Critical thinking practices in mathematics education and beyond

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INTRODUCTION

Critical thinking in mathematics can be envisioned as an aspect of a wide range of mathematical activities, such as problem solving, problem posing and reasoning. Moreover, critical thinking contains the element of dispositions, which stress the human agent, specifically the critical thinker. Additionally, some approaches stress the critical aspect, by focusing on the critical thinker as a responsible citizen, who is not only able to solve problems, but also to be aware of the societal impact of the suggested solutions. The multifaceted nature of critical thinking has led to various definitions and approaches on its study. This fact is showcased in the current volume.

Critical thinking in mathematics adheres to the above characteristics, therefore it can be related to affective components of mathematics learning, but also to critical mathematics. Additionally, one should not ignore the affordances and the constraints for developing critical thinking in the mathematics classroom.

The works in this volume contribute to the above considerations, by presenting approaches that focus either on teachers or on students. Following this, the volume contains two main parts. Part 1, entitled *Teachers promoting critical thinking in the mathematics classroom* presents works that focus on preservice and inservice teachers' views and actions towards enhancing their own or their students' critical thinking skills. Part 2, entitled *Students manifesting critical thinking in the mathematics classroom* presents works that focus on analysing students' critical thinking skills, usually in relation to a learning environment designed by the teacher. The volume begins with a study on mathematics education researchers' views on critical thinking in mathematics.

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The Editors

CRITICAL THINKING IN MATHEMATICS EDUCATION: THE RESEARCHERS' PERSPECTIVES

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Bearing in mind the importance of critical thinking in education and, particularly in mathematics education, we designed a study in order to examine the ways that researchers in mathematics education perceive critical thinking. An online questionnaire was distributed, containing four questions on critical thinking, its relationship with mathematics and the possibilities for developing it within education. The results from the thirteen respondents adhere to those of relevant studies, but also add some new elements of critical thinking, such as being an effective communicator.

CRITICAL THINKING IN (MATHEMATICS) EDUCATION

The notion of critical thinking has been extensively used in the literature of education to denote particular skills and dispositions. Many definitions for critical thinking have been suggested by researchers (Ennis, 1989; Facione, 1990; Paul & Elder, 2002). One of the most encompassing ones is the following:

> We understand critical thinking to be purposeful, self-regulatory judgment which results in interpretation, analysis, evaluation, and inference, as well as explanation of the evidential, conceptual, methodological, criteriological, or contextual considerations upon which that judgment is based. (Facione, 1990, p. 2)

It is important to note that critical thinking consists of skills but also of dispositions. Abrami et al. (2015) invite us to consider "a person who possessed the cognitive skills associated with CT but who lacked the disposition to learn about or discuss social issues–it would be difficult to call this individual an effective critical thinker" (p. 277). Making informed judgements about social issues can be related to social responsibility as one of the '21st century skills', which are commonly mentioned in the relevant literature (Ananiadou & Claro, 2009).

In mathematics education, critical thinking can be viewed:

> as a by-product of mathematics learning, as an explicit goal of mathematics education, as a condition for mathematical problem solving, as well as critical engagement with issues of social, political, and environmental relevance by means of mathematical modeling and statistics. (Jablonka, 2014, p. 160)

Additionally, as we have noted in an earlier work (Maj-Tatsis & Tatsis, 2021), critical thinking has been associated with specific mathematical activities, such as problem solving and posing, reasoning, identifying the suitability of problem solutions, and metacognition. At the same time, there seems to be no agreement among researchers on whether critical thinking is subject-bounded and/or subject-specific (Ennis, 1989). Additionally, there have been varying approaches on the types of teaching that contribute the most to the development of critical thinking in mathematics (Aizikovitsh & Amit, 2010), but also in general education. This fact, together with the inconclusive nature of the discussion on *what is critical thinking in mathematics*, have led us to conduct a study among researchers in mathematics education. Our aim was to examine their views on the nature of critical thinking, its relationship with mathematics and the possibilities for developing it within education. The questions posed to the researchers were the following:

- What is critical thinking in mathematics? Can you give some examples of critical thinking in mathematics?
- How would you describe a person who is a "critical thinker in mathematics"?
- Do you believe that critical thinking in mathematics can be developed? If yes, how? What are the challenges for the teachers?
- Do you think that critical thinking in general can occur without mathematics? If yes, what could be its characteristics?

The questions were chosen in such a way to enable us to investigate the dual nature of critical thinking (abilities and dispositions), the teaching approaches for critical thinking and whether it is viewed as subject-bounded and/or subject-specific.

METHODS

An online questionnaire was delivered to mathematics education researchers in 2022; it contained the four questions mentioned in the previous section, together with the following:

- How many years of experience do you have as a researcher in mathematics education?
- How many years of experience do you have as a mathematics teacher in primary and secondary education?

The responses to the above questions, together with the countries of residence and the gender of the responders are presented in Table 1:

Researcher	Country	Gender	Research experience	Teaching experience
R1	England	F	25	6
R2	Netherlands	M	35	12
R3	Portugal	M	40	6

R4	Italy	F	50	0
R5	Latvia	F	20	6
R6	Italy	M	10	3
R7	Israel	F	17	28
R8	Poland	F	17	1
R9	Poland	F	40	10
R10	Poland	F	4	8
R11	Poland	F	10	1
R12	Greece	M	18	0
R13	Poland	F	20	10
	M		23.5	7

Table 1: Participant data.

The responses to the questions on critical thinking were mainly analysed by performing a thematic analysis (Boyatzis, 1998). Particularly, we categorised the emerging themes within the main topics of our interest, namely: *Definition* (i.e., characteristics of critical thinking in mathematics), *Thinker* (i.e., characteristics of a critical thinker in mathematics), *Development* (i.e., features of teaching focused on critical thinking in mathematics), and *Subject* (i.e., relationship between critical thinking and mathematics). For each theme, we identified the relevant categories, as appeared in the responses. In most cases, each utterance was assigned to one category; in few cases a single utterance was assigned to two categories. The responses could contain more than one category.

RESULTS

Definition: characteristics of critical thinking in mathematics

Five overarching categories were identified in the responses: problem solving, evaluation, reasoning, mathematical knowledge, and reflection. Table 2 presents the frequencies of all nine categories:

Category	*f*
Problem solving	10
Evaluation	6
Reasoning	4
Mathematical knowledge	4
Reflection	3
Citizenship	2

Noticing mistakes	2
Autonomy	1
Motivation	1

Table 2: Definition/characteristics of critical thinking in mathematics.

The above results provide a glimpse of the researchers' views on critical thinking; by looking at each response, we identified the qualitative properties of each category. *Problem solving* possessed various properties. For instance, the most prevailing one was the mere solving of given tasks. In other cases, though, the researchers referred to real-life problems or, to ways to solve a problem:

> critical thinking is the process of application [of] the appropriate models/methods/mathematical tools in solving real-life problems (R10)

> I would consider instances of critical thinking: […] neglecting information (entry data of a problem) to transform the problem in an easier one (R6)

Evaluation was mainly expressed in the form of evaluating a problem's solutions (or solution paths), in order to decide on their plausibility based on the given contextual demands. At the same time, some researchers expanded the evaluation to more aspects of mathematical activity:

> Critical thinking is a focused analysis of information. Student analyzes and evaluates different types of information and situations (R5)

> critical thinking in mathematics is the ability to make an evaluation of a response, of mathematical proof, of a mathematical text, of the value or beauty of a mathematical problem, or of any other piece of mathematical text (R3)

Some researchers claimed that the evaluation of solutions or solution paths should be made on the basis of mathematical *reasoning*:

> A child, or a teacher, who is able to think critically in mathematics will have knowledge of an appropriate number of mathematical facts and procedures but will also be able to reason about which one would be appropriate to use in a given context and to use mathematics to solve problems in that context (R1)

Mathematical knowledge was manifested in different ways in the researchers' responses; from being a prerequisite of critical thinking (see, e.g., the excerpt above) to being able to connect or work with mathematical theories:

> critical thinking involves the identification of gaps in existing mathematical theories and/or discerning the need for formulating new theories and/or realizing the need to reformulate mathematical theory to be in line with the mathematical idea (e.g. the history of the various definitions of the curve) (R12)

Reflection was mainly identified as reflective thinking, which could be in turn associated with metacognition:

> Critical thinking is also related to the ability to choose the way of proceeding, the method of solving the task. For me, critical thinking in mathematics is related to reflective thinking (R8)

> the ability to self-observe mathematical activity and the ability to analyze one's own errors (R13)

Citizenship, or precisely critical citizenship, was manifested either directly as the awareness of the 'why?' of a problem, or indirectly as a result of the ability to verify information (which also falls in the category of Evaluation):

> critical thinking is that you are aware of the reasons for the problem to be solved and what is the mathematical use of the theory in focus. Also what are the consequences of doing this kind of job (calculations, problem solving activities, ...) (R2)

> In order for critical thinking to be effective, in addition to the motivation mentioned above, you need to have the tools to be able to verify the information. (The main way to manipulate society is precisely to provide information that cannot be verified.) (R13)

Noticing mistakes in our study was associated with students noticing mistakes or contradictions (see also the above excerpt from R13), sometimes by examining the validity of a solution (cf. the Evaluation category):

> Critical thinking allows you to notice mistakes. (R11)

> Critical thinking in mathematics is revealed when a student notices contradictions. (R11)

> Critical thinking manifests itself in a situation where the student, solving a word task after committing a computational error, notes that the result obtained does not make sense in relation to the context of the task. (R11)

Autonomy was manifested as independent thinking, which sometimes emerged from the need to take one's own position, when, e.g., one does not agree with the existing results or an existing theory:

> First of all - independent, with reflection on the procedures performed, with reference to "self-confirmation" (how do I know it, maybe some earlier experiences or proven methods). Often - in surprising situations, with new results, but also when we do not agree with the results or ways of working of others. Generally - when I am faced with a situation that provokes me to take my own position. (R9)

Motivation was manifested as the disposition to engage into critical thinking activities:

> motivation and attitude (towards mathematics and towards life) play a big role here. Students and teachers, including mathematics teachers, are struggling with the lack of motivation to undertake critical thinking (R13).

Thinker: Characteristics of a critical thinker in mathematics

Eight categories were identified in the responses. They are shown in Table 3.

Category	*f*
Problem solving	5
Interpreting information	4
Flexible thinking	4
Reasoning	3
Mathematical thinking	3
Reflection	3
Applications	2
Discourse	2

Table 3: Characteristics of a critical thinker in mathematics.

Problem solving was manifested in most responses, just like in the Definition theme. Since the question contained the human agent, the responses were sometimes slightly altered (compared to those given to the previous question), in order to better represent an element of action or even affect. Additionally, the feature of the novelty of a problem – apparently in the sense of the non-triviality of a problem (Schoenfeld, 1992) – came to the fore:

> A person that can deal with a problem or question even if it is new and unexpected for him (R4)

> person who is not afraid to "take" tasks which is new for himself (R9)

Interpreting information was manifested in many responses. This category resembles 'Evaluation' of the Definition theme:

> A person who applies critical thinking in mathematics: […]
>
> - assesses the correctness of arguments,
>
> - verifies whether there are no errors in processes (actions, reasoning),
>
> - can assess the correctness of proofs,
>
> - takes into account the criteria,
>
> - analyzes the information received and verifies its correctness (R11)

Flexible thinking was manifested as flexibility in applying mathematics or as following an inquiry-oriented approach:

> The person should develop his natural curiosity (inquires), knowledgeable, thinker, communicator, risk-taker, open-minded (R10)

Reasoning was manifested by explicit reference to it, or by the term 'thinker' mentioned in the previous excerpt by R10. *Mathematical thinking* was mainly manifested in the following ways:

> Logical, strong number sense, capable of reasoning and problem solving, creative and flexible in applying mathematics (R1)

> A practicing mathematician should be a critical thinker in mathematics and, thus, anyone who practices mathematics should include critical thinking (R12)

Reflection was associated with metacognition, just like in the Definition theme. It is noteworthy that the first appearance of the term 'disposition' has been identified in a response to this question:

> I would say that a person who interrupts frequently their mathematical activity to check what has been done, its outcomes, and what remains to do is acting as a critical thinker. A critical thinker would evaluate the process and the products and modify them accordingly to their assessment. Thus, a disposition to self-evaluation and self-regulation appears as important feature of a critical thinker (R6)

Applications was mainly associated with applying mathematics in real-life situations. Finally, *discourse* was associated with the disposition of discussing with others and, if needed, revising one's point of view:

> […] such a person is not afraid to discuss his ideas, he can ask about inaccuracies (or about what seems unclear, incomprehensible to him) in the reasoning presented by others (R8)

Development: Features of teaching focused on critical thinking in mathematics

Three categories were identified in the responses. They are shown in Table 4:

Category	*f*
Discourse	9
Contextualised problems	8
Flexibility	2

Table 4: Features of teaching focused on critical thinking in mathematics.

Discourse was the most prevailing strategy, since it appeared in nine out of thirteen responses. Its manifestations varied from general claims about classroom discussions to specific claims on the teacher's attitude during these discussions and the challenges they are facing:

> the way to do is to have once a while a classroom discussion about the question for what reason we are doing mathematics in school? (Because we want to build society with critical thinkers). Challenges for teachers are: how to organise this in classrooms with many different perspectives and cultural backgrounds (R2)

> Critical thinking *must* be developed in mathematics, may be promoting the use of the language, through the request to explain the reasoning and in which way the problem was solved (R4, emphasis in the original)

> You just need to give students the opportunity to think for themselves, without rushing and criticizing every mistake (R9)

Contextualised problems appeared to be another effective teaching strategy, according to the respondents. Just like in the previous category, we have identified general claims, but also more specific guidelines on how to implement such an approach:

> I think that critical thinking in mathematics can be developed by offering learners problems and scenarios which they can address meaningfully in ways that make sense to them. Learners need to be exposed to problems that demand creative thinking and the capacity to connect and apply a range of mathematical concepts and methods. The challenge for the teacher is to identify appropriate contexts and scenarios for the learners and so build pathways to understand mathematics (R1)

> The challenges are as following: sometimes it is very hard to find appropriate problem to solve, there are problems in formulating appropriate questions (asking good questions is not very easy), sometimes the level of mathematics of students varies a lot (R10)

Flexibility was manifested in two responses, and referred to teacher's readiness to adopt her teaching, according to the needs of the classroom and/or particular students:

> […] requires from her [the teacher] a very flexible and reflective approach, the ability to react quickly to emerging situations (R8)

> More than once I hear from both active [in-service] and future teachers: "there is a tight program in high school, we do not have time for it, why introduce five methods – just one – why do students need to mix in their heads with five methods, they will still be confused. You have to teach certainties for the final exams." Teachers reject solutions that they do not know – and therefore the need for continuous substantive improvement on the part of teachers (R13)

Subject: Relationship between critical thinking and mathematics

The vast majority of respondents answered that critical thinking can occur without mathematics; seven of them gave a clear 'yes', and two of them noticed that their answer in the Definition theme would fit to any subject. Five respondents claimed that critical thinking can occur in subjects other than mathematics, but only if particular conditions were met:

> Critical thinking can occur in any discipline: science, languages, art… but the characteristics of applying problem solving and reasoning are key (R1)

> If we define critical thinking as the ability to reflectively approach a given situation, analyze it and draw appropriate conclusions, then critical thinking can be found outside of mathematics (R8)

Only one respondent excluded the possibility of critical thinking out of mathematics:

> I think that without logic there is no critical thinking. Logic is a part of mathematics. (R10)

DISCUSSION

Our results, in general, go in line with those that appear in the literature; especially reasoning and problem solving prevailed in the responses of the researchers, to both the first two questions on the characteristics of critical thinking and of the critical thinker. We have also identified the categories of noticing mistakes, which resembles (although in a narrower sense) the activity of *noticing* (Sherin et al., 2011), which seemingly has not been associated with critical thinking.

It is noteworthy that the main 'critical' aspect of critical thinking, 'Citizenship' was explicitly manifested in only one response in the first question, while in the second question it could be remotely related with 'Applications'. We may have to agree with Jablonka (2014) that:

> Notions of CT in mathematics education with a focus on argumentation and reasoning skills have in common that the critical competence they promote is directed toward claims, statements, hypotheses, or theories ("texts") but do include neither a critique of the social realities, in which these texts are produced, nor a critique of the categories, in which these texts describe realities. (p. 161)

Most scholars in the relevant literature stress the duality of abilities and dispositions that constitute critical thinking (Facione, 1990). In our study, the second question was designed to elicit the characteristics of these dispositions, although 'Motivation' did already appear in the Definition theme. What we found in the Thinker theme was mainly a repetition of the categories of critical thinking, such as problem solving and reasoning. However, the category of 'Discourse' in the sense of a disposition towards communicating one's ideas, appeared in two instances, which may reflect the general trend towards communicative views of learning (Sfard, 2008).

Discourse was expressed as the most prevailing way to develop critical thinking in the classroom. Most researchers agreed on the importance of engaging students in meaningful discussions, with the teacher not exercising their authority (Tatsis, Wagner, & Maj-Tatsis, 2018) in these discussions. It seems that, according to the respondents, the teacher's ability to 'step back' and give room to the students – while, at the same time, monitoring and preparing oneself for the next move – is of crucial importance to the development of students' critical thinking. This reminds us of Rowland, Huckstep and Thwaites's (2005) notion of *contingency*, which can be defined as "the readiness to respond to children's ideas and a consequent preparedness, when appropriate, to deviate from an agenda set out when the lesson was prepared" (p. 263).

The fact that all but one respondent agreed on the possibility of developing critical thinking out of mathematics can contribute to the ongoing discussion on the subject-specificity or subject-dependence of critical thinking (Ennis, 1989). We agree with those respondents who claim that the development of critical thinking should be based on activities that require reasoning and reflection.

Generally, our study adds to the rather limited literature on critical thinking in mathematics education, by offering some researchers' views on it; it also brings to the fore aspects of critical thinking, which were not stressed until now, especially the communicative one. We believe that any approach on critical thinking, including the teaching approaches should bear in mind the need for effective and meaningful communication among students and among the mathematics teachers and the students. The teacher's role is crucial to the development of critical thinking and, according to the results of our study, this can be achieved not only within mathematics, but in most – if not all – school subjects.

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Teachers promoting critical thinking in the mathematics classroom

Part 1

MATHEMATICAL CREATIVITY IN THE CLASSROOM: TEACHERS' BELIEFS AND VALUES

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In this chapter I focus on promoting mathematical creativity in the classroom. First, I discuss different views of what it means to foster mathematical creativity and ways for occasioning mathematical creativity in the classroom. Next, I review studies of teachers' beliefs related to mathematical creativity. Finally, I will introduce the issue of values, and describe initial findings from a study concerning how values may influence teachers' choices when their aim is to occasion mathematical creativity in the classroom. These studies may help teacher educators plan professional development that can encourage more teachers to promote mathematical creativity in their classroom.

INTRODOUCTION

Along with promoting critical thinking, fostering mathematical creativity is one of the major aims of mathematics education. One of the challenges to promoting creativity in the classroom is that educators do not agree on how to define, promote, or evaluate mathematical creativity. The first part of this chapter will present researchers' views regarding these issues. A second challenge is that teachers may hold various beliefs related to creativity that may or may not coincide with educational goals. For example, do teachers believe that we can foster mathematical creativity among all students, or do they believe that creativity is an inborn trait? A third challenge is that teachers' values may also interact with their intention to foster creativity. For example, if a teacher values originality, he may promote individual creativity as opposed to collective creativity. This chapter will present results from studies which investigated teachers' beliefs and values related to mathematical creativity, and discuss how beliefs and values may impact on the ways teachers foster mathematical creativity in their classrooms.

MATHEMATICAL CREATIVITY: SOME THEORY

Fostering mathematical creativity is one of the major aims of mathematics education. As stated by Sriraman (2009) "mathematical creativity ensures the growth of the field of mathematics as a whole" (p. 13). While there is no single accepted definition of creativity or creative thinking (e.g., Runco & Jaegaer, 2012), many researchers from several domains, including mathematics, note that creativity involves at least two attributes: originality and appropriateness (e.g., Runco & Jaeger, 2012). Runco (1996) stated that creativity is "manifested in the intentions and motivation to transform the objective world into original interpretations, coupled with the ability to decide when this is useful and when it is not" (p. 4). Thus, critical thinking may be considered a component of creativity.

As educators, we are less concerned with the creativity of a few eminent persons (Big-C creativity) and more concerned with creativity as it manifests itself in the classroom (mini-c creativity) (Kaufman & Beghetto, 2009). As students learn new concepts, they may come up with "novel and personally meaningful interpretation of experiences, actions, and events" (p. 3). With regard to mathematics classrooms, this aspect of creativity may manifest itself when a student examines many solutions to a problem, methods or answers, and then generates another that is different (Silver, 1997). As such, the product of mathematical creativity in the classroom may be original ideas that are personally meaningful to the students and appropriate for the mathematical activity being considered. Notice that originality and novelty are used almost interchangeably. In a previous paper (Levenson, 2013), I claimed that the two are not synonymous. An idea that is novel may be "new," while an idea that is original refers to an idea that is "one of a kind" or "different from the norm." An idea, especially one raised in the classroom, may be new to a student, but if other students have the same idea, it may not be original. When measuring originality, Leikin (2009) considered the level of insight and conventionality of a solution in comparison to the learning history of the participants. For example, a solution based on a concept learned in a different context would be considered original but maybe not as original as a solution which was unconventional and totally based on insight.

Originality is not the only consideration when characterizing creativity; fluency and flexibility are also components of creative thinking. In general, fluency refers to the total number of unduplicated ideas generated (Jung, 2001), and in mathematics, the total number of unduplicated valid mathematical responses. Flexibility is evident when different solutions employ different strategies, or are based on different representations (e.g., algebraic and graphical representations), properties, or branches of mathematics (Leikin, 2009).

Flexibility may also be considered in relation to its counterpart, fixation. In problem solving, fixation manifests itself in mental rigidity (Haylock, 1997) or self-restrictions (Krutetskii, 1976). Flexibility is then shown by overcoming fixation or breaking away from stereotypes. In a previous study (Levenson, 2011), I demonstrated this type of flexibility in a fifth-grade classroom, when a student ventured to suggest that a sequence may have unequal jumps, in contrast to the rigid adherence of equal jumps. Haylock (1997) differentiated between content-universe fixation and algorithmic fixation. Overcoming the first type of fixation requires the thinker to (intentionally) consider a wider set of possibilities than at first is obvious and extend the range of elements appropriate for application. For example, elementary school students asked to find two numbers whose sum is 18 may miss 18 and 0 or may not think of fractions because they do not consider the possibility of numbers which are not natural. The second type of fixation relates to when an individual adheres to an initially successful algorithm even when it is no longer efficient. This type of fixation relates to the familiar case of a student who is requested to calculate 20×20 and resorts to the long multiplication algorithm, though it is clearly unnecessary in this case.

Promoting mathematical creativity in the classroom

Several factors need to be considered when discussing the promotion of mathematical creativity in the classroom, including the types of tasks implemented, the classroom environment, and the teacher's actions. Regarding tasks, Torrance (1965) was one of the first to suggest the use of divergent production tasks in promoting fluency, flexibility, and originality. In order for such tasks to actually produce creativity, Haylock (1997) claimed they need to include the possibility of using a wide range of ideas, have at least 20 possible appropriate responses among several of which are obvious to students, some responses which will likely be obtained by only a few students, and they should not be mathematically trivial. Silver (1997) claimed that both problem-solving as well as problem-posing activities may encourage students to generate multiple solutions, in turn encouraging the development of creativity. Kwon, Park, and Park (2006) advocated the use of open-ended problems for developing students' creativity in mathematics. In general, open-ended problems have a clear starting point but may have less clear objectives. This allows students to choose their own paths and come up with diverse and novel solutions. Sheffield (2008) added that a task which can be extended and thus promotes further questioning, can also promote mathematical creativity.

The classroom environment may be another factor influencing mathematics creativity. For example, consider a classroom where students work in groups. This situation may offer additional opportunities for group creativity or may, on the other hand, inhibit creativity. Paulus and Yang (2000) claimed that deficient results of group creativity may be caused by the group members not being attentive to the ideas of the group or there may not be enough incubation time to reflect on those ideas in order to integrate them with one's own ideas. Hadamard (1945) considered the incubation stage, working on a problem at the unconscious level, an intrinsic part of the creative process. In a group, time for incubation is usually scarce. Another perspective on group creativity considers the situation where diverse individuals come together to solve a problem. On the one hand, the different backgrounds and knowledge base of a diverse group may contribute different perspectives for consideration. On the other hand, diversity may be so wide as to hinder individuals as they strive to understand different ideas and come up with an agreed-upon solution (Kurtzberg & Amabile, 2001). Recently, we investigated individual and group creativity among post high-school students (Molad and Levenson, submitted). In that study, we found that the group experience, working collaboratively on open-ended tasks, can impact positively on the group's collective creativity as well as on the group participants' creativity even as they work individually.

Finally, the teacher also has several roles in promoting creativity. In my study of collective mathematical creativity in the classroom (Levenson, 2011), it was found that the teacher's roles included (1) choosing appropriate tasks, (2) fostering a safe environment where students can challenge norms without fear of repercussion, (3) playing the role of expert participant by providing a breakdown of the mathematics behind a process, and (4) setting the pace, allowing for incubation periods. Regarding the third point, Mhlolo (2017) suggested that when a teacher exhibits representational competence, working with and engaging students with varied equivalent representations, and encouraging students to use different modes of representations meaningfully, the teacher is supporting flexibility. Another aspect is acknowledging creative solutions when they arise among students, and prompting further mathematical inquiry with appropriate questions (Hoth, Kaiser, Busse, Döhrmann, König, & Blömeke (2017).

TEACHERS' BELIEFS REGARDING MATHEMATICAL CREATIVITY

As noted above, the teacher has a prominent role in promoting mathematical creativity. Taking into consideration that teachers' beliefs may affect their instructional decisions (Schoenfeld, 2011), as well as students' learning (Cross, 2009), this section focuses on teachers' beliefs related to mathematical creativity. These beliefs include beliefs about mathematics as a discipline, about the teacher's roles in promoting mathematical creativity, and about students' capabilities.

I begin with the question of whether teachers believe that mathematics is a creative domain. When asked to identify school subjects and domains likely to elicit creativity, most elementary teachers list art, theatre, and music, (Diakidoy & Kanari, 1999; Kamplylis et al., 2009). As one prospective teacher stated, "creativity has strong relations with subjects such as art, while mathematics is related with logic" (Panaoura & Panaoura, 2014, p. 6). In a different study, elementary school teachers associated creativity with language and writing (Aljughaiman & Mowrer-Reynolds, 2005) because these subjects offer opportunities for discussing and exploring ideas, where students have freedom, can use their imagination, and where no 'correct answer' exists (Bolden, Harries, & Newton, 2010). Similar sentiments were found among prospective middle and secondary mathematics teachers when investigating the place of creativity in classroom discussions (Beghetto, 2007). Mathematics teachers were significantly less likely to value unique contributions to discussions, than other subject teachers. The future mathematics teachers considered unique responses as potentially disruptive, believing it more important to focus on the problem at hand, and to follow the curriculum.

Teachers' reluctance to associate mathematics with creativity might be due to their beliefs of mathematics as a domain, and not necessarily to their beliefs of creativity. Prospective primary teachers often suggest that mathematics is a body of knowledge, based on facts, figures, and rules, with little room for developing independent ideas (Bolden, Harries, & Newton, 2010). Many participants claimed that English, art, and even science had fewer set goals, and were therefore more conducive to creativity development. Some prospective middle- and high-school mathematics teachers believe mathematics to be a closed domain, where all possible concepts have previously been invented by mathematicians (Shriki, 2010). These participants sometimes based their beliefs on their past experiences. For them, mathematics in school was always about getting the right answer, checking the answers in back of the textbook, and then moving on to the next problem. Another factor related to the subject of mathematics in school is the standardized curriculum for mathematics in many countries.

Another issue investigated by several researchers is whether teachers believe that creativity can be developed. Among general teachers, it was found that approximately 75% of participants believed that creativity is not a characteristic of all people and that some children are more creative than others (Diakidoy & Kanari, 1999). Yet, about 90% still believed that creativity can be facilitated amongst all children. More recently, a study of Greek prospective and practicing elementary school teachers (Kampylis, Berki, & Saariluoma, 2009) found that participants held conflicting beliefs. Half of the participants believed that only a few students have the "gift" of creativity, and yet the vast majority agreed that creativity can be developed in all students. Thus, while teachers may believe that some people are born more creative than others, most believe that everyone can learn to be creative.

Focusing specifically on mathematical creativity in the classroom, studies found that in general, teachers of all levels, from elementary to secondary school, believe that mathematical creativity can be developed (Lev-Zamir & Leikin, 2011; Shriki, 2010). In one comparative study of secondary school mathematics teachers in different countries (Leikin et al., 2013), participants were requested to rate their level of agreement on a scale of 1-6 with the statement "a creative person is born that way." The average across all participants was 4.23, indicating a tendency to agree. Participants from Mexico were less likely to agree with this statement than participants from India, Cyprus, Israel, Latvia, and Romania. Yet, some mathematics teachers believe that not all students can develop creativity (Levenson, 2017), or that relatively few students are capable of being creative (Shriki & Lavy, 2012). Reasons mathematics teachers give for why only some students exhibit mathematical creativity are related to views of individual characteristics, such as mathematical ability.

Taking a closer look at teachers' beliefs, I recently investigated mathematics teachers' perceptions regarding the relationship between mathematical creativity and mathematical excellence (Levenson, 2020). The specific research questions of the study were: (1) Do teachers believe that there is a relationship between mathematical creativity and mathematical excellence, and if so, what types of relationships do they believe exist? (2) What beliefs regarding mathematical creativity surface, as teachers describe the relationship between mathematical creativity and excellence? (3) Are different beliefs regarding creativity associated with different beliefs regarding the relationship between mathematical creativity and excellence?

Forty-five mathematics teachers responded to the following query: "There are those who say that mathematical creativity is related to excellence in mathematics. What is your opinion?" From participants' comments, six different categories of relationships were found (see Table 1). Considering those who believed that mathematical creativity can promote mathematical excellence, as well as those who believed the relationship to be mutual, we find that half of the participants believed mathematical creativity to have some influence on mathematical excellence.

Category	F (%)	Examples of teachers' statements
A: Mathematical excellence precedes mathematical creativity	13 (29)	"When a student is good at mathematics, his self-confidence rises, which causes him to dare more and to try different solution methods without fear of failure."
B: Mathematical creativity precedes mathematics excellence	19 (42)	"Creativity comes from having an open mind, solving problems in many different ways, which leads to excellence." "Mathematical creativity promotes excellence in mathematics… However, excellence in mathematics does not promote creativity because creativity is genetic and cannot be acquired."
C: Creativity and excellence in mathematics are reciprocally related	4 (9)	"The relationship between creativity and excellence is two-way. While it may be that stronger mathematics students exhibit more mathematical creativity, if teachers promote creativity among the weaker students, they will become stronger in mathematics."

D: There is a non-influential relationship between mathematical creativity and excellence	2 (4)	"There is a relationship between mathematical creativity and mathematical excellence, but one is not a sufficient condition for the other."
E: Mathematical creativity and excellence are not related	4 (9)	"Creativity in mathematics can be developed and acquired, even among lower achieving mathematics students. It is dependent mostly on a supportive environment of which the teacher is responsible."
F: Undecided	3 (7)	"I cannot decide. It depends on how one defines excellence in mathematics."

Table 1: Types of relationships between creativity and excellence (Levenson, 2020, p. 167).

A second analysis of the data investigated inferred beliefs specifically related to creativity. These beliefs were related to (1) creative processes (e.g., thinking creatively means thinking out of the box), (2) the product of creativity (e.g., creative products include unconventional solutions), (3) the nature of creativity, and how creativity might be affected by the environment (e.g., opportunities afforded in a classroom), and (4) affective issues (e.g., creativity is enjoyable). Interestingly, the same inferred beliefs related to creativity were sometimes associated with different beliefs concerning the relationship between excellence and creativity. For example, a belief that the environment is a factor in promoting mathematical creativity, may lead one teacher to claim that excellence (because of opportunities given to excellent students) leads to mathematical creativity, while another teacher may claim that since it is up to the environment, mathematical creativity and excellence are unrelated.

Deeping our knowledge of teachers' beliefs related to mathematical creativity can help teacher educators address these beliefs. For example, some teachers believe that mathematical ability leads to greater motivation and less fear of failure, which can then lead to greater creativity. Knowing this aspect of teachers' beliefs, teacher educators can discuss with teachers how to mitigate fear of failure among *all* students, possibly then motivating those same teachers to promote creativity among all students. Closely related to beliefs is the notion of values. This is discussed in the next section.

VALUES AND MATHEMATICAL CREATIVITY

A brief review of past studies

Beliefs and values both belong to the affective domain associated with learning and teaching mathematics, but they are not the same. DeBellis and Goldin (2006) stated that beliefs involve attribution of some external truth to a set of propositions. Values refer to “personal truths or commitments cherished by individuals. They help motivate long-term choices and shorter-term priorities.” (p. 135). Philipp (2007) stated that people hold beliefs to be true of false, with varying degrees of conviction. On the other hand, values come from the word ‘value’, meaning the worth of something, and are thus desirable or undesirable. In addition, beliefs are context-specific, whereas values are less so.

Few studies focused on the question of whether teachers value creativity, although one might infer such values from other studies. For example, in one study, when teachers were asked to describe their views of creative production, none of them mentioned its usefulness (Aljughaiman & Mowrer-Reynolds, 2005). The researchers attributed this oversight to teachers’ stressing about academic achievement and thus viewing creativity as perhaps interesting, but not necessarily of value. It could be, however, that those teachers viewed creativity in the classroom as did Kaufman and Beghetto (2009), who described students’ (mini-c) creativity as being personally meaningful to the student, and not necessarily of value or useful to others. Similar results were found when mathematics teachers were asked to describe creativity (Shriki & Lavy, 2012).

In addition to the question of whether or not teachers value creativity for creativity’s sake, there are other values which teachers may associate with the promotion of mathematical creativity in their classrooms. In a previous study, I investigated mathematics teachers’ choices of tasks when their aim was to occasion mathematical creativity (Levenson, 2013). In that study, teachers were requested to choose a task that in their opinion had the potential to occasion mathematical creativity and state the reasons for their choice. While several teachers chose a task because it had more than one solution or more than one way to solve a task, participants’ values were also in evidence. For example, two teachers mentioned the importance of having every student participate. “One teacher wrote, ‘there isn’t a student who cannot participate in this activity, even special-needs students [can participate].’ Another teacher wrote, ‘Every student can find his own unique solution method’” (p. 286). Some teachers valued group work and wrote that they chose a task because it encouraged students to cooperate. Other teachers valued individuality, stating that their chosen task allowed each child to come up with a unique solution. There was one teacher who seemed to relate the cognitive demand of connecting different mathematical representations (which she calls ‘media’) to the value of allowing for student individuality, “The task promotes the use of different medias such as graphs, algebra, numbers, and words and does not limit the solution to a specific media thus allowing many students the possibility of expressing themselves in the area where they are strongest" (p. 286).

In a follow-up study of one teacher's changing perspectives on tasks that may occasion mathematical creativity (Levenson, 2015), values were again part of the choosing process. When stating why a certain task was chosen, the teacher wrote, "There is an opportunity here for collective creativity with its associated characteristics: searching for help (I didn't find the rule, I don't understand the conclusion…), giving help, and gaining new comprehension." (p. 12). The secondary mathematics teacher in that study raised the value of collective creativity. In her opinion, the value of group work is offering students the opportunity to help each other and to help students who might find the task difficult.

Values and preference

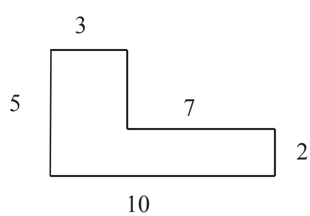
According to Bishop (2012), values are often revealed at decision points in the lesson. Thus, in order to investigate teachers' values associated with mathematical creativity, I began studying how teachers choose among three given tasks, which task has the most potential to occasion creativity. By asking teachers to choose between three tasks, I hypothesized that values would play a role. Participants were 42 teachers who had taken part in a graduate course called "Creativity in mathematics education," which aimed to increase teachers' theoretical and pedagogical knowledge regarding mathematical creativity. As such, participants had some knowledge and experience with mathematical creativity in educational settings. Below, I present the three tasks given to participants, and then give examples of teachers' choices and the reasons for their choices.

The first task was taken from a book written about problem solving by the mathematician Polya:

> To number the pages of a bulky volume, the printer used 2989 digits. How many pages has the volume? (Polya, 1945, p. 234).

The second task was taken from a fourth grade mathematics textbook entitled Geometry for the Fourth Grade (The Center for Educational Technology, 2006):

> Find the area of the polygon. It may be helpful to divide the polygon into rectangles. (Draw the dividing lines on the diagram.)



Computation: _________ The area of the polygon is: ___________________

Discuss: Are there different ways in which you can divide this polygon? If so, what are they? From the different ways of dividing the polygon do you get different areas of the polygon?

The third task was taken from a seventh-grade mathematics book, chosen by a teacher in a previous study as a task that has potential to occasion mathematical creativity (Levenson, 2015, p. 12). The teacher stated that it was intended as an inquiry-based task, where the students come to a new, for them, multiplication rule:

Multiplying signed numbers

Below is a multiplication table:

×	3	2	1	0	-1	-2	-3
3	9	6	3	0	-3	-6	-9
2	6	4		0			
1	3	2		0	0	-2	
0	0	0	0	0		0	0
-1		-2	-1	0	2		
-2		-4		0			6
-3		-6		0		6	

a) What is the rule in the first row?

b) What is the rule in the second column?

c) Find the rule in each column/row and fill in the rest of the empty cells.

d) (i) What is the sign of the solution when multiplying a positive number with a negative number?

(ii) Where in the table are these numbers located?

(iii) Write a multiplication example using a positive number and a negative number?

e) (i) What is the sign of the solution when multiplying a negative number with a negative number?

(ii) Where in the table are these numbers located?

(iii) Write a multiplication example using two negative numbers.

After analysing each task in terms of its potential to occasion mathematical creativity, each participant was requested to state which task, in their opinion, had the most potential to occasion mathematical creativity and state the reasons for their choice. It is this last part that I focus on here. First, I present the case of Adina (all names are pseudonyms), who despite her analysis of the creative potential for each task, finally chose one of the tasks for a reason not even mentioned in her initial analysis. I then present three different teachers, each who chose a different task as the most preferred. Finally, I present the case of a teacher who could not decide which of the two tasks she most preferred.

I begin with Adina's analysis of each task. When analysing the three tasks, Adina stated that each task led the student to search for several solution methods. For Task 1, she stated:

> According to Silver (1997), this task encourages creativity because it encourages fluency and original ideas. Finding the number of pages in the book is not trivial and therefore requires the solver to check <u>several</u> solution paths (fluency), and because questions of this kind are <u>not</u> usually given to students, the question encourages originality. (Underline in the original.)

For Task 2, she stated,

> Because there are a number of ways in which to divide the figure into rectangles, there is an element of fluency and there are many ways to reach the same solution.

For Task 3, she wrote,

> There are many ways to complete the table and to reach a rule. For example, one student can see that a row jumps by three (0, 3, 6, 9, and so on) and so can complete the table. Another will recognize the multiplication table from his previous experience (the positive side) and then fill in the negative side according the example given.

In the end, Adina preferred Task 1 and stated that this task had the most potential to occasion creativity because:

> The task is interesting, even to a person who is very knowledgeable in mathematics, because (as I wrote in the above paragraph), the style of the question does not lead the student to use a clear and known algorithm, like calculating an area or multiplying numbers.

In Adina's case, we see an example of a participant who analysed each task according to its potential to occasion creativity. For each task, she refers to elements of mathematical creativity discussed during the course, such as fluency and being able to connect mathematical topics learned at different times. Yet, when it comes to choosing the task she believes has the most potential, what Adina values is a problem that is interesting.

Orah, Deborah, and Ron each preferred a different task. Orah preferred the first task and wrote the following:

> Each of the tasks has features and cognitive demands that promote creativity. In my opinion, the first task has the greatest potential to occasion creativity. The first task has a different style than what the student is used to. There are no instructions how to solve [the problem]. There are many high cognitive demands and there is potential and challenge in those cognitive demands. The main difference between this task and the other two is in the cognitive demands, which are higher in the first task.

Orah begins by noting that each task has potential to occasion mathematical creativity. In fact, when she analyzed the second task, she specifically noted that it could lead to fluency, flexibility, and originality, characteristics of mathematical creativity mentioned by educators and researchers (e.g., Silver, 1997). She also wrote that the instructions specifically call for finding different ways to solve the problem. When she analysed the third task, she noted that the use of the table format, along with the verbal instructions, allows students to make their own connections between different representations, assisting them to generalize. She specifically wrote that this might help them to think creatively in the future. Yet, Orah still chose the first task as having the most potential to occasion mathematical creativity, despite writing in her initial analysis that it does not call for solving the problem in different ways. Why did she prefer the first task? The first reason is because of its "different style." The second reason is because of its high cognitive demand. It seems that for Orah, these characteristics are valued over others that are also associated with mathematical creativity.

Deborah (T15) preferred the second task. She wrote:

> On the one hand, it (Task 2) is simple, but the discussion which asks if there are additional ways to solve the problem is a challenging question which shakes up the student's knowledge and causes him to investigate other possibilities. The first task is too challenging, and students might just give up. The third task is too structured and directed. The second task is the only one that promotes fluency, flexibility, and originality.

First, we note that when analyzing the first task, Deborah claimed that it could promote flexibility and originality, but not fluency, while in her opinion, the third task did not promote creativity at all. Thus, it makes sense that Deborah chose the second task, because in her opinion, it could occasion all three characteristics of mathematical creativity, characteristics learned and discussed in the course. But this is not the only reason Deborah chooses the second task. She stresses the issue of challenge. Deborah values challenge, but the degree of challenge is also important. The second task, in her opinion, presents a balance between a task being too challenging and not challenging enough.

Ron (T20) preferred the third task and wrote:

> The third task encourages the student to connect between two mathematical domains – numbers and algebra. The students are able to make a generalization

> based on the numbers and their place in the table and find a rule. The task has many ways to solve it, and it is somewhat challenging. The task promotes flexibility – a change in the way of thinking.

Above, we see that when explaining why he preferred Task 3, Ron mentions generalization and finding a rule as reasons for preferring the third task. These attributes were also mentioned in his initial analysis of the third task; they were not mentioned in conjunction with the first two tasks. Ron also states that the third task can promote flexibility. Yet, in his initial analysis he did not note this attribute. In fact, in his initial analysis, Ron did not mention flexibility in relation to any of the tasks. It is possible, that when Ron must compare the tasks with each other, this attribute comes to the fore, and in line with Bishop (2012), this when values come into play.

The last case presented here is of Dorine. Dorine could not decide if Task 1 or Task 2 had the most potential to occasion creativity. She wrote:

> I recognize the second task from textbooks that I use in seventh grade and it promotes creativity. It has fluency, flexibility, and originality. However, because the discussion requests dividing the polygon into rectangles, it limits the students by not suggesting additional polygons, such as triangles and rhombuses. If the question was to divide the polygon into different polygons in different ways, we would see more original solutions.

Dorine then continues to discuss the first task:

> When solving the first task, I did not have fluency nor flexibility. However, the first task is challenging and requires a different way of thinking. That is, it requires using methods that are not the usual solution methods and therefore requires a great deal of thought and trial to find different solution methods. Although I only found one solution method, I did try to find other ways using sequences and functions. That is, I tried not to be fixated on one solution path and tried to find other ways.

Dorine exhibits knowledge of mathematical creativity. She recognizes that the first task can promote fluency, flexibility, and originality, but also points out its limitation. She also points out a way that this limitation could be overcome and how the task could be extended to promote more creativity. Interestingly, despite her experience of solving the second task and finding that it did not elicit fluency, flexibility, or originality, she still believes that the second task has great potential to occasion creativity. Why? When attempting to solve the problem Dorine tried to find other ways to solve the problem, and although she was unsuccessful, she did not become fixated on one method. Note that Dorine's reservation with the first task was that it limited the students' ways of thinking and that her positive review of the second task was due to her not being fixated. Basically, what Dorine found lacking in the first task, was not lacking in the second task. To sum up, what Dorine seems to value most when it comes to promoting creativity is overcoming fixation.

SOME CONCLUDING THOUGHTS

As I discussed in the beginning of this chapter, the challenges to promoting creativity in the mathematics classroom are many. Theories and research point to various views of characterizing creativity, and different means of promoting mathematical creativity. As mathematics teachers, and mathematics teacher educators, we need to ask ourselves what we can do to encourage students' mathematical creativity.

In the sections above, it was shown that teachers hold various beliefs related to mathematics, the teaching of mathematics, and the promotion of mathematics creativity in the classroom. While beliefs may be slow to change (Schoenfeld, 2011), professional development can have an impact. In one of my courses, a secondary school teacher claimed that she had no time to promote creativity because she was too busy preparing students for state exams. She believed that promoting creativity could not go hand-in-hand with enhancing students' mathematical knowledge. By the end of the course, she saw that mathematical knowledge and creativity were not mutually exclusive, and that students can benefit by thinking flexibly. In her case, enhancing her knowledge of characteristics of mathematical creativity, along with her experiences while solving problems in multiple ways, encouraged her to try new methods in her classroom. It also increased for her, the value of promoting creativity. Our values, along with our beliefs and knowledge, influence our aims. If teachers believe that promoting mathematical creativity can assist in students' mathematical growth, they may explicitly plan lessons that do both. According to Schoenfeld (2011):

> Teachers' beliefs and orientations, like students' beliefs and orientations, are built up slowly from experience and are often not consciously held. Thus, they are slow to change, especially if the individuals are unaware of having them (Schoenfeld, 2011, p. 464).

As teacher educators interested in promoting mathematical creativity, let us raise our own awareness, as well the awareness of other mathematics educators, regarding beliefs and values we hold related to mathematical creativity in the classroom. This could be a first step in recognizing and promoting mathematical creativity among students of all ages. The next step is to afford teachers opportunities to experience for themselves mathematical creativity and guidance as they occasion mathematical creativity for the students in their classrooms.

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ARGUMENTATION, EXPLANATION, MATHEMATICAL PROOF

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The proof and the deductive method in mathematics have their origin in the classical model of exposition developed by Euclid in his famous book of elements. The attitude of mathematicians towards this method has certainly evolved over the centuries, but the relationship between understanding and acceptability of mathematical statements has not changed and still constitutes a characterizing element of this discipline. In this article we discuss some aspects that can be considered at the origin of the difficulties related to the mathematical proof. In particular, we focus on the tension between two poles, that of understanding and that of the theoretical arrangement of mathematical knowledge.

INTRODUCTION

Like all human activities, mathematical practice has many facets, and it is for this reason that it is difficult, and perhaps useless, to circumscribe it in rigid schemes or, even worse, in a single scheme. Particular conclusions can determine specific interpretations and beliefs towards mathematics, and the school experience has always had a great influence on that, often contributing to privilege and strengthen one practice and overshadowing others, so as to lose the complexity and richness of Mathematics. Of the many aspects that concern Mathematics and its practices, perhaps the least popular, and certainly not beloved one, concerns mathematical proof, and more generally what commonly, and sometimes with some contempt, is referred to as *rigor*.

