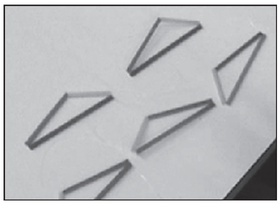
Lesson 3:

Pupils had to create different ordering of 3 coloured straws (Picture 1) then form triangles from them. (Picture 2). This activity helped children identify triangles by rotation and understand the differences between an image and a reflected image.

We had an interesting observation: Pupils sorted triangles in two groups, but they weren't able to conceive the connection between the two groups, e.g. they are reflections of each other (Picture 3).

Teacher: What did you recognise? These two are ……. of each other.
Gábor: Öööööö…
Teacher: Each other…..
Gábor and Fanni: ……….(listen)
Teacher: If I would put here something…..What should I put here? What should I put here to make it the same?
Gábor and Fanni: ………(listen)
Teacher: What?
Fanni: Red?
Teacher: Oh, no! I didn't think on a colour! Look at the triangles! You can see that they aren't the same. What are these two of each other?
Gábor: Pairs!
Teacher: What happen if I put a mirror here in the middle? What can we see in it?
Gábor: The same!
Teacher: Then they are ……
Gábor and Fanni: …….(They listen and don't know the word waiting the teacher.)

Picture 1



Picture 2

Picture 3

Pupils are able to distinguish the image and the reflected image on the level of manipulation. *Gábor* said that they are "pairs", so they are close to each other visually, but he isn't able to express this situation verbally.

The post- and delayed-test

Problem of rotation:

The problem in the post- and the delayed-test was the same as the problem of rotation for Grade 3 in the pilot study. Diagram 4 shows the results of the post- and delayed-test.

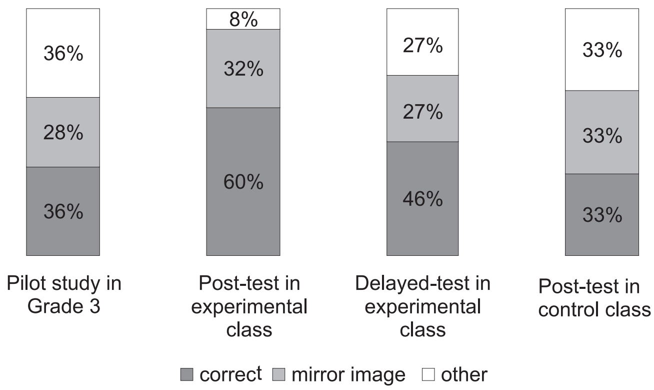
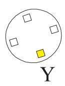


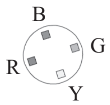
Diagram 4. The solutions of problem of rotation

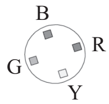
The "correct" answer means that all the 4 squares are well coloured.

The "mirror image" means that the order of colours is correct, but the direction of mental rotation is not.

For example in the case of disk 

:

If the correct answer: 

then the „mirror image”: 

Third graders in the pilot study and the control class achieved near the same result. The experimental class in the post-test was quite successful but the delayed-test shows that although the activities doing through the teaching experiment were useful, the stabile knowledge needs more experience on mental rotation.

Problem of reflection:

The problem of the delayed-test was the same as the problem of reflection in the pilot study for Grade 1–2.

Comparing the correct answers on rotation and reflection problems we see a slight incrase in „our” class in both cases (Diagram 5).

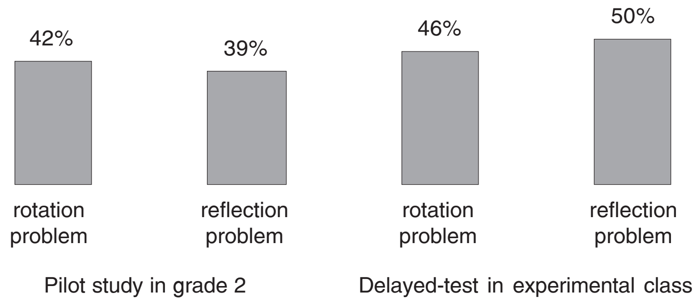


Diagram 5.
Correct solutions percentages in the two problems of pilot study and delayed-test

Conclusion

- Whereas in everyday life we use the cyclic order and cyclic orientation several times, the problem situations linking to them are almost unknown for pupils.
- Cyclic orientation assumes a dynamic situation, a rotation. Mental rotation especially in discrete case is quite difficult, while rotation with some concrete instrument is not.
- Construction of a rotated image is a simpler task than deciding whether an image is the rotated image of the others.
- The concept of cyclic order with different instruments and activities is developable effectively, but the development is a long-term period.
- The second graders are not familiar with construction of a reflected image of an arrangement. If they are experienced in using mirror, it can help drawing the image.
- The construction of the reflected image of an arrangement is more difficult then of the rotated image.
- Lots of different activities are preconditions of successful mental rotation and reflection.

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Building the concept of line symmetry

In his paper I discuss the key processes characteristic to the levels of reasoning in geometry. I have discussed particular of the thinking processes and their functioning by 12–16 year old students in relation to the concept of line symmetry on the plane.

Introduction

The nature of the concept of line symmetry is very composed. At primary school the students start to learn some aspects of this concept by using different tools: mirror (they observe mirror reflections), ink – stains and paper cut – outs. This kind of activities help the students to build the correct understanding of a concept. Van Hiele (1986) expressed important implications of his theory: students cannot show adequate performances at a certain level without having experiences which enable them to reason intuitively at each proceeding level. Hoffer (1986) claims that hands – on activities usually help students to perform at level 1 (Visualisation) and to move towards level 2 (Analysis).

A question arises whether or not using of these different tools (like mirror, paper cut – outs or ink – stains) is sufficient to build a correct properties of the concept of line symmetry? What kind of relationships do students observe while using these tools?

Van Hiele's levels of reasoning integrated by several key thinking processes which are characteristic of the levels may be useful for answering these questions. In order to evaluate a student's thinking level we have to evaluate the way in which the student uses the key thinking process.

Theoretical background

Gutierrez and Jamie have described different processes of reasoning as characteristic of several van Hiele levels (Gutierrez, Jaime 1994, 1998):

1. **Recognition** of types and families of geometric figures, identification of components and properties of the figures.
2. **Definition** of a geometrical concept. This process can be viewed in two ways: as the students formulate definition of the concept they are learning, and as the students use a given definition read in a textbook, or heard from the teacher or another student.
3. **Classification** of geometrical figures or concept into different families or classes.
4. **Proof** of properties or statements, that is to explain in some convincing way why such property or statement is true.

Table 1 summarises the key processes characteristics of each van Hiele level.

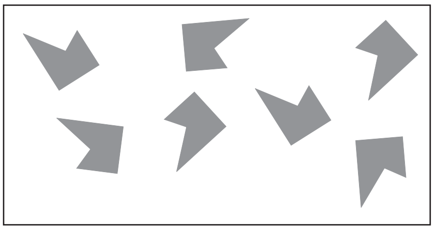
	recognition	Definition	classification	proof
Level 1	+	State	+	
Level 2	+	Read & state	+	+
Level 3		Read & state	+	+
Level 4		Read & state		+

Table 1. The key – processes characteristics of the Van Hiele levels

Each process is a component of two or more levels of reasoning. At each level the students show them in a different way. Based on the Gutierrez and Jaime proposition of a test for evaluating the level of student's thinking (Gutierrez, Jaime 1994, 1998) I prepared a test based on open – ended items that are not pre – assigned to a specific level, but to a range of the levels in which answers can be given (see *Appendix 1)*. Each of the key processes have been verified at least at two items. A test was solved by 15 students 12–13 years old from the 5th and 6th class of primary school and 15 students 16 years old from the 3rd class of junior high school. I thought, that the students participating in my research project could be in the 1st or 2nd van Hiele level of reasoning according to the concept of line symmetry. Therefore I have restricted to analysis of the key – processes characteristic to 1–3 levels. In this paper I would like to consider the key – processes of reasoning characteristic to van Hiele levels but I would not like to establish in which the van Hiele's level of reasoning the students are.

Results – analysis of students' answers

The test began from the task connected with observation and manipulation. From among the congruent figures, children were to choose pairs of figures, which were their own mirror reflection (*picture 1*). Figures should have been cut-out and pasted to the test. For each pair of figures there was a need to draw a line of symmetry. Students were informed that they could paste figures in any way, but not coloured side to the paper (it was not possible to flip any figures). From the mathematical point of view the goal of this task was to focus students' attention on a very important mathematical property of congruent figures: it is not possible to transform a figure on the plane into a figure symmetric to it, only by the movement on the plane (shift or rotation). The results were the following:

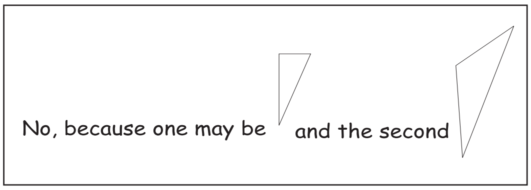


Picture 1

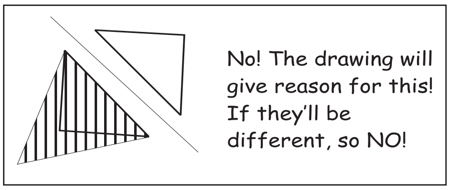
– 70% placements with a vertical line of symmetry with figures having a side parallel or perpendicular to the line of symmetry,
– 13% placements with a slanting line of symmetry with figures having a side parallel or perpendicular to the line of symmetry,
– 10% placements with a horizontal line of symmetry with figures placed oblique to the line of symmetry
– 7% wrong arrangement of figures (often point symmetry).

The second task: "*Is it possible for any two squares in the plane to be mirror reflections of one another?*" and the task no. 5: "*Is it possible for any two triangles on the plane to be mirror reflection of one another?*" concerned the classification of figures among one family of shapes (squares, triangles). In order to two triangles might fulfil the relation of mirror reflection they must be the same shape and size and proper orientation. All the students stated that the relation of mirror reflection on the plane fulfil only the triangles with the same shape and size. Students supported their thesis in the following way: Triangles can be different, because there are different kinds of triangles. Such triangles cannot be their own mirror reflection. Younger students, as opposed to older ones, often made a drawing (*Example 1, 2*).

Example 1

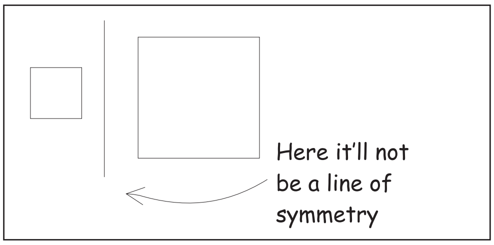


Example 2

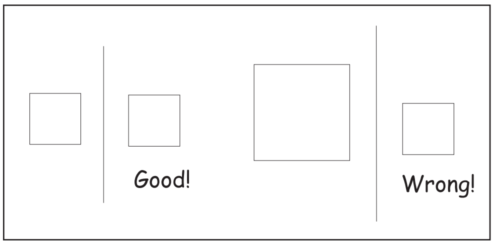


The question concerning squares had a different meaning for students. 2/3 of all students answered that the squares must be of the same sizes in order to fulfil the relation of mirror reflection. Among squares there are not figures with different shape. From that reason students did not mention about the same shape of figures. In this task younger students did the drawing as well. All the drawings were similar but they had different remarks (*Example 3, 4)*.

Example 3

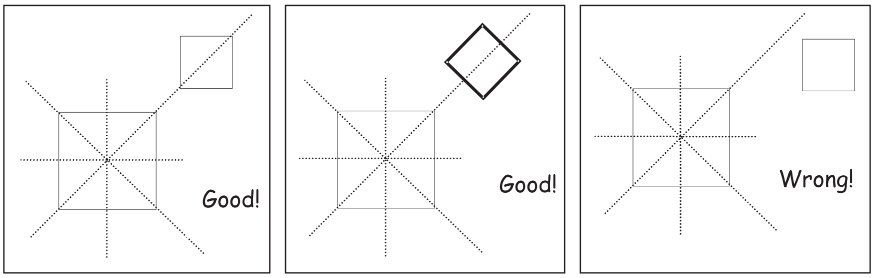


Example 4



In group of 12–13-year old kids, most of students (80%) answered that not each of squares on a plane is the mirror reflection of the other one. Among all remarks, the one stood out: "*I think that not, because if there is one and we shift the second one a bit irregularly that it will not be symmetry*". It was a different argumentation from the others such as: "Yes, because…", "No, because …". The lack of references to the same size of squares focuses our attention. Here "is one" square, so it will be whichever and freely placed. One added the other one, whichever size too. Adding the second square we can damage the symmetry, if "we shift the second one a bit irregularly". The question arises, how the child understand the essence of the task and what he/she expressed by this answer. In this case (maybe it has connection with school's experiences) the child thought about drawing of two squares on the plane and about the situations, in which that drawing has a line of symmetry. He/she focused attention on a specific placement of figures in mirror symmetry. A square has four lines of symmetry. Adding the second one we have to place it on the one of existing lines of symmetry in order to the whole drawing still has a line of symmetry.

Example 5



It is possible that the child have understood the question as follow: “does always the drawing of two squares present mirror reflection?” and gave the exact answer. At school she/he was checking different pictures – more or less complicated – if they have a line of symmetry. She/he had a lot of mental images of symmetric figures and concerned her/his attention of a placement’s relationships.

In group of 16-years old a half of students answered that not each square could be place so they would be mirror reflection of each other, because “*squares would be of different sizes*”. The second half of students answered that each square could be place so they would be mirror reflection of each other because “*all squares are the same*”.

The arrangement of students’ answers was very surprising. All of them focused their attention on a shape and size of figures. As a result of conversations with students took place after the test it turned out that in different ways they understood the question. It shows the dialog placed in appendix 2. At first understanding of the statement „does each square” meant for students participating in the conversation “the square and its reflection about freely placed line of symmetry. In this case there existed two squares. Having one square we always can get the second one as the image of the first one in the line symmetry. With regard for the special shape of square (it is the same from each side) it is possible to draw the line of symmetry in any place and in any direction. The figure after reflection always looks the same like the first one (has the same shape). Students clearly claimed that “each two” means the first square and that, which we received as its image in line symmetry. It is not important which of the squares is “the first”. Always one of them is “its own image”. Statement “is its own image” is used with reference to figures having line of symmetry. Here this statement was understanding in another way. This second figure is “its own reflection” in the same way as my reflection in the mirror is my reflection (not the reflection of other person).

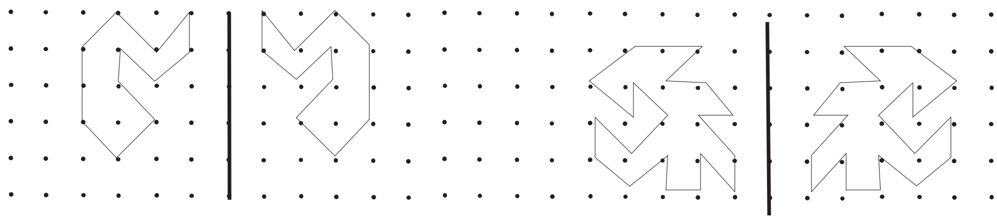
As a result of the conversation, drawing different squares – small and big – the meaning of “each two squares” was questionable. At first it meant two same figures (congruent). Bit by bit it extended to similar figures. One of students quickly understood the meaning of “each”. He claimed that only one answer is correct and it is the answer “not each square”. His college agreed that in the case of congruent squares we can answer “Yes” but in the case of different squares it must be the answer “No”. She did not accept that these two particular answers give the general statement “no”.

Summarize, the main reason of discrepancy between answers for the question “Is it possible to place each square on the plane so they would be mirror reflection of each other” was the language of the task. For 16-years old students the statement “each two squares” had different meaning either. It might concern any squares chosen from the whole family of squares on the plane. It might concern “any” squares that is “as I like to have”.

Van Hiele emphasizes that each level of reasoning has its own language. Moving from one level to the other one manifests in a language. In that task the language referred to the level 3 or

4 and was different understanding by students because it might be a language from another level, which was inaccessible for students at that moment.

In the task no 3 children drew on dot paper any figure and its image in mirror reflection (an axis was not given). All the drawings were correct, even very complicated.



Among figures dominated squares (30%), figures with very complicated shapes (27%), triangles (20%). Therefore the shape of drawing figure does not indicate the level of concept reasoning. At a visual level a child is able to draw very complicated figures and their images if she/he make task on dot paper. She/he knows that she/he has to draw the same figure (keep the same shape and size) but "in the other side" (left – right figures). After drawing symmetric figures children explained why those figures are symmetric:

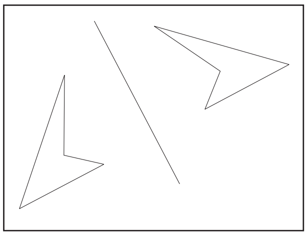
– Because if we put a mirror we will see that they are their own mirror reflection (30% of the whole answers)
– "they are the same" (40% of the whole answers), sometimes with remark „and between them there is an axis of symmetry" (but child did not draw an axis),
– "after folding a sheet of paper about the axis the figures overlaps (10% of the whole answers),
– "because every point in the figure has its own reflection on the other side" or "if we draw a line from vertex A it will touch A', similarly from B – B' etc." (10% of the whole answers), where student appeals intuitively to transforming vertexes.

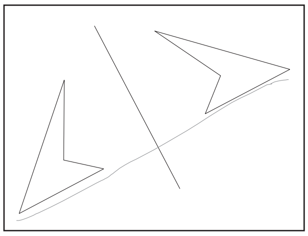
In one more task children were asked about justification how to check if two drawing figures were symmetric about an axis. Most students answered "with a mirror" – 2/3 of whole students. For younger children (12–13 years old) a mirror is a tool using during mathematics lessons. That is why those students know that this tool help to check a drawing, especially if the drawing is complicated. For these students line symmetry is closely associated with the mirror. They do not know the mathematical definition of this transformation. The older got to know this definition. In their justification they did not feel the need of referring to property from definition.

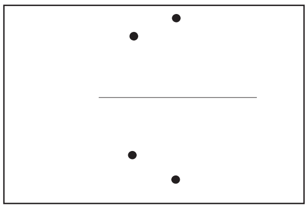
For both younger and older students the mirror still remains important tool. Among 15-years old 40% of them opined that it is need to use a mirror either, 20% that it is need to fold the sheet. The rest of them opined that it is need to measuring something: either the distance between figures or the distance between corresponding vertexes. It points at the students transformed the figure but not points (vertexes). It did not appear references to the point and its image like in the definition. Viener and Hershkowitz (1980) claim that in thinking, people do not use definitions of concepts, but rather concepts images, combinations of all the mental pictures and properties that have been associated with the concept.

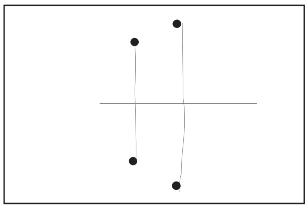
Various types of argumentation show that children function on different levels. However, contrary to expectation the older children (15-years old) did not function on the higher level.

Children recognized figures symmetric about a line very well. Occasional mistakes appeared when the axis was oblique, and figures did not have any side parallel or perpendicular to the axis. Justification why figures are symmetric about the drawing line children mentioned more property than shape and size, for example: “if one put a mirror, the figures would overlap”, “one can fold the sheet”. In a bit complicated situation for children (*picture 2, 3*) the deeper analysis and another argumentation appeared: “figures are placed equally from the line” or “figures begin on the same level (supported by a drawing of a segment connected to corresponding sides or vertexes. – *pic. 2a*). Children were not able to express by mathematical language relationship between a point and its image, although they intuitively felt it. They knew that figures must to have the same distance from the axis (what means the same distance between a point and its image from the axis) as well as figures begin on the same level (what means that point and its image lay on the same line perpendicular to the axis).


Picture 2


Picture 2a


Picture 3


Picture 3a

Similarly, for pic.3 children gave argumentation: “dots are in the same place from axis” or “connected points are in the same distance”.

I observed among all students justifications why figures on the drawing are / are not symmetric two types of argumentation:

I. “the line of symmetry is arranged correctly”

II. “figures are arranged correctly with respect to the axis”

According to the students opinion a part of drawings do not present symmetric figures about given axis for two reasons:

– “because figures were placed wrongly” – but not one figure placed wrongly (wrong image of a given figures about given axis)
– or “the axis of symmetry was placed wrongly”.

Explaining why figures on the drawing were not symmetric about given line children staked out: “the placement of the axis should be changed”, “the axis is placed wrongly”. Children justifying that the drawing is correct: “the axis is good placed”, “the line is perpendicular to the dots”. It was necessary to change the line of symmetry to correct the drawing and to change an image of a figure.

From the research of E. Swoboda (2006) results that in situation when children constructed axis-symmetrical mosaic the axis of symmetry existed in their mind though it was not drawn. Imaginary axis organised surface of a sheet. Children act in a different way when arranging figures on the plane and when analysing a placement of two figures on the plane. If they select figures self-dependently and compose them symmetrically on the sheet of paper, the axis exists in child’s mind and determines the placement of figures. On the other hand when they have two figures on a picture and they have to determine if figures are/ are not symmetric about drawing line a configuration of figures moves forward, to the first position, and dominates. It is possible only to change a position of an axis of symmetry. Mathematical definition is different: first a line is given and then we define the transformation. Farther activities we apply to existing axis of symmetry.

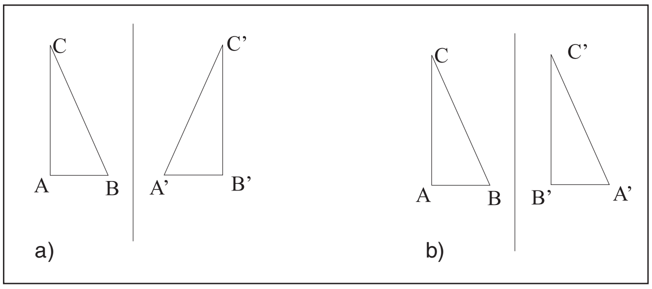
The key – processes characteristic to the van Hiele levels with the reference to the concept of line symmetry

Comparing my research to the Gutierrez and Jaime description of the key – processes characteristic to van Hiele levels I have made some remarks.

Recognition by the students at level 1 is limited to physical, global attributes of figures. They sometimes use geometric vocabulary, but such terms have a visual meaning more than a mathematical one. However students at level 2 or higher, are able to use and recognize mathematical properties of geometric concepts. For that reason the ability of recognition does not discriminate among students in the van Hiele levels 2, 3 or 4.

All the students recognized correctly symmetric figures. Both – the younger (12–13 years old) and the oldest (15 years old) used the geometrical language in a low degree. They used descriptions: "tip", "dot" instead of mathematical terms like vertex, point. They based on the visual assessment of properties but these properties had mathematical meaning for them. The same shape and size they referred to congruent figures.

When the students had two important visual information: one about a shape and the second about the vertexes, they have had difficulties in recognition which one of this information is more important (*picture 4*). Sometimes the information about the vertexes was stronger and they indicate the situation *4b* as correct.



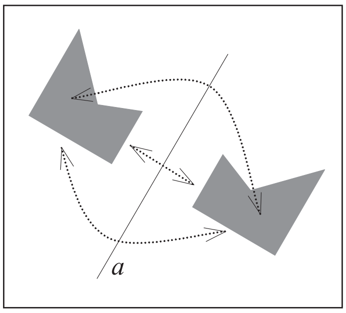
Picture 4

According to the next key – process, students at level 1 are not able to use a given mathematical **definitions**. The only definitions they can formulate consist of descriptions of physical attributes of the figure they are looking at and perhaps some basic mathematical property. The students mentioned generally the mathematical properties: the same / different shape and size.

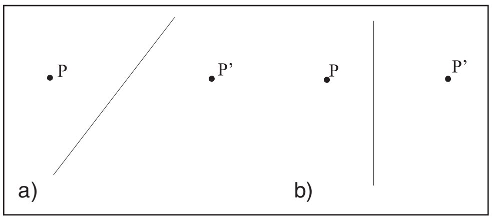
Gutierrez and Jaime discriminate to processes: "**read**" and "**state**" **definition**. Students participated in my research can both, "read" and "state" definition but they experienced difficulties with giving all important properties or only the necessary conditions.

While "state" definition, they referred to the basic mathematical properties: shape, size and distance. Their justification why the figures are/ are not symmetric about the given line referred to the shape and size of figures. It was very difficult to them to name any other property they have observed. They pointed at only one property to one given situation as: "the shape was not changed", "the size is the same", "the same distance between figure and a line", "it is mirror reflection", "if you put the mirror, the figures wholly overlaps". They did not write a complete list of properties. They mentioned sometimes that figures were "in different direction" (that means for them left-right figure). When drawing the image of the figure in mirror reflection they always put the ruler so as it was perpendicular to the line of symmetry and they did not verify if it is really a right angle. When they are asked why the figures are symmetric, they did not mention that property explicitly but they give the answer connected with this property. They stated that the figures "begin at the same place" or "at the same level" and they pointed at the line between appropriated sides or vertexes (the language of gesture was important and made their description easer).

When students at level 2 know every property contained in the definition, they can use it, but they may experience difficulties with understanding of the logical structure of definitions. When they are asked for a definition that has not been learned by rote, their answer may not include some necessary property that the students use implicitly. I observed that in the case of the known but not reminded definition the students also mentioned only one property: the distance from the line. They did not mention the other properties the definition includes: perpendicularity, point and its image, a distance between point and line of symmetry. When "read" definition the students focused very seldom on the necessity of lying of a point and its image on the line perpendicular to the given axis of symmetry. They felt this property intuitively drawing a line connected two corresponding vertexes. More frequently they pointed at corresponding sides (*pictures* 5).



Picture 5. First the student pointed at corresponding sides parallel to the line *a*, second – at sides perpendicular to the line *a* and then at the whole figures

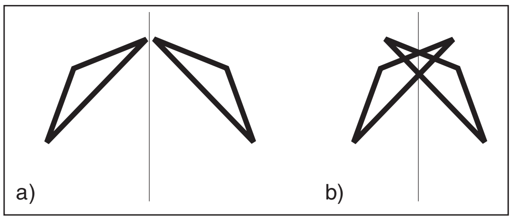


Picture 6

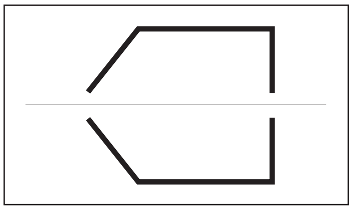
Easier for them was justification why the figures were not symmetric about drawing line then why they were symmetric. Students did not have difficulty in situation shown on pic.6b, were figures are not symmetric because they have different distance from the line. However in situation on pic.6a it was difficult for them to give a proper explanation (the figures do not lay on the line perpendicular to the axis of symmetry).

In "reading" definition students' attention focused on property concerning the distance of a point and its image from the axis. They were able to use it in the process of "state" definition. The second property – laying on the line perpendicular to the axis – functioned in the intuitive range.

Student at level 1 can understand only exclusive **classifications**, since they do not accept nor recognize any kind of logical relationships between classes nor, many times, among two elements of the same class having quite different physical appearance. When they are asked if the figures are symmetric about the given line they have accepted the situation on picture 7a, but they have had difficulties with estimate of the situation on picture 7b. In the case of *picture 8* they need to "close" a figure to the common shape.



Picture 7

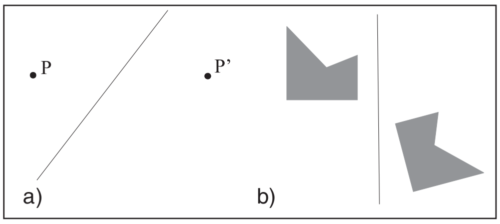

Picture 8

According to the concept of line symmetry or mirror reflection students in level 1 may accept only some of very typical situations: with horizontal or vertical line of symmetry or with figures having some of sides parallel or perpendicular to the line of symmetry. Otherwise they would “change” the position of line of symmetry in order to “fit” the position of figures on the plane.

A more accurate discrimination between students in level 2 or 3 is based on the ability to accept and identify non – equivalent definitions of the same concept and to change one’s mind about the kind of classification, exclusive or inclusive, when the definitions are changed but in this research I did not discriminate among students in level 2 or 3.

Conclusion

The important conclusion from this research is the statement about the role of the line of symmetry which I have observed. For some children the figures were not symmetric about the given line because “the line was placed wrongly”. One had to change the position of the line of symmetry in order to “repair” the given drawing. The students did not think about the figure and its image in this transformation. They consider pare of figures and a line between them. First the figures were placed and then the line of symmetry was added. Probably it may be the influence of the tools used at school during mathematics lessons.



Picture 9

Very common and useful for pupils was a mirror. It seems to me that the use of a mirror may get them accustomed to a pair of figures. There is a necessity to use at school different tools which emphasise the role of a line of symmetry and the relationship between the placement of an image of a given figure and the position of a line of symmetry (Jagoda 2005, 2007).

The research showed that regardless of the age difference between students they hardly used a mathematical terminology and they understood this language in a different manner. Also, regardless of the age difference they referred to basic mathematical properties (shape, size). The most difficult key thinking process for them was to “read and state” the definition.

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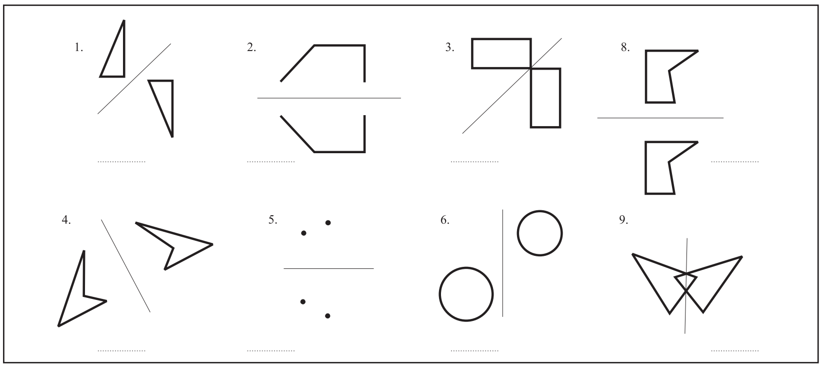
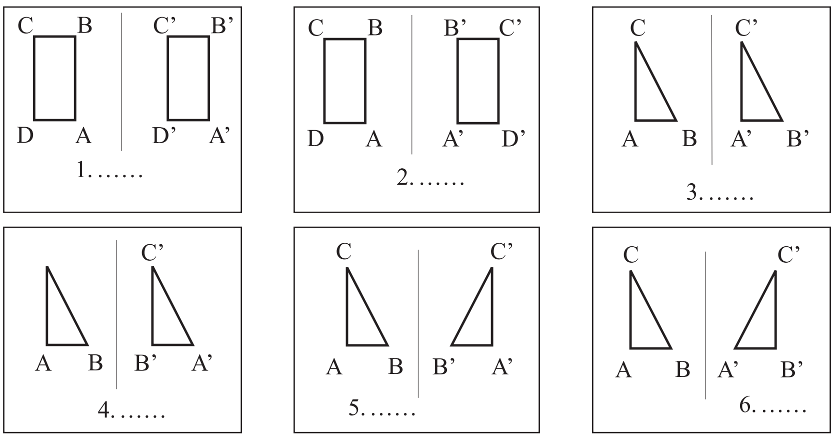
Appendix 1

2. Is it possible to place any two squares on the plane so they would be mirror reflection of one another?
3. Draw any figure and a figure symmetric to the first one.

4. Explain why the figures you drew are symmetric?
5. Is it possible to place any two triangles on the plane so that they are mirror reflection of one another?
6. How can we verify if two figures are symmetric?
7. Write a proper letter under the drawing:
   S – if the figures are symmetric about the given line
   N – if the figures are not symmetric about the given line



8. We adopt a convention: the image of a point X in the line of symmetry about any axis will be denoted by X'. Write a letter **S** if the figures in a picture are symmetric about a given line or a letter **N** if the figures are not symmetric about the given line.



9. Explain, why the points in the drawings below **are not symmetric** about the given line.

Appendix 2

T7: Task 2: *Is it possible for any two squares in the plane to be mirror reflections of one another?*

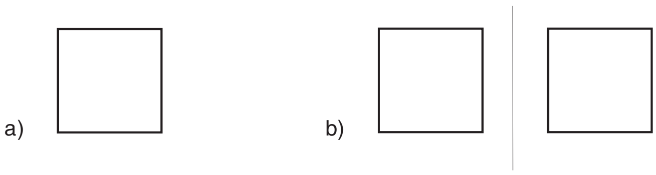
J8: Yes, because a square has equal sides, and all the angles have 90°.

K9: A square looks the same from each side. That is if we reflect it in any way, it would look the same. If it was a concave figure, then it would be a different matter. However a square is the same from each side.

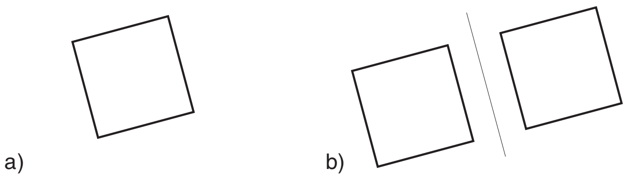
T10: What does it mean for you "any two"?

K11: The first and the reflected one.

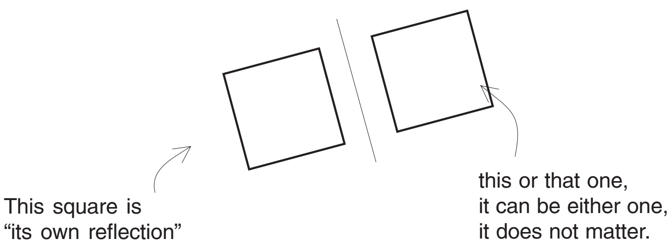
T12: So, if the first would be like this
(The teacher draws a square on a paper. Kasia draws an axis of symmetry and then the second square on the left side of the axis)



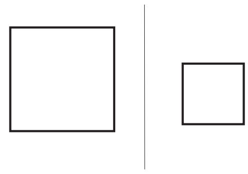
T13: What if you have this:
The teacher draws a square. Kasia draws an axis of symmetry and then the second square on the left side of the axis.