All this corresponds in school practice to the perception of a specific difficulty related to mathematical proofs; in the field of mathematics education, since long ago the issue of *proof* has become a very active research topic, opening a debate among researchers that has not only made it possible to clarify some aspects, but above all has shown how difficult it is to isolate the mathematical proof from other practices that are connected to it (Mariotti, 2006; Mariotti et al., 2018; Stylianides et al., 2016) and how important is to educate students to manage the specificity of each of them.

The objective of this contribution is to support this statement, we will start from some reflections on the nature of the mathematical proof to clarify the complexity of the relationship it has with other practices; I will focus on specific features of a mathematical proof that characterize it and for this very reason can be hardly neglected without a serious loss for mathematics education (Mariotti, 2006).

In the following statement the educational value of proof was clearly stated.

> The concept of proof is one concerning which the pupil should have a growing and increasing understanding. It is a concept which not only pervades his work in mathematics but is also involved in all situations where conclusions are to be reached and decision to be made. Mathematics has the unique contribution to make in the development of this concept [...] (Fawcett, 1938, p. 120, quoted by Reid & Knipping).

SOME EPISTEMOLOGICAL ISSUES

The deep roots of the mathematical proof date back to one of the most famous texts that antiquity has handed down to us: the Euclid's Elements; the specificity of mathematical proof lies in the key features of the way used by Euclid for collecting and exposing mathematical knowledge. Such a way considered by Proclus excellent, provided a solution to a difficult problem.

> It is difficult, as in all science, to choose as much as to arrange in the order due to the elements from which everything else derives... (Heath, 1956 (I), pp. 115-116).

The problem was: how to "organize knowledge in the proper order". The mode of exposition refers to a "style of rationality" that some historians define precisely as "Euclidean" (Arsac, 2007). The nature of style stems from the fact that a deep unity connects organization and understanding, making the organization itself functional to the understanding of content, an understanding that is inextricably linked to the requirement of acceptability and recognition within a scientific community. The aspect that must be emphasized is precisely that of the dual purpose, of understanding on the one hand and of acceptability on the other, unanimously recognized as characteristic of a theoretical corpus (Hanna, 1989, pp. 21-22). That can explain why Euclid's Elements remained for centuries a reference book for those who intended to learn Geometry, and Mathematics in general. Though other possible approaches have been proposed to learn Geometry (Menghini, 2015), the logical-deductive mode introduced by Euclid has remained and reaches us as the paradigm of communication for mathematicians.

Despite the difficult change of perspective, which led to the radical revision of the idea of truth in mathematics, the relationship between understanding and acceptability regarding mathematical statements does not seem to have changed over the centuries. And it continues to be a characteristic element of this discipline. The development of increasingly complex relationships between two fundamental moments in the development of mathematical knowledge: the production of conjectures as the heart of knowledge production and the systematization of such knowledge, has led to a slow elaboration of the idea of rigor that at the end of the nineteenth century perhaps had a turning point (Lolli, 2004).

All this leads us to underline the profound continuity between the construction of knowledge and its systematization into a logic-deductive frame, between typical aspects of communication, such as that of being understandable, and typical aspects of knowledge as a cultural product, such as acceptability. The issue is broad, and it is not possible to address it in the space of this contribution. I have therefore chosen an aspect, on which perhaps not enough attention has been paid, but which I believe is of great interest for the close relationship with school practice: the relationship between proving and explaining.

DIFFERENT APPROACHES TO THE PROBLEM OF MATHEMATICAL PROOF

When the discussion about mathematical proof opens, we are generally faced with two possible positions, often seen in opposition, but which can be considered rather to refer to different and complementary perspectives. On the one hand, it is possible to take a descriptive approach, observing and classifying different answers that students provide after the request of justifying a specific statement. This is the case, for instance, of the seminal work by Harel and Sawder (1998); on the other hand, it is possible to take an approach that we can call cognitive-epistemological. In this latter approach, an a priori analysis is carried out, describing the different ways of determining the truth of a statement, comparing and characterizing them with respect to the cognitive processes involved.

Argue, prove, explain: continuity or rupture? is the title of a seminal article by Duval (1992). The author focused his attention on a crucial point: the difference between the semantic plane, where it is the epistemic value – the truth or falsehood, with all possible intermediate degrees – of a statement that is fundamental, and the theoretical plane where only validity is at stake, i.e., only the logical status of a statement in relation to the theory. Regardless of the truth values attributed to propositions and their combinations – in arguments or explanations – what matters is being an axiom or a theorem, that is, provable with respect to the theory that determines the value of truth and acceptability. The Duval analysis highlights in a very clear way the problem of cognitive distance between proof and argumentation, a distance that can explain many of the difficulties observed and at the same time brings to the centre of the didactic problem the relationship between hypothetical-deductive system and understanding, and in particular the explanatory function of the mathematical proof (Hanna, 1990).

There may be several possibilities. Focalizing on the break between the semantic plane, where it is possible to speak of the truth (epistemic value) of a given statement, and the theoretical plane where one can only speak of validity within a theoretical system, one must also distinguish the different points of view in terms of understanding. Semantically, understanding can refer to links between meanings and not necessarily to links of logical consequence. To focus on continuity, which is so evident in the processes of production of conjectures in which the logical connection between two propositions is unthinkable without referring to the meanings of the propositions at stake?

The fundamental problem therefore seems to be to solve the possible tension, sometimes the conflict, between the different functions that an argument can assume, explain and validate, and acquire a flexible thought that knows how to pass, casually and consciously, from the intuitive level, (that) of truth in terms of the meanings of the statements, to the formal level of validity, in terms of relations of logical dependence between the statements within a theory.

The practice of mathematicians is, in this respect, enlightening; every mathematician is convinced that he proves true theorems, but, at the same time, truth is understood in terms corresponding to a certain theory. This is a relative truth, stemming from the assumption of truth assumed for the axioms and the assumption that accepted rules of inference "transform truth into truth". The relativity of the meaning of truth, however, is fundamental: axioms are not "absolute truths", let alone factual truths.

> A statement B can be a theorem only in relation to some theory; it is nonsensical to say that it is a theorem (or a truth) in itself: even a proposition like '2+2=4' is a theorem in a theory A (e.g., some fragment of arithmetic). (Arzarello, 2000)

One does not spontaneously arrive/acquire to the control – that the expert is automatic and unconscious of the relativity of the meaning of truth; rather, it seems to us that first awareness and its 'atomism' constitute an achievement and, in this sense, a formative objective.

From the didactic point of view, it seems obvious that the persistence of confusion between the two points of view, without clarifying the relationship between the explanatory function and the validation function, has as a consequence the construction of a distorted conception of mathematical proof (Balacheff, 1999).

In this regard, the following answers given by some students to the open question seem significant[1]: "write everything you know about proofs in mathematics and their use".

> The proofs serve to explain more clearly, to make us understand the reason for some mathematical rules. In theory they should facilitate us, but sometimes they are more difficult than the rules! (Luc)
>
> I can't understand what mathematical proofs are for. They are too difficult to do and then you lose too much time for something that you already know at the beginning if it is true or false. (Lu)

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[1] The test was at the heart of an investigation whose main objective was that of shading light on students' conceptions regarding mathematical proof. The students involved attended the first year of the high school (15-16 years old). The names are abbreviated.

> Proofs in mathematics serve to better understand rules or concepts. They are used to explain and to see if we have understood the rules studied. (Ce)
>
> Mathematical proofs have accompanied us since elementary school, but their use is always given almost exclusively in homework. Perhaps such mathematical proofs serve to open the brain to various situations in life, in various decisions. (Ric).
>
> [Proof is] what professors do to explain things that are always simple. (Nic)

In the statements reported above, we find recurrent reference to the function of explanation; we also find considerations regarding the effectiveness unless the proofs with respect to this function, up to the paradox expressed by Nicole, for which the proofs are *what the professors do to explain things that are always simple.* What the students write links mathematical proof and explanation, but some of the proposed distinctions suggest that the relationship between an explanation and a mathematical proof is not completely clear; this leads us to try to better understand what lies behind an explanation especially in reference to what happens in school practice. I will start by briefly introducing the argumentation and proof and the complex relation between them, then I will focus on explanation, formulating a characterization that can allow us to discuss its relationship to both argumentation and proof.

ARGUMENTATION AND PROOF

Contiguity and differences in comparing argumentation and proof has been widely discussed (Mariotti 2006; Reid & Knipping, 2010). According to Duval (2002) difficulties originate precisely in the differences between the justification processes characteristic of mathematics and those typical of other domains.

> Proving processes in mathematics are quite unlike those of the other fields. […] there is a gap between the discursive ways of using arguments and using theorems. (Duval, 2002, p. 3)

At the core there is the distinction between the epistemic value[2] of a statement that in the case an argumentation is based on the interpretation of the arguments provided to support it in, and in the case of a mathematical proof is based on the logic status of the arguments provided to support it. Thus, the control differs in the two cases: the semantic control, for an argumentation and the theoretical control, for a math proof. Nevertheless, this distinction is totally artificial. In any case, the statement and the discourse accompanying it will be interpreted and though in principle validation occurs at the theoretical level, meanings involved will be taken into account. If for the expert such a distinction, between the semantic and the theoretical level, can remain almost unconscious but perfectly

[2] Epistemic value is the degree of reliability of what is stated in the proposition. In the very moment of his apprehension, the content of a proposition appears obvious, or certain or only plausible, or plausible, or simply possible, or impossible, or absurd... [...] Naturally, the same proposition does not necessarily have the same epistemic value for two different people (translated by the author). (Duval, 1995, pp. 218-219)

under control, this is not the case for the students for whom it may become difficult to be grasped.

In the articulation between these two levels, the semantic and the theoretical, we can situate mathematical explanations and their use in school practice; before entering the discussion about the relationship between these three discourse modes, let us clarify what explanation can mean for us in this context.

EXPLAINING, WHAT DOES IT MEAN?

Openly in contrast with the principle that students must be responsible for their learning, explanations remain a persistent didactic method, which any teacher would hardly abdicate, mainly with young students (Leinhardt, 2001). Though often unclear what explanations consist of, teachers seem to acknowledge that explanations have an intrinsic effectiveness, hardly questioned. In school practice, mutual roles and expectations are usually well defined, and rarely admit derogations: *the teacher explains, and the students listen,* while in case of failure, reasons are always to be ascribed to students' responsibility, for instance lack of attention.

If the theme of proof has long been at the centre of attention of many studies in mathematics teaching, the same cannot be said for explanation, despite the fact that explaining is a widespread practice in our classes.

Studies concerning explanation at school, or as some author call it *instructional explanations*, focused on the effect of this method and findings seem not to imply that instructional explanations should generally be regarded as being ineffective, rather show their effectiveness greatly depends on the design and quality of their implementation (Lachner & Nückles, 2015; Wittwer & Renkl, 2008), highlighting some possible elements concerning different elements involved: as teachers naturally resort to their own knowledge when generating instructional explanations (Nückles et al., 2005), focus has almost been on the teacher, thus teachers' Content (mathematical) Knowledge, but also, to great extent, Pedagogical Content Knowledge (Baumert et al., 2010) showed their relevance. Nevertheless, less attention has been devoted to study explanation in itself and with respect to its functioning; in this contribution, I will focus on it with the aim of clarify its relationship with other instructional discourse modes belonging to mathematics class practices, e.g., argumentation and mathematical proof. As commonly shared, we will consider explanation's main goal as that of supporting students' understanding of a specific subject-matter (Wittwer & Renkl, 2008). Specifically, among the few available on this theme, we take the characterization of characterization offered by Mopondi:

> Explanation is a tool used by an interlocutor to make an object of communication understand, or make sense of, [...] the role of an explanation is to clarify the meaning of an object (a method, a term, a task, ...) (Mopondi, 1995, p. 12, my translation).

The explanation has as *its objective to make people understand*, to make sense of one of its objects, which in some cases can be expressed by a statement, though not always.

This characterization of an explanation in terms of its goal: with respect to its object and an interlocutor, allows us to compare it with other types of discourse that involve an interlocutor and the intention to change his/her relationship with an object; specifically, with an argument whose goal is to *convince someone of the truth of the object* and with a mathematical proof, whose goal is to *make someone accept the object within a theory*.

The goal of explanation appearing in the definition formulated above, refers to understanding, but though often used in the field of education this term needs some clarification. Sierpinska (1994, p. 28) proposes a clear and articulated interpretation of what it usually meant with term: *understanding* is a mental process that relate something new, to what is known to the subject and on which understanding is based process; such a process combines *acts of* understanding, or operations that relate to the *object* of understanding, the *basis* of understanding (i.e., representations, mental models, beliefs, personal opinions, etc.). The acts of understanding are of different type and nature: for example, common acts of understanding are generalizing as well as synthesizing (ibid., p. 60). But, as we will see in the following analogy is one of those that are frequently used, together with deductive inferences.

The unity of a process of understanding is determined by the close relationship between the objects of understanding of the individual acts of which it is composed. The links between acts of understanding can be of various kinds among these obviously there may be deductive inferences, though not exclusively (ibid., p. 73).

The definition of understanding as a mental operation that allows the subject to connect the object to his own knowledge, clarifies how understanding is a completely *personal and private* phenomenon, and makes it reasonable to ask to what extent one can make someone understand something.

The analysis of the act of understanding shows us which elements to consider: certainly, on all or some of the mental operations necessary to carry out the act of understanding, but above all *on the system of knowledge of one's interlocutor*, those who will have to be activated to become the base of understanding necessary to found the individual acts and the chains of acts of understanding.

For this reason, the action of a subject who seeks to promote the understanding of an interlocutor must be oriented and tuned to the latter. The person who intends to make someone understand will have to take into account what may be the bases of understanding available to his interlocutor and on which mental acts can be based that lead him to understand the object at stake.

The discursive nature of an explanation and the contiguity between the structure and the forms of the discourse of argumentation, makes explanations interesting to be investigated and in relations to school practice leads us to the question: to what extent a mathematical argument and specifically, a mathematical proof can also make one understand what is proved.

WHEN CAN AN ARGUMENT EXPLAIN?

Though introduced in a very synthetic way, the definition of understanding given above can shade light on the complexity and problematic nature of the relationship between an explanation and an argument, and in particular between an explanation and a mathematical proof of which some authors speak (Hanna, 1989); in fact, two opposite situations are possible regarding the same argument or the same mathematical proof.

On the one hand, it is possible that an argument, and in particular a proof, *can take on an explanatory function*. This will depend on whether the arguments used have a direct correspondence with the base of understanding, that is with the system of conceptions, models, beliefs that the subject has available for interpreting them. In the specific case of a mathematical proof, it can have an explanatory function only if the theoretical elements (axioms, theorems or definitions) used in the deduction, are available as part of the base of understanding; in particular, in the case of a mathematical proof, it is also necessary that the modes of inference used offer the necessary support to identify the connections between the object of understanding, and the base of understanding.

On the other hand, it is possible that a certain argument or mathematical proof, even if considered explanatory by the person (for instance the teacher) presenting it, *can be judged non-explanatory by the interloc*utor. In particular, such a misinterpretation can occur when a teacher or a textbook present a mathematical proof with the intention of providing not only a validation of the theoretical acceptability but also an explanation of a statement; in that case, it can happen that a student who listens at teacher in class, or reads the text on the book, does not perceive the discourse as an explanation, because s/he cannot grasp the necessary connections between the object of the explanation and his own knowledge. In the following section we present an example.

Is this proof explanatory?

Consider the following statement "If two numbers are divisible by 3 then their sum is also divisible by 3".

In the following a possible mathematical proof is reported and analyzed.

Proof

1. if a is divisible by 3 then there exists k such that $a = 3k$
2. if b is divisible by 3 then there exists h such that $b = 3h$

3. the sum $a + b$ will then be equal to $3k + 3k$, from which collecting the factor 3 we get
4. $a + b = 3(k + h)$ then
5. $a + b$ is divisible by 3 by definition.

The proof has as its key element in the following definition that we formulate in its general version belonging to number theory:

x is divisible by y if and only if there exists k such that $x = k \cdot y$

However, this way of characterizing 'divisibility' is not the current way students have in mind: students usually consider that

one number is divisible by another if performing the division, one gets remainder 0.

If we assume that student's knowledge available for interpreting the proof corresponds to such a statement, it will be difficult for the proposed proof to be explicative; although the individual passages might be acceptable, there is a gap (instead of a link) between the available knowledge and the arguments. The occurrence of the word 'divisible' evokes the *fact of dividing and obtaining as remainder 0*, instead of the fact that *there exists k such that $a = b \cdot k$*. The common mode of conceiving divisibility does not fit with the mode of expressing the divisibility in the theory within which this proof is developed and is acceptable as a validation of the statement.

But there is something more, this proof is also delicate for its logical structure, for the logic role that a definition plays in the construction of the guarantee that links one step and the next of the argumentation.

Each definition has the logical structure of a double implication, in the sense that premises and conclusion can exchange their logic roles: one can be a consequence of the other. What happens in this proof is the following: in the first step, being divisible by 3 has the status of premise from which the consequence is drawn to obtain "there exists k such that $a = 3k$" (step 1); in the last step, the same property, being divisible by 3, appears as a conclusion derived from the property obtained in step 4: "$a+b = 3(k+h)$".

This example highlights how the process of understanding can depend on a possible gap between student's conception and corresponding formalization; in this specific case such a gap will be overcome only after the transition from a procedural knowledge of divisibility linked to performing an operation and obtaining zero rest, to a relational knowledge (Skemp, 1971; Sfard, 1991), which expresses divisibility through a multiplicative relationship between a number and its divisor. Until divisibility activates this a procedural knowledge in the reader, s/he will not be able to give meaning to the key arguments of which the proof is constituted, that is, the necessary connection between the object of the explanation and the base of understanding is missing, and no understanding will be possible.

This example can be considered a prototype with respect to a broad category of situations in which it may happen that there is a discrepancy between the system of available knowledge and the guarantees used in the argumentation, when it occurs that the combination of arguments fails to create the link between available knowledge and object of understanding.

What is formalized in the theory and what is known do not fit and that originates a break between explaining and proving, e.g., validating with respect to the theory.

WHEN THE OBJECT OF AN EXPLANATION BECOMES AMBIGUOUS

Let us now consider another situation, which we can consider opposite with respect to the previous one. Consider an *explanatory text*, that is a text aimed at making some mathematical knowledge understandable, which leads to the loss of mathematical meaning.

The following example shows a case of a text in which the explanatory objective of the text is intertwined with the objective of convincing the reader of both the truth and the mathematical validity of what is asserted, without clearly outlining the boundaries of these three different practices with respect to their respective purposes. The object of such an explanatory text is making the 'rule for adding monomial' understandable.

Consider the following passage, taken from a textbook for the first year of secondary school; the text has been divided in two numbered parts for the convenience of the analysis.

1. If in a basket there are 3 apples and we add another 2 apples at the end in the basket we will have 5 apples; if you have 2 notebooks in your backpack and we add another 4 notebooks, in the end you will have 6 notebooks; but if to the 3 apples in the basket we add 4 oranges, we will always have 3 apples and 4 oranges.
2. Therefore, if we have to add $3x$ and $2x$, we can say that the sum is $5x$; but if it should add $3x$ and $4a$, in the end we will still have $3x$ and $4a$.

Moving from the first segment of the text to the second segment we can clearly distinguish a shift between two semantic fields: in the first paragraph, the terms belong to the semantic field of everyday experience – baskets, apples, and notebooks; in the textual segment coinciding with the second paragraph, the terms refer to the semantic field of mathematics, in particular algebra. The presence of a shift of speech from one semantic field to another, poses a problem of understanding and requires the construction of a link between the respective meanings. In the text, the link seems to be ensured by the adverb 'Therefore' that binds the two discourses and that implicitly, introduces the analogy between the description of the situation referred to the real context, and its 'mathematization' referred to the algebraic context. Actually, the analogy, like the metaphor, constitutes an operation among those often involved the process of understanding, and for this commonly used. However, in this case analogy has also the specific objective of supporting the validity of a mathematical statement concerning an algebraic property. It is important, therefore, to question whether the analogy evoked in the text is mathematically acceptable.

The transition by analogy, from the field of everyday experience to the mathematical field of algebra, involves the consequent transition of the criteria of validity for the arguments: the acceptability criteria running in the source field are automatically transferred, 'by analogy', in the new field. However, such an automatic transfer from one field to another not always succeeds, and the transition would require a careful control. In the specific case, the analogy that is established transforms the argument developed in the field of everyday experience, and acceptable according to "common sense", into an argument that, in the field of algebra, is totally meaningless. The way of representing the sum of two monomials, for instance $3x$ and $2x$, into the algebraic expression $3x+2x$, has the algebraic meaning of the *sum between two products*, given that the writing '$3x$', conventionally leaves implicit the multiplication symbol and stands for the product $3 \cdot x$ and similarly, $2x$ stands for $2 \cdot x$; thus, the algebraic interpretations of the symbolic (and conventional) writing '$3x+2x$' is incompatible with the proposed analogy.

In algebra, letters represent variables for which it makes sense interpreting the juxtaposition of the symbols 3 and x, as a multiplication between a number and a variable; on the contrary, in the proposed analogy, the interpretation of the expression $3a+2a$ as "3 apples added to 2 apples", does not make sense: in this case, the letters play the role of abbreviations of the names of the counted objects. At the same time, a multiplicative interpretation does not make sense.

The mathematically correct explanation, which is also a proof, can refer directly to the distributive property of multiplication with respect to the sum:

$3x + 2x = (3 + 2)x$ [for distributive property]

$(3 + 2)x = 5x$ [calculation of the sum]

$3x + 2x = 5x$ [prop. Transitive of equality]

All this makes the explanation provided by the text not only mathematically incorrect, but also misleading with respect to a correct use of the analogy. The correspondence between the mathematical meanings and meanings emerging form everyday experience is not made explicit; moreover, the apparent simplicity of the arguments used, produces an immediate acceptance. All that invites students to an uncontrolled use of analogy and reduces its value with respect to mathematical thinking.

As a matter of fact, besides its persuasion strength, the use of analogy has a strong argumentative power: arguments by analogy generates a high degree of acceptability: precisely, it generates immediate adhesion by virtue of transferring in a new and almost unknow field, the strength of a sound argument produced in familiar a field.

Moreover, and above all, it seems to us that it can be considered a missed opportunity to propose, or perhaps let the students find, a sensible justification that at the same time consolidates important algebraic meanings, such as that of variable, that of the properties of operations and the role that these properties can have in defining what symbolic manipulation consists of (Kieran, 1992).

The example just discussed, can be considered paradigmatic of the risks that the use of analogy and similarly that of metaphor, can hide. From the didactic point of view, it is appropriate to become aware and clarify not only the differences but above all the contiguities between mathematical proofs and explanations, and in particular it is required to analyze in a fine way all those situations where the proximity between explanatory texts, argumentative texts and mathematical proofs raises the risk of confusion, because this confusion hinders not so much the correct understanding of the content that is intended to be explained, but the functioning of the different modalities, in particular the correct development of the mathematical sense of the argument.

CONCLUSIONS

The crucial role that argumentation and proof have in mathematical practice, has as implication for teaching practice: above all, the need to recognize the importance of placing the development of argumentative skills among the primary and transversal objectives of Mathematics education; at the same time, the discussion presented has clearly highlighted that this didactic objective must be harmonized with the need to promote understanding and therefore to develop the mathematical sense of what is proposed in class.

In the restricted limits of this contribution I tried to focus attention only on a very particular aspect of the didactic problem of developing argumentative skills in the Mathematical field, that asks to effectively intertwine in a meaningful net three fundamental components of mathematical thought: producing *arguments*, whose goal is to convince of the truth pf a statement, producing *proofs,* whose goal is to establish its validity within a theory, and *explaining* that is making it understandable for the interlocutor.

All this corresponds to better articulate the primary objective – to develop argumentative competencies in the Mathematical field – highlighting two interrelated dimensions. The first consists in developing in the classroom the "culture of why?", that is, to make 'natural' the *habit of looking for a reason* for each statement that happen to emerge in the mathematical field. The second dimension consists in developing the awareness of the different nature of the possible reasons coming as an answer to the question “Why?”. Reasons that can be produce for explaining, that is for favouring one’s own understanding, reasons that can be produces for proving that is for supporting the acceptability of a statement with respect to Mathematics, and in particular to its theoretical organization. Developing such a complex competence is a demanding task that requires the mindful mediation of the teacher.

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PRIMARY TEACHERS' PEDAGOGICAL DESIGN CAPACITY FOR A SMOOTH MATHEMATICAL TRANSITION FROM PRIMARY TO SECONDARY EDUCATION

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In this article we study two primary school teachers during their interaction with textbook mathematical tasks of the first grade of secondary school in order to investigate their hypothetical didactical use in the last grade of primary school. We explore aspects of their Pedagogical Design Capacity and their beliefs regarding issues of mathematical transition. The results of the study have shown different types of the teachers' interaction with the mathematical tasks, which are justified by their different beliefs on mathematical transition.

MATHEMATICAL TRANSITION FROM PRIMARY TO SECONDARY EDUCATION: THE CASE OF GREECE

The transition in mathematics from primary to secondary education is a topic that has occupied the field of mathematics education in various ways. In the last twenty years, there have been studies that focus on the continuity of the mathematics curriculum (Nicolescu & Petrescu, 2015), on the perceptions and experiences during the transitional period in the mathematics of teachers and students (Attard, 2010), on the knowledge of teachers regarding issues of transition (O'Meara et al., 2020), as well as on the cooperation of teachers of the two educational levels (Soto et al., 2020).

The results of the above studies indicate that there is usually a discontinuity between the mathematics curriculum of the two educational levels (Nicolescu & Petrescu, 2015) and a lack of communication and cooperation between the teachers of the two levels (O'Meara et al., 2020). In addition, students experience a decline in their math performance and motivation (Athanasiou & Philippou, 2006). Attard (2010) maintains that the teacher-student relationship and the differences between the two educational levels (such as the mathematical content, assessment techniques, teaching practices, students' workload and technology integration) influence the students' engagement with mathematics in this period.

However, there is a lack of research focusing on primary school mathematics teachers on the issues of the transitionin mathematics to secondary education. In this paper, we study two Greek primary school teachers in their attempt to modify mathematical tasks from the textbook of the first grade of secondary school for a hypothetical use in the last grade of primary school.

In Greece, the transition in mathematics between the two levels takes place between the 6th and 7th grades, when the students are between the ages of eleven and twelve. Mathematics curriculum, textbooks and teachers' guidebooks are uniform throughout the country, and they are published and approved by the Ministry of Education. In Greek primary schools, mathematics is taught by teachers who possess a degree of education departments. These departments contain courses related to pedagogy, didactics of the courses which are taught in primary education and psychology. In some cases, primary school teachers experience difficulties in the mathematical content they teach, which is further enhanced by the small number of courses in mathematics and mathematics education in the departments of education. Furthermore, in Greek secondary schools, mathematics is taught by teachers who have graduated from mathematics departments, where it is quite possible that they have not attended almost any course related to mathematics education. In this paper, we only focus on primary school teachers and especially on their relationship with the curriculum resources (particularly the tasks), which come from the chapter of the equations in the textbook of the 7th grade (1st grade of Greek high school), as a tool of primary teachers' awareness of issues of transition in mathematics between the two educational levels.

PEDAGOGICAL DESIGN CAPACITY (PDC) OF A MATHEMATICS TEACHER

Brown (2009) perceives teaching as a designing process, which includes both the lesson preparation stage and the stage of practice in the classroom. According to Brown (2002), Pedagogical Design Capacity (PDC) is "teachers' capacity to perceive and mobilize existing resources in order to craft instructional contexts" (p. 70). PDC "describes the manner and degree to which teachers create deliberate, productive designs that help accomplish their instructional goals" (Brown, 2009, p. 29). Brown (2002) developed the framework of Design Capacity for Enactment (DCE) for understanding the interaction between teachers and curriculum resources. Teachers' resources include the teachers' subject matter knowledge, goals, beliefs and pedagogical content knowledge, while curriculum resources contain physical objects, domain representations and procedures. The interaction among teachers and resources is a dynamic process, in which different PDC can be justified by the fact that teachers with similar knowledge and skills interact and use curriculum resources in a different way (Brown, 2009). There are three types of interactions. The *offloading* (when the teacher is based significantly and without pedagogical changes in curriculum materials), the *adapting* (when the teacher shows less dependence on curriculum materials during the design and implementation of teaching) and the *improvising* (when the teacher takes the responsibility for the teaching design and implementation by creating tasks or pedagogical steps without reliance on curriculum materials) (Brown, 2009).

In this paper, we adopt Brown's (2009) approach, considering that the process of design is a crucial and dynamic element of an effective teaching, and it embodies teachers' knowledge, beliefs and skills. According to the literature on the transition in mathematics from primary to secondary education, we found that there is a lack of research focusing on the role of the teaching design during the transition period. Furthermore, the approach to the issue of transition from the perspective of the teaching design of primary education teachers is an issue that has not been sufficiently analyzed in the field of mathematics education. More specifically, we cooperated with two primary mathematics teachers in order to explore their PDC during their lesson preparation for the 6th grade, by using/modifying mathematical tasks from the textbook of the 7th grade. The research questions are:

1. In which ways the two teachers interact with the secondary mathematics tasks at the stage of their preparation of teaching?
2. How these teachers' beliefs about the transition in mathematics from primary to secondary level affect their teaching design?

METHOD

Context of the study and Participants

The present study is a case study (Yin, 1994) and involved two primary school teachers of mathematics, whose pseudonyms are John and Mary. We asked them to use and modify (if they deemed it necessary) seven mathematical tasks from the textbook of 7th grade (from the chapter of equations) (Vandoulakis et al., 2012) as part of their preparation for a hypothetical teaching in 6th grade for a smooth transition in mathematics. We chose tasks from the chapter of equations in the 7th grade because there is a corresponding chapter in the 6th grade. The selection of these mathematics tasks was based on the similarities and differences that they could present in relation to the tasks of the 6th grade, based on the criteria ofthe mathematical solvability, as well as their linguistic and mathematical complexity (Silver & Cai, 1996). The research took place at the end of the school year, in order for the teaching of equations to be completed.

The first primary teacher was John. John was a 6th grade teacher at the time of the study. He had more than 25 years of teaching experience and in the past, he was a 6th grade teacher for five years. The second primary teacher was Mary, with over 13 years of teaching experience and a master's degree in mathematics education. Mary was teaching the 1st grade at the time of the study, but three years ago she was a 6th grade teacher. In total, she was a 6th grade teacher for three years in the past. We chose these two teachers because they both had a lot of teaching experience. Additionally, they displayed interesting differences, such as the fact that Mary held a master's degree in mathematics education and that John had more teaching experience in the 6th grade.

Data Collection

The data was collected from a questionnaire and a semi-structured interview. The questionnaire contained seven mathematical tasks from the 7th grade mathematics textbook (Vandoulakis et al., 2012). The criteria for selecting these tasks were explained above. The main question of the questionnaire was to adapt the mathematical tasks, if deemed necessary, in order to integrate them didactically in the 6th grade for a smooth transition to mathematics between the two educational levels. The mathematical tasks provided were the following (Vandoulakis et al., 2012):

Task 1: The side of a square is a. What is its perimeter and how much is its area? (p. 74)

Task 2: Write the mathematical expressions in a simpler way: (i) $x+x$, (ii) $a + a + a$, (iii) $3 \cdot a + 52 \cdot a$, (iv) $2 \cdot b + b + 3 \cdot a + 2 \cdot a$, (v) $4 \cdot x + 8 \cdot x - 3 \cdot x$, (vi) $7 \cdot w + 4 \cdot w - 10 \cdot w$ (p. 74)

Task 3: If $x \cdot y = \frac{2}{9}$ and $z = \frac{3}{5}$, find $x \cdot (y \cdot z)$. (p. 74)

Task 4: The difference in the age of the daughter from her mother is 25 years. If the daughter is 18 years old, how old is the mother? (p. 78)

Task 5: Christina spent half of her money to buy 2 notebooks and markers. If it is known that each notebook costs 1€ and all markers 3€, what is the amount of money that Christina had before these purchases? (p. 76)

Task 6: A father is four times the age of his son. The two ages together add up to half a century. How old is each one? (p. 78)

Task 7: A worker for a five-day job agreed to get half of his pay in advance and the rest to be paid when the job was done. If the payment in advance was 180€, what was his daily wage? (p. 77)

The semi-structured interview was divided into three parts. The first part contained questions concerning the beliefs and knowledge of teachers on issues of transition in mathematics. The second part concerned the teaching and learning of algebra between the two educational levels. The questions of the first two parts were based on specific categories of mathematical transition (mathematical content, assessments techniques, teaching practices, students' workload and technology integration), as they emerged in Attard's (2010) study. The third part of the interview focused on the changes made in the tasks and their justification by the teachers.

The data collection was realised in two phases. In the first phase, we asked the teachers to complete the questionnaire and in the second phase, the semi-structured interview took place. The aim of the interview was to explore the teachers' resources (according to DCE) regarding the issue of mathematical transition, the teachers' justifications of the modifications of the tasks and the teachers' explanations of their didactic use in the classroom. The first author was the interviewer.

Data Analysis

For the analysis of the data, we relied on the theoretical framework of DCE (Brown, 2009) and the concept of PDC (Brown, 2002). We firstly decoded the teachers' interviews and then identified their beliefs, goals and knowledge about teaching and learning mathematics during the transition period. Subsequently, we identified broader categories of beliefs regarding specific issues of transition (i.e., beliefs about the role of curricula in the transition, beliefs about the role of mathematical tasks in the textbooks of the 6th and the 7th grade).

In the next phase, we studied the task changes made by the teachers, and tried to identify the reasons for these changes from the third part of the interview, and the way of their hypothetical use in the classroom. In this phase we examined whether, which and how the above categories of beliefs are related to the task changes and their hypothetical use in practice.

Finally, we synthesized all data in order to identify how teachers interact with resources. The combination of data regarding teachers' beliefs, their knowledge, their profile and their interaction with the curriculum resources of the 7th grade contribute to the exploration of their PDC.

RESULTS

Firstly, according to our data analysis, both teachers seemed to be concerned with issues of mathematical transition. Both teachers agreed that in Greece the transition in mathematics from primary to secondary education is not completely smooth and that the transitional period covers mainly the 5th and 6th grade from primary school and the 1st grade from secondary school.

We observe that the two teachers displayed a different profile. John was more experienced, both as a teacher and as a 6th grade teacher. In addition, he was teaching at the 6th grade at the time of the study. John considered that the "teacher should build bridges between the two levels" (belief about the role of teacher in the transition period), that is why he used 7th grade mathematical tasks in his teaching. On the other hand, Mary had less teaching experience in the 6th grade, but she showed a greater academic depth in teaching mathematics, probably because of her master's degree.

Task/ Teachers	John	Mary
Task 1	Offloading	Offloading
Task 2	Adapting	Offloading
Task 3	Adapting	Adapting
Task 4	Adapting	Improvising

Task 5	Offloading	Offloading
Task 6	Offloading	Adapting
Task 7	Offloading	Adapting

Table 1: Interaction between primary teachers and 7th grade tasks.

Table 1 shows the ways of interaction that occurred between the 7th grade mathematics tasks and each teacher. We notice that in some tasks the way of interaction is the same (Tasks 1, 3, 5), while in other tasks (2, 4, 6, 7) Mary showed a smaller degree of dependence on these. These differences could be justified by the different approach/beliefs they expressed concerning mathematical transition.

More specifically, John considered that "the students of the 6th grade could respond satisfactorily in the cognitive demands of 7th grade tasks" (belief about the demands of tasks between the two levels). Mary believed that "the tasks in the 6th grade should be linguistically and cognitively more accessible to the primary students in relation to the corresponding ones in the 7th grade" (belief about the demands of tasks between the two levels). Consequently, in some tasks she changed the language and lowered their mathematical complexity. Therefore, we may claim that she created her own pedagogical steps in her hypothetical use of the tasks in a classroom. Maybe that is why she mainly interacted on a greater degree of independence with the tasks than John, who believed that no extensive changes are required.

The case of John

In the interview, John explained that "the curricula as well as the mathematics textbook of the 7th grade should be adapted more to the goals of the 6th grade of primary education" (belief about the role of curricula and mathematics textbook in the transition period). Therefore, we expected that John would show a greater independence in his planning with the given tasks. Actually, we noticed that he designed his lesson as a mixture of 'offloading' and 'adapting' of the given tasks.

John believed that "the equation in secondary education is used more as a mathematical tool, while in primary education it should arise through a verbal problem" (belief about the concept of equation in the two educational levels). In more detail, he emphasized that "I want to go through the textual data to the formation of the equation with the use of a variable, so I focus more on the symbolism than on the solution of the equation" (belief about the didactical approach of equation in primary education). Tasks 5, 6 and 7 were word problems, which are solved with the help of the equation. For this reason, John made almost no modifications, therefore his interaction with the tasks can be categorized as 'offloading'. Furthermore, the fact that he relied closely on the given mathematical tasks of the textbook of the 7th grade is justified by his belief that "there should be some exactly the same tasks in the 6th and 7th grade mathematics textbooks, as a way of a smooth mathematical transition (belief about the role of mathematics tasks in textbook for a smooth transition).

The case of Mary

The analysis of Mary's interaction with the given tasks led us to no particular type of interaction. Mary believed that "the mathematical tasks in the 6th grade should contain an easier linguistic background than the 7th grade mathematics tasks and also the 6th grade tasks should offer questions that will gradually help the student to solve them" (beliefs about the demands of tasks between the two educational levels). As a result, she often adjusted the given tasks with appropriate changes to make them less demanding. Hence, the interaction of Mary with the 7th grade mathematical tasks could be characterized, in some cases, as 'adapting' or 'improvising'.

For example, when Mary analyzed her planning for the hypothetical lesson, she said that she wanted to "adapt the mathematics task to the primary students' pre-existing knowledge and experiences" (belief about the didactical approach of tasks in primary education). Thus, in several mathematical tasks (tasks 3, 6 and 7) she modified them linguistically and lowered their mathematical complexity. In one case the changes were so many, that in fact she modified her original goals and she said that in the class she would approach the task differently than she should in the 7th grade. In other words, she interacted with the task with a high degree of independence ('improvising'). This was Task 4:

> The difference in age of the daughter from her mother is 25 years. If the daughter is 18 years old, how old is the mother? (Vandoulakis et al., 2012, p. 78)

Mary suggested three replacements of the word "difference". For instance, she suggested "if the daughter is 25 years younger or when the mother gave birth to the daughter, she was 25". What is worth noting is that during the interview Mary said that she would motivate her students to solve it with simple operations and not with the use of equations. In other words, she significantly changed the original goal of the task. She claimed that "the equations are one of the most difficult concepts in the 6th grade, because students cannot understand what a variable is, and that one part of the equation is equal to the other" (belief about the concept of equation in primary education). Therefore, it is implied that there is a discontinuity in terms of cognitive requirements and students' pre-existing experience with equations based on the mathematics tasks between the 6th and 7th grades' mathematics textbooks.

DISCUSSION AND CONCLUSION

The present study is part of a wider study on the mathematical transition from primary to secondary education in Greece. The purpose of this paper was to explore aspects of the Pedagogical Design Capacity of two primary school teachers when they interact with mathematical tasks of the textbook of the first grade of high school with the aim of their hypothetical teaching use in the last grade of primary school. Specifically, we focused on the ways of interaction ('offloading', 'adapting' and 'improvising') (Brown, 2009) between teachers and tasks and in the way their beliefs on mathematics transition affect these interactions.

The results of this study suggest that there was no dominant way of these teachers' interactions with mathematical tasks from the 7th grade textbook. This is illustrated by the fact that the teachers' different beliefs on aspects of mathematical transition between the two educational levels also had different impact on their actions on the tasks. The first teacher (John) seemed to support his teaching design with a greater adherence to the 7th grade mathematical tasks, because he believed that students can cognitively succeed in them. The second teacher (Mary) considered that the 7th grade tasks were quite demanding for the students of the 6th grade, thus she demonstrated a greater degree of independence from the tasks.

Furthermore, our study has showed the importance of planning for the mathematical transition as well as the complexity of these in Greece. Issues such as the role of the mathematics curriculum, textbooks, and word problems between primary and secondary education, as well as the demanding nature of the concept of equation in the Greek educational system were also showcased in our study. These issues are aligned with the findings of other studies about issues of mathematics' transition period (Attard, 2010; Katsomitros, 2021; Nicolescu & Petrescu, 2015). Further research is deemed necessary. It would be important to study the implementation of these tasks in the classroom, in order to observe whether there would be any other modifications of these in the classroom and whether teachers would continue to interact with them in the same way.

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A STUDY ON THE USE OF MATHEMATICAL SENSES AND CRITICAL THINKING OF STUDENT TEACHERS

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This work explores, by means of a two-phase qualitative research study, the use of spatial, measurement and number senses of 67 student teachers. The first phase consisted of an instruction of the participants on the mathematical senses from a teaching and learning perspective. In the second phase they were asked to solve a task whose resolution required the use of different components of the mathematical senses. We analyse their responses and errors in their written productions from the second phase. Our findings reveal a low reading comprehension leading to task misinterpretations and a poor use of measurement and spatial sense, producing multiple mistakes on data handling and geometric representations. Conclusions are drawn about the use of critical thinking.

INTRODUCTION

For several decades, international institutions and commissions of programs in Mathematics Education (e.g., National Council of Teachers in Mathematics [NCTM], Organization for the Economic Co-operation and Development [OECD], Programme of International Students Assessment [PISA], etc.) have joined forces to reflect on the notion of mathematical competence. Different procedures and instruments have been designed and used to assess the mathematical competence of students and to promote its development and improvement from the elementary grades. In the framework of PISA, *mathematical literacy* is defined as

> An individual's capacity to formulate, employ and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgements and decisions needed by constructive, engaged and reflective citizens. (OECD, 2018, p. 77)

This conception of mathematical literacy makes it closely related to critical thinking, being this associated with the dimensions of information (as a source and as product), effective communication and social responsibility (Ananiadou, & Claro, 2009).

The design of the elementary school mathematics curriculum, the planning of the teaching and learning processes (Rico, 2016) and the training of prospective schoolteachers turn out crucial factors of the objective of enhancing students' mathematical competence and critical thinking. Linked to these notions and being an operational concept to address them is what other authors (e.g., Flores & Rico, 2015) call *mathematical sense*. Ruiz Hidalgo et al. (2019) perceive mathematical sense as the set of capabilities related to the mastery in context of numerical, geometric, metric and statistical contents, which allow to use these contents in a functional way. This concept encompasses four interrelated mathematical senses: number sense (Castro & Segovia, 2015; Dantzig, 1954; Sowder, 1992), measurement sense, geometric and spatial sense (Clements & Battista, 1992; Flores et al., 2015), and stochastic sense (Batanero et al., 2013; Ruiz Hidalgo & Serrano, 2015; Watson, 2006).

Multiple studies (Clements & Stephan, 2004; Kamii & Kysh, 2006; Lehrer, 2003; Tan Sisman & Aksu, 2016) report the difficulties that some primary school pupils find and experience when dealing with certain mathematical tasks and their poor use of their mathematical senses. Interestingly enough, other works (Baturo & Nason, 1996; Ryan & McCrae, 2005; Tierney, Boyd, & Davis, 1990) reveal that these difficulties persist in later educational levels students and even in student teachers. Since teachers are one of the key bodies of the educational system, it is of great interest to investigate these errors and misconceptions in student teachers, in order to know more about their nature and to consider appropriate actions in this regard.

OBJECTIVES

In the context of the task presented in the Method section, the main objectives of this qualitative research are:

O1. To explore the mathematical errors and misconceptions of student teachers when addressing a paper-based task involving geometric, measurement and number senses; and

O2. To test the value of that task to promote the use of components of the geometric, measurement and number senses in student teachers.

METHOD

We conducted a qualitative and descriptive research based on a two-phase classroom intervention.

Participants

Sixty-seven second year students of the Degree in Primary Education of a public Spanish University participated in the study. They were a class group undertaking their degree under a bilingual (English-Spanish) modality. This means that they were taught in English and had class materials in this language, using Spanish only in specific cases. At the time of the study, it was assumed that they possessed, in general, the mathematical knowledge and skills established in the corresponding Spanish secondary education and high school curriculum (Ministerio de Educación, Cultura y Deporte, 2015) as well as in the syllabus of the subject called “Mathematical Bases in Primary Education” (University of Granada, 2021) of the first year of their university studies.

Implementation

In the first phase of the study, we conducted six interactive instructional sessions of 120 minutes each and six practical interactive sessions of 60 minutes each. This was carried out along 10 weeks during the academic year 2020-2021. Throughout the instructional sessions the spatial, measurement and number senses were thoroughly introduced to the participants, exemplifying their different components (Castro & Segovia, 2015; Flores, Ramírez, & del Río, 2015; Moreno, Gil, & Montoro, 2015) through a wide variety of situations posed in intra-mathematical and extra-mathematical contexts.

During the practical sessions, participants were asked to solve and analyse from a didactic point of view diverse tasks related to the use of spatial, measurement and number senses with the guidance of the researcher. These tasks allowed them to: on one hand, put in practice their own mathematical senses; and, on the other hand, to reflect on the design of mathematical tasks for elementary students based on the mathematical senses demanded in their resolution. Figure 1 shows one of the tasks that the participants were working on during these sessions.

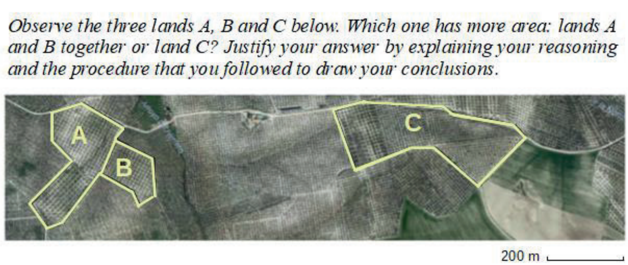

Figure 1: Example of task proposed in a practical interactive session.

Each practical session was structured as follows: (1) presentation of a number of tasks to the whole class group by the researcher; (2) active discussion on the questions of the tasks in small groups upon completion; (3) resolution of the task by one or more volunteer students (with the help of the researcher, when necessary) in front of the rest of the group; (4) sharing of the work, joint argumentation of the validity and invalidity of other given responses and summary of conclusions.

In the second phase participants were presented a task (shown in the following subsection) via paper worksheets to solve individually. The sheets provided to the students to write their answers were deliberately not gridded, in order to encourage the use of their own measurement references (of length and area) when estimating and handling proportional relations in their floor plan drawings. However, they were allowed to use the ruler at their discretion.

Research instrument and information source

According to our research objectives, we presented to the participants the following task.

> Erika and Laure have just bought together a new flat with a total area of 137 m^2 and 3 very spacious rooms. They are determined to do renovation works to modify the distribution of the flat. They would like to maintain as they are the living room —which has an area of 21.8 m^2—, the kitchen —13.3 m^2— and the hallway —7 m^2—. Regarding the bedrooms, they plan to have four: one of them, the main one, with (a) twice the area than any of the other bedrooms (the three of them with the same area) and (b) an access to another room, used as dressing room, with an area equivalent to one third of the area of the main bedroom. In addition, they will expand the two bathrooms, making them equal in area and, jointly, being equal in area to the kitchen.
>
> (1) What will be the area of each bedroom (the main and the others), the dressing room and each bathroom?
>
> (2) Draw a possible floor plan of the flat, respecting the relationships between the dimensions of its rooms.

Additionally, the original task asked the participants for the identification of the mathematical contents and components of mathematical senses involved in their resolution processes. Likewise, it requested a reflective discussion on the suitability of the questions (1) and (2) for primary school students according to the official Spanish mathematics curriculum for this educative stage. These questions have been omitted in the task statement shown above since the present study focuses on the analysis of the participants responses to the items (1) and (2).