J14: It should be "any two squares". And then the answer should be "Yes", because a square has equal sides. *Kuba indetifies this in the picture:*



T15: What does it mean for you "any two"?
J16: The same two figures.
K17: But there would be small and big squares...
K18: It is not possible .., if one is larger than the other, then they won't reflect in the same way.
T19: What if the first is like this: *(The teacher draws a smaller square and Kasia draws a line of symmetry and the second square)*

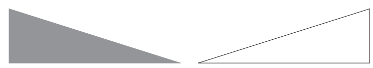


J20: It is "reflection to scale". Does there exist something like that?
J21: The answer should be "Yes", provided both squares have the same size.
T22: Thus, "any two squares"...?
K23: Yes, but only if they have the same size. "No", if they have different sizes.
J24: Nevertheless "every" – but every means we take all squares into account, that is small ones and large ones.
J25: I think the answer should be "No". We have small and big squares, that is different squares. So, not all of them can be paired together.
K26: What about you? I think the answer should be "Yes" if the squares are the same and "No" if they are of different sizes. Here we do not have a specific set.
J27: We have – small and big squares. We talk about the whole family of squares – small and big ones. The answer will be "No". Definitely it should be "No".
…
T33: What about triangles?
K34: Not every triangle because there are different triangles.
T35: If the triangles are the same?
K36: So "Yes".
J37: No, because one has to be flipped. If it's not, it won't work.
K38: What?
T39: If they were cut–out triangles to be glued on the paper, like in the first task?
J39: If we paste them on the paper, one will be grey and the second white.
Kuba draws.

Example 10

a)

If there are triangles like these two, it fails.



b)

If you paste them on the paper, one will be gray and the second white.Well, it won't work.

Teachers' professional development

Part 3

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Investigating mathematics: A challenge for students, teachers, and mathematics education researchers

I start this paper by considering different conceptions of mathematics and showing how exploring and investigating are central to the mathematical activity. Next, I present several examples of students investigating mathematics in the classroom that illustrate important aspects of an exploratory approach to mathematics teaching and learning. Such as approach does not depend only on the nature of tasks, but requires an analysis of the roles of teachers and students, the communication patterns in the classroom as well as the overall organization of content and processes in meaningful mathematics teaching units. I indicate that this kind of teaching is rather demanding and refer three main conditions to help teachers in developing professionally to carry it out: collaborating, researching their own practice and getting involved in the professional community. I close with a brief discussion on the relationships of investigating, teaching, and learning. I argue that as students explore and investigate mathematics, teachers need to investigate their own practice in professional collaborative settings.

1. Conceptions of mathematics

There are many perspectives about mathematics. Most dictionaries present it as the *science of number and form* (Davis & Hersh, 1980). For many mathematicians, mathematics is the *science of proof*. We may recall the famous saying of Bertrand Russell: "mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true" (Kline, 1974, p. 462). Jean Dieudonné put the same idea in a shorter way: "qui dit mathématiques, dit demonstration"[1]. The structuralist movement of the first half of the twentieth century encouraged the view of mathematics as the *science of structures*, and that framed the program of Bourbaki and influenced a deep educational reform in the 1960s known as "modern mathematics". Still another view claims that mathematics is best described as the *science of patterns*, aiming to describe, classify and explain patterns in manifestations such as number, data, forms, organizations, and relations (Steen, 1990).

When we think about mathematics we may focus on the body of knowledge enbodied in articles and books or in the activity of people doing mathematics. Regarded as an activity, mathematics is indeed a dynamic science. That is well captured by George Pólya (1945), who says "mathematics has two faces; it is the rigorous science of Euclid, but it also something else [...] Mathematics in the making appears as an experimental, inductive science" (p. vii). It is also sustained by Irme Lakatos (1978) that states "mathematics... does not develop through monotonous

[1]Who says mathematics, says proof.

growing of the number of theorems unquestionably established but through the increasing improvement of conjectures by speculation and critique, by the logic of proofs and refutations" (p. 18).

Mathematics can be an interesting activity not only for the mathematician but also for the teacher and the student. Singh (1998) refers that Andrew Wiles, now famous for his proof of a long standing theorem, recalls the role of his teacher in getting him involved in mathematical explorations:

> Since I found for the first time Fermat's Last Theorem, when I was a child, this has been a major passion... I had a high school teacher who did research in mathematics and gave me a book on number theory and provided some hints on how to attack it. To begin with, I started from the hypothesis that Fermat did not know much more mathematics than me... (p. 93)

Another mathematician, Jacques Hadamard (1945) states that there is no major difference in the mathematical activity of a student and a mathematician when they are solving problems and exploring mathematical relationships:

> Between the work of the pupil who tries to solve geometry or algebra problem and creative work [of a mathematician], we can say that there is a difference in degree, a difference in level, both having a similar nature (p. 104).

In mathematics, the starting point for an investigation may be a mathematical or a non-mathematical situation from other sciences, technology, social organizations, or daily life. As an activity, a mathematical investigation includes the formulation of questions, the production, testing and refinement of conjectures, proving and communicating results. Carrying out a mathematical investigation involves conscious and unconscious processes, aesthetic sensibility, and connections and analogies with mathematical and non-mathematical situations. It is undertaken by different people with different cognitive styles – analytic, visual, conceptual (Burton, 2001; Davis & Hersh, 1980).

2. Students investigating mathematics in the classroom

Let us consider some examples of students working as mathematics researchers.

Example 1 – Working with numbers. The first example comes from a class led by Irene Segurado, a grade 5 teacher working with 10 year old students (see Ponte, Oliveira, Cunha & Segurado, 1998). The task is the following:

1. Write in column the 20 first multiples of 5.
2. Look at the digits of the units and tens. Do you find any patterns?
3. Now investigate what happens with the multiples of 4 and 6.
4. Investigate with other multiples.

This task was presented at the beginning of a 50–minute class. The teacher had planed group work, but she found the students very agitated at the beginning of the class and decided to redefine her strategy and work as a whole class. A list of multiples of 5 was written on the board and students begun looking for patterns:

> Tatiana, raising her arm, answered quickly: The units' digit is always 0 or 5, and that was accepted by her colleagues, echoing around the room: it is always 0, 5, 0, 5...
> Teacher: What else?
> Octávio, with a happy face: The tens digit repeats itself: 0–0, 1–1, 2–2, 3–3...
> Carlos, agitated: I discovered something else... May I explain at the blackboard? (...)

> As he got to the blackboard, he explained: 0 with 5 is 5, 0 with 0 is 0, 1 with 5 is 6, 1 with 0 is 1, 2 with 5 is 7, 2 with 0 is 2, 3 with 5 is 8, are you getting it? There's a sequence. It's 5, it jumps one, it's 6, jumps one, it's 7... Or it's 0, jumps one, it's 1, jumps one, it's 2... (Ponte et al., 1998, pp. 68–69)

We see that the students were able to identify different kinds of patterns. They noticed simple repetition patterns (such as 0 5 0 5…) and more complex patterns combining linear growth and repetition (such as 1 1 2 2 3 3…). They also combined different elements to identify linear patterns as subsequences of rather complex patterns (0 5 1 6 2 7 3 8 …).

The class also analyzed patterns in the multiples of 4. Then, they turned to the multiples of 6 that were put in a column side by side with the multiples of 5 and 4.

0	0	0
5	4	6
10	8	12
15	12	18
20	16	24
25	20	30
30	24	36
35	28	42
40	32	48
45	36	54
50	40	60
55	44	66
60	48	72
65	52	78
70	56	84
75	60	90
80	64	96
85	68	102
90	72	108

Students' discoveries were coming in bunches. They were rather excited, thus creating some difficulties to the teacher in recording and systematizing their contributions:

> The units' digit is always 0, 6, 2, 8 and 4.
> The units' digit is always even.
> The tens' digit does not repeat from 5 in 5.

The teacher tried to handle this enthusiasm: *Take it easy! Let us verify if what your colleague said is true. Attention! Look! Look how interesting what your colleague discovered!* Suddenly, Sónia said: *There are the same digits that for the multiples of 4.* Even before this statement made any sense to the teacher, Vânia continued: *But they are in a different order.* The teacher figured out that the students were comparing the multiples of 4 and 6, and she explained that to the class. Other students went on:

> It also begins with 0.
> The other digits are in a different order.
> There are multiples of 4 that are also multiples of 6.
> The multiples of 6, beginning at 12, are alternately also multiples of 4.

The students could find again complex repetition patterns (such as 8 2 6 0 4 8 2 6 0 4 8…) and, more interesting, they were able to compare features of different patterns. In this activity they developed their number sense, they got a better grasp of the behaviour of multiples, and they did a lot of mental computation. They expressed their generalizations in natural language.

In her reflection, Irene indicates that the students surpassed all her expectations. She says: "I had not foreseen the hypothesis of comparing the multiples of the different numbers, because I had never put them side by side. Therefore, I experienced their discoveries with great enthusiasm" (p. 71). She also reflects on the implications of working as a class, as compared to small groups: "The contribution of a student was 'picked' by all his colleagues, yielding a greater number of discoveries" (p. 72). It would seem that in curriculum topics such as multiplication facts, multiples, and divisors, at the elementary school level one can just do routine exercises. This experience shows that, on the contrary, these topics allow for much exploratory and investigative work.

Example 2 – How is the typical student in my class? A second example comes from a class of Olívia Sousa, a grade 6 teacher working with students aged 11 (see Sousa, 2002). The task was organized as a statistical investigation: "Imagine you want to communicate to another student in a distant country, or, who knows, to an ET, how students in your class are?…" This was meant to have students taking all kinds of measurements about their bodies and collecting data about their families, which usually raises high levels of students' enthusiasm.

Five 90-minute blocks were planned to carry out this task, with students working in small groups. In practice, six blocks were necessary. The teacher divided the whole task in four main steps: (i) preparation of the investigation questions; (ii) data collection; (iii) data analysis; and (iv) reporting the results. In each step some written instructions were provided to the students. See below, for example, the directions for step 2.

With your colleagues:
- Write as a question each one of the characteristics that you are going to investigate.
- What answers do you expect to obtain for your questions?
- How (through observing, measuring or a questionnaire) can you get the answers to your questions?
- Prepare data sheets to collect the data.

In this class the statistics measures (mean, median, mode) had not been taught before. A major decision in this experiment was to have the students working with their previous knowledge of these measures, instead of teaching them first and after propose that they do application exercises to practice. Therefore, along with several other directions, the students were asked to find the mode (indicated as "the most frequent value"), the median (the "middle" value), and the mean. They had no trouble in finding the most frequent value. To find the median took more time, but when they realized that they could order the values, it became a rather easy task for most students. There were a few problems as some students forgot to count repeated values or took the median as the average of the extremes. The class discussion was a good setting to sort these things out. The students had already a strong intuitive notion of mean as the result of summing two values and dividing by 2:

Inês: Then we put 1 and 35.
Alexandre: 1 and 40.
Prof. How did you get 1 and 35?
Inês: Pardon?
Prof. How did you made that 1 and 35?
Inês and Estelle: It was an estimation!
Inês: It is not as Mauro (1,20 m) nor as myself (1,50 m),. It is in the middle.
Estelle: It is between.
Inês: It is between the two.
Estelle: Mauro and Inês.

With the help of the teacher, they were able to generalize this intuitive notion to find the mean of more than just two numbers.

In her reflection, Olívia considered that carrying out this task was a significant learning experience, of an experimental nature, in which the students worked mathematics contents of two domains, statistics and numbers and computation, in an integrated way. Decimal numbers, obtained from measuring quantities associated to the body, were no longer abstract quantities and acquired a strong meaning. The manipulation of these numbers in a significant context – comparing, ordering, sorting, and operating – contributed towards students improving their global understanding of numbers. Olívia considered that, regarding statistics topics, the contact with different kinds of variables and with different ways of collecting, organizing, and representing relevant and meaningful information, promoted students' understanding of the statistics language, concepts and methods that went much beyond simple memorization. This example shows that undertaking an investigation based on the students' reality can be the starting point to develop investigation competences, to learn new mathematics concepts (in this case, statistics concepts), and to practice and consolidate existing mathematics knowledge.

Example 3 – How to amplify? The next example concerns an experience carried out by João Almiro (2005) a grade 8 teacher. The task is indicated in figure 1.

> The Visual Education teacher wants to amplify the picture below but she put the following condition: the area of the amplified picture must be 400 times larger than this. The teacher is going to do a overhead transparency with the picture and project it in the wall. But she has a big problem: At what distance she must put the overhead projector from the wall? How can we help her? Write a report that includes the description of your investigations, the computations that you made, your conjectures and possible solutions.

(M.C. Escher, 1965)

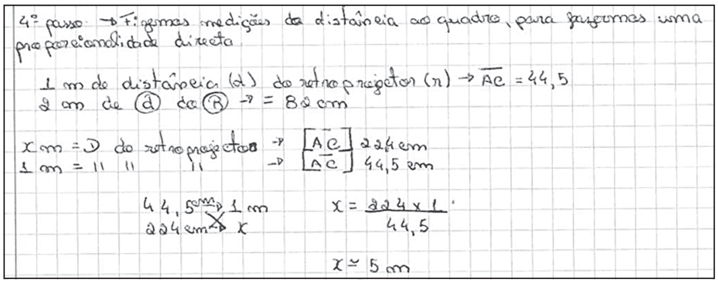
Figure 1. How to make it 400 bigger?

The students had to design their own strategies. The teacher prepared the room with four overhead projectors (each one to be used by two student groups) and gave a metric strip and a ruler to each group. The room was a little small for the projectors but work was possible anyway. The teacher did not provide further instructions, just said that it was a request from the Visual Education teacher. Of course, many students did not believe this.

The reactions from the groups were very different. Some were lost, not knowing what to do. As one student wrote in a final questionnaire: "I felt some difficulties with the overhead projectors since in the beginning we did not know where to start". Others, immediately started trying to find ways of doing the task. The teacher was pleased to notice that all the groups understood that the projected rectangle would need to have length and width 20 times larger that the initial picture, so that the area was 400 times larger. The students had solved problems involving enlargements before and were able to mobilize this previous knowledge.

The big difficulty of the students was finding the distance that they should put the overhead projector from the wall so that the length and the width amplify 20 times. Almost all the groups constructed a rectangle with the dimensions of the picture. They projected, they measured what they found, and then they figured out how many times the length and width were now larger. They quickly understood that they did not have space in the room to enlarge the dimensions 20 times and, therefore, they had to use a computational strategy to know what distance the overhead projector had to be from the wall.

In one of the groups, the students understood that there was a direct proportion between the distance of the overhead projector from the wall and the number of times that the dimensions were amplified and quickly solved the problem. Other four groups, however, had much more difficulty. Helping each other, they went on measuring and arguing and when a group arrived to a conclusion it was shared with the others. They progressed sometimes making conjectures that the other groups refuted and proved that were not correct. They arrived to solutions that the teacher considered acceptable. This is the final part of the solution of one of the groups that used the notion of unit rate and the cross product:



4º passo → Fizemos medições da distância ao quadro, para fazermos uma proporcionalidade directa

1 m de distância (d) do retroprojector (r) → $\overline{AC}$ = 44,5
2 m de (d) do (R) → = 82 cm

x m ⇒ D do retroprojector → $[\overline{AC}]$ 224 cm
1 m = " " " → $[\overline{AC}]$ 44,5 cm

44,5 cm → 1 m
224 cm → x

$x = \frac{224 \times 1}{44,5}$

$x \simeq 5$ m

Figure 2. Unit rate and cross product to solve the amplification problem

Measuring the picture, they found that it was a rectangle with 11,2 cm by 7,9 cm. Enlarging the length by 20 yields 224 cm. As they found that with the projector 1 m from the wall transformed this length in a segment with 44,5 cm, using the cross product, they found the required distance. For three other groups this was a very difficult task, and they were not able to do it, even with frequent help from the teacher.

Some students (about 1/5) reported a negative view of these classes. One of them wrote: "I didn't like these classes (…) I think that I learn more in classes doing exercises and asking questions". However, other students were pleased and recognized that they had significant learning. As one of them said:

> The problems are a bit more complicated that those from other classes, at lest the overhead one, in which we had to think a lot, develop, we had to think different methods, to achieve the ideal method to get the correct result. We had to begin by finding out what was to do. In textbooks, the questions are direct, they tell us immediately what we have to do.

These responses from students show that not all of them will get very excited when the teacher presents challenging tasks. It is not because of "motivation" that these tasks have an important role in mathematics teaching. It is because they may promote significant learning. Working on this problem the students were called to draw on their previous knowledge of similarity, area, and direct proportion. They also had to design a strategy to collect data to figure out the relationship of the distance of the overhead projector to the wall and the size of the image.

Example 4 – Numerical equations. This example is drawn from a teaching experiment carried out by Ana Matos (2007) in her grade 8 class, involving an algebra teaching unit. This teaching unit included the study of numerical sequences, functions, and 1st degree equations. The class included a high number of students that were recent immigrants from countries such as Angola, Brazil, Cap Verde, Guinea, S. Tomé and Prince, and Romania.

The unit was carried out in 16 classes (90 minutes each). It included several kinds of learning experiences. The first part of the unit included exploratory and investigative tasks (Ponte, Brocardo, & Oliveira, 2003), as a mean to foster the construction of new concepts. In the tasks about numerical sequences, the students had to explore numerical patterns, with different levels of difficulty (some of which presented pictorially). These tasks created opportunities for identifying generalizations, which could be expressed in natural language at first but should progressively be expressed using algebraic language. In this part of the unit, letters were mainly used as generalized numbers and as unknowns in simple 1st degree equations. This is the overall plan:

Blocks (90)	Topics	Objectives	Aspects to develop	Tasks
3,5	Numbers yet – Number sequences.	• To discover relationships among numbers; • To continue sequences of numbers: divisors;	• Searching patterns and establishing generalizations; • Representing numerical relationships in natural language, by other means and symbols;	1 2 3
3	Functions – Tables; – Graphics; – Functions defined by an algebraic expression.	• Read, interpret and construct tables and graphics for functions such as, or other simple ones;	• Constructing tables of values, graphics and verbal rules representing functional relationships;	4 5
	Direct proportion as a function. Graphics of the functions and.	• Relate in intuitive way the slope of a line with the rate in a function such as .	• Understanding the use of mathematical models of real world situations.	6 Textbook exercises and problems
6	1st degree equations	• Interpret the statement of a problem;	• To particularize relationships among variables and formulae and solving simple equations;	7 Textbook exercises and problems
	– Equations with denominators and parenthesis;	• Translate a problem by an equation;	• To solve problems represented by equations and to carry out simple algebraic procedures;	8

Blocks (90)	Topics	Objectives	Aspects to develop	Tasks
	– Literal equations	• To search solutions of an equation;	• To construct tables of values, graphics and verbal rules that represent functional relationships and translate information from one to the other.	Textbook exercises and problems
		• To solve 1st degree equations with an unknown;		
		• To solve literal equations, notably formulas used in other disciplines, for one of the unknowns.	• To understand the use of functions as mathematical models of real world situations.	

In the second part of the unit, the study of functions was introduced by two tasks involving relationships between variables. Although the letter is used both as a generalized number and as an unknown, in this part of the unit the focus was on its use as a variable and on the notion of joint variation. In the third part, tasks 7 and 8 continued the study of equations that the students begun at grade 7 and revisited in previous topics, solving new kinds of problems and equations with denominators. In this phase, letters were mostly used as unknowns and as generalized numbers. All of the tasks allowed the students to use different strategies and to draw their own paths of exploration. This approach stimulates students' active participation, providing them multiple entry points, adequate to their ability levels.

Working with sequences and functions became an opportunity to use the algebraic language as a tool for generalizing and sharing meanings. The study of these topics generated the opportunity to solve simple equations, which was important to create a common understanding among students, allowing them to continue learning more complex algebraic ideas. In the first general discussion, the sequence with general term $3n + 5$ was considered. The following dialogue took place:

> Teacher: So, which was the order in which 300 was placed?
> Erica: Teacher, 3 x 100…
> Teacher: OK, but does that give 300?
> Erica: No, that is just with 3n.
> Teacher: Oh, but I can't change the rule like that because we would be working with another sequence, different from this one. We just need to know which is the n that makes this expression yield 300.
> Sofia: 300 – 5? I don't know. [Students talk with each other.]
> Erica: So, we make 3n = 300 – 5.

Some students did not follow the reasoning proposed by Erica, and went on thinking on their own strategies. Pedro claimed with enthusiasm: "3 x 98 + 5 = 299; 3 x 99 + 5 = 302. It will not pass on 300!" This discussion continued with the contributions of Isabel, who solved the equation at the board, using her previous knowledge. The discussion provided a contrast between Erica's idea, the formal solution proposed by Isabel and the intuitive process used by Pedro to see if 300 was a term of the sequence and the advantages of each of the processes.

This example shows how students may be encouraged to design their own strategies and how these may be discussed and contrasted in the classroom. Such discussion helps students to realize more connections and relationships and to become more resourceful to deal with new problems in the future.

An important feature of this teaching unit in the interconnection of sequences, functions and equations. The work with sequences leads itself to formulating generalizations and using the algebraic language to express them. This language may, in turn, be used in functions and equations. And equations may again be used to solve problems concerning functions and sequences.

Example 5 – Investigating a polynomial function. The following task concerns the exploration of the proprieties of polynomial functions. This work was carried out in a grade 10 class by the class teacher, Cristina Fonseca, in cooperation with another teacher, Alexandra Rocha (Rocha & Fonseca, 2005). The task was proposed after the study of the absolute value and quadratic functions.

During some time the students studied the situations posed in the task concerning the proprieties of the function $y = ax^3$, using the graphing calculator. Given their former experience in explorations and investigations, the students worked on a systematic way: they assigned positive values to a and recorded the behaviour of the function; then they used the same procedure for negative values of a.

The students formulated several conjectures based on the variation of the parameters of the families of 3rd degree polynomial functions:

- It is symmetric! These values move to here and these to here [referring to the symmetry of the functions $y = ax^3$ and $y = -ax^3$ relative to the *yy* axis];
- As a increases, the graphic moves way from the *xx* axis and becomes close of the *yy* axis [referring to the behaviour of the function $y = ax^3$];
- In the positive part of the domain, the function [$y = x^3$] is given by $y = 1{,}5x^2$ and in the negative part of the domain is given by $y = -1{,}5x^2$ [the student compares the graphical aspect of the functions];
- The function $y = x^3$ goes through the origin of the Cartesian system.

Some groups, after they identified the simplest properties of the families of polynomial functions went on looking for new challenges, transforming their activity in a real investigation. They asked, for example:

- What is the behaviour of polynomial functions such as $y = x^n$, with $n \geq 4$?
- The points of intersection of the linear and quadratic functions belong to the graphic of the product function of the two functions?
- The 3rd degree polynomial function does cross the points where the linear and quadratic functions intersect the *xx* axis?
- What relation exists between the signal of the linear and quadratic functions and the signal of the 3rd degree polynomial function (product of the former two)?
- How to find algebraically the relative extremes of a 3rd degree polynomial function?

The students from one group, after they assigned the value 2 to all the parameters of the quadratic function g and the linear function h, observed that the 3rd degree polynomial function that is the product of those two "cross the points where the functions g and h cross each other". To validate their conjecture, the students tested other cases, assigning different values to each of the parameters, $g(x) = 2x^2 + 3x$ and $h(x) = 4x + 5$. Walking by the group, Cristina saw that the students seemed rather unhappy and questioned them:

> Cristina – What was your idea?
>
> Ricardo – We gave the same value to all [the parameters]. Then we introduced [in the graphing calculator] the three functions and we saw that the function [3rd degree polynomial] crossed the points were the other two intersected. Now we changed the values and it is not right anymore!

The students had made a conjecture and were quite convinced it was true. They had difficulty in admitting that their conjecture could be false. Using the graphing calculator, they showed the teacher the examples that they had considered. To make them reflect on the information already available, the teacher suggested that they determined the intersection points of the functions, using the facilities of the graphing calculator. The teacher moved way from the group and the students, after new explorations, refined their conjecture:

> Ricardo – I know! The function [3rd degree polynomial] crosses the points were the other functions [linear and quadratic] intersect the *xx* axis.
> Patrícia – Then say it!
> Ricardo – Wait! I am not sure!

To validate this conjecture, the students determined algebraically the roots of the three functions: -1,5, -1,25, 0. They concluded that the roots of the 3 rd degree polynomial function are the roots of the corresponding quadratic and linear functions. This was a proof for a specific case that, of course, could be generalized to the general case of any 3rd degree polynomial function, or any polynomial function, or any function that is a product of two given functions. The work of the students, however, took a different direction, seeking to explore other relationships among the three functions, such as, for example, the signal and relative extremes.

When the task was completed, the final discussion took place. This discussion provided new insights, leading sometimes to the formulation of new problems and new conjectures and assigning a high value to justifications and proofs. The dynamic of the discussion led the students to compare their ideas and the others', to appropriate these ideas, and to put relevant questions, revealing a significant understanding of the topic.

3. Direct teaching and exploratory learning

The examples of the previous section illustrate some key ideas about mathematics teaching and learning that I now address in more general terms.

Tasks. At the core of the former situations there were investigations, explorations and problems. It is important to note the difference between these tasks and exercises. A set of common *exercises* is the following:

Simplify:

a) $\frac{6}{12} =$

b) $\frac{3 \cdot (10-7)}{17-2} =$

c) $\frac{\frac{20}{18-9}}{\frac{(15-10)\cdot 2}{3}} =$

In an exercise, a computational procedure or a straightforward reasoning provides the answer. Furthermore, the question is clear as well as the given conditions.

A *problem* may be a task such as: “What is the smallest integer number that, divided by 5, 6 and 7 all yield 3 as remainder?” A problem clearly states what is given and what is asked, but there is no straightforward way to find the solution.

And this is an example of what we may call an *investigation*:

1. Write the table for 9s, from 1 to 12. Observe the digits in the different columns. Do you notice any pattern?
2. See if you find patterns in the tables of other numbers.

Here the question is somehow open as the reader does not know what kind of “pattern” can be found. Whereas in a problem we begin with a well formulated question, in a investigation, deciding exactly what our question is is the first thing we need to do.

We can differentiate tasks according to two main dimensions: (i) structure, ranging from closed to open, and (ii) complexity, ranging from accessible to complex (figure 3). Explorations and investigations are open tasks but with different complexity. Explorations are most suitable to help with the development of new concepts and representations. Investigations are necessary so that students go through a real mathematical experience of formulating questions, posing and testing conjectures, and arguing and proving statements. Problems are necessary to challenge students with non-trivial mathematics questions. And exercises are important to consolidate students’ knowledge of basic facts and procedures. In consequence, the teacher cannot do his/her job properly using just one kind of task – the issue is to select an appropriate mix, taking into account the students’ needs (Ponte, 2005).

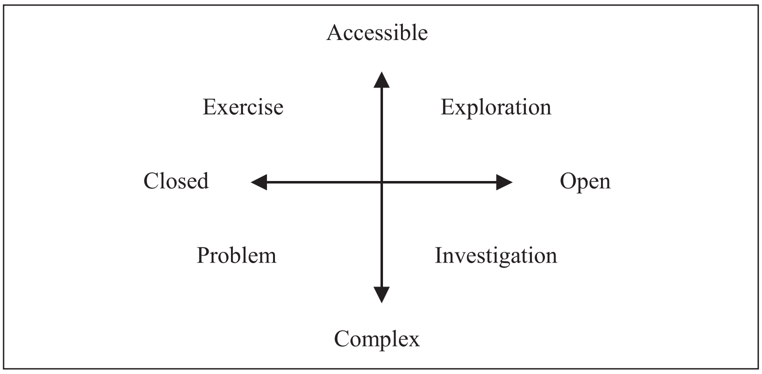


Figure 3. Structure and challenge as dimensions of tasks

Of course, tasks differ in other dimensions, such as the time needed to do them. For example, investigations that take a long time to complete are usually called “projects”. Another dimension of tasks is pure/applied. In our examples, some tasks were framed in “real-life” contexts (Sousa; Almiro) and others in “pure mathematics” contexts (Segurado; Matos; Rocha & Fonseca).

Classroom roles. Usually, a class in which students work on explorations or investigations has three main segments (Christiansen & Walther, 1986): (i) introduction of the task; (ii) development of the work, and (ii) final discussion and reflection about what was done, its meaning, and new questions to study. In the introduction, the task is negotiated between teacher and students; during the development of the work the students have an opportunity to work by themselves; and the final discussion is a key moment of sharing ideas and institutionalising new mathematical knowledge. The roles of teacher and students change along these three segments. However, at each segment, rather than a one way flow of information, centred on the authority of the teacher, we may have a classroom marked by multiple and complex interactions.

In the former examples tasks were proposed to the students who had to discover strategies to solve them. They designed a strategy to solve the task and had the responsibility of using logical arguments to convince the others of the correctness of their solutions. Therefore, the student has right to a voice, not only to ask clarification questions, but also to defend claims as an intellectual authority. This is a quite different picture from the teaching in which students receive "explanations" from the teacher, who shows "examples" and indicates "how to do things". When this happens, the teachers and the textbook are the sole authorities in the classroom.

Controlling the class when the students are more agitated, as in the case of Irene, or leaving them to work with large autonomy, as João Almiro did, that is a decision that the teacher needs to take according to the particular situation. However, in all cases presented, the students are assigned a significant role in their mathematical work as a classroom community**.**

Classroom communication. In a standard mathematics classroom the teacher dominates the discourse, either providing explanations and examples or posing questions and providing immediate feedback. The sequence IRF is well known – the teacher *initiates* with a question, a student *responds* and the teacher *feedback* closes down the issue, confirming or rejecting this response. We must note, however, that not all the questions fall in this pattern. In fact there are many kinds of questions (e.g., focus, confirmatory and inquiry questions) and questioning is one of the main resources that teachers have to lead classroom discourse (Pólya, 1945; Ponte & Serrazina, 2000).

In our examples the students are encouraged to share ideas with their colleagues, often working in groups or pairs. At the end of a significant work, there are discussions with all the class. These are very important moments in which there is negotiation of meanings (Bishop & Goffree, 1986). Different representations may be contrasted and the conventional representations analysed in detail. The proper use of mathematical language is fixed. This is also the moment when the main ideas related to the task are stressed, formalized, and institutionalized as accepted knowledge in the classroom community**.**

During group work, the kind of communication that is established among students may vary a lot. Sometimes, there is a real exchange of ideas and arguments. In other cases, only one or two students conduct all the work and the others remain silent. The way the teacher interacts with the students of a group is also of great importance. If the teacher does not respond to the students' questions, these may loose their motivation in the task. If the teacher provides all the answers, he/she takes out all the possible benefit of the task for the students. This means that the teacher has to deal permanently with many dilemmas in conducting the communication in his/her classroom.

Teaching units. Just by itself, a very powerful task does not much. If the students are going to experience some significant mathematics learning, they have to work on a field of problems for some extended period of time (at least for a couple of classes), where they have the opportunity to grasp the non-trivial aspects of the new knowledge, connect it to previous knowledge, and develop new representations and working strategies.

Teachers have to work through teaching units that, on the one hand, provide a journey that supports students' learning trajectory (Simon, 1999) on a given theme and, on the other hand, support the development of students' transversal aims for mathematics learning, including their representing, reasoning, establishing connections, problem solving, and communicating capacities. As Witmann (1984) indicates, designing these teaching units, according to careful criteria, is a major task for mathematics education researchers and classroom teachers.

Summing up. This consideration of different kinds of tasks, roles and communication patterns provides a characterization of two main styles of mathematics teaching that we find today in classrooms all over the world in different grade levels. We may call them *direct teaching* and *exploratory learning* (figure 4).

Direct teaching	Exploratory learning
Tasks • Standard task: Exercise, • The situations are artificial, • For each problem there is a strategy and a correct answer.	*Tasks* • Variety: Explorations, Investigations, Problems, Projects, Exercises, • The situations are realistic, • Often, there are several strategies to deal with a problem.
Roles • Students receive "explanations", • The teachers and the textbook are the single authorities in the classroom, • The teacher shows "examples" so that they learn "how to do things".	*Roles* • Students receive tasks to discover strategies to solve them, • The teacher asks the student to explain and justify his/her reasoning, • The student is also an authority.
Communication • The teacher poses questions and provides immediate feedback (sequence I-R-F). • The student poses "clarification" questions.	*Communication* • Students are encouraged to discuss with colleagues (working in groups or pairs), • At the end of a significant work, there are discussions with all class, • Meanings are negotiated in the classroom.

Figure 4. Direct teaching and exploratory learning

Challenges to teachers

One must note that a class with exploration and investigation tasks is much more complex to manage than a class based in the exposition of contents and doing exercises, given the impossibility of predicting the proposals and questions that students may pose. In addition, the students do not know how to work on this kind of task and need that the teacher helps them doing such learning. Notwithstanding their difficulties and limitations, this work is essential in a mathematics class that aims educational objectives that go beyond those that are achieved by doing structured activities.

We need to ask what is necessary for a teacher to carry out such exploratory and investigative work in his/her classroom. An analysis of this activity and its contextual requirements leads us to two main areas. The first area concerns the personal relation with mathematical investigations and the second the use of investigations in professional practice.

Personal relation with mathematical investigations

1. To have a good notion about what a mathematical exploration/investigation is, how it is carried out, how results are validated (*What is it/How to do it?*)
2. To feel a minimum level of *confidence* and spontaneity in carrying out a mathematical exploration/investigation;

3. To have a *general view of mathematics* that is not restricted to definitions, procedures and rules, but that values this activity.

Use of investigations in professional practice

4. To know how to *select and adapt* exploratory and investigative *tasks* adjusted to the needs of his/her classes;
5. To know how to direct pupils carrying out investigative work in the classroom, in the phases of *introduction, development of the work* and *final discussion*;
6. To have confidence in his/her capacity to manage the classroom *atmosphere* and the *relations* with pupils to carry out this work;
7. To develop a perspective about his/her role in *curriculum management,* so that mathematical exploration/investigations, in combination with other tasks, have an adequate role according to the needs of the students.

These are not competencies that teachers develop from one day to the next. The teachers involved in the projects that I mentioned developed professionally for an extended period of time. As important as their projects, was the work in communicating their experiences, writing papers, making conferences and communications in professional meetings. This enabled a deeper look at the experiences that become an important resource for mathematics education, showing the path that curriculum development and change of professional practice may take.

The development of this competence stands on three main elements: collaborating, researching on our own practice, and getting involved with the professional community, beginning at the school level.

Collaborating. Joining together the efforts of several people is a powerful strategy to cope with the problems of professional practice. Several people working together have more ideas, more energy and more strength to overcome obstacles than an individual working alone, and they may build on the diversity of competencies. To do that, of course, they need to adjust to each other, creating an efficient system of collective work. When one of the members of the group is going through a difficult time, he/she receives the support from the others. When a member is really inspired, he/she energizes all the group. Collaboration may develop within a homogeneous or heterogeneous group. A heterogeneous group may involve teachers of different generations, as well as mathematics educators, psychologists, sociologists, etc. A heterogeneous group may experience more difficulty in finding a proper working framework and dynamic, but, when this is achieved, it may become very creative.

Researching our own professional practice. Teachers' culture has been essentially that of "knowledge transmitters". The teacher bridges the gap between scholarly knowledge and school students. Today, this is not enough has the defining trait of the professional identity. Teachers, although experts in their subject matter field, are professionals that face very complex problems and need to do research to solve them. This means that teachers need to be able to identify problems, gather information about them, consider all sides of the issues, test solutions, analyse data and interpret results. They have to present their studies to the other members of the profession interested in the same problems. This does not depend so much in learning "research methods" but, most especially in keeping an inquiry stance (Cochran-Smith & Lytle, 1999), in knowing about defining issues and problems, and learning about theoretical notions that help in interpreting data and results.

Investigating is therefore a new element of the teachers' professional culture. It requires an integrative view of theory and practice as two sides of a single coin – where there is a theory, there is a practice, and vice-versa. In every situation, establishing a dialogue between theory and practice is a major step towards understanding and solving problems.

Involvement with the professional community. Valuing a culture of research among teachers does not depend only on an obstinate individual agency. On the contrary, it requires a fundamental role of the collective stances where teachers carry out their professional activity, especially the schools, pedagogical movements and associative groups. In Portugal there is an important tradition of innovative projects carried out by collaborative groups and sharing experiences in associative settings. What is still missing is reflective and transformative activity at the school level. Teachers who want to change things in their schools need to carry out their own projects within the schools, showing the results to the other teachers, stimulating reflection, creating the need to know more, to experiment, and, hopefully to get other teachers involved in common initiatives.

Conclusion

Mathematical explorations and investigations can be a significant part of the mathematics curriculum. This is because of a number of reasons:

- They constitute an essential part of the mathematician's work,
- They favour the involvement of the student in work carried out in the mathematics class, indispensable for a significant learning,
- They provide multiple entry points for students with different levels of mathematical competence,
- They stimulate holistic thinking,
- They can be integrated naturally in every part of the curriculum,
- Although related to complex thinking, they reinforce learning elementary concepts.

With greater or lesser emphasis, either mathematical investigations – or key elements of investigating such as conjecturing, testing, and proving – are recommended in the official curricula in many countries around the world (see Ponte, Brocardo, & Oliveira, 2003).

Investigating, teaching, and learning can be seen as an interconnected triangle. The researcher who teaches benefits from the contact with students, as he or she listens to their questions that may challenge his or her theories and methods. The teacher who investigates can use current examples and open problems, making teaching a lively, stimulating activity. And through investigations, the student may become involved in genuine activities of knowledge construction.

Mathematics teachers and teacher educators have interest to investigate their own professional practice, seeking to understand students' and student teachers' difficulties, the factors from the social and school contexts that influence them, and the power of teaching strategies to promote qualitative changes in students' learning. As I argued elsewhere (Ponte, 2001), students may explore and investigate mathematics, teachers and teacher educators may investigate students' mathematics learning and the conditions that enable it.

In mathematics education there are at present two separate worlds. One is the world of research, as an intellectual elaboration with high rigour but with problematic practical relevance. The other is the world of practice, where problems are felt in a cogent way, but where there is little capacity to theorize and to introduce and sustain innovative solutions. We now have an emerging reality, the world of researching practice. One may expect that it will deal with questions with strong practical relevance, with proper rigor and intellectual elaboration. Working towards such an agenda is a joint task of teachers and teacher educators.

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Developing creative mathematical activities during lessons of mathematics

This paper presents a research project conducted among mathematics teachers. The aim of the project was to improve the teachers' ability to develop creative mathematical activities. For that purpose, diagnostic activities, workshops and lessons' observations were organized. The results show a considerable improvement in the teachers' ability and their attitude towards mathematical activities.

Introduction

Independent and creative thinking of students is an integral part of mathematics teaching. Contemporary mathematics education follows in the direction of mathematical activities. The assumptions of the PISA exams reflect that tendency (Sułowska & Marciniak, 2004). School mathematics, apart from ready-made knowledge (a set of definitions, theorems and procedures) is mostly a domain of a specific human intellectual activity whose product is ready-made knowledge and the tool used is specific mathematical thinking. Thus, that view of mathematics should be formed among the students (Klakla, 2002; Hejny & Kratochvilova, 2005). Mathematics knowledge is not only the main goal of mathematics education, but should also be a tool which enables a student to engage in mathematical activity. As a result of the work on mathematics lessons a student should learn to work like a mathematician – that is s/he should be able to put hypotheses, notice some regularities and relations, argue, justify, etc. Among these, some creative skills should also be included in that work.

However, the results of our research (Maj, 2006) show that:

– Among the mathematics teachers the knowledge and skills regarding creative mathematical activities are insufficient.
– Among the mathematics teachers it is generally erroneously believed that the creative mathematical activities are developed by themselves during the mathematics lesson and do not require any special didactic endeavours, methods or tools to develop them.
– The mathematics teachers do not have experience and skills of undertaking these activities and what goes after that, they cannot provoke these activities and they cannot organize them in the work with students.

It can be noticed that these results have direct influence on the skills and experience of the students for the whole range of creative mathematical activity. The results of PISA 2003 exam confirm these conclusions: only a few more than 10% of Polish students achieved the results from the fifth and sixth level (on a total of six levels) in the scale of mathematical achievement. Sułowska and Marciniak (2004), in their analysis of the solutions of the tasks, pointed out the weak sides of Polish students:

1) The problem of "the upper quarter" – the best of our students are often weaker than the best students in the rest of the world.
2) Difficulties with independent, creative and abstract thinking.

In this connection there is a need of paying special attention on developing creative mathematical activities and elaborating some methods of instruction for these activities. This view appeared and still appears in the literature of mathematics education (Krygowska, 1985, 1986; Polya, 1975, 1993; Mnich, 1980; Klakla, 1982, 2002; Mason, Burton & Stacey, 2005; da Ponte, 2001).

> Mathematics teaching should acquaint the students with all the aspects of mathematical activities so far that it is possible. Particularly, the students should have the opportunity of creative work according to their abilities (Polya, 1975).

The essential condition for the development of the skills needed for different kinds of creative mathematical activities among students is the deep understanding of those issues by the mathematics teachers. To that it is necessary:

– to raise their awareness of the necessity of formation of such activities among their students, and
– to develop their skills of organizing the situations which favour the undertaking of different kinds of such activities.

Only then the teachers would effectively form and develop these activities in their work with the students.

Theoretical framework

Mathematical activity of a student is "a work of mind oriented to the formation of concepts and to mathematical reasoning, stimulated by the situations which lead to formulating and solving theoretical and practical problems" (Nowak, 1989).

It is worth to underline two things, namely that the mathematical activity is firstly a work of mind and secondly that it should be stimulated. Therefore, it is not a work of a student which appears in a natural way.

A conception of forming creative mathematical activities was worked out by Klakla (2002) and is based on two elements. The first of them constitutes the distinction between particular kinds of creative mathematical activities, which are present in an essential way in activities of mathematicians. These are:

(a) hypotheses formulation and verification;
(b) transfer of a method (of reasoning or solutions of the problem onto similar, analogous, general, received through elevation of dimension, special or border case issues);
(c) creative receiving, processing and using mathematical information;
(d) discipline and criticism of thinking;
(e) problems' generation in the process of the method transfer;
(f) problems' prolonging;
(g) placing the problems in open situations.

The second element of that conception refers to the multistage tasks which:

– consist of series of tasks, problems and didactic situations which have a specific structure,
– are based on problematic situations,
– connect different kinds of creative mathematical activity with each other in complex and rich mathematical-didactic situations,
– provide a specific laboratory of creative mathematical activity for the students.

Ponte (2001) states that there is some similarity between research activity of mathematicians and activity of the students in the classroom.

> Of course, there are differences between the knowledge held by both, their degree of specialization, the time they spend, and their relation with the subject. However, their problem-solving activity is of a similar nature. Hadamard (1945), a well-know mathematician, refers to this, for example: "Between the work of the pupil that tries to solve a problem in geometry or algebra and a work of invention [by the mathematician] one can say that there is only a difference of degree, a difference of level, both works being of a similar nature".

That opinion was also shared by Ernest (1991) and Polya (1975). Polya adds that the students should have an occasion to experiment with many aspects of mathematical activities. Based on examples of some problems he shows that the teachers can create the conditions to make the students develop creative and independent work (Ponte, 2001).

Methodology

In this paper we present a short description of the research carried among a group of mathematics teachers and also a description of the lesson conducted by one of the teachers who took part in the research, with the analysis of that lesson from the point of view of creative mathematical activities.

A group of seven teachers of mathematics (of gymnasium and high schools) has taken part in a series of workshops from March to September 2006. The workshops were organized as part of the Professional Development of Teachers Researchers (PDTR) project (226685-CD-1-1-2005–PL-Comenius-C2.1), during the mathematics course. The main content of that course was solving different kind of mathematical problems which were supposed to be challenging for the teachers. The workshops were organized aroung three multistage tasks. They consist of solving some chosen tasks – open-ended problems and discussing the possibilities of introducing the students to the particular problem. The didactical comments produced were related to the organization of the work with the students and some reflections on different kinds of mathematical activities which the students had the chance to undertake in the process of solving those tasks.

The main aims of the workshops were:

- developing the skills of undertaking creative mathematical activities among mathematics teachers,
- raising the mathematics teachers' awareness of the need to develop creative mathematical activities and developing the skills of provoking these activities among students,
- showing the teachers a model of working with the students.

Our purpose was to influence the development of the teachers' skills in organizing situations that – under certain circumstances – can lead to creative mathematical activities which are favourable to be undertaken by the students.

After the end of the workshops the teachers had the task to prepare and conduct a mathematics lesson which main aim was to develop some creative mathematical activities among students. In their previous experience concerning the preparation of the lessons and the determination of the lessons' aims, the teachers were used to focus on the mathematical content enclosed in the curriculum. Now they had to concentrate on mathematical activities which they will form and develop around a theme of a lesson.

The observations of these planned lessons had the aim to show us if the teacher can plan and organize a work of his/her students in such a way that they can have the opportunity to undertake different kinds of creative mathematical activities. However, it was less important what class that lesson is conducted in and what mathematical content it is related to.

We will present a short description and analysis of the lesson of the teacher who participated in the workshops. That analysis was conducted in the direction of the answers to the following questions:

- Does the teacher develop any mathematical activities among his/her students and if yes, what kind of activities does s/he develop?
- Does s/he stimulate his/her students to undertake creative mathematical activities?
- Do the ways the teacher acts favour independent and creative thinking of the students?

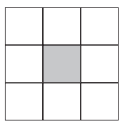
The data collected comprised of audio recording of the lesson (45 min.) and notes. After this a full transcription of the audio recording and analysis of this transcription was made.

Analysis of the lesson

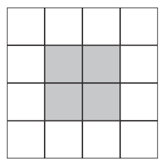
The lesson was conducted on 22nd October 2007 in the second class of gymnasium (16 students, 13–14 years old).

The students were working in four groups (3–5 persons each). The lesson started with the teacher posing the following problem (Swoboda, Turanu & Urbańska, 1997).

> Tang Ming designs square swimming-pools. Each swimming-pool has a middle part which is also square and filled with water. Around each swimming-pool there is an edge made from the white squares. These are the first two swimming-pools designed by Tang Ming:



Swimming-pool 1



Swimming-pool 2

Then the teacher asked the students: "What would you like to ask about? What questions would you like to ask about this task?".

The students worked in groups for about 10 minutes and formulated the questions to that situation. Every group wrote in their exercise books from five to eight questions and then presented their lists of the questions. Some of them were the same (I), some of them were not mathematical (II), some appeared only once (III). For example:

(I) What are the areas of the swimming-pools? What are the perimeters of the swimming-pools?
(II) Why Tang Ming does not design round swimming-pools?
(III) How will the next swimming-pools look like? What per cent of the areas are the white squares?

The teacher chose five questions from all of them:

1. How will the next swimming-pools look like?
2. What are the areas of the swimming-pools?
3. What is the difference between the areas of the water and the white tiles?
4. What part of the whole swimming-pool is the middle part?
5. What part of the whole swimming-pool is the edge?

To systematise the work of the students the teacher gave them sheets with the table:

Number of the swimming-pool	Number of the white squares (the edge)	Number of the blue squares (water)	Total sum of the white and blue squares

The students worked in groups for around five minutes and then they shared their own ideas and filled the table. The students filled the first two lines by counting the number of the squares on the drawings. To fill the next lines they either drew the next swimming-pools, or "drew them in the head" imagining (visualising) the next swimming-pools, or they noticed some arithmetical relations between numbers. During the students' work the teacher did not suggest any ways of solutions, she only motivated them positively and appreciated their exertions:

1. Teacher: /is checking the results in every group/: the third (swimming-pool), the fourth, the fifth, ok, Ela has it already. Do you also have?
2. Some students: We also have the fourth and the fifth.
3. Teacher: You also have the fouth and the fifth? But there are no pictures... the third, fourth and fifth…
4. Kuba: We have just thought up how it would be!
5. Teacher: So? What have you noticed, Kuba? In what way, though there are no pictures, did you calculate the third, fourth and the fifth? What did you observe there?
6. Kuba: That you add the blue one.../he is showing in the straight line/. In the first (swimming-pool) was one blue, then two, three, four etc.
7. Teacher: Ok., Kuba noticed that. We will wait for one moment because I see that you are still working...

The teacher was interested in what way the students calculated the numbers of the squares for the swimming-pools 3, 4, 5, not only the result of that calculation. By asking the question in (5) she suggested the student that he is clever, she motivated him and provoked him to give the explanations. This resulted to the student wanting to share his own idea of creating the next swimming-pools. The student used simple, informal language, referring to the concrete examples, but he also had a general image of the situation, he saw some relations in it. The teacher, by not asking about the result, from the one side encouraged her students in argumentation, from the other side she valued them, because for her their way of thinking was important. She also gave to all students the chance for discovery (statement (7) is evidence of that).

The teacher very often asked the following questions and gave the following instructions:

> How did you calculate this? In what way did you calculate this? What did you notice? How did you get it? How come you know about it? Justify what you wrote. What if somebody calculated the numbers in a different way? Why? Are you sure? What is the rule?

The result of these questions was that the students had to justify every number written on the table. It stimulated them to discover arithmetical regularity and then it "forced" them to make generalisations and formulate these rules. Ipso facto they verified their statements, thus developing the skills of communication about mathematics, together with the skills of agumentation and critical thinking:

8. Teacher: Where does the number 16 come from?
9. Radek: plus 4 to the previous one
10. Teacher: Ola, and you? /she goes to the next group/
11. Ola: also plus 4
12. Teacher: And you? /she asks the last group/
13. Ania: also plus 4
14. Teacher: And are you sure that if it is here, for example, plus 4 /she shows the numbers 8 and 12 of white squares for the swimming-pools 1 and 2/, so maybe here it is plus 5, and then it will be plus 6, plus 7...?
15. Radek: No, because we calculated!
16. Teacher: Did you calculate? Did you check that here it will not be a difference 5?
17. Radek: There has to be plus 4, we calculated those squares!

The students had only two examples, but they formulated a rule. The teacher tried to sow "a grain of uncertainty" (14), but they were convinced about their own thinking, they were independent in thinking (17). The teacher, wanting to provoke them to check and justify their own notices, put a hypothesis that another rule exists. She implied that when you have only two examples it is hard to state that a rule will be also true for the rest cases. Therefore, she gave them another rule which was sensible and could also be true for those two numbers. So the students had to refute that hypothesis.

After filling the first five lines in the table, the teacher asked the students to calculate the values for the tenth swimming-pool.

18. And now I have another question: If you know what rules manage the filling of the next columns and lines, then can you fill the table for the tenth line?
19. Kuba and Jacek raise their hands.
20. Kuba /goes to the blackboard/: I'm substituting here /in the second column – the number of white squares/ a different formula: it is the number of the swimming-pool multiplied by 4 and plus 4.
21. Teacher: Aha, you discovered a different rule, that it is the number of the swimming-pool multiplied by 4 and plus 4… So for this column you discovered two rules… Then try to use the first rule: plus 4 to the previous line… Can we do in that way?
22. Kuba: No, then we would have to calculate in turn every swimming-pool…
23. Teacher: exactly, we would have to calculate in turn every swimming-pool. What about the second rule? What was it?
24. Kuba: the number of the swimming-pool multiplied by 4 and plus 4.
25. Teacher: Is it easier in that way?
26. Kuba: Yes
27. Teacher: So try to do it.

By asking about the tenth swimming-pool (18) the teacher provoked the students to generalise and she "forced" the formulation of the rule for the white squares in another way; instead of "adding 4 to the previous line", the students proposed "the number of the swimming-pool multiplied by 4 and plus 4". Kuba was able to pass easily to the different situation, to see some relations and to generalise. However, the teacher wanted the students to convince themselves that the recurrent relation stopped playing its role at that point. So she asked Kuba to use the first rule (21). The student stated that it is easier to use the second rule, because "then we would have to calculate in turn every swimming-pool" (22). Here he used economisation of the processes by using algebraic language.

The lesson ended with the teacher's request to write the rules of every column in algebraic language or in words (if somebody could not do this symbolically). Some students wrote the rules using algebraic expressions during the lesson.

Conclusions

The choice of the open situation during the lesson let the students pose the questions by themselves. They could show their ability in creative invention. One thing is solving a task which is imposed by the teacher and another thing is to solve a task which was to some extent thought up by yourself. In the second case the engagement of the student in the process of solving the task is usually much bigger, which was what we observed during that lesson. It stimulated them to undertake mathematical activity – to formulate hypotheses, whereas the frequent "how" and "why" questions of the teachers resulted in the necessity of verification, thus enhancing the skills of argumentation and critical thinking. The students proposed the question about the "look" of the next swimming-pools (3, 4 and 5), therefore they prolonged the problem. However, the question of the teacher about the tenth and then the n-th swimming-pool motivated them to generalize and directed their thinking to the more abstract level, at the same time without suggesting anything (the students noticed that the recurrent relation did not play any role here). During that lesson the students had the opportunity to discover and construct algebraic formulas, describe some general rules, so therefore, they had the opportunity to get the sense of these formulas by discovering recurrent and variable relations.

In our observations we mainly focused on the actions of the teacher during her work with the students. During the described lesson the students had the opportunity to undertake a few kinds of creative mathematical activities. The actions of the teacher were limited to the organization of the teaching process, in order to let the students put questions, formulate problems, and then discover some mathematical relations. The teacher stimulated the work of the students in the direction of creative mathematical activities; she did not impose her own way of thinking, she was flexible and reacted positively to the ideas of her students. And all these favoured students' independent and creative thinking.

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Teaching practices revealed through arithmetic misconceptions[1]

The aim of our research was to investigate whether primary school pupils (7–11 year olds) can distinguish between simple arithmetic errors and misconceptions. Data from a specifically designed task are presented. We demonstrate the importance of teachers' practice in the development of pupils' reflexive thinking.

Theoretical framework

There is a huge quantity of literature on mathematical misconceptions using a range of different research strategies and analytical techniques (e.g. Zan, 2000; Swan, 2001; D'Amore & Sbaragli, 2005). For example when analysing of misconceptions associated with Stochastic Thinking, Tversky & Kahneman (1983) suggest that they originate from children's interaction in the classroom or in the physical and social world or from their prior learning. In contrast, Smith *et al*. (1993) refute the 'misconception approach' on empirical and methodological grounds, arguing that misconceptions are in fact context specific and not general and that, for example, they may be due to the absence of appropriate tools to explore the questions. The literature tends to focus on activities for diagnosing and remedying misconceptions. Thus, typically, an adult perspective is adopted and a teacher's or a researcher's view of misconceptions is presented. Here we focus on the child: do primary school pupils perceive there to be a difference between mistakes and misconceptions in Arithmetic? We use the term 'misconceptions' as proposed by Kaldrimidou & Tzekaki (2006): "to identify the difference between the meaning that students construct about a mathematical concept and the concept itself"; and 'mistake' as the term 'slip' proposed by Schlöglmann (2007): "Errors that are based on the incorrect application of a formula, or simple mistakes in a calculation".

The research

This paper stems from a meeting of the first author and some Italian primary teachers[2]. The discussion turned to pupils' apparent boredom when their teachers revisit incorrect work either with a whole class or with a child individually. This prompted Guastalla to share an activity she designed to encourage pupils to become more involved in the marking process and hence their mathematical understanding. Here we have modified Guastalla's idea to investigate pupils' perceptions of what constitutes an arithmetic error as opposed to a mathematical misconception.

[1] The authors gratefully acknowledge the support of the British Academy (Grant no. LRG-42447) which provided a platform for this study.

[2] This meeting was organized in the School Year 2005–2006 for the aims of the project sponsored by British Academy. Teachers Carla Aleotti, Maria Bertagna, Rossella Guastalla, Maura Previdi, Roberta Santelli and Nunzia Seminerio of Primary school of Viadana (MN) – Italy took active part with the proposal of the item used later with children.

We presented 253 children – aged 7–11 years–old – with a set 'children's' work and asked them to adopt the teacher's role. The tasks were designed to include correct and incorrect answers with the latter reflecting arithmetical misconceptions rather than simply careless work on the part of the 'pupil' who completed them. In reviewing the tasks (see below) teachers were given an opportunity to reflect (a research by-product) on the notion of misconceptions for, unlike practitioners in the U.K., it is not customary for their Italian counterparts to make a clear distinction between mistakes and misconceptions. As a result Italian children are therefore not explicitly exposed to this distinction and only few of them appreciate that there is a difference between mistakes and misconceptions. Interestingly, until relatively recently, little attention has been paid to this seemingly subtle – but important – distinction in the academic literature making it a challenging concept for Italian teachers.

Our hypothesis was that a change in the children's perspective, from pupil- learners to pupil-teachers, could influence their attitudes. Cobb's (1985) example of Scenetra showed that her performance improved when she assumed the role of teacher. Moreover as the teacher is – in the pupil's mind – the person who is closest to scientific contents in the classroom, it is possible that, in this position, the child-teacher could identify mistakes and misconceptions more successfully.

The task

We worked with 253 pupils (52 grade 2, 89 grade 3, 65 grade 4 and 47 grade 5). The tasks for all grades were of the following type:

> Look at the task of a pupil of another parallel class. (1) Please mark and assess the work, giving it a grade; (2) Try to explain the line of reasoning s/he applied.

There followed a list of fully completed exercises, some of them correct and others wrong (cf. figure 1 for 4th graders). Teachers adapted the text to their class inserting some of the most typical wrong answers so that they could observe the reactions of their pupils.

The organization of an operation in column is different in different countries. In figure 1 we have used the Italian style of representation, adopted by the Italian teachers. We have, however, translated the original Italian 'da' (*decina*), in the English 't' (*tens*).

Please mark the work of a child of another grade 4 class. You must correct any mistakes and explain why you think they occurred. Please give a grade for the work.

1) Place the sign >, < or = in the following pairs

45 < 54   198 < 189   2t 8u > 1h 3u   2t 8u = 8u 2t   1t 11u = 21u

Calculate in columns

2) 17 + 35 =	202 – 96 =	23 · 14 =
17 +	202 –	23 ·
35	96 =	14 =
412	106	92
		23
		115
3) 3,5 + 4,2 =	7,6 + 3 =	9 – 4,9 =
3,5 +	7,6 +	9,0 –
4,2	3	4,9
7,7	7.9	5,9

Figure 1. The fictitious 4th grader's task

During the preparation of the task, teachers confronted the problem of distinguishing between mistakes and misconceptions as they had to identify the most appropriate examples for the research and, in doing so, consider the most frequent mistakes/misconceptions which occurred in their classrooms.

As specified above, all of the wrong answers presented in the tasks could be explained in terms of arithmetical misconceptions. They also had to be relatively simple so that they were not beyond the pupils' intellectual capabilities and yet rich enough to reveal children's understanding. As such the tasks were designed to provide teachers with the opportunity to clarify how learners face this kind of activity (Schubauer-Leoni *et al,* 2004) as the children explain their 'colleague's' reasoning for incorrect solutions.

Results

The task proved successful in every class: each wrong exercise had at least one pupil suggesting that it was right and for each right exercise there was at least one child stating that it was wrong, supporting the statement of Santi & Sbaragli (2007) that misconceptions are unavoidable and, as stated in David *et al*. (1994), they have a constant presence.

Some pupils took the task very seriously. Others did not. In all cases, however, the children provided some interesting insights into their interests, their involvement with mathematics and their understanding of the subject. We generally found three levels of response:

- The first is the detection of the wrong answers and suggests that the child has an awareness of algorithms. In terms of Bruner's work (1992) this level requires the logical-scientific thinking related to what it is to do mathematics.
- The task also asks for explanation of the thinking of fictitious colleague. This is a second level and not all of the pupils reached it. This level requires narrative thinking – as defined by Bruner (1992) – in order to produce a plausible report. This sort of thinking is related to one's ability to talk about mathematics.
- The third level is the pupils' assessment of the task.

A few children went beyond the task requirements and we found what might be termed a fourth level as they evaluated the task and its goals. For example:

"Goals [*of the task*]: To know the use of <, > or = ; to know computation with natural numbers and with decimal numbers, but the joint aim is to recognize the (place) value of digits." (4 – A.B.)

We could not find any references in the literature on the third and fourth levels thinking we observed.

Analysis of L.'s protocol

The first three levels are evident in the protocol of L., a grade 2 pupil (figure 2). As far as possible the following translation from Italian reflects L.'s expressions:

- she corrects the incorrect answer 37 < 29 saying: "NO you got mixed up";
- she comments on the correct response 23 > 22 with "Yes";
- she incorrectly marks a correct answer 45 < 54 saying: "Not true as 54 is not > than 55";
- she corrects 17 + 35 = 412 with "52" adding, "NO PR. you counted too much. Next time do it this way."
- She also presents a different method for the same computation where she singles out the amount carried.
- She marks 24 + 50 = 70 saying: "NO you did a computation mistake".
- Finally she grades the task as "Distinguished +".

The words “Si” (Yes) and “No” (No) indicate the first level (described above) in which L. – employing logical-scientific thinking - states the exactness of her 'colleague's' answer. The sign “>” in the first exercise and “52” and “4” in the two addition items belong to the same level.

The second level is evident in her comments. It is perhaps surprising that L. deems that the incorrect answer to the first question was the result of 'mixed up' thinking and yet she does not specify the nature of the confusion. Comparing this to L.'s answer to the third question (i.e. 45 < 54), however, suggests that this sort of muddle is familiar to L. With a similar task (i.e. 189 > 198) some grade 4 pupils suggested that confusion is the result of a faulty consideration of place value (misconception), since it could be explained on the basis of comparing the unit digits.

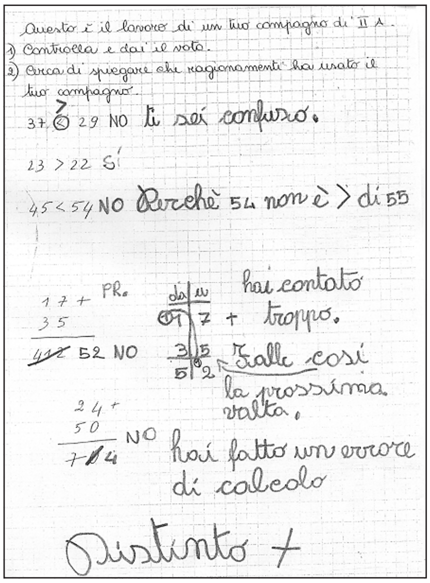

Figure 2. The protocol of L. (Grade 2)

L.'s comment, “you counted too much” to the solution 17 + 35 = 412 suggests that, even if she is able to apply the standard algorithm of addition in columns, she is also aware of the counting-on technique as she wrote 'count' instead of 'add'. By this she could have meant, 'you have gone too far' or 'your result is too big'. G. (also grade 2) wrote something similar which can be translated from Italian in one of two ways: “He made a mistake, because he counted incorrectly and also he was in a hurry” or “He made a mistake, in my opinion, because he counted too much, badly and also he was in a hurry”. Such reference to counting was still evident in three grade 5 responses suggesting that awareness of counting-on may be a strategy for testing the correctness of an addition calculation for some pupils throughout the primary years.

The two letters “PR.” could mean “Prova” (Experience). The girl was interviewed about this, but she did not remember what she meant. L.'s comment on the exercise 24 + 50 = 70 – “NO you did a computation mistake” – is attributable to the second level, comparing her remark to the previous one, shows an evident difference in her thinking.

The third level is exemplified by L.'s mark “Distinguished +”. In Italy grades are not expressed by numerical or literal values. The idea is to give an assessment with formative features. In practice this laudable aim is substituted by a 'scale' in which a task is assessed by an adjective: insufficient, sufficient, good, distinguished, excellent. These grades might be modified by adverbs, such as: seriously (insufficient), only just (sufficient), nearly or fully (sufficient, distinguished, good, excellent), and so on, according to teachers' linguistic creativity. Teachers may further refine grades adding the signs “+” or “–”.

Other findings

Sometimes in mathematics it is sufficient to estimate an approximate answer, without specifying the exact result. Some children worked effectively in this way and demonstrated a good awareness of the properties of operations.

“since 17 + 35 makes a number of two [digits]” (M.B. Grade 2)

“the result 412 is impossible since the numbers are too small.” (L. & E. Grade 4) "over the nine there was nothing and it is as he was putting 0 and then he computed 0–9 and this cannot be done". (A. & R., Grade 4, commenting 202 – 96 = 106)

"the number is too small for the high class of thousands" (A.P., Grade 5, commenting on the task 27030 : 3 = 91)

To conclude thus far; asking pupils to take on the role of teacher looks like a suitable didactic device for exploring their arithmetical misconceptions and their ability to assess the correctness of others' solutions. In such circumstances pupils show, "a different kind of mathematics that is often intolerably hard" (Gray & Tall, 1994, p. 116).

Moving on to the second level – explaining another's thinking – Table 1 shows evidence of this second level across the grades studied.

Grade 2 (No. 52)	Grade 3 (No. 89)	Grade 4 groups (No. 19)	Grade 4 singly (No. 22)	Grade 5A (No. 23)	Grade 5B (No. 24)
36.54	36.54	36.54	36.54	36.54	36.54

Table 1. Presence of second level (in percentage)

The children reacted to the presence of wrong answers in different ways. We would suggest that this is likely to depend both on the teachers' practice, and the number of years of schooling. Significantly Table 1 shows that pupils in grade 4 were more successful than any other group but, given that their teacher, Rossella, usually requires this second level activity, perhaps it is not surprising. Indeed we found some of these 4th graders – working either singly or in groups – to be very detailed in their comments of wrong results. This was particularly true in the case of the wrong answers to inequality exercises.

Having said that, we obtained protocols from all of the classes showing sensitivity to misconceptions. Clearly we did not expect children to use the word 'misconception' although we found some explicit instances of 'reasoning mistake' versus 'slip of pen' among grade 4 pupils and we interpreted the former as the child's term for 'misconception'. In order to provide a more detailed example of such answers, we consider the following exercise given in Class 5A (N = 23): 12506 + 99 =12407. We have included the number of children responding in a particular way in brackets:

1) the addition sign is inappropriate since the result is obtained by subtraction (3);
2) the addition calculation is wrong (19), with:
    - 2.1) (global) attention to the whole calculation modifying the hundreds digits, the tens and the units, eventually with right result (3);
    - 2.2) (local) partial attention to the calculation restricted to the tens and units digits (1)
    - 2.3) (local) only attention to the units digit (5)
    - 2.4) simply the statement that the result is wrong (7)
    - 2.5) the child purposes another wrong result (1)
3) the addition calculation is considered right (3).

All children stating that their 'colleague' had given the wrong result are correct, but closer examination shows that – despite being in the same class – their reasoning differs. For example, some demonstrated a global understanding of this task and, in the case of (1) above, considered the misconception to be due to the wrong interpretation of the addition sign, while those who responded as in (2.1) perceived it to be the result of the incomplete understanding of how to use the addition algorithm. In both cases we can assess these answers as indicating the perception of the presence of a misconception.

The ‘local’ pupils (2.2 and 2.3) show an attitude that can be defined by Gardner (1993), (quoted in Zan (2005)), as the ‘compromise of correct answer’: many teachers and pupils consider that the education act is successful when learner succeeds in giving answers that teacher assesses as right. Here the change of perspective transforms the ‘compromise’ in giving the answer that a colleague is wrong, without studying the reason behind the error in depth.

From another perspective many of the children’s responses demonstrating second level answers to our question appeared to be strongly influenced by their own classroom experience and appeared to mirror teachers’ typical reactions to mistakes, albeit, in some cases, inappropriately. We observed only three pupils giving type (1) responses speaking with their own voices.

Focusing on the third level: sometimes there was a great distance between the assessment and the mark. An anonymous 3rd grader wrote, “You mistook everything, you are not able to do….You disappoint me – Excellent – –”. We might judge this assessment, in itself, severe, but the mark expresses the child’s own hope. There is also a moral blackmail in the judgement. This distance was apparent in the majority of the protocols and many of them illustrate the affective dimension with which pupils understand the mark.

Conclusions

We believe that teachers should combine the teaching of algorithms with reflection of the mathematical activity for, as stated by, Haylock & Cockburn (2003), “the learning of recipes for answering various types of questions is not the basis of understanding in mathematics” (p.1)

This reflective activity could fill the gap between the teacher’s assessment and the child’s understanding of the assessment. Indeed such sharing between teacher and learner could enhance the learning environment. Our data, for example, suggest that the second level activity prepares a fertile soil for understanding the difference between misconceptions and mistakes and thus broadens adults’ and children’s appreciation of mathematical concepts.

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Students discussing their mathematical ideas: Group-tests and mind-maps

In an explorative research project, teachers experimented with new ideas to make their students discuss (i.e. show, explain, justify and reconstruct their work) their mathematical ideas with each other. Two kind of special tasks were developed: group tests and mind maps. Also, the role of the teacher was studied in order to evoke discussions between students. In this workshop, tasks and results will be presented, experienced and discussed.

This paper investigates two examples of how to evoke mathematical discussions between students. With mathematical discussions we mean discussions in which students **show** each other their mathematical (thinking) work, **explain** it to each other, **justify** it and **reconstruct** their (thinking) work, described as *key activities* in the process model of Dekker & Elshout-Mohr (1998). In a new Amsterdam school for secondary education (13 and 14-year-old students), a teacher, a student-teacher and a researcher have been experimenting with two types of tasks for small-group discussions in the mathematics lesson: group tests and making mind-maps.