For our design of the task, we considered the criteria of (a) diversification of geometric, measurement and algebraic contents and components of the mathematical senses involved in its resolution and (b) setting the task in a realistic context. Regarding (a), the task potentially requires handling with numerical relations, use of algebraic language, equation solving, use of geometric concepts and relations, visualisation and representation skills, proportionality considerations, comparing and estimating measures, etc. Concerning (b), making major renovation works in a newly acquired flat (especially an old one) is a common practice among owners and investors. Moreover, the task is considered to include the three components of critical thinking identified by Maj-Tatsis and Tatsis (2021) in their review of studies on this topic: reasoning, problem solving, and identifying the suitability of problem solutions.

Data analysis

During the implementation of both phases, a variety of written participants' productions were collected. In this work we analyse the ones collected in the second phase. Participants' written responses to the previously presented task were considered our units of analysis.

Comprehensive data reviews enabled the examination of the solution processes and embedded proposals of solutions in participants' written responses, being the identification of errors and misconceptions the main focus of the analysis. These reviews led to classifying errors into the following four categories: errors related to reading comprehension and interpretation of the task (L), errors in the use of number and algebraic senses (A), errors in the use of measurement sense (M) and errors in the use of geometric sense (G).

RESULTS

From the 67 participants, 10 (14.9%) of them did not provide any response to the task and 9 (13.4%) initiated their solving processes but did not manage to propose a solution. Part (1) of the task was answered by 48 (71.6%) participants and part (2) by 33 (49.2%). Only 14 (20.9%) participants provided a correct response to part (1) and 8 (11.9%) to part (2). Table 1 shows their most frequent errors and Figure 1 illustrates some of them.

Errors	n_i	f_i (%)
L1. Consider a different number than 4 as the total number of bedrooms	10	14.9
L2. Subtract the dressing room area to the main bedroom area	7	10.5
L3. Consider the main bedroom area twice the joint area of the other 3 bedrooms	7	10.5
L4. Consider 13.3 m² as the area of each bathroom	5	7.5
A1. Formulate wrong relations between areas of different rooms (e.g., x/3 for dressing room area where x stands for the standard bedroom area)	18	26.9
A2. Not taking into account some rooms (dressing room, bathrooms…) when equaling the sum of the known and unknown rooms areas of the flat to its total area	6	9
G1. Draw a non-realistic flat floor (represent the hallway as a room itself and not as a corridor; do not consider access area to rooms)	23	34.3
G2. Do not include some parts of the flat in its representation	12	17.9
M1. Do not respect the proportionality between the area measures of the flat rooms in their representation	16	23.9

Table 1: Most frequent student teachers' errors when addressing the task.

Note: n_i = Absolute frequency of participants that made a specific error. f_i = Relative frequency in percent. Error labels allude to the categories specified in the subsection “Data analysis”.

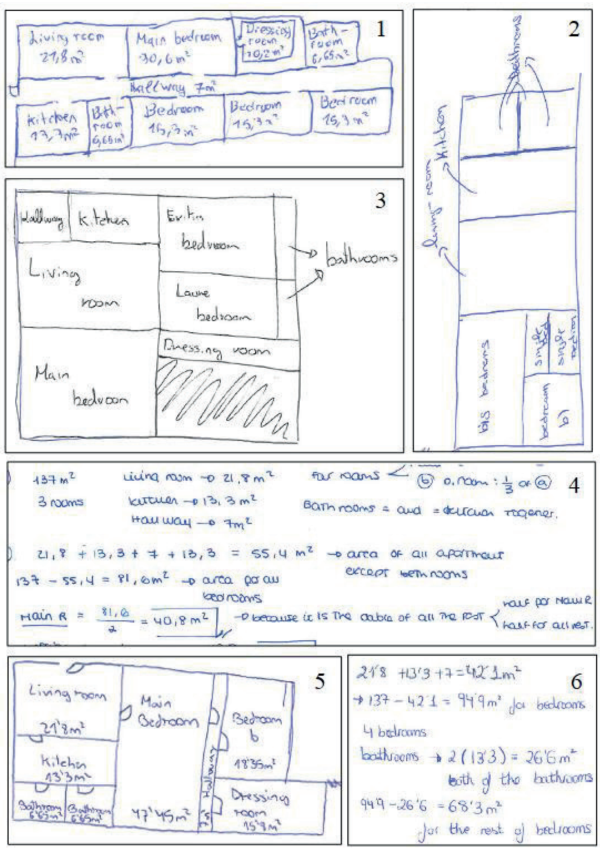

Figure 1: Participants’ productions illustrating errors specified in Table 1. Note. 1: M1; 2 and 3: G1; 4: L3; 5: L1 and G2; 6: L4. Labels refer to errors described in Table 1.

DISCUSSION

Next, we discuss our results according to our research objectives and the categories of errors of our data analysis. For this, it is important to note that part (1) of the task was answered by 48 (71.6%) participants and part (2) by 33 (49.2%).

Firstly, it is noteworthy that the number of participants who made at least one mistake related to reading comprehension and interpretation of the task statement amounts to 24 (35.8%), which is equivalent to the half of the group who responded to the part (1), and that a total of 29 mistakes were registered in this category. The difficulties to properly understand the task statement and being able to autonomously eliminate ambiguity where appropriate could be related to a lack of reading habits on the part of some student teachers, a fact pointed out by research studies on the topic (e.g., Applegate & Applegate, 2004). It would be of interest to collect data on the participants to this respect in future studies in order to explore a possible correlation.

Concerning our first research objective, a total of 22 participants (32.8%) failed in algebraically formulating the problem, appearing in some cases the reversal order error (Rosnick & Clement, 1980), probably due to a direct-translation strategy based on the syntax of statements as pointed by other authors (González-Calero, Berciano, & Arnau, 2020). Both results are aligned with those reported by Taplin (1998).

Most of the participants struggled with the representation part and only few of them (8, representing 11.9%) were able to produce a realistic and valid floor plan of the flat. A large number of participants (23, representing 34.3%) drew non-realistic floor plans by representing the hallway as a room itself and not as a corridor or by not considering any access area to the different rooms. This seems to be aligned with the findings of Kiliç (2017). This together with the inconsistency in the representation of the rooms' areas according to the area measures that they algebraically obtained reveals a poor use of geometric and measurement senses as well as of critical thinking, that enable solvers to detect unrealistic solutions.

Lastly, it should be noted that very few participants provided written evidence of checking and validating their responses. It is needed to continue emphasizing the importance of this practice also at these educational levels, train critical thinking skills to identify inconsistencies and contradictions (including one own's) (Paul & Elder, 2002), and work on the evaluation and internalisation phases in problem solving (Yimer & Ellerton, 2010).

Our main findings show relevant lacks in the comprehension reading and text interpretation as well as in the use of mathematical senses by student teachers. We stress the importance of reinforcing the work on mathematical senses with student teachers during their training, with both foci: their instruction in teaching and learning processes of mathematics through these senses and the strengthening of their own mathematical senses. In this spirit, working on word problems is presented as a suitable and interesting option (González-Calero, Berciano, & Arnau, 2020).

Regarding our second research objective, by virtue of the analysis of the student teachers' performances, we highlight the value and interest of the task for our original purposes. It promoted the use of a diversity of components of their mathematical senses, regardless of whether this use was made appropriately or not by the student teachers participating in the study.

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REFLECTION OF PUPILS' COMPOSITION OF WORD PROBLEMS: A CONTRIBUTION TO THE DEVELOPMENT OF DIDACTIC COMPETENCES OF PROSPECTIVE PRIMARY SCHOOL TEACHERS

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In our research we studied the way pupils prepared several word problems and the way prospective primary school teachers subjectively perceived this process. The paper presents outcomes of this research. We wanted to investigate whether and how prospective primary school teachers are able to analyze the way pupils constructed the word problems and to assess the significance of such composition in primary mathematics education. This competence is considered to be one of the factors that contribute to the formation of the professional identity of the prospective teacher. As a research method, the analysis of authentic statements of prospective teachers expressing their expectations was used. This analysis was complemented by the authentic problems worded by the pupils.

INTRODUCTION

The mathematical and didactic aspects represent an important part of multidisciplinary pre-graduate teacher training for primary school teachers. The prospective teacher training that considers teacher as reflective practisian to be one of the starting points of teacher training (Schön, 1987; Wubbels & Korthagen, 1990) refers to the requirement for a teacher to closely watch the process as well as the results of own teaching activities and think about and reflect the feedback. The reason is to understand own activity better in order to further enhance its impact (Janík et al., 2013, p. 183). The reflection may be concentrated on several aspects of the teacher's thinking and acting. In this research, we focused on reflection of curriculum represented by word problems. Word problems and their solution are traditionally considered one of the most challenging components of school mathematics. Rendl, Vondrová et al. (2013, p. 50) consider this issue one of the critical points of school mathematics as seen by teachers as well as pupils.

THEORETICAL FRAMEWORK

The model of reflective practice often appears in the context of constructivist teaching and learning, in which knowledge acquisition is usually linked to reflection of practical experience (Korthagen & Vasalos, 2005; Janík et al. 2013). Discussions about reflective education of prospective teachers and its effectiveness have led to the formulation of new concepts (Pollard & Anderson, 2008). This way of training teachers develops their knowledge base, which is considered a key aspect of the way to teacher profficiency. It is based on Shulman's (1987) concept of knowledge base for teaching: content knowledge, pedagogical content knowledge and curriculum knowledge. Harel and Kien (2004) emphasize that knowledge of mathematical content significantly affects other components of the knowledge base. The acquisition of knowledge and beliefs of prospective teachers as part of their professional development can be described from different perspectives (Leder, Pehkonen & Törner, 2002). Wilson and Cooney (2002) relate them to the overall development of teacher competencies as a set of personal characteristics and professional dispositions of the teacher. They emphasize that the development of competencies is a challenging didactic skill for teachers, in which they should demonstrate how they develop their competencies in real life and in self-reflection. Our perspective complies with the realistic approach to teacher training (Korthagen et al., 2011). At the beginning, beliefs of prospective teachers regarding what the process of educating pupils should be represent "opinions without any background or rationale, which, however, are rather resistant to changes" (Korthagen et al., 2011, p. 82). In the next phase, prospective teachers acquire first professional experience, solve educational situations and gradually develop their relationship to the profession. Professional beliefs, individual conception of the teaching profession and professional identity are formed (Pajares, 1992). The object of our interest was the way prospective teachers reflected the composition of mathematical word problems. Vondrová et al. (2019) emphasize that word problems show that mastering mathematical operations acquired in mathematics teaching can become an important tool for understanding the surrounding reality and can be a tool for solving real life situations and tasks. Greer, Verschaffel, and De Corte (2002) understand word problems as pieces of text that contain quantitative information and describe a situation with which the reader is familiar. Questions of "mathematical symbolization" (Vondrová et al., 2019) - the ability to express word problems in mathematical (numerical) expressions are often associated with the problem of reading comprehension. Some authors show that we can support successful solution of tasks by letting the solvers to compose their own word problems. (Cai & Leung, 2011; Kovács & Kónya, 2021; Silver, 1997; Singer, Ellerton, Ponte, & Henriques, 2013; Tichá & Hošpesová, 2010). In school teaching, teachers often provide support to pupils solving the problems, for example, by rewording the problems, constructing an easier variant, wording questions of 'secondary', related tasks. Therefore, we agree with Tichá and Hošpesová (2015) and others (Leung & Silver 1997; Pittalis et al., 2004; Sierpinska & Osana, 2012; Osana & Pelczer, 2015), who consider the inclusion of word problem composition in teacher training to be beneficial for development of teacher competencies.

METHODOLOGY

Aim and method of research

In preparing our qualitative research, we used a similar format as in our contribution for CME 2020, which we follow up on. We aim at bringing a deeper insight into the way prospective primary school teachers subjectively percieve the way pupils compose word problems. As a research method, we used the analysis of comments written by prospective teachers, and subsequent semi-structured interviews of the prospective teachers during joint reflection. The advantage of the written statement is that the mediation effect of the researcher is smaller compared to the interview and the respondent works at his own pace (Elisabeth, 2007). We sought answers to research questions through a thematic analysis of individual statements (Braun & Clarke, 2006). The group of research respondents consisted of 45 primary school prospective teachers of the Faculty of Education, Masaryk University in Brno. The research was carried out in autumn 2021.

We worded two research questions:

a) How can prospective primary school teachers analyze and reflect the word problems composed by primary school pupils?[1]

b) What do prospective primary school teachers think about the way primary school pupils compose word problems and how do these opinions change in the course of their education?

Stages of research realisation

The research had the following three stages:

Stage 1 – preliminary: Giving information about the aim of the research, Discussion with prospective teachers about the importance of word problems and their composition as a tool for the overall development of personality of pupils. Three types of activities were carried out:

- Analysis of the language and the context of the 'authentic' word problems, taken directly from real life (relationship between reality and the mathematical model of the word problem);

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[1] One of the expected outcomes in the current curriculum – Framework Educational Programme for Basic Education (FEP BE) – is directed towards the independent composition of problems by primary school pupils: The pupil solves and composes problems in which he/she applies the acquired numerical operations. The standards of education at the end of primary mathematics education state two indicators for a given expected output: the pupil understands the text of a simple problem (distinguishes information important for solving the problem) and solves the problem; the pupil **composes a simple word problem according to the pattern**.

- Own attempts of prospective teachers to construct meaningful word problems of a given structure such that its solution includes a certain calculation. We expressed the structure of the problem by a mathematical expression (notation), for example (a + b) × c;
- Estimates, in which prospective teachers guesses the expected number and quality of word problems that pupils of the 5th year of primary school would compose.

Finally, prospective teachers were asked to hand word problems to their pupils and afterwards to assess them during their pedagogical practice at the primary school.

Stage 2 – teaching: Prospective teachers in the role of the problem givers, commentators and facilitators during the process of solution of the problems by pupils during their own teaching practice. Pupils were given the following problem:

> Create meaningful word problems for the following mathematical expression (real life situations). Create as many different word problems as possible.
>
> a) 50 + □ = 135
>
> b) (10 + 5) × 5 =
>
> Example:
>
> a) Martin wants to buy a gift for 135 CZK to his mother. He already has 50 CZK. How much more does he have to save?
>
> b) There are 5 rows of trees in the garden. There are 10 apple trees and 5 pear trees in each row. How many trees grow in the garden in total?

Stage 3 – analytical: Prospective teachers in the role of assessors and evaluators of pupils' work. At the end of the course, prospective teachers analyzed records written by pupils or their comments and compared them with their own expectations. They individually reflected their own findings and together presented their attitudes to the implementation of the activities.

RESEARCH RESULTS

Attempts to analyze the problem composition aimed at assessing the success of the work, as well as the interpretation of pupils' reactions to the unusual nature of problems during teaching and in written records of solutions. Composing word problems is a suitable motivational tool for pupils' creative activity as well as a tool for diagnosing pupils. This is an opportunity for critical thinking. According to prospective teachers, problem composition is difficult for pupils, mainly because the activity is new for them, which means that they are not acquainted with it.

Some pupils composed creative and funny problems, showed a good understanding of the assignment, although some were very distant from reality:

> My dad promised me 135 CZK, but so far he only gave me 50 CZK. How much does he owe me?
>
> A plot of land costs 135 CZK. Dad has saved 50 CZK. How much more does he have to save?

There were big differences between pupils. Many of them asked the teacher questions during the class. They needed further explanations. If they tried to compose their problems, they started with performing the given numerical calculation, and only then they were usually able to compose the problems by changing the context according to the pattern.

> The farmer planted 5 rows of vegetables. There are 10 carrots and 5 cucumbers in each row. How many vegetables grow on the farm in total?
>
> There are five rows of candies in the sweet shop. There are 10 packs of marshmallows and 5 packs of bompars in each row. How many packs of candy are there?

The prospective teachers were also surprised by some almost illegible text and large number of grammatical mistakes that appeared in the texts.

Here are some authentic statements of prospective teachers:

> The activity only confirmed what I had thought: it was completely unknown to them, incomprehensible, and therefore it was difficult to work with the pupils on it. Word problems are used only to be solved, so that they always have only one numerical solution. Pupils usually only solve word problems with one numerical solution.
>
> I think that it was something they are not used to and that this urge to think differently was a bit painful.
>
> I was glad that there were at least some pupils who tried to compose the problems. The assignment seemed unusual to them, but in the discussion they realized that it was necessary to think hard and not just copy the numbers from the assignment, choose a suitable arithmetic operation and solve the problem.

From the reflections of prospective teachers, we extracted three levels of attitudes towards the word problem composition:

a) Manifestations of reserved attitude towards usefulness of word problem composition done by pupils. Prospective teachers prioritize the way pupils can solve the presented problem, whether it is quick and without mistakes. They do not perceive the activity as an instrument for the development of pupils' personality and mathematical literacy but rather as something unnecessary or even inappropriate in the process of teaching mathematics. Only clear and comprehensible methodological instructions are required, which problems are suitable, how to assess and evaluate their solution.

These prospective teachers remain at the level of their own preconcept, which has been created in the role of a pupil.

b) Mere statement of criticism that pupils were not able to compose the problems. These prospective teachers do not think about the cause of the failure. They do not consider searching for the relationship between mathematical model of the word problem and its language expression in a meaningful context important for their own practise and development.

c) Prospective teachers start to approach word problems, their composition and relationship between the problem and its solver in a different way. They realize the purpose and benefits of utilisation of word problem composition and other similar activities aimed at developing of critical thinking from the personality development of the pupil point of view. He takes into account own experience:

> When I myself learn to compose word problems, I have to combine, trial and error, substitute. I think to myself about possible difficulties of pupils, about what needs to be bewared and similarly. When I just solve the problems, I only shortly think of the problem wording and I can easily miss related enriching context.

DISCUSSION AND CONCLUSION

We used the chance to discuss selected problems with participants of the research in order to give them a chance to think about the issue of word problem composition in broader context. Prospective teachers facing pedagogical practice are beginning to realize that they compose word problems in mathematics teaching practically on daily basis. This can be the reason why they are often convinced that they can easily compose and word suitable problems on the spot in the class. We believe that looking at problems composed by someone else (pupils) can be a chance for critical self-reflection, compared to their own skills to compose problems.

In our research, prospective teachers used the model of the problem structure, expressed by numbers and numerical operations, as a starting point for creating word problems (see FEP BE). In teacher training, other methods can be used as tools for composing word problems. Other research (Pittalis et al., 2004; Nesher & Herskovitz, 1994; Hošpesová & Tichá, 2014) presents other ways of composing word problems, e.g., using graphic schemes similar to branch strings (diagram for schemes) or based on a certain situation loosely described by a story or a picture.

The analysis of the word problems composed by pupils done by prospective teachers indicated that the process of acquiring subject didactic competencies is only initiated during the studies. The comments of prospective teachers regarding the word problem composition by pupils in primary mathematics education is unique. The author believes that this is also related to their perception of the role of problems (not only word problems) in teaching. The nature of our research, the scope of the sample of respondents and a certain degree of subjectivity, given the different approach of prospective teachers to the problem composition, do not entitle to unambiguous, categorical judgments. However, our findings suggest that independent problem composition is not a common phenomenon in school practice, that for many students the required activity was very unusual to the degree that they had never met it before. We believe that the inclusion of word problem composition in teaching offers suitable motivation, work and diagnostic tool for the creative activity of pupils or for the diagnostics of pupils.

The author considers the utilisation of activities connected with the composition of problems of ~~a~~ given structure in the teacher training as one of the instruments of a deeper insight of prospective teachers into the content of school mathematics, represented by the problems. Our experience confirms that this didactic knowledge of content (in the sense of the Shulman concept) is one of the important components of teacher education. It can help the prospective teacher to remove the stereotype of simple, easy-to-solve "textbook" type problems that are uninteresting in their context and mathematical structure. Prospective teachers can realize the structure of problems, gain the necessary insight into their structure and stop perceiving them only as a sequence of calculations that lead to obtaining the result. In teacher training, it appears to be a method with a strong motivational content that respects constructivist approaches.

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(HOW) DO TRAINEE TEACHERS SUPPORT MATHEMATICAL THINKING?

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What do trainee teachers do when they support children to learn mathematics? The presented research project aims for (further) development of mathematical support activities of trainee teachers in primary schools. Pursuing this question observations of teacher actions in everyday teaching situations were conducted. With this paper we present initial analyses.

INTRODUCTION

The demand of the UN Convention on the Rights of Persons with Disabilities for participation for all, has an impact on educational plans and teaching in schools. The heterogeneity of students has received increasing attention in education. For mathematics learning it is important to support the mathematical understanding and knowledge of all children. "What teachers therefore do – matters" (Hattie, 2010, p. 22), and this includes the whole spectrum from mathematically interested and gifted children to children with difficulties in learning mathematics.

THEORETICAL BACKGROUND

Within the theoretical background we (1) state our view on heterogeneity and (2) refer to a constructivist learning and teaching perspective, since we regard these aspects fundamental to support children to learn mathematics.

Heterogeneity in learning mathematics

The perspective of mathematics didactics on heterogeneity takes different aspects into account (Komm & Huhmann in press). Central points are the findings on children's mathematics learning and the related considerations on teaching mathematics.

In mathematics didactics, models have been developed that focus on the prerequisites and learning of mathematically gifted children (Käpnick, 1998) and of children with difficulties in learning mathematics (Kaufmann, 2003). For both areas there is no common definition or modelling, but there is consensus regarding different areas of influence. It can be stated that giftedness and difficulties in learning mathematics are influenced by co-cognitive as well as intra- and interpersonal factors (Benölken, 2014). Spiegel and Walter (2005) concretize this heterogeneity by talking about "vertical heterogeneity", which focuses on the different performance spectrum of the students, and "horizontal heterogeneity", which considers the different ways of proceeding and thinking of the children with regard to mathematical content areas.

Learning and teaching mathematics

Learning and teaching mathematics is based on the constructivist (learning) paradigm (Winter, 1989; Wittmann, 1981). This means that knowledge can be acquired actively through an active-discovering, self-acting approach with the learning content. Acquisition of knowledge and skills cannot happen through information transfer from the outside, one's own perceiving and acting, analyzing, reflecting, and verbalizing are central for this (Huhmann, 2013). There is consensus on how mathematics learning must be designed so that *children* have the opportunity to construct their own mathematical understanding and knowledge: therefore, first the *teacher* has to focus on the development of understanding and the development of basic competencies. Second, referring to the constructivist (learning) paradigm, the principle of active-discovery learning in order to enable the individual exploration of mathematical content, so that children can follow their own ways of thinking (Häsel-Weide & Prediger, 2017). And third, avoiding of "small-step" learning instead focussing holistic learning so that children can recognize connections (Gaidoschik et al., 2021).

With regard to heterogeneity in learning mathematics, it is important to note that students have different needs. To develop mathematical understanding and knowledge it is important to see patterns, structures, and relations. In her study of children's pattern and structure skills at the beginning of primary school, Lüken (2011) was able to show that there is a connection between the ability of identifying and using patterns and structures and the mathematical performance. For example, studies with regard to arithmetical basic competencies showed that the development of numerical competencies and the ability to structure are closely related (Häsel-Weide, 2016) and that recognition of structures and relations is important for children to be able to detach themselves from counting (Rechtsteiner, 2013). Therefore, children with difficulties in learning mathematics need competent learning support when actively exploring holistic contexts in order to open up connections and thus acquire a basis for comprehension-based learning. In contrast mathematically gifted children are good at recognizing and using structures, possess characteristics of good problem solvers, are often able to structure facts themselves, and use heuristic tools in a purposeful way as well as problem-solving strategies (Fuchs, 2006). A teacher has to take these characteristics and aspects into account when planning lessons and accompanying the children to support their mathematical thinking. Therefore, suitable tasks, representations and an adequate cognitive activation have to be considered.

METHODOLOGY

The observation presented here provides insights into a dissertation project. It is linked to a project to the further development of the Second Phase of Teacher Education in cooperation with three different State Seminars for Teacher

Education and In-Service Qualification. The teacher education for the primary level (in the federal state of Baden-Württemberg, Germany) consists of two independent phases: first phase - university studies; second phase - a one-and-a-half-year teacher training at the Seminar for Teacher Education. In this phase, they teach at a school frequently supervised by lecturers.

Research question

Within a qualitative research approach, we name the following question: What are the supporting actions of trainee teachers, when they (intend to) support mathematical learning within a heterogeneous classroom?

To pursue this question trainee teachers of these State Seminars were asked for their participation in the research project. Those who had volunteered to participate were asked to teach a lesson in arithmetic and submit a lesson plan in advance. Due to the voluntary participation, a positive selection can be assumed. Then participatory classroom observations (Boer & Reh, 2012) were conducted, so far in six different cases. Two cameras were used to record the lessons. In addition, the trainee teachers were equipped with a small body camera and a microphone to directly catch the teacher's support actions with individual children. This was followed by semi-guideline-based recall interviews (Baur & Blasius, 2014), in which the participating trainee teachers viewed their own supporting actions in video excerpts. These video excerpts, the corresponding transcripts and transcripts of the recall-interviews will be used for the qualitative content analysis (Kuckartz, 2012).

Excerpts from the data collection and initial analyses are presented in the following.

FIRST FINDINGS

We observed a characteristic form of support actions of (trainee and pre-service) teachers in learning situations with cognitive conflicts between students. The following example illustrates these typical support actions. Typical means that a teacher usually acts in this way in situations of cognitive conflicts. Thus, our focus of the analysis is on the trainee teacher's support actions. Moreover, the children's actions are described because they form the starting point for the trainee teacher's support actions.

The learning situation took place in a first-grade class just before the Christmas holidays in a rural primary school. The focus is on the processing of a group of children to the task "Find terms which make 10". The children worked on this task in groups of two or three. The focus was on adding. The groups of children were given a blank sheet on which they wrote the terms they found. Optionally, the children could get arithmetic chips to work on the task. The teacher approaches a group of three children (George, John, Betty). He looks at the children's sheet and points to the term 10 =1+5+4. (This term is unconnected to any other term on the sheet up to this moment). In the following transcript, gestures and actions are included.

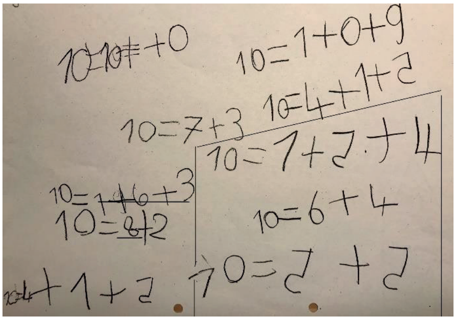

Figure 1: Sheet from this group. Till then they found the terms in the frame.

Teacher: Who came up with that one?

John points to George.

Teacher: George. Great!

George: John said it makes nine

Teacher: Shall we get some chips and try it together?

George: But I do find it makes ten.

[…]

Teacher: Shall we get some chips and try it?

Kids nod.

Teacher: Yes, then I'll go get some.

The teacher fetches arithmetic chips (more than 10).

Teacher: So, what do we calculate?

The teacher places the arithmetic chips in front of the children. George, who has written down the task 10=1+5+4, shows his classmates what he has calculated. He accompanies his action verbally. Successively he takes the required number of chips from the set placed by the teacher. The chips are all placed in blue colour. The subsets are each represented with a corresponding number of chips.

Figure 2: George counting the chips.

George: 1 plus 5 plus 4. makes (chips are counted off by George) one, two, three, four, five, six, seven, eight, nine, ten. (Addresses John directly) See!

The teacher looks at John.

Teacher: George is right, isn't he?

Turns to the girl.

Teacher: Or Betty?

Betty: Yes.

Analysis of the situation

There is a cognitive conflict: is the result 9 or 10? The trainee teacher starts with the children's question. He enables the solving of the problem by initiating the linking of the symbolic and the acting level. Therefore, he offers arithmetic chips to create the different subsets of the term 1 + 5 + 4. The idea is to change the perspective with a representation transfer: from the symbolic level to focus on the quantities by laying quantities of each subset. Based on this the learner can perceive and identify the quantities of each subset. After that, in the concrete situation the learner starts counting successively from the first summand followed by the second and third one, to determine the cardinality of the total set of the three subsets. With this he proves that the result is 10.

The teacher initiates a representation transfer, so that the students have to link different levels of representation. His support is a form of *self-help*, allowing and enabling students to find the answer on their own (Winter 2016). In the situation of the cognitive conflict this supporting action enables a renewed cognitive activation of the students: a process of action with material is initiated. The associated transfer of representation is an indicator to capture and also to develop children's individual understanding.

The trainee teacher’s view of the situation

The excerpt from the recall interview refers to the support situation presented above.

> Then I fetched the arithmetic chips and had George represent his calculation method […] respectively explain his thoughts acting, so that the others could follow. Of course, that was a bit unskilful, because he only used the blue arithmetic chips; it would have been better if he had done blue, red, blue. Betty was a bit absent, I noticed that too. That’s why I asked her again at the end. And then came the classical yes. [....] But otherwise I think, that the question, whether the calculation is correct or not George could illustrate and solve through the arithmetic chips. […] Betty was actually left out.

Analysis of this part of the recall-interview

Overall, the situation of the recall interview can be characterized as a process description by the trainee teacher. He describes that he “fetched the arithmetic chips and had George represent his calculation method […] respectively explain his thoughts acting". He argues, "so that the others could follow". His focus is on playing the ball back to the children to solve the conflict, especially on George explaining it to the others. It remains implicit why he chose the material arithmetic chips. The change of the level of representation is neither named nor argued in terms of didactics or content, as, on the other hand we have shown above in the part analysis of the situation. So, we cannot know, if the trainee teacher chose the representation transfer consciously with respect to the content or if he chose it like an unreflected procedure because it is something to do in such a situation. An alternative procedure only on the symbolic level would have been to start from the term 1+5+4 and to transform it to 1+4+5 and further to the well-known task 5+5=10. The representation George made, is evaluated by the trainee teacher as "a bit unskilful, because he only used the blue arithmetic chips". Why this representation is not suitable, is not reasoned. Instead, the trainee teacher stated, "it would have been better if he had done blue, red, blue", what means, to change the colours blue and red to distinguish the different subsets. Again, it remains implicit why the trainee teacher evaluates this representation as a better one. Why is a spatially perceivable separation of the three subsets only with blue arithmetic chips not suitable? Again, we cannot know, if the trainee teacher chose the feature 'change of colours' consciously with respect to the content or if he chose it like an unreflected procedure because it is something to do in such a situation. Then he describes that he noticed, "Betty was a bit absent". Because of that the trainee teacher "asked her again at the end" with the aim of drawing her attention back to the mathematical activity. But he uses Betty's absent situation to ask a suggestive question, "if George is right". She can only answer with "the classical yes", because "Betty was actually left out". To get her attention back to the content by cognitive activation an appropriate task would be: Use the arithmetic chips to create the for the term 4+5+1.

Analysis of narrower and wider text context

Regarding our analysis of this situation, we will use the narrower and wider text context of the recall interview by taking into account the trainee teacher's view on further support situations, in which he initiates a representation transfer. We use this as further information to sharpen the focus of the analysis of these situations and to consolidate our analysis of his actions. With the recall interview we intend to capture the trainee teacher's view of his support actions. What is the goal of his support activities? Does he perceive the mathematical thinking of the children? Does he identify and describe a connection between his support actions and the children's mathematical thinking?

The following statement from the trainee teacher refers to his general support actions during the lesson independent from specific supporting situations. The italicizations are made by the authors.

> But I offered the arithmetic chips to all children, and if they didn't find anything else themselves, I also encouraged them to throw the arithmetic chips, for example [...], so that they simply have this action again. So I practically offered the [representation] *level further down*, *for those who need it and those who just don't need it* and work on the, what was it, symbolic level, they could also do that. So, practically *I offered all levels* and *then students can choose for themselves* or *use it for themselves the way they need it*.

The trainee teacher offers arithmetic chips to all children *again*. It remains implicit what has happened up to this point until “they didn't find anything else themselves.” He seems to assume that the children’s task processing has been completed temporarily, but it remains unclear which task processes of the children on which levels of representation are meant here. His terms he used “*again*” and “*level further down*” give clues as indicators to this interpretation. His further impulse “*to throw the arithmetic chips*” opens up the possibility to find number decompositions through this random experiment that have not been found yet. Another purposeful possibility to find number decompositions, would be the activity “sorting the found number decompositions”. By sorting, further solutions can be found because relationships and structures are perceived. For this, however, a large number of arithmetic chips is needed, since all number decompositions are placed as sets (i.e., for all number decompositions of 10 in two summands, 110 chips are needed).

The trainee teacher argues his supporting activity “representation transfer with arithmetic chips” with the fact that each child can thus choose and use the level that he or she needs. His description “*offered the level further down*” raises questions: What does “further down” mean? How does the trainee teacher look at the different levels of representation? Is there a hierarchic view? In mathematics education the learner’s own transfer of representations within the same as well as between different levels of representations is an essential indicator to develop understanding. But the levels of representation are not hierarchized, the relations are significant. The complement “for those who need it” also reinforces this interpretations. Furthermore, he notes “then students can choose for themselves or use it for themselves the way they need it”. What does he mean in particular with these descriptions? This remains still implicit.

Considering the narrower and wider text context of the recall interview, in which the trainee teacher was questioned about his support actions, it becomes apparent that he mainly remains on the level of narrative descriptions with his statements and feedbacks to his support actions and the lesson. This is also confirmed by those text passages with further follow-up questions and inquiries on specifications and analysis of his support actions. Moreover, socially desirable responses are made by noting that “of course there would be more suitable options”. Or the trainee teacher refers to learning opportunities “in a following lesson”. This, however, he does not specify further.

Thus, it remains implicit in which way, from his point of view, the representation transfer processes lead to comprehension-oriented learning and are an indicator of understanding. So which content concepts does he have in mind when he noticed the cognitive conflict right in the learning situation? The starting point provides a concept on the symbolic level. There we talk in an abstract way about an arithmetic term, different summands, and the sum. On this level for him there is no suitable way to clarify the cognitive conflict. This concept is linked with a representation transfer to the concept of decompositions of a whole in different parts or a composition of different parts to a whole. So "parts" offer two different perspectives: Parts as whole units, each consisting of uncountable size. And parts as whole quantities, each consisting of countable single elements.

The support situation in this lesson focuses on the second perspective: decompositions of a natural number in two or three or … *parts as subsets* or vice versa on compositions of two or there or … *parts as subsets* to a *whole*, the natural number 10 *as total set* of 10 *single elements*.

However, in both ways it is about comparing a set with quantities of 10 elements with subsets of different amounts of elements by laying, sorting and structuring the arithmetic chips. All in all, this different concepts use different interpretations. And which of these does the trainee teacher have in mind and uses for his support action when noticing the cognitive conflict? It can be stated that George was able to represent and prove his mathematical thinking to his calculation by using the arithmetic chips on the acting level. And also, John's question, which the trainee teacher wanted to clarify with the children, could be solved with the help of the acting level. Despite the analysis of the observed support situation, it still remains unclear, if John and Betty gained understanding during George's representation processes for decompositions of a total set of 10 into three parts, and also if this was even intended by the trainee teacher.

RESULTS

The example illustrates positively that the trainee teacher chose a substantial task for his lesson. Furthermore, with his supportive action he concretizes a constructive understanding of learning: The starting point is the cognitive conflict of the children. He reflects this back to the learners with the change of the representation level for their own processing and solving. However, only two of the three learners deal with this, one child is left out.

On the level of our analysis of the videographed teaching situation we made a description of the situation and mainly we analyzed from a didactic point of view. We pointed out which content-related backgrounds are significant and taken into account in the trainee teacher's support activities.

On the level of our analysis of the recall interview, in which the trainee teacher viewed and commented on his support situation, we identified that he mainly made a process description of the situation. This was also confirmed by the analysis of the narrower and wider text context. So, in what way is critical thinking an element for his professional actions to develop supporting activities? It remains implicit and unclear whether he acted consciously, i.e., he chose his support action and used it purposefully founded on didactical knowledge or whether he chose it like an unreflected procedure because it's something to do in such a situation.

In addition to the fundamental question *what* teachers do when they (intend) to support, our example shows how important it is to look at the underlying reasons, *why* teachers *do what*. This helps on the developmental path from *'intending to support'* to a well-founded assurance *'that they support'*. Only in this way it is possible to implement conscious and goal-oriented support activities, especially with regard to subsequent and further support activities, but also with regard to the availability of a flexible support repertoire for the trainee teachers. For the professionalization of teachers critical thinking and reflecting on (one's own) lesson planning and ways of acting to support children is essential in all phases of teacher education – therefore, opportunities must be created!

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ACTIVITIES SUGGESTED BY ADULTS: COUNTING AND ENUMERATING

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This study focuses on adults' knowledge of activities that can promote early counting and enumerating competencies. Prior to an intervention, and then again afterwards, 18 adults were requested to suggested activities that could promote children's counting and enumerating competencies. Before the intervention, participants did not realize that counting and enumerating are two separate competencies. After the intervention, adults designed activities that were purposeful and that had the potential to actively engage children in specific competencies.

INTRODUCTION

The importance of promoting mathematical development during early childhood is supported by studies that found early mathematics competencies to be a predictor of later school success (e.g., Duncan et al., 2007). In addition to the learning that takes place in preschool, the home environment can have a significant impact (Anders et al., 2012). In a previous study (Levenson et al., 2021a) we investigated adults' beliefs (none were early childhood educators) regarding supporting children's engagement with various numerical activities. Findings indicated that in general, participants had positive beliefs towards early mathematics. Yet, adults may not know how to foster young children's mathematics competencies (Cannon & Ginsburg, 2008).

In an effort to increase adults' knowledge of activities that can support young children's number competencies, an intervention was designed and implemented with a group of 18 adults. This intervention was part of a larger study that investigated adults' knowledge and beliefs related to early years mathematics (e.g., Levenson, et al., 2021a). The aim of this part of the study was to investigate adults' knowledge of counting and enumerating activities before and after participating in the intervention.

COUNTING AND ENUMERATING

Verbal counting and object counting are separate but related competencies. In this paper, we will refer to verbal counting as counting and object counting as enumerating. Counting is the skill of reciting numbers in the conventional order, (Baroody et al., 2006). The relationship to language may be seen in the difficulties of English-speaking (and Hebrew-speaking) children when learning the number words from 11 till 20, and learning that 29 is followed by 30 (Han & Ginsburg, 2001). According to the Israel National Mathematics Preschool Curriculum [INMPC] (2010), before first grade, children should also be able to count backwards, and count by twos, fives, and tens (also called skip counting). Counting backwards is a prelude to learning subtraction, while skip counting lays the groundwork for multiplication (Sarama & Clements, 2009).

Verbal counting can be practiced by reciting the number words in unison with a parent or caregiver, or when playing games, such as "hide and seek," where counting is part of the game (INMPC, 2010). Body movements and sounds can aid when learning the number sequence. For example, The Big Math for Little Kids program (Greenes et al., 2004) encourages children to use different body movements and sounds to represent different decades from 1 to 100, such as making funny faces when reciting numbers from 11 till 19, and twisting their bodies when reciting the twenties. Body movements and sounds may add motivation, which in turn may facilitate the learning of verbal counting.

Enumeration refers to counting objects for the purpose of saying how many. Gelman and Gallistel (1978) outlined five principles of counting objects. The first principle, called the stable-order principle, is based on being able to recite the counting numbers (i.e., verbal counting). The one-to-one principle involves assigning one count word to each object. Common, related mistakes occur when one object is assigned more than one count number, or an object is skipped over, and not counted (Fuson, 1988). The third principle is cardinality, which involves knowing that when counting objects in a set, the last number mentioned represents the number of objects in that set. A child who has not yet understood this principle, may simply state any number when asked how many objects are in a set, or recount the objects which have just been counted (Fluck & Henderson, 1996). The fourth principle is the abstraction principle, indicating that any set of discrete objects can be counted. Finally, the order-irrelevance principle means knowing that one may enumerate the objects in any order (e.g., from right to left, from left to right, etc.) and that enumerating objects in different ways results in the same cardinality. Notably, children may show knowledge of one principle such as understanding of cardinality, while violating another principle, such as one-to-one correspondence principle (Fuson, 1988).

Because enumerating involves several sub-competencies, activities that aim to promote enumeration might consider each sub-competency. For example, although parents may read books to their children that aim to promote children's enumeration, few counting books explicitly or even implicitly emphasize cardinality (Ward et al., 2017). It is up to the adult reading the book to focus on different sub-competencies. Yet, parents rarely provide cardinal labels after counting when reading a number book (Mix et al., 2012). In fact, when specifically choosing such books, more parents choose books based on their assessment of how fun and enjoyable it would be for their children, than on the mathematical content and challenge it would provide (Gaylord et al., 2020)

The types of objects counted may impact on children's acceptance of the abstraction principle (Gelman & Gallistel, 1978). Counting objects that are different in color, size, or function can help children recognize that these other attributes do not affect counting (Greenes et al., 2004). Counting objects in different formations may promote the order-irrelevance principle (Gelman & Gallistel, 1978). In Tsamir et al.'s (2018) study, when 4- to 5-year-old children were requested to count seven bottle caps placed in a circle, two children, who had previously correctly counted the bottle caps when placed in a row, claimed that they did not know what to do and gave up.

RESEARCH AIM AND QUESTIONS

Prior to this intervention, we found that adults believe they have an important role in promoting children's numerical knowledge, yet significantly less believe that they needed guidance to do so (Levenson et al., 2021a). No differences in these beliefs were found between parents, adults who have some other connection with young children, and those who claimed to have no connection with young children. We also found that adults are not aware that verbal counting is a competency that needs to be promoted on its own, and not necessarily as part of object counting (Levenson, et al., 2021b).

The current study focuses on adults' knowledge of activities that can promote both counting and enumerating competencies. The aim is to investigate adults' knowledge of these tasks, before and after taking part in an intervention. Specifically, we ask: Before taking part in the intervention, do adults differentiate between counting and enumerating tasks? After taking part in an intervention, what types of activities do adults offer for promoting children's counting and enumeration principles, and do those tasks take into consideration sub-competencies?

METHOD

Setting and participants

The setting for this study was an elective course entitled *Early childhood numerical thinking: Theory and research,* attended by students working towards their master's degree in mathematics education. There were 18 participants (not the same as in our previous studies), of which six were parents of children between the ages of three and six years, nine had a family relation that age (e.g., grandchildren, nephews, nieces), and three had neighbours with young children. None were early childhood educators. We chose this context for our study, wishing to include at this point in our research only adults whom we knew to have a positive disposition towards mathematics. There were three aims to the entire course: to raise participants' awareness of number competencies developed prior to first grade, to increase their knowledge of children's development of those competencies, to increase their knowledge of tasks that might promote early number knowledge and competencies. Three content areas were reviewed: counting and enumerating, comparing sets, and number composition and decomposition. In this study we report on the first content area.

The course included 13, 90-minute sessions. It was designed by all authors of this study, was taught by the first author of this study, with the second author attending all lessons. All lessons were video recorded and transcribed. In this study we focus on counting and enumerating competencies, the subject of the first four lessons.

During the first lesson, participants watched and then analyzed a video of a three-year old boy and his grandmother, engaging in various counting activities while baking cookies. Analysis of this video, as well as other video clips viewed during the course, focused first on the child's ability to carry out the activity (e.g., Could the child count the cookies on the tray, and what were his difficulties?), as well as the adult's role in the activity (e.g., What exactly did the grandmother ask her grandson to do? How were the cookies arranged on the tray?). During the second, third, and fourth lesson, participants read and discussed related research (e.g., Baroody et al., 2006; Gelman & Gallistel, 1978), and viewed and analyzed together YouTube videos of preschool children counting and enumerating. We discussed children's ability to carry out a particular skill, as well as how a task may be designed and implemented to focus on a specific skill. At the end of the fourth lesson, participants were given a home assignment, to design a task they would implement with a preschool child, that aimed at promoting counting and enumerating competencies. This assignment is one of the research tools of this study.

Tools

Prior to beginning the course, participants were asked to fill out a questionnaire (see Levenson, et al., 2021b, for more details). In this study, we analyze responses to the following open questions: (1) The preschool mathematics curriculum states that by the end of kindergarten, children should be able to count till 30. What counting activities would you implement with children to promote this skill? (2) What enumerating activities would you implement with children to promote their ability to enumerate eight objects? Participants had plenty of empty space to write their responses and took as much time as they needed to fill in the questionnaires.

After the fourth lesson, participants were given the following assignment to complete at home: Plan an activity for a young child that can promote counting and enumerating competencies. State the competencies you wish to promote, what items you will use, how they will be placed, and how you will use them. State what questions you are likely to ask the child.

FINDINGS

Prior to the intervention

Participants' descriptions of counting activities were categorized according to the numerical competencies they would elicit from a child (see Table 1). Although adults were requested to describe counting activities, only four adults focused solely on verbal counting. Three other adult combined verbal counting with other skills, and nearly half described enumerating activities, (i.e., counting objects) indicating that they did not differentiate between counting and enumeration.

Category	F	Examples
Counting only	4	A13: "Counting forwards, backwards." A15: "Counting together out loud."
Counting and recognition of number symbols	1	A6: "Consecutive counting, counting even numbers, counting by tens. Writing numbers till 30 and recognizing the written numbers."
Counting and enumerating	2	A11: "Especially playful activities that require counting. Counting blocks, counting while playing hide and seek, counting the number of children in the kindergarten, etc."
Enumerating only	8	A1: "Hand out up to 3 balloons to each child and ask to count the total amount of balloons." A8: "I would ask children to count objects."
I don't know	3	A14: "I don't know."

Table 1: Frequencies of types of activities suggested for promoting counting (N=18).

When asked to suggest enumerating activities, half of the participants indeed described enumerating activities only (see Table 2). Four adults claimed not to know of any activities; three of which were the same adults who claimed not to know any counting activities. Interestingly, A6 added the skill of recognizing number symbols, both in the counting and enumerating activities. Two adults offered counting only activities when asked for a counting activity and enumerating only activities when asked for an enumeration activity. Notably, none of the adults specifically related to sub-competencies such as one-to-one correspondence or cardinality.

Category	F	Examples
Enumeration only	9	A8: "[I would place objects] in a row, and then in a pile." A15: "I would ask the child to bring me eight

		objects.”
Enumeration and recognition of number symbols	2	A6: “Identifying a written number (numeral) and asking the child to give me that number of objects.”
Ordinal numbers	1	A18: “An activity with Hannukah candles. First candle, second candle, …
Number operations	2	A13: “Addition and subtraction.”
I don’t know	4	A14: “I don’t know.

Table 2: Frequencies of types of activities suggested for promoting enumerating (N=18).

After the intervention

We now describe and analyze the activities designed by three participants, submitted after the fourth lesson of the course. These adults were chosen as they exemplify different trajectories of learning, that is, they started with different degrees of knowledge and ended with different degrees of knowledge.

A14 was the father of baby girl (under a year old) and an uncle to children between the ages of 3 and 6 years. Before the course, he claimed not to know any activities that would be suitable for promoting either counting or enumerating skills. After the fourth lesson, he planned the following activity for a child, he specified as 3.5 years old:

> Aim and setup of the activity: The aim of the activity is for the child to count backward from seven. You need to place seven dolls of different sizes on a bed and start to sing, like the song about five monkeys jumping on a bed and one falls off…, but here there are seven dolls, and one falls off, so how many are left?
>
> Activity: Each time the doll falls off the bed, you ask how many dolls are left. That will promote counting backwards, and the child can count in whichever way she likes.

In analyzing A14’s activity, we first note the stated aim of having the child practice counting backwards. While the song does start with seven and ends with one (or zero, depending on how you sing it), it does not necessarily encourage counting backwards. If there are seven dolls and one falls off, you only need to know the number which precedes seven, to know how many dolls are left on the bed. You do not need to sequentially count down from seven to zero. Furthermore, A14 states at the end that the child can “count” in whichever way she likes. This hints of the possibility of asking the child to enumerate the dolls left on the bed.

To summarize, A14's activity is a playful way to engage children with numbers. It involves jumping, dolls, and singing. A14 is aware that counting backwards is a competency worth supporting in young children. However, the activity focuses more on a different numerical competency, that of knowing which number precedes some other number, than on counting sequentially backwards.

A13 is the father of a young child. Looking back at Tables 1 and 2 we see that A13 described a counting activity when asked to do so, but for an enumeration activity he listed number operations. His planned activity for a child between the ages of three and four was as follows:

> Aim and setup of the activity: The aim of the activity is to encourage the child to count backwards from ten to one. The activity relies on the child's knowledge of number symbols. The activity requires a drawing of hill with numbers going up the hill from 1 to 10 and then numbers going down the hill from 10 to 1 (see Figure 1). At the top of the hill, appears the number ten.
>
> Activity: You tell the child that we have a ball that went over the hill, and so we must retrieve the ball. In order to do so, we have to climb the hill while counting forward, and then climb down the hill while counting backward.

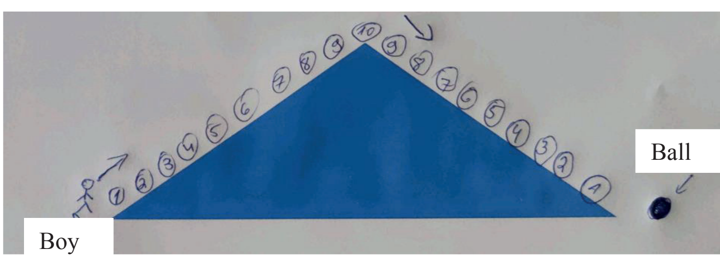

Figure 1: Counting up and counting down.

A13's aim of practicing to count backwards is met in this activity. He assumes that the child will identify the number symbols, and although this is not stated specifically, having the numbers to follow while going down the hill, will indeed support the child who does not yet know the backwards sequence of numbers. Furthermore, if a child does know how to count backwards, but does not yet identify number symbols, then this activity will promote this competency as well. A13 also wrote, "Because the activity is like a story-game, the child will be engaged, and we will have reached our aim through a positive experience." In other words, one of A13's aims is to provide a positive mathematical experience for the child.

A8 was an aunt to young children. As can be seen from Tables 1 and 2, although both of her suggestions were for enumerating activities, when asked specifically for an enumeration activity, A8 adds details regarding the arrangement of the objects.

After the fourth lesson, A8 planned the following activity for a child she specified as between the ages of four and five years.

> Aim and setup of the activity: The aim of the activity is to encourage children to count forwards and backwards from 1 to 10, and to allow children to enumerate amounts from 1 to 10. Objects needed for the activity are large hoops, balls, and a basket (it doesn't matter what size or color the objects are).
>
> Activity: A path on the floor is laid out with 10 hoops. In each hoop there is a number from 1 to 10 in numerical order. At the end of the path there is a ball and in the beginning of the path there is a large basket. The children are requested to jump from one hoop to the next while counting out loud, one, two, three and so on till they reach ten. Then they pick up the ball, and go back down the same path, this time counting backwards 10, 9, 8 till they reach the end, throw the ball in the basket and say zero. After that, each child is requested to count/enumerate the balls in the basket and say how many balls are in the basket. In other words, the first boy will count one ball, the second two balls, and so on.

A8's planned activity is in line with her stated aims of promoting two counting skills – counting forwards and counting backwards – and enumerating. Although not specified in the aims, we recognize that A8 also includes identification of numerals in her activity. The activity is playful, involving jumping and throwing balls. Note that A8 separates the counting activity from the enumerating activity. The child is not asked to say how many jumps took place. The jumping is more of a motivation to keep the counting moving forward. The enumeration part of the activity is separate. A8 specifically writes that the child will be requested to "count/enumerate" the balls. By using the term "count" A8 recognizes that a child may not understand the meaning of the term enumerate because it is not a day-to-day concept. However, she herself knows that this is an act of enumeration. Furthermore, by specifically stating that she would ask the child to say "how many" balls there are, she is promoting the cardinality principle. Finally, stating that the balls can be any size or color can promote the abstraction principle. Thus, although A8 does not state these principles as aims of the activity, we can identify these principles in her plan.

DISCUSSION

In our previous studies, we found that many adults were not aware of the difference between counting and enumerating (Levenson et al., 2021b). In those studies, adults did not necessarily have a mathematical background. Although participants in this study were mathematics education graduate students, prior to the course, they too were not aware that verbal and object counting are separate competencies and that each includes separate sub-competencies. Furthermore, some of the adults in this study mentioned additional competencies, such as ordinal numbers, identifying number symbols, and number operations, when asked to suggest counting and enumeration activities. While those other competencies are noted in the preschool curriculum (INMPC, 2010), and may be related to counting and enumerating, they are separate skills.

While we recognize that this study was limited by the small number of adults who attended the course and cannot be generalized, the activities described above by the three participants exemplify how interventions affect different people in different ways. After four lessons focused on counting and enumerating, some, like A14, may have increased their awareness of early number competencies, but may still not be able to plan an activity that aims to promote a specific skill. Recall, however, that A14 stated in the beginning that he did not know of any counting or enumerating activities. Others, like A13, might have already been aware of a specific competency, but are now able to plan an appropriate activity. Finally, there are those like A8, who can now plan an appropriate activity that purposefully aims to support several competencies.

All three participants added details to their activities that were not present prior to the course. Furthermore, in line with Greenes et al., (2004), all three designed activities that would engage the children in an active way, while allowing them to physically experience the sequence of numbers, either by seeing the decreasing number of dolls on the bed (A14), or by traveling up the hill as the numbers increase (A13). Although they did not state so explicitly, the activities of A13 and A8 could also be used to encourage one-to-one correspondence, as the children go from one number to another, saying one number for each time they climbed or jumped forward. We hypothesize that the increased quality of participants' suggested activities, stemmed from the emphasis on mathematical tasks during the course. Parents are interested in receiving information regarding mathematical activities to do with their children at home (Sonnenschein et al., 2021). Thus, workshops for adults that focus on mathematical activities, may be a way to enhance the mathematical environment at home.

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Students manifesting critical thinking in the mathematics classroom

Part 2

LEARNING TO REASON MATHEMATICALLY WITH MEANING

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The aim of mathematics teaching is to get students not only to learn mathematical concepts and procedures but also to develop the ability to think mathematically. Thinking mathematically involves being able to perform mathematical reasoning, that is, to make inferences in which, from a certain information given, new conclusions are reached. In mathematics, deductive reasoning assumes a fundamental role, allowing the validation of knowledge, but an equally important role is assumed by inductive and abductive reasoning, essential for the creation of new knowledge. Mathematical reasoning has particular characteristics given the nature of the objects of this science, as abstract entities constructed from real-world experiences or from experiences with previously known mathematical entities. The links among mathematical objects and among them and real-world objects allow the attribution of meaning to concepts and reasoning. I present results of recent research work in mathematics education, seeking to highlight key reasoning processes used in mathematics, generalizing and justifying, and show how these processes can develop within the framework of an exploratory approach to mathematics teaching.

INTRODUCTION

One of the main objectives of mathematics teaching is to develop students' ability to reason. This idea raises several issues that we need to consider: What are the key aspects of mathematics reason that we can expect from mathematics students? How can the work in the classroom promote its development? These are issues that I propose to discuss taking into account theoretical issues and analyzing concrete examples.

REASONING AND REPRESENTATION

It is commonplace to say that "mathematics requires reasoning" and also that it "develops reasoning". However, the term "reasoning" is polysemic, as seen by the various meanings given to it by the dictionary:

> To reason: 1. make use of reason to figure out, judge or understand; 2. chain thoughts logically; 3. present reasons; 4. ponder; reflect; think (From Latin ratiocinári) (Porto Editora Dictionary)

To figure out, judge, understand, think logically, present reasons, ponder, reflect... These are many meanings that are far from coinciding! From the outset, the question arises whether "reasoning" will be the same as "thinking" or will be, more specifically, “thinking in a certain way”. In fact, I assume that one should attribute to “reasoning” a more specific meaning than “thinking”. In this perspective, reasoning is to make inferences in a reasoned way, that is, to obtain new information from information given, making it by a justified process. This understanding is in line with another dictionary, which says that reasoning is to establish inferences or conclusions from facts known or assumed to be true. I would just add that this should be done in a reasoned manner and not more or less random. If, in response to a question, a person says the first thing that occurs to him/her without analyzing all the pertinent information, it is not reasoning, it is simply making blind guesses. Thus, all reasoning is thinking, but there is thought that is not reasoning. We think when we describe an object, when we report an event, when we express a feeling, or when we make a wish, but these actions do not require reasoning.

There is reasoning in mathematics and also in other areas of knowledge as well as in everyday life. Of course, the question arises: is reasoning in mathematics different from reasoning in other fields, it has anything specific? Let us look at some general aspects of the reasoning as it develops in the most diverse areas. The study of reasoning is a field of Philosophy, with its roots in ancient Greece, namely with its formalization in the rules of Logic. Aristotle is the first theorist who establishes this discipline and, already in the twentieth century, the development of Mathematical Logic led to great developments and practical applications, particularly in computers. For what interests us here, I will confine myself to note that there are essentially three types of reasoning: deductive, inductive and abductive.

Deductive reasoning is characteristic of mathematics, where it occupies a fundamental place. In this science, we assume a set of statements as true (axioms or postulates) and assume a set of rules of inference, to obtain new valid statements (theorems). Thus, “reasoning deductively involves mainly chaining assertions in a logical way and justifying this chain” (Ponte, Branco & Matos, 2008, p. 89). As long as the chain of deductions is free from errors “deductive reasoning produces conclusions that are necessarily valid” (Oliveira, 2008, p. 7). As Oliveira (2002) points out, deductive reasoning constitutes “the structuring element, par excellence, of mathematical knowledge” (p. 178), and through it mathematical statements are validated. Its importance is such that Davis and Hersh (1995) even claim that deduction is the seal of mathematics.

The fundamental role of deductive reasoning is mainly the validation of knowledge. However, the new discoveries, in most cases, do not arise through deductive reasoning but rather from other types of reasoning, namely inductive and abductive reasoning. George Pólya (1990) eloquently valued the role of inductive reasoning. As he indicates, induction is the inference of a rule from the observation of what is constant in several particular cases. Abduction, in turn, is a process of inference that is based on an unusual fact that seeks an explanation for its occurrence. The great theorist of abductive reasoning is Charles Sanders Peirce (1931–1958), who states that “[abduction] at the end is nothing but conjecture... it is the process of choosing a hypothesis” (Vol. 7, p. 219).

It is true that deductive reasoning has a fundamental place in the validation of mathematical true statements, but it is the inductive and abductive reasoning that leads to the discovery of these statements. Thus, students should learn to reason deductively in mathematics but should also learn to reason inductively and abductively (Rivera & Becker, 2009). Thus, it is of great importance to know how the teacher, in the mathematics classroom, can contribute to students developing their reasoning ability in these three dimensions. To this end, I will focus on three fundamental processes: conjecturing and generalizing, fundamental processes of inductive and abductive reasoning, and justifying, a fundamental process of deductive reasoning. Conjecturing is proposing that something must be true. It may refer to a particular object, or to a whole range of cases – case in which we have a generalization. Justifying is to give reasons why some statement is true.

It should be noted that it is impossible to directly access the mathematical reasoning of students – to know this reasoning it is necessary that they communicate it, which is only possible through different representations. As the NCTM (2000) indicates, “Teachers can gain valuable insights into students' ways of interpreting and thinking about mathematics by looking at their representations” (p. 68). In addition to making the reasoning known, the representations are an essential support for the realization of reasoning. Without representing mathematical concepts in some way, it is impossible to make inferences about them.

As Bruner (1999) indicates, these representations can be active (objects such as manipulated materials or actions such as “counting by fingers”), iconic (images, figures and diagrams) or symbolic (mathematical symbols, other symbols and natural language). Representations play a decisive role in learning because, as the NCTM (2000) points out, “When students gain access to mathematical representations and the ideas they represent, they have a set of tools that significantly expand their capacity to think mathematically” (p. 67).

Models for the study of reasoning in the classroom

For a long time, studies in mathematical education on reasoning processes focused exclusively on deductive reasoning (Balacheff, 1888; Galbraith, 1995; Hanna, 2002). Most of these studies tended to see inductive and abductive reasoning as obstacles to mathematical reasoning. More recently, there has been a significant change in this regard – rather than being seen as conflicting, these different forms of reasoning have come to be seen as complementary.

Lannin, Ellis and Elliot (2011) developed a model in which the “central idea” is that “mathematical reasoning is an evolving process of conjecting, generalizing, investigating why, and developing and evaluating arguments” (p. 12). In this model three poles stand out, the first being “conject and generalize”, the second “investigate why” and the third “justify or refute”. It is a model that combines deductive aspects and inductive and abductive aspects.

A comprehensive discussion about mathematical reasoning was provided by Jeannotte and Kieran (2017). These authors consider three main kinds of reasoning processes. Some processes concern the search for similarities and differences, and include generalizing, conjecturing, identifying regularities, comparing and classifying. Other processes are related to validating, such as justifying, proving and making formal proofs. Finally, there is exemplification that the authors consider having a heuristic value.

Another model is from Ponte, Mata-Pereira and Quaresma (2013), and seeks to frame the reasoning with two other fundamental processes, representing and sense making (Figure 1). The authors stress that mathematical reasoning necessarily relies on representations and requires sense making regarding the objects and actions involved. This model is based on the entire process of conducting an investigation or solving a mathematical problem, beginning with the formulation of questions, moving to the formulation of conjectures and solution strategies (generalization), going to the application of these strategies and the test of conjectures, up to the validation process (through justification). Thus, conjecturing, generalizing and justifying stand out as essential aspects of mathematical reasoning.

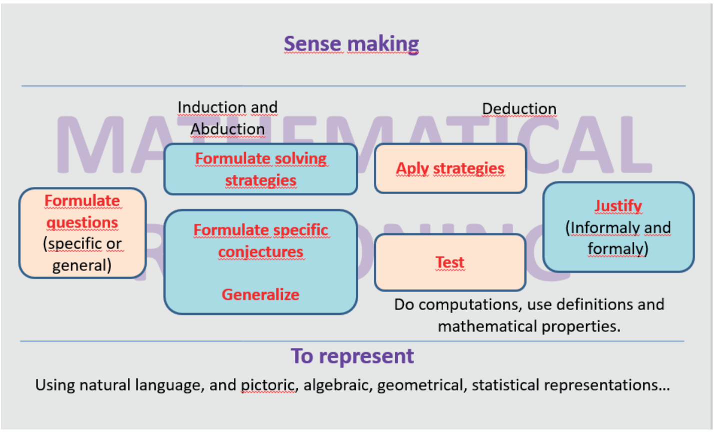
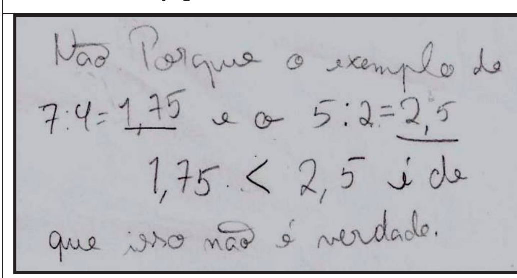


Figure 1: Conceptual framework for analyzing mathematical reasoning of Ponte, Mata-Pereira and Quaresma (2013).

ANALYSIS OF STUDENTS' REASONING

Students' reasoning can be communicated orally or in writing. Let us look at some examples of reasoning that is apparent in students' written responses to mathematical tasks. Figure 2 shows a justification by counter example, an important process of mathematical justification. The student indicates that the answer to the question is negative because a case can be given in which the statement is false. It is interesting to note that the student made several changes of representation to give his answer. First, it converted the fractions $\frac{7}{4}$ and $\frac{5}{2}$ towards quotients (7:4 and 5:2) and then he converted these quotients into decimals (1.75 and 2.5). It is in this representation that the student considers that the justification becomes convincing since 1.75 is undoubtedly less than 2.5.

<table>
<tr><td colspan="2">Question – If a fraction has numerator and denominator greater than another fraction, is it necessarily greater than this second fraction?</td></tr>
<tr><td>
Não Porque o exemplo de
7:4=1,75 e o 5:2=2,5
1,75 < 2,5 é de
que isso não é verdade.</td><td>No. Because the example of...

and

and that is not true</td></tr>
</table>

Figure 2: Answer of Marco (grade 5) to the question.

Another example of students' reasoning is given in the answers indicated in Figure 3. To the first question, is it $\frac{2}{4} = \frac{8}{16}$?, the student answers affirmatively and, without being asked, he immediately proceeds with a justification, based on a change of representation: $\frac{2}{4}$ is equal to 0.5 and $\frac{8}{16}$ is equal to 0.5, therefore $\frac{2}{4}$ is equal to $\frac{8}{16}$, since two quantities equal to a third one are equal to each other. In the answer to the second question, the student further details her justification and presents a curious generalization: "A number divided by its double is equal to 0.5".

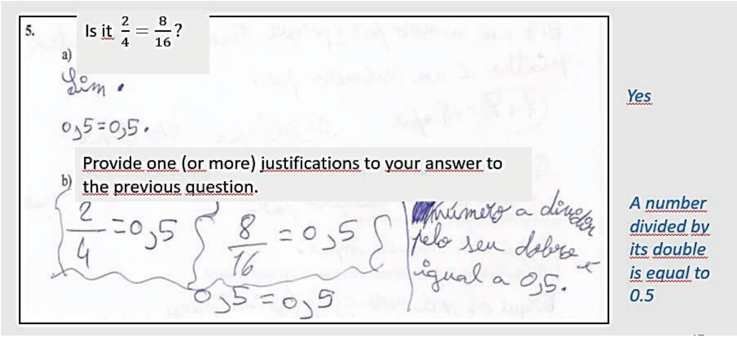


Figure 3: Justification and generalization of Catarina (grade 5).

Let us see now how reasoning may be promoted in the classroom through the exploratory approach (Ponte, 2005; Ponte & Quaresma, 2020). In this approach, students' learning is supported by the work on tasks in which, using prior knowledge, they develop new ideas, concepts and representations, and are prompted to establish new conjectures and generalizations and to justify them. Tasks need to be mathematically fruitful and allow for students' involvement. Usually, the work develops in three phases: (i) Launching of the task; (ii) Students' autonomous work, in which they work in interaction with their colleagues, supported by the teacher; and (iii) Whole-class discussion, in which different students' solutions are presented and discussed, and a final synthesis is made, so that all students in the class appropriate the main ideas.

Let us look now at some situations of an actual mathematics classroom. The examples are drawn from the work on the task "Edges of pyramids and prisms" (Figure 4), proposed to 12-year-old students (grade 7).

1. Do all pyramids have an even number of edges? Justify your answer.
2. And regarding prisms, will they have an odd or even number of edges? Justify your answer.
3. Can you find another property regarding the total number of edges of a prism? What is that property?

Figure 4: Task – Edges of pyramids and prisms.

This task is oriented towards the realization of generalizations and justifications. Question 1 requires a generalization and a justification on the number of edges of pyramids. Question 2 begins by asking for a generalization about the number of edges of a prism and then asks the student to justify the answer. Question 3 calls for an additional generalization of the number of edges of a prism. The first two questions are oriented, specifying the generalizations or justifications to be made, while the third question is open, asking for the formulation of a property, without specifying which one. The students are seated at double tables and, as it is usual in mathematics classes, they work in pairs.

Launching the task

At the launching, the teacher considers important to ensure that the students know the mathematical terms of the statement. He begins by asking a student to read the statement of question 1, which leads to a small discussion about the meaning of “edge”:

> Teacher: OK, this is the first challenge you’re going to have to think about. Is everyone comfortable with what edges are?
>
> Jaime: No...
>
> Teacher: Ana, what is an edge?
>
> Ana: It is this from the pyramid.
>
> Teacher: “It is this.” Can anyone define what an edge is? Other than that, “it is this.” Diogo.
>
> Diogo: That part of the sides.
>
> Several students: That’s the sides.
>
> Teacher: Is the sides of the pyramid? What is it? The faces?
>
> Irina: The lines that determine the sides.
>
> Teacher: The lines that determine the sides…
>
> Bernardo: The segments.
>
> Teacher: The segments, we already approaching a more correct mathematical language. They are the segments that join any vertices of the pyramid. So, when I join one vertex to the other, that line is called edge and it is a straight segment... So, think if the number of edges of a pyramid is always even.

In this dialogue, by asking questions, the teacher sought to lead the students to develop students’ appreciation for a precise description of the concept of “edge”. In the last intervention, he restates the challenge provided by this question.

Students’ autonomous work on a task

During students’ autonomous work (in question 1), the first main goal is that students reach a generalization. The teacher circulates around the room, observing the students’ work and interacting with them. His interventions have different objectives, depending on what he observes. For example, for students who have trouble in formulating a solution strategy, the teacher gives suggestions that help them reach a generalization for themselves:

> Teacher: Give examples, give examples to see what happens.

In the work on question 2, during students’ autonomous work, the way the generalization is made is also a concern for the teacher. In a dialogue with a student, he helps this student to formulate his generalization more clearly, while recalling the need to justify it:

Duarte: It is the triple.

Teacher: What is the triple?

Duarte: Of the edges…

Teacher: Of the number…

Duarte: Of the number of edges of the base.

Teacher: Think about it and try to put that phrase there. The idea is already there.

Duarte: With the formula?

Teacher: You can write the formula too, but first you must justify it.

Still during students' autonomous work, for the students who quickly solve the task, the teacher proposes extensions. Thus, speaking with a pair of students who had already reached a generalization, he formulated a new challenge suggesting the students to formulate this generalization in a more formal language:

Teacher: Do we manage to get here an expression… An algebraic expression?

During the students' autonomous work, the teacher's actions alternate between guiding, when he asks questions that lead students to clarify their statements ("What is triple?" …) and informing/suggesting, when he points out paths that students can follow ("You can write the formula too, but first you must justify"). In the case of students who are able to answer the questions proposed in the task, the teacher formulates new challenges. In this phase of the work, the teacher seeks not to suggest solution strategies that could lead to decrease the degree of challenge of the task.

The teacher begins the whole-class discussion of question 1 by encouraging the students to share their ideas. He requests the participation of a student, Marta, whose answer is represented on the board. The student's justification is based on the analysis of two particular cases:

Whole-class discussion

Teacher: [Let us] begin with Marta. So, first, read the question, so we're all talking about the same thing.

Marta: [Reads the question] Yes, it's correct, because all the edges added up give an even number. Even so, the triangular pyramid has a base with three [edges], odd, the total number of edges is always even.

Teacher: You wrote something else.

Marta: So, in the triangular pyramid the number of edges is six. At the base, the number of edges is three, so, [the total number of edges] is always double.

Teacher: And you concluded that through an example?

Marta: No, two.

The teacher asks Marta to explain her reasoning. The student's generalization is correct. She uses two examples to justify this generalization, which is mathematically invalid, but the teacher, at this moment, decides to accept and value her contribution.

Later, during the whole-class discussion of question 1, the teacher promotes a reflection on the validity of this justification. He asks the students to identify valid and invalid mathematical justifications, highlighting what validates them:

> Teacher: So, we're in mathematics, aren't we? And Marta is saying, I have an example here that works, I have another example here that works, so, yes, it's true. In mathematics two examples are enough to prove that something is true?
>
> Several students: No.
>
> Teacher: It could be two things that work, three, four, five, a thousand . . . But [that is not enough for us].

The argument that justifies this answer for all pyramids is the possibility of associating, in a biunivocal way, to each edge of the base, a side edge. This is intuitive, but the argument could be made explicit. The teacher does not introduce this discussion, possibly taking into account the age level of the students.

Later, in the discussion of question 2, which refers to prisms, the teacher keeps encouraging the sharing of ideas. He begins by asking a student to read the question and then her answer:

> Rita: It can be even or odd. If the [number of] base edges is even, it's even. If the number of base [edges] is odd, it's odd. Thus, it depends on the number of edges of the base.

The student presents a correct generalization. In order to obtain a justification, the teacher asks for the explanation of the "why":

> Teacher: And how did you get to that conclusion?
>
> Rita: Doing edges times three.
>
> Teacher: And why times three?
>
> Rita: Because we have to know, we have to add [the edges of] the base, plus the side edges, plus [the edges of] the other base.

During the whole-class discussion, the teacher asks the students to present their solutions, starting with partial or incomplete solutions, which he seeks to value. However, he also promotes moments of reflection in order to draw attention to the limitations of these responses. His questions highlight generalizations and justifications. For the most part, they are guiding questions ("And have you concluded this through an example?"…), although there are also informing questions ("And that's a property"...). There are also some challenges, particularly when the teacher seeks to lead students to formulate their answer in a more formal language or when he asks the students to justify their answer ("Why?"). In each question, the teacher seeks that the students' contributions lead to the formulation of a correct answer, which he finally synthesizes in a small informing action.

Making the exploratory approach work

Thus, in the exploratory approach the students work on tasks for which they do not have an immediate solution method – to solve them they have to construct their own methods, using previous knowledge. They have opportunities to construct or deepen their understanding of concepts, procedures, representations and mathematics ideas. The students assume an active role in the interpretation of questions, in the representation of information and in the design and enactment of solution strategies. They are called to present and justify their reasoning. In this approach, the teacher, instead of teaching directly procedures and algorithms, showing examples and assigning exercises to practice, proposes students a work of discovery, and promotes moments of negotiation of meaning, argumentation and collective discussion.

The exploratory approach has two main supports: The first is the choice of appropriate tasks that may promote the construction of concepts, the formulation of strategies for solving problems, conjectures and justifications. The second is the establishment of an environment of classroom communication that enables students' participation and reflection, through the launching, autonomous work and whole-class discussion.

This approach stresses the construction of concepts, the modelling of situations and also the use of definitions and properties of mathematical objects for reasoning – conjecturing, generalizing and justifying. It pays attention to computational aspects of mathematics, but values conceptual aspects – that is, considers important to get results, but even more important to understand the general strategy that was used and its justification.

Role of the teacher

The teacher, paying attention to the reasoning processes underlying the solution of mathematical tasks, contributes to the development of students' reasoning. For example, in primary education (grade 1 to grade 6), the teacher may ask students to formulate and test conjectures concerning simple mathematical situations as well as to explain ideas and processes and justify mathematical results. The teacher may make explicit the use of examples and counterexamples and may use an exhaustive of collection cases as justification processes. This can be done, for example, through actions such as:

- To ask for the explanation of mathematical reasoning orally and in writing.
- To request examples, counterexamples, and analogies.

- To propose the investigation of regularities and numerical relationships in tables, seeking the formulation and testing of conjectures.
- To ask, “How did you do it? Why do you think what you did is right?”
- To ask, “What happens if...? Will this always happen?”
- To request the presentation of arguments as well as examples and counterexamples.
- To encourage students to make generalizations by presenting examples and other particular cases and to pose questions such as, “What will happen next? Is this valid for other cases?”

Later on, in secondary school (grade 7 to grade 12), the teacher can carry out these same actions and others such as:

- To ask students to identify particular cases, formulate conjectures and generalizations and test their validity.
- To provide situations in which students reason inductively (formulating conjectures from data obtained in the exploration of regularities) and deductively (demonstrating these conjectures).
- To highlight the role of definitions in the deduction of properties, for example in the study of quadrilaterals.
- To raise questions that lead to the reduction of absurd as a method of demonstration.
- To request the justification of statements through mathematical concepts, properties or procedures, or their denial through counter-examples.

CONCLUDING REMARKS

To provide more attention to the development of mathematics reasoning, through an exploratory approach, is a necessary condition for mathematics learning with understanding. It must be noted that reasoning is a capacity that, although not exclusive of mathematics, may be promoted in an important way by the work in mathematics. It is important to note, also, that the work with mathematics reasoning is not exclusive of the most advanced grades – it may and should begin in the elementary grades.

To promote the students’ mathematical reasoning, the teacher may carry out a practice that creates opportunities for students to make conjectures, generalizations and justifications. These opportunities depend, in essence, on two aspects: on the characteristics of the tasks proposed in the classroom and on how these tasks are tackled in the classroom. Tasks, in addition to explicitly asking for conjectures, generalizations and justification of responses or solution processes, may have different degrees of challenge. To favour the contrast between different strategies and representations, it is useful to propose tasks with questions that allow a variety of solution processes.

The exploratory class allows students a key role in working in tasks and in expressing their reasoning. In this class, the teacher begins by proposing a task (introduction), followed by a period in which the students work in groups, in pairs or individually (autonomous work), and this leads to a collective moment of presentation and justification of results (whole class discussion). Thus, explicit appreciation of mathematical reasoning in the classroom can be done naturally from this type of work. Emphasizing reasoning constitutes a significant change in mathematics teaching, introducing a new emphasis in the work in mathematics, which may lead students not only to develop their reasoning but also to assume a more positive perspective on what mathematics is as a human activity.

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IMPROVING UNDERSTANDING OF LOGARITHMS USING CRYPTOGRAPHY-BASED ACTIVITIES

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We designed an interactive cryptography-based activity requiring students to code using powers of integers and to devise a strategy decoding messages. We analyse the strategies that participants developed and compare their understanding of logarithms with a control group of traditional intermediate algebra students. Our results show that our study group outperforms the control group in understanding logarithms and basic computational skills.

INTRODUCTION

This study is a part of a larger project supported by NSF Noyce 1660521 grant that included the development of active learning activities promoting mathematical engagement, excitement, and students' creativity. In this paper we describe cryptography-based activities targeting early algebra curricular experiences related to introducing and understanding the concept of the logarithm. Even though teachers strive to prepare their students to use mathematics in powerful ways in various contexts often students' ability to transfer abstract knowledge to application is not effective. Often after being taught logarithms, students continue to struggle to recognize or apply logarithms in even simple situations, which suggests that they have not understand this fundamental concept fully. As research shows, logarithms are quite abstract and difficult for the students (Webb, 2011), even though they are presented with many real-world applications including interest rates in finance, earthquake measures (Richter Scale), population growth, etc. Since understanding logarithms is important for more advanced mathematics and science concepts, finding pedagogy to introduce them early in curriculum and in a meaningful way is crucial. In general, problem solving and creative thinking are necessary for professional success in the fast paced, technology intensive global setting of the 21st century. At every level of mathematics education, there have been criticisms about the excessive amount of structure imposed on learners, especially at the school level, where students are rarely encouraged to solve open-ended problems, strategize, or pose their own questions. In 1989, the National Council of Teachers of Mathematics addressed the need for standards that include modeling, creativity, and independent thinking, but over two decades later the situation in American schools is not much better, as mathematics education still concentrates on basic skills and traditional problem solving. While for a long-time problem-solving strategies have been advocated (Polya, 1957), they are not leading pedagogy in our schools (Drew, 2011).

Studies show that contemporary students prefer innovative rather than traditional pedagogy (Star et al., 2008), learning with multiple representations (Ainsworth, 2006), through activities related to their interests in engaging environments (Kuh, 2003). There have been some efforts to implement new pedagogical strategies (inquiry-based or problem-based learning) to improve students' skills and to move them towards discoveries (Vygotsky, 1987). Recently, mathematics education community started to research the issues related to understanding logarithms and developed some activities and examples supporting learning of the concept (see Campo- Meneses et al., 2021; Kenney & Kastberg, 2013; Siebert, 2017; Suerda & Otero, 2019). However, most of them concentrate on computational or technical aspects (Reed, 2016; Weber, 2019) without giving students a chance to explore and formulate their ideas of logarithms (as inverse operations). For this study, we designed and implemented an engaging and conceptually interesting cryptography-based activity that can be introduced even in pre-algebra lessons. This activity supported personal explorations and asked participants to formulate their own definition of logarithms and strategies to calculate them. At the end of the activity, through the group discussion the formal definition and usual notation were introduced to assure the common language with the mathematical community. We analyse data collected during the activity and compare performance of participants with a control group of intermediate algebra students with similar background.

METHODOLOGY

A group of 20 intermediate algebra high school seniors (who did not yet have a lesson on logarithms) participated in the study. To engage learners from the start we posed the problem of sending secret messages in such a way that outsiders have problems decoding them. There were two 50 minutes sessions dedicated to the cryptography-based activity with the study group, and two days later the post-test was administered.

Description of the activity

Twenty students were divided into 10 pairs. In each pair students were called A and B (in cryptography, people sending messages to each other are usually named Alice and Bob). Matching participants into small groups A's and B's works as well. The following four parts were implemented in order.

a. We introduced to students the well-known cypher numbering the letters (A=1, B=2, C=3…, Z=26.) In each pair, A's were asked to send a numerical code for a short message (a word they selected) to B's, who were supposed to decode the message. B also sent a coded message to A to be decoded. All the numerical codes for messages were also posted on the board where all participants could see them. During the discussion that followed all participants decided that this way of communicating while nice and easy is not secure, as others were able to decode the massages of each pair.

b. To make the cyphers harder, each A was assigned a base number, and each B was assigned a different base number (note that each pair had their own set of base numbers, and members of each pair knew both). To code a message, students had to raise their base number to the power that symbolized a letter they wanted to send. For example, if a student A with the base number 2 wanted to send CAT as a message to B, he/she had to compute 2^3, 2^1, 2^{20} (participants could use calculators) and sent these numbers to B, who had to figure out how to decode them. The numbers were also posted on the board where everyone in the study group could see them. To decode the messages, student had to introduce their own methodology (at this time they did not know that they calculate logarithms!). There were several approaches developed to decoding: making a table for all possible powers of the base, calculating consecutive powers of the base up to the number received in the message, guessing and checking the powers by calculating. Students sent and decoded several messages. During the discussion that followed all participants decided that this way of communicating is more secure than the first one, as other participants were usually not able to decode the massages between a specific pair as they did not know the base numbers.

c. During the discussion, the logarithmic notation was introduced, and the messages sent between students now looked like $\log_3 81$ as they could choose different basis for their codes. At this stage, students were allowed to use their own strategy or the logarithmic tables for computations (but not the *log* function on their calculators).

d. Students were asked to graph on the board $y=a^x$ and $y = \log_a x$ for the base a that they used for decoding the messages in Part b, and compare the graphs. All students could see these graphs and they were able to analyze the similarities and reflections, formulating logs as 'inverse functions' to exponential functions.

Note that participants were not given any suggestions how to decode the messages, they had to invent their own strategies to figure out values of exponents used, effectively calculating the logarithms from the beginning. As the activity was introductory on the pre-algebra level, we worked only with positive integers. However, for more advanced students, positive fractions can be used as base numbers. To introduce negative powers, one would need to renumber the alphabet (for example (A = -10, B = -9, C = -8…, Z = 15.) At first, some of the students were frustrated with decoding tasks asking them to work without specific strategy and requested formulas to be used. Gradually, they developed their own methods of figuring out the coded messages. While the activities proved to be quite challenging, learners were fruitfully engaged and tried to simplify their strategies at every stage. We have collected all participants work including white board graphs, the instructor's observations as well as their post-test answers (that was also administered to the control group). The post-test was focusing not only their ability to calculate simple logarithms, but also their understanding of the definition of logarithm and related basic concepts. Here are the post-test questions that we assess later.

1. Calculate $\log_5 125 =$
2. Which number is larger, $\log_2 8$ or $\log_2 16$?
3. Which number is larger, $\log_3 81$ or $\log_9 81$?
4. Does it make sense to use number 1 as a base for the logarithm? Explain your answer.
5. Can we use number 0 as a base for the logarithm? Explain your answer.
6. Calculate $\log_{10} 1000$. Define logarithm $\log_{10} x$ in your own words.
7. Is the concept of logarithm easy to understand for you?

RESULTS

We analysed data from all 20 students taking part in the cryptography activity (Study Group), their strategies to code and decode secret messages and their answers on the post-test. The same post-test was administered to a control group of 24 students, who were introduced to logarithms using the traditional lecture mode presentation. Note that study group participants were receiving (usually large) numbers which encoded secret messages and were not given any directions how to decode them. Since they were encoding their messages first, they gained some experience of how the encoding process works. Hence knowing the base for the code given to their partners, they could develop some methodology for solving the problem of decoding their messages. For example, after receiving 64 as a message from a partner using the number base 2, they could try to raise 2 to different powers to figure out that $2^6 = 64$. Hence the message received was 6, i.e., letter F was sent to them. Initially, students either guessed the power and checked if the answer is right by computing it, or they started calculations by raising the base number to consecutive powers ($2^1=2$, $2^2=4$, $2^3=2\cdot2\cdot2=4\cdot2=8$, $2^4=2\cdot2\cdot2\cdot2=8\cdot2=16$, etc.).

We start by analysing strategies used. While participants had no problems with encoding, their decoding methodologies varied and changed throughout the activity. Initially, about half of them tried to guess the exponent to which the base number was raised and checked if they were correct by calculating it (3^8 too much, 3^5 too little). Most of them noticed that the numbers may repeat and tried to organize their calculations. The other half just systematically raised the base number to consecutive powers till they reached the desired number. This group also noticed that it is useful to organize their calculations and list them in some order. Table 1 shows gradual modification of strategies during the activity. Calculating logs refers to Part 3 of our activity, when students were changing their base number often.

Strategies	Tables for a^x	Guess and Check	Calculations	Combination
Initial decoding	0%	50%	50%	0%
Final decoding	70%	20%	20%	30%
Calculating logs	80%	10%	10%	0%

Table 1: Strategies used to find x to decode a message given by a number a^x.

As stated in Table 1, at the end the majority of students organized their numbers as tables, often including the new codes for the entire alphabet (which help them to decode the messages instantaneously). For example, one of the tables for base number equal to 3 started as follows.

$A = 3^1 = 3 \quad B = 3^2 = 9 \quad C = 3^3 = 27 \quad D = 3^4 = 81 \ldots.$

Hence, decoding 81 as 4 in the case of base 3 was immediate. Note that when decoding the final message in Part b of the activity, a majority of students used their own tables, or a combination of several methods (depending on how complete their tables were). For example, a student may have a table with powers of 3 up to 3^{20} and the number to decode was larger than 3^{20}. He/she may calculate consecutive powers or guess the exponent and check (with the calculator). Since the numbers tended to be very large, the participants were forced to organize their calculations. The 30% of participants using the combination strategy split into 3 almost equal groups each using two of the methods. After learning the definition of logarithms in Part c, students' strategies changed to relay more on tables (with varying base numbers) or on one strategy that they preferred, see Table 1.

Sample size 20	Correct Codding	Decoding	Engagement
Initial tasks	19	15	High
Final tasks	20	20	High
Discussion	18	12	Moderate
Graphs/functions	20	16	Moderate

Table 2: Correctness of tasks and engagement.

The instructor monitored the engagement level during the activity. As shown in Table 2, participants were highly involved in coding and decoding problems throughout the activity, and at least moderately involved during creating exponential and logarithmic graphs and discussions. This may be related to the fact that a new abstraction level was introduced, and some students needed more time to understand it.

Note that almost all students coded the messages correctly at every stage, and one made several computational errors. Only 75% of students decoded all messages correctly, while the remaining 5 (25%) usually did not complete the decoding process on time, especially when their partners changed their base numbers. However, all students in this group decoded at least 50% of the messages received. During the discussion phase, 18 students described the coding process correctly, while 12 (60%) students explained the decoding methodology correctly in precise mathematical language. All students were able to graph the exponential functions using their calculations, while only 16 of them provided proper graphs for logarithmic function. We observed that most of them were confused with the fact that to graph $y = \log_a x$ they had to use many large values of x and did not know values of y for many small x's.

The post-test was given to the study group (20) and the control group (24) and participants were not allowed to use calculators as the numbers were relatively easy to handle. Before the activities both groups were compatible and both improved understanding of logarithms through the activity or through the lecture. Our interesting result shows that the study group performed better on all the questions, and statistically significantly better overall on the post-test as compared to the control group (with the p-value below .001). Which means that their understanding and skills were significantly enhanced by the participation in the cryptography-based activity. Here we present and compare detailed results of student performance.

	Correct	Incorrect	Partial Credit
Study Group (20)	95%	5%	0%
Control Group (24)	50%	41%	8%

Table 3: Correctness of calculating $\log_5 125$.

Note that almost all students in study group were able to figure out $\log_5 125 = 3$, while only 50% of the control group calculated it correctly, with 2 people obtaining partial credit for expressing the result as an exponential equation only.

	Correct	Incorrect	Partial Credit
Study Group (20)	100%	0%	0%
Control Group (24)	67%	12%	11%

Table 4: Comparing $\log_2 8$ with $\log_2 16$.

As table 4 shows, all students in the study group figured out that $\log_2 8 = 3$ is smaller than $\log_2 16 = 4$, hence they considered comparison of two logarithms with the same (small) base as easy. At the same time, only 16 (67%) students in the control group answer the question correctly and 11% did the calculations but did not finalize the comparison and received some partial credit.

	Correct	Incorrect	Partial Credit
Study Group (20)	85%	5%	10%
Control Group (24)	50%	42%	8%

Table 5: Comparing $\log_3 81$ with $\log_9 81$.

The next task was to compare two logarithms with different base numbers, see table 5. The study group outperformed the control group with 17 students (85%) able to figure out that $\log_3 81= 4$ is larger than $\log_9 81=2$, and some with partial credits for proper calculations but no conclusions. On the other hand, only 50% of control group performed the task correctly.

Table 6 shows students' answers to the question: Should 1 be used as a base for the logarithm? The students who had experience with the cryptography provided extensive explanations (such as 'if we use 1 as a base, then coding does not work because $1^x=1$,'; 'cannot use 1, as for example $\log_1 3$ = does not exist, because 1^x always equals to 1, not 3'). 80% gave correct explanations, 15% gave partial explanations, and 5% (one student) gave no explanation. In the control group only 6 students (25%) gave a correct explanation while the rest either did not answer the question (50%) or gave incorrect answer, see table 6.

	Correct	Incorrect	Partial Credit
Study Group (20)	80%	5%	15%
Control Group (24)	25%	75%	0%

Table 6: Do logarithms with base 1 make sense?

However, explaining why number 0 cannot be a base for the logarithm turned out to be a confusing question as a large majority of incorrect answers (70%, i.e., 31 out of total 44 total number of students in both groups) stated that "since any number in power 0 is equal to 1 then $\log_0 1= 1$ follows from $1^0 =1$".

	Correct	Incorrect	Partial Credit
Study Group (20)	50%	50%	0%
Control Group (24)	12%	88%	0%

Table 7: Do logarithms with base 0 make sense?

Still half of the students in the study group gave a correct justification, while only 12% of study group provided any satisfactory explanation. All students perceived this question as unusual, and they had hard time analysing it. Many stated that this 'never happens'. The summary of the results is in Table 7.

The task to calculate $\log_{10}1000 = 3$, was done correctly by 90% of all students. Then they were supposed to define $\log_{10}x$ in their own words. Here are examples of answers accepted as correct:

$\log_{10}x$ is a number n that is used as an exponent in 10^n to obtain number x;

10 in some power equals x. This power is a number called $\log_{10}x$.

Table 8 shows that all but 2 students in the study group came up with some definition, and 80% (16 students) stated it correctly. While only 25% (6 students) from the control group gave an acceptable definition. This was a surprising result, as the study group learn logarithms hands-on through the cryptography activity and did not spend much time on formal definitions. The control group was introduced to logarithms by a formal definition from the very start of the lecture, but they did not master their understanding after two lecture sessions with examples and problem solving.

	Correct	Incorrect	Partial Credit
Study Group (20)	80%	10%	10%
Control Group (24)	25%	75%	0%

Table 8: Definition of logarithm $\log_{10}x$

The last question on the post-test was asking students to assess how hard logarithms are. Only 5% of students having experience with cryptography considered the concept hard, while 63% of the control group thought that working with logarithms is at least medium hard.

	Easy	Medium hard	Very hard
Study Group (20)	70%	25%	5%
Control Group (24)	12%	13%	50%

Table 9: How difficult are logarithms to you?

This huge difference in assessment of the difficulty of the concept of logarithm by the two groups also came as a surprise. It looks like the cryptography-based activity removed some of the mystery from the concept, introduced students to computations with large numbers and asked them to develop strategies to figure out proper exponents to obtain specific large number matching the message send. These help them understand the underlying idea of inverse functions of the exponential functions.

All study group participants (100%) reported that they enjoyed the interactive cryptography activities and found the discussions helpful for understanding the logarithms. The initial coding tasks inspired curiosity in modern applications of mathematics. Participants showed perseverance deriving various strategies to calculate logarithms and expressed concerns about dealing with large basis and large numbers (even when aided by computers).

Some interesting comments from the study group students.

Student 1: I liked coding and decoding with numbers. Now I start to understand how real coding is done with large numbers, so decoding is hard for unauthorized people.

Student 2: I enjoyed the activity. It showed me how calculating powers and finding exponents can be useful for secret messages. Discussions helped me understand the logs.

Student 3: Figuring out the strategy to decode the message was hard. I was guessing at the beginning, but than just made a table for all the options. I never though math can be so fun.

Student 4: I tried to send hard messages (with z's and y's) to make decoding hard and interesting. Logs are ok. I wish we were taught that way!

Student 5: I liked having different basis and debating the strategies. Cryptography made me interested in logarithms and math.

CONCLUSIONS

It is important to provide activities requiring critical thinking and original strategies in various contexts interesting to the students. It is beneficial to them to struggle a bit by developing their own ways and to share them with others to come up with definitions for new concepts. Our study provided participants with personalized tasks, by giving them their own base number and a choice of messages to be send. This made them engaged in the strategy planning for quick decoding and interested in calculations with large integers. They were sharing their ideas during the discussion periods, developing an abstract definition for logarithms. Their comments indicated the suitability of the cryptography-based activities as engaging, modern and relevant. Interestingly, the study group significantly outperformed students that have learned about logarithms at a traditional lecture. They performed better on conceptual and computational questions showing overall better understanding of the subjects. Our results show that activity-based learning providing students with interesting, independent, unpredictable tasks leads to better command of the subject and to development of strong computational skills. Therefore, to improve the overall performance and attitudes of students towards mathematics, there is a need for further development of learning activities in many areas and at various levels.

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OBSERVING CRITICAL THINKING DURING ONLINE PAIR WORK

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The closures of schools caused by the pandemic and the transition to online education put teachers and students in a difficult position. When teachers were asked to select their top three concerns about distance learning on students, common answers were: students' social isolation, decreased student well-being, and potential learning loss. We elaborated a chapter from the curriculum in a problem-based way suitable for online learning with these concerns in mind. We also paid attention to the manifestation of students' critical thinking, learning outcomes, and motivation. In this paper, we aimed to analyze a part of a lesson in which we paid increased attention to observing critical thinking and the teacher's role during online pair work and students' reflections about that.

INTRODUCTION

Flack et al. (2020) claim that teachers' top three concerns about online learning are social isolation, decreased student well-being, and potential learning loss. Therefore, we designed a teaching project in which we transformed a part (one chapter) of the curriculum into a problem-based approach, that is suitable for online learning and addressing these concerns. This paper highlights social isolation; we want to reduce social isolation through pair work and pay attention to students' critical thinking by constantly following their work through video/audio recordings. We focused on only one part of a one-hour lesson, namely pair work completed with whole-class discussions, in which we analyzed students' critical thinking. We aimed to integrate pair work into an online environment and follow these pair discussions during problem-solving. We wanted to find out whether the adaptation of the problem-based learning for the online environment is successful or not in terms of the appearance of critical thinking in pair discussions. We were also interested in the students' views on online pair work, so we analyzed the related issues through semi-structured interviews.