Group tests

Students have been working in groups of three on a test on the subject of 'unities'. The students had prepared for the test individually and were grouped by the teacher on basis of their expected level and preparation time. Groups were not allowed to interact, but within the group, students were encouraged to discuss with each other. Eight classes were making their test in this way. In each class, one group was audio recorded. Both teacher and students were enthusiast about this way of working. Students mentioned that they were better able to think, since they could discuss their ideas with peers. Moreover, some students were surprised by the experience of collaborating with a peer with whom they had never been working before. It appeared that within most groups, students performed key-activities. Retrospective analysis made clear what factors determined the success of this method (group size, preparation at home, assessment, group composition).

Mind-maps

A mind map is a diagram used to represent words, ideas, tasks or other items linked to and arranged radially around a central key word or idea. It is used to generate, visualize, structure and classify ideas, and as an aid in study, organization, problem solving, decision making, and writing. In groups of three or four, students made a mind map starting with 'measurements and unities', in order to evaluate and structure what they learned on this subject. The mind-maps of the groups were shared with the whole class and a joint map was created on the white-board. For the next lesson, each students prepared a card which represented a word from the joint mind-map, or a picture related to this. During a whole-class discussion, the cards were structured on a magnetic board and a definitive version of the joint min-map was created. This map was copied on a small poster for all students. During the lessons, audio recordings were made from all groups and a video recording from the overall classroom. The students mind-maps were collected. The audio recordings were analysed on the occurrence of key-activities.

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Developing teachers' subject didactic competence: Case of problem posing

Problem posing is discussed as one of the ways of empowering subject didactical competence (pedagogical content knowledge). The work with students and/or teacher can start with examples of problems posed by student teachers and elementary teachers.

Introduction

The programme declaration of the 53rd CIEAEM conference stated that the society had become increasingly mathematical and noted that consequently, the importance of mathematical education and the need to acquire mathematical literacy had grown. This puts great demands on teachers' professional competence.

In our previous research, especially the need for a good level of *subject didactical competence* appeared very strongly, i.e., the knowledge of mathematical content and its didactic elaboration as well as its realisation in school practice (Tichá, Hošpesová, 2006). (Let us be reminded of Shulman's idea: if teaching should become a profession, it is necessary to aim at creating a *knowledge base for teaching* which encapsulates, in particular, *subject-matter content knowledge, pedagogical content knowledge, and curriculum knowledge* (Shulman, 1986)).

Many teacher educators (Silver, Cai, 1996; English, 1997; Pittalis et al., 2004) emphasise that the core of mathematical education (in school) is not only problem solving but also (particularly) *problem posing*. Problem posing was specified by Silver as generating new problems and questions (growing from some mathematical or "non-mathematical" situation; Koman, Tichá 1998) as well as the reformulation of given problem, e.g., by "What if (not)?" questions, by releasing parameters, etc.

Problem posing contributes to the development of students' ability to solve problems. The ability to formulate questions and pose problems should therefore be an important component of teachers' competence (knowledge base for teaching). In its development, we see both an *aim* and a *tool for the education* of student teachers and teachers. An analysis of posed problems is also a good *diagnostic tool*; it allows not only to teachers in practice but also to researchers and to teacher educators, to determine the level of understanding as well as causes of mistakes and errors of the student. It is possible to get a lot of didactically interesting information from posed problems (Silver, Cai, 1996; Tichá, 2003). It also enables participants' to realize deficiencies in their own knowledge which leads to the improvement of their competence.

Example of the workshop

The basis of the workshop could be examples of student teachers' work (posed problems and their analysis and assessment).

We can start with the exchange of ideas concerning the role and benefit of problem posing. The core of the work will be a gradual discussion on

- the examples of problems posed especially by student teachers,
- the assessment of posed problems,
- student teachers' analysis of posed problems.

An example of a task for student teachers

14 year old students should solve the following task: Create such a word problem which can be solved by a mere calculation 2/3 x 1/4.

One student formulated the following three problems.

1. There were 2/3 of the cake on the table. David ate 1/4 of the 2/3 of the cake. How much cake did he eat?
2. There were 2/3 kg of oranges on the table. Veronika ate 1/4 kg. How many of oranges were left (kg)?
3. 2/3 of the glass was full. Gabriel drank 1/4. What part of the glass was full?

Which answer (problem) is correct? And, in fact, is any of them correct?

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Thinking to the future: Prospective teachers encouraging children's mathematical thinking Example of a workshop

This workshop is a practical exploration into the lessons of two prospective primary teachers half way through their teacher preparation course. The aim is to develop the quality of their education and, in turn, enhance the mathematical experience and prospects of children in the future.

Background

Recent research has demonstrated that mathematical misconceptions may originate during the earliest years of schooling and yet remain undetected until children encounter more complex mathematical ideas in secondary school or even university (Cockburn and Littler, 2008). There is also a growing understanding of the characteristics of successful teachers (Shulman, 1986; Ball, Bass and Hill, 2004) and, in particular, how they can open up opportunities for advanced mathematical thinking with children as young as 5–6 years-old (Iannone and Cockburn, 2006).

Our knowledge as to how teachers become experts is less well developed, however, Rowland, Huckstep and Thwaites' (2005) 'knowledge quartet' provides a detailed account of the key attributes we might expect of elementary teachers towards the end of their teacher education courses.

This workshop begins with trainee teachers half way through their training and is designed to encourage us to reflect on (a) our own pedagogical practice and (b) how we can encourage future teachers to open up children's opportunities for mathematical thinking while reducing the likelihood of misconceptions arising in the process. The catalysts for discussion arise from a small observational study of 8 volunteers as they teach mathematics to primary pupils in Norfolk, U.K.

Workshop outline

Introduction: Following a brief introduction to the literature and the research, the group will be introduced to Jodie – a student who is half way through her teacher education course – and the first few minutes of a lesson she taught to 5–6 year-olds on permutations *(10 minutes)*[1]

Task 1: In small groups colleagues will discuss how they would have continued Jodie's session had they been asked to take over. *(15 minutes)*

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[1]Timings are very approximate and, should the discussion become particularly lively, adjustments can be made and, for example, task 4 may be omitted.

Task 2: The same groups will compare and contrast their suggestions with what Jodie actually did noting, in particular, opportunities to motivate and extend children's mathematical thinking and any potential for the birth of mathematical misconceptions *(15 minutes)*

Task 3 and 4: Repeat the above processes focusing on a second prospective teacher – Morag – working on combinations with 9–10 year-olds *(25 minutes)*

Discussion and conclusion: A whole group discussion highlighting common themes and idiosyncratic inspirations ending with any implications for both teacher education and, indeed our own practice, as a result of the workshop *(15 minutes)*

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Language and communication in mathematics

Part 4

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Analysing interaction processes with jigsaw during mathematics lessons in elementary school

The interaction processes that stem from the Jigsaw cooperation form can be analysed by decomposing the recipients' roles based on their interactional status and their interpersonal speech acts. In this paper we elaborate on the influence of an individual's participation form for the ongoing interaction process.

Introduction

In the mathematics education discussion during recent years, it has repeatedly been pointed out that the learning of mathematics occurs in the interaction between independent, actively discovering and cooperatively communicative processes. It is thereby emphasised that learning on one's own should be combined with learning from and with each other. In order to implement these requirements into everyday classroom, structured cooperative forms of teaching and learning are therefore increasingly realized when teaching mathematics in elementary school. In this article we deal with processes of interaction that stem from such a realization in the everyday classroom practice. Particularly, we deal with the use of the Jigsaw cooperation form in Geometry lessons in a second grade class (about 8 years of age). This cooperative form was conceived by Elliot Aronson in the early 1970s and has since then been extensively tested and researched (Aronson & Patnoe 1997; Kronenberger & Souvignier 2007). It is a form of mutual teaching and learning, in which the learners first develop a partial area of the topic that is being taught within groups of experts and then mutually communicate this expert knowledge that they have independently acquired in puzzle groups. The puzzle groups thereby consist of learners who have previously developed various partial areas. Our intention is to extend an existing theoretical and methodological framework (Brandt 2006, Krummheuer 2007) and to include some useful sociological insights coming mainly from role theory (Tatsis & Koleza 2006) in order to elaborate the influence of the individual participation form for the ongoing interaction process.

Theoretical framework

In the current discussion, the forms of group work in the pre-school and elementary school are mostly justified on educational goals and constructivist learning theories, whereas in terms of knowledge-oriented learning processes the small groups serve the learner as a place in which his/her own, subjective cognitions can be tested in terms of their viability. On this basis, the communication with others offers the potential for cognitive conflicts that can lead to a further development of the individual cognitions. In this sense, work within small groups especially provides better learning conditions through improved possibilities for active, productive participation. Our perspective on the classroom processes is an interactionistic one (Bauersfeld 1994), in which the interaction serves as a place for joint construction of meaning. The individual cognition is thus bound to the participation in collective generative processes as interpretations that are seen as being shared. From this perspective Brandt and Krummheuer have developed a model of the everyday mathematics classroom situation (Brandt 1998, 2006; Krummheuer 2007): The way students are involved in explaining, reasoning and justifying content-related actions is crucial to the success of their learning – and this involvement includes productive and receptive aspects of the interaction. Referring to Goffman (1981) Brandt and Krummheuer identify these two aspects as the production-design and the reception-design of the processes of argumentation and describe tuition as a meshing of a smooth period of interaction, which is subject to the financial aspects of conflict minimization in communication, and condensed periods of interaction, which provide optimised enabling conditions for (content-oriented) learning. The condensed periods of learning differ from the smooth periods of interaction, on the one hand in terms of the complexity and explicitness of the argumentation that is put forth, and on the other hand in terms of the involvement of the listeners in this argumentation and the requirements for a change from a listening to a speaking form of participation in the process of negotiation. In collaborative learning situations the students adjust the balance between these two interaction modes by managing the turn-taking-problem and the content organisation of the interaction – and by doing this interactively, they shape their own opportunities for content-related learning processes (Brandt 2006).

Additional to these basic conversational tasks, the participants are simultaneously involved in profiling themselves and the others, which is described as face-work by Goffman. For example, in the classroom the teacher is caught between the need to correct a student and the need to respect the student's face. The students are also caught between the need to participate actively in the classroom processes and the need to protect their face from any wrong or unpleasant moves. Face is defined as "the social value a person effectively claims for himself by the line others assume he has taken during a particular contact" (Goffman 1972, p. 5) and is considered one of the most important factors that affects a person's behaviour, or, in sociological terms, a person's *role performance* (Goffman 1971, p. 26). During the condensed interaction periods the students may interpret the ongoing situation differently; this leads to the adoption of different roles which, in turn, affect the argumentation processes in various ways. In the following section we will describe the methodology we have used in order to clarify the interaction processes involved in a Jigsaw cooperation situation.

Methodology

Classroom interactions have a fundamentally different structure of interaction than dyadically organised face-to-face dialogues, whether dealing with teacher-centred class discussions or partner and group discussions among students. All discussions in class are rather *polyadically* structured: the teacher-centred class discussion cannot be described as a dialog between a teacher and the learner as a homogenous counterpart, and in addition dialogs take place in public as determined by the classroom and have characteristics that are caused by the integration into the overall events in this public area (Brandt 1998).

In classrooms it is thus possible, as a speaker, to focus on individuals among a multitude of listeners by addressing them correspondingly. Goffman (1981) suggests the decomposition of the interaction into more detailed analytical elements than 'speaker' and 'hearer'. Brandt and

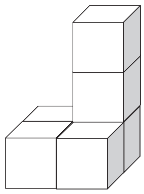
Krummheuer have taken up this view and developed it further into a conceptual net for the description of educational interaction patterns. The starting point is the individual statements or actions that have a certain scope; the allocation of a certain listener role is thus not based on the quality of listening, but on the speaker's way of addressing (Brandt 1998).

In multi-party interactions some of the persons are directly involved in the process of negotiation within the scope of the statement; within this direct participation, individual recipients can be emphasised by being named, through personal pronouns, or else through gestures or facial expressions. This exclusive role is called the *interlocutor* and is for example connected with a special right to speak in the following turn, but also with a certain obligation to react to the current statement. In smaller groups all recipients may be given the status of interlocutor. This must be differentiated from the status of *listener*, who has equal access to the reception of the negotiation, but is not obliged to provide a reaction that goes beyond providing the signals of a listener. In class discussions the differentiation between the interlocutor and the listener can be found for example in the relationship between the teacher and the student who has just been called upon, whereas the rest of the class is often conceived of as a group of listeners. It is, however, possible to fall back on these listeners if there is no reaction on the part of the interlocutor who has been addressed or if his/her reaction is deemed to be insufficient. In public places, in addition to the interlocutor and the listeners, there are usually other persons present within the framework of the scope of a statement, who are not directly involved in the process of negotiation. These recipients can be further differentiated into *overhearers* and *eavesdroppers*. Overhearing is definitely tolerated by those directly involved, but overhearers do not have the right to ask about details that make it possible to gain an understanding access to the process of negotiation. Overhearers are taken into consideration when it comes to the choice of words, the choice of the topics that are discussed or the degree of indexicality. The status of eavesdropper is indicated by the speaker through the posture, the pitch of voice and/or through an inaccessible language code; the understanding reception of the statement is thus made difficult or impossible. In this model, eavesdropping is an allocation that is made by the speaking person and his/her immediate interlocutors, for example when children in class switch from the teaching language (e.g. German in the observed classroom) to their language of origin and thus exclude individuals present from the understanding reception.

These categories are not always selective; there are, especially within each main category, grey zones of addressing that can be clarified only by the subsequent conversation. In processes of group work, interlocutor and listener can be assigned to the interior structure of a condensed period of interaction, whereas overhearer and eavesdropper represent different outer relationships. If a change occurs from the status of an overhearer to that of a listener or even interlocutor, a change occurs in the group affiliation. From the role theory point of view, these shifts between categories do not happen casually; by accepting the fact that students' behaviour may be attributed to 'face-saving' strategies (Brown and Levinson 1987) we are then able to observe the interpersonal and intrapersonal function of each utterance. By doing this, we are then able to draw some conclusions on the group dynamics, its evolution in time and its relation to other groups' dynamics. In the following section, we deal with the inner structure and dynamics of a single study group.

Sample analysis

The teaching unit was integrated into the normal everyday teaching and was documented with two video cameras. The processes of cooperation by two groups of experts and one associated core group were video filmed and from this recording, sections were transcribed for analysis. Before that, the class had gained experience realising the group jigsaw form of cooperation in social studies, whereas more complex cooperative assignments in mathematics education were not yet dealt with. Within the groups of experts and without a preparatory phase, the learners were supposed to find possibilities to jointly or individually record on paper three-dimensional structures made of wooden cubes. As an introductory problem, the teacher had specified a concrete cube structure (see Figure 1) and connected it with a little story. There were various hints for the groups of experts in order to lead to the following solutions: drawings of the various side views, verbal formulation of a construction manual in a construction dictate and notation in building plans, whereby the heights of the structures were recorded in a square matrix (see Figure 1).



1	3	
1	1	

Figure 1. Cube structure and expected solution in the building plan

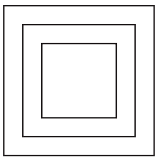
The expert group in focus (Charline, Jamal, Jens and Nele) were to deal with the third solution possibility, i.e. with the notation in the building plan. For this, the children had a worksheet with a 3x3 matrix as a guideline, together with the notice: “This space is sufficient” and the tip: “Place the cubes on the plan and circumscribe them with a pencil”. Due to space limitations we present only a part of the discussion, while our analysis deals with the whole episode (1–55). The cells are named after the letters a, b, c, d, e, f, g, h, i.

[Jens has a lot of cubes on the table in front of him, in front of Nele is the empty construction plan, in front of Jamal the sheet with the group assignment, between Charline and Nele there are more cubes]

9. N: and look/ then we would draw around it here/ [moves with the pen around the construction; only indicates the drawing] then we would make lines in the middle like this/ [indicates lines along the inner edges] well and how we would have to build it higher/ [voice becomes quieter] I wouldn’t know either/
10. J: [raises his right hand] I know it-
11. C: <I got it\
12. N: <how/ [looks at Jamal]
13. J: you you put a few more inside, here, these two [momentarily lifts the two top cubes off of cell a]
14. C: look/ I had the idea
15. C: <look/ [Nele looks towards Charline] here we can/ then this is the top one/ [leans onto the table, shows the top cube cell a] then the second one goes [shows the middle cube cell a] there and then we can draw that/ [removes all cubes from the diagram]
16. J: <so like this\ [draws three concentric squares onto the cover sheet of the group assignment]
17. J: [looks up from his drawing back to Charline and Nele]
18. C: so like this\ these are/ (..) so that they’re [places the two cubes back onto the diagram (1,0,0),(1,0,0),(0,0,0)] (..) on top of one another somehow/ [places the rest of the cubes onto the diagram] and then we can like draw that there too-
19. N: <yeah- but we don’t have enough room for that/
20. C: <[arranges the cubes on the diagram (1,1,1),(1,1,1),(0,0,0)]
21. J: yeah we can do it [points to his drawing] like this-

22. Je: [has been building with his cubes at his seat until now] or we draw that would work too/ if we/
[23 Jens is still presenting his solution; Nele is listening; Charline asks Jamal to erase his
26 drawing; Jamal erases his drawing]
27. Je: #the stones/ for example give them different numbers and then uhm put them on top of each other\ (.) [becoming quieter] and then write down the numbers in the order\#
28. C: #exactly we <can-
29. N: <I didn't really get that
30. C: well look/ we could give numbers to the cubes/ [takes the cubes off the diagram]#
31. N: #oh like like for example [counting the cubes while speaking] this is one that is two that is three that is four that is five that is six#

As a starting point for the group work, Nele draws the depicted building on the empty building plan and circumscribes it with a pencil; she thereby converts the hint for the solution into a concrete action and finally formulates their task (9). She thus focuses on the problem of recording the third dimension ("to build it higher") on the two-dimensional form, which was also not solved by the tip. All other children in the group are uniformly addressed by her as interlocutors and could take the next turn. Nele's initiating act places her in the lead of the discussion and thus makes her a *collaborative initiator* (Tatsis & Koleza, 2006), i.e. a person that takes initiatives but at the same time respects the other participants' opinions. During the next two to three minutes, three different possibilities arise for depicting the third dimension. The entire scene can be described as a chain of condensed periods of interaction, which focuses on the problem of three-dimensional constructions and thus answers the question that Nele directed to everyone as interlocutors. First Jamal replies (10), and fulfils Nele's request to first of all verbally explain the suggestion. Later he draws concentric squares on the sheet in front of him, representing the tower and which can be compared to topographic lines (see Figure 2).



1	2	
4	3, 5, 6	

Figure 2. Jamal's sketch and the group's solution

Charline, however, does not feel obliged to follow his suggestion attentively as an interlocutor but is more intent on gaining their attention as interlocutors by repeatedly saying "look" (14, 15) and by construction activities on the building plan which is lying in front of Nele. The fact that she is so eager to present her view that she ignores Jamal's contribution reveals her intention to maintain her face in the group, by taking a more active stance in the discussion. Her solution can be interpreted as an attempt to draw the floor plan and the upright projection of the building. From a mathematical point of view, a further differentiation of this solution would be quite correct, but it would be not compatible with the notice. This space is insufficient, and Nele comments on the suggestion correspondingly (19). Nele is thus addressed both by Jamal and by Charline as interlocutors regarding possible solutions to her initial question, whereas the other children are only involved as listeners or they feel as if they are addressed as listeners and thus do not feel obliged to comment. Jamal finally focuses his gaze on the negotiations between Charline and Nele and presents his finished drawing to Nele at a suitable moment (21). However, almost simultaneously, Jens uses the opportunity to offer his solution to her (22). Although he makes it clear that he has heard the other suggestions, his contribution is not argumentatively connected with the suggestions that were presented by Jamal and Charline. Instead he seems to bypass these two solution suggestions thematically, and to react directly as a speaker to Nele's initial question; this corresponds to the role of the listener for the previous answers to the initial question. This variety of suggested solutions clearly shows that the interaction patterns in this group are structured in such a way that all participants have the opportunity to express their views; this 'exchange' between the various recipients' roles helps to release the potential tension, since everybody has the right to speak and justify his/her position. Shortly thereafter, Charline is the first to react affirmatively to Jens's idea that his interlocutor Nele has not understood yet (29). Following this unspecific question in a mutual, dyadically oriented discussion partnership, Charline and Nele clarify how Jens's idea for a solution is to be understood (30–33); Jens is thereby simply included as a listener. Finally, Jens and Jamal take turns and clarify in which cell the numbers 5 and 6 should be entered (Figure 2). These four students can be characterised as *collaborative evaluators* (Tatsis and Koleza 2006) concerning Jens's solution, since their mutual involvement is based on the evaluation and enrichment of each other's suggestions. The solution is thereby accepted as a group solution and the boundary to the group of listeners consisting of the other children is again dissolved in favour of a general group of interlocutors within the group. After a further brief clarifying sequence, which is negotiated especially between Nele and Jamal (44–52), Jens finally announces: "we've we've figured it out" (55). With this speech act, people outside the group, and particularly the teachers, are for the first time addressed as interlocutors. The first internal solution process is thereby concluded in a way that is visible and audible to the outside.

Conclusions

The episode that we have analysed here can be described through the participation theory model of the smooth periods of interaction and the condensed periods of interaction as optimised learning situations – at least for individual participants – regardless of a measurable learning success. However, every participant brings very individual orientations into the cooperation, and the group work process is a space of cooperation, which is characterised by the interdependency of individual participation profiles (Brandt 2006). These participation profiles, seen from role theory, may be attributed to 'face-saving' strategies. Sometimes, these strategies hinder the cooperation (Tatsis & Koleza 2006); this was not the case in our focus group and this may be attributed to Nele's moderating participation, being a collaborative initiator at the beginning and a collaborative evaluator later on, a fact that had a positive influence on the group's cooperation. However, to what extent these individual orientations and the resulting dynamics that arise in group interaction are, if at all possible, methodologically controllable, is probably more than only a question of the work material, the assignment or a communicative or cooperative competence that can be trained through questioning behaviour.

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Discussing on the fairness of probabilistic games: The creation of a discursive community with kindergarten children

In the present paper we provide an analysis of the verbal interactions that took place during the realisation of two activities in a kindergarten classroom. Focusing on discussions about the fairness of the games played we observed the strategies that children used to justify their opinion and monitored the development of their intuitions concerning the fairness of a game.

Introduction

The importance of probabilistic thinking is quite high since in everyday life we have to make predictions and then decide under uncertain circumstances while having to evaluate at the same time a large amount of information. This fact is acknowledged by all mathematics educators, researchers and policy makers and is expressed in various papers or other official reports. For example, the guidelines given by NCTM (2000) stress the importance of having all students develop an awareness of probability constructs and applications from an early age. Moreover, research has shown that students have difficulties with these concepts (e.g. Fischbein & Schnarch, 1997; Shaughnessy, 1992). Concerning young children's probabilistic thinking there has been a considerable amount of research (e.g. English, 1993; Jones et al., 1997; Kafoussi, 2004; Piaget & Inhelder, 1975) but the results sometimes seem to be contradictory. For example, the work of Piaget and his colleagues and the work of Fischbein seem to have crucial differences with respect to their educational implications (Greer, 2001; Skoumpourdi, 2003). Langrall and Mooney (2005) in their literature review state that the results of relevant studies show that children eventually can move beyond subjective judgements and animistic behaviours, although it is not always the case. The research described in this paper deals with a number of activities conducted for a European project and its basic aim was to examine the children's and teacher's language during these activities and particularly:

a) The ways young children verbally express their thinking, as they try to perceive the notion of the fairness of a game.
b) The teacher's verbal actions in establishing the notion of the fairness of a game.

In the next section we will describe our theoretical framework related to the functions of language in mathematics education which led us to the adoption of a particular methodology for the analysis of the activities.

Theoretical framework

The role of communication is emphasised by a number of contemporary educational researchers, who sometimes follow distinct approaches. Whether the focus is on the student, the community or the interactions that the student participates in, there is a common agreement on the central role of communication. Concerning mathematics, there are various communicative means: ordinary language, mathematics verbal language, quasi-mathematical language, symbolic language, visual representations and unspoken but shared assumptions (Pirie, 1998). All these characteristics, some of which are unique in the mathematics classroom, eventually lead to the establishment of a discursive community which has its own practices and is regulated by a specific set of norms (Yackel and Cobb, 1996). Relevant research in the field has shown that the "unspoken but shared assumptions" (norms) in mathematics classrooms may influence the content and the structure of the interactions that occur (Tatsis and Koleza, 2008; Yackel and Cobb, 1996). Particularly, they may influence the way a student perceives (and therefore attempts to solve) a problem (Boaler, 1999). Concerning the role of language in this process we believe that it is used to establish mathematical meanings during processes of joint negotiation. These processes can be observed and analysed through the interactions that take place in such settings and by looking for regularities or any kind of pattern that can be connected to the establishment of a discursive community.

Methodology

The approaches that endorse the joint formulation of mathematical meanings or even the establishment of a discursive community focus their analytic lens on the communication that takes place in educational settings. Thus, there is a tendency that the focus should be the verbal (and sometimes also the non-verbal) exchanges of the participants. Sfard (2006) talks about communicational moves and practical actions which continuously affect and shape the interaction. The focus of our paper are these verbal communicational moves and we will use all other non-verbal actions as supplementary to our analysis. Following Austin's (1962) terminology, for the rest of the paper we will call these verbal moves speech acts. The basic assumption underlying the speech act theory is that each verbal expression is used to perform an act, therefore it is interpreted as such by the listener. In our case, after transcribing the discussions into written text we looked for any patterns in the group's discussion that may imply the establishment of a common new probabilistic notion (particularly the notion of fairness) and at the same time looked for patterns in the group's discussion that may reveal the establishment of a shared assumption. We decided not to use any predetermined categories for the participants' speech acts, because our focus was their evolution and their commonalities.

The experiment took place in a typical state kindergarten school of Rhodes in 2007. The children that worked under the guidance of an experienced teacher were five years old. The tasks were designed to examine children's informal knowledge of probability (Skoumpourdi, Kafoussi & Tatsis, in preparation).

Sample analysis

We will present the analysis of two tasks; Task 1 concerns the prediction of the most/least likely event in a random experiment and Task 2 is related to probabilities comparison.

Task 1 – Turtle vs. snail

Nineteen children were separated in two groups (the teacher was member of one team, so that both teams had an equal number of members) and played the following game: the first group had to move a turtle to the end of a ten-stepped green path; the second group had to move a snail to the end of a ten-stepped red path, parallel to the turtle's one. The spinner used had its sectors painted green and red, at the proportion of l' and L' respectively. According to the colour that came out, the respective group made or didn't make one step with their pawn. The winner was the team that will arrive first at the end of the path. The green team won and then the following discussion took place between the teacher and the students. The letter T refers to the teacher and all the other letters refer to the young students (each letter is used once). The symbol ( ) is used when many children speak simultaneously and our notes are in brackets:

1. T: I would like you kids now to tell me how did you make it and finish first?
2. Y: Because it was larger.
3. T: What was larger?
4. Y: The green!
5. S: The green.
6. T: Ah, so what does this mean?
7. P: What does it mean?
8. K: That we won.
9. T: You did it. Because you say that it was larger… Which was larger in the disc?
10. (): The green. [About six children speak]
11. T: So, if we wanted to play fair and complete the game? If we wanted the snail and the turtle to have equal chances to walk, should the disc be like that?
12. (): No! [Almost by all children]
13. T: So Yiannis, you tell us…
14. Y: Because it was, it was little orange. The green was, the green was more.

…

49. T: Nicolas, tell me how should we make the disc so that the game will be fair and we don't, both animals would have the same chances to reach the end?
50. N: The green should be like the orange.

Then the children worked in two groups in order to construct a disc that would make the game fair. We can observe many interesting things in the discussions above. The teacher's speech acts are focused on a basic aim: to help the children understand the relation between the area of the sectors and the fairness of the game. In order to achieve this aim she moves gradually: firstly she tries to relate the area of the sectors to the game they already played (9) and then she tries to introduce the notion of fairness and its relation with the area of the sectors (11). The children seem to realise from the very beginning that the sectors' areas influence the outcome of the game, but they are unable to answer the question what does it mean (6). Although the teacher is probably expecting to hear something about the connection between the sectors' areas and the outcomes of each team's draws, the children do not seem ready to verbally express this relation. Then the discussion moves to the relation of the spinner's sectors with the fairness of the game (11); the children need some time to grasp the idea that equal sectors mean equal chances, but they do realise that the game they already played was not fair (12) and that it has to do with the sectors' size (14). Later on, the children were asked to construct a 'fair spinner'. Figure 1 below shows the evolution of a group's construction; initially, a female student separated the disc in two sectors as shown. Then, after the teacher's query ("Is the spinner fair now?"), another line was added and a student had the idea of creating equal numbers of sectors; the outcome is shown in the fourth image of the sequence.

Figure 1. Constructing a 'fair' spinner

Task 2 – Cherry trees

Fifteen children were separated in two groups (the teacher was member of one team, so that both teams had an equal number of members). Each group used a spinner with the sectors painted green and red as shown in Figure 2.

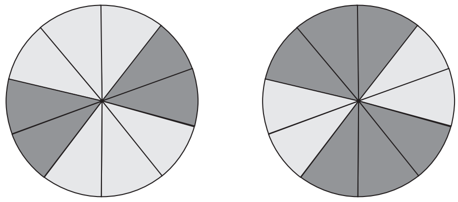

Figure 2. The spinners of the 'Cherry-trees' game.

There was one picture with two cherry-trees and every tree had ten cherries on it. One child from each group turned the spinner and according to the colour that came out, the group picked one cherry from their cherry-tree, when the spinner stopped at the green colour and two cherries from their cherry-tree, when the spinner stopped at the red colour. The winner was the team that will firstly collect all cherries. The game finished before all children had the chance to play. At that moment, the following discussion took place:

1. T: Which team won?
2. R: We did.
3. T: You did.
4. A: But it's not fair.
5. T: Angelica says it's not fair. How could you win so fast?
6. A: Because the red is more times.
7. T: Nicolas, do you agree? Angelica said that in their spinner there is more red and the game is not fair. [Nicolas does not reply]
8. Y: Because it is one, two, three. [Yiannis shows the red sectors in the right spinner]
9. T: Tell us Yiannis.
10. Y: Because it is three pieces. And the green is two.

…

15. T: So? Why it is not fair Yiannis?
16. Y: Because it must have three, three, three green and three red.

…

22. T: Why this team had the red more times?
23. N: Because it's bigger.
24. T: What is bigger? Please explain to us. [Nicolas gets up, goes close to the spinners and starts counting the red sectors of the right spinner]
25. N: One, two, three, one, two, three makes us six and these make four.
26. T: And what did we have on this side? [Nicolas moves towards the left spinner] Let Yiannis tell us. Here? What did we have Yiannis?
27. Y: Three green and two red.

…

41. T: How should these discs be like? Can you tell us Angelica?
42. A: It should be red-green, red-green…
43. T: So, both discs should be what? [Yiannis raises his hand] Tell us Yiannis. [Yiannis moves towards the discs]
44. A: The same.

The discussion went on with the teacher's query "If you were about to play the game again, which spinner would you choose?" Some children initially chose the same disc that they already used, but gradually they agreed that they should pick the spinner that has 'more red'. The previous discussion shows that some children have grasped the notion of the fairness of the game; in (4) Angelica says without being asked that the game was not fair. This utterance reveals an attitude established to the particular student that playing probabilistic games involves a discussion about their fairness. But we cannot conclude that this has become a shared assumption across the children. What looks like a shared assumption is that the fairness of a game is associated with counting and comparing the spinners' sectors; without being explicitly asked to, two children in our case (Yiannis and Nicolas) compare the number of sectors by counting them. Yiannis focuses on half the spinner and sees three green and two red sectors in the left spinner (8, 10), while Nicolas is able to see the spinner as a whole, so he finds the sum of all sectors (25). Angelica, on the other hand, suggests a different approach, probably influenced by their work in the previous task (Figure 1): she suggests that the sectors should be green and red alternately, without any reference on numbers. The teacher chooses to focus not on the different alternatives presented, but on the fact that both discs 'should be the same'. This is probably related to an assumption that there should be a 'general agreement' at the end of the school day on what has been learned.

Conclusions

The basic characteristics of the teacher's verbal acts may be categorised according to their content and their intentions. Concerning mathematical content, in our case the teacher made speech acts that dealt with the notions of chance, fairness, straight lines (when the children were constructing a 'fair' spinner), equality and inequality, sectors, disc, circle, numbers (steps to be made) and space orientation (direction of the pawns). In most cases, she used quasi-mathematical language as well as children's everyday language, which proved to be very efficient, since we did not observe any signs of misunderstanding. The relatively 'new' for the children notion of fairness was introduced in a functional – and not static – manner: a fair game is a counter-example of the unfair games already played and a fair game is made possible through the use of appropriate materials (spinners).

Concerning their intentions, the verbal acts were made in order to ensure that:

a) most (if not all) children will have the chance to talk and express their opinion
b) most (if not all) children will comprehend the concepts involved in this game
c) the 'correct' view will be accepted at least by the majority of children
d) the practical tasks involved (e.g. construction of a 'fair' spinner) will be completed successfully and on time

The above verbal acts did not always achieve their end; there were times – especially in the 'Cherry-trees' game – when many children seemed unable to follow the teacher's guidance and see the difference between the two spinners and the respective outcomes. This in turn affected the establishment of a discursive community in which every participant can speak and has something to contribute on the issue under discussion. Moreover, the teacher sometimes focused too much on the 'correct' opinions, leaving queries and different opinions unanswered. As we already mentioned, this may be attributed to her assumptions on what 'needs' to be learned from these activities.

Most children had the opportunity to express their ideas and we could observe that counting proved to be their most powerful strategy for justifying their opinion. This was more obvious in the 'Cherry-trees' game, where the children used different strategies – based on counting – in order to justify their answer about the fairness of the game. In our opinion, the occurrence of these verbal arguments reveals the ability of kindergarten children to engage meaningfully in probabilistic activities concerning the notion of the fairness of a game.

Langrall and Mooney (2005) suggest that instruction on probability should be two-fold in order to elicit the learner's "awareness of the potential conflict between a primary intuition and the logical structures of probability (p. 113) and to assist the development of "more normative secondary intuitions that can be accessed to override inappropriate or limiting primary intuitions" (p. 113). We believe that in our case, the games the children played together with the teacher's scaffolding assisted them on the development of some secondary intuitions concerning the fairness of game and the role that materials play on that. Moreover, we have witnessed the primary steps of a discursive community based on the premise that each opinion should be justified in order to be accepted by the more experienced members of the community, i.e. the teachers.