LITERATURE REVIEW

In mathematics education, a problem is a task that requires the application of an unknown combination of tools or a novel combination of several known tools to solve a problem and is not obvious to the problem solver (Claus, 1989; Dörner, 1983). Generally, a common feature of problem definitions is that there is an obstacle to achieving a goal in a situation. The way to overcome the obstacle is problem-solving and purposeful reasoning (Polya, 1962). The use of mathematical problems in mathematics education can be achieved through various educational strategies. In this paper, we define problem-based learning in mathematics as requiring students to analyze mathematical problem situations, approach their own and their peers' minds critically, and learn to explain and justify their reasoning (Csíkos, 2010; Kónya & Kovács, 2021).

In the above interpretation, the critical attitude and thinking towards one's thoughts or peers appear. Because critical thinking is a complex concept involving cognitive skills and affective dispositions, we can find many definitions.

Semil (2006) claims that critical thinking makes individuals think, question issues, challenge ideas, generate solutions to problems, and make intelligent decisions when faced with challenges. Therefore, critical thinking skills enable one to analyze and synthesize information to solve problems in a broad range of areas (Facione, 1990). Three components of critical thinking in mathematics were identified: reasoning, problem-solving, and identifying the suitability of problem solutions (Innabi & Sheik, 2007). In other studies, critical thinking is viewed as a more general competence, including metacognition, intellectual perseverance and autonomy, reasoning, and the ability to identify inconsistencies and contradictions (Paul & Elder, 2002). In this paper, we use Mansoor and Pezeshki's (2012) definition, according to which critical thinking involves deep reasoning and consideration of the received information rather than forward acceptance of different ideas.

All these definitions prove that critical thinking is one of the basic components of the skills needed to handle certain situations. According to this view, critical thinking is just as important as information search and organization skills, effective communication, and social responsibility (Ananiadou & Claro, 2009).

Our research questions were formulated based on these defining skills.

RQ1. How does critical thinking appear in the online pair discussions?

RQ2. How do students reflect on the pair work in relation to a task that requires critical thinking?

RQ3. Can the teacher be a critical observer of the pair work in the online environment?

THE SETTING OF THE EXPERIMENT

The investigation included two seventh-grade classes, a total of 61 13-14 years old children from Transylvania, Romania (Class A - 31 students, Class B - 30 students). The same person teaches the two classes. The instruction language is Hungarian, as Hungarian is the mother language of the students. The teacher was herself the researcher, too, delivering action research, aimed to design a unit from the curriculum in a problem-based way. The title of this unit was: Equations and problems that can be solved by equations, containing six lessons.

We used the online pair work method three times during this period. After the experiment, we conducted a semi-structured interview with 12 students. The students were selected randomly from three different categories regarding their learning outcomes: above average, average, below average, two from each category per class.

In order to document the experiment, different research instruments were used. Due to the pandemic, all lessons were held online, using Google Meet and editable shared documents. The shared documents allowed the teacher to monitor the students' works in real-time. The lessons were recorded, which helped the teacher in reflecting and analyzing. The students recorded all the discussions during the pair work and sent them to the teacher. The teacher also made a written record. Every worksheet filled out by the students was photographed, their notebooks were scanned, and every online document they were writing in was saved. These documents helped us to interpret students' oral manifestations.

In this paper, we analyze the pair work connected to the following two tasks (Mason, 1988):

Task 1: The following pattern is given, make a table showing the number of points and line segments in each figure. Find a rule and a correlation between the number of points and line segments. Explain the rule!

Figure 1: Pair work, task 1.

Task 2: The patterns are laid out from square tiles in the figure below. Make a table showing the number of tiles in each figure.

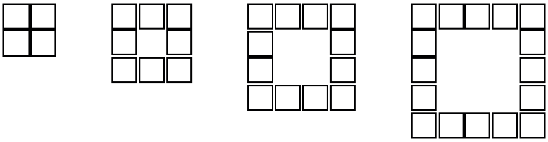

Figure 2: Pair work, task 2.

a. How many tiles will there be in the 9th figure?

b. How many tiles will there be in the 20th figure?

c. Which figure will have 98 tiles?

d. Generalize: how many tiles will there be in the nth figure?

Based on Mansoor and Pezeshki (2012), we examine critical thinking from two perspectives: (1) conjecture and reasoning; (2) consideration of the received information rather than forward acceptance of different ideas.

RESULTS

Conjecture and reasoning

In most cases, pairs managed to identify the rule(s). For example, in Task 1, the standard answer was "the number of line segments increases by five," but in both classes, some pairs formulated a rule such as "$6 \cdot n - (n - 1)$", where n is the number of the points, or even counting further "$5n + 1$". In Task 2, almost every pair came to the rule "$4 \cdot n$", where n is the number of the figure.

It was also interesting that some pairs desired to discover more than one rule. For example, S1 identifies the rule "five times the points plus one", to which S24 replies: "…yes. But I think there is one more. Because it's also pretty clear that 6+5 is 11, 11+5 is 16, 16+5 is 21, and so on […], but your rule is also good". S20 claims, "I have two solutions"; S17 asks, "is there anything else?" or S34: "What other rule could we discover…?".

One of the students' tasks was to explain the rules they found. Analyzing the recordings, we found out that students react to the question "why?" in three different ways, so we sorted the ability of the reasoning into three categories:

1) They explain the rule, and the reasoning is correct.

Pair S19-S4 (Task 1):

S19: […] because the first one has six [line segments], and then every time it increases plus 5 comes, cause…

S4: Because there will be a joint side!

Pair S6-S21 (Task 1):

S6: For three, it is like 6 times 3, that is 18, but there are two joint sides, so minus 2. I think that is how it is.

Pair S14-S29 (Task 2):

S29: You divide it into two parts for long lines and shorter ones…

S14: Well, yes…

S29: You call the top two long lines; those are the full rows. And the shorter ones are the vertical ones. […] So twice the number of the figure plus two - so these are long ones - plus twice the number of figure minus two.

2) They would like to explain the rule, but they can't.

Pair S44-S58 (Task 1):

S44: But the point is why? Why does the number of sections increase by five?

S58: That is a very good question…

S44: I don't know why either, but I think the rule is right.

3) They have no need for an answer: after revealing the rule, they leave the breakout room because they think they have finished.

Interestingly, we could not find a recording in which students would like to explain the rule, but the reasoning contained a mistake. We would wonder if this category would be omitted even for a larger sample.

Consideration of the received information

Regarding the consideration of the received information, we found two main patterns.

1) Students with different learning outcomes

Listening to the students' recordings, we have recognized a phenomenon that we were already familiar with (Báró, 2021). Namely, students with lower learning outcomes tended to accept their partners' ideas just because they usually get better marks, even if the students' solution with better results was wrong and their solution was the right one (pairs: S53-S43; S56-S45; S8-S23; S56-S41). Critical thinking failed in these situations because the partner's higher math grade became an influencing factor; hence the review and analysis phase was missing. In a worse case, students with lower grades are not even thinking about the solution or answer because they know their partner will find a solution (pairs: S5-S20; S15-S30; S32-SS48).

2) Students with similar learning outcomes

We observed that pair works are more effective between students with similar learning outcomes. If they are on the same level, they are more willing to correct their partners, add another idea, or question what has been said. The following dialogue exemplifies the appearance of critical thinking, where students correct each other.

Pair S9-S27 (Task 1):

S27:	So how will it be here? So five times the number of points then multiplied by… no…
S9:	No need to multiply!
S29:	We do not multiply…
S9:	Five times the number of points plus one.
S27:	Yes. For example: in the first one, five times 1 plus 1 is 6. Checkmark here.
S9:	Yeah.
S27:	The second one…5 plus the number of the points plus 2… it is not 11!
S9:	But… no, no, no! Five times the number of points plus one.
S27:	Yes, because it will be $2 \cdot 5 + 1 = 11$; $3 \cdot 5 + 1 = 16$; $4 \cdot 5 + 1 = 21$.

Students' reflections on the pair work

Based on the interviews, we explored students' attitudes towards pair work. Analyzing the answers to the question "Do you usually prefer to work alone, in pairs, or in groups?" we found that above-average students have an ambivalent or negative attitude towards pair work, while others have responded positively.

The higher achievers tended to answer "alone" or "it depends on whom I am working with and the type or the quantity of the task." The following extract reveals one of the reasons why an excellent student prefers working alone to working in pairs or groups. This answer is strongly connected to failing critical thinking, i.e., low achievers accept the opinion of high achievers.

T: Do you usually prefer to work alone, in pairs, or in groups?

S32: Alone.

T: Why?

S32: Well, first of all, I don't like to put my opinion on someone else, and I don't like it if someone else wants to convince me of what I don't think is true. It usually happens- not only in math teamwork- that I say something I don't want others to accept, but they accept it […], but I want everyone to add in the meantime.[…]

T: What do you think? Why do they accept your idea straightforwardly?

S32: I don't know. Only because I am a good student doesn't mean I am always right. Maybe someone else's opinion would be much better, and overall the work of more people would be better, and I don't expect it to be what I say. […] And I accept that if someone goes beyond that and says even better, […], but there are times when they just let it be what I say.

In contrast, students with average or below-average learning outcomes preferred working in pairs or groups rather than working alone, saying that "my main characteristic is that I am not confident and if my partner thinks the same, it is good because it is possible that I can get a little more confidence"; or "for example, I knew the answer to one, and she knew the other, and that's why we complemented each other and explained things to each other".

The role of the teacher in and after the pair discussions

The sub-point c of Task 2 requires making an intelligent decision at the end of the problem-solving. After identifying the rule "four times the number of the figure" and calculating how many tiles are needed in the 9^{th} and 20^{th} figure, at sub point c, the students need to think backward at first, dividing 98 by 4. Then, observing that the answer is not an integer, they have to make the decision "this problem does not have a solution".

The teacher was monitoring the pair work through a shared document with the students' answers and she noted the answers that needed correction. Some students (e.g., S38-S59) were not attentive enough, interpreting the question like "how many tiles are needed in the 98^{th} figure?" and answering 392 because 98 times 4 is 392. A common mistake was not drawing the conclusion, just dividing and answering "24.5".

Pair S10-S16 (Task 2c):

S16:	So which figure will have 98 tiles?
S10:	Well…
S16:	I think we have to divide 98 by 4.
	[…]
S10:	Yes, I think so; we have to divide by 4. That is not a decimal number?
S16:	Indeed. It is 24.5.
S10:	I think so too.
S16:	But it cannot be otherwise. Let's write it down.
S10:	Ok, it's 24.5 then.

Thanks to the class discussion after the pair work, we had the chance to clarify these situations and correct the eventual mistakes. However, it was not even the teacher's job to correct them; the students reacted to each other's pair work. During the class discussion, one aspect of problem-based learning became more emphasized; they approached their peers' minds critically. For example, when one of the pairs showed their solution of "392 tiles" mentioned above, the answer came right away from another student: "No…the question is not about how many tiles will be! But in which figure will be 98 tiles?! So, you have to divide 98 by 4". These pairs, that solved the problem correctly also reasoned correctly, explaining and justifying their reasoning as the second aspect of problem-based learning: "and there won't be a figure with 98 tiles, because 98 cannot be divided by 4"; "This is not an integer, […] so there will not be a figure like that."; "24.5. So there is no such figure. That's our luck!"

What is the teacher's role in such a lesson? As it seems, the teacher does not participate in the pair conversation (only monitoring), not even in the class discussion that follows the pair works. However, it does not mean that the teacher is not part of the lesson; she just goes into the background, who coordinates the conversation. She listens to the discussions critically, analyses and synthesizes the information coming from the different pairs, and, when necessary, intervenes. If a problematic situation appears, she asks clarifying questions or clarifies them herself. In other words, the teacher is a critical observer who will only intervene if the situation is troubling. In this way, if the pair work slips away, it can be repaired in the class discussion phase.

SUMMARY

In conclusion, we can claim the adaption of pair work for the online environment was successful. Audio records of pair discussions allowed us to listen to students' problem-solving processes. We also had the opportunity to map their critical thinking or even discover why critical thinking could fail in certain situations by analyzing their conversations. In these two classes, we observed that lower achievers tend to accept their partners' ideas, even if that is wrong, but they are willing to correct their partners of the same performance. We also found out that the interviewed high achievers prefer working alone than in pairs, mainly because they feel they are working alone even if they are in pairs. This fact is worth addressing, along with many research deals with the topic of whether high-performance students prefer working alone (e.g., Walker & Shore, 2015). We also observed through class discussions that generally, students were happy to correct each other in a larger group, analyze their partner's or classmate's solution and evaluate the received information. The teacher had to intervene only in troubled situations, being her role the critical observer. In this paper, we mainly highlighted the critical situations because we want to pay attention to the deficiencies and try to eliminate them in the future.

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FIRST EXPERIENCE WITH PROBLEM-POSING: WHAT CAN BE DONE WITH A MULTIPLICATION TABLE?

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Critical thinking has been identified as a principal skill to face the world. Thus, it should be encouraged in all educational levels, including teacher preparation programs. This research is guided by a question: How does problem-posing promote prospective teachers' critical thinking? The participants were directed to observe patterns in the multiplication table and pose problems to their peers. Data analysis was carried out by observing the dialogues of prospective teachers, which might be in the form of either mathematical or non-mathematical questions, simple responses, and evaluations of their peers' oral manifestations. As a result, this research implies that problem-posing promotes the emergence of manifestations stimulating critical thinking.

INTRODUCTION AND RESEARCH AIMS

One of the factors contributing to improved education quality in Indonesia is a change in the curriculum, from a teacher-centred rote learning method to an active student-centred method. Since 1994, the adjustment has been aimed at preparing students with some essential skills such as critical thinking to face a variety of challenges in their life (Mailizar et al., 2014).

Even so, while the study by the Indonesian ministry of education and culture (MOEC) and World Bank revealed several positive aspects of teaching and learning, it also highlighted the teacher-centred nature of many classrooms (Tobias et al., 2014). Transition efforts to a student-centred method are still shadowed by the persistence of the prior teaching tradition. As students grow up to become teachers, it is likely that prospective teachers (PTs) will continue the tradition unless they are properly assisted (Chapman, 2012). Thus, to support the progression and to reach the aims of the current curriculum, providing PTs with several approaches that lead to active student-centred learning through their empirical experiences is considered noteworthy. One such approach is problem-posing.

This paper reports an introductory lesson on problem-posing which utilized a well-known material, the multiplication table. As several scholars stated that problem-posing can stimulate critical thinking (Bonotto, 2013; Maj-Tatsis & Tatsis, 2021), it is worth knowing how it might happen. To give a clear picture of it, it is investigated through the following research question: How does problem-posing promote PTs' critical thinking?

THEORETICAL UNDERPINNINGS

Mathematical problem-posing exists within the central importance of mathematics discipline and the nature of mathematical thinking (Bonotto & Santo, 2015). One of notable perspectives on mathematical problem-posing comes from Silver (1994) who defined it as generating new problems and reformulating new problems which can occur prior to, throughout, or after the solution of a problem. By examining some existing definitions, Papadopoulos et al. (2021) organized the meaning of problem-posing in detail: only generating new problems, only reformulating the existing or given problems, both generating and reformulating problems, raising questions, and modelling.

Some scholars connect problem-posing with critical thinking. After contemplating numerous studies, Maj-Tatsis and Tatsis (2021) even classified problem-posing and problem-solving as substantial components of critical thinking. According to Sternberg (1986), "Critical thinking comprises the mental processes, strategies, and representations people use to solve problems, make decisions, and learn new concepts" (p. 3). The critical approach leads to the cultivation of reason to foster rationality (Siegel, 2010) through analysis, interpretation, inference, explanation, evaluation, monitoring, and correcting reasoning (Facione et al., 2000). Therefore, the approach to infuse critical thinking in mathematics education should focus on understanding rather than memorization.

According to Bonotto (2013), problem-posing appears as a promising approach to identify and stimulate critical thinking in mathematics. By observing specific situations, questions and conjectures can emerge. The process necessitates analysing the available data, determining if the problem or question is solvable, devising a proper solution, and considering whether it makes sense or not. Thus, the enormous richness of this approach extends to the opportunity to discuss questions or problems that arise from peers, which then provides an opportunity to evaluate and discuss the problem itself and its solution. In this case, it supports the practice of three interwoven critical thinking phases, i.e., analyse, evaluate, and improve thinking (Paul & Elder, 2014) and contributes to the active learning which consists of intellectual, social, and physical activities (Edwards, 2015). Figure 1 demonstrates the framework between problem-posing, critical thinking, and active learning.

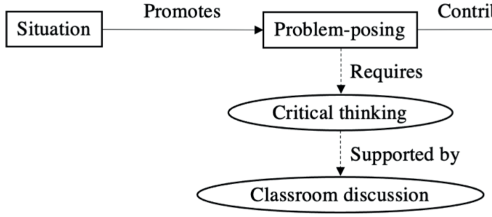

Figure 1: The interconnection of problem-posing, critical thinking, and active learning.

THE STUDY

The lesson reported here is an introductory lesson to problem-posing which was delivered through a workshop. It is part of the larger research project: investigating the role of problem-posing and problem-oriented teaching for active mathematics learning. The participants are three Indonesian PTs who had no experience with problem-posing prior to the workshop.

The whole activities consist of looking for patterns in the multiplication table, posing problems to the class related to the identified patterns, discussing the problems proposed by peers and the instructor, and posing a problem based on the current calendar as the homework. The lesson adheres to the three properties of problem-oriented learning proposed by Kónya and Kovács (2021), since it encourages PTs to analyse mathematical problems presented by their peers or the instructor, to critically evaluate their own and their classmates' thinking, and to express and justify their own thinking.

The lesson was held online, and video recorded. To figure out the role of problem-posing in promoting critical thinking, the dialogues were analysed and at certain points, interviews were conducted for clarity. PTs' manifestations might be non-mathematical and mathematical which can be classified as simple and valuable. Referring to Ennis (1989) that problem-posing includes evaluation, critiquing, and drawing a reasonable conclusion, both valuable mathematical and non-mathematical questions and responses with reasoning, comments, or evaluations are considered critical thinking. Figure 2 portrays the category of the emerging manifestations.

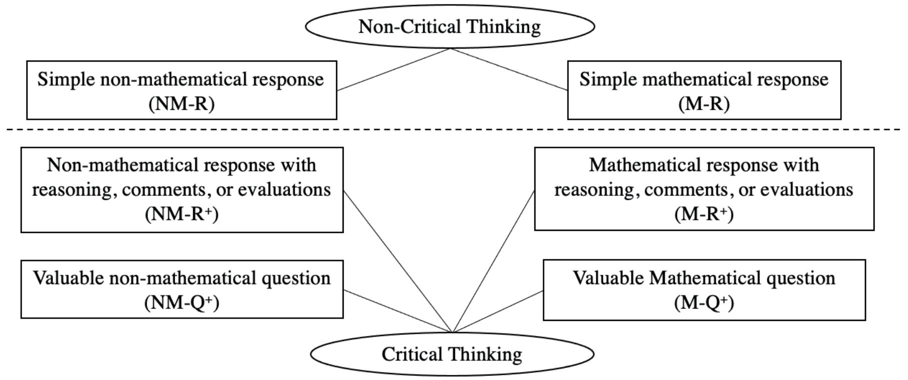

Figure 2: Manifestations which belong to critical or non-critical thinking.

THE LESSON AND DISCUSSION

The first activity was looking for patterns in the multiplication table. The PTs found some patterns which then proposed questions based on their curiosity. I and PT represent the instructor and prospective teacher, respectively.

1	I:	Now, let's play with this 10×10 multiplication table. Please observe it and tell me the pattern you found.
2	PT1:	I found a square number pattern. $1, 4, 9, 16, 25$, and so on. (M-R)
3	PT2:	I found a multiple-of-two pattern on vertical and horizontal directions. $2, 4, 6, 8, 10, 12$, and so on. (M-R)
4	PT3:	I found a multiple-of-three pattern. I mean $3, 6, 9, 12$ and so on, on vertical and horizontal directions. (M-R)
5	I:	OK. After realizing those patterns, are there any questions or curiosities in your mind?
6	PT2:	Why is the multiplication table arranged like this? I mean, why are 1 to 10 placed on top and the left? Why are they not at the top and bottom? (NM-Q$^+$)
		Maybe it follows the Cartesian coordinate system. Eits… but not really. Oh… It maybe. (M-R$^+$)
7	PT1:	Maybe because Indonesians read the text from left to right and from top to bottom. So, it starts from the top left corner. (NM-R$^+$)
		Now, I am wondering. Are there any other patterns besides multiples and square numbers? (M-Q$^+$)
8	I:	Anybody found another pattern?
9	PT1:	Oh, I found $n^2 + n$. The numbers can be $2, 6, 12, 20, 30, 42, 56, 72, 90$. (M-R$^+$)

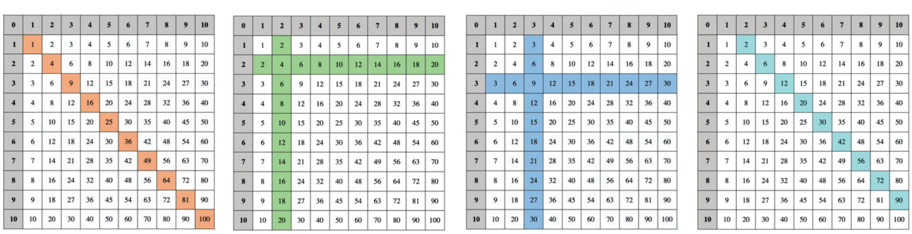

Figure 3: The found patterns.

Figure 3 depicts the found patterns. The first appeared question concerns the arrangement of the table (line 6), which is not mathematical in nature. It is categorized as a non-mathematical question since it is only related to the way to represent the table visually; however, there is no indication of mathematical reasoning. Regardless of non-mathematical, the question is critical and interesting to discuss, directing the class to consider the possible reason behind the number arrangement. During the interview, the PT expressed what she was thinking. When considering the principal axis in the horizontal direction, the arrangement seems to follow a Cartesian diagram. But the idea was dismissed because the numbers below 0 are positive, contrary to what is in the Cartesian diagram. Then, if she looked at the entirety of the extended table, it looked like a Cartesian diagram with modifications to the quadrant positions. The question brings up a reasonable response related to everyday fact from the other PT. But basically, there is no rule about the arrangement, and we may rearrange the rows and columns. As an example, when we change the second and third columns, the header changes, the multiplication rule remains true, but the figural pattern does not. Those manifestations exemplify critical thinking since part of it is questioning, understanding the logical connection, and carefully examining something.

After discussing the previous question, another PT subsequently expressed his curiosity about the other existing patterns which then the PT himself noticed $n^2 + n$ pattern. During the interview, he stated that the number sequence he was referring to is located on the right side of the n^2 sequence. Those ideas are declared into mathematical expression as a form of representation, which is a component of critical thinking (Sternberg, 1986). The class discussion then proceeded to the following question which is explicable from a mathematical standpoint.

10 PT2: Why don't prime numbers have patterns? I mean the pattern is random, scattered. It's bad to see. (M-Q^+)

11 I: She said the prime numbers pattern looks bad. It's not beautiful when she looks at it. Ehm… Can you mention the prime numbers?

12 PT1: $2, 3, 5, 7, 11, 13, 17, 19$, and so on. But in this multiplication table, the prime numbers are only up to 7. (M-R^+)

13 I: Why? Can you find the rest of the prime numbers?

14 PT1: They don't exist. Because prime numbers only have two factors, the number itself and 1. So, they must be in multiples of 1. (M-R^+)

15 I: If so, what can you say about the place of the prime numbers in this multiplication table?

16 PT1: They must be in the first row or the first column. (M-R^+)

17 I: Then, does prime numbers have a pattern?

18 PT2: If I take a glance, they have a pattern. [It represents] a reflection. (M-R)

0	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Figure 4: The place of prime numbers.

One of the PTs was wondering about the position of prime numbers which is scattered according to her (see Figure 4). Examination of the properties of prime numbers and their implication for their location in the table indicates mental processes that lead one of her peers to explain the issue by utilizing an existing mental scheme (line 14). After realizing the prime numbers must be in the first column and row, the PT changed her mind that prime numbers have a pattern representing a reflection. Can we see it as beauty in a mathematical pattern? Nevertheless, the beauty is due to the commutative role in the table, not from the pattern of prime numbers because prime numbers basically have no pattern. Likewise, the symmetric property has nothing to do with prime numbers. The lesson moved on to the next step, posing a problem based on the found pattern.

19	I:	After observing the patterns, can you make questions according to the patterns you found earlier? Please generate questions for your friends. The question can be a riddle.
20	PT1:	My turn to ask. How can you guys convince me, if this table is expanded, will 1001 appear in the [row/column header of] two or not? (M-Q^+)
21	PT2:	Of course not. The last digit is 1, while the numbers divisible by two must be even. A multiple of two means it is divisible by two. It requires the number must be even. 1001, the last digit is 1, odd. So, it can't be divided by two. (M-R^+)
22	PT3:	I agree with PT2. 1001 is not divisible by two. So, even if the table is expanded, it doesn't fall into the pattern of multiples of two. (M-R^+)

The proposed question directed the other PTs to answer along with the reason to convince him. The lesson facilitated interpretation and critical analysis through the opportunity to discover the relationships between data either to ask or to respond to questions. The atmosphere increasingly made them comfortable and enthusiastic for the next step of the lesson.

23	PT2:	I want to ask PT1. If 1 continues to the left to negative infinity, and in the vertical direction 1 continues to negative infinity, does the sequence you found: $1, 4, 9, 16, 25$, and so on still has 1 as the first term? Or is there another number that starts the sequence? (M-Q^+)
24	PT1:	It will return to 1, right? Because we know that the product of a negative number and a negative number will be a positive number. So, the extension, if -1×-1, the result will still be 1. So, the answer is still 1 [even though it is extended to negative]. Since the number closest to 0 is -1, the sequence will still start with 1. (M-R^+)
25	PT2:	In my opinion, this is a collection of positive and negative integers. If we put 0 there, so, it's $0 \times 0 = 0$. It should start with 0, the a (first term), followed by 1 and so on. So, the first term changes. (M-R^+)
26	PT2&1:	(Laughing)

27 I: Oh, I think it was a misunderstanding. You both observe this case from a different point of view, but the essence of your interpretation is in fact the same.

9	6	3	**-3**	-3	-6	-9
6	4	2	**-2**	-2	-4	-6
3	2	1	**-1**	-1	-2	-3
-3	**-2**	**-1**	**0**	**1**	**2**	**3**
-3	-2	-1	**1**	1	2	3
-6	-4	-2	**2**	2	4	6
-9	-6	-3	**3**	3	6	9

X	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**
-3	9	6	3	0	-3	-6	-9
-2	6	4	2	0	-2	-4	-6
-1	3	2	1	0	-1	-2	-3
0	0	0	0	0	0	0	0
1	-3	-2	-1	0	1	2	3
2	-6	-4	-1	0	2	4	6
3	-9	-6	-3	0	3	6	9

Figure 5: PT1 and PT2 Perspectives.

Although it is hard to understand at first, the problem includes considerations of shifting conditions, such as what happens if the table is expanded. Expanding the table to examine the conditions that follow implies a "what-if" or "what-if-not" strategy by Brown and Walter (2004). As there is no clear picture of the expanded table, since they have a clearer idea of the multiplication list presented structurally rather than procedurally, this part also came up with different perspective of the table appearance and different pattern interpretation of the sequence, as illustrated in Figure 5. Following up on the PTs' problem, the instructor asked what about the other patterns.

28 I: OK. When the table is expanded, will the table still hold the previous arrangement? What do you think?

29 PT1: It seems to keep holding the order [of the number], in which the diagonal numbers become the symmetry line. (M-R)

30 PT2: In square numbers sequence, the structure is still valid, but on the other pattern found by PT1: $2, 6, 12$, and so on, which is $n^2 - n$ or $n^2 + n$ you mean, PT1? I forgot it. It might be different. (M-R)

31 PT3: Yes. The structure in the $n^2 + n$ [pattern] will change but the structure in the n^2 [pattern] remains valid. (M-R)

32 PT2: [The structure] will continue. Let's say the initial table that we discuss is in quadrant IV, and above in quadrant II there will be $1, 4, 9$, and so on because the square of a negative number is a positive number. (M-R^+)

33 PT3: The structure in the n^2 pattern will remain because from 0 upwards, the numbers will return to $1, 4, 9$, and so on. In the $n^2 + n$ pattern, the previous structure does not apply because n will be negative. It seems the structure is not valid anymore (in quadrant II according to PT2). (M-R^+)

34 PT2: I think [the table expansion] will form a new structure. This one: $2, 6, 12$, if the table is expanded, a new structure will appear which cannot be connected to the previous structure. (M-R)

35 PT1: I just investigated, trying to observe the pattern by considering the extension. Evidently, the pattern is still valid. But I still checked for small numbers. I mean the numbers closed to 0. I haven't checked for the numbers above -2. For $n^2 + n$, until $n = -2$, the pattern is still valid. The rest, I haven't checked it. (M-R$^+$)

36 PT2: Oh yeah, still holds [the structure]. (M-R)

9	6	3	-3	-3	-6	-9
6	4	2	-2	-2	-4	-6
3	2	1	-1	-1	-2	-3
-3	-2	-1	0	1	2	3
-3	-2	-1	1	1	2	3
-6	-4	-2	2	2	4	6
-9	-6	-3	3	3	6	9

Figure 6: The expanded table.

The question provokes conjecture, stimulates PTs to provide a reason behind it, and leads them to examine their conjecture. As part of critical thinking, they must consider the accessible strategy when investigating their conjecture (Sternberg, 1986). Thus, analysing problems that arise as a result of the modification they generated may facilitate their critical thinking as they feel free to discuss the validity of the problem and consider different assumptions (Bonotto, 2013). Investigating the extended table, the PTs noticed that the rule "$n^2 + n$ numbers on the right side of the n^2 numbers" also applies in the extended multiplication table (lines 35 & 36). Given an extended table and geometric rule, the geometric position remains valid. Drawing the inference, the existing structure in the initial table applies also in other parts of the expanded table, which is followed by considering positive and negative signs. In addition, what the PT mentioned (line 30), "$n^2 - n$ or $n^2 + n$", drives the author to clarify it through interview. Focusing on the right position of the n^2 in which the other PTs considered as $n^2 + n$, PT2 saw it as $n^2 - n$ (see Figure 6). Both interpretations can be considered right, but the starting number is different.

PEDAGOGICAL IMPLICATIONS

Referring to the problem-posing category proposed by Papadopoulos et al. (2021), the activity in this study can be thought of as raising questions based on a fixed starting point, which is observing patterns in the multiplication table. The activity stimulates the emergence of questions and responses accompanied by reasoning, comments, or evaluations of peers' manifestations which reflect intellectual and social aspects of active learning (Edwards, 2015). Hereby, some of the wealthy situations that problem-posing learning has in its merits are noteworthy: (1) PTs direct the lesson towards critical discussion by asking questions that emerge spontaneously from their curiosity, (2) Although the proposed question is probably not mathematical in nature, it might stimulate critical attitude and thus, it is still worth discussing, and (3) In contrast to the teacher-centred approach, which places the teacher as the sole authority, the atmosphere in this lesson brings the PTs as students to be more relaxed in expressing their critical attitude or responding to their peers' manifestations.

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MANIPULATION POSSIBILITIES AND MANIPULATION REALITIES WITH DIGITAL MEDIA BY LEARNING MATHEMATICS

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Learning with different media offers different manipulation possibilities and different possibilities of media use for learners. Thereby, the question arises: Which manipulation possibilities are implemented in the application and which manipulation realities can be identified during individual task processes with the application? Based on a digital learning environment for elementary school students in geometry lessons we will analyze manipulation possibilities and realities with the help of the model Representation-Transfer-Spectrum.

INTRODUCTION

With digital media, the possibilities of representation have developed fast with regard to temporal and local availability. They can be manipulable, dynamizable, connected and synchronous. In mathematics education, learning with representations and especially the learner's own transfer processes of representations within the same and between different levels of representations, are seen as essential to develop understanding in the process of learning mathematics (Wittmann, 1981). For this purpose, new forms of representations, levels of representations, possibilities of combinations and representation transfer processes in teaching and learning contexts have to be 1. identified, 2. analyzed and 3. evaluated from a mathematical educational point of view. Thereby, these questions arise: Which manipulation possibilities are implemented in the media and which manipulation realities show learners? Which cognitive demands are supported or replaced by digital media or are still placed on learners by learning mathematics (Huhmann & Müller, 2020, 2022a in press, 2022b in press)?

THEORETICAL BACKGROUND

Representing and representations pursue two basic intentions. In representing the focus is on doing, on externalizing one's own thinking for communication with oneself and with others. *Representing for oneself* takes place in order to relieve and support one's own thinking processes through what is (visually) represented, in order to orient oneself in one's own thinking and to shape the further thinking process. *Representing for others* is done to communicate one's own thoughts through what is (visually) represented. It also helps to explain where words are missing for oneself and for others. The (visual) information can support to get into exchange with others, to communicate about one's own thoughts and to justify findings about relationships and regularities with the help of what is represented (Duval, 2006). Representations serve for the process of representing as a tool to present one's own perceptions and ideas externally. So, the intention of representations is to document information which is volatile (Huhmann, 2013; Wollring, 2006). Representation transfer processes are always required when at least two representations are given. They have to be compared on one or between different levels of representation – the levels acting, iconic, symbolic (see Bruner, 1971). A representation transfer is also required when a new representation has to be constructed based on a given representation. In both cases, given elements of one representation have to be related to given or to be constructed elements of another representation. A purposeful use of different representations can create learning opportunities to explore relationships between representations and to recognize basic structures (Kuhnke, 2013).

In summary, these areas, representing, representations, and representation transfer processes lead to comprehension-oriented learning and are an indicator of understanding. In the following we will use the term *TripleR* for these three areas – representing, representations, and representation transfer processes.

Based on the models of acquisition and representation of knowledge (Bruner, 1971; Piaget, 1972), further models can be found for identifying and analyzing representation possibilities and representation transfer processes (Johnson, 2018; Ladel, 2009; Lesh et al., 1987). However, these models do not take into account which (cognitive) demands are placed on learners by analogue and digital media during TripleR or which are supported or replaced by media.

In summary, we see a research desideratum in the model-theoretical identification and analysis of possibilities of TripleR in learning with analogue and digital media.

THE REPRESENTATION-TRANSFER-SPECTRUM

The model development is based on the fundamental models of Bruner (1971) and Piaget (1972) and takes into account the extended possibilities of representation that have occurred through digital media. On this basis, we have developed an extended model as a representation transfer spectrum, as shown in Figure 1. Learning in terms of perceiving and acting with analogue and digital media shall hereby be identified, analyzed, and evaluated from a mathematics educational point of view (Huhmann & Müller, 2022a in press, 2022b in press):

1. *Identify*: In which levels are representing, the representations and the representation transfer processes located?
2. *Analyze*: Which cognitive demands are associated with representing, the representations and the representation transfer processes? Which cognitive demands are placed on learners and which are replaced or supported by media?

3. *Evaluate*: Which representing, representations and representation transfer processes are suited from a mathematic educational perspective?

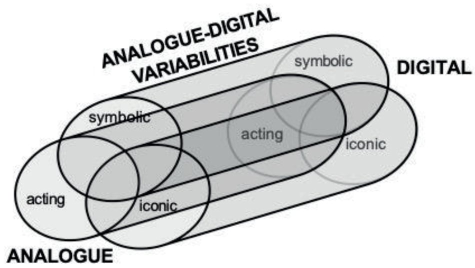


Figure 1: Representation-Transfer-Spectrum (Huhmann & Müller, 2020, 2022a in press, 2022b in press).

Learning objects and associated activities can be located in their representations in the analogue area, digital area or analogue-digital area. Within these areas, a further assignment to the different levels of representation takes place. Between these levels we see no hierarchic arrangement. The focus is on the reciprocal transfer processes of representations within and between these levels of representation. Build on the levels of representation according to Bruner (1971), the intersections of the levels of representation are new elements of the model. We include these intersections under the term levels of representation because representations cannot always be assigned to just one level. If representations contain elements from different levels e.g., iconic elements (depictions) and at the same time symbolic elements (descriptions) (Schnotz & Bannert, 2003), they are to be placed in the corresponding intersection.

Analogue area: The characteristics of the three levels – acting, iconic, symbolic correspond to the known levels from Bruner. If actions are verbally accompanied, verbal expressions are supported by gestures, or symbolic representations are used to act, we identify these as representations of the acting-symbolic level. Representations that contain both depictions and descriptions (e.g. tables, diagrams, function graphs) belong to the iconic-symbolic level. Actions with inherently unchangeable depictions are located in the acting-iconic intersection. These can be ordering, sorting and comparing processes of images. Actions such as ordering, sorting and comparing processes with iconic-symbolic images are assigned to the acting-iconic-symbolic intersection.

Digital area: The model extension by the digital area is identical to the analogue area in its structure, but it differs in specific characteristics. On the acting level, there is the fundamental characteristic of the manipulability of objects of action. However, these are no longer haptically tangible and movable. Objects are manipulated and moved merely by wiping and tapping movements. Objects that are digitally represented as inherently unchangeable images are assigned to the iconic level. Objects that are represented as audio or written text in a symbolic way in the digital area are assigned to the symbolic level. The acting-iconic intersection covers both the user’s own digital actions with inherently unchangeable images and representations of digital actions in the form of animations and movies based on images. The acting-symbolic intersection covers the user's own creating, manipulating, and acting with symbolic representations. This involves audio texts and written texts that can be accessed, duplicated, and combined with digital media. The iconic-symbolic intersection covers inherently unchangeable representations that include both depictions and descriptions. The acting-iconic-symbolic intersection covers the user's own actions with manipulable representations as well as representations of digital actions in the form of animations and videos that contain both depictions and descriptions.

Analogue-Digital area: The analogue-digital area forms a spectrum between the analogue and the digital area. This area is to be explored with regard to the representations of learning objects and activities in terms of their characteristics and possibilities. This involves the identification and analysis of analogue-digital variabilities - in the sense of variable portions closer to the analogue or digital areas as well as with variable focal points in or between the three diameters of the respective areas. In this area, augmented reality and virtual reality applications, among others, are to be considered in a future-oriented manner.

With regard to the digital and analogue-digital area, there is an urgent need for research into these areas, the possible representations of the learning objects, and the suitability of the representation-transfer-spectrum to identify, analyze and evaluate from a mathematic educational perspective.

Representation transfer processes become visible in this model by connecting the activities located on the representation levels with arrows, so that the transfer from an initial representation, which is given to learners, to a final representation, which learners are supposed to construct independently or relate both given ones, is recognizable. Representations become visible by dots in the levels.

METHODOLOGY

This project is part of a qualitative study in which the influence of digital media on the learners own transfer of representation is researched. The focus of this paper is on the question: which manipulation *possibilities* are implemented in the application and which manipulation *realities* can be identified during individual task processes with the application? With the help of the model, *possibilities* and *realities* of use are to be identified and made visible.

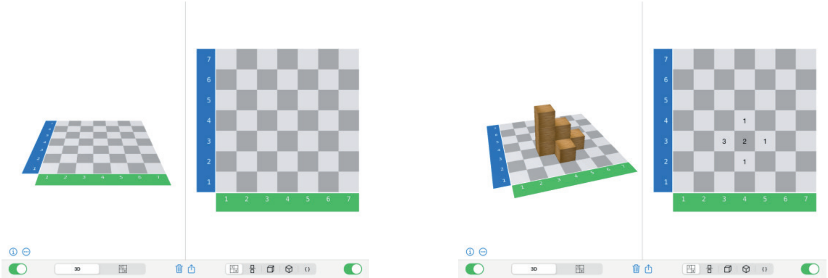
RESEARCH DESIGN

For the analysis of *manipulation possibilities* and *manipulation realities* we use the activity *Architect and Bricklayer* (Thöne & Spiegel, 2003) for elementary school students in geometry lessons, which is realized with the app *Cubes (Klötzchen)* (Etzold, 2015).

Description of the task: Learner 1 builds a cube building in the App and describes it verbally to learner 2. Without seeing the cube building, learner 2 must build a cube building in the app based on the description from learner 1. Afterwards, both cube buildings are compared with each other.

This activity was explored in individual interviews with elementary school students of 3rd grade. In a pre-workshop with the entire class, different analogue representations of cube buildings were introduced, explored and discussed. All learners had the opportunity to build cube buildings, make construction plans and create shadow pictures of a cube building with analogue media. In addition, there were activities that required the transfer of representations between the other representations mentioned above. The aim of this pre-workshop was that the learners get to know the meaning and the construction of the different analogue representations of cube buildings. After this pre-workshop, there was a special orientation lesson for the entire class for using the app cubes. The learners had the opportunity to get to know and try out the functions of the app and to explore the different digital representations of cube buildings within the app.

Using the app for the activity *Architect and Bricklayer* the focus is on the 3D view, as shown in Figure 5 a) and b) on the left side, and on the construction plan, as shown in Figure 5 a) and b) on the right side.



a) Home view of the app cubes b) App cubes with a built cube building

Figure 2: App Cubes.

When learners start the app, they see the screen as shown in figure 5a. If learners have built a cube building, it will look like Figure 5b. It is possible to build the cube building in the 3D view or by using the construction plan. The 3D view can be rotated 360° in all directions, so that the cube building can be viewed from all perspectives. The special feature of this App is that if something is changed in one view, for example when a cube is added in the 3D view, this change is automatically shown in the other view, for example the number at the according position in the construction plan.

Manipulation Possibilities

First we will visualize the manipulation possibilities that learners have in the activity Architect and Bricklayer, using the app Cubes. As you can see in figure 3, the activity is separated in its individual steps. These are made visible in the model Representation-Transfer-Spectrum in terms of representations on the corresponding levels and representation transfer processes to be carried out. A description of these steps follows under the heading manipulations realities.

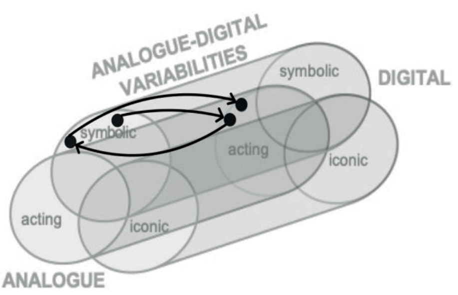
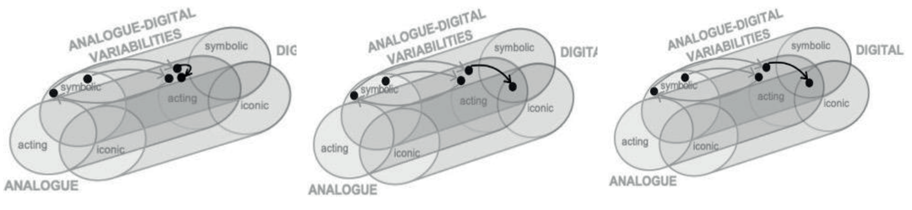


Figure 3: Possibilities of action - Architect and Bricklayer (Huhmann & Müller, 2020, 2022a in press, 2022b in press).

In particular, different manipulation possibilities become visible when the final cube buildings need to be compared with each other.



a) Comparison of 3D and 3D b) Comparison of 3D and construction plan c) Comparison of construction plan and construction plan

Figure 4: Possibilities of action - result verification.

Manipulation Realities

In the following, two examples are chosen, which show the manipulation realities of learners by performing the task Architect and Bricklayer. Both learners (Example 1 and Example 2) received the same introduction as described above to the learning content of cube buildings. The analyses are based on the observations made in the video clips of the interviews.

Both learners were given the task "Build a cube building with 8 cubes and describe it afterwards, so that someone can build it without seeing it" verbally.

Example 1:

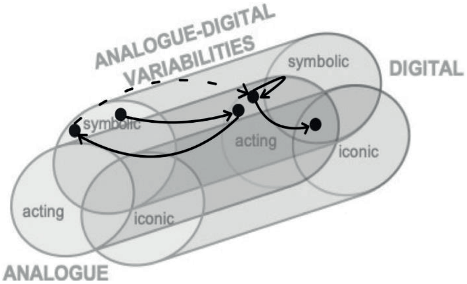


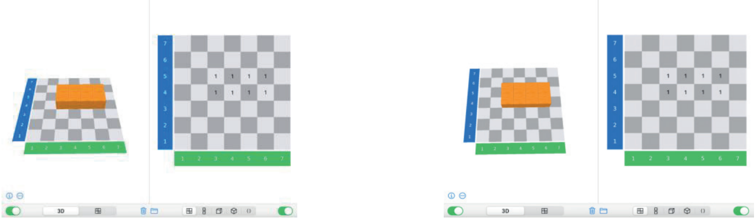
Figure 5: Example 1 - Manipulation Realities.

Based on the task presentation, on the symbolic level in the analogue area, the learner made a representation transfer to the acting level in the digital area in order to build the cube building in the app. Next, the learner uses the built cube building to describe it verbally and thus performs a transfer of representation to the symbolic level in the analogue area.

The dashed arrow shows the representation transfer of the interviewer who, based on the verbal description, performs a representation transfer to the acting level in the digital area in order to build the cube building.

The learner now performs the verification of the results. In the first step, the learner has compared the two created construction plans, although he has acted the whole time in the 3D view and has not used the construction plan either to construct or to describe the cube building.

Even if the learner has already come to the statement, "It fits", he checks in addition both 3D views. During this task, the learner did not move the 3D image at any time.



a) Picture of screencast Learner b) Picture of screencast interviewer

Figure 6: Pictures of screencast.

Example 2:

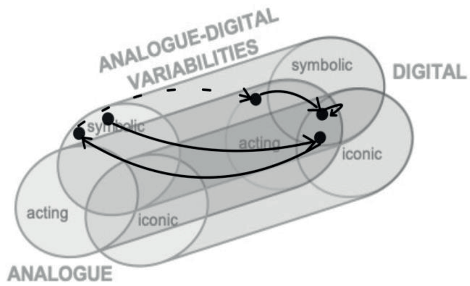
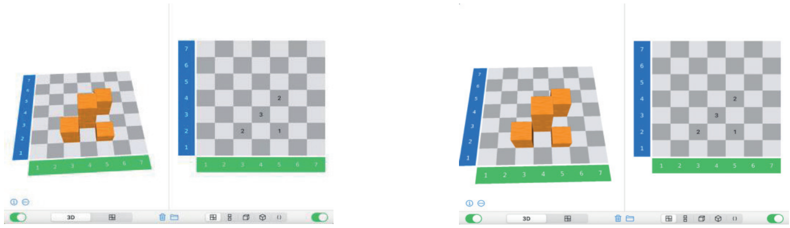


Figure 7: Example 2 - Manipulation Realities.

Based on the task presentation, on the symbolic level in the analogue area, the learner made a transfer of representation to the action-iconic-symbolic level in the digital area, as he created his cube building in the construction plan. With the help of this, he describes the cube building verbally and thus performs a transfer of representation to the symbolic level in the analogue area.

The dashed arrow shows the representation transfer of the interviewer who, based on the verbal description, performs a representation transfer to the acting level in the digital area in order to build the cube building.

The video shows that the view of the learner first moves to the 3D views and compares them. In the second step, he looks at the construction plans and compares those.



a) Picture of screencast Learner b) Picture of screencast interviewer

Figure 8: Pictures of screencast.