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Change: What change? Considering whether a national initiative has had an impact on the quality of classroom talk in primary mathematics

This paper will consider the transcripts of two mathematics lessons in a primary school as 'telling cases' (Mitchell 1984) which illustrate the nature of communication in primary mathematics classrooms in the United Kingdom at present. Using qualitative analysis and a theoretical framework developed as part of an earlier study, the quality of the talk is examined and the nature of the social and mathematical dimensions of the talk analysed. Quantitative analysis explores the proportions of the talk that are attributed to the pupils and to their teachers and comparisons are made with earlier findings. The examples are selected to throw light on the quality of mathematical communication and the nature of participation in the classroom talk by the teachers and pupils involved.

Introduction

This paper arose as a result of observations made of primary mathematics classrooms during this academic year after a gap in making such observations of a period of over ten years. During this period in English schools, the National Numeracy Strategy was introduced and implemented. The National Numeracy Strategy came with an attendant programme of professional development for teachers delivered by Primary Numeracy Consultants and various changes in the structure and content of mathematics lessons were recommended. Resources were also made available to the teachers to support their planning including a comprehensive collection of ready made lesson plans following the recommendations of the Strategy. However the immediate impression that we gained of the lessons observed after this gap was that not a lot had changed in terms the quality of talk or its structure and so we decided to examine this impression in greater detail.

Background

The original study, with data collected between 1996 and 1998, was part of Jenni's doctoral research project which sought to examine the quality of communication in primary mathematics classrooms. The data were gained through participant observation in such classrooms over extended periods of time and were analysed using grounded theory techniques and frameworks derived from discourse analysis. One of the key findings of the research was of a connection between the social component of the classroom talk and the mathematical challenge of the activities being offered to the children. Evidence of the children's engagement in mathematical thinking and reasoning characterised by actions such as offering justifications, generalising and proving occurred in classrooms in which the children were able to voice their own ideas in an open social context and in which the activities offered to them were highly mathematically challenging.

The transcripts that we consider here were chosen from a selection of classroom observations collected in the autumn term of 2007. They came from classrooms in which the teachers were participating in a research project undertaken with support from the National Centre for Excellence in Teaching Mathematics and a north London Local Authority. The project involved the teachers in a continuing professional development course which comprised six half day workshops and follow up activities with the children in their schools. The transcripts and observation data formed part of the preliminary data gathering before the commencement of the course. So the agenda for collecting the second batch of data was unrelated to the first and the similarities which we observed came as a surprise to us. It is these surprising similarities that we felt needed examining and which have led to the analysis presented in this paper.

Theoretical framework

In Jenni's original doctoral study she focused on the talk of the primary mathematics classroom and paid considerable attention to social and mathematical aspects of this talk. She based her analysis on ideas about language games and forms of life outlined by Wittgenstein (Wittgenstein, 1968). In essence she is interpreting language games as the use of language together with the actions that are woven into it. She also takes a form of life to be an established human social practice, established within a community and involving its own purposes, rules and behaviours as well as its own special language games. Mathematical forms of life are characterised by thinking and reasoning that emphasise exemplifying, specialising, changing, varying, altering, completing, deleting, correcting, generalising, conjecturing, comparing, sorting, organising, explaining, justifying, verifying, convincing and refuting about number, data, shape and space. They also involve making connections between mathematical ideas and concepts in a variety of contexts as part of the process of generalising mathematically. This process of generalising comprises conscious mathematical thinking and reasoning and the development of mathematical argument and includes notions of proof.

In her study Jenni focused on the talk of the primary mathematics classroom and paid considerable attention to social and mathematical aspects of this talk. She considers the social and the mathematical to be key to the development of children's ability to participate in mathematical forms of life.

Jenni's model suggests that every utterance in the talk can be analysed in relation to its social and mathematical components and we want to suggest that these components can be viewed as dimensions of the talk as illustrated by the following diagram:

useful sociological insights coming mainly from role theory (Tatsis & Koleza 2006) in order to elaborate the influence of the individual participation form for the ongoing interaction process.

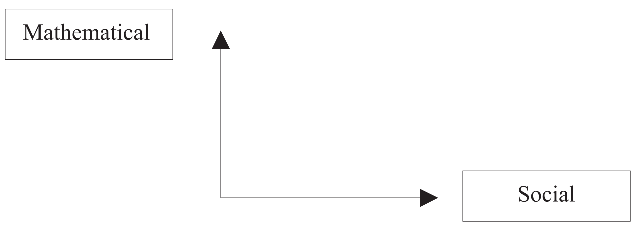

Figure 1. Dimensions

The social dimension of talk is connected with building and maintaining the social relationships within the class, between teacher and pupils and between pupils. There is a sense in which all the talk is social: it involves social interaction between the participants. However we are interested in the contribution of the talk to the social contexts of the learning environment that the teacher and pupils are creating.

The mathematical dimension is concerned with the mathematical component of the talk and relates to the way in which the talk contributes to mathematical forms of life, particularly those that support mathematical thinking and reasoning. We are interested in the contribution of the talk to the mathematical contexts of the learning environment that the teacher and pupils are creating.

In order to develop some sort of scale along these dimensions, we would suggest that the social dimension can vary from open to closed depending on the emphasis of the utterance in terms of its contribution to the social relationships within the class. Openness on the social dimension would suggest contributing to open relationships that encourage pupils and teachers to view themselves as joint participants in the learning and teaching processes. Closedness would be linked with rigid interpretations of the participants' involvement and force them to follow predetermined patterns of contribution to the talk. Jenni developed this scale from her observations that the social dimension of the utterances varied between open and closed.

The mathematical dimension can vary from low to high depending on the emphasis of the utterance in terms of its contribution to the mathematical contexts of the learning environment in the classroom. A high mathematical dimension would suggest that the utterance was closely linked with mathematical forms of life that take account of mathematical thinking and reasoning. A low mathematical dimension would suggest little relationship to these forms of life and might possibly reflect an instrumental understanding of mathematics.

It is clear that this identification of mathematical and social dimensions of classroom communication is not unique and other researchers have investigated their relationship. For example, research into the sociomathematical norms of classroom practice is described by Paul Cobb and Heinrich Bauersfeld (1995). The findings presented here differ from those of these researchers in that they were gathered from 'normal' classrooms rather than an experimental setting. As such they offer the opportunity to observe practices that are occurring with minimal researcher intervention.

Another researcher who has explored mathematics teaching in ordinary classrooms, albeit in secondary schools, is Barbara Jaworski (1994). Her findings make strong connections between the social and the mathematical dimensions of classroom events. Jaworski examined the talk in a number of lessons in which children were engaged in mathematical investigations and came to the conclusion that there were three important components to teaching and learning which comprised what she called the 'teaching triad'. These components were management of learning, sensitivity to students and mathematical challenge. We would like to suggest that, in each of these components, social and mathematical elements play a part.

Our work is similar to that of Jaworski to the extent that it focuses on teaching and learning in 'ordinary' classrooms and explores the factors involved. In primary schools the main focus has tended traditionally to be on the learner and learner development rather than the mathematics that needs to be covered. This had implications for our analysis of the transcripts as it made the social as important as mathematical. However as a proviso to this the National Numeracy Strategy with its emphasis on learning objectives has possibly tended to change this focus away from the learner and towards curriculum content.

We feel that there is some gain to be made by separating the social and mathematical dimensions. This enables us to explore those strategies that are common across teaching and learning situations generally and those that are special to teaching and learning mathematics and that may be related to mathematical forms of life. In identifying the social and mathematical dimensions of the talk we hope to disentangle some of the complex issues involved in classroom talk and present them more clearly in the contexts of primary school mathematics classrooms generally.

Methodology

As described earlier this paper presents a detailed study of talk between teachers and pupils in two mathematics lessons in primary school classrooms collected in September 2007. It is taken from a sample of observations of lessons with a small number of teachers and classes from schools situated in London. The analysis focused on the detailed study of the transcripts of these lessons, two of which have been selected as 'telling cases' (Mitchell 1984).

The data collected comprised audio recordings of the lessons and extensive field notes. After this a full transcription of the audio recordings was made. Analysis of this transcription identified the nature of each utterance in terms of the speaker and also its purpose using the traditional initiation, response, feedback descriptors (Edwards & Westgate, 1994). We also examined the social and mathematical dimensions of the utterances exploring the level of mathematical challenge and the opportunities afforded to the pupils to voice their mathematical thinking as well as the 'answer' to the teachers' questions.

The findings

The two classes were from the same school and were parallel Year 4 groups of 27 and 28 pupils respectively. In each class the children were arranged around groups of tables with 4, 5 or 6 pupils in a group. There were two teaching assistants present in one of the classes and one in the other.

Each lesson was nominally of one hour's length. The conduct of the lessons was tightly controlled by the teachers, though the pupils had short periods of activity during which they essentially practised the techniques taught in the more didactic part of the lesson. These periods of 'independent' pupil activity totalled 42 minutes out of the 2 hours (35%) and were characterised by continued question and answer between the teachers, moving round their classes, and individual pupils. Thus for some of the time the 'independent' working became an extension of whole class teaching though with the additional interactions between pupils.

We analysed the talk in two ways, as explained above, in terms of the social dimension and the mathematical dimension. While to some extent the classification is subjective in most cases the talk could be categorised quite clearly.

The following example of how we classified a piece of a teacher's talk from one of the lessons will serve to illustrate the point:

> …this time I am going to give you only one minute. *(socially closed)*
>
> After that minute I am going to ask anyone to explain to me so I need to hear a lot of talking while you are doing your working out. *(socially open)*
>
> Ok so we are going to have another subtraction, so this time 126–12. *(mathematically low)*
>
> Let's try that one. One minute. Before you start, of course, everyone listen to me. *(socially closed)*
>
> The most important thing is I'm not really interested in the answer, so it's not a race to get the answer the quickest, it's actually how you are doing it and how are you explaining to your friends ok. that's what the most important thing is. So you've got a minute to talk… *(mathematically high)*

The socially closed parts indicate that the teacher is very much in control and imposing restrictions on the children. However, she does encourage interaction between pupils (socially open) albeit within a much restricted context. The task set is another example of a process to be applied that has been explained and learnt earlier and as such is low in mathematical thinking. On the other hand asking for an explanation of the method used and playing down the importance of the answer is an encouragement to think mathematically and could lead to high mathematical involvement and discussion.

So, we categorised the classroom utterances within the two lessons under the same four headings. The result of this analysis is shown in the table below:

Dimension	Number of utterances
Socially Open	68 (34%)
Socially Closed	132 (66%)
Mathematically High	42 (14%)
Mathematically Low	253 (86%)

This result becomes more interesting if it is compared with data collected from before the advent of the NNS. Jenni (Back, 2004) found in the study for her thesis that the lesson with the highest levels of mathematical thinking and reasoning had a social and mathematical profile as follows:

Dimension	Number of utterances
Socially Open	30 (64%)
Socially Closed	17 (36%)
Mathematically High	150 (61%)
Mathematically Low	95 (39%)

The profile for the lesson that had the lowest levels of thinking and reasoning within her sample had quite a different profile with the dimensions tending towards the socially closed and mathematically low categories:

Dimension	Number of utterances
Socially Open	6 (15%)
Socially Closed	35 (85%)
Mathematically High	19 (19%)
Mathematically Low	80 (81%)

If we compare the profile for the lessons observed for this study we find that they match more closely with the second profile. The preponderance of teacher dominated talk in these classrooms has not encouraged social openness and has failed to stimulate mathematical thinking. The mathematical talk was very much of a procedural nature and lacked any mathematical challenges.

A word count of the transcriptions of the whole class teaching part of the lessons revealed that for each class the number of words uttered by the teachers amounted coincidentally to

92.6% of the total. Pupils' utterances, making up the other 7.4%, were on average 10 words for each pupil who spoke. All but one pupil in the two classes contributed to the dialogue though many pupils' responses consisted of one word or one number in answer to a closed question. Again we may compare this with the corresponding data from Jenni's earlier study:

	Proportion of total word count	
Study	Teachers	Pupils
Current study	92,6%	7,2%
Earlier study (higher mathematical)	80,9%	19,1%
Earlier study (lower mathematical)	88,7%	11,3%

Taken together, these and the results above imply that the NNS initiatives that have been adopted by the school have not resulted in any improvement in mathematical thinking and reasoning levels. On the other hand it is clear that the NNS has had a large effect in the style of teaching. The lessons followed the pattern of an introductory mental section followed by a main part to the lesson and usually finished with a plenary session checking openly with the pupils whether they had met the criteria for success stated at the outset. There was a substantial proportion of whole class teaching and the teachers set great store by the explicit use of learning objectives (LO) and their attainment. For example in one class LO was written on the whiteboard and the children were asked if they could remember what LO stood for. Similarly, an awareness of the prescribed structure necessary for a 'good' lesson is shown when thirteen minutes into the lesson we hear "Now I'm going to give you one last one then we are going to start with the main lesson." Then five minutes later again, "Well done, everybody sitting up straight ready for the main lesson." The main lesson that ensued was similar in style and content to the introductory section but had at least been clearly delineated to the children.

Brown and Millett (Brown & Millett, 2003) have suggested that it is the quality of the whole class teaching that is the crucial factor in improving attainment in mathematics rather than whole class teaching per se. We found in our observations that the format of the lessons and much of the structure required in the NNS was being faithfully followed in schools. Our small scale study resonates emphatically with other studies, for example Smith, Hardman et al (2004) and Iannone and Cockburn (2008) and reinforces the notion that the changes taking place may be only superficial. All the indications point to an urgent need for CPD for teachers that will change the mathematical quality and openness of teacher-pupil interactions.

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The pupils' interpretation of mathematical writings

Communication in the teaching of mathematics is distinguished by using various systems of semiotic representation. Their usage is often given by customs or conventions in mathematics. The acquirement of rules how to form, interpret, and use them properly plays an important role in communication and cognitive processes. The problems in communication tend to be caused by not only different mathematical images or communication contexts but also non-accepting of conventional mathematical writing by the pupils or insufficient acquaintance of the pupils with rules of their usage. Some cases of pupils' inventional representations are presented and analysed.

Introduction

Usage of various semiotic systems is not only characteristic for mathematics but also necessary for it enables us to communicate and cognize mathematical laws. Signs help us to grasp and understand abstract mathematical objects and relationships among them. In this context, signs mean not only mathematical symbols but rather everything used to represent (i.e. substitute) objects and relationships in mathematics, thus also words, figures, graphs, etc. In didactics of mathematics, it is common to speak of representations (although the meaning of this concept is wider). For communication in mathematical classes, it is important to acquaint the pupils not only with various forms of mathematical knowledge representation, but also with the rules how to form, interpret, and use them properly (compare Ferrari, 2006).

It is stated in curriculum documents that mathematical training leads the pupils to accurate and concise expressing themselves by using the mathematical language including symbolism, by performing analyses and writing statements in problem solving, and to perfecting their graphical expression. It may seem that pupils achieve this level of communication competences by applying the courses mentioned automatically. However, practical experience shows that if the mathematical language is not developed on all of its levels and if analyses and writing of problem solutions are performed formally only, numerous problems appear in the communication between the teacher and the pupil.

In this paper, we described some cases of usage of mathematical symbols in pupils' writings that is not in accord with prevalent customs.

Theoretical framework

Bertrand (1993) indicated that the majority of pupils with problems in mathematics at the elementary school do not develop any type of representations of problems set. The pupils do not gain the impression of understanding from the teacher's explanation but based on transformation performed by the pupil when listening to the teacher. Duval (2006) added that understanding in mathematics implies coordination of at least two semiotic representations.

The difference of images (mental representations) and contexts tends to be one of the most frequent causes of misunderstanding in the communication between the teacher and the pupil. Difference of images of the teacher and the pupil is entirely natural; it namely stems from different level of their knowledge. Therefore, it is especially up to the teacher to take this fact into account during communication, and to eliminate the misunderstanding by choosing suitable language means. Speech of the teacher should be characterized by the sociometric nature, i.e. by orientation at mental structures of the pupil (to say things in such a manner so that the pupil understands the teacher). The teacher may make understanding of the pupils and creating of adequate knowledge structures more difficult precisely by the way of communicating with them. When the teacher's expressions are exact technically but not understandable for the pupils, mutual communication is disabled. A common language in which the concepts used have a very close content for the teacher and the pupils, and in which the words connotate similar meanings in their minds, is very important for the communication. Sometimes, a part of the language needs to be delimited, explained to each other, and agreed upon.

In communication, we rely on knowledge adopted as correct. We seek to incorporate our knowledge in the existing knowledge structure. When two pieces of knowledge are not compatible, tension is formed, and the given individual seeks to eliminate it by changing one of the pieces of knowledge or by integration of both. An important role in communication between the teacher and the pupil is played by pupils' preconceptions, i.e. the pupils' concepts characterized by certain immaturity, imperfection, preliminarity, provisionality. When explaining a new subject, the existing preconceptions do not disappear but rather form certain symbiosis with the new subject matter.

A significant role in words and symbols interpretation is played by the communication context, i.e. the framework within which the communication takes place. The context is determined by images of the pupil and the teacher that affect their understanding and usage of communication means, by the social environment, and cultural customs. The context plays an important role in intrepreting the sign; for example, the sign "**N**" may denote the set of natural numbers (symbol) or vertex of a geometrical figure (index) or it can be viewed as figure symmetrical with respect to a central point (icon). And also the meaning of some mathematical concepts is given by the context in which they are used.

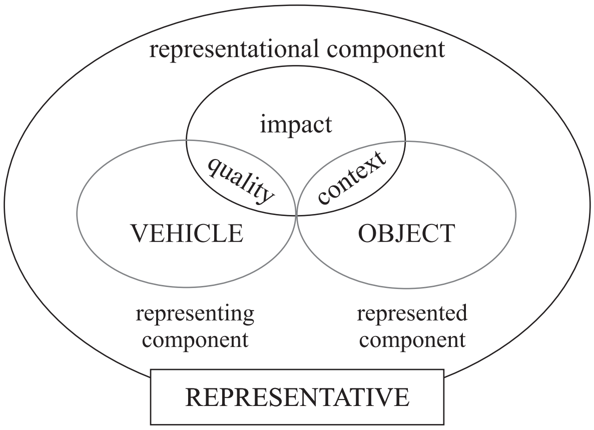
The phenomenon when the pupil uses an expression that does not correspond with the communication context or when the pupil uses such an expression in two different semantic contexts is called the communication confusion. When confusion in communication is caused e.g. by usage of an improper term and if the semantic context is not lost by doing so, the communication is not usually threatened markedly. Confusion caused by non-uniformity or ambiguity of the context tends to be more serious as it leads to communication dissonance if prolonged. Context non-uniformity of the communicants arises from sudden or repeated change of the context not registered by the other party or by using an expression interpreted differently by the teacher and the pupil. Communication dissonance is a phenomenon caused by communication confusion that leads to discordance or disagreement between the communicants. The source of dissonance is usually hidden and not realized by the communicants.

It is evident that the process of representation and understanding of concepts is purely individual. The need to communicate and make understood led to establishing of social conventions for the use of representation means. Representations as social conventions are expressions of intersubjectivity. Thompson (2002) defines the term intersubjectivity as the state where each participant in a socially-ongoing interaction feels assured that others involved in the interaction think pretty much as does he or she. We meet at pupils in the mathematics classroom with the conventional representation systems which are proposed to them by teacher and also with the pupils' individual representation means, so called inventional representation.

Methodology

During The semiotic approach is one of approaches used in investigation of communication in mathematics education (see Hoffmann, 2003, Presmeg, 2006, Steinbring, 2006). The main reason why the semiotic approach finds its use in didactics research relates probably to the relation between semiosis and communication. Semiotics is used as an analytic tool for the didactics of mathematics which is applicable in the cognitive, the social or the cultural level of investigation (Winslřw, 2004). In our methodology we use the triadic model of a sign: the sign is interpreted as a relationship between a sign vehicle and an object that it stands for in some way (Presmeg, 2006), and the triadic, relational view of communicatin (Ongstad, 2006): any communication will have a structural (form), referential (content), and an addressive (use).

During analysis of pupils' work or statements, we seldom work with an integrated representation system. We usually deal with individual elements of such a system. This fact has led me to delimiting of the concept of the representative (Roubíček, 2006). This term denotes an element of the semiotic representation system or partial product of the representation process. Representative is a triad formed by three components: representing, represented, and representational.



The representing component or the vehicle is what represents the represented object. A mark, line, item, sound etc. can be the representation vehicle. The represented component or the object is what is represented. In mathematical training, the representation object usually means a mathematical concept. The representing component and the represented component of the representative correspond with the indicating and the indicated in de Saussure's diadic conception of the sign.

The relationship of representation between the vehicle and the object is determined by the representational component of the representative that includes (1) a qualitative property of the vehicle identical with the object property; (2) context accompanying the representation process and delimiting the represented object; (3) impact of the vehicle-object relation on the interpreter. The type of effect of the vehicle-object relation on the interpreter is affected by the interpreter's experience and achieved knowledge. In his triadic semiotic concept, Peirce uses the term interpretans to denote this component.

The representative is not only a key concept of the theoretic framework of the representations problems exploration but also an important methodological starting point. The semiotic analysis method is based precisely on identification of representatives and on observing of relationships among them. Representing components of the representatives and their mutual relationships, the so called syntax, are explored on the syntactic level of the semiotic analysis. The vehicle-object relations, i.e. meaning of the representatives, are analysed on the semantic level. Pragmatic level of the semiotic analysis is focused on exploring the representative component of the representatives, i.e. the representatives usage.

Exploring of the representatives is one of the ways how to diagnose the pupils' understanding of mathematical concepts. Representatives that appear in the pupils' communication when describing mathematical situations provide represent means of perceptible representation of mathematical objects. However, at the same time, they provide certain information on mental representation of these objects (i.e. on ideas created about them by the pupils). Based on these indicators, the level of the pupils' understanding of mathematical concepts can be assumed.

Some results

The following text includes a summary of partial results of a number of observations of communication at mathematical classes at an elementary school (10–15 years old pupils), as well as of didactic experiments focused on the communication processes exploration.

The phenomena as a communication confusion and a communication dissonance appear in oral as well as written communication. While the communication dissonance can be eliminated by clarifying the context in a discussion or possibly by using other means of representation, in the written communication the teacher stems from the pupil's writing, usually without having an opportunity to clarify the semantic context. Writing of problem solutions tends to be the source of numerous communication problems. For example, wrong statements of equality are seen in written solutions of calculation of an expression value using multiple operations, in which cases the pupil gradually adds parts of the expression to a partial result and uses the equal sign irrespective of the actual value of the expressions. For example:

$$2(3+4)-5 = 7 \text{ x } 2 = 14-5 = 9$$

$$2(3+4)-5 \neq 7 \text{ x } 2 \neq 14-5 = 9$$

$$2(3+4)-5 \rightarrow 7 \text{ x } 2 \rightarrow 14-5 = 9$$

$$3+4 = 7$$
$$2 \text{ x } 7 = 14$$
$$14-5 = 9$$

It follows from the analysis of further situations that the equal sign is a multifunctional symbol for many pupils, that denotes almost any relationship among the data. For example, the following statements appear in the pupils' solutions:

3 kg = CZK 24	…cost
1/2 = CZK 250	…be
1 cm = 2 km	…correspond
cube = 5 cm	…have

Errors of this type the reeducation of which is usually not an easy task are caused in a certain extent by the teachers' approach. Pupils acquainting themselves with a certain semiotic system are not usually aware of their mistake, and it is therefore up to the teacher to correct the pupil in using the system. If not doing so, the teacher indicates to the pupil that the pupil's usage of the system is correct which may mean improper acquisition of the semiotic system rules as a consequence.

Another problem that can be seen in solving of verbal tasks is represented by senseless usage of the unknown variable *x* in written statements. The letter *x* is used to denote an unknown datum, although not appearing in the calculation at all, or it denotes several different unknown variables or partial results. For example:

Width ... 5 m
Length ... 14 m
Area ... *x*
S = a x b
S = 5 x 14 = 70 m^2

By 1/10 more
CZK 350 per m^2
Total ... CZK *x*
x = 70 : 10 = 7 m^2
x = 70 + 7 = 77 m^2
x = 77 x 350 = CZK 26 950

The mistake can be found on part of the teachers again who have not acquainted the pupils with these forms of written statements. It must be realized that the exemplary statement or the solution course using an unknown variable, provided in the textbook, is not always the only correct one, that other possibilities exist, as well, and it is up to the teacher to acquaint the pupils with them. Verbal tasks can be solved not only by means of an equation but also by means of judgment, by listing of the elements (table), in the graphical form (graph, scheme). No unknown variable may thus appear in the writing. However, it should be apparent from the writing what and how the pupil has calculated (or how the pupil has thought) and to what solution the pupil has come.

Conclusions

Functional communication in the class represents an important precondition of an effective teaching process. It is apparent that it is never possible to arrive at complete agreement in communication between the teacher and the pupil for their knowledge of mathematics is different. It shows in some class situations that the pupils communicate on a given mathematical problem more easily among themselves than with the teacher. The teacher namely attends to the formal aspect of the communication while the pupils use also unconventional expressions in mutual communication, vague from the teacher's point of view, however, understandable for the pupils. Communication in the mathematics classes is thus about seeking for a common language in a certain extent.

The occurrence of inventional representations in pupils' mathematical writing is natural and frequent in particular at non-conforming and inventive pupils. Some of these deviations from prevalent forms of communication in the teaching of mathematics are inconsiderable, another ones present a didactical problem because they conflict with mathematical laws or procedures. In these cases it is necessary to intervene in due time for that reason the postponed rectification or reeducation is an exacting and lengthy process.

Practical experience from the classes shows that precisely the mathematical terminology and symbolism tends to represent an obstacle for some pupils in understanding mathematics. The problem does not usually consist in terms and symbols themselves but in ways in which they are introduced and used in mathematical classes. When introducing the symbolism of mathematics, it is necessary to acquaint the pupils not only with the form of individual symbols but also with the rules to create admissible combinations of the symbols, their meaning and usage in various contexts. If the pupils do not know such rules, statements written using the symbols become a formal matter for them in the better case, or a communication and cognitive obstacle in the worst.

Acknowledgement

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Problem posing, problem solving

Part 5

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Approaches to solving processes of one fuzzy problem by primary school children

We have focused on description of different pupils' ways of solving the fuzzy problem and to monitor development of mathematical thinking depending on the children's age. This is based on the process of solving the problem. Results of this experiment are obtained by the method of detailed analysis of pupils' solutions.

Fuzzy problems are a new tool for math teachers. The researchers of mathematics education have not dealt with this matter so far. Therefore we have examined it in our research. We have studied the subject from different point of views. In this paper we have described the primary school pupils' attitude to fuzzy problem. In our study we have concentrated on primary school children because their mathematical thinking is already developed at that age.

Impulses in our real life are ambiguous as well as mathematic problems. The most suitable way to clarify this subject is to explain it by using an example. This assignment was used by M. Hejný and F. Kuřina to describe and show the different ways of teaching (Hejný, Kuřina, 2001). The assignment is originally from the forth year's schoolbook. The example of assignment of fuzzy problem:

Count all even numbers between the numbers 5 and 21.

The Slovak language enables two different meanings and understandings of the word "to count". One is with the meaning "to sum" (it gives a result of 104) and the second meaning is "to count up = to find out the quantity" (it makes the result of 8).

In our study we used the expression "the fuzzy problem" to define a mathematic problem expressed in words whose assignment can be interpreted in different ways.

The formulation of the aims of the work

It is obvious that child's understanding of the world around him/her (as well as mathematics) is different than the view of an adult. Therefore solutions of the pupils in the first four grades at the primary school enable us to see the fuzzy problems from different points of view.

The aim of this study is to describe different pupils' ways of solving the fuzzy problem and to monitor a development of mathematical thinking with the age of children. This is based on the process of solving the problem. We suppose that pupils of the first four grades are not bound by a following didactic contract: "I do not say what I consider to be right but what I think the teacher thinks is right."

Methodology

To fulfil the aims of this study we consider the qualitative research method to be the most appropriate. In the experiment we used an individual discussion with pupils and also a discussion in the group.

The experiment was carried out in two parts: in April 2007 in the second grade (three children of age 7–8 years) and in May 2007 in the fourth grade (three children of age 9–10 years) in one of the primary school in Prague. At the very beginning of the experiment we had an introductory conversation to get to know pupils better. This conversation was used as a start of co-operation between pupils and us. We told them in advance what was going to happen. We sent two of them back to the class and the third one was given the assignment on the paper that he/she should solve. After he/she had solved it the second child came back and was told the assignment by the pupil who already solved it. Then the second child solved the problem. The same situation happened when the second child told the last pupil the assignment. After the last solution was given all of them were asked to work on other four differently reformulated assignments of the same original task. They solved all of these problems.

As an example of the fuzzy problem we used an assignment with twelve cubes. The same assignment was also used in the research carried out with secondary and college students (Kovárová, 2006) so it enables us to compare the ways of solving the task by primary and secondary schoolchildren. Of course, it was necessary to make a little modification of the original task for second grade pupils because they may not know the meaning of the word "block" yet. For the fourth year pupils we can use the original task with the word "block" (expression used in fourth grade is in brackets):

How many blocks of flats (blocks) can you build from twelve cubes?

The words "block of flats" substituted the word "block" for second grade children because we do not want to define new concept. In the first part of the experiment (introductory talking) we were discussing the meaning of the word "blocks of flats" to make the idea clear (block of flats = block).

As it was mentioned above the elder students in the secondary school and the college students have solved this problem. From the output of research with the secondary school and college students we reformulated the original assignment and we got four different interpretations. These interpretations were given to the pupils at primary school in the last part of the experiment. By these reformulations we wanted to monitor the abilities to read and understand mathematical text.

The reformulations were given in this order:

1. How many different blocks of flats (blocks) can you build from twelve cubes using all cubes for each block of flats (block)?
2. What is the maximum number of identical blocks of flats (blocks) that you can build from twelve cubes using all cubes?
3. Using all twelve cubes you have built several identical blocks of flats (blocks). What different types of blocks of flats (blocks) could you build?
4. How many blocks of flats (blocks) can you build from twelve cubes if you do not have to use all twelve cubes and each time you build different block of flats (block)?

For better understanding of the assignment there are attached complete answers for each interpretation (Tab. 1). Explanation for signs used in the table: for example 6*1x1x2 means six blocks in dimension 1 width x 1 thickness x 2 height.

Quantity of used cubes	Interpretation No. 1	Interpretation No. 2	Interpretation No. 3	Interpretation No. 4
1 cube		12*1x1x1	12***1x1x1**	1x1x1
2 cubes		6*1x1x2	6***1x1x2**	1x1x2
3 cubes			4***1x1x3**	1x1x3
4 cubes			3***1x1x4**	1x1x4 1x2x2
5 cubes			2***1x1x6**	1x1x5
6 cubes				1x1x6 1x2x3
7 cubes				1x1x7
8 cubes				1x1x8 1x2x4 2x2x2
9 cubes				1x1x9 1x3x3
10 cubes				1x1x10 1x2x5
11 cubes				1x1x11
12 cubes	1x1x12 1x2x6 1x3x4 2x2x3		1***1x1x12**	1x1x12 1x2x6 1x3x4 2x2x3
result	**4**	**12 or 6**		**21**

Table 1

We supposed that it is difficult for primary schoolchildren to write down the process of solving the problem. Pupils could use cubes (more than 12, approximately 100). They showed us their solution manually so we could see their work and also the processes of thinking and solving the problem. The processes of the solving were recorded with the video camera and written down into the protocols (the notes of all that was done and said). Afterwards we deeply analyzed the protocol with the detailed analysis method. We can find this kind of analysis published in paper (Kratochvílová, Swoboda, 2003). We tried to describe all phenomena that we have observed. The phenomena were divided into three groups: social, emotional and cognitive (Kratochvílová, Swoboda, 2003). Social phenomena include all behaviour which is influenced by human interaction with each other. Emotional phenomena are mental and physiological states of a human being associated with a wide variety of feelings. Cognition phenomena are associated with the development of concepts or individual minds. In this paper we focused only on the cognitive phenomena. The emotional and social phenomena were not described in this paper but they are also important. We used these cognitive phenomena to describe pupils' ways of thinking during the process of solving problems.

The results of the experiment

It is natural that the pupil for example solving interpretation No. 1 did not build all 4 blocks that is the correct answer. We cannot expect complete solutions from primary pupils, as they are not mathematically mature enough. We can estimate in which interpretation they are solving the assignment just according to one block of flats (block) they have built and according to their process of building this block of flats. For better understanding of the following text we attached pictures of the 3D solution. Abbreviation used below for marking is in a form of Number and Letter. The number stands for the grade (2 for second grade, 4 for fourth grade) and the letter stands for an order of the pupils (A for first pupil, B for second pupil, C for third pupil).

We analyzed pupils' solutions with the detailed analysis method. This method is very long and takes a lot of space. For that reason we showed this method only on work of girl 2A. The other pupils' work is mentioned only in brief comments through the paper.

The understanding of the problem by the second year pupils

We worked with a group made up of two girls and one boy:

2A

As an example we show here only a part of the protocol 2A and a part of its analysis concentrated on cognitive phenomena (Tab. 2). It is clear from the analysis, what has convinced us about the pupil's choice of interpretation. Cognitive phenomena are described as an interior monologue of pupil (interior monologue is in inverted commas "").

PROTOCOL		2A
Conversation	Activity	Cognitive phenomena
$2A_1$: *Does it have to be 12 cubes exactly?*		Cognitive phenomena "The assignment does not say that you have to use all 12 cubes."
Ex_1: *Here is the assignment; on it is written how many cubes you should use.*	She has made a block 2x1x6 (she used the block 2x2x4 which she had built previously) and she went on.	"The assignment might state the need for using all 12 cubes (in other case my teacher would not correct me)."
$2A_2$: *For example, one.*	She suddenly realised that she used too many cubes (15 cubes). She destroyed segment of block and she has made a tower 1x1x8 (Fig. 01).	"My idea of a block of flats looks like a tower."
Ex_2: *Well, so leave the block as it is and make another one.*		
$2A_3$: *Next? Next block of flats, yes? From twelve cubes ...*	She has built the same tower as before (Fig. 02). When it was 8 cubes height it broken down and it destroyed the first tower.	"The next block of flats … they can be the same. Make up from 12 cubes... I would like to make up a tower from 12 cubes but it is not so easy. I will build a tower that I manage to do."