In summary, it should be emphasized in both examples that the transfer of representation from the 3D view to the evaluated floor plan or vice versa was taken over by the app and was not performed by the learners.

Compared to the example given here, in an analogue implementation of the activity Architect and Bricklayer with only analogue media, all representation transfer processes have to be done by the learners on their own. The analogue media cannot provide any support with regard to the representation transfer processes and the cognitive requirements.

If the analogue implementation is not explicitly constructed in such a way that, in addition it is also possible to construct cube buildings with the help of other representations, only the transfer of representation between the actual cube building and the verbal description of the cube building would be possible. Therefore, the way of solving the task as in example 2 would not be possible.

FINDINGS AND PERSPECTIVES

With the help of the model, possibilities of manipulation and use and realities of manipulation and use can be made visible, as shown in the example. This provides a basis for the second step, the analysis of the cognitive requirements that are placed on learners or are supported or replaced by digital media.

Our findings show, that on the one hand, expected manipulation possibilities appear as manipulation realities, as well as combinations of manipulation possibilities, as shown in examples one and two - and this became visible in the model: Both learners used a combination of the possibilities to verify the cube building. They have compared both - the 3D views and the construction plans - with each other. Neither of them has used the 3rd option of comparing 3D view and construction plan.

On the other hand, manipulation realities became visible that were not expected in advance. In example 2, the learner created the cube building in the construction plan and used the synchronicity and interconnection of the representation levels to accomplish the task.

Furthermore, by detailing the individual steps of the activity, we were able to identify the cognitive demands and compare them with regard to an implementation analogue.

Future research will focus on identifying the variety of usage differences and the factors that influence this. In relation to the example presented here, the view on the individual competences of the learners regarding spatial perception as well as the characteristics and complexity of the cube buildings must be analyzed and brought into connection with each other. Only *manipulation possibilities implemented in media* do not lead automatically to *manipulation realities for the learner*.

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CRITICAL THINKING IN EARLY ARITHMETICS: DISCOVERING AND REFLECTING ON TASK SOLUTIONS WITHIN RECIPROCALLY DESIGNED LEARNING ENVIRONMENTS

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Reciprocally designed learning environments in which opportunities for perception, action and documentation are closely connected can support critical thinking about task solutions and solution sets. In the context of a discovery-based approach and an understanding of mathematics as an activity, perceptions, actions with various objects on the one hand and dealing with documentations on the other hand creates numerous possibilities to reflect in and on action regarding task solutions and solution sets. How this manifests itself in individual learning trajectories of young learners is shown and exemplarily made visible by a model focusing on the connectivity of action and documentation in substantial learning environments.

INTRODUCTION

Against the background of (i) a constructivist and discovery-based approach to learning (Freudenthal, 1991; Winter, 2016) as well as a conception of (ii) the nature of mathematics as a process-oriented science of patterns and structures rooted in common sense, reflecting as a form of doing mathematics is very important for critical thinking even in early arithmetics. However, designing discovery-friendly and reflection-rich learning environments is a complex task, but in the concept of substantial learning environments these aspects can be concretized for practical teaching and learning. In this respect, within substantial learning environments (Wollring, 2008) the so-called ‘play room’ refers to opportunities for physical actions with mathematical objects and the ‘docu room’ for dealing with documents. In addition, the ways of connecting play and docu rooms influence the possibilities for reflection as mathematical activity and can be characterized and modeled by different settings (Huhmann & Komm, 2022). Through a reciprocal design setting of play and docu rooms (ibid), which repeatedly enables interrelated perception, action and documentation, new opportunities for reflection *in* and *on* action (cf. Schön, 1983) are thus repeatedly created (1. designing). Building on this, the question arises as to how created *opportunities* for reflection and discovery in learning environments translate into actual *realities* in practice (2. exploring realities in practice). Within a qualitative research project, the aim is to analyze processes of “doing mathematics” and reflecting in reciprocally designed settings regarding two selected arithmetic learning environments for first graders. The overall research goal is to enable, design and accompany discovery learning in a sustainable way. In an exemplified analysis of an individual learning trajectory mathematical activities and reflections on task solutions and solution sets are reconstructed and visualized. This reveals how the reciprocal design and connection of play and docu rooms in combination with the use of appropriate forms of documentations translate into actual reflection and discovery realities concerning task solution(set)s (3. analyzing relations of 1 and 2).

THEORETICAL BACKGROUND

Based on a constructivist understanding of the nature of mathematics that suggests a discovery-based learning approach the concept of substantial learning environments with a design focus on the connection of play and docu rooms embodies an appropriate way for teaching and learning mathematics.

On the nature of mathematics: Activity-centred understanding of mathematics developing

This understanding was internationally shaped by Freudenthal (1979, 1991). He contrasts it with that of finished, ready-made mathematics. Thus, in addition to the finished, deductively oriented nature of mathematics, there also exists a form of mathematics as activity, which is individually recreated by thinking (Devlin, 2002). A more precise description of the nature of mathematical activity can be derived from the cross-content understanding of mathematics as a science of patterns and structures[1] (Devlin, 2002). Concrete or abstract activities therefore involve the process-oriented exploring, recognizing, discovering, creating desirably using patterns and structures. A central and basal cognitive prerequisite of such mathematical activity is found in the “elementary logical structures” of classification and seriation (Piaget, 1973). The broadly recognized process-related competencies such as problem solving or reasoning and proof (NCTM, 2000) also emphasize the processual character of mathematics. Thereby doing mathematics includes reflecting in and on these activities in the sense of critically questioning current thought structures, whereby common sense (cf. Freudenthal, 1991) can be seen as a starting point to be reflected on again and again. According to Korthagen (2001, p. 58) and applying his understanding of doing mathematics we understand reflection as “... mental process of trying to structure or restructure an experience, a problem, or existing knowledge or insights”.

[1] The terms ‘pattern’ and ‘structure’ overlap in meaning and are often not used distinctly. We use the concept of structure to refer to the way in which an entity is composed of its parts and the relationship between the parts and the whole (Hoch & Dreyfuss, 2004). The concept of pattern also refers to the relational structure of different objects, concrete or abstract in nature, but focuses on regularities, repetitions, and thus on generalizing validity.

On the nature of learning mathematics: Discovery-based approach

The described understanding of mathematics implies a constructivist understanding of learning. Different learning approaches based on this understanding can be interpreted as discovery learning. We consider discovery learning as the acquisition of knowledge and skills, which is not characterized by the transfer of information from the outside, but by one's own perception and action as well as the analysis and reflection, with constant reference to already existing knowledge structures rooted in common sense (cf. Neber, 1981 as cited in Winter, 2016; Huhmann, 2013). In this process, learners' observing, exploring, trying, and asking questions are central (Winter, 2016), and we see discovery as a (mental) activity in the sense of Bruner (1961) and in line with Korthagen's definition of reflection as

> a matter of rearranging or transforming evidence in such a way that one is enabled to go beyond the evidence so reassembled to additional new insights. It may well be that an additional fact or shred of evidence makes this larger transformation of evidence possible. But it is often not even dependent on new information (Bruner, 1961, p. 22).

On teaching and learning mathematics: Substantial learning environments

What implications can be derived from this view of the nature of mathematics and of learning mathematics for actual teaching? The concept of substantial learning environments reflects these basic attitudes and provides guidelines for their practical implementation. One crucial aspect to this is an articulation-rich design with opportunities for volatile and non-volatile representations. In this context Wollring (2008) introduces two terms: The 'play-room' opens possibilities to actions and creations with physical mathematical objects and thereby offers opportunities to reflect *in* action (cf. Schön, 1983), thereby the play room is characterized by the representational volatility of actions (Huhmann, 2013). In the 'docu(ment) room', action processes and products are kept and represented in a non-volatile way, which encourages reflection *on* action (cf. Schön, 1983). Furthermore, Wollring points out that documentations should provide opportunities for reframing. Following up on this, we focus on the *connection between* play and docu rooms: From the play room there can be a connection to the docu room by documenting concretely performed actions, processes, and created products. On the other hand, there can be a connection from the docu room to the play room, by providing impulses for further (and new) actions through dealing with documentations in the docu room. The following results of different degrees of connectivity between play and docu rooms (see Figure 1) were empirically identified and have a central influence on the opportunities for doing and discovering mathematics (Huhmann & Komm, in press) within substantial learning environments:

(i) *Unconnected*: Documentations cannot arise from actions with physical objects in the play room and do not provide impulses for further actions in the play room. Documents that occur in this setting are created without direct reference to the self-executed previous action with physical mathematical objects.

(ii) *Unilaterally connected*: There is the possibility of documenting processes and products created or arisen in the play room. However, according to the unconnected setting, these documents are not intended to be used for further activity in the play room.

(iii) *Reciprocally connected*: There is the possibility of documenting the performed processes and created products from the play room. Moreover, these documents can stimulate further, repeated or new actions in the (new) play room.

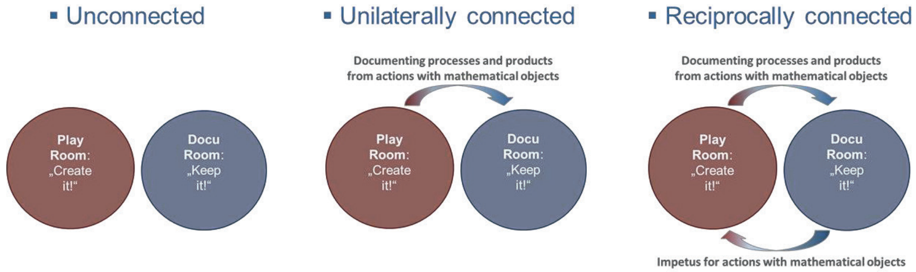


Figure 1: Different degrees of connectivity regarding play and docu rooms.

The degree of connectivity can be influenced by the use of different types of documentations (Huhmann & Komm, 2022) e.g., by documentations that can be dynamized. Dynamizable means, that the physical carrier medium of the document can be broken up into smaller (documentation) units. As a subset of dynamizable documentations, *easy to dynamize documentations* contain special potential for a reciprocal design. They consist of various, changeable and therefore individual documentation units (“notes”), that can be combined to create a larger document and easily be resolved, i.e., dynamized, again. We adopt Wollring’s idea of the degree of fixation of documents (2006) and apply it to documentation units as objects to be fixed. Reversible fixations, e.g., removable adhesives, enable easy dynamization as the effort to disassemble and (re)assemble again is low. That is why these documentation units can easily be used as mathematical objects for actions in the play room to be (re) structured and (re) arranged similar to a concept map. The model in Figure 2 illustrates a reciprocal discovery setting within substantial learning environments involving easy to dynamize documents. In the play room, new possibilities for perceiving and acting can open up again and again, be it with concrete physical objects or with documentation units gained through dynamization. Particularly significant are opportunities for action such as classifying and serializing documentation units, which enable repeated and new (re)structuring in the sense of reflection *in* action. In the docu room, activities such as repeated and new recognition and explanation can promote discoveries and stimulate reflection *on* former actions in the play room. In this way, actions that promote elementary logical structures and thus mathematical activity are made possible, and at the same time there are possibilities for documentation and visualization of these actions with little effort. The interface on play and docu rooms results from the double use of documentation units: on the one hand, as records in the document room and, on the other hand, as objects of action in the play room.

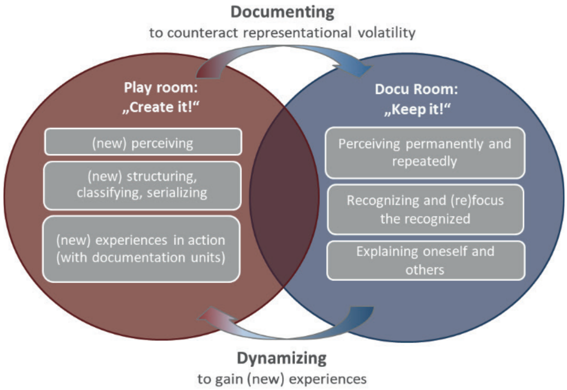


Figure 2: Model for discovery learning in reciprocally designed substantial learning environments with dynamizable documents (Huhmann & Komm, 2022).

METHODOLOGY

Research design and background

The project is part of a larger research study concerning (i) the (re)design of substantial learning environments and, building on this, (ii) the analysis of individual learning trajectories, both focusing on the connectivity of action and documentation and the use of documentations for critical thinking and discovery. Within a qualitative orientation and a design-based approach different data collections and analysis have taken place and are still going on. The results (i) were incorporated in the theoretical background (see Figure 1 and 2) and currently we focus on research questions regarding (ii). In this context we word the following research question: How do first grade students discover and reflect on arithmetical task solutions and solution sets within reciprocally designed learning environments containing easy to dynamize documents?

We made teaching experiments with reciprocally designed substantial learning environments and focused on the use of easy to dynamize documentations. Therefore, we analysed the individual cases of learning trajectories. For the teaching experiments pairs of students worked on the tasks including phases of individual as well as partner work. During this time each student was filmed from various perspectives. For the data analysis we use qualitative content analysis in a deductive-inductive setting (Kuckartz, 2018). Deductive categories are in particular derived as elements from the model of discovery learning in reciprocal settings (Figure 1). In addition, the suitable distinction of reflection in and on actions, focusing on the individual use of (easy to dynamize) documentation, and the discovery and reflection realities that arise within the given opportunities due to the specific reciprocal design, serve as a deductive starting point. So far, our data base refers to eight case studies with first grade students aged six to seven. Always two pairs had to deal with a specific task of a substantial learning environment. In the following the two chosen substantial learning environments and the selected task as a common initial problem situation are described.

Description of substantial arithmetical learning environments

Calculating squares (with "ears") (see Figure 3) are a known substantial task format (Huhmann, 2008). Depending on the abilities of the students they can be used with or without ears. Considering the fact, that the test persons were students in the middle of the first grade, it was decided to work with the inner square only, i.e., without ears. There the following rule has to be considered: $a + b = c + d$. With regard to the intended challenge to reflect not only on task solutions but also on solution sets the following task has been chosen as a common starting point for the students from the overall great variety of possible tasks: Find solutions for a square with two given numbers *a* and *b*! In a next step the students are asked to reflect on the solution set: How many solutions can be found and why?

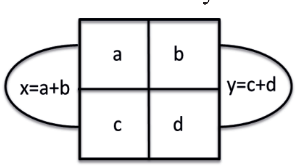


Figure 3: Calculating squares with ears.

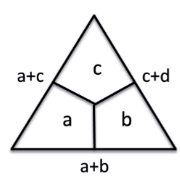


Figure 4: Calculating Triangles.

Calculating triangles (Wittmann & Müller, 2000) consist of three inner numbers *a*, *b* and *c* and three outer numbers, each forming the sum of two inner numbers as shown in Figure 4. The chosen task is: Find triangles with identical outer numbers! Building on this the students were asked to reflect on the number of solutions and corresponding justifications.

For the reciprocal setting both environments are designed with easy to dynamize documentations to create opportunities for acting in the play room. Therefore, the students had arithmetic chips and a blank format of the square respectively triangle to place them within. In addition, the documentation units contained always one square or triangle to be filled in. For reversible fixation possibilities the students could use removable adhesive.

FINDINGS

We divide the findings in two perspectives: First, an overall look on the observed mathematical activities to discover and reflect on task solutions and solution sets from a cross-case perspectives regarding both presented learning environments. Second, a specific inner case perspective of how an individual learning path is shaped within the reciprocal setting dealing with the presented task with calculating squares.

Cross-case perspective

From a cross-case perspective we can state that in both learning environments learners used the opportunities named in the model in Figure 2, with the individual variety concerning the intensity and frequency and occurrence. In particular, the following activities with regard to various objectives could be observed:

- repetitive actions with arithmetic chips to find solutions and to document them,
- perceiving and refocusing documented solutions to find further ones (also by new actions) or to compare newly found solutions with already found ones,
- intuitive or stimulated structuring (serializing and classifying) with documentation units to find more or missing solutions,
- dynamizing documents for (re)structuring to find more or missing solutions,
- serializing documentation units to reflect on the properties of the discovered solution set and to justify its cardinality.

The structuring activities with documentation units could inductively be specified into subcategories. Classifying, serializing and a combination of both could be observed. The most common and intuitive type of structuring were classifications. Significant and frequent classification took place along the criteria of correctness of documented solutions, with “correct” and “incorrect” as two mutually exclusive categories regarding calculations or applications of the rule of the task format. Often this was practically realized by sorting out wrong solutions. Classifying according to solvable or non-solvable number constellations of a triangle (see Figure 5a) top and bottom) supported reflection on characteristics of solution sets. Serialization took place according to the chosen outer numbers of the triangles or of one of the bottom numbers (c or d) of the square (see Figure 6c)). Figure 5 shows a final document combining different classifications and serialization: (i) Classification categories along the criteria of solvability with the two categories “solvable” (upper area) and “non-solvable” (lower area), (ii) Classification along decadic analogies between outer numbers of different triangles (triangles placed on top of each other in the upper part) and (iii) Serialization within the categories “solvable” and “non-solvable” in ascending order of the outer numbers. The reflection of the serialization within the categories "solvable" and "non-solvable" in the docu room led the student to the discovery that there are infinite sets: "Ah, it always goes on, like…like normal numbers. There is always one and then...like, like in counting."

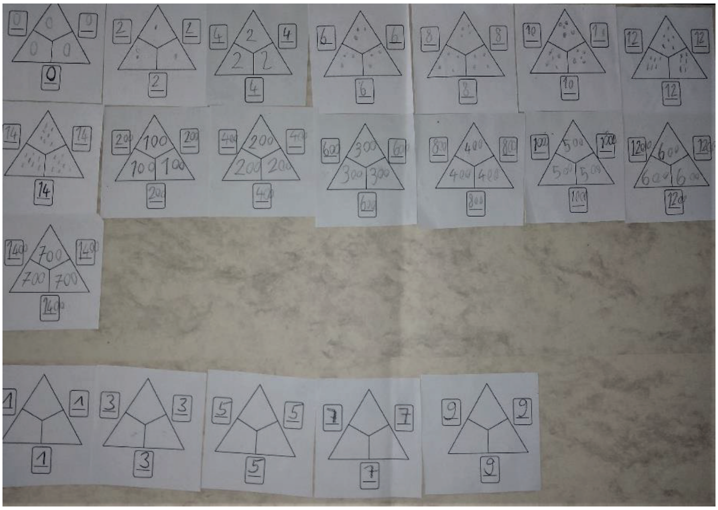
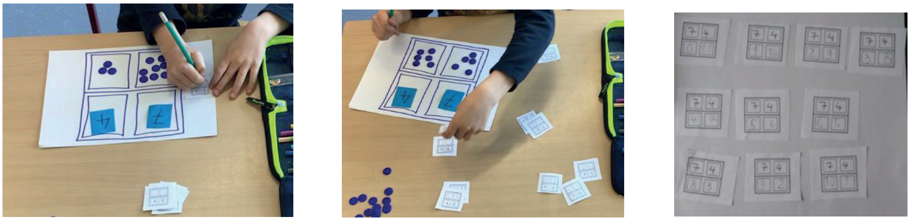


Figure 5: Final document combining various classifications and serializations.

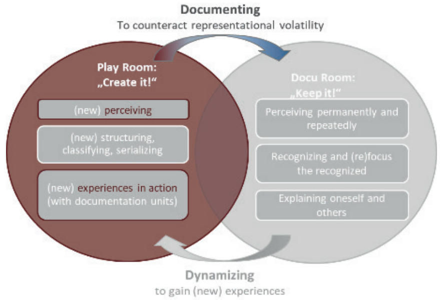
Inner-case perspective

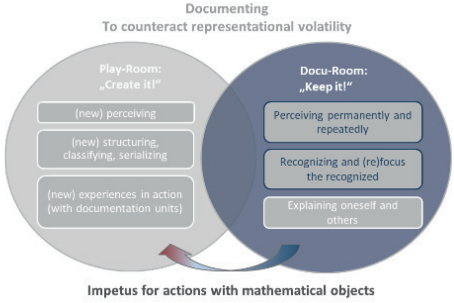
Considering an inner perspective on cases an individual learning trajectory is to exemplify the use of the opportunities created in a reciprocally designed learning environment in the following table (cf. Huhmann & Komm, in press). Tim's learning path is reconstructed in different phases according to dominant activities and discoveries. In the left column, the respective iconic representation of the activity is colored in the presented model of discovery learning (see Figure 2). The entire learning process thus becomes apparent as a complex, reciprocal interplay of individual model components. In the table on the right, the individual phases of the learning processes are explained. The analysis focuses on the influences and effects of the reciprocal design in connection with (re)structuring and reflective elements. The actual numbers for the described task, finding (all) solutions for *a* given combination of *a* and *b* were $a = 7$ and $b = 4$. The pictures in Figure 6 show documentations and actions that arose in the course of the learning process and are used to illustrate the explanations.



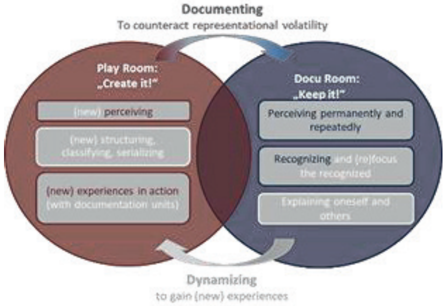
a) b) c)

Figure 6: Actions and documentations within Tim's learning trajectory.

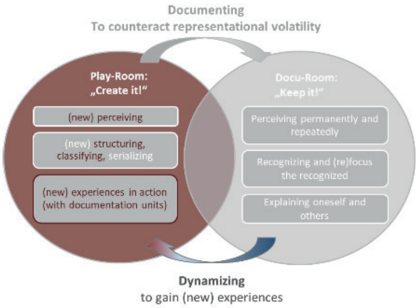
<table>
<tr><th colspan="2">Finding and documenting first solutions</th></tr>
<tr><td>


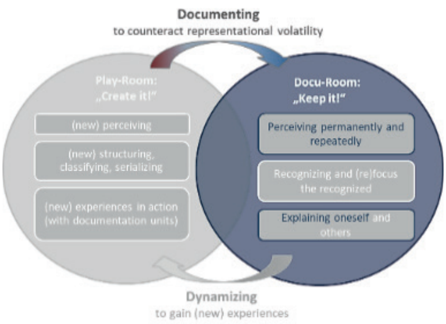


</td><td>Tim explores different squares by acting with arithmetic chips, and thus finds different, not systematically related solutions, which he documents on a blank template and collects them loosely (see Figure 6a)). In this phase, he discovers five squares with solutions for c and d.

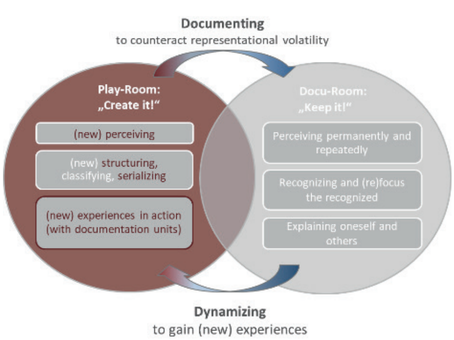
Tim reflects on the documented solutions e.g., by visibly laying out the individual documentation units for perceiving and recognizing previously found squares. And with this information he creates own ideas for further actions and solutions with the arithmetic chips.</td></tr>
<tr><th colspan="2">Finding and documenting further solutions</th></tr>
<tr><td>


</td><td>Tim explores further squares by acting with the chips and documents the new solutions. So, he discovers four more squares. Now that he has found a total of nine squares Tim looks at the documentation and recognizes pairs of solutions that resemble commutative swap tasks.</td></tr>
</table>

<table>
<tr><th colspan="2">Structuring process I: Classifying commutative solutions</th></tr>
<tr><td>




</td><td>
…and intuitively begins to dynamize and act with the documentation units: he classifies squares, characterized by the commutative structure of the numbers c and d (c in one square is used as d in the other square and vice versa), and puts them on together (see Figure 6b)). This classifying activity leads to further activities with the chips and to the discovery of one more, the tenth solution.

When he documented it, Tim calls out “Ready!” and seems to be sure to have found all solutions now.
</td></tr>
<tr><th colspan="2">Structuring process II: Serializing</th></tr>
<tr><td>


</td><td>
Following an impulse from the teacher Tim begins to dynamize and structure his document units. He chooses a seriation of the squares in ascending order by number c, and in descending order by number d (see Figure 6c), horizontal lines), and documents this by fixing his arrangement with removable adhesive.

In a common reflection, Tim explains his order with pointing on the numbers c and d: “That it becomes more here and less here”. During the joint analysis of his created order, it occurs to him that he has not yet found a final missing solution, namely the one that would mark the end of his order: $c = 0$
</td></tr>
</table>

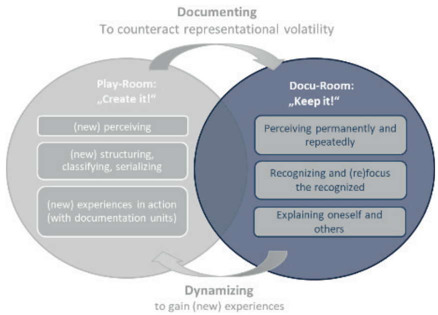
<table>
<tr>
<td>
</td>
<td>and d = 11. Due to this specific position, Tim adds the missing square to his serialization without the need to dynamize his document again. When asked for further solutions, Tim now answers with conviction: “Because there are no more than 11 solutions, because I also have the 11 number!” (In the previous reflection of his document, Tim had pointed out that he was counting the pair of numbers $c = 4$ and $d = 7$ not as a solution, since they are the numbers of a and b.)</td>
</tr>
</table>

Table 1: Reconstruction and modelling of Tim’s learning trajectory.

CONCLUSION

Our results show how these young learners use the created opportunities in the reciprocal learning environments to discover and reflect on arithmetical task solution(set)s. The structuring activities with documentation units led to arithmetical reflections and insights e.g., concerning an infinite solution set (see p.7: “Ah, it always goes on”) or a specific number of possible solutions (see p. 9: “… there are no more than 11 solutions ...”). The inner-case perspective revealed the closely interwoven interplay between the central activities and reflections ‘in action’, the respective ‘documenting’, the ‘reflecting on the actions on the basis of the created documents’ and the ‘dynamizing of documents in order to (re)act’, and could be made visible with the exemplified modelling of individual learning processes. The reconstruction of Tim’s learning trajectory shows how he reflects ‘in and on his actions’ in the reciprocally designed learning environment and therefore how the created *opportunities of discovery* and reflection can lead to *individual realities*. Based on the use of the model to analyse learning trajectories, the question of viewing it as a thinking, analysis, and reflection tool for instructional development and research can be addressed. Beyond that, the connection of play and docu rooms seems to offer chances for various challenges in early mathematics like dealing with heterogeneity or combining individual and collaborative learning.

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CRITICAL THINKING IN MODELLING REAL-LIFE PHENOMENA BASED ON STUDENTS' EXPLORATIONS

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Critical thinking is one of the most important abilities of International Baccalaureate (IB) students. The present study is based on the works of IB students that are called explorations. Exploration is a part of mature exam, and it is a project, in which students need to solve some real-life problems using mathematical tools. The study focuses on the reasons why the exploration seems to be so hard to create. In the current study, three groups of students' explorations were analyzed. Particularly, I examined the role of critical thinking in the process of writing the exploration and what kind of difficulties students meet in modelling real-life phenomena. The analysis has shown that regardless of results, the students had difficulties with formulating reflections and stating, and proving hypotheses.

INTRODUCTION

The notion of critical thinking is not easy to define. There are several features that characterize critical thinking in mathematics. Innabi and Sheikh (2007) conducted studies that identify three components of critical thinking in mathematics, namely reasoning, problem posing and problem solving, and identifying the suitability of problem solutions. Moreover, critical thinking can be interpreted as a complex concept that involves cognitive skills and affective dispositions. It may also involve logical reasoning and ability to separate facts from opinion, examine information critically with evidence before accepting or rejecting ideas (Irfaner, 2006). Mansoor and Pezeshki (2012) claim that critical thinking involves reasoning and consideration of what we have received rather than forward acceptance of different ideas.

Due to the fact that critical thinking consists of many activities such as problem solving through information analysis, formulating and verifying hypotheses, evaluating evidence and arguments, and searching for an action strategy, it can be present not only in mathematics but also in different areas of life (Firdaus et al., 2015; Huitt, 1998; Krulik and Rudnick, 1999; Pyzara, 2021; TC2, 2013).

Another approach was presented by Facione (1990), who defined six cognitive abilities as central to the concept of critical thinking: interpretation, analysis, explanation, self-regulation and inference. Moreover, he claimed that these skills enable people to analyse and combine information to solve problems in many areas.

The present study focuses on the notion of critical thinking in the process of writing projects related to mathematics course in an IB Programme.

THEORETICAL FRAMEWORK

The International Baccalaureate Diploma Programme (IBDP) is a programme that is devoted for students aged 16-19 in countries around the world. The programme aims to develop students who have knowledge on modern problems and is divided into six subject groups, with mathematics being one of them. Apart from the theoretical concept of IBDP, there are also specific skills that characterize all IBDP students. Each of IBDP's subjects is committed to the development of students according the IBDP learner profile. The profile aims to develop learners who are: inquirers, knowledgeable, thinkers, communicators, principled, open-minded, caring, risk-takers, balanced and reflective.

Regarding the concepts of critical thinking introduced at the beginning of this paper, it is easy to notice that those concepts and definitions commensurate with the skills promoted by IBDP organization. Moreover, according to Schoenfeld (1992), there are many aspects of mathematics like abstraction, symbolic, representation and symbolic manipulation, and being trained in the use of them can be developed only with appropriate features. All of them are required in IBDP.

Focusing our attention on mathematics, there are several courses available in IBDP: analysis and approaches (standard or higher level), and application and interpretation (standard or higher level). The aims of these courses are to:

- develop mathematical knowledge, concepts and principles,
- develop logical, critical and creative thinking,
- employ and refine students' skills in abstraction and generalization,
- take action to apply the transfer skills to alternative situations, to other areas of knowledge and to future developments in real life problems,
- appreciate how developments in technology and mathematics influence each other,
- appreciate the contribution of mathematics to other disciplines,
- independently and collaboratively extend the students understanding of mathematics.

The current study was based on the theory and the methods related to mathematics application and interpretation higher level (AIHL). According to AIHL, the mathematics education should encourage the development of solid written, verbal and graphical communication skills, and also critical and complex thinking. Within this concept, in the classroom, students should regularly learn mathematics by being active participants in learning activities.

Teachers should provide students with regular possibilities to learn through mathematical inquiry, by using strategies that stimulate students' critical thinking and problem-solving skills (Huang, Ricci, & Mnatsakanian, 2016). One of the goals of mathematics education is mathematical modelling.

Mathematical modelling is a technique used in problem solving, to make sense of the real world. Based on research, engaging students in mathematical modelling enables them to be successful in many non-mathematical as well as mathematical courses and careers. Moreover, this process requires critical reflection throughout the process (Ferri & Mousoulides, 2017).

Skills allowing for solving problems related to everyday life are the most desirable in this course. The main assessment objectives are knowledge and understanding, problem solving, communication and interpretation, technology, reasoning and inquiry approaches. Students are assessed in four components: Papers 1, 2, 3 and exploration. In my research I focused on exploration.

Mathematical exploration is a component that is internally assessed by the teacher and externally moderated by the IBDP examiners at the end of the 2-year course. The explorations analysed in this paper were assessed by the author of the paper. As a part of mature exam, it is marked according to five assessment criteria:

A. Presentation (0-4 points): coherence, conciseness, and organization of the project.

B. Mathematical communication (0-4 points): usage of appropriate mathematical language, key terms, multiple forms of representation and deductive method.

C. Personal engagement (0-3 points)

D. Reflection (0-3 points): can be limited or meaningful, as well as provided by substantial evidence of critical thinking.

E. Use of mathematics (0-6 points): the mathematics demonstrated in exploration has to be relevant and commensurate with the level of mathematics.

The aims of the exploration are to encourage students to discover and appreciate the power of technology as a mathematical tool and provide opportunities for them to demonstrate their mathematical development. The role of the teacher is to support students in the process of creating the exploration. She should provide oral or written advice on how the work could be improved. Only one feedback is allowed, therefore the second draft is the final one.

RESEARCH QUESTIONS

The aims of the study were to examine the role of critical thinking in the process of writing the exploration and what kind of difficulties students meet in modelling real-life phenomena. Many activities involved in writing the exploration are related to critical thinking. The study aimed to answer the following questions:

a) Is the exploration coherent, well-organized and concise?

b) Have the students chosen reliable resources?

c) Have the students shown the required knowledge, skills and understanding?

d) Have the students used the deductive method and were the proofs logically valid?

e) Is there evidence of substantial or critical reflection?

f) What were the main difficulties that have arisen?

These questions have guided the analysis of the explorations through individual students' works.

METHODOLOGY

The research was conducted at the beginning of 2022. The participants were 21 students of mathematics, aged 17-19, who attended Mathematics Standard Level in years 2019-2020, and Applications and Interpretations Higher Level IBDP courses in years 2020-2021. The students' explorations were internally assessed and evaluated by the teacher (the author of this paper), and externally by the IBDP examiner. These two assessments commensurate.

The research tool consists of examples of explorations of these students that require mathematical presentation, mathematical communication, and reflection. These requirements concerned the assessment criteria A, B and D. The results of these students were divided into three groups:

- Group 1: 15-20 points (5 students)
- Group 2: 9-14 points (10 students)
- Group 3: 2-8 points (6 students)

In this paper fragments of explorations from each group are presented.

ANALYSIS OF STUDENT SOLUTIONS

The analysis concerned the students' works that have different topics and concerned a variety of mathematical tools. Some choices of mathematical tools in explorations of each group were the following:

- Group 1: optimisation (one and two variables' functions), calculus and applications of integrals, graph theory.

- Group 2: bivariate statistics (including linearization), statistics and statistical tests (e.g., t-Test).
- Group 3: modelling with applications of different kinds of models (linear, quadratic, cubic, logistic), geometry and trigonometry, statistics.

Some exemplary topics were the following:

Group 1: "Applications of Graph Theory in problems regarding transport management."

Group 2: "Optimisation of areas of boxes for Christmas presents using two approaches."

Group 3: "Using mathematics to predict the winner of NBA playoffs."

The analysis was conducted by considering the research questions in the explorations in each group.

Analysis of explorations from Group 1

The plan of the exploration was detailed, coherent and organized. It contained the contents of exploration and bibliography. The sources were reliable and presented many scientific items. Students presented knowledge and mathematical skills that were relevant and commensurate with level of mathematics (see Figure 1).

Creating a graph and the representation of data

The first thing that is necessary to do it so set up graph. I achieved it by using an online map tool – Google My Maps. There, I marked every single McDonalds in my city and created a map of roads between them. The rule I followed was to choose the relatively shortest routes that were not too restrictive in the weight and size of the vehicle. I also avoided using one-way streets for the sake of convenience.

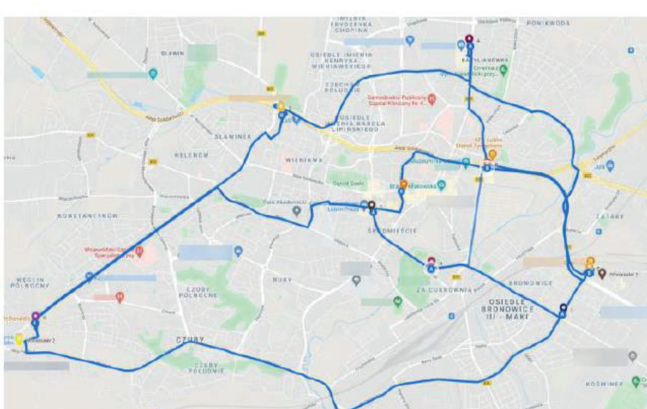


Figure 1. The map with representation of all McDonald's restaurants in my city (shown as markers with letters) and the connections between them using roads without weight limits.

The Second Problem

Sometimes, the product is not licensed, and other restaurants also suffer from the shortage – then, a visit to a wholesaler is unavoidable. In this case, making a choice is more complex, as not only the distance between the restaurant and the wholesaler matters – the prices of products are going to differ, too.

Both wholesalers are located approximately 300meters from the restaurants E and G. From now on, we will refer to them as Wholesaler 1 and Wholesaler 2, respectively.
Let us say that the restaurant A needs restocking on five different products: tomatoes, onions, cheese, lettuce and coffee beans. The prices at each wholesaler are as follows:

Wholesaler 1		Wholesaler 2	
Product	**Price**	**Product**	**Price**
Tomatoes	4.30zł/kg	Tomatoes	4.14zł/kg
Onions	1.25zł/kg	Onions	1.04zł/kg
Cheese	14.95zł/kg	Cheese	17.84zł/kg
Lettuce	2.19zł/kg	Lettuce	2.37zł/kg
Coffee Beans	8.95zł/100g	Coffee Beans	6.15zł/100g

Table 4. Prices of products at the chosen wholesalers in my city. The data originates from the wholesaler's internet sites.

Figure 1: Fragments of the exploration from Group 1.

The proofs were conducted in a logical and simple way, but some parts were incoherent and needed to be better prepared. An example of proof is shown in Figure 2.

"In any graph $G = (V(G), E(G))$, the sum of the degrees in the degree sequence of G is equal to one half the number of edges in the graph, that is $\sum_{v\in V(G)} deg(v) = 2|E(G)|$."

Proof: Whenever we make a graph (G), each edge will always be attached to two vertices. Because of that, each edge is going to contribute 1 to each of the vertices it is incident to.
Therefore $\sum_{v\in V(G)} deg(v) = 2|E(G)|$.

Figure 2: Fragments of the exploration from Group 1.

There was evidence of substantial reflection, but it was rather descriptive than creative. There were some open questions but sometimes they were not reliable or valid (Figure 3, Figure 4).

After finishing the exploration and reflecting over it, I found a certain room for improvement. First, the second problem should be investigated more thoroughly, especially when it comes to buying certain products at one wholesaler and other ones at the other. The point of it would be to find whether the fuel cost would extend over the profit made by buying cheaper products. Perhaps it would end up as the most profitable way money wise?

Figure 3: Reflection of an exploration from Group 1.

Conclusions (Group 1)

There were two difficulties that arose for the students from the group 1. The first concerned the logical justification in the proofs. The ability of using mathematical terms and concepts is not fully developed in this age. Good students very often see the solution or justification of a phenomena, but they do not know how to present their thoughts in terms of mathematics. Another observation is related to reflections. In most of the explorations from Group 1 there was substantial evidence of critical reflection. They often developed their critical thinking in exploration addressing the mathematical results and their understanding of topic. They discussed the implications of their results and the applications of mathematical tools.

> **Conclusion and evaluation**
>
> My main goal of this investigation was finding the minimum total area of the set of boxes, using two different approaches, to waste as little paper as possible. I used two methods of calculations. The first one was finding the dimensions of star-shaped boxes with minimum possible area, from which the dimensions of the third, the biggest box A_3were derived. The second one was focused on finding the minimum total area of the whole set of boxes altogether. This method required additional work, because the topic of two-variable functions is not within the Mathematics AI HL IB programme.
>
> The second method is a much more precise one, because it included the area of the big collective box, unlike the first method. The Total Area calculated from this method was in fact smaller, hence 9960 cm^2, while in first method it was 10 700 cm^2. Therefore it should be concluded that the more precise, complicated method was more appropriate for this investigation.

Figure 4: The conclusion and evaluation of an exploration from Group 1.

Analysis of explorations from Group 2

Most of the students in this group prepared explorations that were well-organized, planned and contained some bibliography. The number of items was smaller than in the explorations of students in Group 1. Not all the resources were reliable (e.g., webpages that were not referenced). Most of the students showed the required knowledge and used different forms of mathematical presentations like tables, graphs and pictures (see Figure 5).

> **3. Fractal Dimensions**
>
> 3.1. Complexity of real-world objects
>
> At the beginning, I wanted to investigate how irregular and complexed shapes of natural objects may be. In order to do so, I used Google Maps and investigated shape and coastline of one of German islands on the North Sea by comparing island's map in various magnifications.

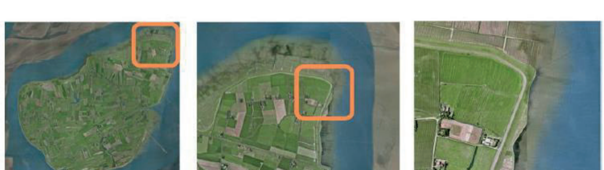


Fig 1. Pellworm island

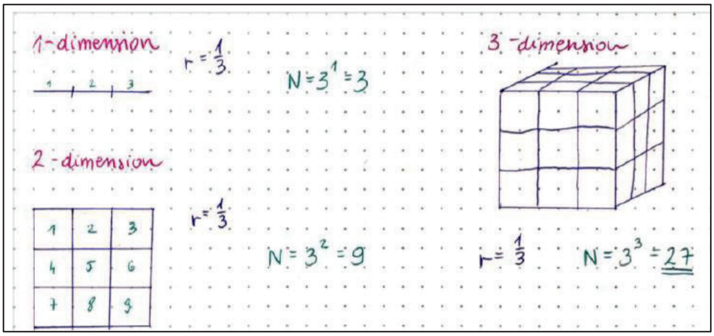
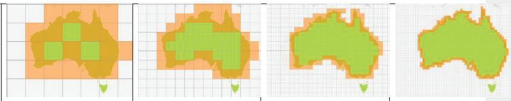


Figure 5: Mathematical presentation of students' exploration from Group 2.

The methods used by the students were deductive and logically valid. Some observations and calculations were presented (see Figure 6).

4.1.1. Box-Counting Method

Firstly, using graphic software I placed a grid over a map of Australia[1] mainland and highlighted boxes that touched the coast of the mainland island. Then I counted all highlighted boxes and wrote down the result as N.

Tab.1				
Scale factor (s)	0	2	4	8
$\ln s$	0	0.693	1.386	2.079
Boxes (N)	17	40	84	163
$\ln N$	2.833	3.689	4.431	5.094

In order to find the value of gradient of $\ln s$ to $\ln N$ plot, I need to separate and put relevant data together to prepare for calculations.

n	$\ln s$	$\ln N$	$\ln s * \ln N$	$(\ln s)^2$
1	0	2.833	0	0
2	0.693	3.689	2.556477	0.480249
3	1.386	4.431	6.141366	1.920996
4	2.079	5.094	10.590426	4.322241

Figure 6: Deductive method in the explorations from Group 2.

Students presented the reflections throughout the exploration. Some works presented meaningful reflection linking to the aims of project, commenting on what they had learned, considering some limitations, or comparing different mathematical approaches.

Conclusions (Group 2)

Most projects from Group 2 presented a logical development and they were partially well-organized. The explorations contained some relevant mathematical communication that was mostly consistent. There was evidence of application of deductive method. Reflections were not critical but meaningful and notified what students have learned, by considering some limitations. Some difficulties appeared in the use of mathematics and by wrong interpretations of the chosen tools to the real-life phenomena. Sometimes instead of approximation sign they used equality sign, or some theorems without considering limitations and constraints.

Analysis of explorations from Group 3

Most of the students presented coherent and organised explorations. The bibliography (if does exist) was not detailed and contained mostly not reliable items (see Figure 7)

Bibliography

1. (n.d.). Retrieved from https://www.britannica.com/topic/Quadrature-of-the-Parabola
2. The Quadrature of the Parabola. (2021, January 10). Retrieved from https://en.wikipedia.org/wiki/The_Quadrature_of_the_Parabola

Figure 7: Example of not reliable bibliography from an exploration from Group 3.

Students showed some relevant knowledge that commensurate with the level of the course, but the demonstrated understanding was limited. The students did not use deductive method and performed only simple calculations. Moreover, the evaluation was mostly descriptive and did not contain reflections related to the conducted calculations. The results were not analysed in detail.

Conclusions (Group 3)

If we look at the bibliography and organization, we can notice that the students were not aware of the scale of reliable source. Their critical thinking abilities were only bounded to their own observations. The mathematics used by the students was mostly correct, but there are some serious mistakes. Moreover, the evaluation was descriptive, and reflections did not exist, or were limited. Most justifications were far from critical reflections.

DISCUSSION

After considering the explorations from different groups of development, several errors in reasoning that repeat in each group were observed: use of unjustified or wrong argumentation, wrong approximation and usage of mathematics, descriptive reflections and conclusions, not enough proficiency in the application of mathematical language.

The analysis of explorations has shown that there are students, in examined sample that have some difficulties in formulating critical reflections, applications of deductive method, stating and proving hypotheses, separate facts from opinion, and applications of mathematical tools to modelling real-life phenomena. More detailed research is required to show if this problem is more general.

It is worth emphasizing that there are some elements of critical thinking that were used correctly by the students. The basic concepts of the personal development of the students were satisfied. The students acquired mathematical knowledge, new concepts and principles especially during the course of the study. They developed logical and creative thinking, which were presented partially in the conclusions (see Figure 4 as an example). The most crucial things in students' self-development were the application of new mathematical tools and new skills in real-life problems, which is one of the main goals of mathematical modelling. They also needed to find the contribution of mathematics to other disciplines, and they independently had to understand the mathematics used.

Summing up, critical thinking is one of the most demanding skills in modelling real-life phenomena. There are many areas that must be developed when an exploration is created. The crucial advance of producing the projects is the self-development of the students. Despite the fact they make mistakes, they shape their mathematical skills and ability of critical thinking.

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METACOGNITIVE ACTIVITIES AS A MEANS TO ENHANCE STUDENTS' CRITICAL THINKING

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Metacognition is an important mechanism involved in critical thinking. The paper sheds light on metacognitive activities in teaching mathematics. It exemplifies how teachers can promote their students' metacognition as a means for enhancing their critical thinking. In doing so the paper aimes at bridging a gap between the theoretical considerations underlying the importance of critical thinking and the practical implications for supporting it in mathematics education.

INTRODUCTION

Despite differences between various conceptualisations of critical thinking, researchers agree on its importance in school education (Halpern, 1998; Ku & Ho, 2010; Kuhn & Dean, 2004; Magno, 2010). Since critical thinking enable students to use "those cognitive skills or strategies that increase the probability of a desirable outcome" (Halpern, 1998, p. 450), it is considered both as an important goal of teaching and as a means to improve students' learning (Hu & Ku, 2010). There is a variety of cognitive abilities involved in critical thinking in learning mathematics, for instance analysing arguments, claims, and evidence, making inferences using deductive or inductive reasoning, identifying assumptions, asking and answering questions for clarification, and generating and selecting alternatives and judging among them (Halpern, 1998). This thinking, however, cannot be seen as merely a composition of such abilities. Critical thinkers must be able to spontaneously make use of these abilities to regulate their learning and understanding. This regulation is a metacognitive mechanism.

Thus, to enhance students' critical thinking, teachers need to support students' metacognition. The purpose of this paper relates to this requirement. The paper explains metacognitive activities involved in learning, and particularly in critical thinking, and exemplifies how they can be supported in teaching and learning mathematics. In doing so, the paper aims at bridging a gap between the theoretical considerations underlying the importance of critical thinking and the practical implications for supporting it in mathematics education.

In the following, the relationships between metacognition and critical thinking are explained first. The next section clarifies the meaning of metacognitive activities. Afterwards, three aspects of class discussions considered favourable to promote students' metacognition in classrooms' natural settings are explained and exemplified based on a transcript from a mathematics lesson in Grade 6. Finally, challenges in supporting students' metacognition are discussed.