PROTOCOL		2A
Conversation	Activity	Cognitive phenomena
$2A_4$: *I would rather build them near to each other.*	She had built the block 2x1x8 (Fig. 03). Then she tried to add another cube but the stability was poor. A part of the block has torn down and the block 1x1x8 was left.	"Two blocks of flats can be near to each other and they are still two blocks of flats."
Ex_3: *From how many cubes is the block of flats built?*		
$2A_5$: *From eight now.*	She is building the same tower as the one she has built before.	
$2A_6$: *Wider block of flats…*	A new block 2x1x6 was created (Fig. 04).	"So far, I imagined the block of flats as a tower. But why couldn't it be a block of flats with wider side?"

Table 2


Figure. 01

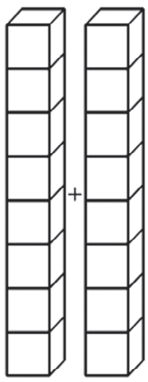

Figure. 02

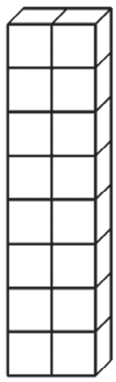
Figure. 03

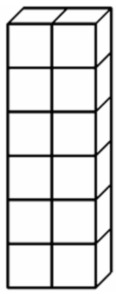
Figure. 04

From the way she was solving the task we can see the progress of her thinking. At first she was trying to build "a tower as tall as possible". Because of poor stability she put the two towers together. In the end she realized that she could build wider block. She considered the problem as the interpretation No. 1.

Her assignment of problem forwarding to the next girl was:

> I read from the paper that from twelve cubes…, from how many cubes…, from how many cubes I can make…. How many blocks of flats can YOU build from twelve cubes?

2B

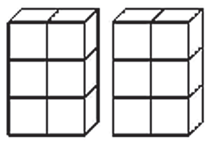
Figure. 05

The second girl built two blocks 2x1x3 (Fig. 05) without hesitation and she told the result: "Two". She did not say a word about her process of solving and she did not want to build any other block. She considered solving the problem as the interpretation No. 3.

Her assignment of problem forwarding to next boy was:

I happened to know how many blocks of flats you can make from twelve cubes. You have to build more blocks.

2 C

The third boy after hearing the assignment built three blocks 2x1x2 straight away (Fig. 06). Despite the fact that they were told not to interfere with other pupils one of the girls laughed at him and shook her head in negative way. The boy broke down the buildings in anger. Later on he built a block 2x1x6 (Fig. 07).

The solution of the third pupil is interesting. He has reacted immediately to criticism and changed his solution. At first he solved the problem as the interpretation No. 3 and after the change he solved it as the interpretation No. 1.

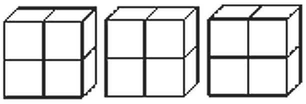

Figure. 06

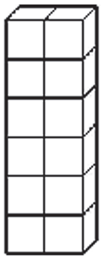

Figure. 07

In the last part of the experiment in this group there was a little disturbance because the pupils could not concentrate any longer. The pupils did not notice any differences in the four modified interpretations.

The understanding of the problem by the fourth year pupils

There were two boys and one girl in this group:

4 A

The first boy's solution was surprising. He came up with new understanding of the assignment and therefore we fill up new interpretation:

5. How many different groups of blocks consisting of all 12 identical cubes exist?

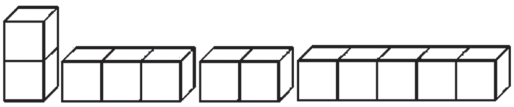

Figure. 08

Figure. 09

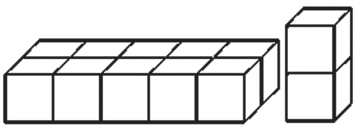

Figure. 10

1.	**1x1x2**	**3x1x1**	**2x1x1**	**5x1x1**
2.	**3x2x1**	**2x1x1**	**4x1x1**	
3.	4x2x1	1x2x1	1x2x1	
4.	3x2x2			
5.	3x1x1	5x1x1	4x1x1	
6.	7x1x1	5x1x1		
7.	4x1x1	2x1x1	3x1x1	3x1x1
8.	4x1x1	5x1x1	3x1x1	
9.	5x1x1	5x1x1	2x1x1	
10.	**5x2x1**	**1x1x2**		

Table 3

He had built ten different groups of blocks. Tab. 3 represents exactly boy's process of building. Fig. 08 shows what he built in the first phase, fig. 09 shows what he built in the second phase and fig. 10 shows what he built as last, creating his tenth group of blocks (in Tab. 3 in bold).

4B

The second girl built, at first, blocks 2x3x1 and 3x2x1 (Fig. 11) so she solved the problem as the interpretation No. 3.

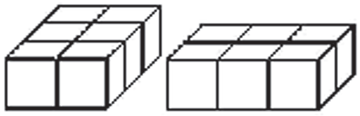

Figure. 11

Immediately, without any explanation, she started to solve the problem as the interpretation No. 1 (she built blocks 1x3x4 – Fig. 12, 1x6x2 – Fig. 13 and 2x3x2 – Fig. 14).

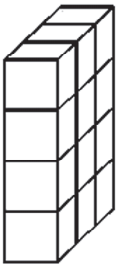

Figure. 12

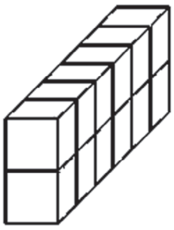

Figure. 13

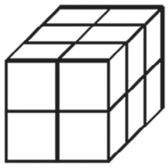

Figure. 14

The girl has changed her interpretation after her questioning. Following dialogue is a part of the protocol 4B put into the text. Change of her interpretation is evident from the protocol. For example expression 4B$_2$ demonstrate that girl didn't feel necessity of using all of twelve cubes.

4B$_1$: I can build 13 blocks … first build one then break it down and after build the next one.

She had built more than 13 similar blocks and then she started asking:

4B$_2$: Is it necessary to use all the 12 cubes?
Ex$_1$: And how many would you like to use?
4B$_3$: Less, for example six.

After this short conversation she built blocks 1x3x2 (Fig. 15), 2x5x1 (Fig. 16) and 1x5x1 (Fig. 17). So she considered the problem as the interpretation No. 4.

Figure. 15

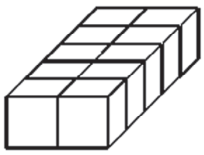

Figure. 16

Figure. 17

4C

The last boy solved the problem as the interpretation No. 3. He built two blocks 3x2x1 (Fig. 18), two blocks 2x3x1 (Fig. 19), four blocks 3x1x1 (Fig. 20), six blocks 2x1x1 (Fig. 21) and one block 3x2x2 (Fig. 22) and 4x3x1 (Fig. 23).

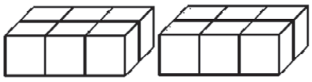

Figure. 18

Figure. 19

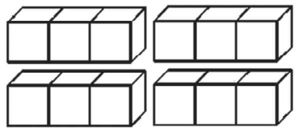

Figure. 20

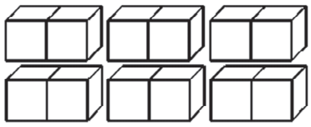

Figure. 21

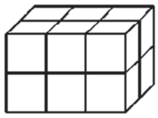

Figure. 22

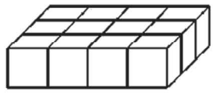

Figure. 23

The last part of the experiment in the fourth grade was completely different from the last part in the second grade. The fourth graders considered all four interpretations as completely different from the original assignment. Each of them was a totally new problem for them and they solved it.

The fourth graders formulated the assignment almost without change in original assignment.

Conclusions

It is interesting to note that 3 out of 6 pupils changed their understanding and reinterpreted their solutions. Complete change in interpretation is evident in solutions 2C and 4B. Gentle progress is in solution 2A. This is the advantage of qualitative method of research which enables us to monitor changes in strategies or interpretations of assignment.

The second year pupils did not see any differences in the four interpretations. But the fourth year pupils could read and understand mathematical text and they noticed all the differences. These results are the proof of the increased mathematical thinking of the fourth grade pupils than the second grade pupils.

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Problem stories in the education for numeracy and literacy

"Problem stories" are a teaching strategy for problem posing. This work is about the analysis, through an interdisciplinary approach, of the implementation of this teaching strategy that joins Mathematics with Portuguese Language and promotes an education for numeracy and literacy.

We present and discuss some studies stressing the concern, at an international level, like the Organisation for Economic Co-Operation and Development (OECD) showed in the Programme for International Student Assessment (PISA), on the levels of numeracy and literacy of the whole population, as well as on the development of competencies. We also present and discuss definitions central to the understanding of this teaching method, particularly of problem posing and solving, and about the construction of stories.

In the first year the data was collected in a private school, from a 5 students group (group A) of eight years old average on their 3rd year of schooling. In the second year, we collected the data in a public school where the environment is a little bit problematic and poor, the group had 4 seven years old students on their 2nd year of schooling (group B). In the third year of data collection, we analyzed the group B, in their 3rd year of school, and also another group (group C), a 4 students group of eight years old average in their 3rd year of school, experiencing this method for the first time. From the analysis, we draw some conclusions reflecting on the development of numeracy and literacy competencies.

Apparently, the implementation of this interdisciplinary method, in the classroom, allows the development of numeracy and literacy competencies.

Therefore, it is possible to conclude that problem stories promote the sharing and articulation of knowledge and the development of critical sense through the cooperative work that is fulfilled.

Introduction

There are educational concerns, worldwide, expressed in institutional and research reports, that there's a permanent need to educate for numeracy and for literacy. Those reports show that the importance given to literacy and numeracy is increasing. According to the OECD – Organisation for Economic Co-operation and Development (2003), it's necessary to improve:

Mathematical Literacy – "An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen." (p.15);

Reading Literacy – "An individual's capacity to understand, use and reflect on written texts, in order to achieve one's goals, to develop one's knowledge and potential to participate in society." (p.15);

Problem Solving Skills – “An individual’s capacity to use cognitive processes to confront and resolve real, cross-disciplinary situations where the solution path is not immediately obvious and where the literacy domains or curricular areas that might be applicable are not within a single domain of mathematics, science or reading.” (p.15).

On the other hand, Bush and Fiala (1993) proposed the creation of stories as a new way of posing problems. They tried, with future teachers of elementary schools and with pupils of the 5th and 6th grades of elementary school, to apply the task of creating a story with contextualized problems.

Connecting the two together, our research problem was: “Does the creation of Problems Stories improve the development of numeracy and literacy competencies?”

Research questions

As to a definition of problem we follow the one of Krulik and Rudnick (1993) [and Posamentier and Krulick (1998)] that defines problem as a situation that students engage on, requiring a solution and, for which, the way to the solution is not known previously. We however agree with Kantowsky who points the difficulty in defining what is a problem, since one determined situation can be a problem for an individual, in a moment, and for the same individual, in another moment, only an exercise or a known fact (Kantowsky, 1977).

Bush and Fiala (1993) defend that stories with problems allow the creation of original stories and non-routine problems, and these may contribute to the logical plot of the story.

According to Brown and Walter (1990) posing problems can help students to see a topic or a pattern in a new perspective, giving them the opportunity to comprehend deeply at the same time. The authors also consider that posing problems is intimately linked to solving problems for two different reasons. First, because it’s impossible to solve a new problem without reconstructing the task. To do it we formulate new problem(s) during the whole process of resolution trying to break the original problem. Second, because after we had solved a problem, sometimes, we don’t understand fully what we have done if we don’t generate and try to analyse a new set of problems.

Lesh and Zawojewski (2007) define problem solving as a process of interpreting a situation mathematically, the authors also refer that problem solving is about seeing situations mathematically, interpreting, describing and explaining, and not simply about executing rules, procedures or skills.

As research questions, we intended to answer:

The method used in this investigation allows pupils of the 2nd and 3rd grade to build stories with problems?

The creation of stories with problems helps to educate for numeracy?

The creation of stories with problems helps to educate for literacy?

Methodology

To develop this study, we created a specific methodology, divided in three phases, and in each one, we worked to get different goals for both areas developed, always respecting the interdisciplinary approach.

In the first year the data was collected in a private school, from a 5 students group (group A) of eight years old average on their 3rd year of schooling. In the second year, we collected the data in a public school where the environment is a little bit problematic and poor, the group had 4 seven years old students on their 2nd year of schooling (group B). In the third year of data collection, we analyzed the group B, in their 3rd year of school, and also another group (group C), a 4 students group of eight years old average in their 3rd year of school, experiencing this method for the first time. From the analysis, we draw some conclusions reflecting on the development of numeracy and literacy competencies.

In the initial phase, in three sessions, we worked the resolution of non-routine problems, the expansion of problems statements and also the macro-textual analysis of stories.

The first activity proposed was the problem story "Raspel the misfortunate". It was a story about a gnome, with a lot of bad luck, which needed to find the wish-tree to end with his misfortune. In his adventure he was always accompanied by a goat, a cabbage and a wolf. At some point of the voyage he came across a river and a problem arose, how could he cross the river to get to the tree if he only had a boat with two places. Apart from this, if they were left alone, the goat would eat the cabbage and the wolf would eat the goat. How many trips must he do to cross to the other side? We analysed the story and asked to solve the problem. In the second and third activities we posed the problems: "Riddle of St. Mathias" (When I was going to St. Mathias I met a boy with seven aunts. Each aunt had seven bags and each bag had seven cats and each cat had seven kittens. Kittens, cats, bags and aunts how many of them were going to St. Mathias?) and "Sebastian the Crab" (Sebastian the crab decided to go to the beach. He was in the sea, twenty meters off the beach. Each day, he walks four meters towards the beach. But at night, while he rests, the tide throws him back two meters. How many days will it take him to get to the beach?).

We asked them to create a story including the two problems we used.

In the development phase, in two sessions, we created a version of the Snow White Story, in which she is a participant narrator. In this phase, from the traditional story, pupils had to pose coherent problems with it.

In the final phase, in two sessions, students had to create a story and formulate coherent problems with the story. This methodology was used with group A.

With group B we introduced a few changes: first we posed another problem, "The Squirrel" (A box has nine cabbage eyes. The squirrel leaves with three eyes per day, however he takes nine days to transport all the cabbage eyes. How do you explain this fact?); besides, in their second intervention, we posed new problems: "Indians paths"(The chiefs of the Indian tribes of Sioux, Oglala, Comanche, Apache, Mescaleros and Navajos gathered for a big Pow-Wow (it's how the Indians call their meetings). In the top of the hill, they put their tents making a circle. Each tent had a path to the others. How many paths were there?) , "Jealous boyfriends" (Two couples of jealous boyfriends, John and Joana, Antony and Antonia, wanted to cross a river were there was a little boat. This boat took only two persons. The problem was that the boys were so jealous that they would not leave their girlfriends with the other boy, even if the others girlfriend was there. How could they do to cross the river?) and "The thinker" (André thought of a number. Then he multiplied it by two. After that he subtracted five. The result was thirteen. Which number was it?) and we asked again for the creation of a story including the problem we used. These last problems were used also with group C, however this group didn't have any contact with problem stories like groups A and B.

In the final phase, in five sessions, students had to create their own story and formulate coherent problems with the story, like group A had made but in just two sessions.

Data analysis

The tools we used to collected data were videos and artefacts produced by the students, direct observations and field notes.

The data collected was analyzed according to the research questions, was given attention to possible theory that could emerge as well as the new possibilities that were not contemplated in the questions, new issues.

From this analysis, we can draw some conclusions.

They felt some difficulties in the application of mathematical concepts or using mathematical competences, showing:

Absence of contact with non-routine problems – groups A and C revealed lack of contact with non-routine problems, since in their first attempt to solve the problems they referred and used arithmetic operations. The group B, maybe because they were used to solve non routine problems from the beginning of 2nd grade, they were always open-minded in the problem solving and didn't use one of the four arithmetic operations as the only strategy, referring to drawings, schemes and charts, and including them in their problem posing;

Group A – Riddle of St. Mathias:

> Guidinha: It must be a multiplication, because they are too many.
> Vilela: Each aunt had seven cats, so seven multiplied...

Group B – Sebastian the crab:

> Bernardo: It's easy, we draw a beach, a beach and...
> Interviewer: You do a beach and...
> Bernardo: And we put the meters. One, two, three,... We are going to draw a ruler.

Group C – Problem posing in the Problem Stories:

> ..."He took two weeks to travel across two roads. Per day he walked 467Km.
> How many Km's he walked in two weeks?"...

Group B – Problem posing in the Problem Stories:

> ..."After fifteen minutes, the police arrived in three cars and of each car came out
> three policemen that arrested the three mans.
> David after ponder a while, ask his colleagues:
> In how many ways can the policemen transport the bandits?"...

Difficulty in carrying out measurements – they revealed difficulties in the application to real-live situations. These difficulties compromised problem solving because even when the reasoning and the solving strategies were correct, the solution was often wrong. Pupils didn't considerer the unit, they counted the numbers, so for them, zero was one, and the one was two and so on;

Group A – Sebastian the crab (drawing a ruler and counting):

> Anita: So, 1, 2.
> Interviewer: It's that way you count the units?
> Anita: 0, 1, 2.
> Interviewer: Or you count spaces 1, 2 or you count units 0, 1, 2.

Group B – Sebastian the crab (drawing a ruler and counting):

> Daniel: I've done it!
> Interviewer: Measure it. I want to see if it has the twenty. Count it that I want to see.
> Bernardo: 1, 2...
> Interviewer: Its starts there Daniel?
> Daniel: 1, 2...
> Interviewer: You do not understand. Show me a ruler Daniel.... Look at the ruler, where does it start?
> Daniel: In 0.

Difficulties in locating relevant information and to separate it from accessory information – students in some problems weren't able to separate relevant from accessory information, which stopped them reaching the solution. This situation was more obvious in the riddle of St. Mathias where there was a lot of accessory information;

Group A – Riddle of St. Mathias (after a few attempts to solve the problem):

Vilela: I was going with Guidinha to St. Mathias.

Interviewer: Guidinha was going to St. Mathias and found Vilela. Vilela had seven aunts and, each aunt had seven bags, each bag had seven cats and each cat seven kittens. But at the end of all this who was going to St. Mathias?

Vilela: Guidinha.

Interviewer: Why do you think that it's only Guidinha?

Vilela: Because it was Guidinha who is going to St. Matias.

Difficulty in verbalizing the reasoning used – this reveals a lack of development activities. These activities must be performed after problem solving because they allow pupils to broaden reasoning skills that can be used in future situations. In all groups, students revealed initially difficulties in describing their reasoning and the strategy used to reach the solution. They appeared to consider enough writing the obtained result.

Group A – Sebastian the crab:

Interviewer: Then you have to show me in the paper how have you done it, even if you have to make a drawing.

Vilela: I was doing 2-4-6-8.

Tobias: I was making 4 minus 2.

Interviewer: He has to walk 20 meters.

Classroom teacher: How many meters he walked?

Vilela and Guidinha: 4.

Vilela: But the sea drags him back 2.

Interviewer: But how many does he walk per day?

Vilela: 2.

Guidinha: 2 and 2 are 4.

Vilela: 2 days.

Guidinha: 4 and 2 are 6.

Vilela: 3 days. Carry on.

Guidinha: with more 2, 8. And more 2, 10.

Vilela goes counting with his fingers the days that Guidinha is counting.

This methodology promoted the use of problem solving and posing strategies. Pupils used and articulated their mathematical knowledge in a more conscientious way, which took them to initiate metacognitive actions. Every time students posed a problem they had to solve it and relate it to the plot of the story. So they had to predict the consequences of the solution to the story. Pupils became aware of the various components of a problem while formulating it, which strengthened their problem solving skills. Problem posing developed their capacities of identification and articulation of knowledge. That allowed them to develop the right strategies to take decisions as well as collaborative skills.

The construction of problem stories promoted an education for numeracy by giving students a new way of interpreting, understanding, solving and posing problems. The situations from the stories developed in students a critical attitude towards text problems and allowed them to communicate, share and explore ideas and strategies of problem solving in an effective way.

The construction of problem stories allowed educating for literacy, because it made students adopt a critical reader attitude – developing, in a conscious way, text construction processes in which the information was carefully and strategically managed in order to cause in the model reader some perlocutionary effects. This way students did learn to use language in its pragmatic function; develop skills to understand, to use and to reflect on written texts to reach their objectives – these made students to ponder about the text by expressing their ideas, their experiences and their own thoughts.

Final reflection

We considerer that problem stories promote the development of numeracy and literacy competences. They demand the development of writing and interpretation skills, creative thinking and the ability of posing and solving problems. By working them in an interdisciplinary way, they develop themselves mutually and establish bridges between the disciplinary knowledge, changing positively their performance in these two different areas of knowledge.

We would like to leave a few final thoughts: first, this study allowed us to observe interesting connections that had been established between mathematics and Portuguese language, bringing up interdisciplinarity; second, the complementarities that all areas possess are a vehicle to reach new competences in the future, through an interdisciplinary approach.

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The differences in mathematical thinking of children in the in-school and out-of-school situations

The report refers to the preliminary diagnosis of the difficulties that children have while adapting their mathematical skills acquired at school in real-life situations. As the primary school students and their parents stated, they rarely find school knowledge useful in real-life situations. Simultaneously, they do not try to use their out-of school experience in math classes. The problem was also noticed while analyzing the results of the competence tests conducted annually in the sixth class of primary schools. Thus we can observe a divergence between the "school" and "life" mathematics, which results in the students' unwillingness to learn, their lack of effectiveness and problems at tests and external exams.

Introduction

I am an English teacher, I work mostly with primary school students in Wielogłowy. I am not in charge of any class, and I take care of the Students' Board, so the students often come to me to share their problems and ask their questions. Surprisingly often the question they ask is "What do we learn this for?"

This is a really good question: what practical use may the students make of all the theoretical knowledge they gain – often with difficulties – at school? Is there any connection between the school subjects and real life? I tried to look closer at the problem.

What do the core curriculum and teaching standards say about preparing students to dealing with real-life situations?

The core curriculum for the primary and lower secondary schools states clearly: "in school the students develop their ability to use the knowledge gathered, so that they can be fully prepared to work in conditions of the modern reality". And more: "teachers (…) aim at giving their students the awareness of the real-life usage of each school subject, as well as the whole education at a given stage" [1].

In every curriculum of mathematical education one of the main aims of education as such is preparing the students to dealing effectively with the real-life situations.

For example, in the introduction to the "Mathematics made to measure" curriculum the authors share a very private statement: "we want our students to rise mathematical knowledge and skills, which they will find useful in their lives. (…) We also want to use the chance given by mathematics and develop their minds." [13]

The authors of the "Aiming at sunny future" curriculum of early education go even further saying: "the fundamental aim of math education is (…) using the knowledge in practical circumstances."[6]

We can see the concern in fulfilling this aim in the choice of taught issues. Especially the texting tasks are considered to be the best way to prepare the students for the real-life challenges, in which they may need mathematical skills to succeed.

The students' and their parents' opinions and the report of the central examination board

That is what the officials and the authors say. I asked my students and their parents to tell me, if math classes knowledge may, in their opinion, be useful out of school. These are some of their answers:

– Well, she won't need that in the greenhouse… (Ms Urszula B., Paulina's mother)
– If only you knew what stupid things we've got to learn… (Robert D., year 6)
– Miss Daria, what do we learn these equations for? (Dawid K., year 6)

These opinions prove that, in the students' and parents' judgments, the primary aim of education is not even close to be fulfilled. Similarly the annual evaluation of the students' achievements shows, that the area of "practical use of gained knowledge" is the weakest point of the system. The Central Examination Board report after the exam in 2004 says: "the achievements in the area of practical use of knowledge seem to be low. The students could get 8 points and got the average of 3,95" [5].. In the next years some improvements in this area could be seen, but it is still the most difficult part of the test for the students.

Not only Polish problem

It is worth stressing that it is not only Polish primary school problem. The fact that school education is not congruent to the challenges of real life was also seen in other countries. The problem was recognized as really serious and the attempts were made to adjust school teaching to changing needs of out-of- school life. The effect of research in this area in Holland was the Realistic Math Education (RME) theory created by H. Freudenthal [9]. It is known in Poland from the beginning of the 1990s [12, 10].

But the deliberations over this matter are mainly theoretical. I did not get to any analysis or research showing the range of correlation between the school teaching and the real needs of the students. So we do not really know in which points those two trends of education meet and in which points they part. But precise assignation of those meeting and parting areas is vital, if we want to move school mathematics closer to reality and make it understandable and useful for students. This will surely cause a raise in the students' motivation and will enable them to have better knowledge of the learning process.

Over the world

For now, I got only to one research material presenting the connections of in-school and out-of-school math learning.. To be exact, it is about using a real-life materials to present mathematical issues in class. The research on that matter was held by Cincia Bonotto from Padova [2].. She tried to introduce new mathematical issues using real-life materials like bills, recipes or bus stop timetables. Her work, however, tends to concentrate more on the differences in the students' experience, which are caused by their coming from different social or culture groups.

Other materials I got at are meant to show the advantages of the RME over the more traditional methods of teaching. Such research works were held in Holland [7], the USA [11], the UK [8], China [3], Taiwan [4] and other places. Quite a lot of them were conducted as parts of the "Mathematics Education into the 21st century" project, in which Poland also took part[1]. In Holland, there was even an idea of changing the teacher training in such a way, which would prepare the teachers better to using the RME.

[1]A conference entitled *The Future of Mathematics Education* was a part of *The Mathematics Education into the 21st Century* Project. It was held in Ciechocinek in June 2004.

Where is the problem the most intense?

The divergence between the "school" and "life" mathematics can be most obviously observed in the primary school. At the same time it is the cause of our students loosing motivation to learn and their work being less effective.

The primary education is divided into two stages:

⇒ the early education, where math education is integrated with other subjects; at this stage there is almost no distance between the in-school and out-of-school learning, because of the idea of integrated teaching. It is also psychologically motivated, as for children of 7–9 years of age only the real situations can serve as a basis for education.

⇒ primary school (years 4–6), where mathematics becomes one of many subjects and gradually, because of more and more abstract issues and tasks, moves away from out-of-school reality.

As the connection between life and education is strongest in the first years of school learning, I decided to concentrate my work on the graduates from the first stage of primary school. I want to establish the touch points and the divergence points of those two streams of education.

The competence of a ten-year-old

Finishing the third year at school children are about ten years old. A child of this age is usually quite independent and the parents expect him or her to deal with quite a lot of problems like shopping, or planning nearby travels. The child's success in such situations is often related to his or her mathematical knowledge and the ability to use it properly.

We cannot forget the significant differences in the upbringing of a child in a city or town and in a village. Children in villages are expected to be much more mature.. The parents want them to take care of younger siblings, often when they are ill, to help with housework, to do everyday shopping, also in quite distant shops. Again, to succeed in such situations the child needs a lot of mathematical competence.

Examples

A lot of people in Wielogłowy have greenhouses and grow different plants in them. Most of them grow chrysanthemums. One of my students, Paulina, comes from such a family. Paulina is no school genius, for every "passable" mark (a three) she needs to work really hard, and every "good" (a four) is a holiday. From the fourth grade onwards, math has been Paulina's horror. She has never managed to do anything right, never understood anything, never could do her homework without help. But every year, from about mid- October, you can meet Paulina in the afternoon near flower shops close to cemeteries. She is surrounded with lots of pots with chrysanthemums of each and every possible kind and accompanied by her older sister. Paulina has no slightest problem with remembering prices, counting how much the customer needs to pay or with changing money. She never mixes orders, even if somebody wants to buy a couple of flowers, different kinds, colors and sizes. Her parents told me, that she is the best flower seller of all the family.

How is it possible, that a child so brilliant in out-of-school reality has such problems with learning math? Or maybe – how come that a child with such math problems is so good at real-life math?

I decided to talk to Paulina about this. She told me, that she somehow "sees" the colour and type of flowers her customers are about to buy. She calls it "premonition" – she just knows. This feeling usually occurs to be right. When it comes to remembering the prices, Paulina uses a simple trick. She chooses the most common type of chrysanthemum and writes down its price on a rectangular piece of cardboard. On other pieces of cardboard she notes down only the distinctions from that "basis".. She puts the cardboard pieces under the flower pots and as the day passes she knows the basis price by heart and checks the differences only.

But sometimes the problem is opposite. Darek, for instance, is a quiet, peaceful boy in year 5. He is not a genius either, but he manages to thrive quite well. In school. Out of school subjects and homeworks Darek gets lost immediately. As his mother says, she would never send him on a mission of shopping. Even if he managed to get to the right shop, which is highly improbable, he would probably buy everything but the items listed by his mother.

When I was talking to Darek's mother I learnt that he is very shy and doesn't like asking questions, especially to strangers. On the other hand, he has some problems with concentrating and remembering complex instructions. Probably this is the reason why the formal class language seems easier to him and clear written instructions are easier to follow.

How does it happen?

Undoubtedly, the children's learning, also math learning happens in two parallel ways:

⇒ in everyday situations, when children help their parents, observe them, follow their instructions and so gather knowledge and abilities (like measuring or changing and counting money in shops)

⇒ in school, where suitable information and abilities are improved in unnatural, abstract situation created and guided by the teacher.

Mathematics, and quite a lot of it, is present in everyday life. For example:

⇒ illness – a child has to know how to check temperature, what medicine is to be bought and where, how much medicine is to be taken and when

⇒ nearby travels, to the shop or school – a child needs to deal with the timetable and read it correctly, needs to deal with money and tickets, needs to estimate how much time the journey will take and how much money he or she can spend in the school kiosk

⇒ everyday washing – a child needs to be able to group things of similar colors, needs to know how much washing powder to use and estimate how long the washing will take.

In all the above situations, and lots of others, the child's success depends on how much math he or she knows and applies.

Research program and methods

I intend to concentrate my research on the group of 10-year-olds, who live outside cities and towns. First of all, I need to find out to what extend math education worked over in the early school education prepares the student for dealing with real-life situations. In order to do so, I need to concentrate on two tasks:

⇒ register the everyday situations in which a 10-year-old may need to use math skills; then isolate mathematical activities conditioning the child's success in those situations;

⇒ identify the issues in the curriculum which directly prepare the child to dealing with real-life problems.

I will analyze the curricula, textbooks and workbooks used by the teachers and I will list all the mathematical activities necessary for a child to succeed in everyday situations. Comparing the list to the curricula and the outcome of the analysis, I will be able to find out the extent to which school math matches the children's real needs.

At this point, a problem occurs. An analysis of the contents of education will not tell me why our students have problems with applying their knowledge in out-of-school reality.

Therefore, the second aim of my research is to explain the causes of such difficulties in applying school knowledge in real-life situations and vice versa.

In order to know this, I need to find out what mental strategies children use while facing similar tasks in school and out of it. So I shall concentrate the next stage of my research on answering three questions:

⇒ how do the children cope with real-life situations needing math skills, especially the ones who do not achieve much at the math lessons?

⇒ what strategies do they use out of school?

⇒ what strategies are used in school while dealing with texting tasks in school?

By observing children in school and by talking to them I will get to know what paths they follow dealing with texting tasks in usual lessons and tests. By talking to the parents I will learn what housework the children do, if they cope with them well and what mistakes they make. In this way I will find out what mental strategies the children use at home and at school.

Then I will compare the strategies used by children in and out of school, so I will be able to state the divergences and suggest reparation activities.

Sum-up

Considering the contents of math education, the core curriculum as well as the specific math curricula are designed to match the children's real-life needs well. We can assume that the difficulties the students have, which occur at external tests and the divergence between "school" and "life" math that they and their parents see have a different source. The problem may rise from the fact, that in everyday situations the children build their strategies on "I know how to do it", and in school they build them on "I know the method I should use" basis.

At the same time we need to consider the fact, that the author of textbooks and texting tasks, as well as math teachers, are adults. Their ways of thinking and coping with reality differ from those of children. They often pass over some content thinking it is obvious. For a child, a task with such "hidden assumptions" may be very difficult to understand.

In everyday life the children often try things out and change their ways if necessary. They try different methods of resolving problems and choose the best one. In school they are expected to follow the teacher's plan, use given methods on a given level of complexity.

While at home a child knows exactly the benefits coming from doing a task correctly and punishments for not doing so, at school it not that simple. The child's success, or lack of it, is measured by abstract marks, which may be important for teachers or parents, but do not have a practical meaning for a child.

What is more, in everyday life a child is given clear, understandable instructions. If something is not clear, a child can always ask for further guidance. At school, especially doing tests, the language of instructions is artificial, highly abstract and unclear, and there is no chance of clarifying. Coming from reality to abstract issues, although gradual, is very difficult for a young mind.

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Designing mathematics walks

There is a lot of mathematics to be found in the streets. The world around us provides many opportunities to come up with 'mathematical' tasks based on everyday situations. Realistic problems lead to authentic presentations of questions to be tackled by pupils of various levels. In maths classes in school, pupils are confronted with descriptions of everyday situations. They have read and realise what is the situation in order to use their mathematical knowledge to solve those everyday situations. These descriptions of every day situations are not needed in mathematics walks, because you meet the situation and that makes the questions more obvious. Moreover, pupils are encouraged by the discovery that maths knowledge and skills actually help them to cope with many real-life problems. In mathematics walks we aim to bring pupils into daily-life situations in which they see that you can recognise, use and discover mathematics. Maths turns out to be versatile, interesting, enjoyable and sometimes surprising. Reason enough to get out of the classroom and take a walk in the school's surroundings.

Realistic mathematics education

The most important point of the mathematics walks is to learn to recognize mathematics in the world around us. As soon as something can be seen or calculated in realistic situations, it becomes clear how mathematics can be used. In mathematics walks there is plenty of scope for the use of knowledge and skills learnt in maths lessons. These have to be applied in new situations. The pupil does not know beforehand what knowledge is useful in a particular situation; therefore this can lead to a great diversity of approach. The development of individual ways of solving problems is an important objective of realistic mathematics education, and maths walks give an opportunity to practise this skill.

During realistic mathematics education meaningful tasks are formulated in a realistic context. By examining a number of different situations, the pupil discovers what is always 'the same' in these situations. In this way an abstract concept develops from the reality and is understood. Although the mathematics walks offer a great variety of contexts, it is not sensible to develop new concepts from it (on the street). However, after a mathematics walk it is possible to go back to the experience in a lesson. It is then quite possible to use the real contexts again to develop new mathematical concepts. For instance, counting the number of people going in and out of a shop can forecast the number of visitors for this shop during one week, month or year.