CRITICAL THINKING AND METACOGNITION

Despite the fact that there is no shared conceptualisation of critical thinking, researchers seem to agree that it is a higher order type of thinking that requires an active control of one's own thinking processes (Kuhn & Dean, 2004; Halpern, 1998; Magno 2010). This control is an aspect of metacognition, i.e. of thinking about one's own thinking and regulating one's own thinking (Flavell, 1979). For instance, Halpern (1998) states that when "engaging in critical thinking, students need to monitor their thinking process, checking whether progress is being made toward an appropriate goal, [and] ensuring accuracy" (p. 545). To enhance students' critical thinking, Halpern suggests engaging students in metacognitive activities, making these activities explicit and public so that they can be examined and feedback can be provided. Similarly, Kuhn and Dean (2004) claim that critical thinking "entails awareness of one's own thinking and reflection on the thinking of self and other as an object of cognition" (p. 270). To foster critical thinking in regular classes, they recommend encouraging students to reflect on and evaluate their thinking, particularly in class communication.

> If students participate in discourse where they are frequently asked, "how do you know?" or "What makes you say that?" they become more likely to pose such questions to themselves. Eventually, we hope, they will interiorize the structure of argument as a framework for much of their own individual thinking. (p. 270)

Researchers' recommendation that teachers need to support students' metacognitive activities in the classroom in order to enhance their critical thinking are in line with results from empirical studies. For instance, Ku and Ho (2010) showed that metacognitive activities are an important mechanism involved in critical thinking and that their effective use contributes to critical thinking performance. Moreover, Magno (2010) provided empirical evidence that metacognitive activities significantly improve critical thinking. His study indicates that when "learners are able to control their cognitive process, the more likely they become critical to facts presented to them. [...] once metacognition is activated, learners become more likely to make inferences, deduce conclusions, interpret accurately, evaluate arguments, and recognize assumptions." (p. 250-251).

The following section clarifies the meaning of metacognitive activities. The focus is on metacognitive activities that can be observed and promoted in class discussions in mathematics lessons.

METACOGNITIVE ACTIVITIES

Metacognition is traditionally defined as a person's own cognition about cognition and regulation of cognition (Flavell 1979). It is generally accepted that a distinction can be made between metacognitive knowledge and metacognitive skills (Veenman et al., 2006). Metacognitive knowledge refers to the knowledge one has about the interplay between person, task, and strategy characteristics. Metacognitive skills, on the other hand, refer to the actual regulation of and control over one's learning (Flavell 1979; Veenman et al., 2006).

When thinking metacognitively, learners make use of their metacognitive knowledge and skills to direct their thought processes. This thinking manifests itself in their *metacognitive activities*, i.e. *planning* and *controlling* their thought processes, and *reflecting* on them (Cohors-Fresenborg & Kaune, 2007; Nowińska, 2016). For instance, when providing arguments to prove a mathematical hypothesis, solving a mathematical problem or trying to understand a definition of a mathematical concept, learners who think in a metacognitive way actively plan how to proceed, control the accuracy of their cognitive activities, and reflect on the given task or problem and on the achieved results or experienced difficulties. Metacognitive planning activities aim at finding an appropriate approach to answer a given question or task, or to solve a problem. They include thinking about strategies and methods that could be useful in a given situation. Metacognitive monitoring refers to an ongoing process of checking the appropriateness and correctness of cognitive activities, in particular checking the correctness of an argumentation or the appropriateness of methods used in a given situation. Reflection, on the other hand, involves activities aimed at deepening and evaluating one's own understanding of the actual object of thinking. Examples include thinking about the meaning of mathematical objects, about the usefulness of particular methods, and about the difficulties or important decisions experienced in a particular situation.

Code	description
P1a	(*planning*) one-step planning activity, e.g. indication of the focus of attention with regard to strategies or methods to be used or of (intermediate) results or representations to be achieved
M4	(*monitoring*) controlling correctness or adequacy of tools or methods used in a particular situation
R4	(*reflection*) thinking about how and when to use particular strategy, tool or method

Table 1: Examples of metacognitive activities.

Table 1 provides examples of the metacognitive activities related to learning mathematics and the codes to capture these activities in transcripts. These examples are based on the coding system published in Nowińska (2016). Prefixes can also be used: *r* and *rd* to code *r*easoned metacognitive activities (including mathematical justifications or explanations concerning one's own thinking) or a *r*easoned *d*emand to engage in a metacognitive activity, and *d* or *dr* to code a request (*d*emand) for a metacognitive activity or for a *r*easoned metacognitive activity.

PROMOTING STUDENTS' METACOGNITIVE SKILLS

Given the important role of metacognition in critical thinking, the question arises of how students' metacognitive skills, i.e. their ability to use metacognitive activities in order to regulate their thinking, can be fostered in teaching mathematics. In the following, three aspects of class discussions regarded as favourable for fostering students' metacognitive skills are explained. They are chosen for the practical purposes of this paper and will be illustrated in concrete classroom situations. When considering these aspects, teachers must be aware of the fact that the challenge goal of fostering students' metacognition is to enable them to use metacognitive activities in a self-determined way, i.e. to help them develop a habit of mind to spontaneously and adequately regulate one's own cognitive activities and comprehension (Hasselhorn, 1992). Therefore, metacognitive activities must be established as a natural feature of class culture, and students must feel obligated to engage in these activities.

First of all, as metacognitive skills develop with practice, teachers are expected to facilitate students' metacognitive engagement (Mevarech & Kramarski, 2003; Veenman et al., 2006), e.g. by challenging them to consider whether they know how to proceed in a particular situation, whether they understand the goal of a particular learning activity, and whether the answers and solutions collected during this activity make sense for them.

However, to develop metacognitive habits of mind, students also need to internalize the questions (Kramarski & Mevarech, 2003). Teachers should therefore not only tell students in a straightforward way what to do, but also provide them with opportunities to regulate their learning activities in a self-determined way, e.g., by motivating students to interact with each other in class discussions, externalize their metacognitive thinking, listen to, hear, and precisely respond to their peers' metacognitive activities (Iiskala et al., 2015).

Finally, teachers should also challenge the quality of students' metacognitive activities (Kramarski & Mevarech, 2003; Van der Stel et al. 2009). During class discussions, this quality is reflected in the extent to which metacognitive activities are well reasoned, i.e., combined with explanations and justifications. Well-reasoned metacognitive activities enable students to adequately plan, control, and evaluate their learning and recognize the need for a reorganization of their actual cognitive activities.

To make these recommendations clear, particularly for teachers who might want to use them to improve their teaching, more details or specific examples from teaching practice could be helpful. The following section provides examples of metacognitive activities as part of class discussions during mathematics lessons.

METACOGNITIVE ACTIVITIES DURING CLASS DISCUSSIONS

The following transcripts document a class discussion videotaped in Grade 6 in Germany. The goal of the lesson was to foster students' problem-solving skills. The teacher claimed to be familiar with the construct of "metacognition" and wanted to promote metacognition in her class. The task she chose for the videotaped lesson is the following "7 Gates Task":

> A man goes apple picking. To get to the city, he has to pass through 7 gates. At each gate, there is a guard who demands a half of his apples and one apple more from him. At the end, the man has only one apple left. How many apples did he have at the beginning?

Episode 1 shows the beginning of the class discussion just after each student had the opportunity to work on the task on their own for 5 minutes. There, the teacher performs various actions to engage her students in metacognitive planning, particularly to make them consider how to approach the given task. The codes used to capture metacognitive activities in the transcript are explained in Table 1.

Episode 1 Metacognitive planning

1	T:	In this discussion that follows, I would like you to not declare the number of apples you consider as your solution. To be honest, […] it is not at all exciting to know whether there were 200, 300 or 500 apples at the beginning. Who needs to know that? What is exciting here is: How do you figure this out? And that's what we're talking about now. So now try to focus on giving tips on what would be a good approach to solve the task. [...] Elena. (*d*P1a)
2	Elena:	So, he always gives one more than a half. Therefore, I would first calculate the number, uhm… so for example one, uhm, plus the one apple. Then you have half and then multiply that by two. (*r*P1a)
3	T:	Let's focus on what Elena said. I think she didn't mention her very first thought. Instead, she went further ahead and mentioned her second thought. What would be the very first clue for someone who has no idea how to solve the task? […] Domi. (*rd*P1a)
4	Domi:	Instead of starting at gate one, we need to start at gate seven. (P1a)
5	T:	This thought is important. Can you justify why it makes sense for this task to start at gate seven and not at gate one? (*dr*P1a)
6	Domi:	In principle, you have to calculate backwards. He had only one apple left after passing gate seven, which means you have to calculate backwards from there, not from gate one. […] (*b*P1a)

7	T:	I think when you say you have to calculate backwards, there might be one intermediate step still missing here that could be helpful [...]. (M4)
		I would actually say you have to think backwards. (P1a)
		What is the difference between calculating backwards and thinking backwards? Alina. (*d*R4)
8	Alina:	I think calculating backwards is when you calculate the result back from the last gate to the starting gate, so to say, and thinking backwards is when you go from the last gate to the first gate. […] (R4)

In her first contribution, the teacher makes it clear that the way of organizing one's own thinking to figure out the solution is the interesting part for discussion instead of the solution itself. In doing so, the teacher indirectly stresses the need for engaging in thinking about one's own thinking – here in planning one's own cognitive activities required to solve the task. Finally, she directly *d*emands students' metacognitive planning by asking them to provide tips on how to approach the task (*d*P1a). In her reaction to this demand, Elena tries to describe her own approach. However, instead of providing general tips, she describes her calculation without explaining how she came up with it. Her contribution indicates a *r*easoned metacognitive planning activity, which is explained by referencing the task (*r*P1a). Since Elena's metacognitive thinking which led her to start with this calculation is not visible in her contribution, the teacher repeats her request for metacognitive planning, justifying the need for explaining the approach to solve the task in a more precise way. Therefore, her request can be seen as a *r*easoned *d*emand for a metacognitive planning activity (*rd*P1a).

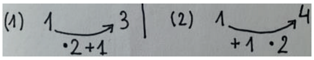
Domi responds to this request. His tip that one needs to start solving the task by focusing on the situation at the last gate is an indicator of his metacognitive planning activity (P1a). The teacher stresses the importance of Domi's tips and uses it as the reason behind her next request. Her contribution can be interpreted as a *d*emand for a *r*easonable metacognitive planning activity (*dr*P1a). In his response, Domi justifies his previous tip. His contribution is therefore a *r*easoned planning activity (*r*P1a). So far, Elena and Domi justified where the focus should be when making calculations. The teacher, however, still sees the need for further clarifications concerning the calculation suggested by Elena. The first statement in her contribution in line 7 indicates that the teacher verbalizes her monitoring concerning the appropriateness of the planning steps suggested by the students (M4). Afterwards, she suggests an additional planning step (P1a), being metacognitively engaged in the process of solving the task. Finally, she engages the students in a metacognitive reflection on the two different cognitive activities required to solve the task – making a calculation backwards and thinking backwards. To this end, she *d*emands a reflection on the mathematical cognitive "tools" (*d*R4). Alina's reaction is a metacognitive reflection on differences between one's own thinking involved in making calculations backwards from gate seven to gate one and in "going" backwards from the last to the first gate (R4). In the latter case, she probably means "going" in the sense of imagining the actions described in the task from the very last to the first.

To sum up, the transcript provides a teacher's exemplary actions aimed at engaging students in metacognitive activities and at challenging their quality. The teacher stresses the importance of providing precise explanations for how to organize the cognitive process of solving the tasks. Thereby, she uses students' answers as a starting point for deeper, reasonable metacognitive activities. It is worth noting that students' responses become more precise in reaction to the teacher's requests. Episode 1 also exemplifies the meaning of making students' metacognitive planning activities visible in the classroom. This involves more than just describing and checking calculations. Calculations are a result of metacognitive thinking about how to approach a given task. Here, the teacher specifically aims at making this thinking visible for others.

Episode 2 presents the class discussion after the students worked in pairs to finish their solutions. This episode provides examples of the teacher's actions aimed at engaging students in metacognitive monitoring.

Episode 2 Metacognitive monitoring

1 T: Before we discuss your solutions, [...] I would like to discuss how many apples there were in front of the seventh gate. [...] Here, Fin and Rico listed two possibilities, and that is actually the exciting part. Franz, can you explain how you came up with the three apples in front of gate seven? (*d*M4)



2 Fin: […] I first doubled the number that is there, and then I added one apple.

3 T: […] How did you come up with the other solution? Rico. (*d*M4)

4 Rico: We first added one and then doubled.

5 T: They wrote down both solutions because they said that both are correct. Can you please comment on this? Elena. (*d*M4)

6 Elena: I would say no, because he always gives a half and then one more. If he has three, he can't give a half and then one more. Then he would only have a half of an apple. (*r*M4)

7 T: Pass it on, please.

8 Elena: Mati.

9 Mati: I agree with Elena. Since it says there that first half of the apples were taken and then additionally one more was taken, I would also say that the second solution is likely correct. (*r*M4)

10 T: Pass it on, please. […]

11	Fin:	Now I also agree with the second solution. Because [...] it is called a backwards calculation, which means you have to change something in the calculation. It is said that he first gives a half and then he gives one, but calculating backwards always involves changing the calculation [...] so that you don't multiply first, but you first add and then you multiply. (*r*M4)
12	T:	You describe it on the algebraic level, I would like to get into the story with you. [...] Who's going to tell the story backwards? [...] (*d*R4)
13	Domi:	The man walks backwards through the gate, he gets an apple from the guard, and then gets as many as he has in his car at the moment. (R4)

In her first contribution, the teacher precisely describes what the students need to control: two different calculations made to determine the number of apples in front of gate seven. Afterwards, she asks Fin and Rico to explain how they came up with their calculations. Both requests can be seen as *d*emands that have the potential to initiate students' metacognitive monitoring (*d*M4), particularly their thinking about the correctness or adequacy of their calculations. This would require for both students not to explain how they calculated the result to be 3 or 4, but rather explain the thought process that led them to their calculations. This explanation would be considered metacognitive monitoring with regard to the correctness of the suggested solutions. Unfortunately, Fin and Rico focus only on their calculations. Their contributions do not indicate any metacognitive activities. In line 5, the teacher explicitly makes both calculations the object of thinking for all students, *d*emanding metacognitive monitoring aimed at clarifying the appropriateness of both solutions (*d*M4). In her reaction to this request, Elena justifies the incorrectness of the calculation with the result of 3. Her contribution can be seen as a *r*easoned monitoring activity (*r*M4). In lines 7 and 10, the teacher motivates the students to take responsibility for the process of clarifying the correctness of both solutions. "Pass it on" means that the student has to choose a classmate to continue the discussion.

Mati and Fin (lines 9 and 11) engage in *r*easoned monitoring activities (*r*M4). Both, however, refer only to calculations they consider correct. The metacognitive thoughts that led them to think that the chosen calculation does indeed provide the correct way to calculate the number of apples is not visible in their explanations. The teacher seems to notice this. That might be the reason why she *d*emands to tell the story backwards (line 12). In doing so, she motivates the students to engage in metacognitive reflection on how one's own thinking works backwards in the given case (*d*R4). This is needed as a means to showcase how to organize one's own thinking before writing down a calculation that matches this thinking and correctly describes the changes in the number of apples. Domi's contribution provides the result of his thoughts on how to organize each step in thinking about the story backwards (R4).

To sum up, Episode 2 exemplifies two kinds of activities used by the teacher to foster students' metacognitive skills. Firstly, she provides opportunities to engage in metacognitive activities, using the students' answers and solutions as a starting point for her requests to think about the cognitive aspect involved in solving the given task. Secondly, she challenges the students to take responsibility for their shared learning process. Interestingly, the students seem to be used to doing this. For instance, they seem to be able to hear what their classmates say and take into account their classmates' answers. The result of this thinking is visible in the agreement the students communicate at the beginning of their own contributions: "I agree with Elena.", "Now I also agree (…)".

DISCUSSION

The practical purpose of this paper is to shed light on metacognitive activities, which are a means to enhance students' critical thinking, and to exemplify how students' metacognitive skills can be fostered in regular teaching. Both episodes from the class discussion provide examples of a teacher's actions aimed at engaging students' in metacognitive activities and challenging their quality. Establishing a discourse-based classroom which obligates students to engage in metacognition is one approach to fostering metacognitive skills. This can be done not only in the case of discussions focused on problem-solving, but also, more specifically, when discussing the meaning of new concepts or relations between concepts, or the appropriateness of students' representations and internal conceptions. Another means to promote metacognitive skills in regular teaching, not mentioned before, is the use of mathematical tasks that explicitly require students to externalize their metacognitive thinking in written form. One such task developed by the teacher, as observed in Episode 1 and 2, is the following: "Cornelia claims: 'Once one figures out how many apples there are in front of gate 7, the task is almost solved!' Do you agree with Cornelia? Explain your reasoning.". The task requires the students to reflect on the complexity of their approach, involving thinking backwards and calculating backwards, and on the difficulties experienced while applying this approach. Two students' answers captured in the class discussion indicate that this task does indeed lead to well-reasoned reflection: "I agree with Cornelia, because then you know how to do the calculation. (...) And then the calculation for the whole task is clear."; "I would say yes and no! (…). Yes, since if you figure it out, you know how to calculate the final result. But making the calculations is the reason for "no", because the calculations are not so simple."

The challenging goal of promoting students' metacognitive skills is to enable them to engage in metacognition in a self-determined way. Empirical studies indicate that teachers often do not know how to promote metacognition. They focus more on training students' cognitive than metacognitive abilities (Dignath & Veenman, 2021). Due to the crucial role of students' metacognitive skills for enhancing their critical thinking (Ku & Ho, 2010), and the relevance of critical thinking for a productive participation in a democratic society, this is an alarming finding. In this sense, supporting students' metacognitive skills must be regarded as a pedagogical task of a higher relevance in school education.

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DIFFICULTIES OF STUDENTS WITH CRITICAL THINKING DURING PROVING

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The ability to think critically is one of the basic competences of the modern person. Critical thinking involves many activities, including formulating and verifying hypotheses, evaluating evidence and arguments, and searching for an action strategy. The study presented here concerns checking the difficulties (concerning critical thinking) that students of mathematics have with tasks requiring the making of a proof. The reasoning presented by students was analysed, which was a solution to two tasks – standard and non-standard. The study has shown that students have made good use of certain elements of critical thinking, such as: explaining reasoning, incorporating assumptions, and applying learned strategies. At the same time, they have shown difficulties in assessing the correctness of evidence and arguments.

INTRODUCTION AND THEORETICAL FRAMEWORK

People learn not only during school attendance, but also acquire knowledge and skills throughout their lives. The Council of the European Union has compiled a list of the most desirable competences of modern person. They should be developed throughout life. The expected skills have been classified into 8 categories, referred to as key competences. The journal of laws (ERF, 2018) states: "Key competences are those competences everyone needs for self-fulfilment and personal development, employment, social inclusion, sustainable lifestyles, successful lives in peaceful societies, healthy life management and active citizenship […]. All key competences are considered to be of equal importance; each contributes to a successful life in society." One of the skills that are part of all key competences (in particular: mathematical competences and competences in the field of life sciences, technology and engineering, competences in the field of understanding and creating information) is critical thinking. Thus, the ability to think critically is an indispensable competence of a modern person, especially a mathematician.

Critical thinking can manifest itself in different areas of life. This is due to the fact that critical thinking consists of many activities such as: problem solving through information analysis, formulating and verifying hypotheses, evaluating evidence and arguments, and searching for an action strategy (Firdaus et al., 2015; Huitt, 1998; Krulik & Rudnick, 1999; Pyzara, 2021; Sukmadinata, 2004; TC, 2013).

According to Facione (2011), the most basic element of critical thinking is the ability of interpretation, analysis, evaluation, inference, explanation and self-regulation. Chance defines critical thinking as the ability to analyse facts, generate and organize ideas, defend opinions, make comparisons, draw conclusions, evaluate arguments and solve problem (Chance, 1986). Similarly, Sukmadinata (2004) states that critical thinking is a skill of reason on a regular basis, systematic skills in assessing, solving problems, appealing the decision, give confidence, analysing assumptions and scientific inquiry.

This means that when people use critical thinking (especially in mathematics) they: make sound decisions, solve problems by analysing information, are able to draw conclusions and evaluate arguments, are able to evaluate the correctness of evidence, take into account criteria or grounds for an informed decision, and do not apply a rule without assessing its usefulness (Ennis, 1996; Firdaus et al., 2015; Huitt, 1998; Krulik & Rudnick, 1999; Pyzara, 2021; Sukmadinata, 2004; TC, 2013).

The assessment of critical thinking skills in mathematical problem solving consists of three parts (Firdaus et al., 2015; Krulik & Rudnick, 1999):

- the identification and interpretation of information,
- information analysis,
- the evaluation of evidence and arguments.

Many of the skills that are part of critical thinking are essential in making mathematical proofs. These are, among others the ability to reason regularly, analyse facts, evaluate arguments, analyse assumptions, take into account criteria or grounds for making informed decisions, the ability to draw conclusions and clearly explain the reasoning. It is also important to be able to evaluate proofs and arguments, allowing you to accept a given reasoning as true or reject it.

Therefore, we can study the critical thinking of students by analysing their actions in carrying out mathematical proofs. These types of tasks require many aspects of critical thinking (Mingla, 2020). Moreover, Otten et al. (2021) report that the proving process has the potential to foster critical thinking in students' current and future daily, real-world experiences.

Let us recall what we mean by a mathematical proof. "A *proof* is a deductive argument that claims to show a conclusion is a logically necessary consequence from agreed upon assumption" (Ciosek et al., 2017, p. 47, emphasis in the original). When carrying out a proof, not only deductive reasoning can be applied, but also reductive, indirect proof or induction. These methods are often combined (Siwek, 2005).

Proving theorems and carrying out tasks that require proving is not a simple activity. It is difficult for both students and sometimes teachers (Ciosek et al., 2017; Dąbrowski, 1993; Mingla, 2020; Otten et al., 2021). Ciosek et al., on the basis of the analysis of numerous studies, state that "pre-service and in-service mathematics teachers have various difficulties in understanding the essence of mathematical proof" (Ciosek et al., 2017, p.45). Interestingly, Dąbrowski (1993) noted in his research that older students, more often than younger, less experienced colleagues, are able to recognize erroneous reasoning as true if they used methods known to students (even if there are errors in them). The younger students tried to understand the logic of the proof and rely less on the methods used.

The difficulties in understanding the essence of the mathematical proof described in the literature include:

- problems in deciding whether an argument can be considered a mathematical proof,
- considering the empirical argument sufficient to justify the truthfulness of the general statement,
- adopting erroneous arguments,
- lack of awareness of the adopted basis of reasoning,
- recognizing correct evidence as incorrect,
- recognition of the validity of the evidence and simultaneous verification of the truthfulness of the statement on the basis of examples,
- recognition of the truthfulness of a proof based on the recognition of known methods or forms of writing, and not on the basis of mathematical content (Ciosek et al., 2017; Dąbrowski, 1993).

Some of these errors may be related to a lack of critical thinking in these areas, e.g., in evaluating arguments.

RESEARCH QUESTION

The aim of the study was to check what difficulties students have in critical thinking when taking mathematical proofs. Being aware of the fact that many activities are involved in critical thinking, we focused on those aspects of critical thinking that can occur in the performance of mathematical proofs. In particular, answers were sought to the following questions:

a) Have the students completed the mathematical proofs correctly and are they using the arguments correctly?
b) Do students use the previously learned strategy or create their own?

c) Are the presented reasonings properly justified?

d) Do students consider entry criteria?

e) Do the respondents provide an explanation of how they reached the presentation of the evidence?

f) Do the same difficulties arise when performing a typical task and a non-routine task?

The above detailed questions will help to analyse the mathematical evidence presented by the respondents from the point of view of critical thinking and indicate the difficulties.

METHODOLOGY

The study was conducted at the beginning of 2022. The participants were 19 undergraduate students of mathematics (22 years old), which enrolled in arithmetic and number theory classes. The course lasted one semester and ended with a written exam. The exam consisted of 10 tasks, two of which are a research tool. Thus, the students tried to write the best solutions. The exam lasted 90 minutes, so each task took 9 minutes.

The research tool consisted of tasks that require a mathematical proof. The tasks concerned the issues that were discussed and practiced during the classes, which means that the respondents had the opportunity to acquire the knowledge and skills needed to perform the analysed tasks.

The first task was a standard task – analogous examples were performed during the classes, so it was enough to use the learned method. The second task was non-routine but could be solved with the methods used in the classroom (it was necessary to apply the known methods in a situation similar to the one analysed during the class). The content of the tasks is presented below.

Task 1. Prove that the number $2^{6n+1} + 9^{n+1}$ is divisible by 11 for every natural n.

Task 2. Show that for every natural $n \geq 1$ number $\sqrt{8n+3}$ is irrational.

The study was based on the analysis of students' written solutions. The mathematical proofs they presented were considered in terms of those activities that are part of critical thinking. In particular, answers were sought to the research questions posed. The study did not intend to analyse statistically, but to examine individual instances of critical thinking difficulties encountered in carrying out mathematical proof.

Each student's solution was analysed in terms of difficulties or errors. A description of the coding process of the difficulties noted is given below:

- use of incorrect mathematical transformations – incorrect application of properties of mathematical operations or use of false properties of operations;

- application of the scheme (mathematical methods) without understanding it – referencing the remembered dependencies, although in a given case they are not true; using memorized arguments, even though in this case they do not make sense (they are untrue);
- use of empirical calculations as a basis for drawing general conclusions – the use of empirical checks as a basis for justifying divisibility in general cases or drawing general conclusions based on the calculations performed for a few initial natural numbers;
- not all stages of the proof are shown – a fragment of the proof (beginning or some elements) is given;
- use of false argument – use of false, incorrect dependency or a false statement.

ANALYSIS OF STUDENT SOLUTIONS

The results will be presented firstly in relation to the tasks, and then collectively.

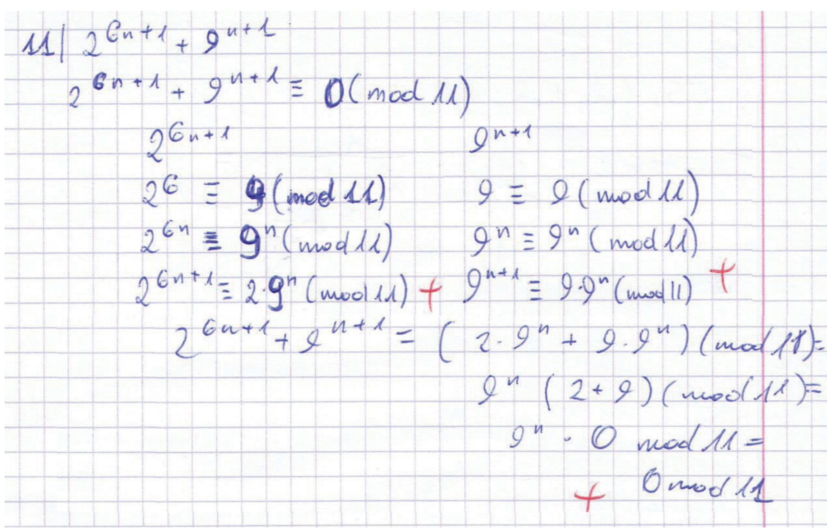

Figure 1: Correct solution of task 1.

Task 1 results

Task 1 requires showing the divisibility of natural numbers. This is a typical problem. To demonstrate divisibility, the properties of the congruence had to be used. An example of a correct solution is shown in Figure 1. The respondents coped with this task quite well. Eleven (out of 19) students presented a fully correct proof – they used the correct reasoning and justifications, took into account the initial criteria and explained how they came to present the evidence. All these students applied the solution strategy learned during the course (ten deductive proofs and one inductive proof). It can be concluded that in these cases no problems with critical thinking were detected.

The next two solutions present the evidence with minor errors (e.g., a mistake in changing the sign of a number in the notation), but nevertheless show the correct line of reasoning and logical justifications.

There were significant errors in six of the responses. Their authors also used the strategies of solutions learned during the classes (mainly deductive proofs), but there were two attempts to combine or change the method of carrying out the proof. Despite the errors, only two people left the proof unfinished, while the others presented a solution, the punch line of which is to prove the thesis (unfortunately, there are errors in the reasoning).

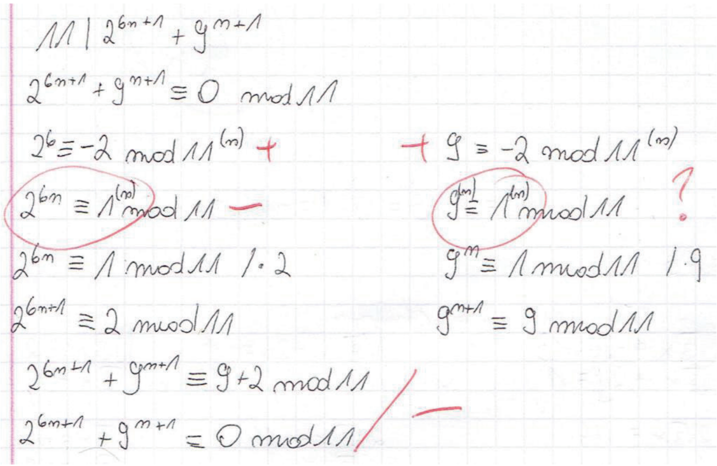

Figure 1: Example of applying a rule without understanding it.

Examples of significant errors are shown in Figure 2 and Figure 3.

In both cases, the students applied the previously learned rule without understanding it (critical thinking mistake). These people remembered that when using the congruence property, it is advantageous to find the congruence of a given number to 1. Then you can take advantage of the fact that $\bigwedge_{n,p\in\mathbb{N}} \bigwedge_{a,b\in\mathbb{Z}} (a \equiv b \; mod \; p \implies a^n \equiv b^n \; mod \; p)$. Thus, the students at one point put a 1 in the place for another number, although there is no logical justification for doing so. They made further transformations using congruence with the number 1, because they simply demonstrated the desired thesis, but did not take into account that the argument they refer to is false.

$$
\begin{aligned}
2^{6m+1} + 9^{m+1} &\equiv 0 \bmod 11 \\
64^m \cdot 2 + 9^m \cdot 9 &\equiv 1^m \cdot 2 + 1^m \cdot 9 \bmod 11 \\
2 + 9 &\equiv 0 \bmod 11 \\
11 &\equiv 0 \bmod 11
\end{aligned}
$$

Figure 2: Example of incorrect mathematical transformations.

Such actions also show that the properties of the congruence are not fully understood, which results in incorrect application of some properties (using wrong mathematical transformations). Thus, the students refer to false arguments, which is a critical thinking mistake.

The reasoning presented by 19 students contains the following errors (the number in parentheses indicates how many student solutions the error has occurred):

- use of incorrect mathematical transformations (5);
- application of the scheme (mathematical methods) without understanding it (4);
- the use of empirical checks as a basis for justifying divisibility in general cases (1);
- not showing all steps of proof (1);
- calculation error (1).

The errors in the students' answers testify to the incorrect presentation of the proofs, the use of unjustified arguments and the making of wrong decisions. Moreover, it can be assumed that the student, when returning such (incorrect) solution, considered it correct, which may also indicate an incorrect evaluation of the evidence.

The indicated difficulties concerning critical thinking were demonstrated by the students when solving a standard task requiring the presentation of a mathematical proof.

It is worth emphasizing that even in incorrect solutions, the respondents tried to explain how they came to present the evidence. No errors were found in taking the assumptions into account in the student solutions. Thus, there are elements of critical thinking that did not pose difficulties to the respondents in this task.

Task 2 results

In task 2, the students had to prove that the numbers in the form $\sqrt{8n+3}$ are irrational for every natural n. This is not a routine task, but the methods by which this proof can be made were practiced during the arithmetic course.

The students had much more trouble with this task. Only 2 students (out of 19) provided correct evidence. They applied the evidence indirectly, presented the justifications correctly and explained the reasoning used.

Every third respondent (7/19) did not attempt to complete the task, while 10 people presented reasoning that cannot be considered correct. Further analyses will concern these 10 solutions.

Most of these 10 students used deductive proof when doing this task, three people did the proof indirectly and one was a combination of deductive and indirect reasoning. Unfortunately, in half of these arguments (5), the justifications were based on empirical calculations (see figure 4). The examples calculated for n = 1, 2, 3 were used to draw general conclusions.

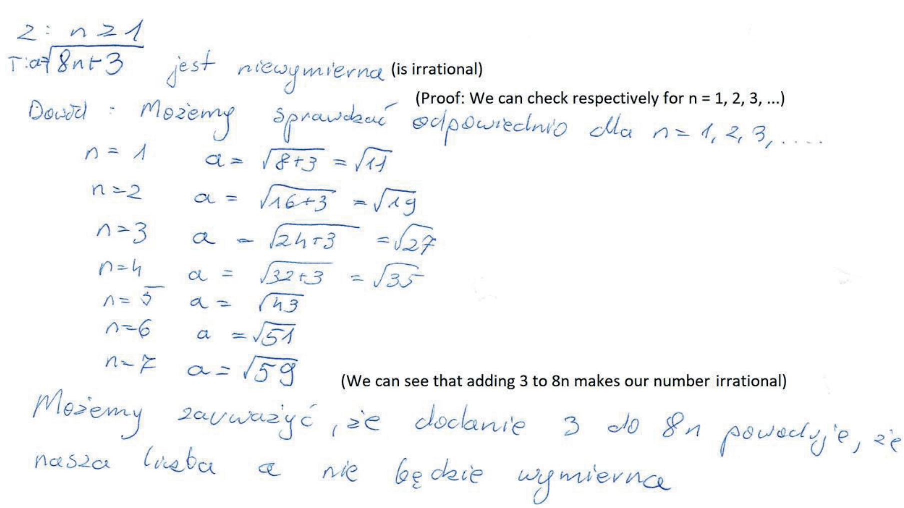

Figure 4: Empirical justification taken as proof.

This shows a misunderstanding of the essence of the mathematical proof, and thus shows a lack of critical thinking in correct proof evaluation.

The type of reasoning used in this task reflects the fact that students performing a non-routine task much more often (7 people) create their own problem-solving strategy. Only 3 people used the method presented in the course based on the analysis of square residuals. Interestingly, 3 out of 7 people using a different method referred to the strategies known earlier, related to the presentation of a rational number in the form of a quotient of two whole numbers.

Unfortunately, regardless of the adopted strategy, these 10 solutions cannot be considered correct.

It is worth noting that out of these 10 students, 4 provided only part of the reasoning, of which 3 presented fragments are correct, contain good justifications and explanations, but their authors were not able to prove further elements of the proof. This is evidence of the critical thinking of these people, as they did not apply false arguments or erroneous generalizations.

The good side of all the answers is that the students take the assumption into account, and they tried to explain the presented reasoning. Thus, these areas of critical thinking did not pose any difficulties to the respondents in this task.

However, difficulties appeared in critical thinking in assessing the correctness of the evidence and the arguments used (7 people).

The reasoning presented by 10 students contains the following errors:

- use of incorrect mathematical transformations (6);
- use of empirical calculations as a basis for drawing general conclusions (5);
- use of a false argument (6);
- not all stages of the proof are shown – a fragment of proof was given (4).

The main problem with critical thinking was the use of unjustified or wrong arguments. Examples of invalid arguments used by students:

- 11, 19, 27 do not have a rational root, so no matter how much 8 we add, it will always give us an irrational number.
- $8n$ is even, 3 is odd, so $8n + 3$ is odd for any $n \geq 1$, and the root of the odd number is irrational.

The students presented a general conclusion not based on deductively justified rationale, but on examples or erroneous arguments. This is a misunderstanding of the idea of a mathematical proof, and thus a difficulty in critical thinking in proof and argument evaluation.

CONCLUSIONS

Research has shown that students of mathematics have some difficulty in critical thinking in correctly evaluating evidence and arguments. Difficulties appeared both in the performance of a typical and non-standard task, but they were of a slightly different nature.

In the case of a typical task, more than half of the respondents completed it flawlessly. The students applied the previously learned strategy, considered assumptions, and presented explanations. Unfortunately, there were solutions that used previously known schemes without understanding them, which resulted in incorrect mathematical operations. Thus, incorrect arguments were used.

The non-routine task was much more difficult. Every third student did not do it, and only 2 people presented the correct evidence. In the case of an unusual problem, students not only used the learned strategies, but also searched for their own new solutions. Unfortunately, there were erroneous justifications in these solutions. The most common problem in critical thinking was the use of false arguments (incorrect assessment of the truthfulness of arguments) and the use of empirical justifications as the basis for demonstrating general conclusions (incorrect assessment of the truthfulness of evidence).

The results of the study conducted are consistent with the research conducted by Harel and Sowder (2007) on understanding the correctness of evidence. They showed that students often used empirical schemes to provide evidence. They also often did not know what characteristics of the argument should be acceptable in mathematics. Current research has confirmed these difficulties and, in addition, pointed to their correlation with the lack of critical thinking in these areas.

It is worth emphasizing that there were elements of critical thinking in which the respondents did not show difficulties, such as: explaining reasoning, taking assumptions into account, and applying learned strategies. Unfortunately, they show difficulties in assessing the correctness of evidence and arguments.

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ONE TASK – DIFFERENT SOLUTIONS

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The aim of mathematics education is not only to equip the student with the necessary mathematical knowledge and skills that can be useful in everyday life; strong emphasis is put on developing creative and critical thinking. This can be achieved by setting various tasks to stimulate students' creativity. Discovering the rules by the students and finding dependencies between them in which there is no pre-imposed solution favours the development of critical and creative thinking. In this paper, the results of students' work on such a task are presented. Students followed different approaches to the same problem, showing a wide spectrum of solutions.

INTRODUCTION

In recent years a lot of effort has been put into improving the educational system. The changes which were made concerned not only the way the system was organized, but also affected the curriculum of teaching. The changing reality in which we are living forces a change in the approach to education. Mathematical education faces new challenges these days. The goal is not only to equip the student with the right mathematical knowledge of concepts and properties, which is important, but there is also something else. It is important to equip the student with skills which are useful in real life (da Ponte, 2008; Krygowska, 1985, 1986). Learning and teaching mathematics is primarily understood as learning to think, act and communicate mathematically (Arzarello, 2016). In addition to substantive knowledge, the students should also have a whole range of mathematical skills, such as the ability to analyse and to make hypotheses, to argument, to have justification ability, and to have creative and critical thinking. Especially critical and creative thinking is particularly important (Oldridge, 2015). It seems that this approach was important for those who are preparing a new core curriculum in the Polish educational system. Changes in the curriculum put the main emphasis in the teaching of mathematics focused on the development of thinking. The idea is to educate in such a way that the student will be a self-thinking person (MEN, 2017). Therefore, an important goal for the teachers is to promote critical thinking by suitable activities. Students who can think critically grow into lifelong problem solvers. Critical thinking with students means that they can take information and analyse it, draw conclusions, formulate opinions, reflect on their work, and approach problems in a systematic way.

Another issue is how the teaching-learning process is organised. The essential aspect of this is the significant role of the student in discovering his or her own mathematical knowledge. The teacher should have a role of an observer and provide some help for the student. Student's self-discovery of knowledge has much more measurable effects.

THEORETICAL FRAMEWORK

Building up the students' mathematical knowledge is a long and complex process. It consists of a number of factors. An important element here is the students' own experience. The more experience they have the better cognitive process the gain.

In mathematics education, two fields are mainly distinguished: arithmetic and geometry. Both are interdependent and interpenetrate and complementary to each other, although they have different roots and approaches. The world of arithmetic is ultimately structured, governed by clear rules. The individual records and symbols used in this world are equally read by all. The situation is different in the world of geometry. As Hejny and Jirotkova (2006) write:

> The world of geometry is a community of individuals or small families and there is a large diversity in the linkages between them. From the didactic point of view, arithmetic is suitable for developing abilities systematically, and geometry is more suitable for abilities such as experimenting, discovering, concept creation, hypothesizing and creating mini-structures. (p. 394)

Thus, geometry is a good starting point for developing mathematical thinking and a creative approach. Especially if we combine geometric ideas with the search for regularity. Studying regularity, discovering rules and dependencies is one of the ways to develop students' mathematical thinking. In literature one can find descriptions of research in discovering and generalizing of noticed rules (Carraher, Martinez, & Schliemann, 2008; García Cruz, & Martinón, 1997; Littler & Benson, 2005a, 2005b; Mason, 1996; Orton & Orton, 1999; Stacey, 1989; Zazkis & Liljedahl, 2002a, 2002b;). Solving tasks related to noticing and discovering regularity has a lot of advantages. It stimulates the development of the students' thinking and creativity, it also teaches the right approach to: analysis, hypothesis, verification. The search for regularity is one of the methods of solving problems. Swoboda (2006) writes about it:

> … noticing the regularity is a skill desired by all means. Activities in which a child is to notice the regularity, act according to the rule – are those stimulating his mental development. They are also the basis of mathematical thinking at each level of mathematical competence. (pp. 51-52)

The tasks related to discovering the rules are interesting and full of challenges for the students. They are also a source of satisfaction for them (Gruszczyk-Kolczyńska, 2001; Urbańska 2003). Used tasks connected with searching and noticing dependency and regularity could support the development of students' reflective, critical thinking. On the other hand, this kind of thinking - critical and reflective thinking - can significantly support the development of mathematical thinking, and in particular algebraic thinking.

METHODOLOGY OF RESEARCH

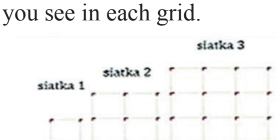
The study described in this paper was carried out as part of a master's thesis by a student of mathematics at the University of Rzeszow (Pikor, 2021). This study concerned the generalization skills of primary school students from grades 4-8. Thirty-four students (grade 4: 12 students, grade 6: 12 students, grade 8: 10 students) participated in this study. It was conducted during one meeting in each class. The research material consisted of students' written works and observations during the research.

In this paper I am presenting the general results from all the classes. However, I particularly focus on the work of students from the fourth grade. My goal was to analyze the strategies used by these students during solving the tasks. I was interested in what way they were looking for solutions and what dependencies they managed to find there. Hence, the research questions were:

- What work strategies did the fourth-grade students use?
- What dependencies did they find in the task?
- Did the way of working on the task allow students to see the dependencies and make a generalization?

The research tool was a worksheet with the following tasks:

1. Draw the fourth and fifth grids. Give the number of matches used to create each grid and the number of squares you see in each grid.



No of grid	1	2	3	4	5
Number of matches					
Number of squares					

2. How many matches and squares will be in the 6th and 7th grids.

3. Is it possible to create a grid with the same number of matches as the number of squares? Justify your answer.

4. Do you see any correlation between the number of the grid and the number of matches and the number of squares? If yes, try to write which one.

Figure 1: The research tool.

The tasks presented on the worksheet did not impose a method of solution. The students had complete freedom in choosing the method of proceeding and interpreting the content of the task. Two possibilities were considered as the correct solution. It was possible to count only unit squares, i.e., squares with a side length of one match. Then the sequence describing the number of squares obtained by the student would look like this: 1, 4, 9, 16, 25, 36, 49, … etc. Therefore, they were squares of consecutive natural numbers (here: numbers of consecutive grids). In the second approach, it was possible to count all the squares present in the grid, i.e., those with the side of one match, two matches, three matches, etc. Then the obtained sequence describing the number of squares would look like: 1, 5, 14, 30, 55, 91, 140. Here, in order to obtain the result, one had to add the next squares of natural numbers to the first grid in the series. Both solutions were treated as correct. As for the number of matches, only the analysis of the results entered in the table made it possible to notice the appropriate relationship and discover the rule. To calculate the number of matches, you had to add successive multiples of 4 to the previous number (e.g., for grid 2: 4 + 8, for grid 4: 4 + 8 + 12). Hence, the general rule was as follows: the number of matches used to build the grid number n is the sum of the consecutive natural numbers from 1 to n multiplied by 4, which can be written as $2n\,(n + 1)$.

THE RESEARCH RESULTS

The analysis of the research material allowed to distinguish several strategies used by students to calculate the number of squares in a given grid (regardless of which class they were in):

A – unit square – students counted only squares with the side of 1 match

B – all squares – students counted all possible squares that were visible in a given grid (i.e., 1×1, 2×2, 3×3, etc.)

C – biggest square and unit squares – students took into account a large square and creating it the unit squares

D – others – different approaches of students, e.g., different for even numbered grids and different for odd-numbered grids

Students usually calculated the number of matches by counting them one at a time on each grid. The results obtained as a result of the analysis of student papers showed that the dominant approach was to count unit squares. In addition, the students tried to see the relationship between the number of squares, the number of matches and the grid number. However, the noticed dependencies did not always lead them to the general rule. The aggregate results of the surveys of all the students taking part in them are presented in the table below:

Grade	Strategy				Number of squares		Number of matches		Noticed dependences	Generalization
	A	B	C	D	Corr.	Incor.	Corr.	Incor.		
4	50%	25%	25%	0%	67%	33%	50%	50%	17%	0
6	80%	17%	25%	8%	75%	25%	75%	25%	60%	30%
8	30%	30%	30%	10%	90%	10%	90%	10%	50%	30%

Table 1: Results from preliminary research.

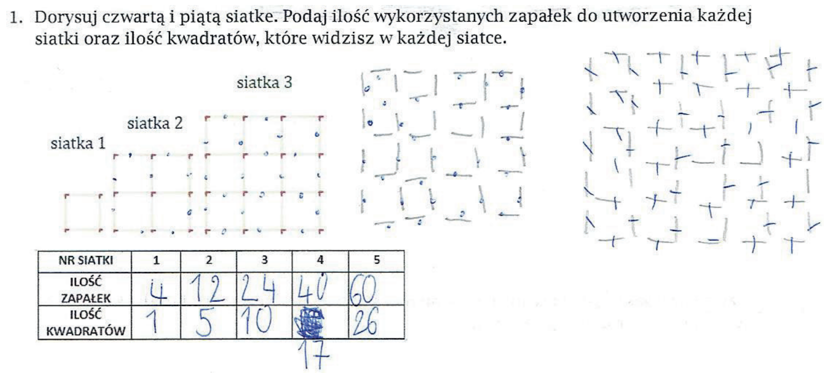
As we can see from Table 1, all students participating in the study undertook to solve the tasks from the worksheet. In addition, most of the first two tasks were completed correctly (more than 80% of students gave the correct number of squares in individual grids, and 75% - the correct number of matches). The students tried to see the relationship between the number of squares, the number of matches and the grid number. However, they had a lot of difficulties in generalizing the observed dependencies and writing it down in the form of a general rule.

EXAMPLES OF STUDENTS' WORK

Fourth-grade students had the least experience in algebra, hence their approach to the solution was the most intuitive. Below I am presenting an analysis of the selected works done by the students from the fourth grade.

Example 1

Student S1 tried to solve all the tasks. The result of his work is shown in Figure 2.



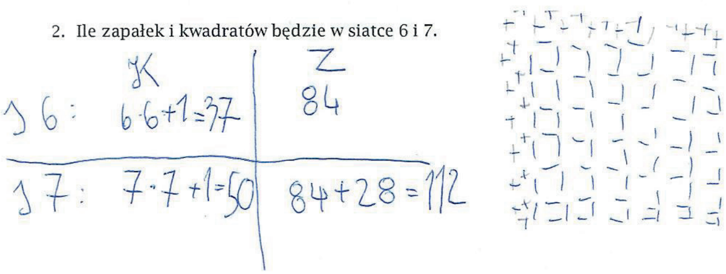


Figure 2: Example of work of student S1.

Analyzing the solutions presented by this student, we can see that in the case of the number of squares, he used a strategy: the biggest square and unit squares. This is evidenced by the recorded calculations for the 6th and 7th grids: 6×6+1=37 and 7×7+1=50. This approach may indicate that the student has noticed that the entire grid has the shape of a square. On the side of a large square there are as many small ones as the number of the grid. Hence, one large square is divided into $n \times n$ smaller, unitary ones. Such perception is strongly related to the visual aspect of the task. Perhaps the visual aspect was the dominant one here. The student consistently applied the noticed dependence in the further part of the task. When counting the number of matches for individual grids for the first task, the student used the help of drawings. He counted 'on foot', marking each counted match, which can be seen in the drawings of the grids. This method worked well with the first five grids. Moving on to the second task, the student began to analyze the successive numbers and noticed that the values were increasing by 8, 12, 16, 20 consecutively. Hence, he concluded that the number of matches in the 6th grid would increase by 24, and in the 7th – by 28. He wrote this discovered relationship in an additional table: 84+28=112. The lines on the matches from the first column and the first row visible in the grid drawing No. 7 may indicate that the student was checking the correctness of his hypothesis – the converted elements are 28 matches. The discovery of this relationship was possible due to the fact that while drawing subsequent grids, the student noticed that each subsequent one is created by adding one row and one column to the previous one. The experience gained during the creation of subsequent illustrations resulted in the discovery of the arithmetic relationship between individual numbers in the 'number of matches' line. The student attempted to answer the remaining questions. However, due to the lack of adequate knowledge in the field of algebra, he was unable to write down the observed dependencies in the form of a general formula. He only noticed that: "a grid is a square composed of small squares, and for one square you need four matches". This is, in a sense, a justification for the rule he used in the task.

Example 2

The student S2 focused his work primarily on the visual aspect of the task. The most important thing for him was fairly accurate mapping of the girds, hence paying attention to the proportions, colours or knots that are matchheads. When starting to draw, he first tried to draw a square correspondingly larger than the previous grid, and then divided it into smaller ones. For this student, it was important to count the unit squares of which the individual grids are composed. He also noticed that the number of squares in each row and column of the next grid is the same as its number. Hence, the number of squares is the number of the grid multiplied by itself. These results were also entered in the individual boxes of the table: 1, 4, 9, 16, 25. Quite surprising in the case of this student is the answer to the question about the number of matches. It seems suggestive to include the first two numbers ‘4’ in the table. This may have resulted in the discovery of a relationship: the number of matches in a given grid is the same as the number of squares in the next grid. And the geometric relationship associated with the number of squares was easier to notice, hence the student first calculated the number of squares, and only then typed the number of matches on its basis. The consequence of this behavior can be seen in the saved solution to task 2: “in grid 6 there will be 49 matches”. For the student, the discovery of these dependencies was so important that he did not see the need to verify them.

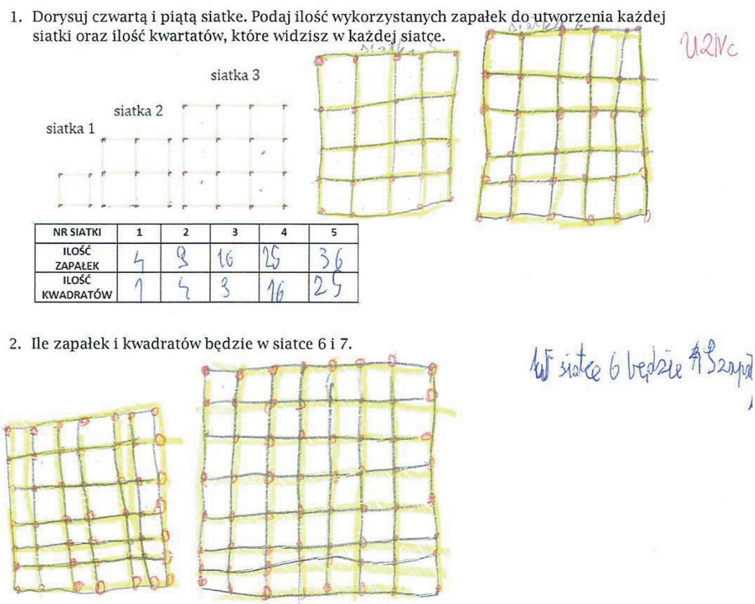


Figure 3: Example of work of student S2.

This approach may indicate that for this student the main goal was to draw more grids. The very way of working on the task – drawing whole squares at once and dividing them into smaller ones – did not give much chance of noticing many dependencies. The series-column system obtained in this way imposed a way of counting squares (a reference to the chocolate multiplication model, a connection with counting the square area). The very understanding of the concept of a square in this situation seems quite narrow: the student treats the square as an independent figure that can be 'filled' with a larger shape.

Example 3

The S3 student used method B in her work – that is, all possible squares. She analyzed the subsequent grids very carefully. She noticed that there are squares of different dimensions in them. She emphasized this fact by outlining squares of the same dimensions with the same colour. In this situation, the total number of squares in a given grid is the sum of the squares of consecutive natural numbers from 1 to the grid number. This discovery was also recorded with the appropriate actions in task 2. Nothing was accidental about this child's work. The girl noticed that subsequent grids are created by drawing one row and one column. She was aware that in order to count the number of matches to be arranged, she did not need to count all the elements on individual grids, but only the 'added' ones. The result obtained in this way is enough to add to the number of matches from the previous grid and in this way, we get the number of matches used to build a given grid.

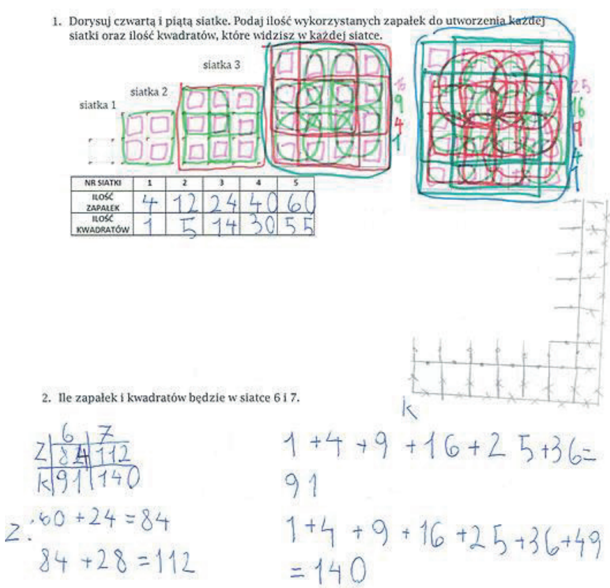


Figure 4: Example of work of student S3.

Confirmation of the use of such a strategy is the drawing made for grid No. 6 and 7. Actually, it is in a sense a ‘complement’ to the drawing of the grid No. 5. The student did not feel the need to draw the entire grid. She drew only those elements that would be added to grid No. 5 to obtain grid No. 6 and No. 7 from it in turn. In addition, the girl wrote down her calculations, thus confirming the strategy which was used by her. This way of working shows that the student thought over the whole solution very carefully. She was able to notice different dependencies in the task. The ability to extract different squares from a given grid indicates that the concept of a square is well developed by her and understood in an abstract level. A square is not a concrete figure, but one that satisfies certain properties. Here it is a quadrangle with the sides of the same length.

SUMMARY

For the students taking part in the study, especially those from the fourth grade, the tasks set before them were a challenge. However, they were able to deal with them quite well. They made an attempt to solve all the tasks from the work card with a great success. Based on the results, we can see that:

- The experience of building subsequent grids resulted into discovering dependencies, allowing to see the relationship between the number of matches and the number of squares in neighbouring grids.
- The students understood the concept of a square differently – for some of them it was an abstract concept, they were able to see different squares in one grid, for others it was a concrete concept, and a square had to be clearly visible and separated.
- Depending on the adopted strategy, students were able to discover different rules. The quickest discovered geometric relationship was the one related to the number of squares in the strategy A or C (that is, that the number of squares is the number of the grid multiplied by itself). This was strongly related to the visual representation of the task.

All students started their work with the concept of a square as a concrete object. Sometimes it was a large square divided into smaller ones (like strategy C), and sometimes small ones arranged into a large square (like strategy A). The visual representation of the task and the focus on the way of drawing subsequent grids could suggest some solutions to the students. Sometimes it was not helpful for them, it even made it difficult to look at the task in a different point of view.

Just counting the number of squares or the number of matches for individual grids did not result in the discovery of the appropriate rule. For this, it was necessary to analyze these results or analyze how to build subsequent grids.

If there was a reflection and critical thinking on the obtained results, there were also attempts to formulate dependencies (as in the case of S1 and S3). The lack of verification of the hypotheses and focusing only on providing the results resulted in the lack of finding appropriate relationships (as in the case of S2). Therefore, it is worth paying attention to the ability to critically approach your actions. Physical experiences also proved to be very important. They definitely made it easier to see the dependencies. This is consistent with Hejny's theory of building up the students' mathematical knowledge (Hejny, 2004).

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AN INVESTIGATION ABOUT THE LINKS OF GEOMETRICAL THINKING WITH SPATIAL ABILITY AND FORMAL REASONING

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In this quantitative study, we investigated whether formal reasoning and the spatial ability of high school students is linked with their level of geometrical thinking. A questionnaire was completed by 203 students attending Grade 9, Grade 10, and Grade 11 in public schools in Greece. The results of the conducted analyses revealed that the students' geometrical thinking was statistically significantly positively correlated with their spatial ability and formal reasoning. Moreover, considering grade of attendance, geometrical thinking and spatial ability statistically significantly differed, while formal reasoning marginally did not statistically significantly differ.

INTRODUCTION

The strong presence of mathematics in the modern curricula seems to be linked with the assumption that learning mathematics is positively linked with various aspects of critical thinking and that it "develops general thinking skills that are useful through life" (Attridge & Inglis, 2016, p. 3). However, it was not until relatively recently that researchers attempted to specify the qualities of the hypothesised positive links (Inglis & Attridge, 2016). Critical thinking refers to both "the ability to reason well and the disposition to do so" (original emphasis; Bailin & Siegel, 2003, p.182). Considering mathematics, critical thinking is linked with the employment of appropriate formal reasoning (both in the generation and the evaluation of a mathematical argument), as well as with mathematical problem solving (Jablonka, 2014). Geometry historically lies at the heart of mathematics, as it was co-developed with arithmetic to facilitate the human societies to cope with everyday difficulties –and beyond– successfully and more efficiently (Moutsios-Rentzos & Spyrou, 2015).

Geometrical problem solving requires the solvers' spatial ability: the ability to "formulate mental images and to manipulate these images in the mind" (Lean & Clements, 1981, p. 267), also referring to "an individual's skill in perceiving fixed geometric/spatial relations and in applying mental transformations such as rotation or reconfiguration to existing spatial relations" (MacLeod et al., 1986, p. 141). Aspects of spatial ability is at work when a solver faces a geometrical problem; to mentally transform a figure, to selectively isolate parts of the figure and the relationships of its parts, to deduce numerical information based on figural relationships and vice versa etc. Duval (2006) notes that "a "geometrical figure" always associates both discursive and visual representations, even if only one of them can be explicitly highlighted according to the mathematical activity that is required" (p. 108). At the same time, geometrical problems require the ability to apply formal reasoning rules to orchestrate the aforementioned complex information and to deductively produce mathematically valid arguments.

In the various educational systems, Geometry has been associated with the introduction of the modern deductive-axiomatic structure of mathematics and with the notion of mathematical proof. In Greece, Geometry, and in particular Euclidean Geometry (partially due to historical, sociocultural reasons), has held a major place in the mathematics curriculum: from pre-school education to high school. The last decade, it appeared that Geometry was not treated in the curriculum as essential as other mathematical topics, which raised the concerns of the Greek mathematicians (including the Hellenic Mathematical Society). Nevertheless, in the latest 2021 comprehensive reform of the mathematics curricula from ages 4 to 18 years old (piloted this academic year and planned to be implemented in the following academic year), the importance of Geometry has been elevated by its being strongly present throughout the school grades; even in the curriculum of the last high school grade (I.E.P., 2021).

Consequently, it is reasonable to pose questions about the broader benefits of learning Geometry, about the nature of geometrical thinking, as well as about its links with other aspects of reasoning. Such questions are scientifically relevant and educationally timely. In this study, we draw upon the van Hiele (1986) theory of geometrical thinking levels, to investigate its development across three different Grades: the last Grade of the Greek Gymnasio (middle school) and the first two Grades of the Greek Lykeio (high school). Considering that spatial ability and formal reasoning are essential in the students' successfully progressing through the different van Hiele levels, we include in our investigations aspects of the students' spatial ability and of their formal reasoning skills.

GEOMETRICAL THINKING, SPATIAL ABILITY AND FORMAL REASONING

The mathematical object, being a mental object, may be communicated through its representations, but it is not identified with any of them; it emerges through the relationships amongst its representations (Duval, 2006). The geometrical object is a mathematical object, characterised by the fact that "in geometry it is necessary to combine the use of at least two representation systems, one for verbal expression of properties or for numerical expression of magnitude and the other for visualization" (Duval, 2006, p. 108). Following these, Moutsios-Rentzos et al. (2014) stress that the complexity of the geometrical object may prove to be particularly challenging for the students, as the co-existence symbolic and visual registers may be linked with different levels of generalisation for each register.

In this study, we acknowledge the complexity and the difficulties the students face when reasoning about geometrical objects and relationships, by explicitly attempting to consider two of the aspects that are at work in geometrical thinking: spatial ability and formal reasoning. For this purpose, we adopt the perspective of Pierre van Hiele and Dina van Hiele-Geldof, who conceptualised geometrical thinking to develop in a hierarchy of five levels (Burger & Shaughnessy, 1986; Hoffer, 1981): 1) Visualization or Recognition; reasoning "primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components" (Burger & Shaughnessy, 1986, p. 31), 2) Analysis; reasoning by employing the properties of the geometrical object, but not the relationships amongst those properties, 3) Abstraction (or Ordering); the student "logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept" (Burger & Shaughnessy, 1986, p. 31), 4) Deduction; reasoning formally within an axiomatic system, 5) Rigor; reasoning about different axiomatic systems and different geometries. It should be added that Clements and Battista (1992) noted that young students show aspects of reasoning of the first level (recognition), which they suggest constitute another level (pre-recognition): students at this level reason based on some of the visual characteristics of a geometrical object, which does not allow them to differentiate between different 'visually-close' objects (e.g., square and parallelogram).

The interplay between the symbolic and the figural, along with the qualities of the reasoning employed are at the heart of the Van Hieles' theory. Hence, it seems reasonable to consider in our investigations those qualities, starting with the students' ability to mentally manipulate geometrical figures, their spatial ability. Spatial ability appears to be challenging to be defined as it is conceptualised to include a variety of factors, which may be differentiated between dynamic and static (Buckley et al., 2018). The main body of the related research seems to focus on the static spatial factors (linked with static stimuli e.g., a drawing), though the dynamic spatial factors (linked with moving stimuli e.g., a moving picture) are also investigated. In this study, we focus on the static spatial factors, which are directly linked with the type of representations mostly employed in middle school and high school Geometry. Some of the spatial factors identified, include McGee's (1979) discussion about spatial visualisation (including the ability to mentally rotate or invert elements; e.g., the Paper-Folding Test) and spatial orientation (including the ability to visualise an object from diverse viewpoint; e.g. the Cube Comparison Test), spatial relations (the ability of mentally rotate two-dimensional objects e.g. various card tests;

Lohman, 1979), spatial factors referring to speed and flexibility (Carroll, 1993) etc.

Considering the need for appropriately applying formal logic rules to produce a valid argument in geometry, we focused on deductive reasoning and, in particular, on conditional inferences: on verbal expressions "If..., then…" (including modus ponens, modus tollens, logical fallacies) and on versions of the Wason Selection Task (e.g., Wason, 1968).

Following these, in this study, we address the following questions:

1) What is the development of geometrical thinking, spatial ability, and formal reasoning as the students progress from Grade 9 to Grade 11?
2) What is the relationship of geometrical thinking levels with spatial ability and formal reasoning?

METHODS AND PROCEDURES

The participants of the study were chosen to be at Grade 9 (the last grade of middle school; Gymnasio), Grade 10 and Grade 11 (the first two grades of high school; Lykeio). Overall, 203 students attending public schools in the region of North-Eastern Greece participated in the study (see Table 1).

	Boys		Girls		Total	
	f	*%*	*f*	*%*	*f*	*%*
Grade 9	38	40.0	40	37,0	78	38.4
Grade 10	29	30.5	35	32,4	64	31.5
Grade 11	28	29.5	33	30,6	61	30.1
Total	95		108		203	

Table 1: The participants of this study.

The rationale of this choice was that at Grade 9 the curriculum devotes two out of the four hours (per week) of mathematics to Geometry and at the same time this grade is the last grade of compulsory education in Greece. In Lykeio, Geometry is for the first time an autonomous course with the same amount of the curriculum time allocated (two hours per week). In Gymnasio, the curriculum includes basic geometric concepts of planar Euclidean Geometry; for example, elements of the triangle and types of triangles, congruency and similarity of triangles etc. The same topics are re-introduced in Lykeio; this time following a Euclidean pseudo-axiomatic format (with definitions, theorems, deductive reasoning, proofs etc).

In this quantitative study, the data were collected through a three-part questionnaire. The first part of the questionnaire focussed on the level of geometrical thinking. We used a version of the questionnaire developed by

Usiskin (1982); translated and adapted for Greek middle school and high school students by Tzifas (2005). For the purposes of the study, we used only the items referring to first four Van Hiele levels. Twenty items (five for each level) assess the students' geometric thinking, with the correct answers scored as follows: 1 point Level 1, 2 points for Level 2, 4 points for Level 3, and 8 points for Level 4. This sums to a maximum of 75 points. The second part identified the spatial ability of the students through nine items: five items investigating spatial relations and in particular the analysis and synthesis of figures (Hidden Figures task and Form Board task, drawing upon Kospentaris, 2011), and four items focussing on mental rotations (drawing upon Vandenberg & Kuse, 1978). Each correct answer was assigned 1 point, summing up to a maximum of 9 points. Finally, the third part investigated the students' reasoning with the conditional inference: verbal expressions (4 items) and Card Selection Task (12 items, drawing upon Moutsios-Rentzos, in preparation; see Figure 1). Each correct answer was assigned 1 point, summing up to a maximum of 12 points.

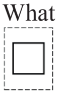
Imagine you're working on the quality control of a toy factory that makes a card game.
Each card in the game has a shape on one side and a color on the other.

The purpose of your work is to ensure that all cards that will enter the game will be subject to the rule:

If a card has a circle painted on one side, then the other side of the card is yellow.

In order to be able to do your work faster, you need to be certain which cards satisfy the rule (in which case they will be put in the game box), so you can *only check the ones you are not sure about*.
The production machine displays four ('4') cards at a time.

What will you do for the card below?



Choose one of the following answers:
A. I know it will get into the box, without turning it over.
B. I know it won't get into the box, without turning it over.
C. I have to turn it to make sure, whether it is put in the box or not.

Figure 1: Sample item of the version of the Selection Task in our study.

The statistical analysis was conducted with SPSS 27, including Pearson correlations and ANOVAs (Games-Howell test for between group comparisons).

RESULTS

In Table 2, we summarise the overall students' geometrical thinking, spatial ability, and formal reasoning, as well as the respective scores for each grade. It should be noted that the students' geometrical thinking was within the range of points of van Hiele Level 3. Their mean score was 26.5, while the points for each level being are respectively 5, 10, 20, 40. Nevertheless, the mean score of all three grades was found to be below 35 (which is the sum of the points of the first three levels), which is in line with the literature (Burger & Shaughnessy, 1986; Usiskin 1982). Moreover, the students appeared to have difficulties in coping with given tasks. The mean score of the students' spatial ability was found to be 4.0 (out of the maximum 9) and of their formal reasoning 5.3 (out of the maximum 12). The students' difficulty was also present when focussing on each grade separately.

	M	*25%*	*50%*	*75%*	*Min*	*Max*
All Grades (N=203)	26.5	16	22.0	33	1	75
Geometrical thinking	26.5	16	22.0	33	1	75
Spatial ability	4.0	2	3.0	6	0	9
Formal reasoning	5.3	3	5.0	7	1	11
Grade 9 (N_{G9}=78)	*M*	*25%*	*50%*	*75%*	*Min*	*Max*
Geometrical thinking	19.1	12	18.0	23	1	71
Spatial ability	3.3	2	3.0	5	0	9
Formal reasoning	4.8	3	4.0	6	1	9
Grade 10 (N_{G10}=64)	*M*	*25%*	*50%*	*75%*	*Min*	*Max*
Geometrical thinking	30.6	16	29.0	45	1	75
Spatial ability	4.6	2	4.0	7	0	9
Formal reasoning	5.5	3	5.0	8	1	10
Grade 11 (N_{G11}=61)	*M*	*25%*	*50%*	*75%*	*Min*	*Max*
Geometrical thinking	31.7	21	29.0	41	2	73
Spatial ability	4.2	2	4.0	6	0	9
Formal reasoning	5.8	3	5.0	8	1	11

Table 2: Geometrical thinking, spatial ability, and formal reasoning.

We investigated the role of grade in the students' scores with three ANOVAs (see Table 3). The conducted analyses revealed statistically differences with respect to grade in the students' geometrical thinking and their spatial ability. The change in their formal reasoning was found to be borderline statistically not significant (P=0.052), though it should be mentioned that the difference in the formal reasoning score between Grade 9 and Grade 11was one correct answer.

	df	*Mean Square*	*F*	*P*
Geometrical thinking	2	3478.585	14.861	<0.001
	200	234.077		
Spatial ability	2	33.138	4.805	0.009
	200	6.897		
Formal reasoning	2	18.577	3.005	0.052
	200	6.181		

Table 3: Geometrical thinking, spatial ability, and formal reasoning: Grades 9 to 11.

The results of the between group comparisons (Games-Howell test) analyses are diagrammatically outlined in Figure 2. It was revealed that for both constructs, the statistically significantly difference was located to Grade 9 and the fact that it was found to be lower than both or one of the other grades. It is hypothesised that this may be related to the way that geometry and mathematics in general is taught in high school in comparison with middle school, and/or an accumulated effect of being mathematically enculturated more time. However, it seems that the additional effect of Grade 11 seems not to be statistically significant.

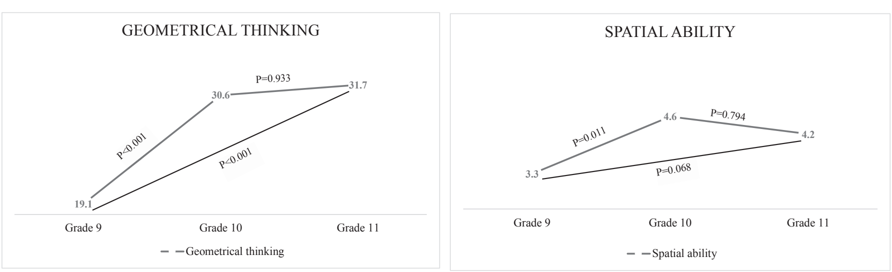


Figure 2: Geometrical thinking and spatial ability: between grades comparisons.

Subsequently, we investigated the relationships amongst the three constructs by calculating their overall correlations. In line with the literature (e.g., Buckley et al., 2018; Xie at al., 2020), all three constructs were found to be statistically significantly positively correlated (see Table 4), which was also evident when focussing on each grade separately.

		Spatial ability	Formal reasoning
Geometrical thinking	*r*	0.593	0.490
	P	<0.001	<0.001

Table 4: Geometrical thinking, spatial ability, and formal reasoning: correlations.

In order to gain deep understanding about these findings, we investigated the links of the scores in the different van Hiele levels with spatial ability and formal reasoning (see Table 5). The results of the conducted analysis showed positive statistically significant correlations for all the van Hiele levels. However, considering the qualitative differences amongst the four levels and the different scores in the three grades, we further pursuit this finding by re-running the analysis for the three grades. It was revealed that in Grade 9, formal reasoning was statistically significantly positively linked only with van Hiele Level 3, whilst no statistically significant correlations were found with the other levels. The students who managed to score comparatively higher in this level were also scoring higher in formal reasoning. It is posited that this may be related to the fact that being successful with the questions at this level requires deductive reasoning skills. Nevertheless, in the following grades, this special link appears to be broadened, which may be due to the fact that the students are scoring higher in all constructs, which results to a broader positive link with their spatial ability and their formal reasoning.

		vH Level 1 Visualisation	*vH Level 2* Analysis	*vH Level 3* Abstraction	*vH Level 4* Deduction
All grades (N=203)					
Spatial ability	*r*	0.528	0.565	0.500	0.438
	P	<0.001	<0.001	<0.001	<0.001
Formal reasoning	*r*	0.300	0.466	0.449	0.360
	P	<0.001	<0.001	<0.001	<0.001
Grade 9 (N_{G9}=78)					
Spatial ability	*r*	0.534	0.383	0.416	0.286
	P	<0.001	0.001	<0.001	0.011
Formal reasoning	*r*	0.220	0.101	0.278	0.204
	P	0.053	0.379	0.014	0.073
Grade 10 (N_{G10}=64)					
Spatial ability	*r*	0.483	0.630	0.529	0.580
	P	<0.001	<0.001	<0.001	<0.001
Formal reasoning	*r*	0.268	0.581	0.570	0.551
	P	0.032	<0.001	<0.001	<0.001
Grade 11 (N_{G11}=61)					
Spatial ability	*r*	0.499	0.544	0.449	0.295

	P	<0.001	<0.001	<0.001	0.021
Formal reasoning	r	0.349	0.528	0.394	0.164
	P	0.006	<0.001	0.002	0.205

Table 5: Van Hiele levels, spatial ability, and formal reasoning.

CONCLUDING REMARKS

In this study, we add to the existing body of research by concurrently investigating the links between geometrical thinking, spatial ability, and formal reasoning. First, we investigated the development of the students' scores for the three constructs across the three grades. The findings suggested that the students were on average on Level 3 ("Abstraction"), whereas their spatial ability and formal reasoning were below or at par with the hypothetical half of the potential maximum. Moreover, it seemed that there was a leap between Grade 9 and Grade 10, whereas the students' advancing to Grade 11 did not seem to statistically significantly add to the identified scores. This may be linked with the fact that, there is a qualitatively significant change in the way that the content is delivered as the students progress to Lykeio (Grade 10): deductive, drawing upon definitions, axioms, theorems etc. Hence, it is posited that it may be this change of delivery, rather than the higher level of mathematical complexity of the content taught in Grade 11, that is linked with the identified differences.

Regarding the links amongst the three constructs, we hypothesised that geometrical thinking would be positively linked with spatial ability and in particular with the lower van Hiele levels, whilst formal reasoning was expected to be linked with higher van Hiele levels. When considering the whole sample, statistically significantly positive links were found amongst spatial ability, formal reasoning, and all the van Hiele levels. When focussing on each grade separately, spatial ability and geometrical thinking remained statistically significantly positively linked. Nevertheless, considering formal reasoning and geometrical thinking within the different grades, a stark difference in the noted pattern was revealed for Grade 9: a specific to Level 3 ("Abstraction") positive link was found, but not in other levels. The students of this grade are in relatively lower geometrical thinking level in this grade (M=19.1; with 15 being the sum of the first two levels), while they answered correctly only one out of the five Level 3 questions (M_{G9}=1.4). Thus, it is hypothesised that for these students being more successful in the task of the more demanding level is strongly linked with being more successful in the formal reasoning tasks. However, we posit that in Lykeio, this peculiarity disappears as the way of delivering and/or an accumulated educational effect appears to reinforce the broad positive links between geometrical thinking, spatial ability and formal reasoning.

Consequently, considering the limitations of the sample, we argue that this study added to our knowledge about the complexity of the links amongst the three constructs, both with respect to the peculiarities of the Greek educational system and beyond. Our current research is focussed on broadening the sample size and its qualitative characteristics (thus, allowing our further investigating specific aspects of the noted links; for example, geometrical thinking levels with specific Wason task items and conditional inference items), whilst qualitative studies employing a systemic perspective are designed to gain deeper understanding about the temporal development of the identified relationships.

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CRITICAL THINKING IN OVERCOMING A FAULTY DECISION-MAKING SYSTEM WHEN SOLVING MATHEMATICAL TASKS

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In this paper we explore the issue of examining and shaping critical thinking in secondary school students. The research was conducted to analyse the students' ability to overcome a fast decision-making system influenced by intuition when solving mathematical tasks. The research tool used tasks geared towards provoking fast and faulty answers. A total of 46 people were interviewed. Preliminary answers were provided to several research questions as a result of the study. The number of respondents who overcame the fast decision-making system increased in each successive stage of the study, despite it dominating the first stage of the study. Task hints provided to the students played a significant role in activating critical thinking. A conclusion is drawn that critical thinking, used to overcome a faulty decision-making system, can be shaped and effectively taught through the use of task hints.

INTRODUCTION

Critical thinking is a key skill in everyday human life and a skill that should be intensively shaped in students, also during mathematics lessons. Our study addresses this issue from a particular perspective, showing one possibility for preparing instruction with secondary school students that is oriented toward developing critical thinking by overcoming the imposition of incorrect and quick responses.

DECISION-MAKING SYSTEM MODELS

System 1 and System 2 in cognitive psychology

Our study refers to the interrelationship between reasoning and intuition in decision-making. Kahneman (2011) presented a model of human cognition based on two modes or 'systems' of thinking: *System 1*(S1) and *System 2*(S2). The author defines it as follows:

> System 1 operates automatically and quickly, with little or no effort and no sense of voluntary control.
>
> System 2 allocates attention to the effortful mental activities that demand it, including complex computations. The operations of System 2 are often associated with the subjective experience of agency, choice, and concentration. (Kahneman, 2011, p. 22)

Leron and Hazzan (2006) stress that these modes operate in different ways, are activated by different parts of the brain, and have different evolutionary origins

(S2 being evolutionarily more recent and, in fact, largely reflecting cultural evolution). In describing how S1 works, Kahneman (2011) points out, among other things, that:

> Several of the mental actions in the list are completely involuntary. You cannot refrain from understanding simple sentences in your own language or from orienting to a loud unexpected sound, nor can you prevent yourself from knowing that 2 + 2 = 4 or from thinking of Paris when the capital of France is mentioned. (Kahneman, 2011, p. 23)

Dual process theory vs. mathematics education

Many researchers are exploring the relationships between the development of cognitive psychology and mathematics education. Leron and Hazzan (2006), for example, described the *dual-process theory* (DPT) for the field of mathematics education and made a comparative summary on intuition vs. analytical thinking in mathematics education, based, among others, on Fischbein (1987) and Vinner (1997), and in psychology. Their analysis was summarized in Figure 1.

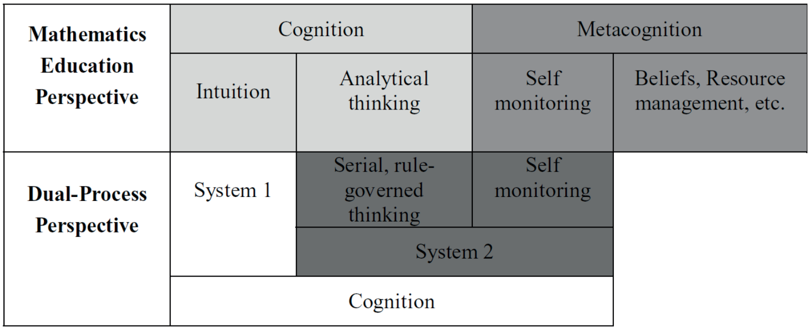

Figure 1: A comparison of terminology between mathematics education and dual-process theory (Leron & Hazzan, 2006, p.112).

They stress the similarity between S1 on the one hand and intuition on the other, which can be seen in the left column of Figure 1.

> Examination of the mathematics education literature shows that the definition and function of intuition in it are also similar to those of S1. Both are characterized by immediacy, high accessibility, automaticity and effortlessness. Also both are considered mostly useful and reliable under normal everyday conditions but are prone to errors under more complex and abstract conditions, especially due to distraction by irrelevant external clues of high accessibility. Here, for example, there are a few quotes from two important books on intuition in mathematics and science education:
>
>> *Intuitive knowledge is immediate knowledge; that is, a form of cognition which seems to present itself to a person as being self-evident.* [...] In all these

> instances, one deals with apparently *immediate* forms of cognition. (Fischbein, 1987, p. 6; italics in the original)
>
> (Leron & Hazzan, 2006, pp. 111-112)

The second similarity the authors consider is between the self-monitoring function of S2 and the same part of metacognition. They stress:

> However, as can be seen from the figure, both S2 and metacognition consist of more that this monitoring, or self-regulating, function, though those additional parts are different in the two frameworks.
>
> The third similarity is in the use of the term "cognition"; however, this is only a partial similarity, which is expressed in the left half of the figure. In the cognitive psychology literature, cognition encompasses all thought processes, indeed all information processing. This includes S1 (including unconscious processes) and S2 (including the monitoring component). In mathematics education the use of "cognition" varies somewhat. Most seem to include intuition within cognition, as the above quotes from Fischbein demonstrate, but compare this with the position of Vinner (1997) who seems to reserve cognition for analytical thinking only. (Leron & Hazzan, 2006, p. 113)

The last statement is described by Vinner (1997) in the following words:

> [. . .] much effort is devoted to find cognitive interpretations for many types of behavior for which, perhaps, a different type of interpretation is more suitable. Furthermore, much didactic effort is invested in 'cognitive corrections' where perhaps a different type of correction would be more effective. By saying this, I am not denying the importance of cognitive research. I am asserting, however, that not every event in a mathematics learning can be explained in cognitive terms, and that it is a fallacy to assume that the cognitive approach is adequate for almost every situation in mathematics learning. (Vinner, 1997, pp. 97-98)

Vinner (1997) defined *pseudo-analytic processes* in which students superficially select elements in the problem and apply a procedure relevant for a typical question due to superficial similarity with previous problems. The pseudo-processes are "simpler, easier, and shorter than the true conceptual processes" (Vinner, 1997, p. 101), thus many students unconsciously apply them.

In our study we used psychological terminology (S1 and S2) because we have been able to resolve whether cognition (analytical thinking) or metacognition (self-monitoring) would account for overcoming the intuition that activates S1.

METHODOLOGY

Aim of the study

The main aim of the study is to find out whether secondary school students, after reading the content of the given task, manage to overcome the action of the intuitive system S1 that suggests an incorrect answer and, as a result, activate the S2; and whether, by giving clues to the tasks, we can teach independent activation of the S2 system in students. More precisely, the aim of the study is to try to acquire preliminary answers to the following questions:

(1) After reading the selected task, does S2 system spontaneously switch on in the students?

(2) Can the analogous tasks affect the responses in stage II of the study when compared with the results from stage I?

(3) Are students able to overcome S1 and activate S2 on their own after being prompted?

Research tool

As a research tool especially prepared tasks were chosen, expecting to provoke quick and incorrect answers, thus activating S1. The tasks did not exceed the level of mathematical requirements of elementary school according to the Polish curriculum. Similar tasks can be found on the internet, where they are called puzzles and it is emphasised that they are at an elementary level, but "most adults will fall for them!".

The questionnaire was divided into three parts. Tasks from the first part of the questionnaire were designed to check which system: intuitive or rational, students switch on spontaneously when solving them under time-limited conditions. Then, in part II of the questionnaire there was a set of analogical tasks, imposing the same solution scheme and provoking the possibility of making an analogical error as in tasks from part I. Analogical tasks, by contrast, were set in a different context. They were designed to test whether students' answers would be consistent and similar to those in Part I. In Part III of the questionnaire, each student received feedback on the correctness of his or her solutions and was given a hint in turn for each incorrectly solved task from Part II. This part was to check whether the hint would be effective and at the same time whether it would activate critical thinking in the students.

Both Part I and Part II consisted of a total of 6 tasks, two analogous tasks each in six categories (A-F). The contents of these tasks are given below.

A. Required activation of reductive reasoning

(I). Task1. The movement speed of a certain object was registered on measuring instruments. It was noted that its speed doubled every two hours. After 64 hours, it reached 640 km/h. How much time elapsed when it reached 160 km/h? (author: W. Błasiak, Research Group of Cognitive Didactics operating at the Pedagogical University of Krakow)

(II). Task 1. The pond is overgrown with duckweeds. Every two days, its area doubles. The whole pond became overgrown in 64 days. How many days passed when ¼ of the pond was overgrown? (Sajka & Rosiek, 2015)

B. An imposed proportion that is not true

(I). Task 2. 5 machines make 5 objects in 5 minutes. How long will it take for 50 such machines to make 50 items? (https://brainly.pl)

(II). Task 2. If five cats eat five mice in five minutes, how many cats does it take to eat one hundred mice in one hundred minutes? (www.matemaks.pl)

C. Use of fractions

(I). Task 3. Eric drinks a barrel of juice in 6 days and George drinks it in 3 days. If they both drink the juice from one barrel - each at his own pace - how long will it take them to empty the barrel? (www.matemaks.pl)

(II). Task 3. Marek mowed a lawn in 3 hours, while Kamil mowed the same lawn in 2 hours (at the same growth rate of grass). How long would it take them to mow the lawn together? (own)

D. Application of a system of equations - differential comparison

(I). Task 4. A baseball bat and a ball cost 101 PLN in total. The bat costs 100 PLN more than the ball. How much does the ball cost? (www.wiemy.to)

(II). Task 5. A hat and a feather cost 110 PLN in total. The hat costs 100 PLN more than the feather. How much does the feather cost? (Pisarski, 2017, p. 6)

E. Use of percentages while imposing an incorrect proportion

(I). Task 6. Before drying, the kiwi weighed 100 grams and was 80% water. After drying, the kiwi contains 50% water. How much does the kiwi weigh after drying? (own)

(II). Task 4. The watermelon weighed 3 kg before drying and contained 99% water. After drying, it contained 98% water. How much did the watermelon weigh after drying? (Pisarski, 2017, p. 7)

F. Stereotypical description (disinformation noise) attached to the data

(I). Task 5. The wedding reception was attended by 200 people. Among the participants there were 10 people who came without an accompanying person. From the group of all the guests, a person with the nickname Andy was drawn. Andy is 40 years old, single, does not like to go out with friends, is a typical homebody. In his free time, he likes reading books. Which is more likely: Andy coming with or without a companion?

(II). Task 6. 10,000 people participated in a study. The participants included 4 men and 9996 women. A random person with the nickname Jo was selected from this group. Jo is 23 years old and graduating from a polytechnic institute. She likes to go out with her friends on Friday nights to listen to loud music and drink beer. Which is more likely: Jo being a man or a woman?

The hints for the tasks in Part II were as follows:

1. How many days did it take for ½ of the pond to be overgrown?
2. How much time does it take for one cat to eat one mouse?

3. How much of the plot does Marek mow in an hour? And how much does Kamil mow?

4. There is a constant proportion of flesh in a watermelon which does not change its volume. What percentage of the watermelon is flesh?

5. What is the total cost of a hat and a feather? Check your solution.

6. Is all the information given in the task relevant? How many women and men participated in the study?

Study procedures

The research involved 46 students from two classes of a secondary school in Krakow, Poland. The students solved the tasks from Parts I and II independently at school during the mathematics lesson in the presence of the mathematics teacher and the researcher, noting the solutions and answers on the printed study sheets (at the end of the school year, shortly after returning from long-term remote learning, in June 2021).

Part I of the survey was conducted during an earlier lesson, 15 minutes before the end of the lesson, while Parts II and III were conducted during a full consecutive lesson of mathematics (45 minutes). This format was kept for organizational reasons, as one lesson might have been too short to conduct all three parts of the survey. We wanted the students to have enough time in solving the tasks in Parts II and III, in which the slow S2 system was to be activated, so that time pressure would not affect the results. Part I of the study, on the other hand, as it was performed in the final minutes of previous lesson, was expected to be all the more rushed due to the short time available, which was in line with the objectives of the study. We assumed activation of the fast decision-making system (S1) in the first approach to solving the tasks.

After solving the tasks from Part II and collecting the results on a separate answer sheet, Phase III of the study began. In order to improve the organization of the test, it was made available individually through a Microsoft Forms form. In this form, questions from stage II were repeated, but in the form of multiple choice tasks, with different answers given, from which the student selected the answer (s)he obtained (among the answers, the ‘other answer’ was included, in case the wrong answers suggested by us did not include all the students’ answers). In this way, students individually checked the correctness of their answers. If the student’s chosen answer was correct, he or she was redirected to the next Stage II task; if the answer was incorrect, the student was given a hint for the task. After reading the hint, the student once again attempted to solve the task and tried to answer the question correctly, thus activating the critical thinking (S2). The student then wrote changes in his solution to the task on the answer sheet (s)he had received earlier. The users of the hints then rated their usefulness on a 5-point scale.

All respondents completed Part III unhurriedly, before the end of the second lesson period.

SELECTED RESULTS

The results overview from each stage of the study are shown in Figure 2.

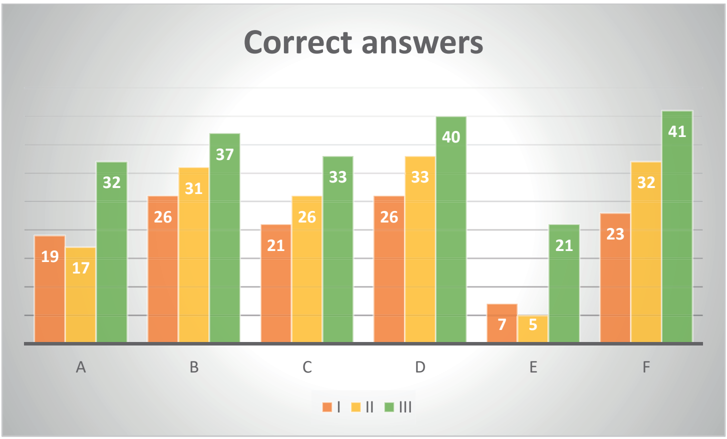

Figure 2: Correct answers (n=46) in the context of the task category (A-F) and the stage of the study (I-III).

The percentage of all correct answers in the entire Part I of the survey is 44%, which is less than half, in Part II it is already above half, 52% to be exact, and in Part III it was 74%.

It can be clearly stated that the results of Part III were by far the best. On the other hand, the results in Part II were better than in Part I in terms of their general correctness, but they differed significantly in individual tasks.

In four types of tasks analogous to those in Part II, students achieved better results than in Part I of the study (types B, C, D, F), but in two types of tasks (A and E) results in Part II were slightly worse

In stage I the best and at the same time the same result was achieved by students in type B and D, obtaining more than half of the correct answers (26 people, 57%). In stage II in both types of tasks the number of correct answers has increased even more, at the same time task II.5 (D) has achieved the best result of part II, namely 33 correct answers (73%).

The task with information noise (type F) turned out to be the task with exactly 50% of correct answers in Part I, and in Part II the analogous task achieved a significantly better result of 32 (70%) correct answers. The hint to this task made this task achieve the best result in Part III in the whole set of tasks - 41 (89%) correct answers.

By far the most difficult tasks were tasks from category E about drying kiwis and watermelon, with the task from Stage II of the study (II.4) being the most difficult in both sets, receiving only 5 correct answers. The hint to this task made its solvability quadruple, to 46%.

The second most significant, nearly doubling of correct answers (from 17 to 32), was the hint to the category A task with the need for reductive reasoning. These were the second most difficult tasks for students in this set.

In the remaining tasks of III stage, the increase in correct answers appeared to be proportional.

The effectiveness of the hints was not fully appreciated in the subjective evaluation of the students who used them (see Table 1). The hint for category D task received the best rating from the students (mean 4.15), but it resulted in an increment of only 7 correct answers, while the hint for category E task resulted in an increment of 16 correct answers and the students rated it average (mean 3.07). The hint to Task A resulted in an increment of 15 correct answers and was also appreciated by students - mean 3.52.

Category task hint	A	B	C	D	E	F
Students' rating	3.52	2.87	2.95	4.15	3.07	3.29

Table 1: Average rating of hints effectiveness (scale 1-5).

RESULTS' ANALYSIS AND DISCUSSION

The analysis of the results obtained from the conducted research allowed to obtain preliminary answers to the posed research questions.

(1) After reading the selected task, does S2 system spontaneously switch on in the students?

We assume that this question can be answered positively only in those situations in which the student gives the correct answer. A detailed answer to this question is therefore provided by Figure 2.

It is worth noting the students' first approach to the research tasks. In stage I, the students achieved an average of 44% correct answers, so the students obtained less than half of the correct answers. However, we can look at the data optimistically and conclude that the students, despite the short working time, in total as many as 122 times independently and spontaneously overcame the imposing S1 of decision-making in a given task and initiated critical thinking. The best answers were given to the tasks in categories B and D, in which the vast majority (26 persons, 57%) coped with giving the correct answer. It can be concluded that these students were able to immediately activate S2 system while working on these tasks. The second stage of the study confirms this ability - in both types of tasks in Part II the number of correct answers increased even more.

Overall, in stage II of the study, students spontaneously and independently beat the imposing S1 144 times, and as many as 204 times in part III of the study.

(2) Can analogous tasks affect responses in stage II of the study when compared with the results from stage I?

Analogue tasks from Part I of the study may have supported the activation of the rational system in the students in the tasks from Part II in the categories B, C, D, F, because in these categories the students obtained better results in the analogue tasks. The second factor of obtaining better results in Part II in these categories could have been the organization of Part II of the study and the awareness of a large amount of time to solve the tasks and the greater concentration of the students at the beginning of the lesson. A factor related to a different task context also cannot be excluded.

(3) Are students able to independently overcome S1 and activate S2 on their own after being prompted?

The answer to this question is positive. Figure 2 shows vividly how the number of correct answers increased after launching the hints (from Part II to Part III), where the number of correct answers increased by a total of 60. The most spectacular role was played by the hints to the most difficult tasks, which at the same time were even more difficult for students than the corresponding tasks in Part I. The solvability of one task increased four times (category E) and the other two times (category A).

SUMMARY

The overall tone of the presented study is positive in our opinion, as the number of respondents who activated S2 increased in each successive stage of the study, despite the fact that in stage I, they were dominated by S1. Recall that this averaged increase is of the order of 44%(I) - 52%(II) - 74%(III). Thus, the solvability of tasks increased by 30 percentage points and reached a satisfactory level of ¾ correct answers.

At this point we would like to emphasize one more fact - before the study was conducted on a group of secondary school students, a preliminary study was conducted on a group of university students of the final year of the teacher's mathematical master's degree. The correct answer to the watermelon task (E) was given by only one student, who commented as follows:

> I have now managed to solve the task correctly, but I think only because I once encountered a similar task in the game 'Wild Logic'. Back then, I didn't have a piece of paper and pencil at my disposal. After all, it was a game, not a school assignment. I gave a quick, seemingly logical answer and it turned out to be wrong. I saw the correct answer, but I didn't understand at the time where it came from. I didn't delve into the topic without the paper and focused on the subsequent puzzles. I now associated this type of task, although I did not know the solution or even the exact content and answer from the game. I knew there was a trap there, but

> I had nothing to compare and use the analogy with. I approached the task with great caution and made a drawing at the beginning and did all the calculations very slowly. After completing the calculations, I was not sure if I was sure everything was correct. After all, I had already made a mistake once during the game. After checking it slowly and carefully three times, I became convinced that everything was correct.

The university student's feedback emphasizes the importance of realizing the error associated with giving a quick and intuitive answer and points out the key role of self-monitoring in achieving success in the task. The lived experience of this situation effectively taught the student to activate S2 in a similar task situation.

This opinion reinforces the most important conclusion of the preliminary research presented in this chapter: Activating S2 can be effectively taught, and one way to do so is to provide effective task hints that activate critical thinking in students.

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