Using realistic contexts

Many mathematics teachers make use of realistic contexts to make mathematical concepts as comprehensible as possible. This is sometimes best done by placing the pupils in situations they can recognise. After all real-life situations appeal more to the imagination, so that abstract concepts gain meaning and are better absorbed. Realistic contexts are then deployed as examples. There are, however, other possibilities. A context with questions in which problems can be solved by mathematical means shows how mathematics can be applied. Contexts may also be used for their value in motivation, to show that mathematical skills are useful for making decisions in daily life. In short, in today's mathematics teaching, realistic contexts are incorporated for a variety of reasons.

Presentation of contexts in the classroom

The presentation in the classroom of a realistic context is generally provided by the authors of the book. They take a great deal of trouble to make the situation as clear as useful. After all, pupils must be able to envisage the situation to be able to decide 1) what information is relevant, 2) what information is needed to answer the question, or even 3) what question you could ask in such a situation which could be solved mathematically. Obviously the teacher can use other means to make the reality appeal as vividly as possible to the imagination of the pupils in the classroom. An (imaginary) story, a photograph, drawing or video could also lead to the development of mathematical problems. In addition material can also form the context from which mathematical questions appear. For the purpose to bring a realistic situation into the classroom it have to be described and if possible supported by visual evidence. There is, however, always the drawback that the description of a situation already provides a selection of the relevant information offered by the practical situation.

Contexts during a mathematics walk

The advantage of a realistic situation in a mathematics walk is that the situation virtually presents itself. What questions can be asked? What is relevant here? Can the question be answered easily and quickly here, or is it important to go to work accurately and with precision? What are the possible ways of solving it? Such questions don't rise separately but more or less they force themselves on you. Pupils have to find relevant measurements and data for calculations for themselves. That makes the application of mathematics more authentic.

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Mathematics through investigations and problem solving

The paper target is to present an idea how mathematical thinking can be developed through investigations and problem solving and still covering traditional curricula.

Good problem solver need to be willing to explore, to be persistent and tolerant to certain amount of frustration. For these characteristics learning environment need to be safe and simulative what would be also discussed.

A picture of school mathematics

Mathematics teaching is usually didactically and pedagogically alike to teachers' personal experiences as mathematics learners. The content structure of teaching at elementary level is still deductive on certain indirect way, for example in geometry starting with concepts (point, line, segment …), continuing with properties and consequences. Teaching does not start with 'all' shapes but it starts with triangles, quadrilaterals etc. Traditional teaching is also influenced by textbooks content structure. Linear, bottom up content development seems to be very safe teaching approach. Investigations and problem solving are detached from usual maths teaching, realized only occasionally. Widely spread beliefs are that problem solving is only for gifted students, and that mathematical content is more important than teaching student to think, solve and investigate problems. Students and teachers share the conviction that problems need to be solved quickly and by that reason students need to solve before similar problems which amount depends on their ability. Solving 10 problems in one lesson is much better than one problem, what causes typical conclusion that investigations and problem solving are time consuming activities. It also happens that problems are correlated with word problems and an exercise is meant as a problem if it is complex, although with linear step by step solving procedure.

Further consideration

What we need to do that investigations and problem solving become a part of every day maths class? One of the answers is changing teachers' beliefs what is long term task as problem solving skills and successful investigative learning environment. The second answer is that mathematical university education needs to be redefined at least for prospect maths teachers.

A seminar gives to teachers only some aspects of this complex didactical situation. Seminars combined with projects are better solution, if well planned work in the class includes appropriate evaluation. It is important that teachers understand and appreciate that while investigate mathematics students still learn mathematics even though it is unlike anything that they experienced when they were at school. For a teacher, the first step to investigative approach to teaching mathematics is to feel like a learner in given starting situation while trying to understand stated problem and looking for appropriate strategy which gives the solution. For this task we need to choose a not routine problem for teachers. When teachers change their role from learners to teachers and they compare the content of the problem with curricula, they would like to find a content connection with it. If they can not find it some of them doubt that thinking processes and strategies are valuable target achievements for their students, because processes are long-term and difficult to assess. It is promising if we can find for seminars an unknown starting situation to teachers which still fit maths curricula and can be confidently used in their classes.

Planning investigations and problem solving is an important step. How to get an idea for a starting point? One possibility is to use the starting situations from the handbooks, articles and seminars. Teachers prefer to develop their own ideas what also shows their personal understanding of problem solving and investigations.

Usually problems and investigations are at the end of certain chapters – in the role of synthesis. Investigations and problems can be also starting points for developing new concepts.

Teacher's role in the mathematics classroom when students are exploring is very different to that in normal lessons. Teacher becomes a listener, a fellow researcher and a questioner. What kind of questions we can use during certain process of exportation?

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Information technology as an educational environment

Part 6

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Development of abstract thinking in mathematics in computer enviroment

This paper is summarizing ongoing research into the use of computer games as a tool for developing abstract thinking in a mathematics classroom. We will discuss the results of three educational gaming environments supplementing mathematics instruction for teenagers that were designed and student tested not only for improvement of content knowledge, but also for innovative thinking and abstraction development. The study shows that a carefully chosen digital learning environment stimulates a higher-level thinking in the majority of players.

Introduction

The commercial development of technology in the last three decades improved our everyday lives and made computerized gadgets more accessible and user friendly. As a consequence, digital games have become highly interesting to educators and researchers since their sophistication has improved considerably, not to mention that even the youngest students are familiar with computers [2, 5, 7]. We all agree that computer technology has unique capabilities for enhancing teaching and learning on every level and subject matter. Moreover, many commercial games and web sites, especially those geared toward younger kids, are already popular among parents and teachers as they address needs to master certain skills or content such as reading and counting [9, 16]. Studies indicate that computer based algorithmic repetitious activities have educational benefits and are common in American classrooms. Some games also support explorations, complexity and various contexts, including social, historical, economical and futuristic environments [8, 12, 14]. In this paper we address the need for developing games which would emphasize the development of abstract thinking [19] supporting mathematics curricula especially for students ages 12–19. We will also show three recently designed examples, went through initial testing stage as educational tools, and impacted student learning. Specific cases of games that promote higher level thinking were treated in some recent research [2, 6, 15], however due to the fact that software development is time consuming and costly, we do not have a common set of useful activities that are widely used.

The aim of the research

Over the last two years, with the cooperation of EGRIS, we designed three prototype computer games that support mathematical abstraction. The activities inspire independent thinking, creative thought through pattern recognition, categorization, modeling, strategy development, analysing and conjecturing using logical rules and previously accumulated mathematical knowledge. Our preliminary research supported by a small pilot laboratory study shows that a proper digital environment can support independent explorations and inspire higher-level thinking.

Note that our games are significantly different from games targeting language skills as humanists consider language hierarchy (of parenthesis) as their primary structure. We, on the other hand, use the time, the space and geometric and algebraic structures as basic concepts, and stress logical inferences, data analysis, mathematical modeling and proving as methodology. While the choice of the social context for the learning environment is important and makes the activities more interesting, higher level thinking skills (especially in mathematics) can be developed in context free abstract situations and be even context free.

The methodology

Since this is an early stage of our research project (especially in terms of the game development), we had 10 randomly selected students aged 12–19 play each game individually for several hours each. They were being observed, and their strategies and thinking skills evaluated by the researchers. Post survey of the activities evaluating their perception of their development skills and learning was administered. The descriptions of the three games/activities addressing different mathematical thinking skills are provided below and the results are summarized later in this paper.

1. *Ace Altro*- is a crime drama logic game, where the ultimate goal is to solve a mystery, 'Who done it?'. A player/detective can evaluate statements of witnesses that are given in everyday language describing what they know about the crime. Using symbolic logic the player translates the statements into logical sentences and evaluates the outcome against the collected evidence and the set of suspects. The complexity of the statements, suspects' diversity and evidence varies depending on the age/level of the player. Witnesses may have speaking patterns (for example giving only 'or' or 'and' statements) that can be noticed by going through different cases, so the long term success of the detective depends on the development of questioning strategy and the ability to categorize the set of suspects.

This game supplements instructions in logic: false, true statements, truth tables, logical inferences, operations on sets, categorization, conjecture making, argumentation, proofs, etc.

Example. *Introductory activity*: *Who broke the window?*

Suspects – four boys with the following characterization:
Boy A (blond hair, green shirt, black pants)
Boy B (blond hair, blue shirt, brown pants)
Boy C (black hair, white shirt, black pants)
Boy D (black hair, green shirt, green pants)

First witness: I have seen the boy who broke the window from far away. My eyes are not so good, but I am sure his shirt was not blue.

Translation into logic:
p: The shirt was blue.
True statement:
Not p

True only for boys A, C, D. So we can eliminate B from the suspect pool.

Second witness: His hair was not black or his pants were not black.

Translation into logic:
q: The hair was black.
r: The pants were black.
True statements by the second witnesses:
Not q or not r

True for boys A and D, since C has black hair and black pants.

Do you need more witnesses? Yes.

Third witness: If his hair was black, his pants were black.

q implies r

True only for boy A, even though boy D has black hair his pants are not black (his pants are green, so the implication is false).

Even in this introductory scenario the student has built the logical structures, analyzed them, and applied them to categorize and re-categorize elements of the set (of suspects) into groups depending on consecutive testimonies. As the cases become more complex the witness interviewing strategies play an important role in reaching the solution.

2. *MonsterSort* – is a combinatorial game that asks students to develop their own sorting strategies and compete with each other, or the computer character-sorting monster Gork. Several parameters are being evaluated such as random initial distribution of the numbered boxes, as well as memory requirements, number of steps, complexity, etc. Students have full access to the data, and can make their decisions to maximize the chances of success in the scientific way. This game supplements school curricula in combinatorics, probability, algorithm development, and decision-making strategies.

Simple example. Put boxes numbered 5, 4, 1, 2, 3 in increasing order.

Possible solution: move 5 to the side (store in the memory). Put 1 in the first spot, now the third spot is empty. Move 3 to the third spot. Move 5 to the last spot. Put 4 to the side, move 2 to the second spot, and move 4 from storage.

Number of steps: 7
Memory (storage) use: 1 unit

Second possible solution: move 4 and 5 to the side (store in the memory). Put 1 in the first spot, 2 in the second spot and 3 in the third spot. Move 4 and 5 to fourth and fifth spots respectively.

Number of steps: 7
Memory (storage) use: 2 units

Depending on initial distribution of numbers, the choice parameters, such as storage costs, number of steps costs, time, and different combinatorial solutions may optimize the best solution function, and the student will develop different strategies. The game may be played against the computer, that uses standard assorting algorithms such as, QuickSort (minimizes memory space usages), MargerSort (minimizes time), HipSort (minimizes complexity function) [see 11]. The higher levels of the game will lead students to algorithm design questions.

3. *Rock, Paper, Scissors* – this game is based on the well-known kids' game, where with the uniform probability of 1/3, two players pick simultaneously a rock, paper or scissors. Rock wins over scissors, paper wins over rock, and scissors win over paper. If both players pick the same object, there is a tie. The game becomes more interesting when the probabilities of each choice are independently controlled, i.e. can be changed. Then the decision-making strategy becomes and interesting issue for the players as the environment varies. The game supports probability and statistics curriculum, data analysis, decision trees, strategic choices, etc.

Example 1: Classical case: both players pick their object with a probability of 1/3. Long-term prediction is that there will be a tie, which can be supported by data collection and analysis. So this is a fair game.

Example 2: One object case: One of the players always chooses one object, say scissors, the other stays with 1/3 strategy. Who wins in the long run?

Example 3: Different probabilities: One of the players (for example the computer) chooses his probabilities. Can the second player (the student) choose his probabilities to beat the first? Explorations, data analysis and decision trees help students choose the right strategies.

Note. These games have no specified winning strategy rather they support independent thinking, making decisions based on evidence, cooperation, discussion, and social interactions. They are set in a friendly, non-judgmental and humorous setting, so students get easily engaged and enjoy playing.

The results and their analysis

All students became deeply involved in the decision-making and the role-playing aspects of activities. Our games have no specified 'right solution' and this stimulated students to freely make decisions. After the initial period of random choices, they started to use collected data, experiments and conjecture testing to improve their results. Since the social setting of the project was supporting cooperation and discussions were encouraged, all students started to analyze the evidence and make categorized decisions while building various models for successful strategies. In surveys, students positively commented on the interactive nature of the games and the learning process itself. They liked the simulations and graphics, and praised instant progress feedback that positively impacted their outcomes and thinking processes. They also cited increased self-confidence in analyzing various situations using mathematics.

These results are quite impressive, as all students showed ability to develop abstract thinking skills and have learned game specific mathematical ideas. So far, we have conducted all our experiments in the small laboratory setting and the hope is that the statistical data will be similar for classrooms, small groups and homework assignments.

Specific Comments about the Games:

Ace Altro- the logic game. Most of the students did very well at the introductory levels of the game and quickly learned symbolic logic language set-up and making inferences from the data analysis. With increased difficulty, they developed various strategies to deal with data and decision-making. Most of them organized and reorganized data sets, and tried to eliminate suspects and optimize witness-questioning strategies. Since this game is cartoon-like, and there is not much interaction, it was hard to keep the younger students focused on their tasks for longer than an hour per sitting.

MonsterSort – this combinatorial/sorting game has interesting three-dimensional graphics, and students enjoyed giving orders to and competing with the Gork character. They all developed their own sorting strategies and their own sorting algorithms trying to maximize their chances to win. Initially students found the sorting questions interesting, but after several hours they wanted more variety. To make the tasks more interesting it is our plan to design various additional combinatorial problems that lead to algorithm development.

Rock, Paper, Scissors – students were familiar with the game and used the uniform probability of 1/3 for fun and to get familiar with the environment. When the probabilities were changed they got very involved in decision-making strategies while competing with each other. They carefully analyzed datasets to improve their odds of wining. Three of them made general conjectures, and tried to test them. All showed improved understanding of statistical data analysis.

Conclusions

The preliminary results of this project are very positive as all students did data analysis and were stimulated to creative mathematical thinking on an appropriate level. Various techniques were developed and used including data categorization, modeling, strategizing for decision making, conjecturing, arguing and proving that their statements are true. Logical rules become the common communication platform amongst the players, and optimization issues become the central element of conversations. To build the connections with their previous mathematical knowledge they tried mapping and remapping structures and models they had built. We will further develop the prototypes and make them available to the larger learning community. These games are designed to support an existing pre-college curriculum and engage students in activities requiring abstract mathematical thinking.

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Teaching computer-assisted mathematics to primary students

The subject of the research report is the natural pedagogical experiment which was to show an innovative didactic situation and changes in arithmetic knowledge and skills learnt by first grade elementary students.

The research aim

The cognitive aim of the empirical diagnostic and verifying study was to estimate the effectiveness of using computers, together with intentionally and carefully chosen computer programs, in developing mathematical skills and active arithmetic knowledge construction of first grade elementary students.

The practical implementation aim was to formulate conclusions and implications for elementary teachers and primary school headmasters, useful in adding computers into the teaching aids range helpful in developing mathematical skills.

References to significant literature and publications

Constructivism is nowadays the most significant educational trend, which describes the process of school learning, its dynamics and the relationship between the activities of the teacher and the student.

According to the constructivist approach, the student should be an active and creative action subject because it is him who constructs his own knowledge. The teacher is the one who supports the student in his activities and development. The support is based on aware creation of optimum conditions encouraging the student self-action incorporated in the process of shaping attitudes, skills and constructing own knowledge.

The great role of the child's self action was stressed, among others, by J.S. Bruner [1], J. Piaget [2], L.S. Vygotsky [3]. New (constructivist) schools refer to real activity and spontaneous work resulting from the student's personal need and interest. Active education underlines the fact that students should express their will to do what they are involved in. The need and the interest that results from it is a factor that produces the reaction of authentic activity [4].

Methodology

The main research question was formulated as follows: In what way does using educational computer programs in first grade elementary education affect the level of students' arithmetic knowledge and skills? The main question was complemented by more detailed questions. One of them was as follows: In what way does computer-assisted mathematics teaching to first grade students of primary school help in developing the ability to use the knowledge in problem situations?

The global dependent variable was the level of students' knowledge and skills in the area of elementary mathematics teaching, whereas the global independent variable was the methodology of using carefully chosen educational computer programs assisting elementary mathematics teaching.

The dominant research method was the pedagogical experiment carried out with the technique of parallel groups. The experiment went according to John S. Mill's canon [5]. It took the form of the natural experiment, which means that the students did not know they were the subjects, and the research was carried out in conditions typical of first grade education.

The complementing research methods were: diagnostic soundings carried out with primary school headmasters and elementary education teachers, skills tests for students of both groups, observing students while learning, the documents analysis and the dialogue method. All the research tools were verified in pilot research.

The experiment was conducted partially in the first (pre-tests) and the second term of school year 2003/2004 in two purposely chosen schools situated in the Silesian province. The schools were nominated due to diagnostic soundings and the approval of the head and teachers to carry out the research.

The post-tests were carried out in June 2004, whereas in the second week of September 2004 distance research was done. Altogether 72 students of four first grade classes of primary school took part in the research, including 37 students of two classes making the experimental group and 35 students of the other two classes making the control group. The statistic analysis of pre-test results revealed that characteristic features of the students, both in the experimental and control group, were of not great difference, and that means the groups came from the same general population and they could participate in the research.

The pedagogical experiment was preceded by thorough analysis of curriculum material in the area of arithmetic, planned for the first grade of primary school in the second term, because the experiment was intended to be carried out exactly during that term. Afterwards, the computer programs' contents were analysed and regarded as positive by the teachers in the diagnostic soundings [6]. The educational material mentioned above also influenced the choice of the computer programs. Besides the technical and didactic quality of the programs and the possibility to use them were judged on the basis of educational computer programs criteria [7]. Then the lesson plans and the schedule of the experimental activities were prepared.

Data collection tools meant to measure the level of mathematical knowledge and skills of first grade students in the area of arithmetic were also prepared. The tools were based on standardized tests by A. Cheba and A. Andrzejewska [8], verified in the pilot research, and then used in the pre-experimental, post-experimental and distance research.

Mathematical Skills Test contained 24 exercises in the area of arithmetic. It was divided into three parts (natural numbers from 0 to 10, from 11 to 20 and from 0 to 100 – denominations of ten and the complete number range), according to the stages of numbers learning [9] and curriculum material in arithmetic planned for the first grade of primary school in the second term. In *the Mathematical Skills Test* the students could score from 0 to 2 points for each exercise, apart from exercises 10 and 20 (in which you could score from 0 to 4 points). The level of arithmetic skills achieved by students was estimated as follows: a very high level (A), a high level (B), a medium level (C), a low level (D) and a very low level (E). All students of a particular class were tested simultaneously, in conditions ensuring that every student solved the test individually. The test took 1 hour 30 min. with one ten-minute break excluded from the overall time of the test. *The Mathematical Skills Test* included the following arithmetic skills: the knowledge of natural numbers in ordinal aspect; the ability to count the missing numbers in addition and subtraction equations, the ability to apply number properties and symbols to make addition and subtraction equations i.e. the ability to make addition and subtraction equations using the numbers given and writing the equations with '+' and '–'; the ability to compare the results of addition and subtraction; the ability to compare natural numbers; the ability to add and subtract natural numbers within and outside denominations of ten; the ability to add and subtract heard natural numbers in memory; the ability to distinguish units and tens in one- and two-digit numbers.

The ability test on applying addition and subtraction in problem solving was made of five word problems. In the test the student could score from 0 to 4 points for each exercise. The levels of applying arithmetic skills in problem solving were estimated in the same way as in the natural numbers equations. The test took 45 minutes. Every student completed the exercises individually, whereas all the students of a particular class were tested simultaneously. The test aimed to estimate computer influence on the ability level of applying the arithmetic knowledge and skills in solving simple and complex word problems.

During the following classes, done according to the lesson plans, the students practised and consolidated the previously introduced arithmetic knowledge and skills meant for the first grade of primary school in the second term. However, the classes in the experimental group were done in the computer laboratory using three selected programs, i.e. *Click teaches to count in the green school* (WSiP program, 1999, recommended by the Ministry of Education), *Virtual School. Mathematics* (YDP program, Interactive Publications, recommended by the Ministry of Education) and *Mathematics. Addition and subtraction* (Aidem Media program). The control group practised and consolidated the same arithmetic material as the experimental group, but using selected textbook publications [10].

Data presentation and analysis

Figures 1–3 present the achievements of students in the control and experimental groups measured in the pre-experimental, post-experimental and distance research. Figure 1 shows that the students in the experimental group made considerable progress during the experimental classes in computer-assisted mathematical skills teaching and constructing arithmetic knowledge. In the post-experimental test 73% of the students reached a very high level of mathematical knowledge and skills (A), 14% high level (B), 8% medium level (C) and only 5% low level (D). Among the students who achieved a very high level there were those who got the maximum number of points (50).

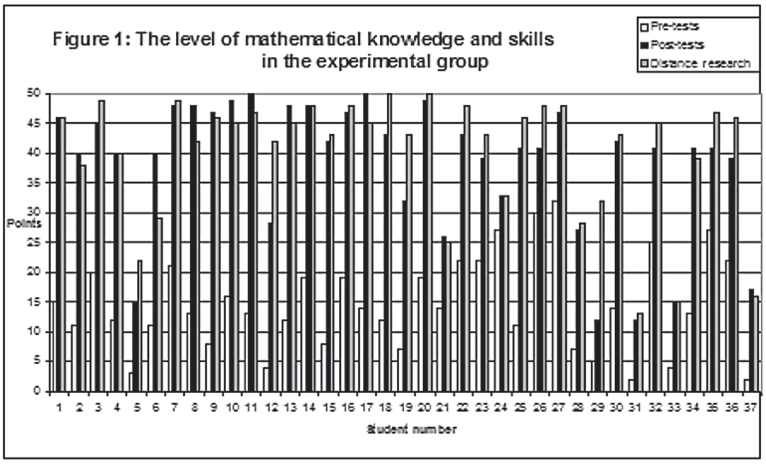


Figure 1: The level of mathematical knowledge and skills in the experimental group

Figure 2 shows increase in mathematical knowledge and skills of the students in the control group. Almost all the students in the control group showed a considerable increase in arithmetic knowledge and skills, however their results in the post-experimental test are lower than those of the students in the experimental group. Only 26% of them reached a very high level (A), 43% a high level (B), 17% medium level (C), 11% low level (D) and 3% very low level (E).

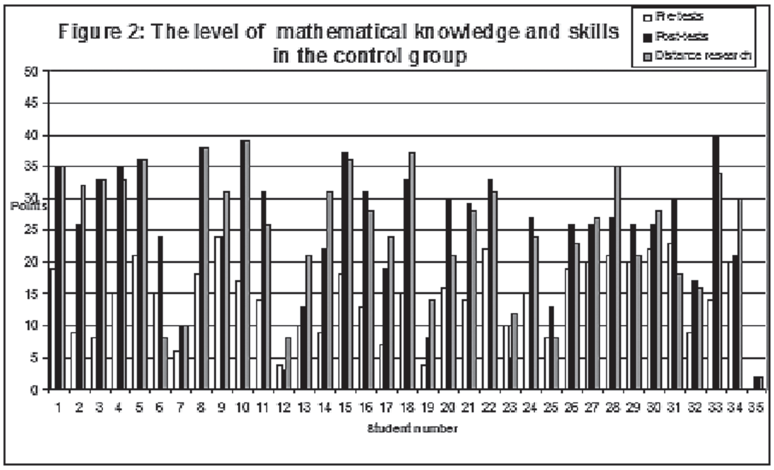
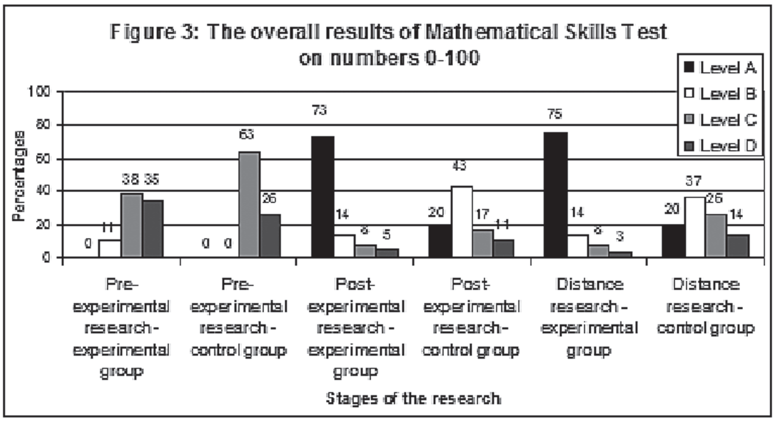
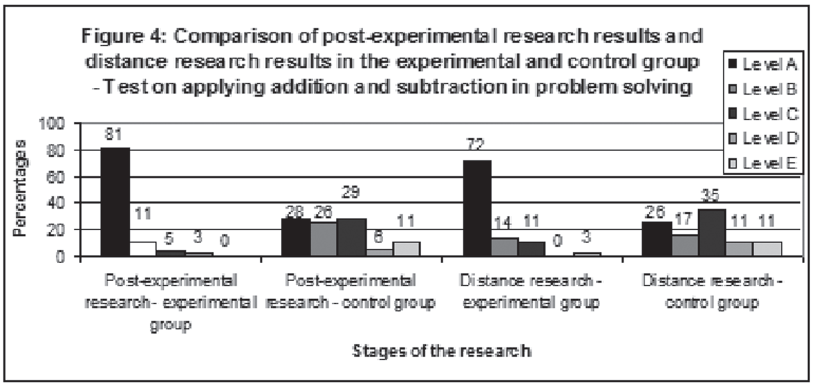


Figure 3 presents the overall results of *Mathematical Skills Test on numbers 0–100*, used in every stage of the research. The proportional data shown in the figure enable you to compare the experimental group students' achievements with the achievements of the students in the control group as far as all the research stages are concerned. The levels of arithmetic skills reached by the students were marked as follows: very high level (A), high level (B), medium (C), low (D) and very low level (E).



In order to check the difference significance between the two groups a Chi-square test was applied (χ^2) [11]. The students in both groups progressed in arithmetic skills in numbers 0–100, whereas the differences in the pre- and post-experimental research in the development were statistically significant both in the experimental and in the control group. However, the progress turned out to be greater in the experimental group than in the control group, and the difference between the groups was statistically significant. The tendency was also confirmed in the distance research. The experiment confirmed the veracity of the major hypothesis that assumed that using educational computer programs in the first grade of elementary education would result in high level of arithmetic knowledge and skills achieved by the students. The major hypothesis was estimated with the 0.99 probability; for the significance level á = 0.01.

The research also confirmed the assumption that computer assisted mathematics teaching in the area of arithmetic has a great impact on the development of the ability to use the learnt notions and arithmetic operations in solving simple and complex word problems (see figure 4). The levels of applying the arithmetic skills in problem solving are marked like in the *Mathematical Skills Test*.



The difference in the test results between the experimental and the control group in the two research stages, i.e. in the post-experimental test and distance research was statistically significant. The students learning in a traditional way achieved poorer results that the students learning with the help of computers. With the 0.99 probability (for the significance level á = 0.01) a hypothesis which assumes that there are statistically significant differences between the groups in applying addition and subtraction in problem solving was suggested.

Conclusions

The proper development of mathematical thinking demands the right organisation of the student's activity using well-chosen teaching aids and materials, among which the computer with educational software appears more and more frequently. The results of the empirical research described above allows to suggest computer-assisted mathematical skills teaching as an integral part of education in the first, second and third grade of elementary education. Well-thought, planned and deliberate using of the new interactive teaching aid that the computer constitutes promotes considerable development of mathematical skills. Besides, it has a great impact not only on the cognitive zone, but also on the emotional and motivational one.

The right course of education based on the idea of constructivism is bound to the necessity to use wide specific material. A computer program including specific material, can make the teacher's work easier, save not only time but also the energy and effort put into preparing specific material for every student in the class. Apart from that, the computer with substantively well composed educational programs and methodologically well used in the process of learning and teaching constitutes an attractive and student-friendly teaching aid [12].

An educational computer program that is well constructed and intentionally used to assist the development of elementary students' mathematical skills has a positive impact on rising interest in mathematics. It creates new possibilities to increase the level of students' engagement in achieving learning and teaching goals. Computer assisted learning becomes more attractive, draws students' attention and changes the relationship between the teacher and students in a positive way. It creates many opportunities to construct knowledge and practise students' mathematical skills.

Success in using computers in mathematics teaching also depends on the level of integrating them with teaching contents and methods, as well as with other teaching aids and materials. It is necessary to use them according to a plan, deliberately and systematically in every aspect of education in order to achieve a positive result in mathematics teaching. That is why the role of the teacher is to combine the traditional way of instruction with computer assisted teaching. On the other hand, the role of textbook authors is to construct educational computer programs which would complement elementary education textbooks, and to include computers and educational software in textbooks by annotations referring to computer assisted learning and teaching.

Primary school head teachers, helped by parents, elementary education teachers and local authorities should take action to provide schools not only with modern computers but also the best educational software. They should open new computer classrooms or adapt the existing ones for computer assisted elementary education and compensation classes.

Elementary teachers should be encouraged to fully exploit the educational value of the computer, and broadly speaking of information technology. It seems necessary to organise methodology workshops in which they would be familiarised with the available educational software, useful in assisting the introduced and/or practised mathematical knowledge and skills, and with computer assisted teaching methodology.

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Procedural and paving way of building-up of geometrical object in concepts and strategies of primary school pupils

This presentation concerns geometry perceptions of primary school pupils in computer applications involving turtle graphics. The main goal was to analyze which strategies children use by discovering which part of a given shape they are capable of drawing by applying an algorithm built from commands of several given types. The application called Obkreslovačka (Tracer) has been created for assessment in schools. Pupils' strategies and procedures were analyzed, with particular attention to how they divide a figure into parts within their turtle procedures, and how strategies and procedures depended on the specific geometrical shapes involved. This presentation reports on a set of tests of about 11-years-old pupils in 28 South Bohemian schools.

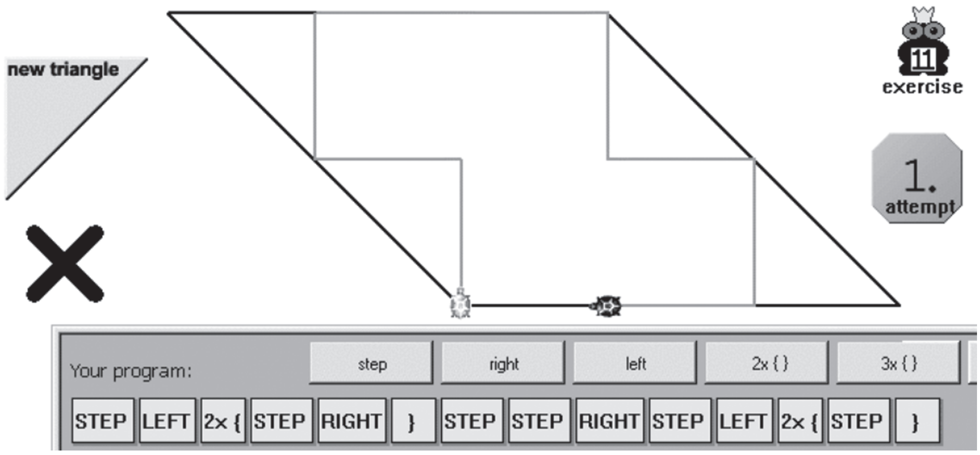

Figure 1. The environment of the testing application of Tracer. The turtle is creating the procedural part of the given figure according to the programme

Introduction

Turtle graphics contributes opportunities for modelling and drawing to the teaching and learning of geometry, both for pupil activities and for the expansion of the teacher's repertoire of tasks. This leads to the more precise pupil conceptions. Our concern is how primary pupils are able to divide geometrical figure in two parts: a 'procedural parts' inscribed into the given figure and created through a procedural process (in terms of turtle graphics commands); and a part paved with given basic tile objects (typically a remainder of the given figure). In what ways do pupils discover those parts of a figure that can be drawn with the aid of turtle procedures, and with which methods do pupils discover this shape in each specific task? What strategies do pupils employ to solve such tasks?

We have started from the requirements of Czech and Slovak experts who declare that geometry teaching in primary school should be based on the pupils' understanding which is developed through various problem-solving experiences. For example, František Kuřina says: "Geometry in primary school has not even a preparatory deductive character, but it is markedly operational." (Hejný and Kuřina, 2001) Hejný argues similarly: "Experimentation is a basic and irreplaceable way to obtain geometrical experience. Pupils in primary schools have a natural ambition to investigate the world by their own activity" (Hejný et al., 2004). Kuřina worked out a theoretical analysis of primary pupil geometrical experience and suggested four didactical principles that should form the didactical structure for geometry teaching in primary schools (Kuřina, 2001). The presence of the following four principles is evident in the described investigative project.

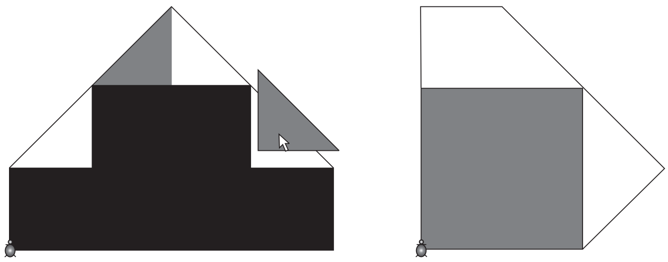

Figure 2. Procedural part and the process of paving the figure (left), wrong solution of the procedural part – it is impossible to pave the remainder (right)

- **Space Partition** is applied when introducing the new notions into geometry (e.g., a line divides a plane into two parts). Pupils meet the space partition principle as early as babyhood. The space partition principle is included in that part of the project where a given figure is divided into parts according to the way a pupil chooses to create it, i.e. into the part that can be modelled by the use of the set of turtle graphics commands, and into the part that can be paved by predetermined basic tile objects.
- **Space Filling.** The fact that the figure contains other points of the space in addition to its sides is emphasized by the space filling principle. The areas of geometrical figures can be measured by filling the space. Calculations traditionally form a part of geometry teaching, but it is also important to teach the pupils to observe and to guess. Although the testing tasks require a pupil to trace a given figure, the pupil fills the shape with objects created by procedure or paves the shape with tile objects.
- **Constructions** mean not only solving the geometrical tasks by means of drawing but also various ways of representing geometrical figures (for example, constructing a building from two, three, or four bricks of the children's building set). Modelling the figure through writing the programme brings the possibility of a deeper understanding of geometrical figure attributes. Above all, the use of a cycle within the programme leads to the idea of symmetry and its discovery in both shapes of parts of objects and in the shapes of their perimeter lines.
- **Movement in the space.** The idea of the direct movement and experience with direct movement is typically well-developed by a child of six years. It can be observed during the tasks in which we search the traces on the plan. The movement in the space can be interpreted as the imagination of turtle movement when tracing the geometrical figure. Here the procedural construction of a figure means the path created with turtle commands (Fig 2).

We distinguish the procedural part of the figure from the algorithm for tracing it. The procedural part depends on the set of used drawing commands, but it has only one shape for given set of commands. There are several ways how to trace procedural part and several ways how to describe the chosen trace by the turtle programme.

Main research objectives

Aspects of the process of conceptual understanding of polygons

- Does the use of the computer and turtle graphics affect conceptual understanding of polygons?
- Understanding of the polygon as a part of the plane (it is not an outline demarcating it, but it includes also points within he plane itself)
- Propaedeutics of the symmetry conception as one of the polygon attributes

Geometrical shape from a dynamical point of view

- How a procedural part of a given polygon will be found (i.e. the ability to find an inscribed part of the given figure, which pupils are able to model by means of given set of motion commands)?
- Which shapes seem to be simple, which difficult to children, and why?
- Which strategies do children typically employ in discovering a procedurally created shape?

The methodology

An application called *Obkreslovačka* (*Tracer*) for researching pupils in schools has been created in the Imagine Logo environment (Blaho and Kalaš, 2001). The commands of turtle painting have been limited to only STEP (go ahead a constant distance), LEFT and RIGHT (turn 90 degrees), because these primary school pupils are not yet familiar with the concept of angle. Some steering commands were added for creating a cycle: REPEAT, and brackets for defining lists of repeated commands. So the limited commands only allowed tracing the rectangular shapes with vertices which lay on a base grid and with horizontal or vertical lines. When the turtle finished the drawing at the start point, the *Tracer* filled up the drawn shape (compare Figures 1 and 2).

Assigned tasks challenged pupils to trace a given geometrical shape in the *Tracer* environment. That meant to inscribe the given figure with a maximal connected polygon of a suitable kind created by a “turtle programme” procedure. Some given figures contained skewed line segments and it was impossible to fill up the whole shape using only allowed drawing commands, so the user could add some tile objects to the figure to fill gaps in the given figure. All these tiles had the shape of right isosceles triangle and the same size. They could be dragged and rotated in steps of 90° by right click (Figure 2). The pupils were prevented from paving the whole shape with these tiles (it was impossible to fill up a basic square of the grid only with tiles). The turtle executed the programme on request of the user.

The testing tasks were divided into four groups of varying difficulty, based on the given objects and the rules for tracing:

- The lowest level included only figures of simple shape made of base grid squares, e. g. rectangles or shapes of L letter of different size and different placing of start point. This level served mainly as a means of familiarization with the application environment and the type of testing tasks.
- The shape used in the second level involved horizontal, vertical and skewed sides with a slant angle of 45 degrees (Figure 3), so that it was necessary to add tile objects to the procedural part in order to fill in the whole shape.

- The third level used very similar shapes as the second level, but required different working processes. Children had to complete the programme first and after confirming their programme they were not allowed to edit the programme during paving. This level was pivotal and the most important for the research because children were forced to imagine the procedural part without the initial help of manipulation with tiles.

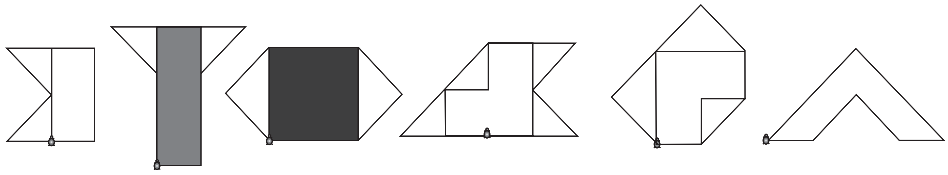

Figure 3. Shapes used in second and third level with procedural parts marked

- The fourth level was the most challenging. The shapes were more difficult to construct using the turtle programme for description of their procedural part. The pupils had to use more complicated structures, i. e., nesting cycles, to fit their programme into the command line.

After finishing each task level, some data usable to later analysis was saved so that the researcher could see the same situation on the screen as the pupil whenever the pupil asked the turtle to depict the programme. This enabled us to find common mistakes made by children as well as their strategies for tracing the given figure.

Course of the testing

The duration of the testing 90 minutes must not be exceeded. The test consisted of an introductory training (20 minutes) and four levels of the game-like activity. At each level, the tested pupil had to pass six tasks to fill a given figure. The main purpose of the training was to acquaint pupils with the environment of the *Tracer* application and to introduce them to turtle graphics and geometry through a simple way of using experiments and manipulations.

After a pre-test to confirm the testing application in school conditions, the organization of the test, and to verify that the data obtained within the testing would be sufficient, about 40 testers were prepared and the test was realized in 28 South Bohemian schools with 212 children. None of the tested pupils had any prior experience with Logo, and most had never used computers in school.

Outcomes of the testing

Programme creation and editing

Some pupils verify their programme very often, especially in the initial level. After each change, they verified the programme by letting the turtle draw an outline. Some of them evoked the feeling that they were randomly choosing any new command for the end of the programme, verifying it immediately, and then erasing it or leaving it in the programme. A typical example was turning the turtle in a concave angle of rectangle (Figure 6).

It appeared that pupils were not initially well-informed about their programs because they corrected longer parts more often than it was essentially necessary. Another strategy was to delete the programme and to begin creating it again from the beginning.

Strategy of polygon creation

While observing them we found that those pupils who seldom verified their programme by the drawing the outline helped themselves with the illustration of an imagined turtle motion with their finger on the monitor. These manners were less frequent at higher levels.

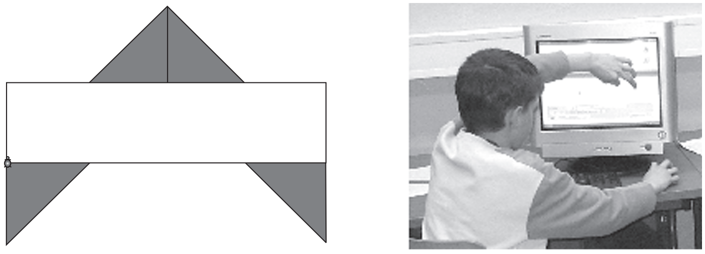

Figure 4. A strategy of paving tile objects first to get the shape for turtle painting (left). Finger helping when the turtle is crawling downwards (right).

Observing the pupils in the main test resulted in the conclusion that pupils moved after approximately two experiments to the strategy of first paving a figure with tile objects (some pupils immediately from the first task, one group at the last one). In this way, they used the computer for drawing a rectangle as the remainder of a figure after paving with tile objects, and for the ensued rest they wrote a programme to be drawn by the turtle.

Some children when unsuccessful tried the opposite direction of a turtle walk through a figure (clockwise/anticlockwise).

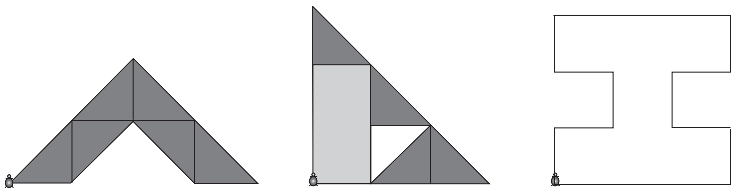

Figure 5. Interesting shapes. From the left: task No. 12, 15, 19.

Overall, pupils have greater difficulties writing a programme than discovering the division of a figure into a procedural and a tile object part. Generally, children have no problems in the third level when they first have to finish the programme and then to pave the figure (No. 13–18 at the Table 1).

We could see many times that tile objects were placed correctly, but that the pupils were not able to write a programme to fill the remaining rectangle. It appears that a longer time period within which the pupils would handle the turtle and write a programme should transpire before allowing the pupils to solve tasks of the main test. This can be supported with a longer and more goal-directed training, and a longer first level of testing.

The statistical evaluation showed us several interesting shapes for analysis (Tables 1, 2 and Figure 5). A very small number of requests for executing the programme before checking at the task No. 12 doesn't correspond with the poor success rate on this task. The shape used in this task might explain this discrepancy; it has no procedural part. Children were probably confused by this, giving up very quickly. – The task No. 15 is the first one with non-trivial procedural part at third level. When children don't imagine the procedural part correctly, they aren't allowed to correct it after checking (see Figure 5). A combination of these two used conditions makes this task difficult which corresponds with a low rate of correct answers. – Maximum of requests for executing is at the task No. 19, which is the first shape in the fourth level, with complicated shapes difficultly described by the turtle commands. – Decreesing number of chidren solving level No. 4 could be explained by lack of time.

Task level	1						2					
Task number	1	2	3	4	5	6	7	8	9	10	11	12
Number of answers	212	212	212	212	212	212	208	208	208	208	208	208
Median of request	5	4,5	5,5	5	4	4	3,5	2	2	4,5	2,5	0
Rate of correct answers	98	97	97	95	93	91	96	95	95	90	88	50

Table 1. Task levels No. 1, 2. A number of answers for each task, a median of a number of requests for turtle drawing before checking and a percentual rate of correct answers.

Task level	3						4					
Task number	13	14	15	16	17	18	19	20	21	22	23	24
Number of answers	184	184	184	183	184	184	128	122	90	76	64	58
Median of request	3	2	4	2	2	2	8	4	4	5	3	5
Rate of correct answers	85	91	52	83	93	58	25	28	42	29	45	26

Table 2. Task levels No. 3, 4. A number of answers, a median of a number of requests for turtle drawing and a percentual rate of correct answers.

Cycle and symmetry

As described by Tržilová, a participant of the research (Tržilová, 2007), only a very small number of pupils did not use the cycle at all when writing a programme. These others used the cycle only for repeating one command (**Repeat** 3x [**Step**]). Children who inserted more commands into a cycle (for example, … **Repeat** 2x [**Step Left Step Right**] …) could find a repeating part of a figure or the repeating sequences of programme.

Some pupils discovered the connection between the central symmetry and the programme which was all written as a cycle (for example **Repeat** 2x [**Step Step Left Step Left**]). These pupils tried to discover symmetry in further pictures, and to use a cycle as a general approach to writing a programme for such kind of picture.

Conclusion

Some findings for further investigation: how do children find a way to trace a given shape, and how do they develop an understanding of the turtle movement and its description? Several general strategies can be described.

- It is easier to fill in the shape with tiles than to describe it by a set of movement commands – it may be because the pupils have less experience with this type of expression.
- When children were not successful in tracing clockwise, they often started from the beginning in the opposite direction.

Pupils begin to use a cycle for notation of programme only when they are forced to shorten their programme.

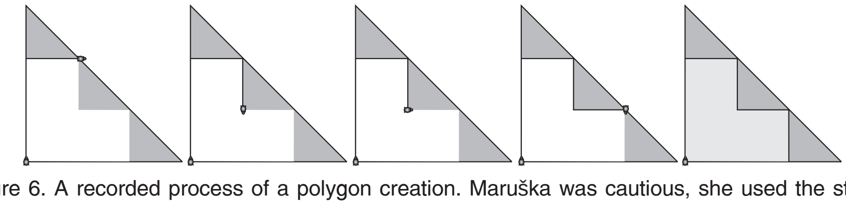

Figure 6. A recorded process of a polygon creation. Maruška was cautious, she used the strategy "paving first" and checked her programme after any change.

Some pupils discovered the connection between central symmetry and the use of a cycle within the writing of a programme. There is a question whether children discovered the procedural part as an inscribed rectangular part of the shape. A new version of the *Tracer* application with a new set of tests has been prepared in which commands for only 60-degree-turns of the turtle are implemented. We hope that it allows us to manage an assessment of non-rectangular procedural parts of a shape and to decide whether children of this age can use not only right angle, whether a way to the concept of angle could go from right angle through familiarization with a concrete non-right angle.

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Promoting thinking through a modern approach for problem solving

The paper introduces the participants to a unique way of problem solving involving algorithmic thinking in the mathematics classroom. Writing different algorithms to a given problem enhances a variety of thinking types one of which is algorithmic thinking and evokes several solving problem strategies. Writing the algorithms serves as a trigger for mathematical exploration, supports number sense ability and intensifies independent thinking of students.

Constructivists believe that the learner must be active, constructing his own knowledge, integrating new information into his schema thus promoting new thinking abilities. Thus, not only ready-made software is to be used for solving mathematical problems but, that the algorithmic computational programs leading to the solution should be prepared by the students (Breuer & Zwas, 1993). The students write Excel programs for their implementations and then run them on the computer.

I. Polya (1957) describes four stages for problem solving:

understanding the problem	$\longleftrightarrow$	devising a plan	$\longleftrightarrow$	carrying out a plan	$\longleftrightarrow$	looking back

We adapted this model throughout our course for pre-service teachers in the following way enhancing different thinking types as seen below:

problem	$\longleftrightarrow$	algorithm	$\longleftrightarrow$	computer program	$\longleftrightarrow$	solution analysis
	intuitive thinking analytic thinking independent thinking algorithmic thinking heuristic thinking recursive thinking				intuitive thinking critical thinking reflective thinking	

It is possible to show how this model is carried out throughout a technology based course which deals with solving algorithmic problems using computers. We will discuss the various types of thinking mentioned above in connection with the four stages of the model. We can focus on problems for pre-service teachers for the elementary school. (For pre service teachers who are intended to teach at the middle school or high school we developed two other courses making suitable adjustments).

This course was developed as a result of our efforts to integrate computers into mathematics classes in teacher training colleges, and as a result, to promote teaching algorithmic problem solving as early as elementary school when they become practicing teachers. Former studies (Hoffmann & Klein, 2007) showed that the subject can be adjusted for elementary school pupils. Most pupils were able to write an adequate algorithm to a given problem and to 'run' it on a computer using an Excel spreadsheet.

We elaborate on the four parts of our adaptation to Polya's model:

The problems and the algorithms-

The problems were chosen either from the materials taught in the elementary-school (pre algebra stage) or higher level problems targeted for the pre-service teachers. The students learn to write (and read) various kinds of algorithms for infinite sequences followed by algorithms for finite sequences taken from Discrete Algorithmic Mathematics, such as: The triangular numbers, the square numbers, factorial numbers, Fibonacci numbers, the golden section, division with remainder, Ancient Egyptian multiplication, or Euclid's algorithm for finding the Least Common Denominator of two given numbers.

In the advanced courses other problems are introduced where students need to develop numerical methods to deal with new and vital subjects, ordinarily absent from the regular school programs, such as: approximate solutions and calculating to a desired accuracy, computing the square and cubic roots for a positive number, or various numerical methods for computing the digits of Pi.

The computer program and the solution-

After writing the algorithms students translate them to computer language using Excel spreadsheets. While using computers student expand their knowledge in Computer Science and are exposed to computational errors which lead to mathematical contradictions. They deal with the influence of a small deviation of the input on the output, rounding errors, rate of convergence etc.

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Brief texts on independent thinking in mathematical education

Part 7

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From a caterpillar to a butterfly: a learning project for kindergarten

Geometry takes origin from motor, visual and tactile experience (Enriques, 1901). The geometrical concepts are generated from spatial investigation, observation and manipulation of tangible objects. Using photos and other materials, the poster presents a learning project, prepared with the aim to introduce kindergarten's children to geometrical concepts and not only. At the beginning, a simple tale, named "The tale of Pelù-pillar" about a furry caterpillar is employed. The tale is wrote in a particular book[1], "illustrated" with pipe-cleaners glued on the pages. In this way the reader can distinguish by touch different shapes, that correspond to different positions, described on the book, assumed from the caterpillar: segment lines, curves, circles and so on. After the reading, the teacher propose a linguistic activity: use appropriated expressions with the aim to identify the different positions of the grub. Afterwards pupils are asked for reproduce with their body the positions of the caterpillar. This implies a dramatisation activity and a planning of behavioural patterns. From the mathematical point of view, the transformations of Pelů-pillar (concerning the position in the page or its length or its shape) suggest the idea of geometrical transformation, an important concept of modern geometry. Children can explore the concepts of deformation and closed or open line (topology), of rotation (isometry), of "horizontal" and "vertical" line, of congruence, and so on. Moreover the positions like "C" or "U" suggest a link with alphabetical letters. The path continues with an iconic activity: the pupils draw the different shapes of caterpillar, copying them from the book. In this way each child have a pile of papers for play. The teacher propose various kind of game: 'found the right paper', composition of sequences, game of 'straight' or 'curved', game about left or right, on or under.

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[1]The book is conceived from E. Alberici (Association of Social Promotion "Contatto" di Traversetolo (PR–Italy).

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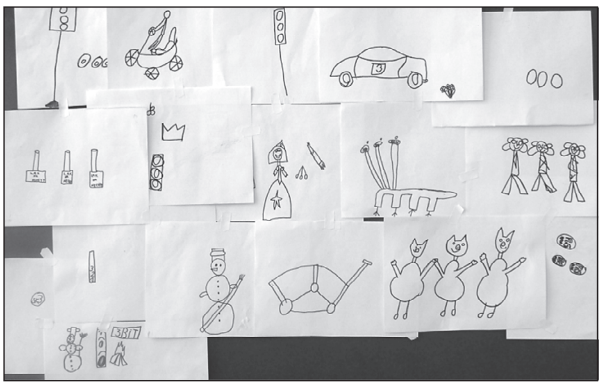
Children´s models of number 3 at the age of 6–7

This paper is result of one experiment which we focused on the 6 and 7 years old children attending the first class at the elementary school. We explored the level of their number conceptions and the ability to use them. For the realization of the experiment we chose number 3, because it is the number from the numerical scope 1–10 and as it is freely distributed in fairy tales, advertisements and other areas of ordinary life.

The experiment was realized in January 2008 with the group of 17 children in the 1.B class at the primary school in Nábrežie Mládeže Street in Nitra. Children were given blank papers and were asked to draw what they imagine, when we say number three. **We pursued the ability of children to assign the specific model to a given number.** All the pictures were fixed to the board. Children were asked to arrange the pictures into groups according to some similarity and to explain their choice. Thus the concept maps of children concerning their images of number three were created. Children created the fairytale with number 3 from their models.

Ternary Fairy Tale

Princess Cindy III. *(the Third) lives on the* ***third floor*** *of her castle. She likes to wear her golden* ***triangular crown*** *on her head, and most of all she likes to drive her* ***car with number 3****. One day, she took a* ***three-coin*** *from the piggy bank and went to buy her favorite chocolate bar* ***3BIT****. On the way back, she had to stop at the* ***traffic lights*** *and there she met her friends –* ***three pigs****. They arranged to build a snowman together. They took a* ***three-wheel barrow*** *and carried* ***three snowballs*** *of the snowman in it. Behind the* ***three mountains****, there lived one bad* ***triple-headed dragon*** *that suddenly flew in and kidnapped* ***Cindy III****. Frightened* ***three pigs*** *ran for the help to* ***three Spidermen****. They did not hesitate, took their* ***three magic stones*** *and flew to save Princess* ***Cindy III****. They slayed each of the* ***three dragon's heads*** *with one of the stones, and flew away with happy Princess* ***Cindy III****. There were already the* ***three pigs*** *waiting for them. After that, they played "PoGs" all together. Te winner obtained* ***three golden "PoGs"****. On the* ***third tower*** *there hung the* ***three bells****, and the fairy tale came to its end.*



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On the numerical competence of six-year-old children

The investigations covered by the poster presentation focus on the numerical competence of six-year-old children, one year prior to their undertaking compulsory education at school. The aim of the research was finding answers to the following questions:

- What do the children who have not begun systematic education at school yet associate the word 'count' with?
- To what extent they have mastered the sequence of initial natural numbers?
- If and how they use this skill to determine potency of a set, to create a set equipotent with another one, to compare potency of two sets, and to divide a set into two equal subsets.
- What difficulties do the children face when using counting competence skills?

The research was conducted with the method of individual interviewing in December 2007. It was a repetition of the research conducted in September 1983. It allowed for comparing numerical competence of six-year-old children in the two periods.

The results of the research conducted in the two rounds (1983 and 2007) are similar and they show quite a wide, although rather differentiated, numerical competence of the investigated children:

- When answering the question "Can you count?" all the children reel off successive numerals at least up to ten, almost half of them up to one hundred, and some of them have consciousness that "this counting will never end";
- A majority of the children can correctly use a known sequence of numerals to count particular things;
- Normally, children are critical and correct in evaluating their own arithmetic skills, and often (apart or instead of the procedure of counting, they apply non-numerical procedures.

Numerical competence of six-year-old children is differentiated: in many of them it includes some knowledge and skills comprised in the syllabus of the first class (and sometimes higher), but also in many, in certain situations, it reveals immaturity for learning mathematics in a "school manner".

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On mathematical education of 6 year old children in Ukraine

Preliminary courses (children of age before 6) in Ukraine are devoted mostly to the notions “more-less”, “higher-lower”, “left-right”, counting up to 10, solving simple problems concerning addition and subtraction up to 10 like: “3 apples + 2 apples = 5 apples”, “3 apples – 2 apples = 1 apple”.

Words “good”, ‘very good” etc are used instead of marks.

There is no special program for this period. These courses are not obligatory in Ukraine and there is no data what number of children enter the courses (this number depends on regions).

The first form (class, 6 year old children): there are two basic subjects, namely, mathematics and language (reading).

The program of mathematics for the first form pupils includes:

- numbers from 1 to 10;
- properties of objects;
- geometric figures;
- quantities;
- addition and subtraction of numbers from 1 to 10;
- numbers 11–20. Quantities.

At the end of the first form, the pupils should count to 20, add and subtract up to 10, know the terms for the components of the addition and subtraction, geometric figures (point, curve, line), polygon, triangle, rectangle etc., the length, mass and volume units, their notations.

There are two basic textbooks for mathematics in the first form.

Because of importance of mathematical education, the number of teaching hours for mathematics increased 1 hour per week during the academic year 2006/07. This allowed for distributing the teaching time in a more rational way. In particular, the topic at the beginning of the first form consists of the material which has been already studied. This will ensure more effective understanding of the material.

The increasing of number of teaching hours allows for wider using game and practical teaching methods, which, in turn, will simplify the process of adaptation of pupils to school. The teacher will have a possibility to pay more attention to the level of mathematical knowledge of pupils and ensure the individualization of the process of studying. The teaching time can therefore be varied in accordance to the level of pupils.

The lesson of mathematics in the first form is usually combined and consists of several parts. Each of these parts has its own logic of studying and methodology of teaching. Such a structure of lessons is relatively new and is a subject of pedagogical investigations. A small number of parts allows for avoiding chaos in problem solving and for deep considerations of program tasks. In addition, thematic lessons are now more and more often used in pedagogical practices.

One of the main aims in studying mathematics by 6 year old children is forming of knowledge how to solve mathematical problems. The twofold essence of problems consists in the fact that they are a pert of the program in mathematics and, on the other side, that they are a didactic tool for studying of mathematics.

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Percentages are not another name for fractions

One of the commonly used mathematical subjects in every day life, and in sciences, is percentage. Therefore, it is our duty, as educators, to take care that our students will possess a well-founded understanding of this subject. Unfortunately, we found that not the case (Sheffet & Bassan, 2004).

There are two main misconceptions held by students.

The first is: Percentages are another name for fractions.

Percent is one-hundredth. Percentages describe part of a quantity and are not numbers such as fractions. Fractions have many functions, only one of which is describing part of a quantity. A percentage can be replaced by a fraction and vice versa, but only when the fraction describes part of a quantity.

When a person believes that fractions and percentages are the same, he may use percentages in the same manner he uses fractions. For example: It is true that if one number is bigger then the other by 1/2 than the other one is smaller then the first one by the some 1/2. Following this knowledge, this person may believes that if one number is bigger then the other by 50% than the other one is smaller then the first one by the some 50%. But the last one is false.

The second misconception is: In percentage problems, 100 is always the denominator.

Elementary percentage problems are of three principal types: 1. Calculating the value of the percentage; 2. Calculating the percentage; 3. Calculating the fundamental size (van-Dijk et. al. 2003). Only at first type problems the 100 is the denominator. At the other two types 100 is the nominator. Still, students use to hold the idea that 100 is the denominator. Whenever they meet an equation with 100 as nominator they believe it is a false answer.

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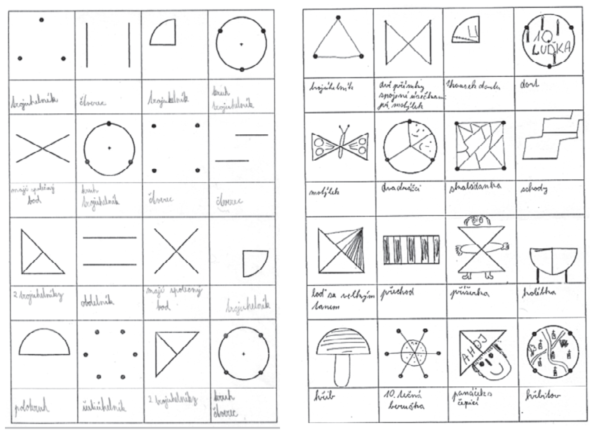
The method of incomplete pictures and pupils´ ideas

Language allows us to see the cognitive process of children (Gray, Pitta, 1996).

Method of incomplete pictures is based on this idea
The pupils (age 10–11) were tested separately without a time limit. Instruction:
Write what you see in the pictures. If you need to finish a picture, you can do that.



A B

The pupils see each of the incomplete pictures as properties of objects: *real* and *imaginative* (the outside world and experiences of a child), *mathematical* – geometric and non-geometric, *symbolical*.

The answers show – first, the attitude of children to mathematics and its teaching process – secondly, the ways how to motivate them to learn mathematics. Real or imaginary objects prevail with some pupils (see sample B) in the other cases mathematical objects are more frequent (A). The link between mathematical images on one side, and the real world and the children's inner perception of the world on the other side, make for the forming of a correct mathematical concept and its true position in the cognitive structure. Resulting from a carefully chosen set of incomplete pictures this method can be used to diagnose a pupils´ idea of a particular concept. This is the reason why the method deserves further development.

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From research on understanding a letter symbol by students of a junior high school

The research deals with understanding a letter symbol by students of the second grade of a junior high school. Its aim was to examine how a sample group of 25 students understands certain issues related to letter symbolism and what difficulties they have in this respect. The goal was also to determine hypothetical causes of observed difficulties.

As a research tool a set of adequately modified tasks which come from a teaching proposal of a functional approach to a letter symbol (Kusion, 1993) was applied. The tasks regarded the linear function and, in the context of this subject matter, they were intended to provide the possibility of recognizing the understanding of the following issues related to letter symbolism by the students:

– Different letters can denote the same number;
– Symbol x does not have to necessarily denote a number from a set of real numbers;
– The formula $y = \ldots$ does not always have to applied to describe a constant function; depending on the denotations used, $t = 5$ can be an example of a correct notation;
– A variable can be substituted by not only a specific number, but also any other variable.

An analysis of the solutions provided by the students, followed by individual interviews with the subjects, suggests certain conclusions (Michalska, 2007). Among others, it indicates student's attachment to the letter symbolism as used during their classes, a mechanical use of a symbol and difficulties in understanding the conventionality of symbols.

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Language structures of pupils within problem posing and problem solving

Language structures are understood as a complex of semiotic systems of representation and rules which guide their construction, interpretation and application. Existing results of investigations of pupils' representations and their transformations in the teaching of mathematics motivate us to look into semiotic systems of representation and rules which guide their construction, interpretation and application, not individually but in a complex way as a language structure. At the same time it is necessary to consider the context which the language structures are grounded in. The quality of language structures is the key factor of mathematical literacy. On the basis of representatives and relationships between them is one of the ways to diagnose pupils' understanding. The pupils' representatives characterise their semiotic systems of representation of mathematical objects. At the same time, they provide some information about the mental representation (images) of these objects.

A typical aspect of the teaching of mathematics is problem solving. Researchers in mathematics education, however, also study problem posing which together with their solution represents an important diagnostic tool (Silver and Cai, 1996). Based on the problem which was created by the pupil, we can infer evidence about his/her understanding of the mathematical concepts and relations in question. Problem posing is connected to the transformation of the given problem into different systems of representation (Leung, 1997), therefore, it is a suitable didactic situation for investigating language structures.

Acknowledgement

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Some geometric problems which help to overcome a pupil's tendency to use formulas

The paper introduces such tasks that would help pupils apply geometric skills rather than algebraic ones. In particular, the tasks are constructed to be easily solved by using a geometric view. Applying algebra can lead to a more sophisticated level of mathematics or may not result in a solution at all. The majority of both pupils and teachers is more prone to use algebraic tools that can be learnt by heart or looked up in the tables. That is why we feel that it would be beneficial to point out such problems that would develop teachers' and pupils' ability to see in geometry. Three problems will be presented.

The first task is called "Garden Fence". Pupils are asked to "straighten" a bent line that represents a fence between two gardens without changing their areas. Since no parameters are provided, it is not possible to apply any simple algebraic procedure. Pupils are encouraged to show their knowledge of triangles and resolve the problem using a geometric insight.

The problem number two, "Path Across a Field", depicts a rectangular field across which pupils are to design a pathway. Two possible paths are available and pupils need to choose the one with a smaller area. Again, the task cannot be solved by algebra because all the data but the width of the path are unknown. The task is based on the knowledge of the congruence of triangles. In the end, pupils should be able to conclude that both pathways are identical in their areas.

The last problem deals with a relationship between line segments within a circle. Even though the radius of a circle is given, the algebraic solution is once again not available. On the other hand, a geometric answer is very clear once a pupil determines a rectangle and its diagonals in the circle.

Similar problem solving activities should help overcome a pupil's tendency to use algebra and apply geometry more often. In addition, geometry should be perceived as a tool for solving problems and not only as a straightforward drawing using a ruler and a compass.

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Development of functional thinking

Functional thinking is an important component of the teaching of mathematics at primary and secondary schools. The paper shows a series of tasks which focus on the development of functional thinking of lower secondary pupils and illustrate their work by examples. The tasks were developed within a Socrates-Comenius project IIATM and tested in the third grade of Prague Eight-Year Secondary Grammar School in standard lessons of mathematics with 31–33 pupils (13–14 years old).

The series of tasks is as follows:

1. *Growth curves of population.* The pupils solved the task: What do the graphs in the picture mean? (Describe them.) They were given three graphs taken from biology without any indication of what the axes might mean.

The pupils discussed a wide range of interpretations which enabled the teacher to diagnose what their current understanding of graphs of functions is.

2. *Graphs of real events.* The second task was assigned as homework: Find examples of graphs which describe some real events or situations.

The pupils found different kinds of graphs and diagrams in newspapers and on the internet.

3. *Properties of functions.* This stage consisted of two tasks: 1) Brainstorming – What properties can a function have? 2) Investigate the given graphs and diagrams from the point of view of their properties.

The goal of the first part was to find out what concepts concerning functions the pupils have and whether they know their proper mathematical names. In the second task, the pupils investigated some graphs and diagrams (which the teacher chose from among those which they brought earlier) from the point of view of their mathematical properties.

4. *"What does the graph say?"* The task was: Describe the relationship in the given picture.

This task was used again as a diagnostic one and the pupils were to use their acquired knowledge and show what they had learned.

During the above process, the pupils started from their real life knowledge which they were to apply in mathematics. Then, the mathematical properties were discussed and correct terms introduced. Finally, the pupils reasoned within mathematics only and applied the knowledge on a concrete graph again. The teacher could see the differences between their initial interpretation in the first task and their interpretation in the final task.

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Developing the teaching of mathematics: Looking at continuing professional development (CPD) initiatives

The data that form the basis of this paper are being gathered as part of the Researching Effective Continuing Professional Development in Mathematics Education Project (RECME). The overarching aim of RECME is to explore the interrelated factors contributing to effective professional development, where we see CPD as a 'patchwork of opportunities – formal and informal, mandatory and voluntary, serendipitous and planned' (Ball & Cohen 1996). The intention is that the findings of the RECME project will inform the recommendations for future initiatives of NCETM and so help to develop opportunities for teachers to increase their expertise in the teaching of mathematics at all levels.

Over thirty ongoing CPD initiatives, which include more than 250 teachers, form our sample. The CPD initiatives that form the basis of this paper are a subset of this group and have been selected on the basis that they show contrasting approaches to working with teachers of pupils in the 5 to 14 age group.

However they are all engaged in working with teachers over an extended period of time and they are all running this academic year. They have been developed by different groups of people for different purposes and use a variety of different models. The purpose of the paper is to consider how these different models work and how the teachers taking part in these initiatives respond to the purposes of the initiatives. In particular the paper will address the ways in which the different initiatives work with teachers to develop the teachers' ways of working with children that help the children to engage more deeply with mathematics.

Data for the case studies will be collected in a number of ways from the leaders of the professional development initiatives and the teachers involved. These will include the observation of sessions in which leaders and teachers are engaged; the collection of data about the projects through online questionnaires; interviews with the teachers and leaders and observations of some of the teachers' mathematics lessons in their own classrooms.

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How do practising teachers use projects in their teaching of mathematics?

One aspect of the on-going reform of teaching in the Czech Republic concerns the change of teaching approaches (not only in mathematics). Among others, practising teachers are encouraged to use *projects* in their mathematical lessons to make them more lively and enjoyable for pupils and to connect pupils' learning of mathematics with their life experience. However, when we look into literature, we can see that the term project is understood differently by different authors. The question is how practising teachers without any formal introduction into teaching with projects cope with them in their teaching.

Last year, we organised a series of courses (within ESF programme) a part of which was a module on project teaching prepared by Marie Kubinova and Miroslav Hricz. The participants, elementary and lower secondary teachers, underwent 6 lessons of teaching and then were to write a report to get an official certificate. The majority of reports concern a suggestion of a project together with some practical experience from using it in the classroom. We have used this unique opportunity to investigate how project teaching is understood and used not by educationalists but by practitioners.

Namely, we have analysed 85 written reports of teachers to answer the following questions:

1. What is the teacher's primary motivation for using a project?
2. What do practising teachers consider to be a project? Do they refer to any authority or do they understand it intuitively?
3. Can a project be a «purely» mathematical one or is it always interdisciplinary? What school subjects are mostly used? What kind of pupils' life experience is mostly required?
4. In what ways are projects actually organised in the class?
5. Are the projects used to introduce new mathematical knowledge or to revise it? What mathematical knowledge is «prone» to be used in a project?
6. How do the teachers solve the problem of evaluating pupils' work? Do they evaluate the use of projects as for their impact on pupils' mathematical learning?

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